Chapter 5

The Thread Problem

The problem to be studied in this chapter is another generalization of the isoperimetric problem which is related to minimal surfaces. Consider a fixed arc Γ with endpoints P_1 and P_2 connected by a movable arc Σ of fixed length. One may conceive Γ as a thin rigid wire, at the ends of which a thin inextensible thread Σ is fastened. Then the *thread problem* is to determine a minimal surface minimizing area among all surfaces bounded by the boundary configuration $\langle \Gamma, \Sigma \rangle$. The particular feature distinguishing this problem from the ordinary Plateau problem is the movability of the arc Σ.

In Section 5.1 we shall describe several variants of the thread problem, and we shall depict some experimental solutions. Most of these questions have not yet been treated mathematically; that is, no existence proof can be found in the literature. We shall state the mathematical formulation of the thread problem in the simplest case, and in Section 5.2 we shall outline the existence proof given by H.W. Alt for this case. The main difficulty to be overcome is that one can no longer preassign the topological type of the parameter domain on which the desired minimizer will be defined. The regularity of the movable part Σ of the boundary of the area-minimizing surface will be investigated in Section 5.3. The main result is that Σ is a regular real analytic arc of constant curvature.

5.1 Experiments and Examples. Mathematical Formulation of the Simplest Thread Problem

Imagine N points $P_1, P_2, \dots, P_N$ in $\mathbb{R}^3$ which are connected by k fixed arcs $\Gamma_1, \dots, \Gamma_k$ and by l movable arcs $\Sigma_1, \Sigma_2, \dots, \Sigma_l$ in such a way that the resulting configuration $\langle \Gamma, \Sigma \rangle := \langle \Gamma_1, \dots, \Gamma_k, \Sigma_1, \dots, \Sigma_l \rangle$ consists of n disjoint closed curves $C_1, C_2, \dots, C_n$ of finite length. The lengths of the arcs Σ_j are thought to be fixed. Experimentally we can realize the points $P_1, \dots, P_N$ as small holes in a plate or as endpoints of thin rods stuck in a plate. The arcs

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Γ_i are made of thin rigid wires, and the curves Σ_j can be realized by thin and essentially weightless synthetic fibres. Into such a boundary configuration we want to span a surface of minimal area, which can experimentally be achieved by dipping the array into a soap solution and then withdrawing it. This way a soap film will be generated which models a surface of minimal area within the configuration. The following figures show a few such experiments. One obtains particularly attractive and surprising results if all arcs are flexible, and one may very well assume that several of the threads Σ_j form closed loops which, by flexible connections, are attached to the ends of supporting rods. The resulting soap films will often be *multiply connected* minimal surfaces.

We may also conceive boundary configurations consisting of wires Γ_i, of threads Σ_j, and of supporting surfaces $S_1, \dots, S_m$ on which parts of the boundary of the soap film are allowed to move freely.

Still different soap film experiments can be carried out by using threads as supporting ridges. This leads to a kind of mathematical questions which are to be viewed as *obstacle problems* with movable thin obstacles. Apparently such questions have not yet been treated.

We want to mention that thread experiments are used by architects to design light weight structures such as roofs and tents. Beautiful models are depicted in the publications of Frei Otto and collaborators (cf. Otto [1], Glaeser [1]).

Let us now consider the simplest case of a thread problem that was already mentioned in the introduction. Here we want to minimize area among all surfaces spanned in a boundary frame that consists of a fixed rectifiable Jordan arc Γ and of a movable curve Σ of given length L, having the same endpoints P_1 and P_2 as $\Gamma, P_1 \neq P_2$. We note that the thread experiment may lead to solutions which are no longer connected surfaces but disintegrate into several components, even if Γ is a smooth arc. One can even envision boundary configurations $\langle \Gamma, \Sigma \rangle$ for which the solution of the thread problem decomposes into countably many components since the movable arc Σ may in part adhere to the fixed arc Γ. The existence result to be described in the next section will take this phenomenon into account. We shall obtain solutions that are parametrized on a compact connected parameter domain B, the interior $\mathring{B}$ of which consists of at most countably many components.

Let us now specify *the mathematical setting of the thread problem* $\mathcal{P}(\Gamma, L)$ that will be solved in the following section.

Notational Convention. In Sections 5.1 and 5.2 we shall, *deviating from our usual notation, denote a disk of center* z_0 *and radius* r *by* $B(z_0, r)$ *instead of* $B_r(z_0)$.

An *admissible parameter domain for the thread problem* is defined to be a compact set B which can be represented in the form

$$B = [-1, 1] \cup \bigcup_{\nu=1}^{\nu_B} B_\nu, \quad 1 \leq \nu_B \leq \infty. \tag{1}$$

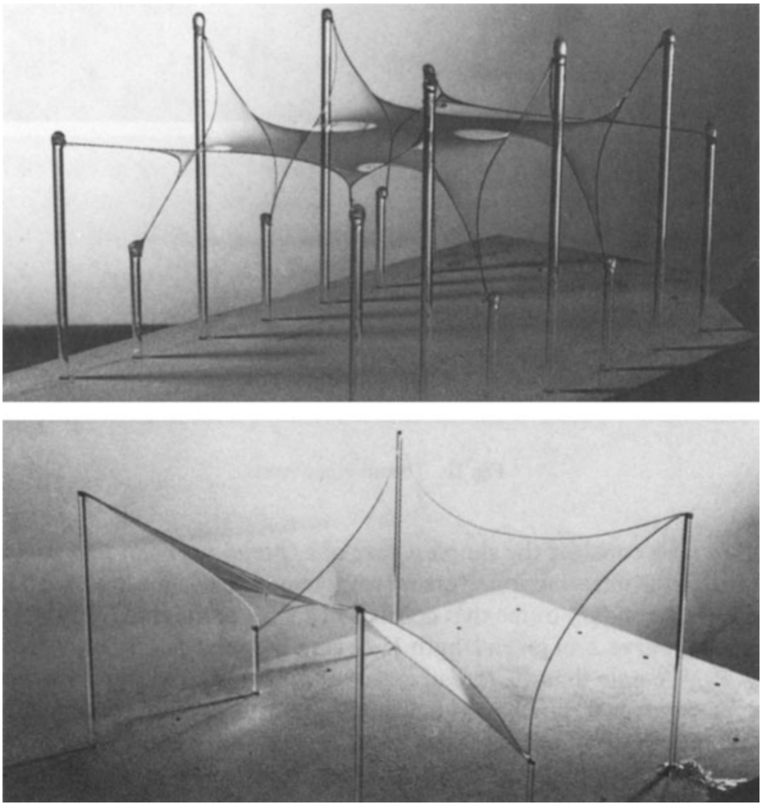

Fig. 1a. Thread experiments. Courtesy of Institut für Leichte Flächentragwerke, Stuttgart – Archive

Here the sets B_ν with $\nu \in \mathbb{N}$ and $\nu \le \nu_B$ denote the closures of mutually disjoint disks $B(u_\nu, r_\nu), r_\nu > 0$ whose centers u_ν are contained in the open interval $\{u\colon -1 < u < 1\}$ on the real axis. Moreover, all disks B_ν are supposed to be contained in the unit disk $B(0,1)$.

Introducing the numbers a_ν and b_ν by

$$a_\nu := u_\nu - r_\nu, \quad b_\nu := u_\nu + r_\nu, \tag{2}$$

we then have

$$a_\nu, b_\nu \in [-1, 1]. \tag{3}$$

Let us denote the set of all admissible parameter domains B by $\mathcal{B}$.

For every $B \in \mathcal{B}$, we introduce the two mappings p_B^+ and $p_B^-\colon [-1,1] \to \partial B$ by

$$p_B^\pm(u) := \begin{cases} u & \text{if } u \in \partial B \cap [-1,1] \\ u \pm i\sqrt{r_\nu^2 - (u-u_\nu)^2} & \text{if } |u - u_\nu| \le r_\nu. \end{cases} \tag{4}$$

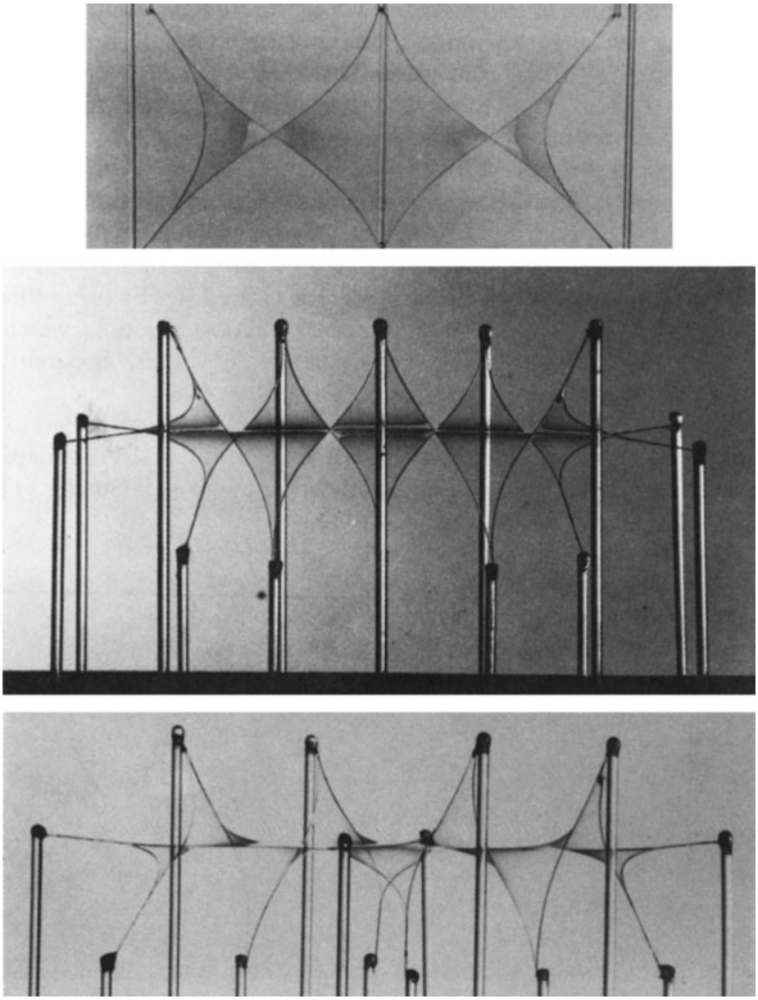

Fig. 1b. Thread experiments. Courtesy of Institut für Leichte Flächentragwerke, Stuttgart – Archive

Let c be a curve mapping a subinterval $I' = [\alpha, \beta]$ of $I = [-1, 1]$ into $\mathbb{R}^3$,

$$c \colon I' \to \mathbb{R}^3.$$

Then the length of c is given by

$$l(c, I') = \sup \sum_{j=1}^{n} |c(t_j) - c(t_{j-1})| \tag{5}$$

where the supremum is to be taken with respect to all possible decompositions $\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$ of I'.

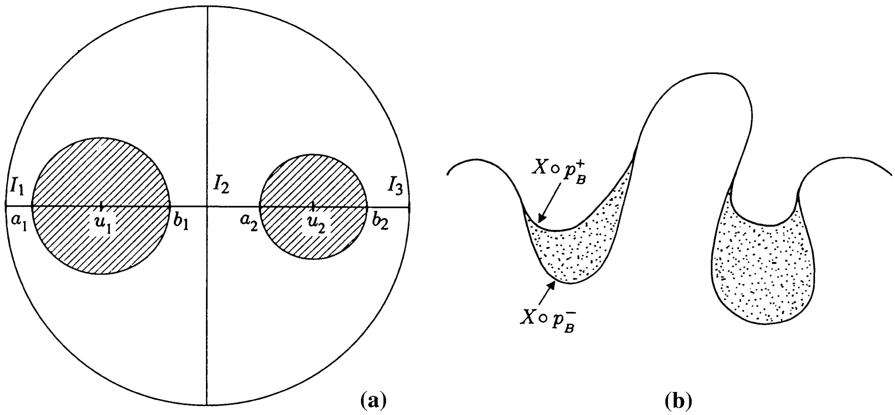

Fig. 2. (**a**) A parameter domain, and (**b**) a corresponding solution to the thread problem consisting of two components

If $I' = I$ we shall write

$$l(c) := l(c, I).$$

For any two intervals I_1 and I_2 in $\mathbb{R}$ we introduce the set $\mathcal{M}(I_1, I_2)$ of continuous, nondecreasing mappings $\theta\colon I_1 \to I_2$ of I_1 onto I_2, and we set $\mathcal{M}(I) := \mathcal{M}(I, I)$.

We observe that the length L of the movable curve Σ is bounded from below by the distance of its endpoints P_1 and P_2,

$$|P_1 - P_2| \le L. \tag{6}$$

Given a rectifiable Jordan curve Γ with endpoints P_1, P_2, and a number L satisfying $0 < |P_1 - P_2| < L$, we are now going to define the *set* $\mathcal{C}(\Gamma, L)$ *of admissible surfaces* X *for the thread problem* as follows:

Definition 1. *The set* $\mathcal{C}(\Gamma, L)$ *consists of the mappings* $X \in C^0(B, \mathbb{R}^3) \cap H_2^1(\mathring{B}, \mathbb{R}^3)$ *with* $B \in \mathcal{B}$ *which satisfy the following two conditions:*

(i) $l(X \circ p_B^+) \le L$;

(ii) *there exists some mapping* $\theta \in \mathcal{M}(I), I = [-1, 1]$, *such that* $\theta|_{\partial B \cap I} = \mathrm{id}|_{\partial B \cap I}$ *and* $X \circ p_B^- = \gamma \circ \theta$ *where* γ *denotes a fixed Lipschitz continuous representation of* Γ *which maps* I *bijectively onto* Γ.

In other words, a function X is admissible if it is parametrized on some domain $B \in \mathcal{B}$, if it is continuous and has a finite Dirichlet integral, if the length of the free part $X \circ p_B^+$ is less or equal to L, and if $X \circ p_B^-$ yields a weakly monotonic parametrization of Γ. Note that Γ and Σ may have one or more interior points in common, that is, Σ may in part adhere to Γ.

The *thread problem* $\mathcal{P}(\Gamma, L)$ now consists in finding some surface $X \in \mathcal{C}(\Gamma, L)$, defined on some parameter domain $B \in \mathcal{B}$, such that X minimizes the Dirichlet integral

$$D(X, \mathring{B}) = \frac{1}{2} \int_{\mathring{B}} |\nabla X|^2 \, du \, dv \tag{7}$$

among all surfaces of $\mathcal{C}(\Gamma, L)$.

The solution of this problem will be carried out in two steps. First we shall single out a set $B \in \mathcal{B}$ which can serve as a parameter domain of a solution of $\mathcal{P}(\Gamma, L)$; this is the nonstandard part of the construction. We shall obtain such domains B as minimal elements with respect to inclusion. In a second step we shall construct a minimizing mapping X parametrized over B.

Let us now introduce the following three infima d, d^+, and d^-:

$$\begin{aligned}
&(8) \quad d &&= d(\Gamma, L) := \inf\{D(X, \mathring{B})\colon X \in \mathcal{C}(\Gamma, L)\};\\
&(9) \quad d^+ &&= d^+(\Gamma, L) := \inf\{D(X)\colon X \in \mathcal{C}(\Gamma, L), B = \overline{B}(0,1)\},\\
& && \qquad \text{where } D(X) := D(X, B(0,1));\\
&(10) \quad d^- &&= d^-(\Gamma, L) := \inf\{\delta\colon \delta \text{ has the approximation property } (\mathcal{A})\}.
\end{aligned}$$

The approximation property $(\mathcal{A})$ is defined as follows: *There exists some decreasing sequence of real numbers $\lambda_n > 0$ with $\lambda_n \to 0$ and a sequence of surfaces $X_n \in \mathcal{C}(\Gamma, L + \lambda_n)$ with parameter domains $B_n \in \mathcal{B}$ such that $D(X_n, \mathring{B}_n) \to \delta$ as $n \to \infty$.*

An obvious consequence of these definitions is the relation

$$d^- \le d \le d^+. \tag{11}$$

We shall prove that

$$d^- = d = d^+$$

holds provided that we assume

$$|P_1 - P_2| < L.$$

In what follows we have to characterize a minimal parameter domain B among all domains in $\mathcal{B}$. To this end it will be convenient to single out a certain subclass $\mathcal{B}^*(\Gamma, L)$ of $\mathcal{B}$ which is defined as follows:

Definition 2. $\mathcal{B}^*(\Gamma, L)$ *is the class of admissible parameter domains $B \in \mathcal{B}$ with the following property: There exists a decreasing sequence of positive numbers λ_n with $\lambda_n \to 0$ and a sequence of surfaces $X_n \in \mathcal{C}(\Gamma, L + \lambda_n)$, parametrized over B, such that $D(X_n, \mathring{B}) \to d^-$ as $n \to \infty$.*

5.2 Existence of Solutions to the Thread Problem

Consider now the particular case $\mathcal{P}(\Gamma, L)$ of the thread problem that was formulated at the end of the previous section. Our main goal is the proof of the following existence result which is formulated as

Theorem 1. *Suppose that* $|P_1 - P_2| < L < l(\Gamma)$*. Then we obtain*

$$d^-(\Gamma, L) = d(\Gamma, L) = d^+(\Gamma, L).$$

Moreover, there exists an admissible parameter domain B *and a surface* $X \in \mathcal{C}(\Gamma, L)$ *parametrized over* B *such that*

$$D(X, \mathring{B}) = d(\Gamma, L).$$

This minimizer X *is a minimal surface, that is,* X *is of class* $C^2(\mathring{B}, \mathbb{R}^3)$ *and satisfies the equations*

$$\Delta X = 0,$$
$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0,$$

in $\mathring{B}$*, and furthermore, the free boundary of* X *is of maximal length, i.e.,*

$$l(X \circ p_B^+) = L.$$

The proof of this theorem is divided into two parts. The first one is concerned with the existence of a minimal parameter domain $B \in \mathcal{B}$. In the second part of our discussion we will show that such a parameter set B is the domain of a solution X for the thread problem $\mathcal{P}(\Gamma, L)$. This will be achieved by establishing the existence of a minimizing sequence $\{X_n\}$ whose elements are defined on B and converge to a solution X of $\mathcal{P}(\Gamma, L)$.

PART I. *Construction of a Minimal Parameter Set* $B \in \mathcal{B}$*.*

We begin our discussion with the following

Lemma 1. *Suppose that* X *is a surface of class* $\mathcal{C}(\Gamma, L)$ *which is defined on* $B \in \mathcal{B}$*, and let* ε *be an arbitrary positive number. Then there exists some* $X_\varepsilon \in \mathcal{C}(\Gamma, L + \varepsilon)$*, parametrized over* $\overline{B}(0,1)$*, such that*

$$|D(X_\varepsilon) - D(X, \mathring{B})| < \varepsilon.$$

(Recall that $D(X_\varepsilon)$ denotes the Dirichlet integral with the unit disk $B(0,1)$ as domain of integration.)

Proof. An admissible domain B is of the form given by formula (1) of Section 5.1. Since $l(\Gamma) < \infty$ and

$$D(X, \mathring{B}) = \sum_{\nu=1}^{\nu_B} D(X, \mathring{B}_\nu) < \infty,$$

we can find a number $\nu_0 \in \mathbb{N}$ such that

$$\sum_{\nu>\nu_0} D(X, \mathring{B}_\nu) < \varepsilon$$

and

$$\sum_{\nu>\nu_0} l(\gamma, [a_\nu, b_\nu]) < \varepsilon.$$

Set

$$B' := I \cup B_1 \cup B_2 \cup \cdots \cup B_{\nu_0}, \quad I = [-1, 1],$$

and

$$X_1(w) := \begin{cases} X(w) & \text{if } w \in B_1 \cup B_2 \cup \cdots \cup B_{\nu_0}, \\ \gamma(w) & \text{if } w \in \partial B' \cap [-1, 1]. \end{cases}$$

Then we infer $X_1 \in \mathcal{C}(\Gamma, L + \varepsilon)$ and

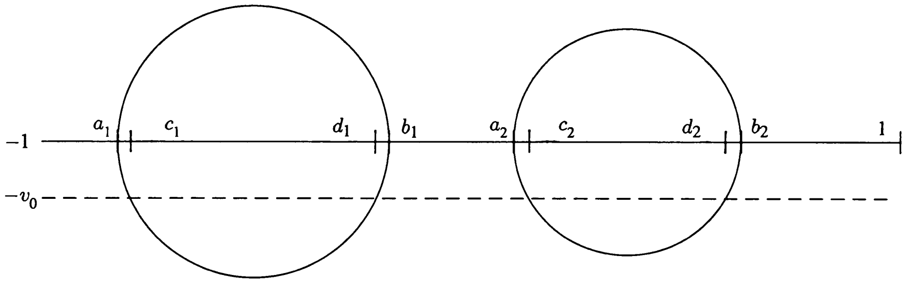

Fig. 1. A parameter domain with $\nu_0 = 2$, and the numbers $a_\nu, c_\nu, d_\nu, b_\nu$

$$(1) \qquad |D(X, \mathring{B}) - D(X_1, \mathring{B}')| \le \varepsilon.$$

For each v_0 with $0 < v_0 < \min\{r_1, r_2, \ldots, r_{\nu_0}\}$, there exist numbers c_ν, d_ν with $a_\nu < c_\nu < d_\nu < b_\nu$ such that $p^-_{B'}(c_\nu) = c_\nu - iv_0$, $p^-_{B'}(d_\nu) = d_\nu - iv_0$; cf. Fig. 1.

Now we choose v_0 so small that also the following conditions are fulfilled:

(i) $X_1(u - iv_0)$ is absolutely continuous with respect to $u \in \bigcup_{\nu \le \nu_0} [c_\nu, d_\nu]$ and has a square integrable first derivative;

(ii) $l(X_1 \circ p^-_{B'}, [a_\nu, c_\nu]) + l(X_1 \circ p^-_{B'}, [d_\nu, b_\nu]) \le \frac{\varepsilon}{2\nu_0}$.

For some arbitrary number $\delta > 0$, we define the set

$$\mathcal{D} = \mathcal{D}(\delta) := \overline{\mathcal{D}}_+ \cup \overline{\mathcal{D}}_- \cup \overline{\mathcal{Q}}$$

by

$$\begin{aligned} \mathcal{Q} &:= \{w = u + iv \colon |u| < 1, 0 < v < \delta\}, \\ \mathcal{D}_+ &:= \{w = u + iv \colon w - i(\delta + v_0) \in \mathring{B}' \text{ and } v > \delta\}, \\ \mathcal{D}_- &:= \{w = u + iv \colon w - iv_0 \in \mathring{B}' \text{ and } v < 0\}. \end{aligned}$$

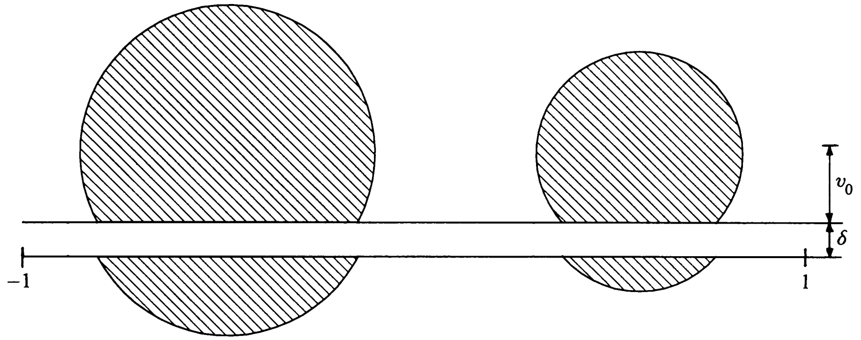

Fig. 2. The domain $\mathcal{D} = \mathcal{D}(\delta)$

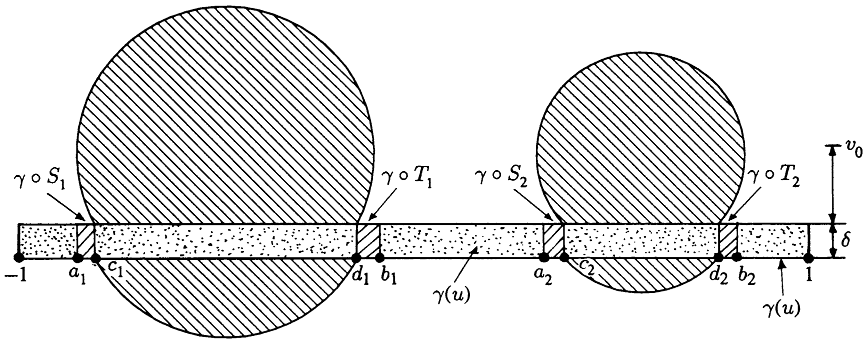

Fig. 3. The definition of X_2

We note that $\mathring{\mathcal{D}}$ is conformally equivalent to the unit disk. Thus, in order to prove the assertion of the lemma, we shall construct a suitable comparison function X_2 defined on $\mathcal{D}$. This function is defined as follows:

$$X_2(w) := \begin{cases} X_1(w - i(\delta + v_0)) & \text{if } w \in \overline{\mathcal{D}}_+, \\ X_1(w - iv_0) & \text{if } w \in \overline{\mathcal{D}}_-. \end{cases}$$

For $0 \le v \le \delta$, we set

$$X_2(w) := \begin{cases} X_1(u - iv_0) & u \in [c_\nu, d_\nu],\ 1 \le \nu \le \nu_0 \\ \gamma(u) & u \in [-1,1] \setminus \bigcup_{\nu \le \nu_0} (a_\nu, b_\nu) \\ & \text{if} \\ \gamma(S_\nu(u)) & u \in [a_\nu, c_\nu] \\ \gamma(T_\nu(u)) & u \in [d_\nu, b_\nu]. \end{cases}$$

Here S_ν is a linear mapping from $[a_\nu, c_\nu]$ onto $[a_\nu, \theta_1(c_\nu)]$, and T_ν is the linear map from $[d_\nu, b_\nu]$ onto $[\theta_1(d_\nu), b_\nu]$, where $\theta_1 \in \mathcal{M}(I)$ is the transformation $I \to I$ that corresponds to X_1. In other words, $\gamma \circ \theta_1 = X_1 \circ p^-_{B'}$.

We infer from the construction that X_2 is of class $C^0(\overline{\mathcal{D}}) \cap H_2^1(\mathring{\mathcal{D}})$. Furthermore, we have

$$D(X_2, \mathring{\mathcal{D}}_+ \cup \mathring{\mathcal{D}}_-) = D(X_1, B') \tag{2}$$

and

$$\begin{aligned} D(X_2, \mathring{\mathcal{Q}}) &\le \frac{1}{2}\int_{\mathcal{Q}} |\nabla X_2|^2 \, du \, dv = \frac{\delta}{2}\int_{-1}^{1} |D_u X_2|^2 \, du \\ &\le \delta \sum_{\nu=1}^{\nu_0} \int_{c_\nu}^{d_\nu} |D_u X_1(u - iv_0)|^2 \, du + \delta \int_{-1}^{1} |\dot{\gamma}(t)|^2 \, dt. \end{aligned} \tag{2'}$$

We, moreover, note that the mapping

$$X_2 \colon \partial\mathcal{D} \cap \{\operatorname{Im} w \le 0\} \to \Gamma$$

is weakly monotonic, and from (ii) we derive the estimate

$$\begin{aligned} &l(X_2, \partial\mathcal{D} \cap \{\operatorname{Im} w > 0\}) \\ &\quad \le l(X_1 \circ p^+_{B'}) + 2\sum_{\nu \le \nu_0} \{l(X_1 \circ p^-_{B'}, [a_\nu, c_\nu]) + l(X \circ p^-_{B'}, [d_\nu, b_\nu])\} \\ &\quad \le l(X_1 \circ p^+_{B'}) + \varepsilon \le L + 2\varepsilon. \end{aligned}$$

Now let $\tau \colon \overline{B}(0,1) \to \mathcal{D}$ be a conformal mapping of $B(0,1)$ onto $\mathring{\mathcal{D}}$, leaving the two points $w = \pm 1$ fixed. Then $X_\varepsilon := X_2 \circ \tau$ is of class $\mathcal{C}(\Gamma, L + 2\varepsilon)$, and we infer

$$\begin{aligned} |D(X_\varepsilon) - D(X, \mathring{B})| &= |D(X_2, \mathring{\mathcal{D}}) - D(X, \mathring{B})| \\ &\le |D(X_2, \mathring{\mathcal{D}}) - D(X_1, \mathring{B}')| + |D(X_1, \mathring{B}') - D(X, \mathring{B})| \\ &\le \delta \cdot \text{const} + \varepsilon, \end{aligned}$$

taking (1), (2) and (2′) into account. Since we can choose $\delta > 0$ arbitrarily small, the assertion of Lemma 1 is proved. □

The following result is an easy consequence of Lemma 1.

Proposition 1. *The class $\mathcal{C}(\Gamma, L)$ is nonvoid, and $\overline{B}(0,1) \in \mathcal{B}^*(\Gamma, L)$.*

Proof. Define

$$\gamma^*(e^{it}) := \begin{cases} \gamma(1) + \frac{t}{\pi}[\gamma(-1) - \gamma(1)] & 0 \le t \le \pi \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} & \\ \gamma\left(-1 + 2\frac{t-\pi}{\pi}\right) & \pi \le t \le 2\pi, \end{cases}$$

where $\gamma(-1) = P_1$ and $\gamma(1) = P_2$. Then $\gamma^* \colon \partial B \to \mathbb{R}^3$ is Lipschitz continuous, and a straight-forward computation shows that $X^*(w) := |w|\gamma^*(\frac{w}{|w|})$ is of class $\mathcal{C}(\Gamma, L)$. Hence $\mathcal{C}(\Gamma, L)$ is nonempty.

It follows from the definition of d^- that there is a sequence of surfaces $X_n \in \mathcal{C}(\Gamma, L+\frac{1}{n})$ parametrized on domains $B_n \in \mathcal{B}$ such that

$$|D(X_n, \mathring{B}_n) - d^-| < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

By virtue of Lemma 1, we can choose a sequence of mappings $X_n^* \in \mathcal{C}(\Gamma, L+\frac{2}{n})$ which are parametrized over $B(0,1)$ and satisfy

$$|D(X_n^*) - D(X_n, \mathring{B}_n)| \le \frac{1}{n}, \quad n = 1, 2, \dots.$$

Thus we infer

$$|D(X_n^*) - d^-| \le \frac{2}{n}, \quad n = 1, 2, \dots,$$

and it follows that $\overline{B}(0,1) \in \mathcal{B}^*(\Gamma, L)$. □

In the next lemma we prove the existence of sets $B \in \mathcal{B}^*(\Gamma, L)$ which are minimal with respect to an ordering of sets defined by inclusion.

Lemma 2. *Suppose that $L < l(\Gamma)$. Then any set of elements $B \in \mathcal{B}^*(\Gamma, L)$ which is totally ordered with respect to inclusion possesses an infimum in $\mathcal{B}^*(\Gamma, L)$.*

Proof. Let $\{B_\alpha^*\}_{\alpha \in A}$ be an arbitrary set of elements $B_\alpha^* \in \mathcal{B}^*(\Gamma, L)$ with the index set A which is totally ordered with respect to inclusion, and set

$$B := \bigcap_{\alpha \in A} B_\alpha^*.$$

We have to show that B is an element of $\mathcal{B}^*(\Gamma, L)$. The first step will be to prove

(i) $$\mathring{B} \neq \emptyset.$$

In fact, if $\mathring{B}$ were empty, we would have

$$I = \operatorname{clos}\left(\bigcup_{\alpha \in A} (I \setminus \mathring{B}_\alpha^*) \right), \quad I := [-1, 1].$$

Then, for any partition

$$-1 = t_0 < t_1 < t_2 < \cdots < t_k = 1$$

of I, there exist numbers $t_j^n \in \bigcup_{\alpha \in A} (I \setminus \mathring{B}_\alpha^*)$ with $0 \le j \le k$ and $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} t_j^n = t_j \quad \text{for } j = 0, 1, \dots, k.$$

Since the set $\{B^*_\alpha\}_{\alpha\in A}$ is totally ordered, we infer that, for every $n \in \mathbb{N}$, there exists an index $\alpha_n \in A$ such that $t^n_j \in I \setminus \mathring{B}^*_{\alpha_n}$ holds for all $j = 0, \dots, k$. As all domains $B^*_{\alpha_n}$ are contained in $\mathcal{B}^*(\Gamma, L), n = 1, 2, \dots$, there exist surfaces $X_n \in \mathcal{C}(\Gamma, L + \frac{1}{n})$ parametrized over $B^*_{\alpha_n}$. This implies

$$\sum_{j=1}^{k} |\gamma(t^n_j) - \gamma(t^n_{j-1})| = \sum_{j=1}^{k} |X_n(t^n_j) - X_n(t^n_{j-1})|$$
$$\leq l(X_n \circ p^+_{B^*_{\alpha_n}}) \leq L + \frac{1}{n} \to L \quad \text{as } n \to \infty.$$

Since

$$\lim_{n\to\infty} \sum_{j=1}^{k} |\gamma(t^n_j) - \gamma(t^n_{j-1})| = \sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})|,$$

we arrive at

$$\sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})| \leq L.$$

As the partition $t_0, t_1, \dots, t_k$ of I may be chosen arbitrarily, we conclude that

$$l(\Gamma) \leq L$$

which contradicts our assumption $l(\Gamma) > L$.

Now we turn to the proof of

(ii) $$B \in \mathcal{B}^*(\Gamma, L).$$

We have to find surfaces defined on B whose Dirichlet integrals converge to d^-, and whose free boundaries (threads) exceed L only by an arbitrarily small amount.

First of all, for every $\varepsilon > 0$ there exists some $\nu_0 \in \mathbb{N}$ with $1 \leq \nu_0 \leq \nu_B$ such that

(3) $$\sum_{\nu\geq\nu_0} l(\gamma, [a_\nu, b_\nu]) \leq \varepsilon.$$

For $\nu \geq \nu_0$ we define

$$Q_\nu := \{w = u + iv \colon a_\nu \leq u \leq b_\nu, |v| \leq \varepsilon 2^{-\nu-1}\}$$

and choose conformal mappings $\tau_\nu \colon \mathring{B}_\nu \to \mathring{Q}_\nu$ from $\mathring{B}_\nu$ onto $\mathring{Q}_\nu$ with fixed points a_ν, b_ν. Here $B_1, B_2, \dots$ denote the components of the domain B (cf. Section 5.1, (1)). Then the surfaces

$$X_\nu := \gamma(\operatorname{Re} \tau_\nu), \quad \nu \geq \nu_0,$$

are continuous and have the Dirichlet integrals

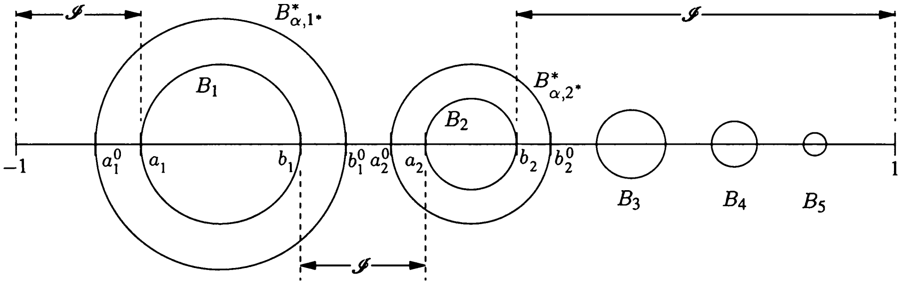

Fig. 4. The case $\nu_0 = 2$

$$(4)\quad D(X_\nu, \overset{\circ}{B}_\nu) = \frac{1}{2}\int_{B_\nu} |\nabla X_\nu|^2\, du\, dv$$
$$= \frac{1}{2}\int_{a_\nu}^{b_\nu}\int_{-\varepsilon 2^{-\nu-1}}^{\varepsilon 2^{-\nu-1}} |\dot\gamma(u)|^2\, du\, dv = \varepsilon 2^{-\nu-1}\int_{a_\nu}^{b_\nu} |\dot\gamma(u)|^2\, du.$$

Moreover, $X_\nu \circ p_B^-$ is monotonic on $[\alpha_\nu, b_\nu]$.

For $\alpha \in A$ and $\nu \in \{1, 2, \ldots, \nu_0\}$ there is a uniquely determined $\nu^* = \nu^*(\nu, \alpha)$ with $1 \le \nu^* \le \nu_{B^*_\alpha}$ such that $B_\nu \subset B^*_{\alpha,\nu^*}$. Here B^*_{α,ν^*} is the ν^*-th component of the domain B^*_α; cf. Section 5.1, (1). Since $\{B^*_\alpha\}_{\alpha\in A}$ is totally ordered, we infer from the definition of B that there is an index $\alpha_0 \in A$ such that the disks $\overset{\circ}{B}{}^*_{\alpha_0,\nu^*}$ are mutually disjoint, and that

$$(5)\qquad \int_{a^0_\nu}^{a_\nu} |\dot\gamma|\, dt + \int_{b_\nu}^{b^0_\nu} |\dot\gamma|\, dt \le \frac{\varepsilon}{\nu_0}$$

holds. Here a_ν and b_ν are the numbers associated with B which are defined in formula (2) of Section 5.1, and a^0_ν, b^0_ν are the corresponding numbers for $B^*_{\alpha_0,\nu^*}$, i.e. $[a^0_\nu, b^0_\nu] := I \cap \overline{B}{}^*_{\alpha_0,\nu^*}$.

We have finitely many (at most $\nu_0 + 1$) open intervals $\mathcal{J} \subset I$ such that

$$(6)\qquad \int_{-1}^{1} |\dot\gamma|\, dt - \sum_{\nu\le\nu_0}\int_{a_\nu}^{b_\nu} |\dot\gamma|\, dt = \sum_{\{\mathcal{J}\}}\int_{\mathcal{J}} |\dot\gamma|\, dt.$$

For any such interval $\mathcal{J}$, there is a partition

$$t_0 < t_1 < t_2 < \cdots < t_k, \quad t_j \in \mathcal{J},$$

such that

$$(7)\qquad \int_{\mathcal{J}} |\dot\gamma| dt \le \frac{\varepsilon}{\nu_0 + 1} + \sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})|.$$

Passing to a suitable refinement of this partition, we may also assume that there is a subset $\mathcal{K}$ of $\{1, 2, \ldots, k\}$ with the following properties:

(I) $j \notin \mathcal{K}$ if and only if $(t_{j-1}, t_j) \subset (a_\nu, b_\nu)$ for some $\nu > \nu_0$;

(II) $j \in \mathcal{K}$ if and only if $[t_{j-1}, t_j] \subset \mathcal{I} \setminus \mathring{B}$.

Moreover we can choose an index $\alpha_0 \in A$ such that the following can be achieved:

The values $t_j \in \mathcal{I} \setminus \mathring{B}$ are replaced by values $t'_j \in \mathcal{I} \setminus \mathring{B}^*_{\alpha_0}$; all other values t_j remain unaltered and will be called t'_j; we have $t'_0 < t'_1 < \cdots < t'_k$ and

(8) $$\sum_{j \in \mathcal{K}} |\gamma(t_j) - \gamma(t_{j-1})| \le \frac{\varepsilon}{\nu_0 + 1} + \sum_{j \in \mathcal{K}} |\gamma(t'_j) - \gamma(t'_{j-1})|.$$

We infer from (3), (7), and (8) that

(9) $$\begin{aligned} \sum_{\{\mathcal{I}\}} \int_{\mathcal{I}} |\dot\gamma|\,dt &\le \varepsilon + \sum_{\nu > \nu_0} \int_{a_\nu}^{b_\nu} |\dot\gamma|\,dt + \sum_{\{\mathcal{I}\}} \sum_{j \in \mathcal{K}(\mathcal{I})} |\gamma(t_j) - \gamma(t_{j-1})| \\ &\le 3\varepsilon + \sum_{\{\mathcal{I}\}} \sum_{j \in \mathcal{K}(\mathcal{I})} |\gamma(t'_j) - \gamma(t'_{j-1})|. \end{aligned}$$

After these preparations, we proceed as follows: Since $B^*_{\alpha_0} \in \mathcal{B}^*(\Gamma, L)$, there is some surface $X \in \mathcal{C}(\Gamma, L + \varepsilon)$ defined on $B^*_{\alpha_0}$ such that

(10) $$D(X, \mathring{B}^*_{a_0}) \le d^- + \varepsilon.$$

Furthermore, for each ν with $1 \le \nu \le \nu_0$, there exists a conformal mapping $\tau_\nu \colon B_\nu \to B^*_{\alpha_0,\nu^*}$ such that $X \circ \tau_\nu \circ p_{B^-_\nu}$ furnishes a monotonic parametrization of that subarc of Γ which corresponds to $[a_\nu, b_\nu]$. Then $X'_\nu := X \circ \tau_\nu$ defines a continuous surface defined on B_ν satisfying

(11) $$D(X'_\nu, \mathring{B}_\nu) = D(X, \mathring{B}^*_{\alpha_0,\nu^*}).$$

By virtue of (5) we obtain

(12) $$l(X'_\nu \circ p^+_B, [a_\nu, b_\nu]) \le l(X \circ p^+_{B^*_{\alpha_0}}, [a^0_\alpha, b^0_\nu]) + \frac{\varepsilon}{\nu_0}.$$

Let us introduce the surface X_ε by

$$X_\varepsilon(w) := \begin{cases} X'_\nu(w) & w \in B_\nu, 1 \le \nu \le \nu_0; \\ & \text{if} \\ \gamma(w) & w \in I \setminus \bigcup_{\nu=1}^{\nu_0} B_\nu. \end{cases}$$

Then we have $X_\varepsilon \in H^1_2(\mathring{B}, \mathbb{R}^3) \cap C^0(B, \mathbb{R}^3)$, and it follows from (4), (10) and (11) that

(13) $$\begin{aligned} D(X_\varepsilon, \mathring{B}) \;&\le\; D(X, \mathring{B}^*_{\alpha_0}) + \sum_{\nu > \nu_0} \varepsilon 2^{-\nu-1} \int_{-1}^{1} |\dot\gamma|^2\,dt \\ &\le\; d^-(\Gamma, L) + \varepsilon + \varepsilon \int_{-1}^{1} |\dot\gamma|^2\,dt. \end{aligned}$$

The length of the movable part of the boundary of X_ε is estimated by

$$
\begin{aligned}
(14) \quad & l(X_\varepsilon \circ p_B^+) \\
&\le l \sum_{\nu \le \nu_0} l(X'_\nu \circ p_B^+, [a_\nu, b_\nu]) + \sum_{\{\mathcal{I}\}} \int_{\mathcal{I}} |\dot{\gamma}|\, dt \\
&\le \varepsilon + \sum_{\nu \le \nu_0} l(X \circ p^+_{B^*_{\alpha_0}}, [a^0_\nu, b^0_\nu]) + 3\varepsilon + \sum_{\{\mathcal{I}\}} \sum_{j \in \mathcal{K}} |\gamma(t'_j) - \gamma(t'_{j-1})| \\
&= 4\varepsilon + \sum_{\nu \le \nu_0} l(X \circ p^+_{B^*_{\alpha_0}}, [a^0_\nu, b^0_\nu]) + \sum_{\{\mathcal{I}\}} \sum_{j \in \mathcal{K}} |X(t'_j) - X(t'_{j-1})| \\
&\le 4\varepsilon + l(X \circ p^+_{B^*_{\alpha_0}}) \le 5\varepsilon + L,
\end{aligned}
$$

on account of (12), (9) and of $X \in \mathcal{C}(\Gamma, L + \varepsilon)$.

The relations (13) and (14) yield $B \in \mathcal{B}^*(\Gamma, L)$. □

Applying Zorn's lemma we infer from this lemma that the following result holds true:

Proposition 2. *The set $\mathcal{B}^*(\Gamma, L)$ possesses minimal elements with respect to inclusion, provided that $L < l(\Gamma)$.*

PART II. *Existence of a Solution of $\mathcal{P}(\Gamma, L)$.*

Let $B \in \mathcal{B}^*(\Gamma, L)$ be a minimal element the existence of which was established in Proposition 2. We want to prove that B is the parameter domain of some minimizer X.

Lemma 3. *If X is a function of class $H_2^1(B(0,1), \mathbb{R}^3)$ with a trace $\xi \in L_2(\partial B(0,1), \mathbb{R}^3)$ on the circle $\partial B(0,1)$ which is of finite total variation $\int_{\partial B(0,1)} |d\xi|$, then the boundary values $\xi \colon \partial B(0,1) \to \mathbb{R}^3$ actually are continuous.*

The *proof* of this result is an immediate consequence of the Courant–Lebesgue lemma and has essentially been carried out in part (iii) of the proof of Proposition 3 in Section 4.7 of Vol. 1. In fact, we even know that ξ is absolutely continuous (see Theorem 1 of Section 4.7 of Vol. 1).

Now we turn to the crucial step in proving Theorem 1, which is to prove

Theorem 2. *Let $B \in \mathcal{B}^*(\Gamma, L)$ be a minimal parameter domain with respect to inclusion. Then there exists some $X \in \mathcal{C}(\Gamma, L)$, parametrized over B, such that*

$$D(X, \mathring{B}) = d^-(\Gamma, L) = d(\Gamma, L);$$

thus X is a solution of the minimum problem $\mathcal{P}(\Gamma, L)$.

Proof. Since $B \in \mathcal{B}^*(\Gamma, L)$, there is a sequence of surfaces $X_n \in \mathcal{C}(\Gamma, L + \frac{1}{n})$, $n \in \mathbb{N}$, satisfying

$$D(X_n, \mathring{B}) \leq d^-(\Gamma, L) + \frac{1}{n} \leq M$$

for some constant $M > 0$. Denote by $\theta_n \in \mathcal{M}(I)$ the mappings associated with X_n, i.e.,

$$\theta_n|_{\partial B \cap I} = \mathrm{id}|_{\partial B \cap I}, \quad X_n \circ p_B^- = \gamma \circ \theta_n.$$

Moreover, let $u_\nu, r_\nu, a_\nu, b_\nu$ be the numbers corresponding to B, and let $B_\nu, 1 \leq \nu \leq \nu_B$, be the components of B (see Section 5.1, (1) and (2)). Applying suitable conformal reparametrizations, we can achieve that

$$\theta_n(u_\nu) = u_\nu \quad \text{for } n \in \mathbb{N} \text{ and } 1 \leq \nu \leq \nu_B.$$

We claim that *the mappings* $\theta_n|_{[a_\nu, b_\nu]}, n \in \mathbb{N}$, *are equicontinuous for every* $\nu \in \mathbb{N}$ *with* $\nu \leq \nu_B$.

Otherwise we could find some $\varepsilon_0 > 0$, some $\nu \leq \nu_B$ and two sequences $\{t_n\}, \{t'_n\}$ with $a_\nu \leq t_n < t'_n \leq b_\nu$, converging to some point $t_0 \in [a_\nu, b_\nu]$, such that

$$|\theta_n(t_n) - \theta_n(t'_n)| \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N} \tag{15}$$

is satisfied. (Actually, this would hold true for some subsequence of $\{\theta_n\}$. However, by renumbering this subsequence we could achieve that (15) is fulfilled.) We want to show that (15) leads to a contradiction. In order to do so, we distinguish the two cases (i) $a_\nu < t_0 < b_\nu$, and (ii) $t_0 = a_\nu$ or b_ν. Case (i) can be excluded by the discussion given in Chapter 4 of Vol. 1, where we have proved that the boundary values of a minimizing sequence for the ordinary Plateau problem are equicontinuous. By this reasoning we obtain that the functions $\gamma \circ \theta_n|_{[c_\nu, d_\nu]}$ are equicontinuous for every interval $[c_\nu, d_\nu] \subset (a_\nu, b_\nu)$. The injectivity of γ then implies that also the functions $\theta_n|_{[c_\nu, d_\nu]}$ are equicontinuous, which contradicts (15). In fact, there is some $n_0 \in \mathbb{N}$ such that

$$a_\nu < c_\nu \leq t_n < t'_n \leq d_\nu < b_\nu$$

holds for $n > n_0$ and for suitably chosen numbers c_ν and d_ν. Then it follows from (15) that

$$|\gamma(\theta_n(t_n)) - \gamma(\theta_n(t'_n))| \geq c(\varepsilon_0) > 0$$

for some fixed number $c(\varepsilon_0) > 0$ and for all $n > n_0$, which contradicts the equicontinuity of the sequence $\gamma \circ \theta_n|_{[c_\nu, d_\nu]}$. Thus case (i) cannot occur.

Now we want to exclude case (ii) as well.

It suffices to show that $t_0 = a_\nu$ is impossible since the case $t_0 = b_\nu$ can be handled analogously. Thus let us assume that $t_0 = a_\nu$.

We can choose sequences of numbers δ_n, r_n, and s'_n with $\delta_n \in (0,1), \delta_n \to 0, 0 < r_n < \delta_n, t'_n < s'_n \leq u_\nu, p_B^-(s'_n) \in \partial B(a_\nu, r_n)$, and with

$$\left\{ \int \left| \frac{\partial}{\partial \varphi} X_n(r_n, \varphi) \right| d\varphi \right\}^2 \le \frac{2\pi M}{\log 1/\delta_n}.$$

Here r, φ denote polar coordinates around a_ν, and the integral on the left-hand side is extended over the φ-interval in $[-\pi, \pi]$ corresponding to the arc in $B \cap \partial B(a_\nu, r_n)$ which contains $\varphi = 0$; cf. Section 4.4 of Vol. 1, Lemma 1.

There is a subsequence of $\{\theta_n(s'_n)\}$ converging to some value u_0; renumbering this sequence we may assume that $\theta_n(s'_n) \to u_0$ as $n \to \infty$. By virtue of (15) we have $u_0 \ge a_\nu + \varepsilon_0$.

Choose values s_n with $a_\nu \le s_n \le u_\nu$ and $\theta_n(s_n) = u_0$, and consider the two closed disks $\mathcal{D}_1$ and $\mathcal{D}_2$ defined by

$$\mathring{\mathcal{D}}_1 := B\left(\frac{a_\nu + u_0}{2}, \frac{u_0 - a_\nu}{2}\right), \quad \mathring{\mathcal{D}}_2 := B\left(\frac{u_0 + b_\nu}{2}, \frac{b_\nu - u_0}{2}\right).$$

Our aim is to define surfaces Y_n on $\mathcal{D}_1 \cup \mathcal{D}_2$ such that the surfaces $X_n^*\colon B^* \to \mathbb{R}^3$, given by

$$X_n^*(w) := \begin{cases} X_n(w) & \quad w \in B \setminus B_\nu, \\ & \text{for} \\ Y_n(w) & \quad w \in \mathcal{D}_1 \cup \mathcal{D}_2, \end{cases} \tag{16}$$

$$B^* := (B \setminus B_\nu) \cup \mathcal{D}_1 \cup \mathcal{D}_2 \in \mathcal{B},$$

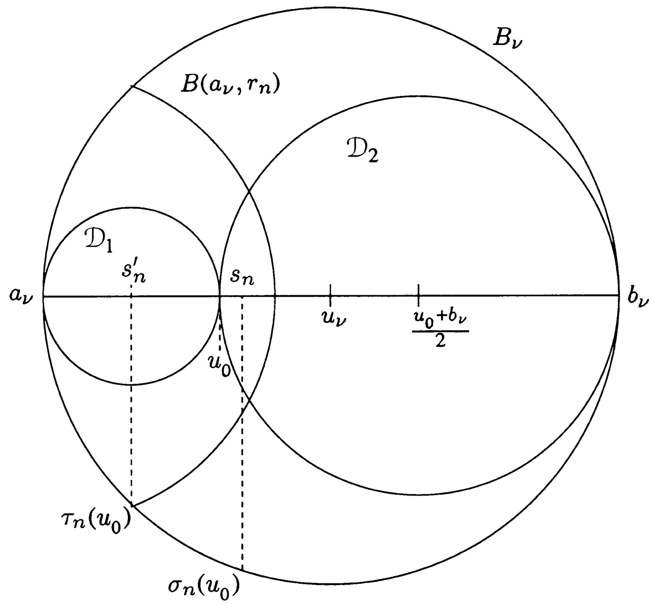

Fig. 5. The disks $\mathcal{D}_1, \mathcal{D}_2$, and B_ν

are of class $\mathcal{C}(\Gamma, L + \lambda_n)$ with $\lambda_n \to 0$ and satisfy

$$D(X_n, \mathring{B}^*) \to d^-(\Gamma, L)$$

as $n \to \infty$. This, clearly, would contradict the minimality of B, and therefore we would also have ruled out $t_0 = a_\nu$ (or b_ν), i.e. case (ii) cannot occur either.

Passing to a subsequence and then renumbering, we can achieve that either

$$s'_n \le s_n \quad \text{for all } n \in \mathbb{N}$$

or else

$$s_n \le s'_n \quad \text{for all } n \in \mathbb{N}$$

holds true. We only treat the first case; the second one can be dealt with in an analogous way.

Consider topological mappings

$$\tau_n \colon \mathcal{D}_1 \to \overline{B}(a_\nu, r_n) \cap B_\nu, \quad \sigma_n \colon \mathcal{D}_2 \to B_\nu \setminus B(a_\nu, r_n)$$

with $\tau_n(a_\nu) = a_\nu, \tau_n(u_0) = p_B^-(s'_n), \sigma_n(b_\nu) = b_\nu, \sigma_n(u_0) = p_B^-(s_n)$ such that $\mathring{\mathcal{D}}_1$ is conformally mapped onto $B(a_\nu, r_n) \cap \mathring{B}_\nu$ by τ_n, and that σ_n maps $\mathring{\mathcal{D}}_2$ conformally onto $\mathring{B}_\nu \setminus \overline{B}(a_\nu, r_n)$.

Note that

$$(X_n \circ \tau_n)(u_0) = X_n(p_B^-(s'_n)) = \gamma(\theta_n(s'_n)) \to \gamma(u_0),$$
$$(X_n \circ \sigma_n)(u_0) = X_n(p_B^-(s_n)) = \gamma(\theta_n(s_n)) \to \gamma(u_0).$$

If we had $(X_n \circ \tau_n)(u_0) = \gamma(u_0)$, we would simply define

$$Y_n := \begin{cases} X_n \circ \tau_n & \text{in } \mathcal{D}_1, \\ X_n \circ \sigma_n & \text{in } \mathcal{D}_2, \end{cases}$$

and the proof would be complete. As we only know $X_n \circ \tau_n(u_0) \to \gamma(u_0)$ as $n \to \infty$, we have to adjust the data correctly. The idea is the same as in the proof of Lemma 1: we have to fill in the missing parts of Γ, thereby slightly changing the Dirichlet integral and the length of the free boundary of X_n. This way we obtain from $X_n \circ \tau_n \colon \mathcal{D}_1 \to \mathbb{R}^3$ a new surface $(X_n \circ \tau_n)_{\delta_n} =: Z_n$ with $Z_n(u_0) = \gamma(u_0)$ such that

$$Y_n(w) := \begin{cases} Z_n(w) & \text{for } w \in \mathcal{D}_1, \\ X_n \circ \sigma_n(w) & \text{for } w \in \mathcal{D}_2 \end{cases}$$

satisfies both

$$D(Y_n, \mathring{\mathcal{D}}_1 \cup \mathring{\mathcal{D}}_2) \le D(X_n, \mathring{B}_\nu) + \delta_n$$

and

$$
\begin{aligned}
&l(Y_n \circ p^+_{\mathcal{D}_1 \cup \mathcal{D}_2}, [a_\nu, b_\nu]) \\
&\quad \le l(X_n \circ p^+_B, [a_\nu, b_\nu]) + \delta_n + 2 \int \left| \frac{\partial}{\partial \varphi} X_n(r_n, \varphi) \right| d\varphi \\
&\qquad + 2l(\gamma, [\theta_n(s'_n), \theta_n(s_n)]) \\
&\quad = l(X_n \circ p^+_B, [a_\nu, b_\nu]) + \lambda_n, \quad \lim_{n\to\infty} \lambda_n = 0,
\end{aligned}
$$

and that the surfaces $X^*_n\colon B^* \to \mathbb{R}^3$ defined by (16) are of class $\mathcal{C}(\Gamma, L + \lambda_n)$, $\lambda_n \to 0$. This finishes the proof of equicontinuity of the mappings $\theta_n|_{[a_\nu, b_\nu]}$, $n \in \mathbb{N}$, for every $\nu \in \mathbb{N}$ with $1 \le \nu \le \nu_B$.

Now we can apply the reasoning of Chapter 4 of Vol. 1 to the sequence $\{X_n\}$ of surfaces $X_n \in \mathcal{C}(\Gamma, L + \frac{1}{n})$ which are defined on the minimal parameter domain $B \in \mathcal{B}^*(\Gamma, L)$ and satisfy

$$
D(X_n, \mathring{B}) \le d^-(\Gamma, L) + \frac{1}{n} \le M \quad \text{for all } n \in \mathbb{N}. \tag{17}
$$

From this inequality, together with

$$
\sup_{\partial B} |X_n| \le M' \quad \text{for all } n \in \mathbb{N}
$$

and some constant M' independent of n, we obtain that $\{X_n\}$ is a bounded sequence in $H^1_2(\mathring{B}, \mathbb{R}^3)$.

Passing to a suitable subsequence of X_n and renumbering it, we can assume that the sequence $\{X_n\}$ tends weakly in $H^1_2(\mathring{B}, \mathbb{R}^3)$ to some limit $X \in H^1_2(\mathring{B}, \mathbb{R}^3)$ such that X_n tends a.e. and also in the L_2-sense on every boundary ∂B_ν to the trace of X. By virtue of the equicontinuity result proved above we can assume that the mappings $\theta_n \in \mathcal{M}(I)$ associated with X_n tend uniformly on I to some limit $\theta \in \mathcal{M}(I)$ such that the relations

$$
\theta|_{\partial B \cap I} = \mathrm{id}|_{\partial B \cap I}, \quad \xi \circ p^-_B = \gamma \circ \theta
$$

hold true for some continuous, weakly monotonic mapping ξ from $p^-_B(I)$ onto Γ, with the property that ξ and X coincide a.e. on $p^-_B(I) \setminus I$. Thus we can use ξ to define X on $p^-_B(I)$ by setting $X(w) := \xi(w)$ for $w \in p^-_B(I)$, and we have $X \circ p^-_B = \gamma \circ \theta$.

Moreover, on account of Helly's selection theorem[1] and of the assumption $l(X_n \circ p^+_B) \le L + \frac{1}{n}$, we can assume that $X_n \circ p^+_B$ tends to $X \circ p^+_B$ everywhere on I, and that $l(X \circ p^+_B) \le L$.

By Lemma 3 we conclude that X has continuous boundary values on every ∂B_ν, and consequently X is continuous on ∂B.

Recall that Dirichlet's integral is weakly lower semicontinuous on $H^1_2(\mathring{B}, \mathbb{R}^3)$, that is, the weak convergence of X_n to X implies

$$
D(X, \mathring{B}) \le \liminf_{n\to\infty} D(X_n, \mathring{B}).
$$

[1] Cf. for instance Natanson [1], p. 250.

Then, by (17), we arrive at

$$D(X, \mathring{B}) \le d^-(\Gamma, L).$$

Consider now mappings $H_\nu \in C^0(B_\nu, \mathbb{R}^3) \cap H_2^1(\mathring{B}, \mathbb{R}^3)$ which are harmonic in B_ν and coincide with X on ∂B_ν. Then we also have

$$D(H_\nu, \mathring{B}_\nu) \le D(X, \mathring{B}_\nu).$$

Set

$$X^* := \begin{cases} X & \text{on } B \setminus \mathring{B}, \\ H_\nu & \text{on } B_\nu,\ \nu \in \mathbb{N}, 1 \le \nu \le \nu_B. \end{cases}$$

The surface X^* is of class $C^0(B, \mathbb{R}^3) \cap H_2^1(\mathring{B}, \mathbb{R}^3)$ and satisfies

$$D(X^*, \mathring{B}) \le D(X, \mathring{B}),$$

$$l(X^* \circ p_B^+) = l(X \circ p_B^+) \le L$$

and

$$X^* \circ p_B^- = \gamma \circ \theta, \quad \theta|_{\partial B \cap I} = \mathrm{id}|_{\partial B \cap I}.$$

Consequently we have $X^* \in \mathcal{C}(\Gamma, L)$, whence

$$d(\Gamma, L) \le D(X^*, \mathring{B}).$$

Thus we obtain

$$d^-(\Gamma, L) \le d(\Gamma, L) \le D(X^*, \mathring{B}) \le D(X, \mathring{B}) \le d^-(\Gamma, L),$$

and therefore

$$d(\Gamma, L) = d^-(\Gamma, L) = D(X^*, \mathring{B}) = D(X, \mathring{B})$$

which implies that $X = X^*$ holds, and that X is a solution of $\mathcal{P}(\Gamma, L)$. □

Theorem 3. *Suppose that $L < l(\Gamma)$. If $X \in \mathcal{C}(\Gamma, L)$ satisfies $D(X, \mathring{B}) = d(\Gamma, L)$, then X is a minimal surface, that is, X is nonconstant, the equations*

$$\begin{aligned} \Delta X &= 0, \\ |X_u|^2 &= |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \end{aligned}$$

are satisfied in $\mathring{B}$, and it follows that

$$l(X \circ p_B^+) = L.$$

(That is, for any solution of $\mathcal{P}(\Gamma, L)$, the movable part of the boundary is taut.)

Proof. The minimal-surface property of X can be derived as in Chapter 4 of Vol. 1, since each of the mappings $X|_{B_\nu}$ solves a Plateau problem with respect to the boundary curve $\Gamma_\nu := X(\partial B_\nu)$. (Here B_ν denotes the disk-components of the parameter domain B of X.) Thus we only have to prove

$$l(X \circ p_B^+) = L.$$

Suppose that this inequality were not true. Then, because of $X \in \mathcal{C}(\Gamma, L)$, we would have

$$l(X \circ p_B^+) < L.$$

We recall that $\mathring{B}_1 = B(u_1, r_1)$, and we set $w_0 := u_1 + ir_1$. Then we can find some $r_0 \in (0, r_1)$ such that

$$l(X \circ p_B^+) + l(X, B_1 \cap \partial B(w_0, r_0)) < L.$$

Let τ be a topological mapping of $B_1 \setminus B(w_0, r_0)$ onto B_1 with $\tau(a_1) = a_1$ and $\tau(b_1) = b_1$ that maps the interior of B_1 conformally onto $B_1 \setminus \overline{B}(w_0, r_0)$. We use τ to define the comparison map $X^* \in \mathcal{C}(\Gamma, L)$ by defining

$$X^*(w) := \begin{cases} X(\tau(w)) & \\ & \text{for} \\ X(w) & \end{cases} \quad \begin{matrix} w \in B_1, \\ \\ w \in B \setminus B_1. \end{matrix}$$

Then it follows that

$$d(\Gamma, L) \le D(X^*, \mathring{B}),$$

and because of

$$\begin{aligned} D(X^*, \mathring{B}) &= D(X, \mathring{B}) - D(X, \mathring{B}_1 \cap B(w_0, r_0)) \\ &= d(\Gamma, L) - D(X, \mathring{B}_1 \cap B(w_0, r_0)) \end{aligned}$$

we infer that $X|_{B(w_0, r_0)} = \text{const}$, whence $X|_{B_1} = \text{const}$, as $X|_{B_1}$ is harmonic and therefore real analytic. The relation $X|_{B_1} = \text{const}$ is a contradiction to $X(a_1) \ne X(b_1)$. □

Proposition 3. *If* $|P_1 - P_2| < L$ *then it follows that*

$$d^-(\Gamma, L) = d^+(\Gamma, L).$$

Proof. Case (i). Suppose that $L \ge l(\Gamma)$. Then we define the surface $Z\colon Q \to \mathbb{R}^3$ on $Q = \{u + iv\colon |u| \le 1, |v| \le \delta\}$ by setting $Z(u + iv) := \gamma(u)$. It follows that

$$D(Z, \mathring{Q}) = \delta \int_{-1}^{1} |\dot{\gamma}(u)|^2 \, du.$$

Consider a homeomorphism of $\overline{B}(0,1)$ onto Q which maps $B(0,1)$ conformally onto $\mathring{Q}$. Then $X := Z \circ \tau$ is of class $\mathcal{C}(\Gamma, L)$, and we have

$$d^+(\Gamma, L) \le D(X, B(0,1)) = D(Z, \mathring{Q}) = \delta \int_{-1}^{1} |\dot\gamma(u)|^2 \, du.$$

As we can make $\delta > 0$ arbitrarily small, it follows that $d^+(\Gamma, L) = 0$ whence

$$d(\Gamma, L) = d^-(\Gamma, L) = d^+(\Gamma, L) = 0.$$

Case (ii). Assume now that $l(\Gamma) > L$. By Theorem 2, there is some $X \in \mathcal{C}(\Gamma, L)$ such that

$$d^-(\Gamma, L) = d(\Gamma, L) = D(X, \mathring{B}),$$

where B is the parameter domain of X.

For given $\varepsilon > 0$ there exists some surface

$$X_\varepsilon \in \mathcal{C}(\Gamma, L + \varepsilon^2) \cap H_2^1(B(0,1), \mathbb{R}^3)$$

with

$$D(X_\varepsilon, B(0,1)) \le D(X, \mathring{B}) + \varepsilon = d^-(\Gamma, L) + \varepsilon,$$

if we take Lemma 1 into account.

Consider now the surface $X^* \in \mathcal{C}(\Gamma, |P_1 - P_2|) \cap H_2^1(B(0,1), \mathbb{R}^3)$ which was constructed in the proof of Proposition 1. We define the 1-parameter family of surfaces

$$X_\varepsilon^* := \varepsilon X^* + (1-\varepsilon) X_\varepsilon, \quad 0 < \varepsilon \le L - |P_1 - P_2|.$$

Then we infer $X_\varepsilon^* \in \mathcal{C}(\Gamma, L_\varepsilon)$ where L_ε is estimated by

$$\begin{aligned} L_\varepsilon &\le \varepsilon |P_1 - P_2| + (1-\varepsilon)(L + \varepsilon^2) \\ &= \varepsilon |P_1 - P_2| + L + \varepsilon^2 - \varepsilon L - \varepsilon^3 \le L - \varepsilon^3 < L. \end{aligned}$$

It follows that

$$d^+(\Gamma, L) \le D(X_\varepsilon^*) \quad \text{for } 0 < \varepsilon \le L - |P_1 - P_2|.$$

Furthermore we have

$$\begin{aligned} D(X_\varepsilon^*) &= \varepsilon^2 D(X^*) + \varepsilon(1-\varepsilon) \int_B \langle \nabla X^*, \nabla X_\varepsilon \rangle \, du \, dv + (1-\varepsilon)^2 D(X_\varepsilon) \\ &\le d^-(\Gamma, L) + \varepsilon K \end{aligned}$$

for some number $K > 0$ which does not depend on ε with

$$0 < \varepsilon \le L - |P_1 - P_2|.$$

Letting $\varepsilon \to +0$, we arrive at the inequality

$$d^+(\Gamma, L) \le d^-(\Gamma, L).$$

On the other hand, we have

$$d^-(\Gamma, L) \le d(\Gamma, L) \le d^+(\Gamma, L)$$

whence

$$d^-(\Gamma, L) = d(\Gamma, L) = d^+(\Gamma, L).$$

□

Theorem 4. *If X minimizes the Dirichlet integral $D(X,B)$ in the class $\mathcal{C}(\Gamma,L)$, then X also furnishes the minimum of the area functional $A(X,B)$ with $\mathcal{C}(\Gamma,L)$.*

Proof. This result can be derived from Morrey's lemma on ε-conformal mappings that we have described in Section 4.5 of Vol. 1. One can proceed in the same way as in the proof of Theorem 4 in Section 4.5 of Vol. 1. The proof can also be obtained by the method described in Section 4.10 of Vol. 1. □

5.3 Analyticity of the Movable Boundary

In this section we want to investigate the regularity of the movable part Σ of a solution X of the thread problem. Let us begin by considering a special case. We assume that Γ is a planar curve. By a projection argument it can easily be seen that X has to be contained in the plane E determined by Γ. In fact, if we assume without loss of generality that E is the plane $\{z=0\}$, and that $X(w)=(x(w),y(w),z(w))$ is a solution of $\mathcal{P}(\Gamma,L)$, then also $X^*(w):=(x(w),y(w),0)$ is a surface of class $\mathcal{C}(\Gamma,L)$, and we have

$$D(X^*,\mathring{B})\le D(X,\mathring{B}).$$

The equality sign holds if and only if $D(z,\mathring{B})=0$, and $D(z,\mathring{B})$ vanishes if and only if $z(w)=0$ holds for all $w\in\bigcup_{\nu=1}^{\nu_B}B_\nu$. As X is an absolute minimizer for the thread problem, there cannot be any surface in $\mathcal{C}(\Gamma,L)$ with a Dirichlet integral smaller than $D(X,\mathring{B})$. Thus we infer that $z(w)=0$ on $\mathring{B}$. Since $z\in H_2^1(\mathring{B})$, we also have $z(w)=0$ a.e. on $B\setminus I$. Finally, on $B\setminus\bigcup_{\nu=1}^{\nu_0}B_\nu$ the function $z(w)$ coincides with the z-component of Γ so that $z(w)$ vanishes identically on I and therefore on all of B.

Thus, X is in fact a planar surface, and by a classical result of analysis, every part of the movable curve Σ not attached to Γ must be a circular arc, that is, a regular real analytic curve of constant curvature.

It is the aim of this section to show that the same result holds true for any solution X of $\mathcal{P}(\Gamma,L)$, even if Γ is not a planar curve. As by-product of our investigation we shall also obtain that all free (i.e. nonattached) parts of Σ are asymptotic curves of constant geodesic curvature on X, and it can be proved that the curvature is the same for all free parts of Σ.

Clearly we can restrict our discussion of X to any part $X|_{B_\nu}$ where B_ν is an arbitrary disk-component of the parameter domain B of X. Thus we shall assume that X is a solution of a thread problem which is parametrized on a disk, say, the unit disk. For this reason we shall from now on abolish the notation of Sections 5.1 and 5.2 and, instead, return to another notation similar to that used in previous chapters. To be precise, we now denote by B the open disk

$$B=\{w=u+iv\colon |w|<1\}$$

in the u, v-plane $\mathbb{C} \mathrel{\hat{=}} \mathbb{R}^2$, and by C^+ and C^- its boundary parts

$$C^+ = \{w = u + iv\colon |w| = 1, v \geq 0\},$$
$$C^- = \{w = u + iv\colon |w| = 1, v \leq 0\}.$$

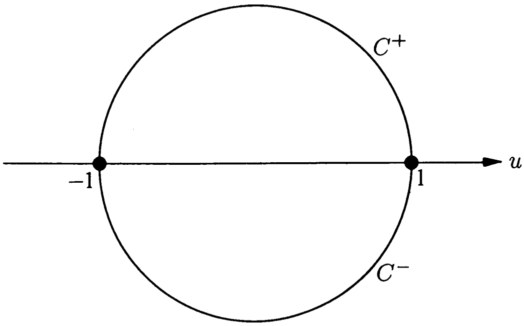

Fig. 1.

The set $\mathcal{C}(\Gamma, L)$ of comparison functions $X(w), w \in \overline{B}$, now consists of all surfaces of class $C^0(\overline{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$ which map C^- in a weakly monotonic way onto a given rectifiable Jordan arc Γ, and whose total variation on C^+ is equal to a fixed number L,

$$l(\Sigma) := \int_{C^+} |dX| = L. \tag{1}$$

Here Σ denotes the movable part $X\colon C^+ \to \mathbb{R}^3$ of the boundary of any $X \in \mathcal{C}(\Gamma, L)$. We assume that

$$|P_1 - P_2| < L < l(\Gamma), \tag{2}$$

where P_1 and P_2 denote the endpoints of Γ, and $l(\Gamma)$ stands for the length of the fixed arc Γ.

Let $X \in \mathcal{C}(\Gamma, L)$ be a minimizer of the Dirichlet integral

$$D_B(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv$$

among all surfaces in $\mathcal{C}(\Gamma, L)$. Such a minimizer will now be called *a solution of the thread problem* $\mathcal{P}(\Gamma, L)$. We already know that any such solution has to be a minimal surface. That is, the equations

$$\Delta X = 0,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

hold true in B, and $X(w) \not\equiv \text{const}$ on B.

Now we state the main result of this section.

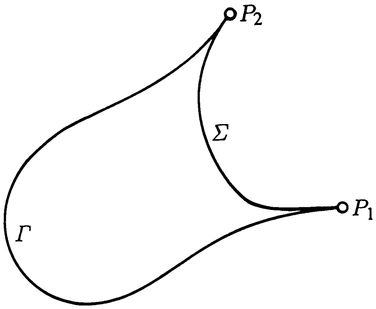

Fig. 2.

Theorem 1. *Let $X \in \mathcal{C}(\Gamma, L)$ be a minimal surface, that is, X satisfies*

$$\Delta X = 0 \quad \text{in } B \tag{3}$$

as well as the conformality relations. Introducing polar coordinates r, θ around the origin by $w = re^{i\theta}$, these relations can be written as

$$r^2|X_r|^2 = |X_\theta|^2, \quad \langle X_r, X_\theta \rangle = 0. \tag{4}$$

Moreover, suppose that X minimizes the Dirichlet integral within the class $\mathcal{C}(\Gamma, L)$. Then $X(w)$ can be continued analytically as a minimal surface across the arc C^+, and it has on C^+ no branch points of odd order nor any true branch points of even order. If, moreover, the boundary mapping $X\colon \partial B \to \mathbb{R}^3$ is assumed to be an embedding, then $X(w)$ has no false branch points of even order on C^+ either. Correspondingly, in this case, the free trace Σ defined by $X\colon C^+ \to \mathbb{R}^3$ is a regular, real analytic curve of constant curvature $\kappa \neq 0$.

For the following we recall some results on the boundary behaviour of minimal surfaces with a finite Dirichlet integral and with boundary values of bounded variation. The assumption $D_B(X) < \infty$ implies that $X(r, \theta) = X(re^{i\theta})$ possesses L^2-boundary values $X(1, \theta)$ on ∂B which are assumed in the L^2-sense as $r \to 1 - 0$. From $\int_0^{2\pi} |dX(1, \theta)| < \infty$ we conclude that $X(1, \theta)$ depends continuously on θ (cf. Lemma 3 of Section 5.2). More subtle results have been derived in Section 4.7 of Vol. 1. For the convenience of the reader, we collect the pertinent statements in the following lemma.

Lemma 1. *Let $X\colon B \to \mathbb{R}^3$ be a disk-type minimal surface, i.e. let (3) and (4) be satisfied, and denote by $X^*\colon B \to \mathbb{R}^3$ the adjoint minimal surface to X which, up to an additive constant, is uniquely determined by the equations*

$$X_r = \frac{1}{r} X_\theta^*, \quad \frac{1}{r} X_\theta = -X_r^*. \tag{5}$$

Assume that $D_B(X) < \infty$ and $\int_{\partial B} |dX| < \infty$. Then we have:

(i) *X and X^* are of class $C^0(\overline{B}, \mathbb{R}^3)$ and*

(6) $$D_B(X) = D_B(X^*), \quad \int_{\partial B} |dX| = \int_{\partial B} |dX^*|.$$

(ii) *The boundary values $X(1,\theta)$ and $X^*(1,\theta)$ are absolutely continuous functions of θ, and $X_\theta(r,\theta), X^*_\theta(r,\theta)$ tend in the L^2-sense to the derivatives $X_\theta(1,\theta), X^*_\theta(1,\theta)$ of the boundary values $X(1,\theta)$ and $X^*(1,\theta)$ respectively as $r \to 1-0$. Then, on account of (5), we deduce that also $X_r(r,\theta)$ and $X^*_r(r,\theta)$ converge in L^2 to boundary values as $r \to 1-0$, and we set*

$$X_r(1,\theta) = \lim_{r\to 1-0} X_r(r,\theta), \quad X^*_r(1,\theta) = \lim_{r\to 1-0} X^*_r(r,\theta).$$

It follows that a.e.

(7) $$X_r(1,\theta) = X^*_\theta(1,\theta), \quad X_\theta(1,r) = -X^*_r(1,\theta),$$

(8) $$|X_r(1,\theta)| = |X_\theta(1,\theta)|, \quad \langle X_r(1,\theta), X_\theta(1,\theta)\rangle = 0.$$

(iii) *If C is an open subarc of ∂B, and ξ is a test function of class $H^1_2(B,\mathbb{R}^3) \cap L_\infty(C,\mathbb{R}^3)$ with $\xi = 0$ on $\partial B \setminus C$, then*

(9) $$\int_B \langle \nabla X, \nabla \xi\rangle r\, dr\, d\theta = \int_C \langle X_r, \xi\rangle\, d\theta.$$

(iv) *If $X \not\equiv \text{const}$ on B, then $X_\theta(1,\theta)$ and $X^*_\theta(1,\theta)$ vanish at most on a subset of $[0,2\pi]$ of one-dimensional measure zero.*

Now we turn to the proof of Theorem 1 which we want to break up into three parts. In the first one we consider a *stationary version of the thread problem*; here the existence of a Lagrange multiplier is supposed. Thereafter we prove that every minimizer in $\mathcal{C}(\Gamma, L)$ is in fact a solution of the stationary problem by establishing the existence of a Lagrange multiplier, and in the third part we sketch how branch points can be excluded by using the minimum property.

Definition. *A minimal surface $X\colon B \to \mathbb{R}^3$ is said to be a* **stationary solution of the thread problem** *with respect to some open subarc C of ∂B if the following holds:*
(i) *$D_B(X) < \infty, \int_{\partial B} |dX| < \infty$;*
(ii) *there is a real number $\lambda \neq 0$ such that*

(10) $$\int_B \langle \nabla X, \nabla \xi\rangle r\, dr\, d\theta + \lambda \int_C \left\langle \frac{X_\theta}{|X_\theta|}, \xi_\theta \right\rangle d\theta = 0$$

holds for all $\xi \in C^1(\overline{B}, \mathbb{R}^3)$ with $\xi = 0$ on $\partial B \setminus C$.

Taking the identity (9) into account we arrive at

$$\int_C (\langle X_r, \xi\rangle + \lambda |X_\theta|^{-1}\langle X_\theta, \xi_\theta\rangle)\, d\theta = 0,$$

and (8) yields

$$\int_C \langle X_r, \xi\rangle\, d\theta = \int_C \langle X^*_\theta, \xi\rangle\, d\theta = -\int_C \langle X^*, \xi_\theta\rangle\, d\theta.$$

Thus (10) is equivalent to

$$(11) \qquad \int_C \langle X^* - \lambda |X_\theta|^{-1} X_\theta, \xi_\theta\rangle\, d\theta = 0$$
$$\text{for all } \xi \in C^1(\overline{B}, \mathbb{R}^3) \quad \text{with } \xi = 0 \text{ on } \partial B \setminus C.$$

DuBois–Reymond's lemma now implies that (11) – and therefore also (10) – is equivalent to the following property of X:

There exists a constant vector $P \in \mathbb{R}^3$ such that

$$(12) \qquad X^* = \lambda |X_\theta|^{-1} X_\theta + P \quad \textit{a.e. on } C$$

holds.

We now prove

Theorem 2. *Let $X\colon B \to \mathbb{R}^3$ be a minimal surface which is a stationary solution of the thread problem with respect to the open arc $C \subset \partial B$. Then, for some $P \in \mathbb{R}^3$ and some $\lambda \in \mathbb{R}, \lambda \neq 0$, equation (12) is satisfied. Moreover, X and its adjoint X^* are real analytic on $B \cup C$, and X^* intersects the sphere*

$$S = \{Z \in \mathbb{R}^3 \colon |Z - P|^2 = \lambda^2\}$$

orthogonally along its free trace Σ^* defined by $X^*\colon C \to \mathbb{R}^3$. Both X and X^* have no boundary branch points of odd order on C. Finally, $\Sigma = X|_C$ has a representation $\mathfrak{X}(s), 0 < s < 1$, by its arc length s as parameter, which is of class C^2 and satisfies $|\dot{\mathfrak{X}}(s)| \equiv 1$, $|\ddot{\mathfrak{X}}(s)| \equiv \frac{1}{|\lambda|}$. Thus Σ represents a regular curve of constant curvature $\kappa = \frac{1}{|\lambda|}$.

Proof. As we have noticed, the assumption on X implies that (12) holds for some $P \in \mathbb{R}^3$ and some $\lambda \in \mathbb{R}, \lambda \neq 0$. Taking the continuity of $X^*(1, \theta)$ into account, we infer that

$$(13) \qquad |X^* - P|^2 = \lambda^2 \quad \text{on } C.$$

In other words, the trace Σ^* lies on S. Moreover, equations (12) and (7) yield

$$X^* - P = -\lambda |X_r^*|^{-1} \cdot X_r^* \quad \text{a.e. on } C. \tag{14}$$

Therefore the vector X_r^* is normal to S a.e. on C. Thus for almost all $w \in C$ the surface X^* has a tangent plane which meets S at a right angle. By the reasoning of Section 1.4 (cf. Theorem 1) we conclude that the adjoint surface X^* is a critical point of Dirichlet's integral within the boundary configuration $\langle \Gamma, S \rangle$ consisting of the arc $\Gamma^* = \{X^*(w) \colon w \in \partial B \setminus C\}$ and of the surface S. We can therefore apply Theorem 2′ of Section 2.8 to X^* and obtain that X^* can be continued analytically across C as a minimal surface. (Note that for this regularity theorem it is not necessary to assume that Γ^* be a Jordan arc which does not meet S except in his two endpoints.) By virtue of (5) we infer that both X and X^* are real analytic in $B \cup C$, as we have claimed.

We furthermore note that, because of (5), X and X^* have the same boundary branch points $w_0 \in C$. Since $X + iX^*$ is a nonconstant holomorphic mapping $U \to \mathbb{C}^3$ of some full neighbourhood U of each branch point $w_0 \in C$, we have the asymptotic formula

$$X_w(w) = A(w - w_0)^\nu + O(|w - w_0|^{\nu+1}) \quad \text{as } w \to w_0, \tag{15}$$

for some integer $\nu \geq 1$ and some vector $A \neq 0$. Since $X_w^* = -iX_w$, we also have

$$X_w^*(w) = -iA(w - w_0)^\nu + O(|w - w_0|^{\nu+1}) \quad \text{as } w \to w_0. \tag{15′}$$

That is, the order of w_0 as branch point of X equals its order as branch point of X^*. We moreover infer from (15) and (15′) that the boundary branch points of X and X^* are isolated. In addition, the conformality relations (5) imply $\langle A, A \rangle = 0$. Thus A is of the form $A = \frac{1}{2}(a - ib)$, where $a, b \in \mathbb{R}^3$, $|a| = |b| \neq 0$, $\langle a, b \rangle = 0$.

If $w = e^{i\theta}$ is not a branch point of X on C, we can define the unit tangent vectors

$$T(\theta) = \frac{X_\theta(1,\theta)}{|X_\theta(1,\theta)|}, \quad T^*(\theta) = \frac{X_\theta^*(1,\theta)}{|X_\theta^*(1,\theta)|}$$

of the curves Σ and Σ^* at $X(w)$ and $X^*(w)$, respectively.

Let $w_0 = e^{i\theta_0} \in C$ be a branch point of X (and of X^*). Then we infer from (15) and (15′) that the one-sided limits

$$T_\pm(\theta_0) = \lim_{\theta \to \theta_0 \pm 0} T(\theta), \quad T_\pm^*(\theta_0) = \lim_{\theta \to \theta_0 \pm 0} T^*(\theta)$$

exist. Moreover, we have

$$T_+(\theta_0) = T_-(\theta_0), \quad T_+^*(\theta_0) = T_-^*(\theta_0) \tag{16}$$

if the order ν of the boundary branch point w_0 is *even*, whereas

$$T_+(\theta_0) = -T_-(\theta_0), \quad T_+^*(\theta_0) = -T_-^*(\theta_0) \tag{16′}$$

if ν is *odd*.

We note that the limits $T_\pm(\theta_0), T^*_\pm(\theta_0)$ are unit vectors. Equation (12), on the other hand, yields that

$$T(\theta) = \frac{1}{\lambda}\{X^*(1,\theta) - P\} \tag{17}$$

holds for all θ satisfying $0 < |\theta - \theta_0| < \varepsilon$ where ε is a sufficiently small number and, moreover, the right-hand side depends continuously on $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. Therefore, $T_+(\theta_0) = T_-(\theta_0)$, and ν must be of even order. Hence X and also X^* can only have even order branch points on C, as we have claimed. If we define $T(\theta_0)$ by $T_+(\theta_0)$ at a branch point $w_0 = e^{i\theta_0} \in C$ of even order, we infer from (16) that $T(\theta)$ is a continuous function on C with $|T(\theta)| \equiv 1$, and (17) holds everywhere on C.

Suppose now that $C = \{e^{i\theta} \colon \theta_1 < \theta < \theta_2\}$ and set

$$l = \int_{\theta_1}^{\theta_2} |X_\theta(1,\theta)|\, d\theta = \int_{\theta_1}^{\theta_2} |X^*_\theta(1,\theta)|\, d\theta.$$

We furthermore introduce

$$s = s(\theta) = \int_{\theta_1}^{\theta} |X_\theta(1,\theta)|\, d\theta = \int_{\theta_1}^{\theta} |X^*_\theta(1,\theta)|\, d\theta,$$

$\theta_1 \le \theta \le \theta_2$, which is the arc length parameter of Σ as well as of Σ^*. Since $s'(\theta) = |X_\theta(1,\theta)| \ge 0$ has only isolated zeros, the function $s(\theta)$ can be inverted. Let $\theta(s), 0 \le s \le l$, be its (continuous) inverse. For $0 < s < l$ we introduce

$$t(s) = T(\theta(s)), \quad t^*(s) = T^*(\theta(s)),$$
$$\mathcal{X}(s) = X(1,\theta(s)), \quad \mathcal{X}^*(s) = X^*(1,\theta(s)).$$

So far, we only know that $\theta(s)$ is continuously differentiable in s-intervals corresponding to θ-intervals free of branch points. We already know that $t(s)$ and $t^*(s)$ are continuous for $0 < s < l$, and that $\dot{\mathcal{X}}(s) = t(s), \dot{\mathcal{X}}^*(s) = t^*(s)$ holds at values of s which do not correspond to branch points on C. Then a simple argument employing the mean value theorem yields that $\mathcal{X}(s)$ and $\mathcal{X}^*(s)$ are of class C^1 for $0 < s < l$, and that

$$\dot{\mathcal{X}}(s) = t(s), \quad \dot{\mathcal{X}}^*(s) = t^*(s) \quad \text{for } 0 < s < l. \tag{18}$$

(In these formulas as well as in the following ones, the dot denotes differentiation with respect to the arc length: $\dot{} = \frac{d}{ds}$.) Thus Σ and Σ^* are representations of regular curves of class C^1.

From (17) and (18) we derive the equation

$$t(s) = \frac{1}{\lambda}\{\mathcal{X}^*(s) - P\} \tag{19}$$

and

(20) $$\dot{t}(s) = \frac{1}{\lambda} t^*(s)$$

for $0 < s < l$. Thus $\mathcal{X}(s)$ is actually of class C^2 on $(0, l)$, $|\dot{t}(s)| = |\ddot{\mathcal{X}}(s)| = 1/|\lambda|$. This means, Σ represents a regular C^2-curve of constant curvature $1/|\lambda|$. This concludes the proof of Theorem 2. □

The first of Frenet's equations yields

(21) $$\dot{t}\,(s) = \kappa n(s), \quad \kappa = \frac{1}{|\lambda|},$$

where $n(s)$ is the principal normal of the curve $\mathcal{X}(s)$. On the other hand, differentiating (17) with respect to θ and employing (7) and (8), we arrive at

(22) $$\dot{t}\,(s) = \frac{1}{\lambda} \cdot \frac{X_r}{|X_r|}(1, \theta(s)).$$

Hence $n = \pm |X_r|^{-1} X_r$, and thus the normal curvature of Σ vanishes. Thus as a by-product of our discussion we obtain the following

Corollary 1. *Under the assumptions of Theorem 1 the free trace Σ of X is an asymptotic line of the surface X of constant geodesic curvature $\pm\,\kappa$.*

Remark 1. In general, stationary solutions of the thread problem will have boundary branch points of even order. In fact, one can easily construct examples of planar minimal surfaces $X^* \colon \overline{B} \to \mathbb{R}^3$ that satisfy (14) for some nonempty open subarc C of ∂B and have a branch point w_0 of second order on C. The adjoint surface X of $-X^*$ will then satisfy (12) or, equivalently, (10). Hence X is a stationary solution of a thread problem with respect to C that has a branch point of second order on C.

Next we come to the second part of the proof of Theorem 1. We shall prove that, for each solution of the real thread problem, there exists a Lagrange multiplier. This is not totally trivial since the applicability of the standard Lagrange multiplier theorem (which requires continuous differentiability of the involved functions) is not clear. The following result provides an appropriate substitute.

Lemma 2. *Let $\varphi(\varepsilon, t)$ and $\psi(\varepsilon, t)$ be real-valued functions of*

$$(\varepsilon, t) \in [-\varepsilon_0, \varepsilon_0] \times [-t_0, t_0], \quad \varepsilon_0 > 0,\ t_0 > 0,$$

which split in the form

$$\varphi(\varepsilon, t) = \varphi_0 + \varphi_1(\varepsilon) + \varphi_2(t), \quad \psi(\varepsilon, t) = \psi_0 + \psi_1(\varepsilon) + \psi_2(t).$$

Here it is assumed that φ_0 and ψ_0 are constant, and that

$$\varphi_1(0) = \varphi_2(0) = \psi_1(0) = \psi_2(0) = 0.$$

We also suppose that ψ_2 *is continuous on* $[-t_0, t_0]$*, that the derivatives* $\varphi_1'(0), \varphi_2'(0), \psi_1'(0), \psi_2'(0)$ *exist, and that* $\psi_2'(0) = 1$*. Finally, let the inequality* $\varphi(\varepsilon, t) \geq \varphi(0, 0)$ *hold for all* (ε, t) *in* $[-\varepsilon_0, \varepsilon_0] \times [-t_0, t_0]$ *with* $\psi(\varepsilon, t) = \psi_0$*.*

Then the relation

$$\varphi_1'(0) + \lambda \psi_1'(0) = 0 \tag{23}$$

is satisfied for $\lambda = -\varphi_2'(0)$*.*

Proof. The assumptions imply that there is a function $\eta(t), -t_0 \leq t \leq t_0$, which satisfies

$$\lim_{t \to 0} \eta(t) = \eta(0) = 0$$

and

$$\psi_2(t) = t\{1 + \eta(t)\}.$$

Then we choose a number δ_0 with $0 < \delta_0 < \frac{t_0}{2}$ such that $|\eta(2t)| < \frac{1}{2}$ for $|t| < \delta_0$, and infer that

$$\psi_2(-2t) < -t < t < \psi_2(2t) \quad \text{for } t \in (0, \delta_0).$$

The continuity of ψ_2 now implies the relation

$$[-\delta, \delta] \subset \psi_2([-2\delta, 2\delta]) \quad \text{for all } \delta \in (0, \delta_0).$$

We also note that $\lim_{\varepsilon \to 0} \psi_1(\varepsilon) = 0$ holds. Therefore we can find a number ε_1 with $0 < \varepsilon_1 \leq \varepsilon_0$ such that $|\psi_1(\varepsilon)| < \delta_0$ is satisfied for each $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. Consequently there exists a real-valued function $\tau(\varepsilon), -\varepsilon_1 \leq \varepsilon \leq \varepsilon_1$, with the properties

$$\tau(0) = \lim_{\varepsilon \to 0} \tau(\varepsilon) = 0, \quad \psi_2(\tau(\varepsilon)) + \psi_1(\varepsilon) = 0,$$
$$|\tau(\varepsilon)| \leq 2|\psi_1(\varepsilon)| < t_0,$$

whence also $\psi(\varepsilon, \tau(\varepsilon)) = \psi_0$ for $-\varepsilon_1 \leq \varepsilon \leq \varepsilon_1$. From the identities

$$\frac{\tau(\varepsilon)}{\varepsilon} = \frac{\tau(\varepsilon) - \tau(0)}{\varepsilon} = -\frac{\psi_1(\varepsilon) - \psi_1(0)}{\varepsilon} \cdot \frac{1}{1 + \eta(\tau(\varepsilon))}$$

for $0 < |\varepsilon| \leq \varepsilon_1$ we infer that the function $\tau(\varepsilon)$ is differentiable at $\varepsilon = 0$, and that

$$\tau'(0) = \lim_{\varepsilon \to 0} \frac{\tau(\varepsilon)}{\varepsilon} = -\psi_1'(0). \tag{24}$$

Moreover, the minimum property

$$\varphi(\varepsilon, \tau(\varepsilon)) \geq \varphi(0, 0) \quad \text{for } 0 < \varepsilon \leq \varepsilon_1$$

implies the inequality

$$0 \le \frac{\varphi_1(\varepsilon)}{\varepsilon} + \frac{\varphi_2(\tau(\varepsilon))}{\varepsilon}. \tag{25}$$

Suppose now that we would have $\tau(\varepsilon) \equiv 0$ on some interval $(0, \varepsilon']$, where $0 < \varepsilon' \le \varepsilon_1$. Then we obtain

$$0 \le \frac{\varphi_1(\varepsilon)}{\varepsilon} \quad \text{for } 0 < \varepsilon \le \varepsilon'$$

on account of (25), and therefore $\varphi_1'(0) \ge 0$. By virtue of (24) we furthermore have $\tau'(0) = 0$ and $\psi_1'(0) = 0$, whence

$$0 \le \varphi_1'(0) - \psi_1'(0)\varphi_2'(0). \tag{26}$$

If, on the other hand, there is no $\varepsilon' > 0$ such that $\tau(\varepsilon) \equiv 0$ on $(0, \varepsilon']$, then there exists a sequence of numbers $\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots$ tending to zero, with $0 < \varepsilon_i \le \varepsilon'$ for $i \ge 2$ and $\tau(\varepsilon_i) \ne 0$. Set $\tau_i = \tau(\varepsilon_i)$. We then infer from (25) that

$$0 \le \frac{\varphi_1(\varepsilon_i)}{\varepsilon_i} + \frac{\varphi_2(\tau_i)}{\tau_i} \cdot \frac{\tau_i}{\varepsilon_i}, \quad i = 2, 3, 4, \ldots,$$

holds. For $i \to \infty$ we once again arrive at the inequality (26) which thus is established. Similarly we can verify the opposite inequality

$$0 \ge \varphi_1'(0) - \psi_1'(0)\varphi_2'(0),$$

and the Lemma is proved. □

In order to apply the previous lemma, we will introduce *the class* $\mathcal{F}(C^+)$ *of test functions* defined in the following way:

A function ζ *is said to be of class* $\mathcal{F}(C^+)$ *if it lies in* $C^1(\overline{B}, \mathbb{R}^3)$*, and if there are a point* $w_0 \in C^+$ *and a number* $r \in (0,1)$ *such that* $\partial B \cap \overline{B}_r(w_0)$ *is contained in the open arc* C^+ *and that* $\zeta(w) = 0$ *for all* $w \in \overline{B} \setminus B_{r/2}(w_0)$.

Lemma 3. *Suppose that* (2) *holds and that* X *is a mapping of class* $\mathcal{C}(\Gamma, L)$ *which satisfies the assumptions of Theorem* 1*. Then there exists some* $\zeta \in \mathcal{F}(C^+)$ *such that*

$$\int_{C^+} |X_\theta|^{-1} \langle X_\theta, \zeta_\theta \rangle \, d\theta = 1. \tag{27}$$

Proof. It clearly suffices to establish the existence of some $\zeta \in \mathcal{F}(C^+)$ for which the integral in (27) is nonzero. To this end, let us suppose that the integral vanishes for all $\zeta \in \mathcal{F}(C^+)$. Then, by DuBois–Reymond's lemma, there would exist a unit vector $e \in \mathbb{R}^3$ such that

$$|X_\theta(1, \theta)|^{-1} X_\theta(1, \theta) = e$$

for almost all $\theta \in (0, \pi)$. Hence $X(C^+)$ would be contained in some straight line $\mathcal{L}$, and since $X : \partial B \to \mathbb{R}^3$ is a continuous mapping, $\mathcal{L}$ would have to be the straight line connecting the two points P_1 and P_2. Applying the reflection principle we could extend X analytically and as a minimal surface across C^+. Hence X is real analytic on $B \cup C^+$ and possesses at most denumerably many isolated branch points on C^+. Then we infer from the equation

$$X_\theta(1, \theta) = |X_\theta(1, \theta)| e \quad \text{for all } \theta \in (0, \pi)$$

that $X(1, \theta)$ yields a strictly monotonic mapping of $[0, \pi]$ onto the straight segment on $\mathcal{L}$ with the endpoints P_1 and P_2, whence we would get

$$L = \int_{C^+} |dX| = |P_1 - P_2|.$$

But this contradicts the assumption required in (2). □

Lemma 4. *Suppose that* (2) *holds and that* $X \in \mathcal{C}(\Gamma, L)$ *satisfies the assumptions of Theorem* 1. *Then* X *is a stationary solution of the thread problem with respect to the arc* $C^+ = \{e^{i\theta} : 0 < \theta < \pi\}$.

Proof. By Lemma 3 there is a test function $\zeta \in \mathcal{F}(C^+)$ such that (27) holds. By definition of $\mathcal{F}(C^+)$, there exist $w_0 \in C^+$ and $r \in (0, 1)$ such that $\zeta(w)$ vanishes for all $w \in \overline{B} \setminus B_{r/2}(w_0)$ and that the closed arc $\gamma := \partial B \cap \overline{B}_r(w_0)$ is contained in C^+. Then $C^+ \setminus \gamma$ consists of two non-empty open arcs C_1 and C_2. We first want to show that X *is a stationary solution of the thread problem with respect to* C_1 *as well as to* C_2. Since the reasoning will be the same for both arcs, it suffices to verify the assertion for, say, C_1.

Firstly, the assumptions of Theorem 1 imply that

$$D_B(X) < \infty, \quad \int_{\partial B} |dX| < \infty, \quad \text{and} \quad X(w) \not\equiv \text{const}.$$

Secondly we have to prove that

$$\int_B \langle \nabla X, \nabla \xi \rangle r \, dr \, d\theta + \lambda_1 \int_{C_1} \langle |X_\theta|^{-1} X_\theta, \xi_\theta \rangle \, d\theta = 0 \tag{28}$$

holds for some real number $\lambda_1 \neq 0$ and for all $\xi \in C^1(\overline{B}, \mathbb{R}^3)$ that vanish on $\partial B \setminus C_1$.

Clearly, it suffices to verify (28) for all $\xi \in C_c^1(B \cup C_1, \mathbb{R}^3)$. We shall, in fact, see that (28) only has to be established for an even smaller class of test functions. For this purpose, we choose some open disk B' with the property that $\partial B \cap B' = C_1$, and that $\Omega := B \cap B'$ does not meet the disk $B_{r/2}(w_0)$. By virtue of some appropriate partition of unity, each element $\xi \in C_c^1(B \cup C_1, \mathbb{R}^3)$ can be written as the sum $\xi = \xi_1 + \xi_2$ of a function $\xi_1 \in C_c^1(\Omega \cup C_1, \mathbb{R}^3)$ and of another function $\xi_2 \in C_c^1(B, \mathbb{R}^3)$. We now note that both integrals appearing in (28) vanish separately if ξ is of class $C_c^1(B, \mathbb{R}^3)$. Thus it remains to prove the following:

There is some number $\lambda_1 \neq 0$ *such that* (28) *holds for all test functions* $\xi \in C_c^1(\Omega \cup C_1, \mathbb{R}^3)$.

This will be achieved by employing Lemma 2. To this end we choose some arbitrary $\xi \in C_c^1(\Omega \cup C_1, \mathbb{R}^3)$ which in the sequel is thought to be fixed, and set

$$X_{\varepsilon,t} = X + \varepsilon\xi + t\zeta, \quad |\varepsilon| \le \varepsilon_0, \ |t| \le t_0$$

for some number $\varepsilon_0 > 0, t_0 > 0$. (At present, the subscripts ε and t indicate the dependence of the 2-parameter family $X_{\varepsilon,t}$ on the parameters ε and t and do, deviating from the previous way of notation, not stand for partial derivatives.)

Let us introduce the functions

$$\varphi(\varepsilon,t) := D_B(X_{\varepsilon,t}), \quad \psi(\varepsilon,t) := \int_{C^+} |dX_{\varepsilon,t}| = \int_{C^+} \left| \frac{d}{d\theta} X_{\varepsilon,t}(1,\theta) \right| d\theta$$

of $(\varepsilon,t) \in [-\varepsilon_0, \varepsilon_0] \times [-t_0, t_0]$. Then we have the representations

$$\varphi(\varepsilon,t) = \varphi_0 + \varphi_1(\varepsilon) + \varphi_2(t), \quad \psi(\varepsilon,t) = \psi_0 + \psi_1(\varepsilon) + \psi_2(t),$$

where we have set

$$\begin{aligned}
\varphi_0 &:= D_B(X), \quad \psi_0 := \int_{C^+} |X_\theta(1,\theta)| \, d\theta, \\
\varphi_1(\varepsilon) &:= D_\Omega(X + \varepsilon\xi) - D_\Omega(X), \quad \varphi_2(t) := D_{\Omega_0}(X + t\zeta) - D_{\Omega_0}(X), \\
\Omega &:= B \cap B_{r/2}(w_0), \\
\psi_1(\varepsilon) &:= \int_{C_1} |X_\theta + \varepsilon\xi_\theta| \, d\theta - \int_{C_1} |X_\theta| \, d\theta, \\
\psi_2(t) &:= \int_\gamma |X_\theta + t\zeta_\theta| \, d\theta - \int_\gamma |X_\theta| \, d\theta.
\end{aligned}$$

(We now have once again used: $X_\theta = \frac{\partial}{\partial\theta} X$, etc.) The functions φ_1 and φ_2 are quadratic polynomials, and clearly

$$0 = \varphi_1(0) = \varphi_2(0) = \psi_1(0) = \psi_2(0).$$

Moreover, the function $\psi_2(t)$ is continuous on $[-t_0, t_0]$. We also claim that the derivatives $\psi_1'(0)$ and $\psi_2'(0)$ exist. In fact, the formula $a^2 - b^2 = (a+b)(a-b)$ yields

$$\frac{1}{\varepsilon}\{|X_\theta + \varepsilon\xi_\theta| - |X_\theta|\} = f(\varepsilon) + g(\varepsilon),$$

where

$$f(\varepsilon) = \frac{2\langle X_\theta, \xi_\theta \rangle}{|X_\theta + \varepsilon\xi_\theta| + |X_\theta|}, \quad g(\varepsilon) = \frac{\varepsilon|\xi_\theta|^2}{|X_\theta + \varepsilon\xi_\theta| + |X_\theta|}.$$

Hence we infer that

$$|f(\varepsilon)| \le 2|\xi_\theta|, \quad |g(\varepsilon)| \le |\xi_\theta| \quad \text{a.e. on } C^+$$

and for $|\varepsilon| > 0$. By Lebesgue's theorem on dominated convergence the derivatives $\psi_1'(0)$ and $\psi_2'(0)$ exist, and

$$(29)\quad \psi_1'(0) = \int_{C_1} |X_\theta|^{-1}\langle X_\theta, \xi_\theta\rangle\, d\theta, \quad \psi_2'(0) = \int_{C^+} |X_\theta|^{-1}\langle X_\theta, \zeta_\theta\rangle\, d\theta = 1.$$

Thus the assumptions of Lemma 2 are satisfied, and we obtain

$$\varphi_1'(0) + \lambda_1\psi_1'(0) = 0, \quad \text{where } \lambda_1 = -\varphi_2'(0).$$

On the other hand, we infer from (29) and from

$$\varphi_1(\varepsilon) = \varepsilon\int_\Omega \langle \nabla X, \nabla\xi\rangle r\, dr\, d\theta + \frac{\varepsilon^2}{2} D_\Omega(\xi)$$

that (28) is true for an arbitrarily chosen $\xi \in C_c^1(\Omega \cup C_1, \mathbb{R}^3)$, and hence (28) holds for all $\xi \in C^1(\overline{B}, \mathbb{R}^3)$ that vanish on $\partial B \setminus C_1$. Because of the equivalence of relations (10) and (12) we conclude that

$$(30)\qquad X^* = \lambda_1 |X_\theta|^{-1} X_\theta + P_1$$

holds a.e. on C_1 for some constant vector $P_1 \in \mathbb{R}^3$. If $\lambda_1 = 0$, we would get $X^* = P_1$; i.e. $X_\theta^*(1,\theta) = 0$ a.e on C_1, and this contradicts Lemma 1, (iv). Hence we have indeed $\lambda_1 \ne 0$, and it is proved that X is a stationary solution of the thread problem with respect to C_1 (and to C_2). By Theorem 2, the mappings X and X^* are real analytic on $B \cup C_1 \cup C_2$ and have at most isolated branch points.

In order to complete the proof of Lemma 4 we now assume w.l.o.g. that $C_1 = \{e^{i\theta}\colon 0 < \theta < \theta_1\}$ for some $\theta_1 \in (0, \pi)$. Then we introduce the two arcs

$$\gamma_1 = \{e^{i\theta}\colon 0 < \theta < \tfrac{1}{2}\theta_1\}, \quad \gamma_2 = \{e^{i\theta}\colon \tfrac{1}{2}\theta_1 < \theta < \pi\}.$$

Let us choose two disks B_1 and B_2 with centers outside of B such that $\gamma_1 = \partial B \cap B_1, \gamma_2 = \partial B \cap B_2$, and that the open sets $\Omega_1 = B \cap B_1$ and $\Omega_2 = B \cap B_2$ are disjoint. We claim that there is a function $\zeta_1 \in C_c^1(\Omega_1 \cup \gamma_1, \mathbb{R}^3)$ such that

$$\int_{\gamma_1} |X_\theta|^{-1} \left\langle X_\theta, \frac{\partial\zeta_1}{\partial\theta}\right\rangle d\theta = 1.$$

Otherwise we would have

$$|X_\theta|^{-1} X_\theta = \text{const} \quad \text{on } \gamma_1,$$

whence by (30) $X^*(1,\theta) = \text{const}$ for $0 < \theta < \frac{1}{2}\theta_1$, i.e. $X_r^* = X_\theta^* = 0$ on γ_1. This would be impossible since the branch points of X^* on γ_1 are isolated. In addition, we choose an arbitrary function $\xi \in C_c^1(\Omega_2 \cup \gamma_2, \mathbb{R}^3)$. Then we apply the previous reasoning to the 2-parameter family

$$X_{\varepsilon,t} = X + \varepsilon\xi + t\zeta_1, \quad |\varepsilon| \le \varepsilon_0, \ |t| \le t_0.$$

By the same arguments as before we can establish the existence of a constant vector $P \in \mathbb{R}^3$ and of a number $\lambda \in \mathbb{R}, \lambda \neq 0$, such that

$$X^* = \lambda |X_\theta|^{-1} X_\theta + P \tag{31}$$

holds on γ_2, and we also know that X and X^* are real analytic on $B \cup \gamma_2$. On the other hand, equation (30) is satisfied on C_1. Since

$$C_1 \cap \gamma_2 = \{e^{i\theta} \colon \tfrac{1}{2}\theta_1 < \theta < \theta_1\},$$

we may infer that $\lambda = \lambda_1$ and $P = P_1$. Thus we have proved that X and X^* are real analytic on $B \cup C^+$, and that (31) is satisfied on all of C^+. This in turn yields

$$\int_B \langle \nabla X, \nabla \xi \rangle r \, dr \, d\theta + \lambda \int_{C^+} \langle |X_\theta|^{-1} X_\theta, \xi_\theta \rangle \, d\theta = 0$$

for all $\xi \in C^1(\overline{B}, \mathbb{R}^3)$ with $\xi = 0$ on $\partial B \setminus C^+$, and Lemma 4 is proved. □

Resuming the results of Theorem 2 and of the Lemmata 2–4, we see that all assertions of Theorem 1 are proved, except for the claim that *Σ is a regular curve*. The proof of this fact will be sketched in the third and last part of our discussion. We shall proceed by proving that no minimizer X can have branch points of even order on C^+. Recall that branch points of odd order were already excluded in Theorem 2; they cannot even occur for stationary solutions of the thread problem. On the other hand, stationary solutions may very well possess branch points of even order, as we have noted in Remark 1. Thus we now really have to employ the minimizing property of X if we wish to exclude branch points of even order. In what follows we shall describe some of the main ideas that lead to the exclusion of true branch points of even order for minimizers X. For this we use some of the reasoning of Gulliver–Lesley and of Osserman [12]. The impossibility of false branch points of even order will not be discussed since we have already described the pertinent ideas in Section 1.9. For further information and for filling in all details we refer the reader to the Scholia of Chapter 6 (see Section 6.4).

It will be convenient to choose the parameter domain of any minimizer X as the semi-disk.

$$B = \{w = u + iv \colon |w| < 1, v > 0\},$$

and C^+, C^- will be replaced by

$$C = \{w = u + iv \colon |w| = 1, v \ge 0\}$$

and

$$I = \{u \in \mathbb{R} : |u| < 1\}.$$

We now assume that $X: C \to \mathbb{R}^3$ yields a monotonic parametrization of Γ, and $X: I \to \mathbb{R}^3$ describes the free trace of X, i.e., its movable part Σ of the boundary. It follows from the previous discussion that X can be continued analytically as a minimal surface across I. Let u_0 be an arbitrary branch point of even order for X with $u_0 \in I$. We want to show that the existence of such a branch point contradicts the minimizing property of X.

Without loss of generality we can assume that $u_0 = 0$ and that $X(0) = 0$ because we can always transform $u = u_0$ into $u = 0$ by a conformal self-mapping of B that keeps the points $u = \pm 1$ fixed, and $X(0) = 0$ can be achieved by a suitable translation of $\mathbb{R}^3$. Performing an appropriate rotation of $\mathbb{R}^3$, we can also accomplish the asymptotic representation

$$\begin{aligned} x(w) + iy(w) &= aw^{m+1} + O(|w|^{m+2}), \quad a \neq 0, \\ z(w) &= O(|w|^{m+2}) \end{aligned}$$

for the Cartesian coordinates $x(w), y(w), z(w)$ of $X(w)$ in the neighbourhood of $w = 0$, where a denotes some positive constant and $m = 2\nu, \nu \geq 1$, is the order of the branch point $w = 0$. By a suitable scaling it can also be arranged that

$$\begin{gathered} x(w) + iy(w) = w^{m+1} + O(|w|^{m+2}), \\ z(w) = O(|w|^{m+2}) \end{gathered}$$

holds true for $w \to 0$. Because of the power-series expansion of $X(w)$ at $w = 0$ we may write

$$\begin{aligned} x(w) + iy(w) &= w^{m+1} + \sigma(w), \\ z(w) &= \psi(w), \\ \nabla^k \sigma(w), \nabla^k \psi(w) &= O(|w|^{m+2-k}) \quad \text{for } 0 \leq k \leq 2 \end{aligned} \tag{32}$$

with $m = 2\nu > 0$.

We will now show that this representation can be simplified even further.

Lemma 5. *Let $X: B_R(0) \to \mathbb{R}^3$ be a minimal surface with the representation (32) at $w = 0$. Then there exist two neighbourhoods $\mathcal{U}, \mathcal{V}$ of 0 in $B_R(0)$, a function $\varphi \in C^2(\mathcal{V})$ with*

$$\nabla^k \varphi(w) = O(|w|^{m+2-k}) \quad \text{for } 0 \leq k \leq 2$$

and a C^1-diffeomorphism $F: \mathcal{U} \to \mathcal{V}$ of $\mathcal{U}$ onto $\mathcal{V}$ such that the formulas

$$\begin{aligned} x(w) + iy(w) &= F^{m+1}(w), \\ z(w) &= \varphi(F(w)) \end{aligned} \tag{33}$$

hold true for $w \in \mathcal{U}$.

(Note that we use the complex notation $\omega = F(w) \in \mathbb{C}$; thus ω^{m+1} is the $(m+1)$-th power of ω.)

Proof. Define

$$F(w) := w\{1 + w^{-m-1}\sigma(w)\}^{1/(m+1)}$$

on a sufficiently small neighbourhood of $w = 0$. Because of $\sigma(w) = O(|w|^{m+2})$, this definition is meaningful if we choose the $(m+1)$-th root to be one at $w = 0$. Moreover, we have

$$\lim_{w \to 0} \frac{F(w)}{w} = 1.$$

Hence $\nabla F(0)$ exists, and $\nabla F(0) = \text{id}$. Moreover, we have

$$(D_u + iD_v)F(w) = 1 + o(1) \quad \text{as } w \to 0,$$

whence

$$\nabla F(w) \to \nabla F(0) \quad \text{as } w \to 0,$$

and this implies $F \in C^1$. By the inverse function theorem, there exists a C^1-inverse f of F on a neighbourhood $\mathcal{V}$ of the origin; set $\mathcal{U} := f(\mathcal{V})$. Since $F \in C^2(\mathcal{U} \setminus \{0\})$, we see that $f \in C^2(\mathcal{V} \setminus \{0\})$, and it is not difficult to prove that

$$\nabla^2 F(w) = o(|w|^{-1}).$$

In order to be able to use the summation convention, we write $w = u + iv = u^1 + iu^2, u^1 = u, u^2 = v$. Then the identity

$$f^\alpha(F(w)) = u^\alpha, \quad \alpha = 1, 2,$$

implies

$$f^\alpha_{,\beta}(F(w))F^\beta_{,\gamma}(w) = \delta^\alpha_\gamma \quad \text{in } \mathcal{U},$$

that is,

$$f^\alpha_{,\beta}(\tilde{w})F^\beta_{,\gamma}(f(\tilde{w})) = \delta^\alpha_\gamma \quad \text{in } \mathcal{V}.$$

Moreover, we obtain

$$f^\alpha_{,\beta\sigma}F^\beta_{,\gamma}(f) + f^\alpha_{,\beta}F^\beta_{,\gamma\tau}(f)f^\tau_{,\sigma} = 0 \quad \text{in } \mathcal{V} \setminus \{0\}.$$

Multiplying this identity by f^γ_ρ we infer

$$f^\alpha_{,\rho\sigma} = -F^\beta_{,\gamma\tau}f^\tau_{,\sigma}f^\alpha_{,\beta}f^\gamma_{,\rho}$$

whence we derive that

$$\nabla^2 f(\tilde{w}) = o(|\tilde{w}|^{-1}) \quad \text{as } \tilde{w} \to 0, \ \tilde{w} = F(w).$$

Now we define $\varphi \colon \mathcal{V} \to \mathbb{R}$ by $\varphi(\tilde{w}) = \psi(f(\tilde{w}))$. Then φ is a well defined function of class $C^1(\mathcal{V}) \cap C^2(\mathcal{V} \setminus \{0\})$ which satisfies

$$\varphi_{,\alpha} = \psi_{,\gamma} f^{\gamma}_{,\alpha} \quad \text{in } \mathcal{V} \tag{34}$$

and

$$\varphi_{,\alpha\beta} = \psi_{,\gamma\rho} f^{\rho}_{,\beta} f^{\gamma}_{,\alpha} + \psi_{,\gamma} f^{\gamma}_{,\alpha\beta} \quad \text{in } \mathcal{V} \setminus \{0\}. \tag{35}$$

The assumptions of the lemma in conjunction with (34) imply that $\nabla\varphi = O(|w|^{m+1})$. Thus $\nabla^2\varphi(0)$ exists and is equal to zero. On the other hand, we infer from (35) that $\nabla^2\varphi(\tilde{w}) = O(|\tilde{w}|^m)$ holds. Altogether we arrive at $\varphi \in C^2(\mathcal{V})$, and the lemma is proved. □

Lemma 5 permits the introduction of a new independent variable $\tilde{w} = F(w) \in \mathcal{V}$ such that $X = (x, y, z)$ can be written as

$$\begin{aligned} x(\tilde{w}) + iy(\tilde{w}) &= \tilde{w}^{m+1} \\ z(\tilde{w}) &= \varphi(\tilde{w}) \end{aligned} \quad \text{for } \tilde{w} \in \mathcal{V}, \tag{36}$$

where $\varphi \in C^2(\mathcal{V})$ and $\nabla^k\varphi(\tilde{w}) = O(|\tilde{w}|^{m+2-k})$ for $0 \le k \le 2$. (The reader will excuse the sloppy notation $X(\tilde{w})$ for the transformed surface; actually we should write $X(F^{-1}(\tilde{w}))$.)

Now we want to describe some local properties of the function φ which appears in the representation formula (36).

Lemma 6. *Let φ be the function that appears in (36), and let $w = u^1 + iu^2$. Then we obtain*

$$D_\alpha \left\{ \frac{\varphi_{u^\alpha}}{\sqrt{1 + c^{-2}|\nabla\varphi|^2}} \right\} = 0 \quad \text{on } \mathcal{V}, \tag{37}$$

where $c(w) := (m+1)|w|^m, w = u^1 + iu^2$.

(Here, we were even more careless and renamed $\tilde{w}$ as w. Thus the reader should bear in mind that $X(w)$ actually means the transformed surface $X(F^{-1}(\tilde{w}))$. The advantage of our sloppiness is that the following formulas become less cumbersome to read.)

Proof. From (36_1) we see that every point $p \in \mathcal{V} \setminus \{0\}$ has a neighbourhood $\mathcal{V}_1(p)$ which is mapped in a regular way onto a neighbourhood $\mathcal{V}_2$ in the x, y-plane. We write $x^1 = x, x^2 = y$. On $\mathcal{V}_2$ the function $\varphi(u^1, u^2)$ obtains a new representation $\psi(x^1, x^2)$, i.e.,

$$\varphi(u^1, u^2) = \psi(x^1, x^2).$$

As X is a minimal surface, we infer that

$$z = \psi(x^1, x^2)$$

provides a nonparametric representation of this minimal surface. Therefore $\psi(x^1,x^2)$ must satisfy the minimal surface equation

$$D_\alpha \left\{ \frac{\psi_{x^\alpha}}{\sqrt{1+\psi_{x^1}^2+\psi_{x^2}^2}} \right\} = 0 \quad \text{in } \mathcal{V}_2.$$

From $\varphi(w) = \psi(\text{Re } w^m, \text{Im}\, w^m)$, we conclude by a straightforward computation that (37) holds in $\mathcal{V}_1$ and therefore also in $\mathcal{V} \setminus \{0\}$.

Now we claim that $\sqrt{1+c^{-2}|\nabla\varphi|^2}$ is of class $C^1(\mathcal{V})$. In fact, let $\lambda_\alpha := c^{-1}\varphi_{u^\alpha}$. Then we see that $\lambda_\alpha(w) = O(|w|)$, whence we can extend $\lambda_\alpha(w)$ in a continuous way to $\mathcal{V}$ by setting $\lambda_\alpha(0) = 0$. It follows that $\lambda_\alpha(w)\lambda_\beta(w) = O(|w|^2)$, and therefore $\nabla(\lambda_\alpha\lambda_\beta)(0) = 0$. Finally we derive from

$$\lambda_{\alpha,\beta} = \varphi_{,\alpha\beta}\, c^{-1} - c^{-2}\varphi_{,\alpha}\, c_{,\beta} = O(1)$$

that $\nabla(\lambda_\alpha\lambda_\beta) = O(|w|)$, whence $\lambda_\alpha\lambda_\beta \in C^1(\mathcal{V})$. This concludes the proof of (37). □

It follows from the representation (36) that selfintersections of X occur at points which are images of points $w \in \mathcal{V}$ with $\varphi(w) = \varphi(\eta^j w)$ where η denotes some primitive $(m+1)$-th root of unity, and $j \not\equiv 0 \mod(m+1)$. Note that $\varphi^*(w) := \varphi(\eta w)$ again satisfies (37). Hence the difference $\Phi(w) := \varphi(w) - \varphi^*(w)$ is a solution of a linear elliptic differential equation. To be precise, we have

Lemma 7. *The difference function Φ satisfies*

$$\{a_{\alpha\beta}(w)\Phi_{u^\alpha}\}_{u^\beta} = 0 \quad \text{in } \mathcal{V}, \tag{38}$$

where $a_{\alpha\beta}$ is of class $C^1(\mathcal{V})$ and uniformly elliptic on $\mathcal{V}$, and $a_{\alpha\beta}(0) = \delta_{\alpha\beta}$.

Proof. Set $T_\alpha(w,q) := q_\alpha/\sqrt{1+c^{-2}(w)|q|^2}$ with $|q|^2 = q_\alpha q_\alpha$, and observe that

$$\begin{aligned} T_\alpha(w,\nabla\varphi^*) - T_\alpha(w,\nabla\varphi) &= \int_0^1 \frac{d}{dt} T_\alpha(w, t\nabla\varphi^* + (1-t)\nabla\varphi)\, dt \\ &= \left(\int_0^1 T_{\alpha,q_\beta}(w, t\nabla\varphi^* + (1-t)\nabla\varphi)\, dt \right) \Phi_{u^\beta}. \end{aligned}$$

Then one sees that the assertion follows for

$$a_{\alpha\beta}(w) := \int_0^1 T_{\alpha,q_\beta}(w, t\nabla\varphi^*(w) + (1-t)\nabla\varphi(w))\, dt. \qquad \square$$

It will be useful to obtain an asymptotic representation for the difference function Φ. This can be achieved by the technique of Hartman and Wintner (cf. Section 3.1), which yields the following *alternative*:

Either $\Phi(w) \equiv 0$, or there exists some integer $n \geq 1$ and some number $a \in \mathbb{C}$, $a \neq 0$, such that

$$\Phi_{u^1} - i\Phi_{u^2} = aw^{n-1} + \rho(w) \tag{39}$$

holds with $\rho(w) = o(|w|^{n-1})$ as $w \to 0$.

Integrating (39), we arrive at

$$(40)\qquad \begin{aligned} \Phi(w) &= \operatorname{Re}\left\{\frac{a}{n}w^n\right\} + \sigma(w), \\ \sigma_{u^1}(w) - i\sigma_{u^2}(w) &= \rho(w), \quad \sigma(w) = o(|w|^n) \quad \text{as } w \to 0. \end{aligned}$$

Applying once again the reasoning used in the proof of Lemma 5 we obtain the existence of some diffeomorphism T defined on some open disk $B_R(0)$ such that

$$(41)\qquad \Phi(w) = \operatorname{Re} T^n(w),$$

and that $T(0) = 0$ and $T'(0) \neq 0$ hold.

Then we derive from the alternative above the following result:

Proposition 1. *Let $X\colon B \cup I \to \mathbb{R}^3$ denote some solution of the thread problem, and suppose that $0 \in I$ is a branch point of X of order $m = 2\nu$. Furthermore, let φ, φ^*, Φ and T be the mappings which we have defined before. Then there exists some neighbourhood $\mathcal{V}_0$ of the origin 0 in $\mathbb{C} \mathrel{\hat{=}} \mathbb{R}^2$ such that the following alternative holds true:*

(i) *Either $X|_{\mathcal{V}_0}$ can be reparametrized in such a way that it becomes an immersed surface,*

(ii) *or else, there exist two simple C^1-arcs $\gamma_1, \gamma_2 : [0, \varepsilon] \to \mathcal{V}_0 \cap \overline{B}$ with $\gamma_j(0) = 0$, $|\gamma_j'(0)| = 1, \gamma_1'(0) \neq \gamma_2'(0), X(\gamma_1(t)) = X(\gamma_2(t))$ for all $t \in [0, \varepsilon]$ and such that the vectors $X_u(\gamma_1(t)) \wedge X_v(\gamma_1(t))$ and $X_u(\gamma_2(t)) \wedge X_v(\gamma_2(t))$ are linearly independent for all $t \in [0, \varepsilon]$.*

Proof. Suppose first that $\Phi(w) \equiv 0$. Then, as in Lemma 5, we can show that (i) holds with $\mathcal{V}_0 = \mathcal{U}$. In fact, the system (36) assigns to each $\tilde{w} \in \mathcal{V}$ or to each $w \in \mathcal{U}$ a unique point $X(w) = (x^1(w), x^2(w), x^3(w))$, and the surface X may locally be written as $x^3 = \psi(x^1, x^2)$ with $\psi(x^1, x^2) = \varphi(\tilde{w}), \tilde{w} = F(w)$ and $\psi \in C^1$ since $D\varphi(\tilde{w}) = O(|\tilde{w}|^{m+1})$.

Now we want to settle the case $\Phi(w) \not\equiv 0$ using the expansion (40). We note that $n \geq m + 2$ since $\Phi(w) = O(|w|^{m+2})$. Since $m = 2\nu \geq 2$, we find that $n \geq 4$. Define $\mathcal{V}_0 := F^{-1}(B_R(0))$ with a sufficiently small number $R > 0$, and consider the mapping $T \circ F\colon \mathcal{V}_0 \to \mathbb{C}$ which is conformal at the origin. Let $\zeta := T \circ F(w)$, and denote by $\mathcal{R}_j, 1 \leq j \leq 2n$, the $2n$ rays in the ζ-plane which emanate from $\zeta = 0$ and are defined by $\operatorname{Re} \zeta^n = 0$. The rays $\mathcal{R}_j$ correspond to $2n$ curves γ_j in $\mathcal{V}_0$ via the mapping $T \circ F$. Moreover, since $n \geq 4$, at least one of the curves γ_j meets the positive real axis at an angle which is between 0 and $\pi/3$. We can assume that $\gamma_j(t)$ is such an arc, and we can also assume that t is the parameter of arc length along γ_1. Then we have

$$0 = \Phi(F \circ \gamma_1(t)) = \varphi(F \circ \gamma_1(t)) - \varphi(\eta F \circ \gamma_1(t)).$$

Setting $\gamma_2(t) := F^{-1} \circ (\eta F \circ \gamma_1(t))$, we arrive at $X \circ \gamma_1(t) = X \circ \gamma_2(t)$.

Moreover, because of conformality, $\gamma_2(t)$ hits the positive real axis under an angle which is strictly between $\pi/3$ and $\pi/3 + \frac{2\pi}{m+1} < \pi$. For sufficiently small $\varepsilon > 0$, the mappings γ_1 and γ_2 will map $[0, \varepsilon]$ into $\mathcal{V}_0 \cap \overline{B}$. Since Φ describes the difference of two branches of X and because of (41), it immediately follows that the two surface normals along γ_1 and γ_2 respectively are linearly independent. $\square$

Let us now recall the definition of true and false branch points given in Section 1.9.

Definition. *The branch point $w = 0$ of the minimal surface $X(w)$ is called a* false branch point *if case* (i) *holds true; otherwise $w = 0$ is called a true branch point.*

Concerning true branch points, we shall prove:

Proposition 2. *If $X: \overline{B} \to \mathbb{R}^3$ is a solution of the thread problem, then there are no true branch points on the interval $I = \{u \in \mathbb{R}: |u| < 1\}$, which is mapped by X onto the movable boundary Σ.*

Proof. We first recall that X not only minimizes Dirichlet's integral within $\mathcal{C}(\Gamma, L)$ but also the area functional

$$A_B(X) = \int_B |X_u \wedge X_v| \, du \, dv;$$

cf. Theorem 4 of Section 5.2.

We may again assume that the true branch point $w \in I$ under consideration is the point $w = 0$.

Choose a neighbourhood $\mathcal{W}$ of 0 in $\mathbb{C}$ such that $\overline{\mathcal{W} \cap B}$ is diffeomorphic to $\overline{B}$. Suppose that the curves γ_1 and γ_2 first leave $\mathcal{W}$ at $\gamma_1(2\delta)$ and $\gamma_2(2\delta)$ transversally to $\partial\mathcal{W}$. Moreover, let $h: \overline{\mathcal{W} \cap B} \to \overline{B}$ be some C^1-diffeomorphism under consideration which, in addition, maps $\partial\mathcal{W} \cap B$ onto $\partial B \setminus \overline{I}$ and $\overline{\mathcal{W} \cap I}$ onto $\overline{I}$. Furthermore we may assume that

$$h \circ \gamma_1(t) = \frac{t}{\delta}\xi, \quad h \circ \gamma_2(t) = -\frac{t}{\delta}\overline{\xi}$$

for $0 \le t \le 2\delta$, where $\xi \in \mathbb{C}$ denotes some number with $|\xi| = \frac{1}{2}$.

It is now possible to construct a mapping $G: \overline{B} \to \overline{B}$ with the following properties:

(I) G is continuous and one-to-one on $\overline{B} \setminus [0, \frac{i}{2}]$;

(II) $G|_{\partial B} = \mathrm{id}|_{\partial B}$;

(III) For $\zeta \in \mathbb{C}$ with $\operatorname{Re} \zeta > 0$ and $0 \le t < 1$, the following relations are fulfilled:

$$\lim_{\zeta \to 0} G\left(\frac{i}{4}(1 \pm t) + \zeta\right) = (1 - t)\xi,$$

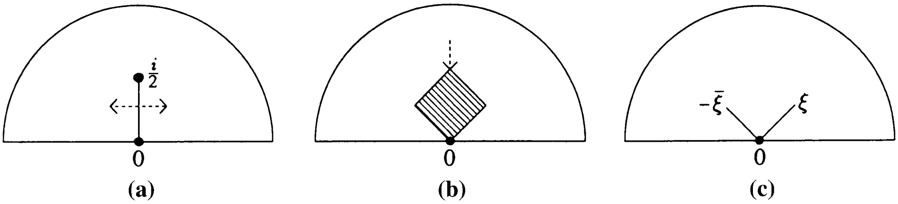

Fig. 3.

$$\lim_{\zeta\to 0} G\left(\frac{i}{4}(1\pm t)-\zeta\right) = -(1-t)\overline{\xi};$$

(IV) G is piecewise C^1 and extends to a C^1-diffeomorphism on each edge of the slit $[0, \frac{i}{2}]$.

We refrain from constructing G explicitly by formulas; Fig. 3 describes the topological action of G.

Now we define a comparison function $X^*: \overline{B} \to \mathbb{R}^3$ by

$$X^*(w) := \begin{cases} X(w) & \text{for } w \in B \setminus \mathcal{W}, \\ (X \circ h^{-1} \circ G \circ h)(w) & \text{for } w \in \mathcal{W}. \end{cases}$$

It is clear that $X^* \in \mathcal{C}(\Gamma, L)$ and that $A_B(X) = A_B(X^*)$. Hence X^* minimizes $A_B(X)$ within $\mathcal{C}(\Gamma, L)$. This leads to a contradiction, since any point $w_0 \in \mathcal{W}$ satisfying $h(w_0) \in (0, \frac{i}{2}]$ possesses some neighbourhood which is mapped onto a surface with two portions intersecting along $X(\gamma_1)$. In view of (ii) this surface has an edge, and by "smoothing out" one can construct from X^* a new surface $X^{**} \in \mathcal{C}(\Gamma, L)$ with $A_B(X^{**}) < A_B(X^*) = A_B(X)$, a contradiction to the minimizing property of X. □

To exclude false branch points we assume that $X|_{\partial B}$ is an embedding of ∂B into $\mathbb{R}^3$. The pertinent reasoning is sketched in Section 4.7 of Vol. 1. A detailed discussion can be found in the paper of Gulliver, Osserman, and Royden [1].

By these remarks we conclude the proof of Theorem 1. □

5.4 Scholia

1. The existence of solutions of the thread problem in its simplest form was first proved by H.W. Alt [3]. Except for minor modifications we have presented Alt's existence proof in Section 5.2. Without any changes the proof can be carried over to 2-dimensional surfaces in $\mathbb{R}^N, N \geq 2$. A different proof has been given by K. Ecker [1], using methods of geometric measure theory; it even works for the analogue of the thread problem concerning n-dimensional

surfaces in $\mathbb{R}^N$. In the framework of integral currents, Ecker has proved the existence of a minimizer, the movable boundary of which has prescribed mass.

2. It seems to have been known for a long time that the *unattached* part of the movable boundary Σ consists of space curves of constant curvature; cf. van der Mensbrugghe [1], Otto [1]. A satisfactory proof was given by Nitsche [21] under the assumption that the free part of Σ is known to be regular and smooth; cf. also Nitsche [28], pp. 435–437 and pp. 706–707.

3. The first results concerning the boundary regularity of solutions for the thread problem were found by Nitsche [23–25]. He proved that the open components of the non-attached part of the movable boundary have a parametrization of class $C^{2,\alpha}$, for some $\alpha \in (0,1)$. Between branch points (the existence of which was not excluded by Nitsche) these parametrizations turn out to be of class C^∞.

The sharper regularity results, presented in Section 5.3, and their proofs are taken from Dierkes, Hildebrandt, and Lewy [1]. We have quite closely followed the presentation given in their paper.

4. By completely different techniques, K. Ecker [1] has established C^∞-regularity of the free part of the movable boundary Σ in the context of his integral-current solutions; the analyticity is in this case still an open question.

5. It is not known whether the *thread* of the solution constructed in Section 10.2 can have self-intersections; we are tempted to conjecture that this cannot occur. In the context of rectifiable flat chains modulo 2 this was in fact proved by R. Pilz [1]. He showed that the free boundary of a minimizer of this kind has no singular points in $\mathbb{R}^3 \setminus \Gamma$, Γ being the fixed part of the boundary.

6. Alt [3] has also proved that the movable arc Σ must always lift off Γ in a tangential way whenever it adheres to Γ in a subarc of positive length provided that Γ is supposed to be smooth.

7. As Alt [3] has pointed out, all pieces of the movable boundary Σ not attached to Γ have the same constant curvature κ. This can easily be proved by the reasoning given in the proofs of Lemmata 2–4 of Section 5.3.

8. In excluding branch points on the free parts of Σ we have used arguments of Gulliver and Lesley [1] and of Gulliver, Osserman, and Royden [1]. This part of our reasoning is restricted to $\mathbb{R}^3$ and cannot be carried over to $\mathbb{R}^n, n \geq 4$, according to an example by Federer [2].

9. A new existence proof for the thread problem was given by E. Kuwert in Section 4 of his Habilitationsschrift [5], pp. 51–52. This proof is a by-product of Kuwert's work on the minimization of Dirichlet's integral $D(X)$ among surfaces $X : B \to \mathbb{R}^n$ whose boundary curves $X\big|_{\partial B}$ represent a given homotopy class α of free loops in a closed configuration $S \subset \mathbb{R}^n$. We refer to the Scholia of Chapter 1 in this volume and to Kuwert [6,7].

10. Recently, the thread problem was anew studied by B.K. Stephens [1–3]. In Section 2 of [1], a new proof of Alt's theorem is given, and Section 3 presents two quantitative bounds on the nearness of minimizers to the wire in

the case that the thread length L is not much less than the length $\ell(\Gamma)$ of the wire Γ: *Suppose that* $\kappa_{\max}$ *is a bound on the curvature of* Γ*, and let* $0 < \lambda \ll 1$ *(relative to the* C^3*-data of* Γ*). Then there is a constant* $R(\Gamma, \lambda) > 0$ *with the following property: If* X *is a minimizer of* $\mathcal{P}(\Gamma, \ell(\Gamma) - \lambda)$*, then the image* M *of* X *lies in a "normal* $\mathcal{R}(\Gamma, \lambda)$*-neighbourhood of* Γ*" whose radius is estimated by*

$$R(\Gamma, \lambda) \leq 2\lambda^{1/2}/(\pi\kappa_{\max})^{1/2} + o(\lambda^{1/2}),$$

and the area of M *is bounded by*

$$A(M) \leq \lambda/\kappa_{\max} + o(\lambda^{1/2}).$$

Consider now the situation studied in Theorem 1 of Section 5.3 (cf. also Fig. 2 of 5.3), and as Stephens [2], call minimizers of this kind *"crescents"*. In [2], several geometric properties of *"near-wire crescents"* are proved. For instance, the representation of such a crescent X as a graph of a Lipschitz function f with $\mathrm{Lip}(f) \leq \mathrm{const}(\Gamma)R^{1/12}$ is established if X lies in an R-tubular neighbourhood of Γ, $0 < R \ll 1$. The main tool is a sophisticated generalization of a result due to Radó (see Vol. 1, Section 4.9, Lemma 2), which Stephens calls "Free Radó Lemma", as it is an adjustment of the original Radó Lemma to the situation available in the thread-problem case.