Chapter 4

Enclosure and Existence Theorems for Minimal Surfaces and H-Surfaces. Isoperimetric Inequalities

In this chapter we shall discuss certain quantitative geometric properties of minimal surfaces and surfaces of prescribed mean curvature.

We begin by deriving *enclosure theorems*. Such results give statements about the confinement of minimal surfaces to certain "enclosing sets" on the basis that one knows something about the position of their boundaries. For example, any minimal surface is contained in the convex hull of its boundary values. All of our results will in one way or another be founded on some version of the maximum principle for subharmonic functions.

Closely related to these theorems are *nonexistence theorems for multiply connected surfaces*. Everyone who has played with wires and soap films will have noticed that a soap film catenoid between two coaxial parallel circles will be torn up if one moves the two wires too far apart. Section 4.1 supplies a very simple proof of the corresponding mathematical assertion which again relies on the maximum principle for subharmonic functions.

A comparison principle for solutions of the equation of prescribed mean curvature is employed in the study of points where two (parametric) surfaces of continuous mean curvature H ("H-surfaces" for short) touch without crossing each other. The resulting *touching point theorem* (Section 4.2) implies further enclosure and nonexistence theorems. Since the proofs are nearly identical for minimal surfaces (where $H \equiv 0$) and for surfaces of continuous mean curvature H, we shall deal with the latter.

We have chosen to extend these principles to submanifolds of arbitrary dimension and, if possible, of arbitrary codimension as well (Section 4.3). In Section 4.4 we discuss a *"barrier principle"* for submanifolds of $\mathbb{R}^{n+k}$ with bounded mean curvature and arbitrary codimension k. Furthermore, a similar argument is used to prove a *"geometric inclusion principle"* for strong (possibly branched) subsolutions of a variational inequality, which is later used (Section 4.7) in a crucial way to solve the Plateau problem for H-surfaces in Euclidean space. Additionally we present some existence theorems for sur-

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faces of prescribed mean curvature with a given boundary in a Riemannian manifold (Section 4.8).

The enclosure theorems of this chapter also serve to find conditions ensuring that the solutions of the free (Chapter 1) or semifree (Chapter 4 of Vol. 1) variational problems for minimal surfaces remain on one side of their supporting surface. Only such solutions describe the soap films produced in experiments because these can evidently never pass through a supporting surface made of e.g. plexiglas, whereas in general we cannot exclude this phenomenon for the solutions of the corresponding variational problems (unless we consider problems with obstructions; see Vol. 1, Section 4.10, no. 5).

Moreover, if the minimal surface remains on one side of the supporting surface, then there are no branch points on the free boundary, as follows from the asymptotic expansions in Chapter 3 (see also Section 2.10). This will be of importance for some of the trace estimates proved in Section 4.6.

The two Sections 4.5 and 4.6 deal with the relationship between the area of a minimal surface and the length of its boundary. In particular, *isoperimetric inequalities* bound the area in terms of the length of the boundary and, possibly, of other geometric quantities. It is a surprising fact that minimal surfaces satisfy the same isoperimetric inequalities as a planar domain Ω for which the relation

$$4\pi A \leq L^2$$

holds true, A being the area of Ω and L the length of $\partial\Omega$.

In Section 4.6 we shall derive upper and lower bounds for the length $L(\Sigma)$ of the free trace Σ of a stationary minimal surface X in a semifree or a free boundary configuration $\langle\Gamma, S\rangle$ or $\langle S\rangle$ respectively. These bounds will depend on geometric quantities such as the area of X, the length of the fixed part Γ of its boundary, and of parameters bounding the curvature of the supporting surface S. We shall close this section by discussing analogous questions for solutions of a *partition problem* which turn out to be stationary surfaces X of constant mean curvature with a free boundary on the surface S of a body U which is partitioned by X.

4.1 Applications of the Maximum Principle and Nonexistence of Multiply Connected Minimal Surfaces with Prescribed Boundaries

Our first result is the prototype of an *enclosure theorem*; it will be obtained by a straight-forward application of the maximum principle for harmonic functions.

Theorem 1 (Convex hull theorem). *Suppose that* $X \in C^0(\overline{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$ *is harmonic in a bounded and connected open set* $\Omega \subset \mathbb{R}^2$. *Then* $X(\overline{\Omega})$ *is contained in the convex hull of its boundary values* $X(\partial\Omega)$.

Proof. Let A be a constant vector in $\mathbb{R}^3$. Then $h(w) := \langle A, X(w)\rangle$ is harmonic in Ω, and we apply the maximum principle to h. Hence, if for some number $d \in \mathbb{R}$, the inequality

$$\langle A, X(w)\rangle \le d$$

holds true for all $w \in \partial\Omega$, it is also satisfied for all $w \in \overline{\Omega}$. As any closed convex set is the intersection of its supporting half-spaces, the assertion is proved. □

Throughout this section, let us agree upon the following *terminology*:

A finite connected minimal surface is a nonconstant mapping

$$X \in C^0(\overline{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$$

which is defined on the closure of a bounded, open, connected set $\Omega \subset \mathbb{R}^2$ and satisfies

$$\Delta X = 0 \tag{1}$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0 \tag{2}$$

in Ω. We call Ω the parameter domain of X.

Then, on account of Theorem 1, we obtain

Corollary 1. *Any finite connected minimal surface X with the parameter domain Ω is contained in the convex hull of its boundary values $X|_{\partial\Omega}$, that is,*

$$X(\overline{\Omega}) \subset \textit{convex hull } X(\partial\Omega). \tag{3}$$

In fact, we can sharpen this statement by inspecting the proof of Theorem 1. Suppose that

$$h(w_0) := \langle A, X(w_0)\rangle = d$$

holds for some $w_0 \in \Omega$, in addition to

$$h(w) \le d \quad \text{for all } w \in \partial\Omega.$$

Then the maximum principle implies

$$h(w) = d \quad \text{for all } w \in \overline{\Omega}.$$

Thus we obtain

Corollary 2. *If a finite connected minimal surface X with the parameter domain Ω touches the convex hull $\mathcal{K}$ of its boundary values $X(\partial\Omega)$ at some "interior point" $X(w_0), w_0 \in \Omega$, then X is a planar surface. In particular, X cannot touch any corner of $\partial\mathcal{K}$ nor any other nonplanar point of $\partial\mathcal{K}$.*

The reader will have noticed that, so far, we have nowhere used the conformality relations (2). In other words, all the previous results are even true for harmonic mappings. Thus we may expect that by using (2) we shall obtain stronger enclosure theorems which will better reflect the saddle-surface character of nonplanar minimal surfaces. In fact, we have

Theorem 2 (Hyperboloid theorem). *If $X(w) = (x(w), y(w), z(w))$ is a finite connected minimal surface with the parameter domain Ω, whose boundary $X(\partial\Omega)$ is contained in the hyperboloid*

$$\mathcal{K}_\varepsilon := \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 \le \varepsilon^2\right\},$$

$\varepsilon > 0$, then $X(\overline{\Omega})$ lies in $\mathcal{K}_\varepsilon$. Moreover, we even have $X(\Omega) \subset \operatorname{int} \mathcal{K}_\varepsilon$.

Proof. Note that $\mathcal{K}_\varepsilon$ is the sublevel set

$$\mathcal{K}_\varepsilon = \left\{(x, y, z) : f(x, y, z) \le \varepsilon^2\right\} \tag{4}$$

of the quadratic form

$$f(x, y, z) := x^2 + y^2 - z^2.$$

Let us therefore compute the Laplacian of the composed map $h := f \circ X = f(X)$. We obtain

$$\Delta h = \langle \nabla X, D^2 f(X) \nabla X \rangle + \langle Df(x), \Delta X \rangle. \tag{5}$$

Because of (1) and

$$D^2 f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

it follows that

$$\Delta h = 2(|\nabla x|^2 + |\nabla y|^2 - |\nabla z|^2) \quad \text{in } \Omega. \tag{6}$$

Moreover, we can write (2) in the complex form

$$\langle X_w, X_w \rangle = 0, \tag{7}$$

that is,

$$x_w^2 + y_w^2 + z_w^2 = 0,$$

whence we obtain

$$|\nabla z|^2 \le |\nabla x|^2 + |\nabla y|^2 \quad \text{in } \Omega. \tag{8}$$

From (6) and (8) we infer that

$$\Delta h \ge 0 \quad \text{in } \Omega,$$

i.e., h is subharmonic, and the assumption yields $h(w) \le \varepsilon^2$ for all $w \in \partial\Omega$, taking (4) into account. Then the maximum principle implies $h(w) \le \varepsilon^2$ for all $w \in \overline{\Omega}$ whence $X(\overline{\Omega}) \subset \mathcal{K}_\varepsilon$.

Suppose that $X(w_0) \in \partial\mathcal{K}_\varepsilon$ for some $w_0 \in \Omega$. Then we would have $h(w_0) = \varepsilon^2$, and the maximum principle would imply $h(w) \equiv \varepsilon^2$, i.e., $X(w) \in \partial\mathcal{K}_\varepsilon$ for all $w \in \overline{\Omega}$. As $X(w) \not\equiv$ const, we know that $X(w)$ has zero mean curvature (except for the isolated branch points) which contradicts the relation $X(\Omega) \subset \partial\mathcal{K}_\varepsilon$, since no open part of $\partial\mathcal{K}_\varepsilon$ is a minimal surface. □

Let us take one step further and assume that the boundary of the minimal surface X is even contained in the cone

$$\mathcal{K}_0 = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) \le 0\} = \bigcap_{\varepsilon>0} \mathcal{K}_\varepsilon.$$

Then in view of the hyperboloid theorem the whole surface $X(\overline{\Omega})$ is contained in the cone $\mathcal{K}_0$.

Can it be true that, in addition, the boundary $X(\partial\Omega)$ intersects both cones

$$\mathcal{K}_0^\pm := \mathcal{K}_0 \cap \{z \lesseqgtr 0\}?$$

If so, then there is some $w \in \Omega$ such that the point $X(w)$ of the minimal surface lies in the vertex of the cone $\mathcal{K}_0$, that is, $X(w_0) = 0$ for some $w_0 \in \Omega$.

On the other hand, as $X(w) \not\equiv$ const, the minimal surface X has a (possibly generalized) tangent plane T at $X(w_0) = 0$; cf. Section 3.2 of Vol. 1. Clearly, there is no neighbourhood U of 0 in $\mathbb{R}^3$ such that $T \cap U \subset \mathcal{K}_0$. Then one infers that the relation $X(w_0) = 0$ is impossible, taking the asymptotic expansion

$$X_w(w) = A(w - w_0)^m + O(|w - w_0|^{m+1}) \quad \text{as } w \to w_0$$

with $A \in \mathbb{C}^3$, $A \ne 0$, $m \ge 0$, into account.

Hence, except for a suitable congruence mapping, we have shown the following result:

Theorem 3 (Cone theorem). *Let $\mathcal{K}$ be a cone congruent to $\mathcal{K}_0$ which consists of the two half-cones $\mathcal{K}^+$ and $\mathcal{K}^-$ corresponding to $\mathcal{K}_0^+$ and $\mathcal{K}_0^-$. Then there is no finite connected minimal surface the boundary of which lies in $\mathcal{K}$ and intersects both $\mathcal{K}^+$ and $\mathcal{K}^-$.*

The cone theorem can be used to prove nonexistence results for Plateau problems, or for free (or partially free) boundary value problems. Instead of formulating a general theorem, we shall merely consider a special case that illustrates the situation. The reader can easily set up other – and possibly more interesting – examples, or he may himself formulate a general *necessary* criterion for the existence of stationary minimal surfaces within a given boundary configuration $\langle \Gamma_1, \dots, \Gamma_l, S_1, \dots, S_m \rangle$.

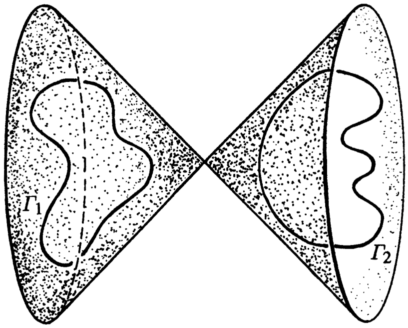

Fig. 1. Two suitable cones give a nonexistence result

Consider two closed Jordan curves Γ_1 and Γ_2 which can be separated by some cone $\mathcal{K}$ as described in Theorem 3. That is, we can move the *test cone* $\mathcal{K}_0$ into such a position $\mathcal{K}$ that Γ_1 lies in the half-cone $\mathcal{K}^+$ and Γ_2 is contained in $\mathcal{K}^-$. *Then there is no connected solution of the general Plateau (or Douglas) Problem for the boundary configuration* $\langle \Gamma_1, \Gamma_2 \rangle$. This corresponds to the experimental fact mentioned in the introduction to this chapter: A soap film spanned into two closed (non-linked) wires Γ_1 and Γ_2 will decompose into two parts separately spanning Γ_1 and Γ_2 if Γ_1 and Γ_2 are moved sufficiently far apart.

We shall show at the end of the next section that the "test cone $\mathcal{K}_0$ for non-existence" may even be replaced by a slightly larger set.

Further results about enclosure and nonexistence of minimal surfaces can be obtained by an elaboration and extension of the ideas used in the proof of the Theorems 1–3, some of which will be worked out in the next three sections. Note, however, that the use of the maximum principle was by no means the first way to obtain information about the extension of minimal surfaces and about nonexistence of solutions to boundary value problems, though the maximum principle is certainly the simplest tool to obtain such results. Concerning other methods we refer to Nitsche's monograph [28], Kap. VI, 3.1, pp. 474–498, and pp. 707–708 of the Appendix (=Anhang).

4.2 Touching H-Surfaces and Enclosure Theorems. Further Nonexistence Results

In the sequel we shall look for other sets $\mathcal{K}$ enclosing any finite connected minimal surface whose boundary is confined to $\mathcal{K}$. Since nothing is gained if we restrict our attention to minimal surfaces, we shall more generally study surfaces of continuous mean curvature H (or "H-surfaces").

To avoid confusion we recall our notation from Chapter 1 of Vol. 1: $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ denote the coefficients of the first and second fundamental form of a surface X; H and K stand for its mean curvature and Gauss curvature respectively.

Assumption. *Throughout this section we will assume that H is a continuous real-valued function on $\mathbb{R}^3$.*

Definition 1. *An H-surface X is a nonconstant map $X \in C^2(\Omega, \mathbb{R}^3)$ defined on an open set Ω satisfying*

$$\Delta X = 2H(X) X_u \wedge X_v \tag{1}$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{2}$$

We call Ω the parameter domain of the H-surface X. An H-surface X is said to be finite and connected if its parameter domain Ω is a bounded, open, connected set in $\mathbb{R}^2$, and if $X \in C^0(\overline{\Omega}, \mathbb{R}^3)$.

Clearly, minimal surfaces are H-surfaces with $H \equiv 0$.

In order to study touching H-surfaces, we need the following

Lemma 1. *Suppose that $\Phi : B_r(0) \to \mathbb{C}$ is a function of class C^1 which can be written in the form*

$$\Phi(w) = a(w - w_0)^m + \Psi(w), \quad w \in B_r(0), \tag{3}$$

for some $w_0 \in B_r(0)$, some real number $a > 0$, some integer $m \geq 1$, and some mapping $\Psi : B_r(0) \to \mathbb{C}$ with $\Psi(w_0) = 0$ and

$$\nabla \Psi(w) = o(|w - w_0|^{m-1}) \quad \text{as } w \to w_0. \tag{4}$$

Then there is some neighbourhood U of w_0 and some C^1-diffeomorphism φ from U onto $\varphi(U)$ such that

$$\Phi(w) = [\varphi(w)]^m \quad \text{for all } w \in U \tag{5}$$

holds true.

Proof. Clearly, if there exists some function φ satisfying (5), it has to be the function

$$\varphi(w) := (w - w_0) \sqrt[m]{\chi(w)}, \tag{6}$$

where

$$\chi(w) = a + (w - w_0)^{-m} \Psi(w). \tag{7}$$

We shall have to prove that φ is well defined and has the desired properties.

First of all, the relation (4) implies

$$\Psi(w) = o(|w - w_0|^m) \quad \textit{as } w \to w_0$$

since $\Psi(w_0) = 0$. Therefore $\chi(w)$ tends to a as $w \to w_0$, and we set $\chi(w_0) := a$. Hence there is a neighbourhood U_0 of w_0 where a single-valued branch $\sqrt[m]{\ }$ of the m-th root can be defined. Thus the function φ defined by (6) and (7) is a well-defined function near w_0.

Now (4) implies for the derivatives of χ in $U_0 - \{w_0\}$ that

$$\begin{aligned}
\chi_u(w) &= -m(w - w_0)^{-m-1}\Psi(w) + (w - w_0)^{-m}\Psi_u(w) = o(|w - w_0|^{-1}), \\
\chi_v(w) &= -mi(w - w_0)^{-m-1}\Psi(w) + (w - w_0)^{-m}\Psi_v(w) = o(|w - w_0|^{-1}),
\end{aligned}$$

whence

$$\begin{aligned}
\varphi_u(w) &= \sqrt[m]{\chi(w)} + \frac{1}{m}(w - w_0)\chi(w)^{(1-m)/m}\chi_u(w) \\
&= \sqrt[m]{\chi(w)} + o(1), \\
\varphi_v(w) &= i\sqrt[m]{\chi(w)} + \frac{1}{m}(w - w_0)\chi(w)^{(1-m)/m}\chi_v(w) \\
&= i\sqrt[m]{\chi(w)} + o(1),
\end{aligned}$$

and therefore

$$\lim_{w \to w_0} D\varphi(w) = \begin{pmatrix} \sqrt[m]{a} & 0 \\ 0 & i\sqrt[m]{a} \end{pmatrix}.$$

On the other hand, we have

$$\lim_{w \to w_0} \frac{\varphi(w)}{w - w_0} = \lim_{w \to w_0} \sqrt[m]{\chi(w)} = \sqrt[m]{a}.$$

Thus φ is a C^1-function, and the lemma follows from the inverse mapping theorem. □

Let us now describe what we can say about touching points of two H-surfaces, one of which is assumed to be regular.

Theorem 1. *Suppose that G is a domain in $\mathbb{R}^3$ and that $\partial_0 G$ is an open part of the boundary of G with $\partial_0 G \in C^2$. Secondly let X be a finite connected H-surface with the parameter domain Ω whose image $X(\Omega)$ lies in $G \cup \partial_0 G$. Finally, denoting the mean curvature of $\partial_0 G$ at P with respect to the interior normal by $\Lambda(P)$, we assume that*

(8) $$\sup_{\overline{G}} |H| \le \inf_{\partial_0 G} \Lambda$$

holds true. Then $X(\Omega)$ is completely contained in $\partial_0 G$ if $X(\Omega) \cap \partial_0 G$ is nonempty (that is, if $X(\Omega)$ "touches" $\partial_0 G$).

Remark 1. This is, in fact, a local result. Instead of (8), it suffices to assume that every point $P \in \partial_0 G$ has a neighbourhood U in $\overline{G}$ such that

$$\sup_U |H| \le \inf_{U \cap \partial_0 G} \Lambda. \tag{8'}$$

This remark implies the following

Enclosure Theorem I. *Let G be a domain in $\mathbb{R}^3$ with $\partial G \in C^2$, and let H be a continuous function on $\mathbb{R}^3$ satisfying*

$$|H(P)| < \Lambda(P) \quad \text{for all } P \in \partial G,$$

where Λ denotes again the mean curvature of ∂G with respect to the inward normal. Then every finite connected H-surface X with the parameter domain Ω whose image $X(\Omega)$ is confined to the closure $\overline{G}$ lies in G, i.e. $X(\Omega) \subset G$.

Remark 2. Note that the condition $|H(P)| \le \Lambda(P)$ for all $P \in \partial G$ is not sufficient to conclude the assertion of the theorem. Indeed this follows easily by considering a plane with $\Lambda \equiv 0$ and a paraboloid of fourth order lying on one side of the plane and touching it in a single point.

Proof of Theorem 1. Clearly we have $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 := X^{-1}(G), \quad \Omega_2 := X^{-1}(\partial_0 G).$$

Since X is continuous, the set Ω_1 is open. Suppose that $X(\Omega)$ touches $\partial_0 G$; then $\Omega_2 = \Omega \setminus \Omega_1$ is not empty. We show that the assumption "$\Omega_1 \ne \emptyset$" will lead to a contradiction.

In fact, suppose $\Omega_1 \ne \emptyset$. Then also $\partial\Omega_1 \cap \Omega$ is nonempty and we can select a point $z_0 \in \Omega_1$ which is closer to $\partial\Omega_1 \cap \Omega$ than to $\partial\Omega$. Since Ω_1 is open, there is a maximal open disc $B_r(z_0) \subset \Omega_1$ with the property $w_0 \in \partial B_r(z_0) \cap \partial\Omega_1 \cap \Omega$ for (at least) one point $w_0 \in \Omega_2$, i.e. $X(w_0) = P_0 \in \partial_0 G$. Without loss of generality we may suppose that $w_0 = 0$. By the reasoning of Section 2.10, we may assume after a suitable shift and rotation of the coordinate system that, close to $w_0 = 0$, the surface $X(w) = (x(w), y(w), z(w))$ has the asymptotic expansion

$$x(w) + iy(w) = aw^m + o(|w|^m),$$
$$z(w) = o(|w|^m),$$

for some integer $m > 0$ and some $a > 0$. According to the preceding Lemma 1, there is a neighbourhood $U \subset \Omega$ of 0 and a C^1-diffeomorphism $\varphi : U \to \varphi(U)$ such that for $w \in U$

$$x(w) + iy(w) = [\varphi(w)]^m.$$

Next we choose an $\varepsilon > 0$ so small that the disk $B_\varepsilon(0)$ is contained in $\varphi(U)$, whence $B_{\varepsilon^m}(0)$ lies in $\varphi^m(U)$. Therefore all the disks

$$\Omega_\varepsilon(\xi, \eta) = B_{\varepsilon^m/2}(\xi + i\eta) \quad \text{with } \xi^2 + \eta^2 = \left(\frac{\varepsilon^m}{2}\right)^2,$$

which cover $B_{\varepsilon^m}(0) \setminus \{0\}$, are subsets of $\varphi^m(U)$, and their preimages under the mapping φ^m cover a punctured neighbourhood of 0.

Now let $\sqrt[m]{}$ denote an arbitrary single-valued branch of the m-th root defined on $\Omega_\varepsilon(\xi, \eta)$. Then

$$z'(x, y) := z\left(\varphi^{-1}(\sqrt[m]{x + iy})\right)$$

defines a C^1-non-parametric representation of a part of the H-surface X, namely the one defined on $\varphi^{-1}(\sqrt[m]{\Omega_\varepsilon(\xi, \eta)})$. For the construction to follow it is convenient and necessary to choose $\Omega_\varepsilon(\xi, \eta) \subset B_r(z_0)$ such that $w_0 = 0 \in \partial\Omega_\varepsilon(\xi, \eta)$.

The plane $\{z = 0\}$ is the (possibly "generalized") tangent plane of X at $P_0 = X(w_0)$. Thus

$$\lim_{(x,y)\to 0} \nabla z'(x, y) = 0. \tag{9}$$

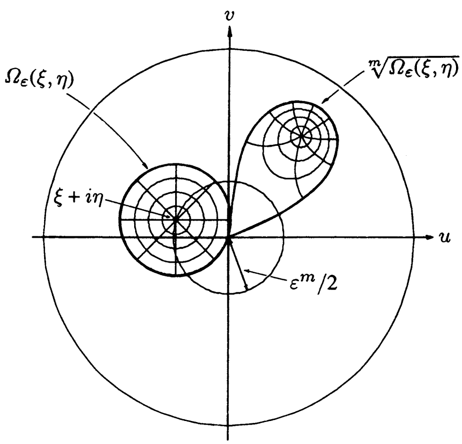

Fig. 1. The domains used in the proof of Theorem 1

Since $X(\Omega)$ lies on one side of $\partial_0 G$ and since $X(w_0)$ belongs to $\partial_0 G$, the set $\{z = 0\}$ is also the tangent plane of $\partial_0 G$ at $X(w_0)$. Therefore (after decreasing ε if necessary) we obtain also a local non-parametric representation of $\partial_0 G$ by means of a function

$$z'' = z''(x, y) \quad \text{for } (x, y) \in \overline{\Omega}_\varepsilon(\xi, \eta).$$

By assumption, we have $z'' \in C^2(\overline{\Omega}_\varepsilon(\xi, \eta))$. If the interior normal of $\partial_0 G$ at $X(w_0)$ points in the direction of the positive z-axis, (the other case is handled similarly), we have by assumption

$$z'' < z' \quad \text{on } \Omega_\varepsilon(\xi, \eta), \quad \text{and also} \quad z''(0) = z'(0). \tag{10}$$

Since $\{z \equiv 0\}$ is also the tangent plane of $\partial_0 G$ at $X(w_0)$ we have

$$\lim_{(x,y)\to 0} \nabla z''(x,y) = \nabla z''(0,0) = 0. \tag{11}$$

Moreover, z' and z'' are solutions of the corresponding equations of prescribed mean curvature (cf. Section 2.7 of Vol. 1), i.e.,

$$Q(z') := \operatorname{div} \frac{\nabla z'}{\sqrt{1+|\nabla z'|^2}} = \pm 2H(x,y,z'(x,y)),$$
$$Q(z'') := \operatorname{div} \frac{\nabla z''}{\sqrt{1+|\nabla z''|^2}} = 2\Lambda(x,y,z''(x,y))$$

for all $(x,y) \in \Omega_\varepsilon(\xi,\eta)$. By assumption, it follows that

$$Q(z') \le Q(z'') \quad \text{in } \Omega_\varepsilon(\xi,\eta).$$

It now readily follows from the theorem of the mean, that the difference $\hat{z} := z'' - z'$ satisfies a linear differential inequality of the type

$$L(\hat{z}) = a_{ij}(x) D_{ij}\hat{z} + b_i(x) D_i \hat{z} \ge 0 \quad \text{in } \Omega_\varepsilon(\xi,\eta),$$

where the coefficients b_i are locally bounded and the a_{ij}'s are elliptic (for a similar argument see e.g. the proof of Theorem 10.1 in Gilbarg and Trudinger [1]).

Now (9) and (11) yield that

$$\lim_{(x,y)\to 0} \nabla \hat{z}(x,y) = 0,$$

and hence also the normal derivative $\frac{\partial \hat{z}}{\partial n}(0,0) = 0$.

However, because of $\hat{z}(0) = 0 > \hat{z}(x,y)$ for all $(x,y) \in \Omega_\varepsilon(\xi,\eta)$, the point $w_0 = 0 \in \partial\Omega_\varepsilon(\xi,\eta)$ is a strict maximum, which contradicts Hopf's boundary point lemma (Lemma 3.4 in Gilbarg and Trudinger [1]). Consequently Ω_1 has to be empty and hence $\Omega = \Omega_2$ or $X(\Omega) \subset \partial_0 G$. This completes the proof of Theorem 1. □

Proof of Enclosure Theorem I. The condition $|H(P)| < \Lambda(P)$ for all $P \in \partial G$ clearly implies that every point $P \in \partial G$ has a neighbourhood U in $\overline{G}$, such that

$$\sup_U |H| \le \inf_{U \cap \partial G} \Lambda$$

holds true. Therefore a local version of Theorem 1 is applicable and we assume, contradictory to the assertion, that some interior point $w_0 \in \Omega$ is mapped onto ∂G, i.e. X touches ∂G at $X(w_0)$. It then follows from Theorem 1 that $X(\Omega) \subset \partial G$. On the other hand X is an H-surface, which in particular means that X has mean curvature H, except possibly at isolated singular points, compare the derivation of the asymptotic expansion near branch points in Section 2.10. Whence, by continuity, it follows that $|H(P)| = \Lambda(P)$ for all $P \in \partial G$, a contradiction to the assumption of the theorem. Enclosure Theorem I is proved. □

The reasoning used to prove Theorems 2 and 3 in Section 4.1, may be generalized to H-surfaces X. In fact consider the quadratic function

$$f(x,y,z) = x^2 + y^2 - bz^2,$$

with $0 \le b < 1$, and compute the Laplacian of the composed map $h := f \circ X$.

We obtain similarly as in Theorem 2 of Section 6.1

$$\Delta h = \langle \nabla X, D^2 f(X) \nabla X \rangle + \langle Df(X), \Delta X \rangle.$$

Because of (1) and

$$D^2 f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2b \end{pmatrix},$$

it follows that

$$\begin{aligned} \Delta h &= 2|\nabla x|^2 + 2|\nabla y|^2 - 2b|\nabla z|^2 + 4H(X) \cdot \langle (x,y,-bz), X_u \wedge X_v \rangle \\ &\ge 2|\nabla x|^2 + 2|\nabla y|^2 - 2b|\nabla z|^2 - 4|H(X)||X_u \wedge X_v| \cdot \sqrt{x^2 + y^2 + b^2 z^2}. \end{aligned}$$

From the conformality condition (2) we obtain

$$|\nabla z|^2 \le |\nabla x|^2 + |\nabla y|^2,$$

whence

$$|X_u \wedge X_v| \le |\nabla x|^2 + |\nabla y|^2.$$

Concluding we find

$$\begin{aligned} \Delta h &\ge 2|\nabla x|^2 + 2|\nabla y|^2 - 2b|\nabla z|^2 - 4|H(X)|\big(|\nabla x|^2 + |\nabla y|^2\big)\sqrt{x^2 + y^2 + b^2 z^2} \\ &\ge 2(|\nabla x|^2 + |\nabla y|^2)\left[1 - b - 2|H(X)| \cdot \sqrt{x^2 + y^2 + b^2 z^2}\right]. \end{aligned}$$

Thus we have proved

Theorem 2. *Let X be an H-surface on Ω and $f(x,y,z) = x^2 + y^2 - bz^2$, $0 \le b < 1$. Then the function $h = h(u,v) = f \circ X(u,v)$, $(u,v) \in \Omega$ is subharmonic on Ω, provided that*

$$b + 2|H(X)| \cdot \sqrt{x^2 + y^2 + b^2 z^2} \le 1 \quad \text{on } \Omega.$$

A consequence of this result and the asymptotic expansion for H-surfaces in singular points is the following

Theorem 3 (Cone Theorem). *Suppose that $X \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is an H-surface on Ω which satisfies*

$$\sup_{w \in \Omega} |X(w)| \cdot |H(X(w))| = q < \frac{1}{2}.$$

Then, for $b = 1-2q \in (0,1]$ *the function* $h(u,v) = x^2(u,v)+y^2(u,v)-bz^2(u,v)$ *is subharmonic on* Ω *and therefore by the maximum principle*

$$\sup_{\Omega} h \leq \sup_{\partial\Omega} h.$$

Moreover let $\mathcal{K} = \mathcal{K}^+ \cup \mathcal{K}^- \cup \{0\}$ *where*

$$\mathcal{K}^{\pm} = \left\{(x,y,z) : x^2 + y^2 - bz^2 \leq 0,\ \pm z > 0\right\}.$$

Suppose that $X(\partial\Omega)$ *is contained in* $\mathcal{K}$, *such that both intersections* $X(\partial\Omega) \cap \mathcal{K}^+$ *and* $X(\partial\Omega) \cap \mathcal{K}^-$ *are not empty; then* Ω *cannot be connected.*

Proof. The asymptotic expansion for H-surfaces, cp. Section 2.10 and Chapter 3, or the discussion in the proof of Theorem 1, imply the existence of a tangent plane for X at every point $w \in \Omega$. Hence the H-surface cannot pass through the vertex of the cone, cp. the discussion in Section 4.1. □

For our next enclosure theorem we need some further terminology which will allow us to give a lucid formulation of the result.

Definition 2. *Let* $\mathcal{J}$ *be an interval in* $\mathbb{R}$. *We shall say that a family of domains in* $\mathbb{R}^3$, $(G_\alpha)_{\alpha\varepsilon\mathcal{J}}$, *depends continuously on the parameter* α, *if for all* $\alpha_0 \in \mathcal{J}$ *the symmetric difference*

$$G_\alpha \Delta G_{\alpha_0} := (G_\alpha \cup G_{\alpha_0}) \setminus (G_\alpha \cap G_{\alpha_0})$$

tends to ∂G_{α_0} *as* α *tends to* α_0, *i.e., if for all* $\alpha_0 \in \mathcal{J}$ *and all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $|\alpha - \alpha_0| < \delta$ *implies that*

$$G_\alpha \Delta G_{\alpha_0} \subset T_\varepsilon(\partial G_{\alpha_0}) := \{P : \operatorname{dist}(P, \partial G_{\alpha_0}) < \varepsilon\}.$$

Definition 3. *If* M *is a simply connected subset of an open set* G *in* $\mathbb{R}^3$, *then a family* $(G_\alpha)_{\alpha\varepsilon\mathcal{J}}$ *of domains depending continuously on its parameter* α *is called an* **enclosure** *of* M **with respect to** G *(or it is said:* $(G_\alpha)_{\alpha\varepsilon\mathcal{J}}$ **encloses** M with respect to G*) if*

(i) $M \subset G_\alpha$ *for all* $\alpha \in \mathcal{J}$;
(ii) *every* $P \in G \setminus M$ *does not belong to at least one of the* G_α;
(iii) *every compact subset* $\mathcal{K}$ *of* G *lies in at least one of the* G_α;

Here are two examples:

[1] Let M be a star-shaped domain in $\mathbb{R}^3$ whose boundary may be considered as a graph of a positive real-valued function $f : S^2 \to (0,\infty)$ of class C^2, i.e. we assume that

$$M = \left\{\lambda P : P \in S^2 \text{ and } 0 \leq \lambda < f(P)\right\}.$$

Then ∂M is the level set

$$\partial M = \{P \in \mathbb{R}^3 \setminus \{0\} : F(P) = 1\}$$

of the function

$$F(P) := |P|/f\left(\frac{P}{|P|}\right) \quad \text{for } P \in \mathbb{R}^3 \setminus \{0\}\,, \; F(0) := 0$$

satisfying $F(cP) = cF(P)$ for $c > 0$. In particular, we have for $c > 0$ that

$$F(P) = 1 \quad \text{if and only if} \quad F(cP) = c,$$

i.e., the level sets of F are homothetic, hence the mean curvature of $\{F = 1\}$ at P is equal to c-times the mean curvature of $\{F = c\}$ at the point cP.

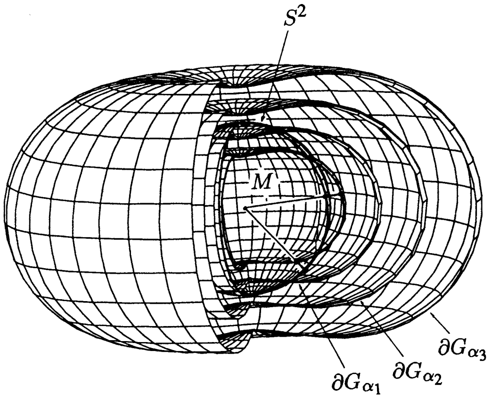

Fig. 2. A star-shaped domain M, whose boundary is the graph of a smooth function $f : S^2 \to \mathbb{R}$ defined on the unit sphere S^2, is enclosed with respect to $\mathbb{R}^3$ by the family of domains $G_\alpha = \{\alpha P : P \in M\}$, $\alpha > 1$, which are homothetic to M

Moreover, the family

$$G_\alpha := \{F < \alpha\} \quad \text{for } \alpha > 1$$

defines an enclosure of M with respect to $\mathbb{R}^3$.

$\boxed{2}$ Let $\tau = 1.199678640257\ldots$ be the solution of the equation $\tau \sinh \tau = \cosh \tau$. Then, for any $c > 0$, the cone

$$\mathcal{K}^c := \left(\mathcal{K}^+ \cup \{0\} \cup \mathcal{K}^-\right) \cap \{|z| < c\}$$

with

$$\mathcal{K}^\pm := \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < (\sinh^2 \tau) z^2,\; z \gtrless 0\right\}$$

is enclosed by the domains

$$\mathcal{K}^c_\alpha := \mathcal{K}_\alpha \cap \{|z| < c\}\,,$$

$$\mathcal{K}_\alpha := \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 < \alpha^2 \cosh^2 \frac{z}{\alpha}\right\}, \quad \alpha > 0.$$

Note that the $\mathcal{K}_\alpha$ have catenoids, i.e. minimal surfaces, as their boundaries $\partial\mathcal{K}_\alpha$, cf. Osserman and Schiffer [1].

By the way, the angle of aperture of the cone $\mathcal{K}^+$ is $\alpha = \arctan(\sinh^2 \tau) \widehat{=} 56.4658\ldots$ degrees whereas the angle of the cone $\mathcal{K}^+$ appearing in the *cone theorem* of Section 6.1 is 45°.

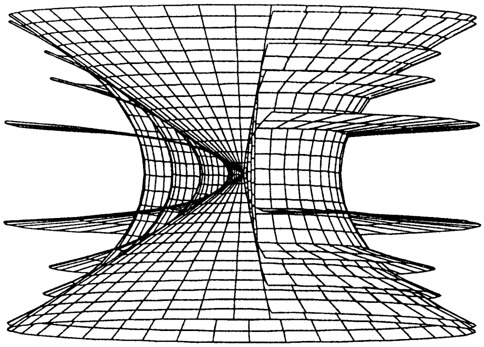

Fig. 3. Let τ be the solution of the equation $\tau \sinh(\tau) = \cosh(\tau)$. Then the cone $\{x^2+y^2 < \sinh^2(\tau) z^2, |z| < c\}$ is enclosed by the family of domains $\{x^2 + y^2 < \alpha^2 \cosh^2(z/\alpha), |z| < c\}$ having catenoids as parts of their boundaries

Assumption. *In the sequel let M be a simply connected subset of a domain G in $\mathbb{R}^3$ which possesses an enclosure $(G_\alpha)_{\alpha\varepsilon\mathfrak{J}}$ with respect to G such that each subset $\partial_0 G_\alpha := G \cap \partial G_\alpha$ of ∂G_α is of class C^2.*

Denote by Λ_α the mean curvature of ∂G_α with respect to the inward normal of ∂G_α.

Recall that $H \in C^0(\mathbb{R}^3)$, and *suppose that we have*

$$\sup\nolimits_{\overline{G}_\alpha} |H| \le \inf\nolimits_{\partial_0 G_\alpha} \Lambda_\alpha \tag{12}$$

for every $\alpha \in \mathfrak{J}$.

Under this assumption we can formulate the

Enclosure Theorem II. *Let $X \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a finite connected H-surface with the parameter domain Ω whose image $X(\Omega)$ lies in G, and whose boundary $X(\partial\Omega)$ is contained in M. Then the image $X(\Omega)$ must, in fact, lie in M.*

Proof. If $X(\Omega)$ is not contained in M, then, according to the definition of an enclosure $(G_\alpha)_{\alpha\in\mathfrak{J}}$, there is an α_1 such that $X(\Omega)$ does *not* lie in G_{α_1}, and an α_2 (without loss of generality greater than α_1) such that $X(\Omega)$ remains in G_{α_2}. Therefore the number

$$\alpha_0 := \sup\left\{\alpha \in \mathcal{J} : \alpha < \alpha_2 \text{ and } X(\Omega) \not\subseteq G_\alpha\right\}$$

is well defined and finite. We shall presently show that

$$\text{(I)} \qquad X(\Omega) \subset G_{\alpha_0} \cup \partial_0 G_{\alpha_0},$$

$$\text{(II)} \qquad X(\Omega) \cap \partial_0 G_{\alpha_0} \neq \emptyset.$$

Then, on account of Theorem 1, we obtain that $X(\Omega)$ lies in $\partial_0 G_{\alpha_0}$; in particular, $X(\partial\Omega)$ is confined to ∂G_{α_0}. This contradicts the assumption that

$$X(\partial\Omega) \subset M \subset G_{\alpha_0}.$$

Now, as for (I), let us assume that for some $w \in \Omega$, the point $X(w)$ lies at a distance $d > 0$ from $\overline{G}_{\alpha_0}$. Then the continuity of the family G_α with respect to α implies that, for some small $\varepsilon > 0$, the point $X(w)$ is not contained in $G_{\alpha_0+\varepsilon}$ either. This, however, contradicts the definition of α_0.

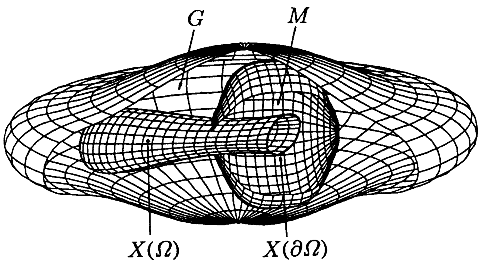

Fig. 4. A simply connected set M which has an enclosure G_α as shown before with respect to an open set G, and an H-surface X whose image $X(\Omega)$ is confined to G and whose boundary even lies in the smaller set M. If the H-surface would satisfy the curvature condition of the enclosure theorem II, then all of $X(\Omega)$ would remain in M

As for (II), since $X(\Omega)$ is contained in G, it will suffice to show that $X(\Omega)$ does not lie in G_{α_0}. Otherwise, as follows from the compactness of $X(\overline{\Omega})$, we have

$$d' := \operatorname{dist}\big(X(\overline{\Omega}), \partial_0 G_{\alpha_0}\big) > 0,$$

which also implies that α_0 is not the supremum since, once again, in view of the continuity of G_α with respect to α, the set $X(\Omega)$ lies in $G_{\alpha_0-\varepsilon}$ for some small $\varepsilon > 0$. □

As an illustrative application of the last enclosure theorem, we have the following

Enclosure Theorem III. *Let $f : S^2 \to (0,\infty)$ be some C^2-function on S^2, and let $F : \mathbb{R}^3 \to (0,\infty)$ be its homogeneous extension to $\mathbb{R}^3$ defined by*

$F(0) := 0$ and $F(P) := |P|/f(\frac{P}{|P|})$ for $P \neq 0$. Denote by M the star-shaped domain $\{F < 1\}$ and assume that the mean curvature of ∂M with respect to the inward normal is everywhere nonnegative. Then every connected finite minimal surface X with the parameter domain Ω satisfies $X(\Omega) \subset M$ if we assume that $X(\partial\Omega) \subset \overline{M}$ and if the intersection of $X(\partial\Omega)$ with M is nonvoid.

This result follows from Theorem 1 and from the remarks about Example 1 in connection with the Enclosure Theorem II. Instead of going into the details we shall state a nonexistence result that follows from the Enclosure Theorem III; it can be proved like the nonexistence result in Section 4.1.

Nonexistence Theorem. *Assume that M, G, G_α satisfy the assumptions stated above, and suppose in addition that there are finitely many points $P_1, \dots, P_m$ in M such that $M \setminus \{P_1, \dots, P_m\}$ decomposes into $n \geq 2$ simply connected components $M_1, \dots, M_n$. Then there is no finite connected H-surface with a parameter domain Ω which has the following properties:*

(i) $X(\Omega) \subset G$;
(ii) $X(\partial\Omega) \subset \overline{M}$;
(iii) $X(\partial\Omega)$ *intersects at least two of the components* $M_1, \dots, M_n$.

Applying the last theorem to Example 2, we obtain the following improvement of the cone theorem of Section 4.1:

Corollary 1. *Set*

$$\mathcal{K}^{\pm} := \{(x, y, z) \in \mathbb{R}^3 : z \lesseqgtr 0 \text{ and } x^2 + y^2 < z^2 \sinh^2 \tau\},$$

where $\tau = 1.199678640257\dots$ is a solution of the equation

$$\tau \sinh \tau = \cosh \tau,$$

and define $\mathcal{K}$ by

$$\mathcal{K} := \mathcal{K}^+ \cup \{0\} \cup \mathcal{K}^-.$$

Then there is no connected finite minimal surface with boundary which intersects both $\mathcal{K}^+$ and $\mathcal{K}^-$.

This "nonexistence test-cone" $\mathcal{K}$ cannot be further increased as one can see by means of catenoids between suitable circles as boundary curves, see Fig. 1 in the introduction of this chapter.

4.3 Minimal Submanifolds and Submanifolds of Bounded Mean Curvature. An Optimal Nonexistence Result

It is the aim of this section to generalize the results of Sections 4.1 and 4.2 to higher dimensions and codimensions. To accomplish this, we first define a concept of n-dimensional surfaces or submanifolds in $\mathbb{R}^{n+k}$. It turns out that, for the present purpose, it is not necessary to develop the complete differential geometric notion of submanifolds in arbitrary ambient manifolds, as e.g. described in Gromoll, Klingenberg, and Meyer [1], do Carmo [3], Jost [18] and Kühnel [2], but rather the more elementary concepts of submanifolds in $\mathbb{R}^{n+k}$ (although later in Section 4.8 we shall also treat surfaces of prescribed mean curvature in Riemannian manifolds). We start with the following

Definition 1. *A subset $M \subset \mathbb{R}^{n+k}$ is called an n-dimensional submanifold of class C^s, if for each $x \in M$ there are open neighbourhoods U, $V \subset \mathbb{R}^{n+k}$ of x and 0 in $\mathbb{R}^{n+k}$ respectively, and a C^s-diffeomorphism $\varphi : V \to U$, such that $\varphi(0) = x$ and $\varphi(V \cap \mathbb{R}^n \times \{0\}) = U \cap M$. Here $\varphi_{|V\cap\mathbb{R}^n\times\{0\}}$ is a local parametrization and φ^{-1} is called a local chart for M. In case that $k = 1$, $M \subset \mathbb{R}^{n+1}$ is also called a* **hypersurface** *(of class C^s).*

Given M, x and φ as in Definition 1 we have

Definition 2. *The tangent space T_xM of M at x is the n-dimensional linear subspace of R^{n+k} which is spanned by the independent vectors $\varphi_{x^1}(0), \dots, \varphi_{x^n}(0)$.*

One easily convinces oneself that the tangent space T_xM is given by all vectors $\xi = \dot\alpha(0)$, where $\alpha : (-\varepsilon, \varepsilon) \to M$ is a regular curve in M with $\alpha(0) = x$. That is we have

Proposition 1. *The tangent space of M at x is given by*

$$T_xM = \Big\{\dot\alpha(0) : \alpha : (-\varepsilon, \varepsilon) \to M \text{ is a regular curve with } \alpha(0) = x\Big\}.$$

Now consider a function $f : M \to \mathbb{R}^m$. One way of defining differentiability of f is to consider all possible compositions of f with parametrizations φ and to requiring the composition $f \circ \varphi : V \cap \mathbb{R}^n \times \{0\} \to \mathbb{R}^m$ to be differentiable, see e.g. Chapter 1. Here we define differentiability somewhat different (but equivalently)

Definition 3. *Let $M \subset \mathbb{R}^{n+k}$ be a submanifold of class C^s and $f : M \to \mathbb{R}^m$. f is differentiable of class C^r, $r \le s$, if there exists an open subset $U \subset \mathbb{R}^{n+k}$ with $M \subset U$ and a C^r-function $F : U \to \mathbb{R}^m$ such that $f = F|_M$.*

In other words, $f : M \to \mathbb{R}^m$ is differentiable, if it is the restriction of a differentiable map from an open set $U \subset \mathbb{R}^{n+k}$. Of particular interest are the cases $m = 1$ *(scalar functions)* and $m = n + k$ (vector fields). If $f : M \to \mathbb{R}$ is differentiable we define the *(intrinsic)* gradient of f as follows

Definition 4. *The gradient of f on M, in symbols $\nabla_M f$, is defined by $\nabla_M f = (Df)^\top$, where $Df = (f_{x^1}, \dots, f_{x^{n+k}})$ denotes the usual (Euclidean) gradient and $(\xi)^\top$ stands for the orthogonal projection of the vector $\xi \in \mathbb{R}^{n+k}$ onto the tangent space of M at x. (Note that here and in the discussion to follow we tacitly assume, that f coincides with its differentiable extension F, cp. Definition 3).*

Definition 5. *The normal space of M at x is given by*

$$T_xM^\perp := \{n \in R^{n+k} : \langle n, t\rangle = 0 \text{ for all } t \in T_xM\}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{n+k}$.

Let $N_1, \ldots, N_k$ be an orthonormal basis of $T_xM^\perp$. Then we obtain $\nabla_M f = Df - \langle Df, N_1\rangle N_1 - \cdots - \langle Df, N_k\rangle N_k$, for the intrinsic gradient of a function $f : M \to \mathbb{R}$.

Equivalently we now consider an orthonormal basis $t_1, \ldots, t_n$ of the tangent space $T_xM \subset \mathbb{R}^{n+k}$ and an arbitrary vector $t \in T_xM$. Recall that the directional derivative $D_t f$ of $f : M \to \mathbb{R}^m$ at x in the direction of t is given by

$$D_t f(x) := \frac{d}{d\varepsilon} f(\alpha(\varepsilon))_{\varepsilon=0},$$

where

$$\alpha : (-\delta, \delta) \to M$$

is a regular curve in M with $\alpha(0) = x$ and $\alpha'(0) = t$. It is easily seen, that this definition is meaningful (i.e. independent of the particular curve α), and furthermore we have by the chain rule

$$D_t f(x) = Df(x) \cdot t.$$

Definition 6. *Let $f : M \to \mathbb{R}^m$ be differentiable. The differential $df(x)$ of f at x is the linear map $df(x) : T_xM \to \mathbb{R}^m$*

$$t \mapsto df(x)(t) := D_t f(x).$$

In fact, it follows immediately from the definition that $df(x)$ is linear. Observe now that the gradient of $f : M \to \mathbb{R}$ is equivalently given by

$$\nabla_M f = (D_{t_1} f)t_1 + \cdots + (D_{t_n} f)t_n = \sum_{i=1}^{n} (D_{t_i} f)t_i, \tag{1}$$

for any orthonormal basis $t_1, \ldots, t_n$ of T_xM. Then equation (1) easily follows from the previous relation by multiplication with the basis vectors $t_1, \ldots, t_n$ respectively.

Note that (1) is already meaningful for functions $f : M \to \mathbb{R}$ which are merely defined on M, whereas Definition 4 assumes f to be defined (locally) on an open neighbourhood of M, however we shall not dwell on this.

The next important notion is that of the divergence on M.

Definition 7. *Let $X : M \to \mathbb{R}^{n+k}$*

$$X(x) = (X^1(x), \ldots, X^{n+k}(x))$$

be a differentiable function on a differentiable submanifold $M \subset \mathbb{R}^{n+k}$, i.e. a—not necessarily tangential—vector field on M. The divergence $\mathrm{div}_M X$ *of X on M is given by*

$$\mathrm{div}\,_M X = \sum_{i=1}^{n} \langle t_i, D_{t_i} X \rangle,$$

where $t_1, \dots, t_n \in T_x M$ is an orthonormal basis of the tangent space $T_x M$.

We observe here that the definition of $\mathrm{div}\,_M$ is independent of the particular orthonormal basis $t_1, \dots, t_n$ of the tangent space $T_x M$. To see this we compute

$$\begin{aligned}\sum_{i=1}^{n} \langle t_i, dX(t_i) \rangle &= \sum_{i=1}^{n} \langle t_i, D_{t_i} X \rangle = \sum_{i=1}^{n} \left\langle t_i, D_{t_i} \left(\sum_{j=1}^{n+k} e_j\, X^j \right) \right\rangle \\ &= \sum_{i=1}^{n} \left\langle t_i, \sum_{j=1}^{n+k} e_j\, D_{t_i} X^j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n+k} \langle t_i, e_j\, D_{t_i} X^j \rangle \\ &= \sum_{j=1}^{n+k} \left\langle e_j, \sum_{i=1}^{n} (D_{t_i} X^j) t_i \right\rangle = \sum_{j=1}^{n+k} \langle e_j, \nabla_M X^j \rangle\end{aligned}$$

by equation (1), where $e_1, \dots, e_{n+k}$ denotes the canonical basis of $\mathbb{R}^{n+k}$ and $\nabla_M X^j$ is the gradient of the j-th component X^j of the vector field X on M.

For later computations we note here

Proposition 2. *Let $X(x) = (X^1(x), \dots, X^{n+k}(x))$ be a differentiable vector field on M. Then the divergence of X on M is given by the relation*

$$\mathrm{div}_M X = \sum_{j=1}^{n+k} \langle e_j, \nabla_M X^j \rangle,$$

where $e_1, \dots, e_{n+k}$ stands for the canonical basis of $\mathbb{R}^{n+k}$.

The next important operator is the *Laplace–Beltrami* operator.

Definition 8. *For $f : M \to \mathbb{R}$ of class C^2 we put $\Delta_M f := \mathrm{div}\,_M(\nabla_M f)$. Then Δ_M is called the* **Laplacian** *on M or* **Laplace–Beltrami operator**.

Note that Δ_M coincides with the Laplace–Beltrami operator on a surface X given in Chapter 1.5 of Vol. 1, equations (15) and (16). Observe also that Δ_M is an elliptic operator on M; this will be used later in this section when we compute the Laplacian of a certain quadratic form.

Finally we have to introduce some curvature quantities for the submanifold M. To this end we choose an orthonormal basis $t_1, \dots, t_n$ of $T_x M$, which together with an orthonormal basis $N_1, \dots, N_k$ of the normal space $T_x M^\perp$ forms an orthonormal basis of $\mathbb{R}^{n+k}$.

Let us initially assume that the codimension k is equal to 1, so that (up to a sign) there is only one unit normal $N = N_1$. Consider $N = N(x)$ as a function of $x \in M$ and assume that $N(\cdot)$ is differentiable, which is true if $M \in C^2$. We then define the **Weingarten map** (cp. Section 1.2 of Vol. 1) of M at $x \in M$ to be the linear map

$$-dN(x) : T_xM \to \mathbb{R}^{n+1} \text{ defined by } t \mapsto -dN(x)(t) = -D_tN(x),$$

where D_t denotes the derivative in the direction of t. Because of $|N|^2 = 1$ it easily follows that $-dN(x)$ is a linear map from T_xM into itself.

The **second fundamental form** $\mathrm{II} = \mathrm{II}_x(\cdot,\cdot)$ of M at x with respect to N is defined to be the bilinear form

$$\begin{aligned}&\mathrm{II} : T_xM \times T_xM \to \mathbb{R} \quad \text{with}\\ &(t,\tau) \longmapsto \mathrm{II}_x(t,\tau) := -\langle dN(t), \tau\rangle = -\langle D_tN, \tau\rangle,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{n+1}$ and $N = N(x)$. It is convenient to consider also the bilinear map $A_x(t,\tau) := \mathrm{II}_x(t,\tau)\cdot N$, which—by a slight abuse of notation—is again called the second fundamental form of M. Observe that for every $x \in M$ the bilinear maps $A_x : T_xM \times T_xM \to T_xM^\perp$ and $\mathrm{II}_x : T_xM \times T_xM \to \mathbb{R}$ are *symmetric*, and that $-dN(x) : T_xM \to T_xM$ is a symmetric endomorphism field. To see this, consider a mapping $\Phi : B_\varepsilon(0) \subset \mathbb{R}^2 \to M \subset \mathbb{R}^{n+1}$ such that $\Phi(0,0) = x$, $\Phi_{x^1}(0,0) = t$, $\Phi_{x^2}(0,0) = \tau$. Differentiating the identities

$$\langle \Phi_{x^1}, N\rangle = 0 = \langle \Phi_{x^2}, N\rangle$$

and putting $x_1 = x_2 = 0$, we infer $\langle \Phi_{x^1x^2}(0,0), N\rangle + \langle t, D_\tau N\rangle = 0$ and $\langle \Phi_{x^1x^2}(0,0), N\rangle + \langle \tau, D_tN\rangle = 0$, whence

$$\begin{aligned}\mathrm{II}_x(t,\tau) &= -\langle D_tN, \tau\rangle = \langle \Phi_{x^1x^2}(0,0), N\rangle\\ &= -\langle D_\tau N, t\rangle = \mathrm{II}_x(\tau, t).\end{aligned} \tag{2}$$

Similarly

$$A_x(t,\tau) = A_x(\tau,t) = \langle \Phi_{x^1x^2}(0,0), N\rangle \cdot N = [\Phi_{x^1x^2}(0,0)]^\perp,$$

where $\xi^\perp$ stands for the orthogonal projection of the vector $\xi \in \mathbb{R}^{n+1}$ onto the normal space $T_xM^\perp$.

As in the case of surfaces in $\mathbb{R}^3$ we define the *principal directions* of M at x to be the unit eigenvectors of the Weingarten map

$$-dN = -dN(x) : T_xM \to T_xM$$

and the *principal curvatures* $\lambda_1, \dots, \lambda_n$ to be the corresponding eigenvalues. Note that there is an orthonormal basis of T_xM consisting of principal directions. Also, if $t_1, \dots, t_n \in T_xM$ are orthonormal principal directions, then obviously the matrix

$$b_{ij} := \mathrm{II}_x(t_i, t_j) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

More generally, we conclude from a discussion similar to the one in Section 1.2 of Vol. 1, that the principal curvatures are the eigenvalues of the matrix $G^{-1}B$, where

$$B = (b_{ij})_{i,j=1,\ldots,n}, \quad b_{ij} := \mathrm{II}_x(\xi_i, \xi_j),$$
$$G = (g_{ij})_{i,j=1,\ldots,n}, \quad g_{ij} := \langle \xi_i, \xi_j \rangle,$$

and $\xi_1, \ldots, \xi_n \in T_xM$ denotes an arbitrary basis of the tangent space T_xM. In particular, the principal curvatures are eigenvalues of the symmetric matrix $b_{ij} = \mathrm{II}_x(t_i, t_j)$ for any *orthonormal* basis $t_1, \ldots, t_n$ of T_xM.

Another description of the principal curvatures might also be of interest: Suppose that near a point $x \in M$, the manifold M is locally defined by a smooth function $\varphi : B_\varepsilon(0) \subset \mathbb{R}^n \to \mathbb{R}$, $x^{n+1} = \varphi(x^1, \ldots, x^n)$ and that $e_1, \ldots, e_n$ are principal directions corresponding to the curvatures $\lambda_1, \ldots, \lambda_n$. Such a coordinate system is called a *principal coordinate system*. Without loss of generality assume that $x = 0$, i.e. $\varphi(0) = 0$, $D\varphi(0) = 0$ or $N(0) = e_{n+1}$. It is not difficult to see that M can locally be represented in this way. Now consider the mapping

$$\Phi : B_\varepsilon(0) \subset \mathbb{R}^n \to M \subset \mathbb{R}^{n+1}$$

given by $\Phi(x^1, \ldots, x^n) := (x^1, \ldots, x^n, \varphi(x^1, \ldots, x^n))$, i.e. Φ is a *local parametrization* of M. By arguments similar to those leading to equation (2) we infer

$$D^2\varphi(0) = (\varphi_{x^ix^j}(0))_{i,j=1,\ldots,n} = \mathrm{II}_x(e_i, e_j) = b_{ij} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

since $e_1, \ldots, e_n$ are principal directions at $x = 0$.

Using the *elementary symmetric functions* of n variables $\sigma_1, \ldots, \sigma_n$, it is now possible to define corresponding curvature quantities K_j by putting

$$K_j(x) := \frac{1}{\binom{n}{j}} \sigma_j(\lambda_1, \ldots, \lambda_n).$$

The cases $j = 1$ and $j = n$ deserve special attention: The *mean curvature* H and the *Gauß(–Kronecker)* curvature K are defined by

$$H(x) := K_1(x) = \frac{1}{n}(\lambda_1 + \cdots + \lambda_n), \quad \textit{and}$$
$$K(x) := K_n(x) = \lambda_1 \cdot \cdots \cdot \lambda_n$$

corresponding to the elementary symmetric functions σ_1 and σ_n.

In other words we have

$$\begin{aligned} H(x) &= \frac{1}{n}\mathrm{trace}(G^{-1}B) = \frac{1}{n}\sum_{j,k=1}^{n} g^{jk} b_{jk} \quad \text{and} \\ K(x) &= \det(G^{-1}B) = \frac{\det B}{\det G}, \end{aligned} \tag{3}$$

where

$$B = (b_{ij})_{i,j=1,\dots,n}, \quad b_{ij} = \mathrm{II}_x(\xi_i, \xi_j),$$
$$G = (g_{ij})_{i,j=1,\dots,n}, \quad g_{ij} = \langle \xi_i, \xi_j \rangle, \quad G^{-1} = (g^{ij})_{i,j=1,\dots,n}$$

and $\xi_1, \dots, \xi_n$ stand for a basis of the tangent space T_xM. Therefore the mean curvature is (up to the factor $\frac{1}{n}$) just the trace of the Weingarten map $-dN(x)$, or—equivalently—of the second fundamental form II_x.

For arbitrary codimension $k > 1$ it is not possible to define principal directions and curvatures. However we can define principal curvatures and directions with respect to a given normal N_j, $j = 1, \dots, k$, and a corresponding second fundamental form, but we shall not dwell on this here (for a further discussion see e.g. Spivak [1]).

Instead we define for arbitrary $k \geq 1$ and $M \subset \mathbb{R}^{n+k}$ the second fundamental form of M at x as the bilinear form $A_x : T_xM \times T_xM \to T_xM^\perp$ given by $A_x(t, \tau) = -\sum_{j=1}^k \langle dN_j(t), \tau \rangle N_j(x)$.

Arguments similar to those mentioned above prove that $A_x(\cdot, \cdot)$ is a symmetric bilinear form.

Motivated by the foregoing discussion, in particular relation (3), we define the *mean curvature vector* $\vec{H}$ of M at x to be $\frac{1}{n}\operatorname{trace} A_x$, i.e.

$$\vec{H}(x) := \frac{1}{n} \sum_{i=1}^n A_x(t_i, t_i), \tag{4}$$

where $t_1, \dots, t_n \in T_xM$ is some orthonormal basis.

In the codimension one case we obtain for the mean curvature vector

$$\begin{aligned} \vec{H}(x) &= \frac{1}{n} \sum_{i=1}^n A_x(t_i, t_i) = -\sum_{i=1}^n \langle dN(t_i), t_i \rangle N \\ &= \frac{1}{n} \left(\sum_{i=1}^n \mathrm{II}_x(t_i, t_i) \right) N(x) \underset{\text{(by (3))}}{=} H(x)N(x), \end{aligned} \tag{5}$$

where $H(x)$ is the mean curvature of M at x with respect to the normal $N(= N_1)$.

We are thus led to

Definition 9. *An n-dimensional C^2-submanifold $M \subset \mathbb{R}^{n+k}$ is called minimal submanifold, if and only if $\vec{H} = 0$ on M.*

A different expression for $\vec{H}$ is obtained as follows:

$$\begin{aligned} \vec{H}(x) &= \frac{1}{n} \sum_{i=1}^n A_x(t_i, t_i) = -\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^k \langle dN_j(t_i), t_i \rangle N_j \\ &= -\frac{1}{n} \sum_{j=1}^k \sum_{i=1}^n \langle D_{t_i} N_j, t_i \rangle N_j = -\frac{1}{n} \sum_{j=1}^k (\operatorname{div}_M N_j) N_j, \end{aligned}$$

taking Definition 7 into account. Thus we obtain

Proposition 3. *Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional C^2-submanifold of $\mathbb{R}^{n+k}$ and $N_1, \dots, N_k$ be an orthonormal basis of the normal space $T_x M^\perp$. Then the mean curvature vector $\vec{H} = \vec{H}(x)$ of M at x is given by*

$$\vec{H}(x) = -\frac{1}{n} \sum_{j=1}^{k} (\operatorname{div}_M N_j) N_j. \tag{6}$$

Remark 1. The mean curvature vector $\vec{H}$ is independent of the particular choice of the (local) orthonormal fields $t_1, \dots, t_n$ and $N_1, \dots, N_k$; in particular independent of the orientation of M.

Remark 2. Using equations (5) and (6) we infer for hypersurfaces the relation

$$H(x) = -\frac{1}{n} \operatorname{div}_M N, \tag{7}$$

where the mean curvature H corresponds to the unit normal N of M.

We should point out here, that (7) also leads to an alternative proof of the Theorem in Section 2.7 of Vol. 1. In fact, suppose that M is the level surface of some regular function

$$S : G \subset \mathbb{R}^{n+1} \to \mathbb{R},$$

say $M = \{x \in G : S(x) = c\}$, $c \in \mathbb{R}$, and $N(x) = \frac{\nabla S(x)}{|\nabla S(x)|}$ denotes a unit normal field along M. Then we claim that

$$H(x) = -\frac{1}{n} \operatorname{div} N(x), \tag{8}$$

where ∇ and div denote the Euclidean gradient and divergence respectively.

Proof of (8). With Definition 4 and Proposition 2 we find for the divergence of $N(x)$ on M the expression

$$\operatorname{div}_M N(x) = \sum_{j=1}^{n+1} \langle e_j, \nabla_M N^j \rangle = \sum_{j=1}^{n+1} \big\langle e_j, \nabla N^j - \langle \nabla N^j, N \rangle \cdot N \big\rangle, \tag{9}$$

where we have put $N = (N^1, \dots, N^{n+1})$. On the other hand by taking partial derivatives $\frac{\partial}{\partial x^i}$ we infer from $|N|^2 = 1$, the relation $\langle N, \frac{\partial N}{\partial x^i} \rangle = 0$ for any $i = 1, \dots, n+1$, or

$$\sum_{j=1}^{n+1} N^j(x) \frac{\partial N^j}{\partial x^i} = 0 \quad \text{for } i = 1, \dots, n+1. \tag{10}$$

Now we get by (10)

$$\sum_{j=1}^{n+1} \langle e_j, \langle \nabla N^j, N \rangle \cdot N \rangle = \sum_{j=1}^{n+1} \langle \nabla N^j, N \rangle N^j = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} \left(\frac{\partial N^j}{\partial x^i} \cdot N^i \right) N^j$$

$$= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \left(\frac{\partial N^j}{\partial x^i} N^j \right) N^i = 0.$$

Therefore (9) yields

$$\text{div}_M N(x) = \sum_{j=1}^{n+1} \langle e_j, \nabla N^j \rangle = \sum_{j=1}^{n+1} \frac{\partial N^j}{\partial x^j} = \text{div}\, N(x).$$

This proves (cf. Vol. 1, Section 2.7, Theorem)

Proposition 4. *If G is a domain in $\mathbb{R}^{n+1}$, and if S is a function of class $C^2(G)$ such that $\nabla S(x) \neq 0$ on G, then the mean curvature $H(x)$ of the level hypersurface $\mathcal{F}_c = \{x \in G; S(x) = c\}$ passing through $x \in G$ with respect to the unit normal field $N(x) = |\nabla S(x)|^{-1} \nabla S(x)$ of $\mathcal{F}_c$ is given by the equation*

$$H(x) = -\frac{1}{n} \text{div}\, N(x).$$

Proposition 4 also permits to carry over the Schwarz–Weierstraß field theory for two-dimensional minimal surfaces to $\mathbb{R}^{n+1}$; compare the discussion in Section 2.8 of Vol. 1. By essentially the same arguments, using Gauss's theorem, we derive

Theorem 1. *A C^2-family of embedded hypersurfaces $\mathcal{F}_c$ covering a domain G in $\mathbb{R}^{n+1}$ is a* Mayer family *of minimal submanifolds if and only if its normal field is divergence free. Such a foliation by minimal submanifolds is area minimizing in the following sense:*

(i) *Let $\mathcal{F}$ be a piece of some of the minimal leaves $\mathcal{F}_c$ with $\mathcal{F} \Subset G$. Then we have*

$$\text{Area}(\mathcal{F}) = \int_{\mathcal{F}} dA \leq \int_{\mathcal{S}} dA = \text{Area}(\mathcal{S})$$

for each C^1-hypersurface $\mathcal{S}$ contained in G with $\partial\mathcal{F} = \partial\mathcal{S}$.

(ii) *("Kneser's transversality Theorem"): Let T be a hypersurface in G which, in all of its points, is tangent to the normal field of the minimal foliation, and suppose that T cuts out of each leaf $\mathcal{F}_c$ some piece $\mathcal{F}_c^*$ whose boundary $\partial\mathcal{F}_c$ lies on T. Then we have*

$$\int_{\mathcal{F}_{c_1}^*} dA = \int_{\mathcal{F}_{c_2}^*} dA$$

for all admissible parameter values c_1 and c_2, and secondly

$$\int_{\mathcal{F}_c} dA \leq \int_{\mathcal{S}} dA$$

for all C^1-hypersurfaces $\mathcal{S}$ contained in G whose boundary $\partial\mathcal{S}$ is homologous to $\partial\mathcal{F}$ on T.

We remark here that a result similar to—but more general than—Theorem 1 has been used by Bombieri, De Giorgi and Giusti [1] to show that the seven-dimensional "Simons-cone"

$$C = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$$

is area minimizing in a very general sense. Indeed they were able to construct a foliation of $\mathbb{R}^8$ consisting of smooth minimal hypersurfaces and the singular minimal cone C. This was also the first example of an area-minimizing boundary in $\mathbb{R}^{n+1}$ with an **interior** singularity, namely the origin, which dashed the hope to prove interior regularity of area minimizing boundaries in arbitrary dimensions.

The *divergence theorem* for a C^2-compact manifold $M \subset \mathbb{R}^{n+k}$ with smooth boundary $\partial M = \overline{M} \setminus M$ states that for any C^1-vector field $X : \overline{M} \to \mathbb{R}^{n+k}$ the identity

$$\int_M \operatorname{div}_M X \, dA = -n \int_M X \cdot \vec{H} \, dA + \int_{\partial M} X \cdot \nu \, dA$$

holds where ν denotes the exterior unit normal field to ∂M which is tangent to M along ∂M. Here $\vec{H} = -\frac{1}{n} \sum_{i=1}^{k} (\operatorname{div}_M N_i) N_i$ denotes the mean curvature vector and integration over ∂M is with respect to the standard $(n-1)$-dimensional area measure (or, equivalently, $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$).

In particular, if X is a tangential vector field, i.e. $X(x) \in T_x M$ for each $x \in M$ or if M is minimal, then we have the formula

$$\int_M \operatorname{div}_M X \, dA = \int_{\partial M} X \cdot \nu \, dA.$$

Similarly, if X has compact support, or if $\partial M = \emptyset$, then the divergence theorem yields

$$\int_M \operatorname{div}_M X \, dA = -n \int_M X \cdot \vec{H} \, dA,$$

and finally

$$\int_M \operatorname{div}_M X \, dA = 0,$$

if X is a compactly supported, tangential vector field on M.

Remark 3. It can be shown that $M \subset \mathbb{R}^{n+k}$ is stationary for the n-dimensional area functional, if and only if $\vec{H} \equiv 0$; see Vol. 3, Section 3.2, for details.

Remark 4. Some authors use trace A_x – instead of $\frac{1}{n}$trace A_x – as a definition of the mean curvature vector. This clearly is irrelevant when working with minimal submanifolds; but it is of importance when $\vec{H} \not\equiv 0$.

Next we shall derive a generalization of Theorem 1 in Section 2.5 of Vol. 1 (compare also Theorem 1 in Vol. 1, Section 2.6). To accomplish this we simply compute the Laplace (–Beltrami) operator of the vector field $X(x) = x$. Assuming that $M \subset \mathbb{R}^{n+k}$ is an n-dimensional submanifold of class C^2, we find for the gradient of X on M the expression

$$\nabla_M X^i = \nabla_M x^i = e_i - \langle N_1, e_i\rangle N_1 - \cdots - \langle N_k, e_i\rangle N_k,$$

$i = 1, \ldots, n+k$, where $e_1, \ldots, e_{n+k}$ stands for the canonical basis of $\mathbb{R}^{n+k}$. Applying div_M to this relation we obtain the identity

$$\begin{aligned}\Delta_M x^i &= \operatorname{div}_M(\nabla_M x^i) = -\langle N_1, e_i\rangle \operatorname{div}_M N_1 - \cdots - \langle N_k, e_i\rangle \operatorname{div}_M N_k \\ &= -\sum_{j=1}^k \langle N_j, e_i\rangle \operatorname{div}_M N_j, \\ \text{since} \quad & \big\langle \nabla_M \langle N_j, e_i\rangle, N_j\big\rangle = 0, \quad \forall i, j.\end{aligned}$$

Thus we have for $i = 1, \ldots, n+k$

$$\Delta_M(\langle x, e_i\rangle) = -\sum_{j=1}^k \langle N_j, e_i\rangle \operatorname{div}_M N_j = -e_i\left(\sum_{j=1}^k N_j \cdot \operatorname{div}_M N_j\right).$$

By Proposition 3 this implies $\Delta_M x^i = n \cdot H^i$, where $\vec{H} = (H^1, \ldots, H^{n+k})$ is the mean curvature vector of M. Thus we have proved

Theorem 2. *Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional C^2-submanifold. Then the position vector x fulfills the identity*

$$\Delta_M x = n\,\vec{H}\,.$$

Corollary 1. *$M \subset \mathbb{R}^{n+k}$ is a minimal submanifold, if and only if $\Delta_M x = 0$ holds on M.*

A straight-forward application of the maximum principle for harmonic functions yields the following enclosure results (cp. Theorem 1 in Section 4.1 for the case $n = 2$, $k = 1$ and its proof).

Corollary 2 (Convex hull theorem). *Let $M \subset \mathbb{R}^{n+k}$ be a compact n-dimensional minimal submanifold. Then M is contained in the convex hull $\mathcal{K}$ of its boundary ∂M. Moreover if M touches the convex hull $\mathcal{K}$ at some interior point, then M is part of a plane. In particular there is no compact minimal submanifold M without boundary.*

We now consider the possibility of obtaining polynomials p which are subharmonic functions on M, i.e. which satisfy

$$\Delta_M p \geq 0 \quad \text{on } M \text{ if } H = 0.$$

To achieve this, we define for any $j = 1, \dots, n-1$ a quadratic function $p_j = p_j(x^1, \dots, x^{n+k})$ by

$$p_j(x) := \sum_{i=1}^{n+k-j} |x^i|^2 - \frac{(n-j)}{j} \sum_{i=n+k-j+1}^{n+k} |x^i|^2.$$

Note that for $n = 2$, $j = k = 1$, we recover the polynomial considered in Theorem 2 of Section 4.1.

We have the following

Theorem 3. *Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional minimal submanifold of class C^2. Then for each $j = 1, \dots, n-1$ the quadratic form $p_j(\cdot)$ is a subharmonic function on M.*

Proof. Fixing $j \in \{1, \dots, n-1\}$ we set $P := p_j$ and compute the Laplace–Beltrami expression $\Delta_M P$ as follows:

$$\begin{aligned}
\frac{1}{2}\Delta_M P &= \frac{1}{2} \operatorname{div}_M(\nabla_M P) \\
&= \frac{1}{2} \operatorname{div}_M \bigg\{ 2x^1 \nabla_M x^1 + \cdots + 2x^{n+k-j} \nabla_M x^{n+k-j} \\
&\quad - \frac{(n-j)}{j} \Big[2x^{n+k-j+1} \nabla_M x^{n+k-j+1} + \cdots + 2x^{n+k} \nabla_M x^{n+k} \Big] \bigg\} \\
&= |\nabla_M x^1|^2 + \cdots + |\nabla_M x^{n+k-j}|^2 + x^1 \Delta_M x^1 + \cdots + x^{n+k-j} \Delta_M x^{n+k-j} \\
&\quad - \frac{(n-j)}{j} \Big[|\nabla_M x^{n+k-j+1}|^2 + \cdots + |\nabla x^{n+k}|^2 \\
&\quad + x^{n+k-j+1} \Delta_M x^{n+k-j+1} + \cdots + x^{n+k} \Delta_M x^{n+k} \Big].
\end{aligned}$$

Since M is minimal this gives

$$\frac{1}{2}\Delta_M P = \sum_{s=1}^{n+k-j} |\nabla_M x^s|^2 - \frac{n-j}{j} \sum_{s=1}^{j} |\nabla_M x^{n+k-j+s}|^2.$$

To compute the terms $|\nabla_M x^i|^2$ we denote by $\mathcal{P} : \mathbb{R}^{n+k} \to T_x M$ the orthogonal projection of $\mathbb{R}^{n+k}$ onto the tangent space $T_x M$. Let $(p_{ij})_{i,j=1,\dots,n+k}$ stand for the matrix of $\mathcal{P}$ with respect to the canonical basis $e_1, \dots, e_{n+k}$ of $\mathbb{R}^{n+k}$. Then we have (by Definition 4)

$$\begin{aligned}\nabla x^i &= \mathcal{P}(e_i) = \sum_{l=1}^{n+k} p_{li} e_l \quad \text{and} \\ |\nabla_M x^i|^2 &= \left(\sum_{l=1}^{n+k} p_{li} e_l\right)\left(\sum_{j=1}^{n+k} p_{ji} e_j\right) \\ &= \sum_{l,j=1}^{n+k} p_{li} p_{ji} e_l e_j = \sum_{j=1}^{n+k} p_{ji}^2 .\end{aligned}$$

Since $\mathcal{P}$ is a projection we clearly have $p_{ij} = p_{ji}$ and $\mathcal{P} = \mathcal{P}^2$, whence

$$p_{ij} = \sum_{l=1}^{n+k} p_{il} p_{lj},$$

in particular

$$p_{ii} = \sum_{j=1}^{n+k} p_{ij}^2 = |\nabla_M x^i|^2 .$$

Again, since $\mathcal{P}$ is a projection, all eigenvalues are either equal to one or zero and the sum of the eigenvalues is equal to n:

$$\operatorname{trace} \mathcal{P} = \sum_{i=1}^{n+k} p_{ii} = \sum_{i=1}^{n+k} |\nabla_M x^i|^2 = n.$$

Concluding we find for $\frac{1}{2}\Delta_M P$ the estimate

$$\begin{aligned}\frac{1}{2}\Delta_M P &= \sum_{s=1}^{n+k-j} |\nabla_M x^s|^2 - \frac{n-j}{j} \sum_{s=1}^{j} |\nabla_M x^{n+k-j+s}|^2 \\ &= \sum_{s=1}^{n+k-j} p_{ss} - \frac{n-j}{j} \sum_{s=1}^{j} p_{n+k-j+s,n+k-j+s} \\ &= \sum_{s=1}^{n+k} p_{ss} - \sum_{s=n+k-j+1}^{n+k} p_{ss} - \frac{n-j}{j} \sum_{s=1}^{j} p_{n+k-j+s,n+k-j+s} \\ &\geq \operatorname{trace} \mathcal{P} - j - (n-j) \\ &\geq n - j - (n-j) = 0.\end{aligned}$$

□

Remark 5. Clearly, for any $j \geq n$ and $n + k - j \geq 1$ the polynomials p_j are trivially subharmonic on M, since $-\frac{(n-j)}{j} \geq 0$ in this case.

Again, by a straight-forward application of maximum principle we obtain

Corollary 3. *Suppose $M \subset \mathbb{R}^{n+k}$ is a minimal submanifold with boundary ∂M contained in a body congruent to*

$$\mathcal{H}_j(\varepsilon) := \Big\{(x^1, \ldots, x^{n+k}) \in \mathbb{R}^{n+k} : p_j(x^1, \ldots, x^{n+k}) \le \varepsilon\Big\},$$

for any $\varepsilon \in \mathbb{R}$*. Then* $M \subset \mathcal{H}_j(\varepsilon)$*,* $j = 1, \ldots, n-1$*.*

In this Corollary one can take $j = 1$ obtaining "*nonexistence cones*" for any dimension n and any codimension k. In other words we consider the cones $C_{n+k} = C^+_{n+k} \cup C^-_{n+k} \cup \{0\}$ defined by

$$\begin{aligned} C^{\pm}_{n+k} :=& \Bigg\{(x^1, \ldots, x^{n+k}) \in \mathbb{R}^{n+k} : \pm x^{n+k} > 0, \text{ and} \\ & \sum_{i=1}^{n+k-1} |x^i|^2 \le (n-1)|x^{n+k}|^2\Bigg\} \\ =& \Big\{x \in \mathbb{R}^{n+k} : \pm x^{n+k} > 0 \text{ and } p_1(x) \le 0\Big\}. \end{aligned}$$

Theorem 4. *Let* $C \subset \mathbb{R}^{n+k}$ *be a cone with vertex* P_0 *which is congruent to* C_{n+k} *and let* $C^{\pm}$ *denote the two disjoint parts which correspond to* $C^{\pm}_{n+k}$*. Then there is no connected, compact, n-dimensional minimal submanifold* $M \subset \mathbb{R}^{n+k}$ *with* $\partial M \subset C$ *such that both* $\partial M \cap C^+$ *and* $\partial M \cap C^-$ *are nonempty.*

Proof. By performing a rotation and translation we may assume without loss of generality that $C = C_{n+k}$. Suppose on the contrary that there is a minimal M satisfying the assumptions of Theorem 4. By Theorem 3 we obtain the inequality

$$\Delta_M \left[\sum_{i=1}^{n+k-1} |x^i|^2 - (n-1)|x^{n+k}|^2\right] \ge 0$$

and by the hypothesis of Theorem 4 we have

$$\left[\sum_{i=1}^{n+k-1} |x^i|^2 - (n-1)|x^{n+k}|^2\right]\Bigg|_{\partial M} \le 0.$$

The maximum principle yields

$$\left[\sum_{i=1}^{n+k-1} |x^i|^2 - (n-1)|x^{n+k}|^2\right]\Bigg|_{M} \le 0,$$

or equivalently, $M \subset C_{n+k}$. Since M is connected and $\partial M \cap C^+_{n+k} \ne \emptyset$, $\partial M \cap C^-_{n+k} \ne \emptyset$, M must contain the vertex 0 of the cone, which clearly contradicts the manifold property of M. □

We remark that Theorem 4 may be used to derive necessary conditions for the existence of compact, connected minimal submanifolds with several boundary components.

Corollary 4 (Necessary Condition). *Let $B_1, B_2 \subset \mathbb{R}^{n+k}$ be closed sets and suppose there exists an n-dimensional compact, connected minimal submanifold $M \subset \mathbb{R}^{n+k}$ with $\partial M \subset B_1 \cup B_2$ and that both $\partial M \cap B_1 \neq \phi$ and $\partial M \cap B_2 \neq \phi$. Then we have:*

(i) *If B_i, $i = 1, 2$ are closed balls with centers x_i and radii δ_i and $R := |x_1 - x_2|$, then*

$$R \leq \left(\frac{n}{n-1}\right)^{\frac{1}{2}} (\delta_1 + \delta_2).$$

(ii) *If B_1 and B_2 are arbitrary compact sets of diameters d_1 and d_2 which are separated by a slab of width $r > 0$, then*

$$r \leq \frac{1}{2}\left(\frac{2n(n+k)}{(n-1)(n+k+1)}\right)^{\frac{1}{2}} (d_1 + d_2).$$ □

Next we consider arbitrary n-dimensional submanifolds $M \subset \mathbb{R}^{n+k}$ with mean curvature vector $\vec{H}$. According to Theorem 2 we have the identity

$$\Delta_M x = n\,\vec{H},$$

and by Proposition 3,

$$\vec{H}(x) = -\frac{1}{n}\sum_{j=1}^{k}(\mathrm{div}_M N_j)N_j$$

for an arbitrary orthonormal basis $N_1, \ldots, N_k \in \mathbb{R}^{n+k}$ of the normal space $T_xM^{\perp}$.

Let $H^1, \ldots, H^k$ be the components of $\vec{H}$ with respect to that basis $N_1, \ldots, N_k$ i.e.

$$\begin{aligned} H &= H^1 N_1 + \cdots + H^k N_k, \quad \text{or} \\ H^i &= -\frac{1}{n}\mathrm{div}_M N_i \quad \text{for } i = 1, \ldots, k, \end{aligned}$$

and put

$$p(x) := \sum_{i=1}^{n+k-j} |x^i|^2 - \frac{(n-j)}{j} b \sum_{i=n+k-j+1}^{n+k} |x^i|^2,$$

where $b \in \mathbb{R}$ and $j = 1, \ldots, n-1$. Defining r_j and s_j by

$$r_j(x) := \sum_{i=1}^{n+k-j} |x^i|^2 \quad \text{and} \quad s_j(x) := \sum_{i=n+k-j+1}^{n+k} |x^i|^2$$

we obtain

$$p(x) = r_j(x) - \frac{(n-j)}{j} b s_j(x).$$

By the same arguments as in the proof of Theorem 3 we conclude

$$\begin{aligned}
\frac{1}{2}\Delta_M p = & \sum_{i=1}^{n+k-j} x^i \Delta_M x^i - b\frac{(n-j)}{j} \sum_{i=n+k-j+1}^{n+k} x^i \Delta_M x^i \\
& + \sum_{i=1}^{n+k-j} |\nabla_M x^i|^2 - b\frac{(n-j)}{j} \sum_{i=n+k-j+1}^{n+k} |\nabla_M x^i|^2 \\
\geq & \ n \left\langle \vec{H}, \left(x^1, \ldots, x^{n+k-1}, -b\frac{(n-j)}{j} x^{n+k-j+1}, \ldots, -b\frac{(n-j)}{j} x^{n+k} \right) \right\rangle \\
& + (n-j)(1-b) \\
\geq & -n|\vec{H}| \left[r_j + \frac{b^2(n-j)^2}{j^2} s_j \right]^{\frac{1}{2}} \\
& + (n-j)(1-b), \quad \text{by Schwarz's inequality.}
\end{aligned}$$

Finally we obtain the estimate

$$\frac{1}{2}\Delta_M p \geq (n-j)\left\{ (1-b) - n|\vec{H}| \left[\frac{r_j}{(n-j)^2} + \frac{b^2}{j^2} s_j \right]^{\frac{1}{2}} \right\}.$$

Thus we have proved:

Theorem 5. *Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional submanifold with mean curvature vector $\vec{H} = H^1 N_1 + \cdots + H^k N_k$, $0 \leq b \leq 1$, $1 \leq j \leq n-1$ and $p(x) = \sum_{i=1}^{n+k-j} |x^i|^2 - \frac{(n-j)}{j} b \sum_{i=n+k-j+1}^{n+k} |x^i|^2 = r_j(x) - \frac{(n-j)}{j} b s_j(x)$. Then $p(x)$ is subharmonic on M, if*

$$b + n|\vec{H}| \left[\frac{r_j(x)}{(n-j)^2} + \frac{b^2}{j^2} s_j(x) \right]^{1/2} \leq 1 \tag{11}$$

holds true, where

$$|\vec{H}| = (|H^1|^2 + \cdots + |H^k|^2)^{1/2}.$$

□

Observe that (11) is satisfied for example if

$$q := \sup_{x \in M} |x||\vec{H}(x)| < \frac{1}{n} \quad \text{and} \quad b := 1 - n\,q. \tag{12}$$

Corollary 5. *Suppose that condition (12) holds true. Then for any $j = 1, \ldots, n-1$ the quadratic polynomial $p(x) = r_j(x) - (\frac{(n-j)}{j}) b s_j(x)$ is subharmonic on M. Therefore, if M is compact the estimate $\sup_M p \leq \sup_{\partial M} p$ is fulfilled. In particular, if $K := K^+ \cup \{0\} \cup K^-$, where $K^\pm := \{x \in \mathbb{R}^{n+k} : \sum_{i=1}^{n+k-1} |x^i|^2 - (n-1)(1-nq)|x^{n+k}|^2 \leq 0, \pm x^{n+k} > 0\}$ and $\partial M \subset K$ such that both $\partial M \cap K^+$ and $\partial M \cap K^-$ are nonempty, then M cannot be connected.*

Alternatively, (11) is fulfilled provided

$$q := \sup_M |x||\vec{H}(x)| < \frac{n-j}{n}, \tag{13}$$

and $b := \min(\frac{1}{n-1}, 1 - \frac{n\,q}{n-j})$.

Corollary 6. *Suppose that (13) holds for some $j = 1, \ldots, n-1$. Then $p(x) = r_j(x) - \frac{b(n-j)}{j} \cdot s_j(x)$ is subharmonic on M. In particular, if this holds with $j = 1$ then there is no connected compact submanifold with mean curvature $\vec{H}$ which satisfies $\partial M \subset K$ and $\partial M \cap K^+ \neq \emptyset$, and $\partial M \cap K^- \neq \emptyset$, where*

$$K = K^+ \cup \{0\} \cup K^- \quad \textit{and}$$
$$K^\pm := \left\{ x \in \mathbb{R}^{n+k} : \sum_{i=1}^{n+k-1} |x^i|^2 - (n-1)b|x^{n+k}|^2 \le 0, \pm x^{n+k} > 0 \right\}. \quad \square$$

4.3.1 An Optimal Nonexistence Result for Minimal Submanifolds of Codimension One

Now we address the question whether the *"nonexistence cones"* C_{n+k} considered in Theorem 4 can still be enlarged. In Section 6.2, Corollary, we have considered the cone

$$\mathcal{K} := \mathcal{K}^+ \cup \{0\} \cup \mathcal{K}^-,$$

where

$$\mathcal{K}^\pm := \left\{ (x, y, z) \in \mathbb{R}^3 : z \gtrless 0 \text{ and } x^2 + y^2 < z^2 \sinh^2 \tau \right\}$$

and $\tau = 1.1996\ldots$ is a solution of the equation

$$\tau \sinh \tau = \cosh \tau.$$

This cone $\mathcal{K}$ is in fact a "nonexistence cone" for $n = 2$, $k = 1$ which cannot be enlarged further, since it is the envelope of a field of suitable catenoids; in other words $\mathcal{K}$ is "enclosed" by the "catenoidal domains"

$$\mathcal{K}_\alpha = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < \alpha^2 \cosh^2 \frac{z}{\alpha} \right\},$$

cp. the discussion in Section 4.2. We generalize this argument as follows: Consider a curve $(x, y(x))$ in the Euclidean plane and its rotational symmetric graph (of dimension $n + 1$)

$$\mathcal{M}_{\text{rot}} := \{(x, y(x) \cdot w) \in \mathbb{R} \times \mathbb{R}^{n+1} : x \in [a, b], w \in S^n\},$$

where $S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$ denotes the unit n-sphere. One readily convinces oneself that the $(n+1)$-dimensional area of $\mathcal{M}_{\text{rot}}$ is proportional to the one-dimensional variational integral

$$\mathcal{J} = \mathcal{J}(y) = \int_a^b y^n(x)\sqrt{1+y'^2(x)}\, dx.$$

In other words, extremals of $\mathcal{J}$ correspond to $(n+1)$-dimensional minimal submanifolds in $\mathbb{R}^{n+2}$, which are rotationally symmetric, the so-called "*n-catenoids*" (or, to be more precise, "$(n+1)$-*catenoids*"). The Euler equation of the integral $\mathcal{J}$ is simply

$$\frac{d}{dx}\left(\frac{y'y^n}{\sqrt{1+y'^2}}\right) = ny^{n-1}\sqrt{1+y'^2}. \tag{14}$$

Since the integrand f of $\mathcal{J}(\cdot)$ does not explicitly depend on the variable x we immediately obtain a first integral of (14), namely

$$y^n = \lambda\sqrt{1+y'^2}$$

for any $\lambda > 0$. A further integration gives the inverse of a solution $y = y(x)$ of the Euler equation (14) as follows:

$$x = x(y) = \lambda \int_{\sqrt[n]{\lambda}}^{y} \frac{d\xi}{\sqrt{\xi^{2n} - \lambda^2}} + c. \tag{15}$$

These inverse functions are defined for any $\lambda > 0$, $c \in \mathbb{R}$ and all $y \geq \sqrt[n]{\lambda}$. Note that (15) with $n = 1$ leads to the classical catenaries, which—upon rotation into $\mathbb{R}^3$—determine the well known catenoids. Of importance in our following construction here, is the one parameter family of "n-catenaries" (or rather of their inverses)

$$x = g(y, \lambda) := \lambda \int_{\sqrt[n]{\lambda}}^{y} \frac{d\xi}{\sqrt{\xi^{2n} - \lambda^2}}, \quad y \geq \sqrt[n]{\lambda}.$$

Claim. *The envelope of the family* $g(y,\lambda)$, $\lambda > 0$, *is the straight line* $y = \tau_0 x$, $x > 0$, *where* $\tau_0 := \sqrt{z_0^{2n} - 1}$, *and* z_0 *is the unique solution of the equation*

$$\frac{z}{\sqrt{z^{2n} - 1}} = \int_1^z \frac{d\xi}{\sqrt{\xi^{2n} - 1}}. \tag{16}$$

Proof. First note that (15) implies that $\frac{d^2x}{dy^2} < 0$, whence the solutions $x = g(y,\lambda)$, $\lambda > 0$, are strictly convex functions when considered as graphs over x. Hence for each $\lambda > 0$ there exist unique numbers $\tau = \tau(\lambda)$, $x = x(\lambda) > 0$ and $y = y(\lambda) > 0$ with the properties

$$x(\lambda) = g(y(\lambda), \lambda) \quad \text{and} \quad \tau(\lambda) = \frac{y(\lambda)}{x(\lambda)} = y'(x(\lambda)),$$

where $y'(x(\lambda))$ denotes the slope of the curve $x = g(y,\lambda)$ considered as a function $y(x)$ at the particular point $x(\lambda)$. Since $y^n = \lambda\sqrt{1+y'^2}$, this last requirement can be written as

$$\tau(\lambda) = \frac{y(\lambda)}{x(\lambda)} = \frac{\sqrt{y^{2n}(\lambda) - \lambda^2}}{\lambda} = \left\{ \left[\frac{y(\lambda)}{\sqrt[n]{\lambda}} \right]^{2n} - 1 \right\}^{\frac{1}{2}} .$$

We now claim that the quotient $q(\lambda) := \frac{y(\lambda)}{\sqrt[n]{\lambda}}$ is independent of λ, i.e. $q(\lambda) =$ const. Indeed we find successively $x(\lambda) = \frac{y(\lambda)}{\tau(\lambda)} = \frac{y(\lambda)}{[q^{2n}(\lambda)-1]^{\frac{1}{2}}}$; on the other hand,

$$x(\lambda) = \lambda \int_{\sqrt[n]{\lambda}}^{y} \frac{d\xi}{\sqrt{\xi^{2n} - \lambda^2}} = \int_{\sqrt[n]{\lambda}}^{y} \frac{d\xi}{\{(\frac{\xi}{\sqrt[n]{\lambda}})^{2n} - 1\}^{\frac{1}{2}}}.$$

Thus $q(\lambda) = \frac{y(\lambda)}{\sqrt[n]{\lambda}}$ satisfies (16). However, there is only one solution of (16), since the left hand side of (16) is monotonically decreasing, while the right hand side monotonically increases, and both sides are continuous. Concluding we have shown that each member of the family $g(y, \lambda)$, $\lambda > 0$, $y \geq \sqrt[n]{\lambda}$, touches the half line $y = \tau_0 x$, $\tau_0 = \sqrt{z_0^{2n} - 1}$ precisely at one point, namely at $x_0(\lambda) = \frac{z_0}{\tau_0} \sqrt[n]{\lambda}$, $y_0(\lambda) = z_0 \sqrt[n]{\lambda}$. Also, each point of the half line $y = \tau_0 x$, $x > 0$, is the point of contact for precisely one member of the family $g(\cdot, \lambda)$, $\lambda > 0$. This proves the claim. □

Let $f(\cdot, \lambda)$ denote the family of inverse functions, that is we have

$$f(g(y, \lambda), \lambda) = y, \quad \text{for } y \geq \sqrt[n]{\lambda} \quad \text{and} \quad g(f(x, \lambda), \lambda) = x \quad \text{for } x \geq 0.$$

We extend f by an even reflection i.e. $f(x, \lambda) = f(-x, \lambda)$ for $x \leq 0$, so as to obtain a smooth function defined on the real axis. Observe that for $n = 1$, these are precisely the catenaries $f(x, \lambda) = \lambda \cosh(\frac{x}{\lambda})$. Put $r := \{\sum_{i=1}^{n+1} |x^i|^2\}^{\frac{1}{2}}$; then for each $\lambda > 0$ the hypersurfaces

$$\mathcal{M}_\lambda = \left\{ x \in \mathbb{R}^{n+2} : r = f(x^{n+2}, \lambda) \right\}$$

are smooth $(n+1)$-dimensional minimal submanifolds of $\mathbb{R}^{n+2}$. Furthermore the foregoing construction shows that the sets

$$\mathcal{G}_\lambda := \left\{ x \in \mathbb{R}^{n+2} : r < f(x^{n+2}, \lambda) \right\}$$

for $\lambda > 0$ *enclose* the cone

$$\mathcal{K}_{\tau_0} := \left\{ x \in \mathbb{R}^{n+2} : \pm\tau_0 x^{n+2} > r \right\} \cup \{0\}$$

in the sense of Section 4.2.

By a straightforward modification of Theorem 1, Section 4.2, i.e. by Hopf's maximum principle and the arguments in the proof of the Enclosure Theorem II, Section 4.2 we conclude the following "*Nonexistence Theorem*".

Theorem 6. *The family of domains* $\{\mathcal{G}_\lambda\}_{\lambda>0}$ *enclose the cone* $\mathcal{K}_{\tau_0}$*, where* $\tau_0 := \sqrt{z_0^{2n}-1}$ *and* z_0 *is a solution of the equation* (16). *Furthermore, if* $\mathcal{C} = \mathcal{C}^+ \cup \{0\} \cup \mathcal{C}^- \subset \mathbb{R}^{n+2}$ *is a cone with vertex* p_0 *which is congruent to* $\mathcal{K}_{\tau_0}$*, then there is no connected, compact* $(n+1)$*-dimensional minimal submanifold* $M \subset \mathbb{R}^{n+2}$ *with* $\partial M \subset \mathcal{C}$ *such that both* $\partial M \cap \mathcal{C}^+$ *and* $\partial M \cap \mathcal{C}^-$ *are nonempty.*

Remark 1. By construction, the hypersurfaces $r = f(x^{n+2}, \lambda)$, $\lambda > 0$ are minimal in $\mathbb{R}^{n+2}$ and intersect the boundary of the cone $\mathcal{K}_{\tau_0}$ in an n-dimensional sphere. Thus there is no "larger" cone with the nonexistence property described in Theorem 6. In particular the corresponding nonexistence cones introduced in Theorem 4 are "smaller" than $\mathcal{K}_{\tau_0}$. This is illustrated in the following table. Observe that the cones $\mathcal{K}_{\tau_0}$ become larger when the dimension increases.

Dimension of the surface: $n+1$	τ_0	Angle of aperture	$\sqrt{n}$
2	1.51	56.46	1
3	2.37	67.15	1.414
4	3.15	72.40	1.732
5	3.89	75.60	2
6	4.63	77.81	2.236
7	5.44	79.59	2.449
8	6.02	80.58	2.645

4.4 Geometric Maximum Principles

4.4.1 The Barrier Principle for Submanifolds of Arbitrary Codimension

Let $\mathcal{S}$ (the "barrier") be a C^2 hypersurface of $\mathbb{R}^{n+1}$ with mean curvature Λ with respect to the local normal field ν. Assume that $M \subset \mathbb{R}^{n+1}$ is another hypersurface with mean curvature H which lies locally on that side of $\mathcal{S}$ to which the normal ν points, and that the inequality

$$\sup_{U\cap M} |H| \le \inf_{U\cap \mathcal{S}} \Lambda \tag{1}$$

holds in a neighbourhood $U = U(p_0) \subset \mathbb{R}^{n+1}$ of any point $p_0 \in \mathcal{S} \cap M$. If the intersection $\mathcal{S} \cap M$ is nonempty (in other words, if M touches $\mathcal{S}$ in some interior point p_0) then, using Hopf's lemma and an argument similar as in the proof of Theorem 1 in Section 4.2, it follows that M must be locally contained in $\mathcal{S}$. In Section 4.2 we have admitted one of the surfaces to be singular in possible points of intersection.

Now we discuss a version of this barrier principle for n-dimensional submanifolds $M \subset \mathbb{R}^{n+k}$ with bounded mean curvature vector $\vec{H}$. The crucial requirement is again a condition of type (1); however, the mean curvature Λ has to be replaced by the "*n-mean curvature*" Λ_n, which is the arithmetic mean of the sum of the n smallest principal curvatures of $\mathcal{S}$, while $|H|$ has to be replaced by the length of the mean curvature vector $\vec{H}$ of the submanifold M.

Let us recall some notations: $\mathcal{S} \subset \mathbb{R}^{n+k}$ denotes a C^2-hypersurface with (local) normal field ν and $\lambda_1 \le \cdots \le \lambda_{n+k-1}$ stand for the principal curvatures of $\mathcal{S}$ with respect to that normal ν. We define the "*n-mean curvature*" Λ_n with respect to the normal ν as

$$\Lambda_n := \frac{1}{n}(\lambda_1 + \cdots + \lambda_n), \quad \text{where } \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \lambda_{n+k-1}.$$

Furthermore let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional C^2-submanifold with mean curvature vector

$$\vec{H} = -\frac{1}{n}\sum_{j=1}^{k} (\operatorname{div}_M N_j) N_j,$$

where $N_1, \ldots, N_k$ denotes an orthonormal basis of the normal space $T_x M^\perp$, cp. Section 4.3 for definition and properties of the mean curvature vector.

Theorem 1. *Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional C^2-submanifold with mean curvature vector $\vec{H}$, and $\mathcal{S} \subset \mathbb{R}^{n+k}$ be a C^2-hypersurface. Suppose that M lies locally on that side of $\mathcal{S}$ into which the normal ν is pointing. Finally assume that M touches $\mathcal{S}$ at an interior point $p_0 \in M \cap \mathcal{S}$ and that in some neighbourhood $U(p_0) \subset \mathbb{R}^{n+k}$ the inequality*

$$\sup_{U \cap M} |\vec{H}| \le \inf_{U \cap \mathcal{S}} \Lambda_n \tag{2}$$

holds true. Then, near p_0, M is contained in $\mathcal{S}$, i.e. we have $M \cap U \subset \mathcal{S} \cap U$.

Corollary 1. *Suppose that M lies locally on that side of $\mathcal{S}$ into which the normal ν is pointing. Then M and $\mathcal{S}$ cannot touch at an interior point $p_0 \in M \cap \mathcal{S}$ if $|\vec{H}(p_0)| < \Lambda_n(p_0)$ holds.*

This theorem implies the following

Enclosure Theorem 1. *Let $G \subset \mathbb{R}^{n+k}$ be a domain with boundary $\mathcal{S} = \partial G \in C^2$ and M be an n-dimensional C^2-submanifold with mean-curvature vector $\vec{H}$ which is confined to the closure $\overline{G}$. Also, let Λ_n denote the n-mean curvature of $\mathcal{S} = \partial G$ with respect to the inward unit normal ν. Finally assume that, if M touches $\mathcal{S}$ at some interior point p_0, then the inequality*

$$\sup_{U \cap M} |\vec{H}| \le \inf_{U \cap \mathcal{S}} \Lambda_n$$

holds true for some neighbourhood $U = U(p_0) \subset \mathbb{R}^{n+k}$. Then M lies in the interior of G, if at least one of its points lies in G.

Remark 1. Clearly, the hypothesis

$$|\vec{H}(p_0)| < \Lambda_n(p_0),$$

implies (2), but excludes e.g. the case $\vec{H} \equiv 0$ and $\Lambda_n \equiv 0$.

Remark 2. Let us consider an example which shows that Theorem 1 is optimal. To see that let $\mathcal{S} \subset \mathbb{R}^3$ be the cylinder $\{x^2 + y^2 = R^2\}$; then the principal curvatures with respect to the inward unit normal ν are given by $\lambda_1 = 0 \le \lambda_2 = \frac{1}{R}$ and the n-mean curvature ($n = 1$ or 2 is possible) are $\Lambda_1 = 0$ and $\Lambda_2 = \Lambda = \frac{1}{2R}$. Take $n = 1$; then Theorem 1 requires $\vec{H} \equiv 0$, and this implies that M is a straight line. This is indeed necessary for the conclusion of Theorem 1 to hold since there are circles of arbitrary small "mean curvature" $|\vec{H}| = \frac{1}{r}$, $r > 0$, which locally are on the interior side of the cylinder $\mathcal{S}$ and touch $\mathcal{S}$ in exactly one point; yet these circles are not locally contained in $\mathcal{S}$.

For the proof of Theorem 1 we need to recall some important facts about the distance function, a proof of which can be found in Gilbarg and Trudinger [1], Chapter 14.6, or Hildebrandt [19], Section 4.6.

Let $\mathcal{S} \subset \mathbb{R}^{n+k}$ be a hypersurface with orientation ν. The distance function $d = d(x)$ is defined by

$$d(x) = \operatorname{dist}(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} |x - y|.$$

Locally we can orient d so as to obtain the *signed* or *oriented* distance function ρ as follows: Choose a point $p_0 \in \mathcal{S}$. Then there is an open ball $B_\varepsilon(p_0) \subset \mathbb{R}^{n+k}$ which is partitioned by $\mathcal{S}$ into two open sets B_ε^+ and B_ε^-. Let B_ε^+ denote the set into which the normal ν points. The oriented distance ρ is then given by

$$\rho(x) = \begin{cases} d(x), & \text{for } x \in B_\varepsilon^+, \\ -d(x), & \text{for } x \in B_\varepsilon^-. \end{cases}$$

It follows easily that d and ρ are Lipschitz-continuous functions with Lipschitz constant equal to one. In fact, let $y \in \mathbb{R}^{n+k}$ and choose $z \in \mathcal{S}$ such that $d(y) = |z - y|$. Then for any $x \in \mathbb{R}^{n+k}$ we have

$$d(x) \le |x - z| \le |x - y| + |y - z| = |x - y| + d(y)$$

and the some inequality holds with x replaced by y, whence we obtain $|d(x) - d(y)| \le |x - y|$. Observe that this holds without any assumption on the set $\mathcal{S}$. Similarly, for $x \in B_\varepsilon^+$, $y \in B_\varepsilon^-$ there exists $t_0 \in [0, 1]$ with $z_{t_0} = t_0 y + (1 - t_0)x \in \mathcal{S}$ and $\rho(x) - \rho(y) = d(x) + d(y) = d(x) - d(z_{t_0}) + d(y) - d(z_{t_0}) \le |x - z_{t_0}| + |y - z_{t_0}| = |x - y|$, whence also ρ is Lipschitz continuous.

Much more is true, if $\mathcal{S}$ is of class C^j, $j \ge 2$.

Lemma 1. *Let $\mathcal{S} \subset \mathbb{R}^{n+k}$ be a hypersurface of class $C^j, j \geq 2$, and $p_0 \in \mathcal{S}$ be arbitrary. Then there is a constant $\varepsilon > 0$ (depending on p_0 in general) such that $d \in C^j(\overline{B}_\varepsilon^+)$, $d \in C^j(\overline{B}_\varepsilon^-)$ and the oriented distance $\rho \in C^j(B_\varepsilon(p_0))$.*

For a proof – which consists in an application of the implicit function theorem – we refer the reader to Gilbarg and Trudinger [1], Section 14.6, or Hildebrandt [19], Section 4.6.

Remark I. Obviously $d \notin C^1(B_\varepsilon(p_0))$ for $p_0 \in \mathcal{S}$, $\varepsilon > 0$.

Remark II. In Gilbarg and Trudinger [1] only the unoriented distance d is considered; however the proofs can be easily modified with almost no alterations.

Remark III. If $\mathcal{S} \subset \mathbb{R}^{n+k}$ is a compact closed hypersurface of class C^j, $j \geq 2$, then it satisfies a uniform interior (as well as exterior) *sphere condition*; that is at each point $p_0 \in \mathcal{S}$ there exists a ball B_{ε_0} of uniform radius $\varepsilon_0 > 0$ which lies in the interior (or exterior) side of $\mathcal{S}$ respectively and such that the closure $\overline{B}_{\varepsilon_0}$ has just one point in common with the surface $\mathcal{S}$, namely p_0. In this case the distance function is of class C^j on a tube T_{ε_0} of uniform width ε_0 where $T_{\varepsilon_0} := T^+_{\varepsilon_0} \cup T^-_{\varepsilon_0}$ with

$$T^+_{\varepsilon_0} := \{x \in \mathbb{R}^{n+k};\ 0 \leq \rho(x) < \varepsilon_0\}, \quad T^-_{\varepsilon_0} := \{x \in \mathbb{R}^{n+k};\ -\varepsilon_0 < \rho(x) \leq 0\}.$$

Then we have $d \in C^j(T^+_{\varepsilon_0})$, $d \in C^j(T^-_{\varepsilon_0})$ and $\rho \in C^j(T_{\varepsilon_0})$.

Choose $p_0 \in \mathcal{S}$ and $\varepsilon > 0$ such that $\rho \in C^j(B_\varepsilon(p_0))$; consider the *parallel surface*

$$\mathcal{S}_\tau := \{x \in \mathbb{R}^{n+k} \cap B_\varepsilon(p_0) : \rho(x) = \tau\},$$

$-\varepsilon < \tau < \varepsilon$, which is again of class C^j, if $\mathcal{S} \in C^j$, $j \geq 2$. The unit normal of $\mathcal{S}_\tau$ at $x \in \mathcal{S}_\tau$ directed towards increasing ρ is given by $\nu(x) = D\rho(x) = (\rho_{x^1}(x), \ldots, \rho_{x^{n+k}}(x))$. (Note that here – for simplicity of notation – we refrain from writing ν_τ instead of ν, so as to obtain a function $\nu \in C^{j-1}(B_\varepsilon(p_0))$ which, on $\mathcal{S} \cap B_\varepsilon(p_0)$ coincides with the unit normal on $\mathcal{S}$.)

For every point $x_0 \in \overline{B}^+_\varepsilon(p_0)$ or $B_\varepsilon(p_0)$ there exists a unique point $y_0 = y(x_0) \in \mathcal{S}$ such that $d(x_0) = |x_0 - y_0|$ or $\rho(x_0) = \pm|x_0 - y_0|$ respectively, in particular $x_0 = y_0 + \nu(y_0) \cdot \rho(x_0)$. We need to compare the principal curvatures $\lambda_1(y_0), \ldots, \lambda_{n+k-1}(y_0)$ of $\mathcal{S}$ at y_0 with the principal curvatures of $\mathcal{S}_{\rho(x_0)}$ at x_0. We recall the following

Lemma 2. *Let $x_0 \in \mathcal{S}_\tau$, $y_0 \in \mathcal{S}$ be such that $\rho(x_0) = \pm|x_0 - y_0|$. If y is a principal coordinate system at y_0 and $x = y + \nu(y)\rho$ then we have*

$$D^2\rho(x_0) = \rho_{x^i x^j}(x_0) = \operatorname{diag}\left(\frac{-\lambda_1(y_0)}{1 - \lambda_1(y_0)\rho(x_0)}, \ldots, \frac{-\lambda_{n+k-1}(y_0)}{1 - \lambda_{n+k-1}(y_0)\rho(x_0)}, 0\right).$$

For a *proof of Lemma 2* we refer to Gilbarg and Trudinger [1], Lemma 14.17. □

We now claim that the Hessian matrix $(\rho_{x^i x^j}(x_0))$, $i,j=1,\dots,n+k-1$ is also given by the diagonal matrix $(-\lambda_i(x_0)\delta_{ij})$, $i,j=1,\dots,n+k-1$, where $\lambda_1(x_0),\dots,\lambda_{n+k-1}(x_0)$ stand for the principal curvatures of $\mathcal{S}_{\rho(x_0)}$ at x_0. To see this consider $\nu(x_0)=D\rho(x_0)=(\rho_{x^1}(x_0),\dots,\rho_{x^{n+k}}(x_0))$ and suppose without loss of generality that $D\rho(x_0)=(0,\dots,0,1)$. Then there is some C^j-function $x^{n+k}=\varphi(x^1,\dots,x^{n+k-1})$ such that $\rho(x^1,\dots,x^{n+k-1},\varphi(x^1,\dots,x^{n+k-1}))=\rho(x_0)$. Differentiating this relation with respect to x^i, $i=1,\dots,n+k-1$ yields $\varphi_{x^i}=-\frac{\rho_{x^i}}{\rho_{x^{n+1}}}$, for $i=1,\dots n+k-1$, and

$$\varphi_{x^i x^j}=-\left(\frac{\rho_{x^i x^j}\rho_{x^{n+1}}-\rho_{x^i}\rho_{x^{n+1}x^j}}{\rho^2_{x^{n+1}}}\right).$$

Hence we get

$$\varphi_{x^i x^j}(\hat{x}_0)=-\rho_{x^i x^j}(x_0),\quad i,j=1,\dots,n+k-1,$$

where $x_0=(\hat{x}_0,x^{n+k})$.

On the other hand we have seen in the beginning of Section 4.3 that the eigenvalues of $D^2\varphi(\hat{x}_0)$ are precisely the principal curvatures $\lambda_i(x_0)$ of the graph of φ, i.e. of the distance surface $\mathcal{S}_\tau$, $\tau=\rho(x_0)$, at x_0. We have shown

Lemma 3. *Let $\mathcal{S}$ and $\mathcal{S}_\tau$ be as above, and $x_0\in\mathcal{S}_\tau$, $y_0\in\mathcal{S}$ be such that $\rho(x_0)=\pm|x_0-y_0|$, i.e. $\tau=\rho(x_0)$. Denote by $\lambda_1(y_0),\dots,\lambda_{n+k-1}(y_0)$ the principal curvatures of $\mathcal{S}$ at y_0 and by $\lambda_1(x_0),\dots,\lambda_{n+k-1}(x_0)$ the principal curvatures of the parallel surface $\mathcal{S}_\tau$ at x_0. Then we have*

$$\lambda_i(x_0)=\frac{\lambda_i(y_0)}{1-\lambda_i(y_0)\rho(x_0)}\quad \textit{for } i=1,\dots,n+k-1.$$

We continue with further preparatory results for the proof of Theorem 1 and select an orthonormal basis $t_1,\dots,t_n$ of the tangent space T_xM of M at x, assuming that x is close to $\mathcal{S}$. Introducing the orthogonal projection

$$t_i^\top:=t_i-\langle t_i,\nu\rangle\cdot\nu$$

of t_i onto the tangent space $T_x\mathcal{S}_{\rho(x)}$ of the parallel surface $\mathcal{S}_{\rho(x)}$ at the point x. Also let $T_xM^\top$ stand for the orthogonal projection of the n-dimensional tangent space T_xM onto the $(n+k-1)$-dimensional tangent space $T_x\mathcal{S}_{\rho(x)}$.

Finally $\mathrm{II}=\mathrm{II}_x(\cdot,\cdot)$ denotes the second fundamental form of the distance hypersurface $\mathcal{S}_{\rho(x)}$ with respect to the normal $\nu=D\rho$ at the particular point $x\in\mathcal{S}_{\rho(x)}$, i.e. (cp. Section 4.3) $\mathrm{II}_x(t,\tau)=\langle -D_t\nu,\tau\rangle$, for $t,\tau\in T_x\mathcal{S}_{\rho(x)}$.

Lemma 4. *Let M and $\mathcal{S}$ be as in Theorem 1 and $p_0\in M\cap\mathcal{S}$. Then the distance function $\rho=\rho(x)$ satisfies the equation*

$$\Delta_M\rho + b_i(\nabla_M\rho)_i - n\langle \overrightarrow{H}, D\rho\rangle + \mathrm{trace}\,\mathrm{II}|_{T_xM^\top} = 0$$

in a neighbourhood $V \subset \mathbb{R}^{n+k}$ of p_0. Here trace $\mathrm{II}|_{T_xM^\top}$ denotes the trace of the second fundamental form II *of $\mathcal{S}_{\rho(x)}$ at x restricted to the subspace $T_xM^\top$ of $T_x\mathcal{S}_{\rho(x)}$, $b_i = b_i(x) := \frac{-\mathrm{II}(t_i^\top, t_j^\top)(\nabla_M\rho)_j}{1-|\nabla_M\rho|^2}$, for $i = 1, \ldots, n$, $(\nabla_M\rho)_i = D_{t_i}\rho$ and $D\rho = (\rho_{x_1}, \ldots, \rho_{n+k})$.*

Proof of Lemma 4. We have $\nabla_M\rho = D\rho - \langle D\rho, N_1\rangle N_1 - \cdots - \langle D\rho, N_k\rangle N_k$, where $N_1, \ldots, N_k$ is an orthonormal basis of the normal space $T_xM^\perp$. Therefore

$$
\begin{aligned}
(3) \qquad \Delta_M\rho &= \mathrm{div}_M \nabla_M\rho \\
&= \mathrm{div}_M D\rho - \langle D\rho, N_1\rangle \mathrm{div}_M N_1 - \cdots - \langle D_M\rho, N_k\rangle \mathrm{div}_M N_k \\
&= \mathrm{div}_M D\rho + n\langle \overrightarrow{H}, (D\rho)^\perp\rangle,
\end{aligned}
$$

where $(D\rho)^\perp = \langle D\rho, N_1\rangle N_1 + \cdots + \langle D\rho, N_k\rangle N_k$ is the normal part of $\nu = D\rho$ relative to M, and $\overrightarrow{H} = -\frac{1}{n}\sum_{j=1}^k (\mathrm{div}\, N_j)N_j$ is the mean curvature vector of M (see Proposition 3 of Section 4.3).

Now equation (3) obviously is equivalent to $\Delta_M\rho = \mathrm{div}_M\, D\rho + n\langle \overrightarrow{H}, D\rho\rangle$ and since the divergence on M is the operator $\sum_{i=1}^n t_iD_{t_i}$ we find, because of $D\rho(x) = \nu(x)$

$$(4) \qquad \Delta_M\rho = \sum_{i=1}^n t_iD_{t_i}\nu(x) + n\langle \overrightarrow{H}, D\rho\rangle.$$

To relate the expression $t_iD_{t_i}\nu$ to the second fundamental form of $\mathcal{S}_{\rho(x)}$ we put $t_i = t_i^\top + \langle t_i, \nu\rangle\nu$ and obtain

$$t_iD_{t_i}\nu = (t_i^\top + \langle t_i, \nu\rangle\nu)D_{t_i^\top + \langle t_i,\nu\rangle\nu}\nu = t_i^\top D_{t_i^\top}\nu = -\mathrm{II}_x(t_i^\top, t_i^\top),$$

where we have used that $\langle \nu, D_{t_i^\top}\nu\rangle = 0$ and $D_\nu\nu(x) = 0$ which is a consequence of the relations $|\nu(x)|^2 = 1$ and $\nu(x + t\nu(x)) = \nu(x)$ for $|t| \ll 1$.

Thus (4) implies

$$(5) \qquad \Delta\rho + \sum_{i=1}^n \mathrm{II}_x(t_i^\top, t_i^\top) - n\langle \overrightarrow{H}, D\rho\rangle = 0$$

in a neighbourhood of $p_0 \in \mathcal{S} \cap M$.

In general the projections $t_i^\top$, $i = 1, \ldots, n$ are neither of unit length nor pairwise perpendicular. Therefore, in order to compute the trace of II_x on $T_xM^\top \subset T_x\mathcal{S}_{\rho(x)}$, we put

$$
\begin{aligned}
g_{ij}(x) = g_{ij} &:= \langle t_i^\top, t_j^\top\rangle = \langle t_i, -\langle t_i, \nu\rangle\nu, t_j - \langle t_j, \nu\rangle\nu\rangle \\
&= \delta_{ij} - \langle t_i, \nu\rangle\langle t_j, \nu\rangle.
\end{aligned}
$$

If $x = p_0 \in M \cap \mathcal{S}$ we have $g_{ij} = g_{ij}(x) = \delta_{ij}$; hence in some neighbourhood V of p_0 we can assume that $\sum_{i=1}^n \langle t_i, \nu\rangle^2 < 1$ and that the inverse matrix $g^{ij} = g^{ij}(x)$ is simply

$$g^{ij} = \delta_{ij} + \frac{\langle t_i, \nu\rangle\langle t_j, \nu\rangle}{1 - \sum\limits_{i=1}^n \langle t_i, \nu\rangle^2} =: \delta_{ij} + \varepsilon_{ij}, \quad \text{for } i, j = 1, \dots, n.$$

Therefore we get for the trace of II_x on the subspace $T_x M^\top \subset T_x \mathcal{S}_{\rho(x)}$

$$\begin{aligned} \operatorname{trace} \mathrm{II}|_{T_x M^\top} &= \sum_{i,j=1}^n g^{ij} \mathrm{II}(t_i^\top, t_j^\top) \\ &= \sum_{i,j=1}^n \mathrm{II}(t_i^\top, t_j^\top) + \sum_{i,j=1}^n \varepsilon_{ij} \mathrm{II}(t_i^\top, t_j^\top). \end{aligned}$$

By virtue of (5) this yields

$$\Delta\rho - \sum_{i,j=1}^n \varepsilon_{ij} \mathrm{II}(t_i^\top, t_j^\top) + \operatorname{trace} \mathrm{II}|_{T_x M^\top} - n\langle \vec{H}, D\rho\rangle = 0 \quad \text{in } V \subset \mathbb{R}^{n+k}.$$

Lemma 4 follows by noting that

$$\varepsilon_{ij} = \frac{\langle t_i, \nu\rangle\langle t_j, \nu\rangle}{1 - \sum\limits_{i=1}^n \langle t_i, \nu\rangle^2} = \frac{(\nabla_M \rho)_i (\nabla_M \rho)_j}{1 - |\nabla_M \rho|^2}$$

and taking $b_i = -(1 - |\nabla_M \rho|^2)^{-1} \sum_{j=1}^n \mathrm{II}(t_i^\top, t_j^\top)(\nabla_M \rho)_j$. □

Lemma 5. *Let* II *be a quadratic form on an n-dimensional Euclidean space V with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$. Then for any k-dimensional subspace $W \subset V$ we have the estimate*

$$\operatorname{trace} \mathrm{II}|_W \ge \lambda_1 + \cdots + \lambda_k.$$

The *proof of Lemma 5* is carried out by induction on $k + n$. The case $k + n = 2$ is trivial. By the induction hypothesis we may assume that the assertion holds for all quadratic forms II and linear spaces $V, W \subset V$ of dimension n and k respectively, $k \le n$, such that $k + n \le N$, $N \ge 2$. For given II, V and W we hence assume that $k + n = N + 1$. By $v_1 \in V$ we denote an eigenvector of II corresponding to the smallest eigenvalue λ_1 and put $V_1 := (\operatorname{span} v_1)^\perp$ to denote the $(n-1)$-dimensional orthogonal complement of v_1. We distinguish between the following two cases:
First case: $W \subset V_1$, then by induction hypotheses we have

$$\operatorname{trace} \mathrm{II}|_W \ge \lambda_2 + \cdots + \lambda_{k+1} \ge \lambda_1 + \cdots + \lambda_k.$$

Second case: $W \not\subset V_1$, then there is a nonzero vector $w_1 \in W$ such that

$$(6) \qquad \begin{aligned} &(w_1 - v_1) \perp W \quad \text{or, equivalently,} \\ &\langle w_1, w\rangle = \langle v_1, w\rangle \quad \forall w \in W. \end{aligned}$$

Select on orthonormal basis $\frac{w_1}{|w_1|}, w_2, \dots, w_k$ of W, then by (6) we find $w_2, \dots, w_k$ perpendicular to v_1, in other words, $w_2, \dots, w_k \in V_1$. Applying the induction hypothesis to the triple $\mathrm{II}|_{V_1}, V_1$ and $W_1 := \mathrm{span}(w_2, \dots, w_k)$ yields the estimate

$$\operatorname{trace} \mathrm{II}|_{W_1} = \sum_{j=2}^{k} \mathrm{II}(w_j, w_j) \geq \lambda_2 + \cdots + \lambda_k$$

and therefore

$$\begin{aligned} \operatorname{trace} \mathrm{II}|_W &= \sum_{j=2}^{k} \mathrm{II}(w_j, w_j) + \mathrm{II}\left(\frac{w_1}{|w_1|}, \frac{w_1}{|w_1|}\right) \\ &\geq \lambda_2 + \cdots + \lambda_k + \mathrm{II}\left(\frac{w_1}{|w_1|}, \frac{w_1}{|w_1|}\right) \geq \lambda_1 + \cdots + \lambda_k. \end{aligned}$$

□

Proof of Theorem 1. We claim that, under the assumptions of the theorem, the inequality

$$(7) \qquad -n\langle \vec{H}, D\rho\rangle + \operatorname{trace} \mathrm{II}|_{T_x M^\top} \geq 0$$

holds true in a neighbourhood of any point $p_0 \in M \cap \mathcal{S}$. To prove this let $y_0 \in \mathcal{S}$, $x_0 \in M$ close to $\mathcal{S}$ and $\lambda_1(y_0) \leq \lambda_2(y_0) \leq \cdots \leq \lambda_{n+k-1}(y_0)$ denote the principal curvatures of $\mathcal{S}$ with respect to the unit normal ν. By Lemma 3 we infer for the principal curvatures of $\mathcal{S}_\tau$, $\tau = \rho(x_0)$, at x_0:

$$\lambda_1(x_0) = \frac{\lambda_1(y_0)}{1 - \lambda_1(y_0)\rho(x_0)} \leq \cdots \leq \lambda_{n+k-1}(x_0) = \frac{\lambda_{n+k-1}(y_0)}{1 - \lambda_{n+k-1}(y_0)\rho(x_0)}.$$

Lemma 5 now implies the estimate

$$(8) \qquad \begin{aligned} \frac{1}{n} \operatorname{trace} \mathrm{II}|_{T_x M^\top} &\geq \frac{1}{n}\left(\frac{\lambda_1(y)}{1 - \lambda_1(y)\rho(x)} + \cdots + \frac{\lambda_n(y)}{1 - \lambda_n(y)\rho(x)}\right) \\ &\geq \frac{1}{n}(\lambda_1(y) + \cdots + \lambda_n(y)) = \Lambda_n(y), \end{aligned}$$

where $y \in \mathcal{S}$ is such that $\rho(x) = |x - y|$. By assumption (2) of Theorem 1,

$$\inf_{U \cap \mathcal{S}} \Lambda_n \geq \sup_{U \cap M} |\vec{H}|,$$

we infer from (8)

$$\frac{1}{n}\,\mathrm{trace}\,\mathrm{II}|_{T_xM^\top} \geq |\vec{H}(x)|$$

for every $x \in M$ close to $\mathcal{S}$. Inequality (7) then follows immediately by Schwarz's inequality. Now Theorem 1 is a consequence of Lemma 4. Indeed by relation (7) and Lemma 4 we conclude the inequality

$$\Delta_M\rho + b_i(\nabla_M\rho)_i \leq 0$$

in a neighbourhood of every point $p_0 \in M \cap \mathcal{S}$. E. Hopf's maximum principle (see e.g. Gilbarg and Trudinger [1], Theorem 3.5) finally proves that $\rho = 0$ in a neighbourhood of any point $p_0 \in M \cap \mathcal{S}$. Theorem 1 is proved. □

Proof of Corollary 1. Assuming the contrary we conclude from Theorem 1 the inclusion $M \cap U \subset \mathcal{S} \cap U$ for some neighbourhood U of $p_0 \in M \cap \mathcal{S}$. Therefore we had $\rho \equiv 0$ on $M \cap U$ and Lemma 4 implied the relation

$$\mathrm{trace}\,\mathrm{II}_{|T_xM} = n\langle\vec{H}, D\rho\rangle = n\langle\vec{H}, \nu\rangle \quad \text{on } M \cap U,$$

since $T_xM^T = T_xM$ and also $\nabla_M\rho = 0 = \triangle_M\rho$ on $M \cap U$. In particular we obtain the estimate

$$\frac{1}{n}\mathrm{trace}\,\mathrm{II}_{|T_xM} \leq |\vec{H}(x)| \quad \text{on } M \cap U.$$

On the other hand, by Lemma 5, this leads to the inequality

$$\Lambda_n(x) \leq |\vec{H}(x)| \quad \text{for all } x \in M \cap U,$$

which obviously contradicts the assumption

$$|\vec{H}(p_0)| < \Lambda_n(p_0).$$ □

Remark 3. We observe here that the estimate (7) is an immediate consequence of an hypothesis of the type $|\vec{H}(p_0)| < \Lambda_n(p_0)$, $p_0 \in \mathcal{S} \cap M$, the continuity of the involved functions, and Lemma 5, without using the explicit estimates in Lemma 3.

4.4.2 A Geometric Inclusion Principle for Strong Subsolutions

We now present a version of Theorem 1 for strong (but not necessarily classical) subsolutions of the parametric mean curvature equation, since they arise naturally as solutions of suitable obstacle problems to be considered later. This will be of importance for the existence proof for surfaces of prescribed mean curvature that will be carried out in Section 4.7.

If $\mathcal{S} \subset \mathbb{R}^3$ is a regular surface of class C^2 with unit normal $\nu = D\rho$ and mean curvature Λ with respect to that normal, we let $\mathcal{S}_\tau, |\tau| \ll 1$, denote the local parallel surface at (small) distance τ and $\Lambda_\tau(x)$ denote the mean curvature of $\mathcal{S}_\tau$ with respect to the normal $\nu(x) = D\rho(x)$ at the point $x \in \mathcal{S}_\tau$. Clearly $\tau = \rho(x)$ and $\Lambda_\tau(x) = \Lambda_{\rho(x)}(x)$.

Theorem 2. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface of class C^2 with unit normal ν and mean curvature Λ (with respect to this normal). Furthermore let H denote some bounded continuous function on $\mathbb{R}^3$ and $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected set. Suppose $X \in C^1(\Omega, \mathbb{R}^3) \cap H^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^3)$ lies locally on that side of $\mathcal{S}$ into which the normal ν is directed, and is a conformal solution of the variational inequality*

$$\delta \mathcal{F}(X, \varphi) = \int_\Omega \{\langle \nabla X, \nabla \varphi \rangle + 2H(X)\langle X_u \wedge X_v, \varphi \rangle\}\, du\, dv \geq 0 \tag{9}$$

for all functions $\varphi \in \mathring{H}^1_2(\Omega, \mathbb{R}^3) \cap L_\infty(\Omega, \mathbb{R}^3)$ with $X + \varepsilon\varphi$ locally on the same side of $\mathcal{S}$ for $0 < \varepsilon \ll 1$. Then the following conclusions hold:

(a) *Assume that $X_0 = X(w_0) \in \mathcal{S}$ and that for some neighbourhood $U = U(X_0) \subset \mathbb{R}^3$ one has*

$$|H(x)| \leq \Lambda_{\rho(x)}(x) \quad \textit{for all } x \in U. \tag{10}$$

Then there exists a disk $B_\varepsilon(w_0) \subset \Omega$ such that $X(B_\varepsilon(w_0)) \subset \mathcal{S}$.

(b) *Suppose that (10) holds for every point $X_0 \in \mathcal{S}$. Then $X(\Omega)$ is completely contained in $\mathcal{S}$, if $X(\Omega) \cap \mathcal{S}$ is nonempty.*

Corollary 2. *The conclusion of the Theorem holds if (10) is replaced by the (stronger) assumption*

$$\sup_U |H| \leq \inf_{U \cap \mathcal{S}} \Lambda. \tag{10$'$}$$

Corollary 3. *Suppose (10) is replaced by the (stronger) hypotheses*

$$|H(P_0)| < \Lambda(P_0) \tag{10$''$}$$

for some $P_0 \in \mathcal{S}$. Then there is no $w_0 \in \Omega$ such that $X(w_0) = P_0$. Clearly, this conclusion holds for a whole neighbourhood U of P_0 in $\mathcal{S}$. In particular if (10$''$) is fulfilled for all points $P_0 \in \mathcal{S}$ then the intersection $X(\Omega) \cap \mathcal{S}$ is empty.

As a further consequence of Theorem 2 we have the following

Enclosure Theorem 2. *Let $G \subset \mathbb{R}^3$ be a domain with $\partial G \in C^2$ and H be a bounded continuous function on $\mathbb{R}^3$. Assume that every point $P \in \partial G$ has a neighbourhood $U \subset \mathbb{R}^3 \cap \overline{G}$ such that*

$$|H(x)| \leq \Lambda_{\rho(x)}(x) \quad \textit{for all } x \in U, \tag{11}$$

where $\Lambda_{\rho(x)}$ stands for the mean curvatures of $\partial G_{\rho(x)}$ with respect to the inward unit normal $\nu = D\rho(x)$. Suppose $X \in C^1(\Omega, \mathbb{R}^3) \cap H^2_{2,\mathrm{loc}}(\Omega, \mathbb{R}^3)$ is a strong subsolution of the H-surface equation, whose image $X(\Omega)$ is confined to the closure $\overline{G}$, i.e. X is a conformal solution of the variational inequality

$$\delta\mathcal{F}(X,\varphi) = \int_\Omega \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\, du\, dv \geq 0$$

for all functions $\varphi \in \mathring{H}_2^1(\Omega,\mathbb{R}^3) \cap L_\infty(\Omega,\mathbb{R}^3)$ *with* $X + \varepsilon\varphi \in H_2^1(\Omega,\overline{G})$ *for* $0 \leq \varepsilon < \varepsilon_0(\varphi)$*. Then* $X(\Omega) \subset G$ *if at least one of the points* $X(w)$ *lies in* G*.*

Corollary 4. *The strong inclusion* $X(\Omega) \subset G$ *holds for example, if, in addition to the assumption of Enclosure Theorem 2,* X *is of class* $C^0(\overline{\Omega},\mathbb{R}^3) \cap C^1(\Omega,\mathbb{R}^3) \cap H^2_{2,\mathrm{loc}}(\Omega,\mathbb{R}^3)$ *and maps one point* $w_0 \in \partial\Omega$ *into the interior of* G*.*

Corollary 5. *Enclosure Theorem 2 is valid if (11) is replaced by the (stronger) assumption*

$$(11') \qquad |H(P)| < \Lambda(P) \quad \textit{for all } P \in \partial G.$$

Remark 4. Suppose $X \in C^1(\Omega,\mathbb{R}^3) \cap H_2^2(\Omega,\mathbb{R}^3)$ satisfies the assumptions of the Enclosure Theorem and that $X(\Omega) \subset G$. Then X is a *strong* (and of course also weak) H-surface in the sense that

$$\int_\Omega \Big\{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v\rangle\varphi\Big\}\, du\, dv = 0$$

for all $\varphi \in \mathring{H}_2^1(\Omega,\mathbb{R}^3) \cap L_\infty(\Omega,\mathbb{R}^3)$. Furthermore, by elliptic regularity results it follows that X is a *classical* $C^{2,\alpha}$-solution of the H-surface equation if H is Hölder continuous. This means that X is an H-surface, i.e.

$$\begin{aligned} \Delta X &= 2H(X) X_u \wedge X_v, \\ |X_u|^2 &= |X_v|^2, \langle X_u, X_v\rangle = 0 \quad \text{in } \Omega. \end{aligned}$$

In Section 4.7 we will see how to find subsolutions X of the kind needed in Enclosure Theorem 2 by solving suitable obstacle problems.

While condition (11) is sufficient to show strong inclusion $X(\Omega) \subset G$ relative to the hypotheses in Enclosure Theorem 2 this is not true under the weaker assumption $|H(P)| \leq \Lambda(P)$ for all $P \in \partial G$, Λ the inward mean curvature of ∂G, see Remark 2 following Enclosure Theorem I in Section 4.2. However this still leaves open the possibility that X might satisfy the H-surface system a.e. in $\overline{G}$. The next result shows that this in indeed the case:

Theorem 3 (Variational equality). *Suppose that* $G \subset \mathbb{R}^3$ *is a domain of class* C^2*,* H *is bounded and continuous with* $H(P) \leq \Lambda(P)$ *for all* $P \in \partial G$*. Let* $X \in C^1(\Omega,\mathbb{R}^3) \cap H^2_{2,\mathrm{loc}}(\Omega,\mathbb{R}^3)$ *satisfy the assumptions of Enclosure Theorem 2. Then* X *is a strong* H*-surface in* $\overline{G}$*, i.e. we have* $\triangle X = 2H(X)X_u \wedge X_v$ *a.e. in* Ω*, and* $|X_u|^2 = |X_v|^2, \langle X_u, X_v\rangle = 0$ *in* Ω*.*

Proof of Theorem 2. Define the sets Ω_1, Ω_2 and Ω_3 by $\Omega_1 := X^{-1}(\mathcal{S})$, $\Omega_2 := \Omega \setminus \Omega_1$, and $\Omega_3 := \{w \in \Omega : |X_u(w)| = |X_v(w)| = 0\}$. Observe that Ω_1 and Ω_3 are closed, while Ω_2 is an open set. Then the function

$$\mathcal{H}^*(w) := \begin{cases} \pm\Lambda(X(w)), & \text{for } w \in \Omega_1, \\ H(X(w)), & \text{for } w \in \Omega_2 \end{cases}$$

is of class $L_{\infty,\text{loc}}(\Omega)$ and we claim that

$$\Delta X = 2\mathcal{H}^*(w)(X_u \wedge X_v) \tag{12}$$

holds a.e. on Ω. In fact, on Ω_3, the (possibly empty) set of branch points of X, we have $X_u(w) = X_v(w) = 0$. Since $X \in H^2_{2,\text{loc}}(\Omega, \mathbb{R}^3)$ this implies that also $X_{uu} = X_{vv} = X_{uv} = 0$ a.e. on Ω_3 (compare e.g. Gilbarg and Trudinger [1], Lemma 7.7); in particular (12) holds a.e. on Ω_3. Again, because of $X \in H^2_{2,\text{loc}}(\Omega, \mathbb{R}^3)$, we infer from (9) using an integration by parts and the fundamental lemma of the calculus of variations that (12) is satisfied a.e. on Ω_2. Finally, to verify equation (12) a.e. on $\Omega_1 \setminus \Omega_3$ we use the same argument as in the proof of Theorem 1 in Chapter 2.6 of Vol. 1, observing that X is a conformal and regular parametrization of $\mathcal{S}$ on $\Omega_1 \setminus \Omega_3$ and that $\mathcal{S}$ has mean curvature Λ.

Now the reasoning of Hartman and Wintner as outlined in Section 2.10 and Chapter 3 yields the asymptotic expansion

$$X_u - iX_v = (a - ib)(w - w_0)^l + o(|w - w_0|^l) \tag{13}$$

in a sufficiently small neighbourhood of an arbitrary point $w_0 \in \Omega$, where $l \geq 0$ is an integer and $a, b \in \mathbb{R}^3$ satisfy the relations $|a| = |b| \neq 0$ and $\langle a, b\rangle = 0$. In particular $\lambda(w) := |X_u(w)| = |X_v(w)| > 0$ on a punctured neighbourhood of w_0 and $\lambda(w_0) = 0$, if and only if w_0 is a branch point of X. Introducing polar coordinates $w = re^{i\varphi}$ around w_0 we infer from (13) the asymptotic relations

$$\begin{aligned} X_u(re^{i\varphi}) &= ar^l \cos(l\varphi) + br^l \sin(l\varphi) + o(r^l), \\ X_v(re^{i\varphi}) &= br^l \cos(l\varphi) - ar^l \sin(l\varphi) + o(r^l), \\ |X_u|^2 = |X_v|^2 &= |a|^2 r^{2l} + o(r^{2l}), \end{aligned}$$

all holding for $r \to 0$. Therefore $\lambda(w) = |a|r^l + o(r^l)$, for $r \to 0$, and consequently the unit normal has the asymptotic expansion

$$\frac{X_u \wedge X_v}{|X_u \wedge X_v|}(w) = \frac{a \wedge b}{|a \wedge b|} + o(1) \quad \text{as } w \to w_0.$$

In particular, the normal $N(w) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(w)$ is continuous in Ω and

$$\lim_{w \to w_0} N(w) = \frac{a \wedge b}{|a \wedge b|} = \frac{a \wedge b}{|a|^2} = \frac{a \wedge b}{|b|^2}. \tag{14}$$

In other words, the surface X has a tangent plane at any point $w_0 \in \Omega$.

Suppose now that $w_0 \in \Omega_1$, i.e. $X(w_0) \in \mathcal{S}$, then since $X \in C^1$ lies locally on one side of $\mathcal{S}$ and because of (14) we obtain

$$\frac{a \wedge b}{|a \wedge b|} = \pm\, \nu(X(w_0)). \tag{15}$$

In fact, (15) can be proved rigorously by the same argument as used in the proof of Enclosure Theorem 1 of Chapter 4.2, namely by invoking a local non-parametric representation of the surfaces X and $\mathcal{S}$ near a punctured neighbourhood of the point $X(w_0)$.

Consider now the oriented distance function $\rho(x) = \operatorname{dist}(x, \mathcal{S})$ which is of class C^2 near $\mathcal{S}$ and put $\nu(x) = D\rho(x) = (\rho_{x_1}, \rho_{x_2}, \rho_{x_3})$, cp. the discussion in the beginning of this section. Recall that $\rho(X(w)) \geq 0$ and "=" if and only if $w \in \Omega_1$ and that $\nu(x)$ is the unit normal of $\mathcal{S}$ at x. For the computations to follow it is convenient to put $u = u_1$ and $v = u_2$, and define for a – sufficiently small – neighbourhood $B_\rho(w_0)$ of an arbitrary point $w_0 \in \Omega_1$,

$$X^t_{u^\alpha}(w) := \frac{X_{u^\alpha}(w)}{|X_{u^\alpha}|} - \left\langle \frac{X_{u^\alpha}(w)}{|X_{u^\alpha}|}, \nu(X(w)) \right\rangle \nu(X(w)),$$

for $w \in B_\rho(w_0) \setminus \{w_0\}$ and $\alpha = 1, 2$, to denote the orthogonal projection of the unit tangent vector $\frac{X_{u^\alpha}(w)}{|X_{u^\alpha}|}$ of X onto the tangent space of the parallel surface $\mathcal{S}_{\rho(X(w))} := \{y \in \mathbb{R}^3 : \rho(y) = \rho(X(w))\}$ to $\mathcal{S}$ at distance $\rho(X(w))$.

The vectors $X^t_{u^\alpha}(w)$ are continuous in $B_\delta(w_0) \setminus \{w_0\}$ but merely bounded on $B_\delta(w_0)$. Define the metric

$$\begin{aligned} g_{\alpha\beta} &= g_{\alpha\beta}(w) := \langle X^t_{u^\alpha}(w), X^t_{u^\beta}(w) \rangle \\ &= \delta_{\alpha\beta} - \left\langle \frac{X_{u^\alpha}}{|X_{u^\alpha}|}, \nu \right\rangle \left\langle \frac{X_{u^\beta}}{|X_{u^\beta}|}, \nu \right\rangle \quad \text{for } w \in B_\delta(w_0) \setminus \{w_0\} \end{aligned}$$

and $\alpha, \beta = 1, 2$, where $\nu = \nu(X(w)) = D\rho(X(w))$.

We assert that

$$\lim_{w \to w_0} \left\langle \frac{X_{u^\alpha}(w)}{|X_{u^\alpha}|}, \nu(X(w)) \right\rangle = 0 \quad \text{holds true.} \tag{16}$$

To see this note that (15) yields the relation

$$\nu(X(w)) = \pm \frac{a \wedge b}{|a \wedge b|} + o(1) \quad \text{as } w \to w_0,$$

which implies (16) by virtue of the asymptotic expansions

$$\begin{aligned} \tfrac{X_u(w)}{|X_u|} &= \tfrac{ar^l \cos(l\varphi) + br^l \sin(l\varphi)}{|a| r^l} + o(1), \quad w \to w_0, \\ \tfrac{X_v(w)}{|X_v|} &= \tfrac{br^l \cos(l\varphi) - ar^l \sin(l\varphi)}{|b| r^l} + o(1), \quad w \to w_0. \end{aligned}$$

On the other hand relation (16) shows that the metric $g_{\alpha\beta}$ is continuous on $B_\delta(w_0)$ and

$$\lim_{w \to w_0} g_{\alpha\beta}(w) = \delta_{\alpha\beta}, \quad \text{for } \alpha, \beta = 1, 2.$$

Hence, by possibly decreasing δ, we can consider the inverse metric $g^{\alpha\beta} \in C^0(B_\delta(w_0))$, $g^{\alpha\beta} = g^{\alpha\beta}(w) = \delta_{\alpha\beta} + \varepsilon_{\alpha\beta}$, where

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(w) = \frac{\left\langle \frac{X_{u^\alpha}(w)}{|X_{u^\alpha}|}, \nu(X(w)) \right\rangle \left\langle \frac{X_{u^\beta}(w)}{|X_{u^\beta}|}, \nu(X(w)) \right\rangle}{1 - \left\langle \frac{X_u}{|X_u|}, \nu \right\rangle^2 - \left\langle \frac{X_v}{|X_v|}, \nu \right\rangle^2}$$

and $\varepsilon_{\alpha\beta} \in C^0(B_\delta(w_0))$ with $\varepsilon_{\alpha\beta}(w_0) = 0$ (by (16)). For $|\tau| \ll 1$ we denote by $\Lambda_\tau(x)$ the mean curvature of the distance surface $\mathcal{S}_\tau$ with respect to the unit normal $\nu(x)$ at x. Also let II_x stand for the second fundamental form of $\mathcal{S}_{\rho(x)}$ with respect to $\nu(x)$ at the point x, cp. Chapter 1 of Vol. 1 or Section 4.3, in particular we have $2\Lambda_{\rho(x)}(x) = \operatorname{trace} \mathrm{II}_x$, and since

$$\mathrm{II}_x(t,\tau) = \langle -D_t\nu, \tau\rangle = -\langle D\nu(x)t, \tau\rangle$$

for $t, \tau \in T_x\mathcal{S}_{\rho(x)}$ this implies for every $w \in B_\delta(w_0) \setminus \{w_0\}$,

$$\begin{aligned}
-2\Lambda_{\rho(X)}(X(w)) &= g^{\alpha\beta}(X(w))\langle X^t_{u^\alpha}(w), D\nu(X(w))X^t_{u^\alpha}(w)\rangle \\
&= \langle X^t_u, D\nu(X)X^t_u\rangle + \langle X^t_v, D\nu(X)X^t_v\rangle + \varepsilon_{\alpha\beta}\langle X^t_{u^\alpha}(w), D\nu(X)X^t_{u^\beta}\rangle.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
(17)\qquad -2|X_u|^2\Lambda_{\rho(X)}(X(w)) &= -(|X_u|^2 + |X_v|^2)\Lambda_{\rho(X)}(X) \\
&= |X_u|^2\langle X^t_u, D\nu(X)X^t_u\rangle + |X_v|^2\langle X^t_v, D\nu(X)X^t_v\rangle \\
&\quad + \varepsilon_{\alpha\beta}|X_u||X_v|\langle X^t_{u^\alpha}, D\nu(X)X^t_{u^\beta}\rangle.
\end{aligned}$$

Now, we look at the term

$$\begin{aligned}
&|X_u|^2\langle X^t_u, D\nu(X)X^t_u\rangle \\
&\quad = \Big\langle X_u - \langle X_u,\nu\rangle\nu, D\nu(X)\big[X_u - \langle X_u,\nu\rangle\nu\big]\Big\rangle \\
&\quad = \langle X_u, D\nu(X)X_u\rangle - \langle X_u,\nu\rangle\langle \nu, D\nu(X)X_u\rangle \\
&\qquad - \langle X_u,\nu\rangle\langle X_u, D\nu(X)\nu\rangle + \langle X_u,\nu\rangle^2\langle\nu, D\nu(X)\nu\rangle \\
&\quad = \langle X_u, D\nu(X)X_u\rangle,
\end{aligned}$$

since $D\nu(X)\nu = 0$ and II_x is symmetric. Similarly, we find

$$|X_v|^2\langle X^t_v, D\nu(X)X^t_v\rangle = \langle X_v, D\nu(X)X_v\rangle$$

and

$$|X_u||X_v|\langle X^t_u, D\nu(X)X^t_v\rangle = \langle X_u, D\nu(X)X_v\rangle.$$

This combined with (17) yields

$$\begin{aligned}
(18)\qquad &\langle X_u, D\nu(X)X_u\rangle + \langle X_v, D\nu(X)X_v\rangle \\
&\quad = -(|X_u|^2 + |X_v|^2)\Lambda_{\rho(X)}(X) - \varepsilon_{ij}\langle X_{u_i}, D\nu(X)X_{u_j}\rangle,
\end{aligned}$$

for all $w \in B_\delta(w_0) \setminus \{w_0\}$. However, for continuity reasons (18) clearly holds on $B_\delta(w_0) \subset \Omega$.

So far we have not exploited the variational inequality

$$\int_{\Omega} \{\langle \nabla X, \nabla \varphi \rangle + 2H(X)\langle X_u \wedge X_v, \varphi \rangle\}\, du\, dv \geq 0 \tag{19}$$

holding for all $\varphi \in \mathring{H}_2^1(\Omega, \mathbb{R}^3) \cap L_\infty$, such that $X + \varepsilon\varphi$ lies locally on the same side of $\mathcal{S}$ as X for all $\varepsilon \in [0, \varepsilon(\varphi)]$. We choose as a test function $\varphi(w) = \eta(w) \cdot \nu(X(w))$, where $0 \leq \eta \in C_c^\infty(B_\delta(w_0))$ is arbitrary and $\nu(x) = D\rho(x)$.

Clearly $\varphi \in \mathring{H}_2^1(\Omega, \mathbb{R}^3) \cap L_\infty$ is an admissible function in (19) and we compute

$$\nabla\varphi = (\varphi_u, \varphi_v) = \nabla\eta \cdot \nu(X) + \eta[D\nu(X)\nabla X]$$

where $\nabla X = (X_u, X_v)$, $\nabla\eta = (\eta_u, \eta_v)$. Plugging this relation into the variational inequality (19) we obtain

$$\int_{B_\delta(w_0)} \{\langle \nabla X, \nabla\eta \cdot \nu + \eta[D\nu(X)\nabla X]\rangle + 2\eta H(X)\langle X_u \wedge X_v, \nu(X)\rangle\}\, du\, dv \geq 0$$

from which we infer, by virtue of

$$\begin{aligned} \langle \nabla X, \nabla\eta \cdot \nu \rangle &= \eta_u \langle X_u, \nu(X)\rangle + \eta_v \langle X_v, \nu(X)\rangle \\ &= \eta_u \frac{\partial}{\partial u}\rho(X(w)) + \eta_v \frac{\partial}{\partial v}\rho(X(w)) = \langle \nabla\eta, \nabla\rho(X)\rangle \end{aligned}$$

and

$$\eta\langle \nabla X, D\nu(X)\nabla X\rangle = \eta\langle X_u, D\nu(X)X_u\rangle + \eta\langle X_v, D\nu(X)X_v\rangle$$

the inequality

$$\begin{aligned} \int_{B_\delta(w_0)} \Big\{ &\langle \nabla\eta, \nabla\rho(X)\rangle + \eta\langle X_u, D\nu(X)X_u\rangle \\ &+ \eta\langle X_v, D\nu(X)X_v\rangle + 2\eta H(X)\langle X_u \wedge X_v, \nu\rangle \Big\}\, du\, dv \geq 0. \end{aligned}$$

In this inequality we replace the expression $\langle X_u, D\nu(X)X_u\rangle + \langle X_v, D\nu(X)X_v\rangle$ using (18) and get

$$\begin{aligned} 0 \leq \int_{B_\delta(w_0)} \Big\{ &\langle \nabla\eta, \nabla\rho(X)\rangle - \eta(|X_u|^2 + |X_v|^2)\Lambda_{\rho(X)}(X) \\ &- \eta\varepsilon_{\alpha\beta}\langle X_{u^\alpha}, D\nu(X)X_{u^\beta}\rangle + 2\eta H(X)\langle X_u \wedge X_v, \nu\rangle \Big\}\, du\, dv, \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 \leq \int_{B_\delta(w_0)} &\big\{\langle \nabla\rho(X), \nabla\eta\rangle + \eta(|X_u|^2 + |X_v|^2)\big[|H(X)| - \Lambda_{\rho(X)}(X)\big] \\ &- \eta\, \varepsilon_{\alpha\beta}\langle X_{u^\alpha}, D\nu(X)X_{u^\beta}\rangle\big\}\, du\, dv. \end{aligned} \tag{20}$$

To see that the function $\rho(X(w))$ satisfies a differential equation of second order we compute the term

$$
\begin{aligned}
(21)\qquad & \varepsilon_{\alpha\beta}\langle X_{u^\alpha}, D\nu(X)X_{u^\beta}\rangle \\
&= \frac{\left\langle \frac{X_{u^\alpha}}{|X_{u^\alpha}|}, \nu\right\rangle \left\langle \frac{X_{u^\beta}}{|X_{u^\beta}|}, \nu\right\rangle}{1 - \left\langle \frac{X_u}{|X_u|}, \nu\right\rangle^2 - \left\langle \frac{X_v}{|X_v|}, \nu\right\rangle^2} \cdot \langle X_{u^\alpha}, D\nu(X)X_{u^\beta}\rangle \\
&= \frac{\frac{1}{|X_{u^\alpha}|}\frac{\partial}{\partial u^\alpha}\rho(X)\left\langle \frac{X_{u^\beta}}{|X_{u^\beta}|}, \nu\right\rangle}{1 - \langle\cdot,\cdot\rangle^2 - \langle\cdot,\cdot\rangle^2} \cdot \langle X_{u^\alpha}, D\nu(X)X_{u^\beta}\rangle \\
&=: b_\alpha(w)\frac{\partial}{\partial u^\alpha}\rho(X),
\end{aligned}
$$

where we have put

$$
b_\alpha(w) := \frac{\left\langle \frac{X_{u^\alpha}}{|X_{u^\alpha}|}, D\nu(X)X_{u^\beta}\right\rangle \left\langle \frac{X_{u^\beta}}{|X_{u^\beta}|}, \nu\right\rangle}{1 - \left\langle \frac{X_u}{|X_u|}, \nu\right\rangle^2 - \left\langle \frac{X_v}{|X_v|}, \nu\right\rangle^2}.
$$

Note that by (16), we have $\lim_{w\to w_0}\langle \frac{X_{u_i}}{|X_{u_i}|}, \nu\rangle = 0$. Thus $b_\alpha(\cdot)$ is continuous in $B_\delta(w_0)$ with $b_\alpha(w_0) = 0$. By assumption (10) we have

$$
(22)\qquad |H(X(w))| - \Lambda_{\rho(X)}(X(w)) \le 0 \quad \text{for all } w \in B_\varepsilon(w_0)
$$

and some positive $\varepsilon \le \delta$.

From (20), (21) and (22) we finally infer the inequality

$$
0 \le \int_{B_\varepsilon(w_0)} \left\{ \langle \nabla\rho(X), \nabla\eta\rangle - \eta \cdot b_i(w)\frac{\partial}{\partial u_i}\rho(X) \right\} du\, dv
$$

which holds for all nonnegative $\eta \in C_c^\infty(B_\varepsilon(w_0))$. Thus the function $f(w) := \rho(X(w))$ is an $H_2^2 \cap C^1(B_\varepsilon(w_0))$ strong (and therefore an almost everywhere) solution of the inequality

$$
\Delta f(w) + b_i \frac{\partial f}{\partial u_i}(w) \le 0 \quad \text{in } B_\varepsilon(w_0).
$$

By the strong maximum principle (see e.g. Gilbarg and Trudinger [1], Theorem 9.6) it follows that $f(w) = \rho(X(w)) \equiv 0$ in $B_\varepsilon(w_0)$. This clearly means that $X(w) \in \mathcal{S}$ for all $w \in B_\varepsilon(w_0)$. Theorem 2 is proved. □

Proof of Corollary 2. We use Lemma 3 to control the mean curvature $\Lambda_{\rho(X)}(X)$ of the parallel surface at X as follows:

$$
\begin{aligned}
\Lambda_{\rho(X)}(X) &= \frac{1}{2}\left(\frac{\lambda_1(y)}{1 - \lambda_1(y)\rho(X)} + \frac{\lambda_2(y)}{1 - \lambda_2(y)\rho(X)}\right) \\
&\ge \frac{1}{2}(\lambda_1(y) + \lambda_2(y)) = \Lambda_0(y) = \Lambda(y),
\end{aligned}
$$

where $y \in \mathcal{S}$ is such that $\rho(X) = |X - y|$ and $\lambda_1(y) \le \lambda_2(y)$ are the principal curvatures of $\mathcal{S}$ at y. By assumption (10′) we have

$$\inf_{U \cap \mathcal{S}} \Lambda \ge \sup_U |H|$$

for some neighbourhood $U = U(X(w_0)) \subset \mathbb{R}^3$. Hence for some $\varepsilon > 0$ there holds

$$\Lambda_{\rho(X)}(X) \ge \inf_{U \cap \mathcal{S}} \Lambda \ge \sup_U |H| \ge |H(X)| \quad \text{on } B_\varepsilon(w_0),$$

that is $|H(X(w))| - \Lambda_{\rho(X)}(X(w)) \le 0$ for all $w \in B_\varepsilon(w_0)$. The proof of Corollary 2 can now be completed in the same way as in Theorem 2. □

Proof of Corollary 3. Assume on the contrary the existence of some $w_0 \in \Omega$ such that $X(w_0) = P_0$. By Theorem 2 there exists a disk $B_\epsilon(w_0) \subset \Omega$ such that X maps $B_\epsilon(w_0)$ into $\mathcal{S}$. Therefore we obtain on $B_\epsilon(w_0)$ the identities

$$\nabla \rho(X(w)) = 0 \quad \text{and} \quad \epsilon_{\alpha\beta} \langle X_{u^\alpha}(w), D\nu(X) X_{u^\beta}(w) \rangle = 0.$$

The variational inequality (20) then yields the estimate

$$0 \le \int_{B_\epsilon(w_0)} \eta(|X_u|^2 + |X_v|^2)[|H(X)| - \Lambda_{\rho(X)}(X)]\, du\, dv$$

for all $\eta \in C_c^\infty(B_\epsilon(w_0)), \eta \ge 0$. However, this contradicts the assumption $|H(P_0)| < \Lambda(P_0)$, since X cannot be constant on $B_\epsilon(w_0)$ and H and Λ are continuous. □

Remark 5. We point out here that the stronger assumption $|H(X(w_0))| < \Lambda(X(w_0))$ (replacing (10) in Theorem 2) leads to a somewhat more straightforward proof of the fact $X(B_\varepsilon(w_0)) \subset \mathcal{S}$, starting from inequality (20); namely we have

$$\begin{aligned}(|X_u|^2 + |X_v|^2)\Big[|H(X)| - \Lambda_\rho(X)\Big] - \varepsilon_{\alpha\beta} \langle X_{u^\alpha}, D\nu(X) X_{u^\beta} \rangle \\ = 2|X_u|^2 \left\{ \big[|H| - \Lambda_\rho\big] - \varepsilon_{\alpha\beta} \left\langle \frac{X_{u^\alpha}}{|X_{u^\alpha}|}, D\nu(X) \frac{X_{u^\beta}}{|X_{u^\beta}|} \right\rangle \right\}.\end{aligned}$$

Put $\sigma(w) := \varepsilon_{\alpha\beta} \langle \frac{X_{u^\alpha}}{|X_{u^\alpha}|}, D\nu(X) \frac{X_{u^\beta}}{|X_{u^\beta}|} \rangle$; then $\sigma \in C^0(B_\delta(w_0))$ with $\sigma(w_0) = 0$, since $\varepsilon_{\alpha\beta}(w_0) = 0$. Thus the assumption $|H(X(w_0))| < \Lambda(X(w_0))$ implies the inequality

$$2|X_u|^2 \Big\{ \big[|H(X(w))| - \Lambda_\rho(X(w))\big] - \sigma(w) \Big\} \le 0$$

on a suitable disc $B_\varepsilon(w_0) \subset B_\delta(w_0)$, whence (20) yields

$$0 \le \int_{B_\varepsilon(w_0)} \langle \nabla \rho(X(w)), \nabla \eta(w) \rangle\, du\, dv,$$

i.e. $f(w) = \rho(X(w))$ is strongly superharmonic on $B_\varepsilon(w_0)$, or $\Delta f(w) \le 0$ a.e. in $B_\varepsilon(w_0)$.

Remark 6. Recall that assumption (10) cannot be replaced by $|H(P)| \le \Lambda(P)$ for all $P \in \mathcal{S}$, see Remark 2 following Enclosure Theorem I in Section 4.2.

Proof of Enclosure Theorem 2. The coincidence set $\Omega_1 = X^{-1}(\partial G)$ is a closed set in Ω. By Theorem 2 Ω_1 is also open, whence either $\Omega_1 = \emptyset$ or $\Omega_1 = \Omega$. However, by assumption there exists a $w_0 \in \Omega$ with $X(w_0) \in G$ and therefore only the alternative $\Omega_1 = \emptyset$ can hold true, i.e. $X(\Omega) \subset G$. □

Proof of Theorem 3. We let $\mathcal{T} := \{w \in \Omega : X(w) \in \partial G\}$ denote the (closed) coincidence set and put, for $\varepsilon > 0$, $\mathcal{T}_\varepsilon := \{w \in \Omega : \operatorname{dist}(w, \mathcal{T}) < \varepsilon\}$. Then $\mathcal{T}_\varepsilon$ is open and $\bigcap_{\varepsilon>0} \mathcal{T}_\varepsilon = \mathcal{T}$. Extend $\nu(x)$ to a bounded C^1-vector field $\tilde{\nu}$ on $\mathbb{R}^3$ which coincides with $D\rho(x) = \nu(x)$ on a neighbourhood of ∂G. Then we take nonnegative functions $\eta \in C_c^\infty(\Omega)$ and $\eta_\varepsilon \in C_c^\infty(\mathcal{T}_\varepsilon)$ with $\eta = \eta_\varepsilon$ on $\mathcal{T}_{\varepsilon/2}$ and put

$$\varphi(w) := \eta(w)\tilde{\nu}(X(w)), \quad \varphi_\varepsilon(w) := \eta_\varepsilon(w)\tilde{\nu}(X(w)).$$

Since both $(\varphi - \varphi_\varepsilon)$ and $(\varphi_\varepsilon - \varphi) \in C_c^1(\Omega \setminus \overline{\mathcal{T}}_{\varepsilon/4}, \mathbb{R}^3)$ are admissible in the variational inequality (19) we have $\delta\mathcal{F}(X, \varphi - \varphi_\varepsilon) = 0$; whence, since also φ and φ_ε are admissible functions we find as in the proof of Theorem 2, cp. (20),

$$
\begin{aligned}
(23) \qquad 0 \le \delta\mathcal{F}(X, \varphi) &= \delta\mathcal{F}(X, \varphi_\varepsilon) \\
&= \int_{\mathcal{T}_\varepsilon} [\langle \nabla\rho(X), \nabla\eta_\varepsilon\rangle + \eta_\varepsilon(|X_u|^2 + |X_v|^2)(|H(X)| - \Lambda_{\rho(X)}(X)) \\
&\quad - \eta_\varepsilon \varepsilon_{\alpha\beta}\langle X_{u^\alpha}, D\nu(X) X_{u^\beta}\rangle]\, du\, dv,
\end{aligned}
$$

assuming that $\varepsilon > 0$ is choosen suitably small. We infer from $\rho(X(\cdot)) \in H^2_{2,\mathrm{loc}}(\mathcal{T}_\varepsilon)$ and an integration by parts

$$\int_{\mathcal{T}_\varepsilon} \langle \nabla\rho(X), \nabla\eta_\varepsilon\rangle\, du\, dv = -\int_{\mathcal{T}_\varepsilon} \triangle\rho(X) \cdot \eta_\varepsilon\, du\, dv.$$

Taking the relations $\triangle\rho(X(w)) = 0$ a.e. on $\mathcal{T}$ and $\varepsilon_{\alpha\beta}(w) = 0$ a.e. on $\mathcal{T}$ for $\alpha, \beta = 1, 2$ and Lebesgue's dominated convergence theorem into account we arrive at

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\mathcal{T}_\varepsilon} [\langle \nabla\rho(X), \nabla\eta_\varepsilon\rangle + \eta_\varepsilon(|X_u|^2 + |X_v|^2)(|H(X)| - \Lambda_\rho(X)) \\
&\qquad - \eta_\varepsilon \varepsilon_{\alpha\beta}\langle X_{u^\alpha}, D\nu(X) X_{u^\beta}\rangle]\, du\, dv \\
&\quad = \int_{\mathcal{T}} \eta(|X_u|^2 + |X_v|^2)(|H(X)|) - \Lambda(X))\, du\, dv.
\end{aligned}
$$

By (23) and the assumption $|H| \le \Lambda$ along ∂G we obtain the variational equality

$$(24) \qquad \delta\mathcal{F}(X, \varphi) = 0$$

for all functions φ of the type $\varphi = \eta(w)\tilde{\nu}(X(w))$, $\eta \in C_c^\infty(\Omega), \eta \geq 0$. Clearly, this also holds for all normal variation $\varphi = \eta\tilde{\nu}(X)$ without assuming the sign restriction on η.

We can now exploit the variational inequality $\delta\mathcal{F}(X,\varphi) \geq 0$ which holds for all $\varphi \in \overset{\circ}{H}{}_2^1(\Omega,\mathbb{R}^3) \cap L_\infty(\Omega,\mathbb{R}^3)$ with $X + \varepsilon\varphi \in H_2^1(\Omega,\overline{G})$. Note that we may hence admit variations $\varphi(w) = \eta(w)\zeta(X(w))$ where ζ denotes a C^1-vectorfield defined on a neighbourhood of ∂G with $\zeta(P) = 0$ for all $P \in \partial G$ or $\langle\zeta(P),\nu(P)\rangle > 0$ along ∂G and $\eta \in C_c^1(\mathcal{T}_\epsilon)$, $\eta \geq 0$, $\varepsilon > 0$ suitably small. By an approximation argument this also follows for C^1-vectorfields as above which are directed weakly into the interior of ∂G, i.e. $\langle\zeta(P),\nu(P)\rangle \geq 0$ along ∂G. In particular we have $\delta\mathcal{F}(X,\varphi) = 0$, if ζ as above is tangential to ∂G along ∂G, since in this case $\langle\pm\zeta,\nu\rangle \geq 0$ on ∂G.

Suppose $\varphi \in C_c^1(\mathcal{T}_\varepsilon,\mathbb{R}^3)$ is arbitrary, then we decompose $\varphi = \varphi^\perp + \varphi^T$ where $\varphi^\perp = \eta(w)\tilde{\nu}(X(w))$ with $\eta(w) := \langle\varphi(w),\tilde{\nu}(X(w))\rangle$ denotes the "normal component" and $\langle\varphi^T(w),\tilde{\nu}(X(w))\rangle = 0$ for all $w \in \mathcal{T}_\varepsilon$. Concluding we find $\delta\mathcal{F}(X,\varphi^T) = 0$ and because of (24) also $\delta\mathcal{F}(X,\varphi^\perp) = 0$, whence $\delta\mathcal{F}(X,\varphi) = 0$ for all $\varphi \in C_c^1(\mathcal{T}_\epsilon,\mathbb{R}^3)$. Since, on the other hand $\delta\mathcal{F}(X,\varphi) = 0$ whenever φ is supported in $\Omega\setminus\mathcal{T}$ we finally conclude the result by virtue of the fundamental lemma of the calculus of variations. □

4.5 Isoperimetric Inequalities

For the sake of completeness we first repeat the proof of the isoperimetric inequality for disk-type minimal surfaces $X : B \to \mathbb{R}^3 \in H_2^1(B,\mathbb{R}^3)$ with the parameter domain $B = \{w \in \mathbb{C} : |w| < 1\}$, the boundary of which is given by $C = \partial B = \{w \in \mathbb{C} : |w| = 1\}$. Recall that any $X \in H_2^1(B,\mathbb{R}^3)$ has boundary values $X|_C$ of class $L_2(C,\mathbb{R}^3)$. Denote by $L(X)$ the length of the boundary trace $X|_C$, i.e.,

$$L(X) = L(X|_C) := \int_C |dX|.$$

We recall a result that, essentially, has been proved in Section 4.7 of Vol. 1.

Lemma 1. (i) *Let $X : B \to \mathbb{R}^3$ be a minimal surface with a finite Dirichlet integral $D(X)$ and with boundary values $X|_C$ of finite total variation*

$$L(X) = \int_C |dX|.$$

Then X is of class $H_2^1(B,\mathbb{R}^3)$ and has a continuous extension to $\overline{B}$, i.e., $X \in C^0(\overline{B},\mathbb{R}^3)$. Moreover, the boundary values $X|_C$ are of class $H_1^1(C,\mathbb{R}^3)$. Setting $X(r,\theta) := X(re^{i\theta})$, we obtain that, for any $r \in (0,1]$, the function $X_\theta(r,\theta)$ vanishes at most on a set of θ-values of one-dimensional Hausdorff measure zero, and that the limits

$$\lim_{r\to 1-0} X_r(r,\theta) \quad \textit{and} \quad \lim_{r\to 1-0} X_\theta(r,\theta)$$

exist, and that

$$\frac{\partial}{\partial\theta} X(1,\theta) = \lim_{r\to 1-0} X_\theta(r,\theta) \quad \text{a.e. } \textit{on } [0,2\pi]$$

holds true. Finally, setting $X_r(1,\theta) := \lim_{r\to 1-0} X_r(r,\theta)$*, it follows that*

$$(1) \qquad \int_B \langle \nabla X, \nabla\phi\rangle \, du\, dv = \int_C \langle X_r, \phi\rangle \, d\theta$$

is satisfied for all $\phi \in H_2^1 \cap L_\infty(B,\mathbb{R}^3)$*. Moreover, we have*

$$(2) \qquad \lim_{r\to 1-0} \int_0^{2\pi} |X_\theta(r,\theta)| r\, d\theta = \int_0^{2\pi} |dX(1,\theta)|.$$

(ii) *If* $X : B \to \mathbb{R}^3$ *is a minimal surface with a continuous extension to* $\overline{B}$ *such that* $L(X) := \int_C |dX| < \infty$*, then we still have* (2)*.*

Proof. Since $L(X) < \infty$, the finiteness of $D(X)$ is equivalent to the relation $X \in H_2^1(B,\mathbb{R}^3)$, on account of Poincaré's inequality. Hence X has an $L_2(C)$-trace on the boundary C of ∂B which, by assumption, has a finite total variation $\int_C |dX|$. Consequently, the two one-sided limits

$$\lim_{\theta\to\theta_0-0} X(1,\theta) \quad \text{and} \quad \lim_{\theta\to\theta_0+0} X(1,\theta)$$

exists for every $\theta_0 \in \mathbb{R}$. In conjunction with the Courant–Lebesgue lemma, we obtain that $X(1,\theta)$ is a continuous function of $\theta \in \mathbb{R}$ whence $X \in C^0(\overline{B},\mathbb{R}^3)$. The rest of the proof follows from Theorems 1 and 2 Vol. 1, in Section 4.7. □

Lemma 2 (Wirtinger's inequality). *Let* $Z : \mathbb{R} \to \mathbb{R}^3$ *be an absolutely continuous function that is periodic with the period* $L > 0$ *and has the mean value*

$$(3) \qquad P := \frac{1}{L}\int_0^L Z(t)\, dt.$$

Then we obtain

$$(4) \qquad \int_0^L |Z(t) - P|^2\, dt \le \left(\frac{L}{2\pi}\right)^2 \int_0^L |\dot{Z}(t)|^2\, dt,$$

and the equality sign holds if and only if there are constant vectors A_1 *and* B_1 *in* $\mathbb{R}^3$ *such that*

$$(5) \qquad Z(t) = P + A_1 \cos\left(\frac{2\pi}{L}t\right) + B_1 \sin\left(\frac{2\pi}{L}t\right)$$

holds for all $t \in \mathbb{R}$.

Proof. We first assume that $L = 2\pi$ and $\int_0^{2\pi} |\dot{Z}|^2\,dt < \infty$. Then we have the expansions

$$Z(t) = P + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt), \quad \dot{Z}(t) = \sum_{n=1}^{\infty} n(B_n \cos nt - A_n \sin nt)$$

of Z and $\dot{Z}$ into Fourier series with A_n, $B_n \in \mathbb{R}^3$, and

$$\int_0^{2\pi} |Z-P|^2\,dt = \pi \sum_{n=1}^{\infty}(|A_n|^2 + |B_n|^2),$$
$$\int_0^{2\pi} |\dot{Z}|^2\,dt = \pi \sum_{n=1}^{\infty} n^2(|A_n|^2 + |B_n|^2). \tag{6}$$

Consequently it follows that

$$\int_0^{2\pi} |Z-P|^2\,dt \le \int_0^{2\pi} |\dot{Z}|^2\,dt, \tag{7}$$

and the equality sign holds if and only if all coefficients A_n and B_n vanish for $n > 1$. Thus we have verified the assertion under the two additional hypotheses. If $\int_0^{2\pi} |\dot{Z}|^2\,dt = \infty$, the statement of the lemma is trivially satisfied, and the general case $L > 0$ can be reduced to the case $L = 2\pi$ by the scaling transformation $t \mapsto (2\pi/L)t$. □

Now we state the isoperimetric inequality for minimal surfaces in its simplest form.

Theorem 1. *Let $X \in C^2(B, \mathbb{R}^3)$ with $B = \{w : |w| < 1\}$ be a nonconstant minimal surface, i.e., $\Delta X = 0, |X_u|^2 = |X_v|^2, \langle X_u, X_v\rangle = 0$. Assume also that X is either of class $H_2^1(B, \mathbb{R}^3)$ or of class $C^0(\overline{B}, \mathbb{R}^3)$, and that $L(X) = \int_C |dX| < \infty$. Then $D(X)$ is finite, and we have*

$$D(X) \le \frac{1}{4\pi} L^2(X). \tag{8}$$

Moreover, the equality sign holds if and only if $X : B \to \mathbb{R}^3$ represents a (simply covered) disk.

Remark 1. Note that for every minimal surface $X : B \to \mathbb{R}^3$ the area functional $A(X)$ coincides with the Dirichlet integral $D(X)$. Thus (8) can equivalently be written as

$$A(X) \le \frac{1}{4\pi} L^2(X). \tag{8'}$$

Proof of Theorem 1. (i) Assume first that X is of class $H_2^1(B, \mathbb{R}^3)$, and that P is a constant vector in $\mathbb{R}^3$. Because of $L(X) < \infty$, the boundary values $X|_C$ are bounded whence X is of class $L_\infty(B, \mathbb{R}^3)$ (this follows from the maximum principle in conjunction with a suitable approximation device). Thus we can apply formula (1) to $\phi = X - P$, obtaining

(9)

$$\begin{aligned}\int_B \langle \nabla X, \nabla X \rangle \, du\, dv &= \int_B \langle \nabla X, \nabla (X - P)\rangle \, du\, dv \\ &= \int_C \langle X_r, X - P\rangle \, d\theta \le \int_C |X_r||X - P| \, d\theta \\ &= \int_C |X_\theta||X - P| \, d\theta = \int_0^{2\pi} |X_\theta(1,\theta)||X(1,\theta) - P| \, d\theta.\end{aligned}$$

Introducing $s = \sigma(\theta)$ by

$$\sigma(\theta) := \int_0^\theta |X_\theta(1,\theta)| \, d\theta,$$

we obtain that $\sigma(\theta)$ is a strictly increasing and absolutely continuous function of θ, and $\dot{\sigma}(\theta) = |X_\theta(1,\theta)| > 0$ a.e. on $\mathbb{R}$. Hence $\sigma : \mathbb{R} \to \mathbb{R}$ has a continuous inverse $\tau : \mathbb{R} \to \mathbb{R}$. Let us introduce the reparametrization

$$Z(s) := X(1, \tau(s)), \quad s \in \mathbb{R},$$

of the curve $X(1,\theta)$, $\theta \in \mathbb{R}$. Then, for any $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, the numbers $\theta_1 := \tau(s_1)$, $\theta_2 := \tau(s_2)$ satisfy $\theta_1 < \theta_2$ and

$$(10) \qquad \int_{s_1}^{s_2} |dZ| = \int_{\theta_1}^{\theta_2} |dX| = \sigma(\theta_2) - \sigma(\theta_1) = s_2 - s_1,$$

whence

$$|Z(s_2) - Z(s_1)| \le s_2 - s_1.$$

Consequently, the mapping $Z : \mathbb{R} \to \mathbb{R}^3$ is Lipschitz continuous and therefore also absolutely continuous, and we obtain from (10) that

$$(11) \qquad \int_{s_1}^{s_2} |Z'(s)| \, ds = s_2 - s_1$$

whence

$$(12) \qquad |Z'(s)| = 1 \quad \text{a.e. on } \mathbb{R}.$$

In other words, the curve $Z(s)$ is the reparametrization of $X(1,\theta)$ with respect to the parameter s of its arc length.

As the mapping $\sigma : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, it maps null sets onto null sets, and we derive from

$$\frac{\tau(s_2) - \tau(s_1)}{s_2 - s_1} = \frac{1}{\frac{\sigma(\theta_2) - \sigma(\theta_1)}{\theta_2 - \theta_1}}$$

and from $\dot{\sigma}(\theta) > 0$ a.e. on $\mathbb{R}$ that

$$\tau'(s) = \frac{1}{\dot{\sigma}(\tau(s))} > 0 \quad \text{a.e. on } \mathbb{R}. \tag{13}$$

On account of

$$\dot{\sigma}(\theta) = |X_\theta(1, \theta)| \quad \text{a.e. on } \mathbb{R}$$

it then follows that

$$|X_\theta(1, \tau(s))| \frac{d\tau}{ds}(s) = 1 \quad \text{a.e. on } \mathbb{R}, \tag{14}$$

and thus we obtain

$$\int_0^{2\pi} |X_\theta(1, \theta)||X(1, \theta) - P| \, d\theta = \int_0^L |Z(s) - P| \, ds. \tag{15}$$

We now infer from (9) and (15) that

$$\int_B \langle \nabla X, \nabla X \rangle \, du \, dv \le \int_0^L |Z(s) - P| \, ds. \tag{16}$$

By Schwarz's inequality, we have

$$\int_0^L |Z(s) - P| \, ds \le \sqrt{L} \left\{ \int_0^L |Z(s) - P|^2 \, ds \right\}^{1/2}, \tag{17}$$

and Wirtinger's inequality (4) together with (12) implies that

$$\left\{ \int_0^L |Z(s) - P|^2 \, ds \right\}^{1/2} \le L^{3/2}/(2\pi) \tag{18}$$

if we choose P as the barycenter of the closed curve $Z : [0, L] \to \mathbb{R}^3$, i.e., if

$$P := \frac{1}{L} \int_0^L Z(s) \, ds.$$

By virtue of (16)–(18), we arrive at

$$\int_B |\nabla X|^2 \, du \, dv \le \frac{1}{2\pi} L^2 \tag{19}$$

which is equivalent to the desired inequality (8).

Suppose that equality holds true in (8) or, equivalently, in (19). Then equality must also hold in Wirtinger's inequality (18), and by Lemma 2 we infer

$$Z(s) = P + A_1 \cos\left(\frac{2\pi}{L}s\right) + B_1 \sin\left(\frac{2\pi}{L}s\right).$$

Set $R := L/(2\pi)$ and $\varphi = s/R$. Because of $|Z'(s)| \equiv 1$, we obtain

$$R^2 = |A_1|^2 \sin^2\varphi + |B_1|^2 \cos^2\varphi - 2\langle A_1, B_1\rangle \sin\varphi\cos\varphi.$$

Choosing $\varphi = 0$ or $\frac{\pi}{2}$, respectively, it follows that $|A_1| = |B_1| = R$, and therefore $\langle A_1, B_1\rangle = 0$. Then the pair of vectors $E_1, E_2 \in \mathbb{R}^3$, defined by

$$E_1 := \frac{1}{R}A_1, \quad E_2 := \frac{1}{R}B_1,$$

is orthonormal, and we have

$$Z(R\varphi) = P + R\{E_1\cos\varphi + E_2\sin\varphi\}.$$

Consequently $Z(R\varphi), 0 \le \varphi \le 2\pi$, describes a simply covered circle of radius R, centered at P, and the same holds true for the curve $X(1,\theta)$ with $0 \le \theta \le 2\pi$. Hence $X : \overline{B} \to \mathbb{R}^3$ represents a (simply covered) disk of radius R, centered at P, as we infer from the "convex hull theorem" of Section 4.2 and a standard reasoning.

Conversely, if $X : \overline{B} \to \mathbb{R}^3$ represents a simply covered disk, then the equality sign holds true in (8′) and, therefore also in (8).

Thus the assertion of the theorem is proved under the assumption that $X \in H_2^1(B, \mathbb{R}^3)$.

(ii) Suppose now that X is of class $C^0(\overline{B}, \mathbb{R}^3)$. Then we introduce nonconstant minimal surfaces $X_k : B \to \mathbb{R}^3$ of class $C^\infty(\overline{B}, \mathbb{R}^3)$ by defining

$$X_k(w) := X(r_k w) \quad \text{for } |w| < 1, \quad r_k := \frac{k}{k+1}.$$

We can apply (i) to each of the surfaces X_k, thus obtaining

(20) $$4\pi D(X_k) \le \left\{\int_0^{2\pi} |dX_k(1,\theta)|\right\}^2.$$

For $k \to \infty$, we have $r_k \to 1-0$, $D(X_k) \to D(X)$, and part (ii) of Lemma 1 yields

$$\lim_{k\to\infty} \int_0^{2\pi} |dX_k(1,\theta)| = \int_0^{2\pi} |dX(1,\theta)|.$$

Thus we infer from (20) that $4\pi D(X) \le L^2(X)$ which implies in particular that X is of class $H_2^1(B, \mathbb{R}^3)$. For the rest of the proof, we can now proceed as in part (i). □

If the boundary of a minimal surface X is very long in comparison to its "diameter", then another estimate of $A(X) = D(X)$ might be better which depends only linearly on the length $L(X)$ of the boundary of X. We call this estimate *the linear isoperimetric inequality.* It reads as follows:

Theorem 2. *Let X be a nonconstant minimal surface with the parameter domain $B = \{w : |w| < 1\}$, and assume that X is either continuous on $\overline{B}$ or of class $H_2^1(B, \mathbb{R}^3)$. Moreover, suppose that the length $L(X) = \int_C |dX|$ of its boundary is finite, and let $\mathcal{K}_R(P)$ be the smallest ball in $\mathbb{R}^3$ containing $X(\partial B)$ and therefore also $X(\overline{B})$. Then we have*

$$D(X) \leq \frac{1}{2} RL(X). \tag{21}$$

Equality holds in (21) if and only if $X(B)$ is a plane disk.

Proof. By Theorem 1 it follows that $D(X) < \infty$ and $X \in H_2^1(B, \mathbb{R}^3)$, and formula (9) implies

$$2D(X) \leq \int_C |X_\theta||X - P| \, d\theta \leq RL(X) \tag{22}$$

whence we obtain (21). Suppose now that

$$D(X) = \frac{1}{2} RL(X). \tag{23}$$

Then we infer from (9) and (22) that

$$\int_C \langle X_r, X - P \rangle \, d\theta = \int_C |X_r||X - P| \, d\theta$$

is satisfied; consequently we have

$$\langle X_r, X - P \rangle = |X_r||X - P|$$

a.e. on C, that is, the two vectors X_r and $X - P$ are collinear a.e. on C.

Secondly we infer from (22) and (23) that

$$|X - P| = R \quad \text{a.e. on } C.$$

Hence the H_1^1-curve Σ defined by $X : C \to \mathbb{R}^3$ lies on the sphere $S_R(P)$ of radius R centered at P, and the side normal X_r of the minimal surface X at Σ is proportional to the radius vector $X - P$. Thus $X_r(1, \theta)$ is perpendicular to $S_R(P)$ for almost all $\theta \in [0, 2\pi]$. Hence the surface X meets the sphere $S_R(P)$ orthogonally a.e. along Σ. As in the proof of Theorem 1 in Section 1.4 we can show that X is a stationary surface with a free boundary on $S_R(P)$ and that X can be viewed as a stationary point of Dirichlet's integral in the class $\mathcal{C}(S_R(P))$. By Theorems 1 and 2 of Section 2.8, the surface X is real analytic on the closure $\overline{B}$ of B. Then it follows from the Theorem in Section 1.7 that $X(\overline{B})$ is a plane disk.

Conversely, if $X : B \to \mathbb{R}^3$ represents a plane disk, then (23) is fulfilled. □

Now we want to state a more general version of the isoperimetric inequality, valid for global minimal surfaces with boundaries.

Definition 1. *A* global minimal surface *(in $\mathbb{R}^3$) is a nonconstant map*

$$\mathcal{X} \in C^0(M, \mathbb{R}^3) \cap C^2(\overset{\circ}{M}, \mathbb{R}^3)$$

from a two-dimensional manifold M of class C^k, $k \geq 2$, with the boundary ∂M and the interior $\overset{\circ}{M} = \operatorname{int} M$ into the three-dimensional Euclidean space $\mathbb{R}^3$ which has the following properties:

(i) *M possesses an atlas $\mathcal{C}$ which defines a conformal structure on the interior $\overset{\circ}{M}$ of M;*

(ii) *for every chart φ belonging to the conformal structure $\mathcal{C}$ the local map*

$$X = \mathcal{X} \circ \varphi^{-1} : \ \operatorname{int} \varphi(G) \to \mathbb{R}^3, \quad G \subset M,$$

is harmonic and conformal, i.e. a minimal surface as defined in Section 2.6.

In other words, a global minimal surface is defined on a Riemann surface M with a smooth boundary ∂M (which might be empty).

Note that $\mathcal{X}$ may have branch points and selfintersections. Moreover we know that, away from the branch points, the map $\mathcal{X} : M \to \mathbb{R}^3$ induces a Riemannian metric on $\overset{\circ}{M}$. With respect to the local coordinates determined by the charts φ of the atlas $\mathcal{C}$ this metric is given by

$$g_{\alpha\beta}(u, v) = \lambda(u, v)\delta_{\alpha\beta},$$

where $\lambda = |X_u|^2 = |X_v|^2$, so that the gradient ∇_M and the Laplace–Beltrami operator Δ_M are proportional to the corresponding Euclidean operators ∇ and Δ with respect to the local coordinates u and v,

$$\nabla_M = \frac{1}{\lambda}\nabla, \quad \Delta_M = \frac{1}{\lambda}\Delta.$$

In particular, the function $|\mathcal{X}|^2 = \sum_{j=1}^{3} |\mathcal{X}^j|^2$ satisfies

$$\Delta_M |\mathcal{X}|^2 = 4. \tag{24}$$

Moreover, if M is compact, $\mathcal{X}$ is of class C^1 up to its boundary, and if $\mathcal{X}$ has only finitely many branch points in M, then $M \setminus \{\text{branch points}\}$ is a Riemannian manifold, and Green's formulas (in the sense of the Riemannian metric) are meaningful and true for smooth functions defined on M; for example, we obtain from (24) the formula

$$\begin{aligned} 4 \operatorname{area} \mathcal{X} &= 4 \int_M d\,\mathrm{vol}_M = \int_M \Delta_M |\mathcal{X}|^2 \, d\,\mathrm{vol}_M \\ &= 2 \int_{\partial M} |\mathcal{X}| \frac{\partial}{\partial \nu} |\mathcal{X}| \, d\,\mathrm{vol}_{\partial M}, \end{aligned} \tag{25}$$

where ν is the exterior unit normal to ∂M in the tangent bundle $TM|_{\partial M}$.

In Chapter 2 we have seen that boundary branch points of $\mathfrak{X}$ on ∂M are isolated. Hence, for reasonably regular surfaces $\mathfrak{X}$, there exist only finitely many branch points in the interior and on the boundary.

Definition 2. *Let $\mathfrak{X} : M \to \mathbb{R}^3$ be a global minimal surface defined on a compact manifold M. Then the boundary $\partial\mathfrak{X} := \mathfrak{X}(\partial M)$ of $\mathfrak{X}$ is called* weakly connected *if there is a system of Cartesian coordinates (x^1, x^2, x^3) in $\mathbb{R}^3$ such that no hyperplane $H := \{x^j = \text{const}\}$, $j = 1, 2, 3$, separates $\partial\mathfrak{X}$, that is, if H is any hyperplane orthogonal to one of the coordinate axes and if $H \cap \partial\mathfrak{X}$ is empty, then $\partial\mathfrak{X}$ lies on one side of H. Moreover $\mathfrak{X} : M \to \mathbb{R}^3$ is called* compact *if M is compact.*

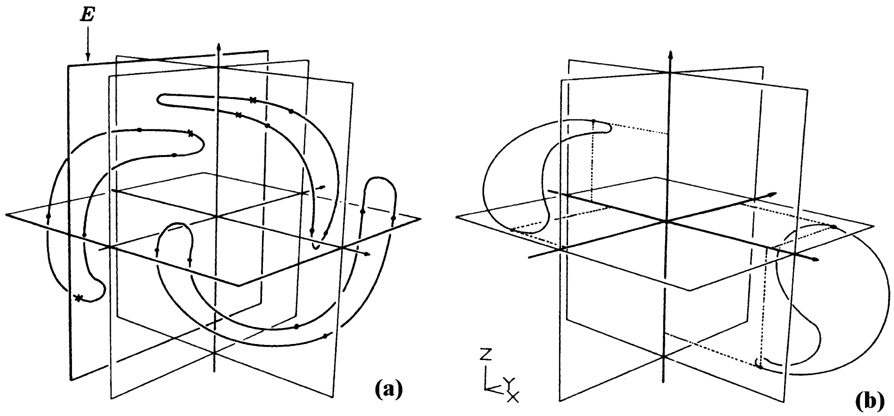

Fig. 1. (**a**) Three weekly connected curves. No plane E parallel to any of the coordinate planes shown separates them. (**b**) Two curves in $\mathbb{R}^3$ which are not weakly connected. It is shown in the text that they lie in opposite quadrants of a suitable coordinate system

Now we can formulate a *general version of the isoperimetric inequality.*

Theorem 3. *Let $\mathfrak{X} : M \to \mathbb{R}^3$ be a global compact minimal surface of class C^1 having at most finitely many branch points defined on a compact Riemann surface M. Suppose also that the boundary $\partial\mathfrak{X}$ is weakly connected. Then the area $A(\mathfrak{X})$ of $\mathfrak{X}$ is bounded from above in terms of the length $L(\mathfrak{X})$ of $\partial\mathfrak{X}$ by the inequality*

$$A(\mathfrak{X}) \le \frac{1}{4\pi} L^2(\mathfrak{X}). \tag{26}$$

Moreover, equality holds if and only if $\mathfrak{X}$ is a plane disk in $\mathbb{R}^3$.

Proof. Let (x^1, x^2, x^3) be the coordinates appearing in the definition of the weakly connected boundary $\partial\mathfrak{X}$. By means of a suitable shift we may even assume that the center of mass of the boundary $\partial\mathcal{X}$ lies at the origin, i.e. that for $j = 1, 2, 3$

$$\int_{\partial M} \mathcal{X}^j \, d\,\mathrm{vol}_{\partial M} = 0, \tag{27}$$

where $\mathcal{X}^j$ is, of course, the j-th coordinate function of the surface $\mathcal{X}$.

On account of (25), it follows that

$$2A(\mathcal{X}) = \int_{\partial M} |\mathcal{X}| \frac{\partial}{\partial \nu} |\mathcal{X}| \, d\,\mathrm{vol}_{\partial M},$$

and it is easily seen that $\frac{\partial}{\partial \nu}|\mathcal{X}| \le 1$. Therefore Schwarz's inequality implies that

$$\begin{aligned} 2A(\mathcal{X}) &= \int_{\partial M} |\mathcal{X}| \frac{\partial}{\partial \nu} |\mathcal{X}| \, d\,\mathrm{vol}_{\partial M} \le \int_{\partial M} |\mathcal{X}| \, d\,\mathrm{vol}_{\partial M} \\ &\le \left(\int_{\partial M} d\,\mathrm{vol}_{\partial M} \int_{\partial M} |\mathcal{X}|^2 \, d\,\mathrm{vol}_{\partial M} \right)^{1/2} \\ &= L^{1/2}(\mathcal{X}) \left(\int_{\partial M} |\mathcal{X}|^2 \, d\,\mathrm{vol}_{\partial M} \right)^{1/2}. \end{aligned} \tag{28}$$

Case (i): Suppose that $\partial\mathcal{X} = \mathcal{X}(\partial M)$ is connected, i.e., $\partial\mathcal{X}$ is a closed curve. Then the proof is essentially that of Theorem 1. In fact, let s be the parameter of arc length of $\partial\mathcal{X}$, and assume that $\partial\mathcal{X}$ is parametrized by s, we write $\mathcal{X}(s)$ for the parameter representation of $\partial\mathcal{X}$ with respect to s. Because of (27) we have $\int_0^L \mathcal{X}(s)\, ds = 0$, where $L := L(\mathcal{X})$, and Wirtinger's inequality yields

$$\begin{aligned} \int_{\partial M} |\mathcal{X}|^2 \, d\,\mathrm{vol}_{\partial M} &= \int_0^L |\mathcal{X}(s)|^2 \, ds \\ &\le \frac{L^2}{4\pi^2} \int_0^L \left| \frac{\partial \mathcal{X}}{ds}(s) \right|^2 ds = \frac{L^3}{4\pi^2}. \end{aligned} \tag{29}$$

From (28) and (29) we derive the desired inequality (26).

Case (ii): $\partial\mathcal{X}$ is weakly connected, but not connected. Hence we are not allowed to apply Wirtinger's inequality, and we have to look for some substitute. Again, we introduce $L = L(\mathcal{X})$ as length of $\mathcal{X}(\partial M)$.

Since M is compact and regular, its boundary ∂M consists of finitely many, say, p closed curves $\partial^1 M, \dots, \partial^p M$. Denote their images under $\mathcal{X}$ by $\sigma_1, \sigma_2, \dots, \sigma_p$, and fix some index $j \in \{1, 2, 3\}$. By assumption, no hyperplane $\{x^k = \text{const}\}$ separates σ_1 from $\sigma_2, \dots, \sigma_p$. Hence, for at least one of these curves, say, for σ_2, we have following property:

There are two points P_1 and Q_1 on σ_1 and σ_2, respectively, whose j-th components P_1^j and Q_1^j coincide. The translation $A_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $P \mapsto P + (P_1 - Q_1)$ leaves the j-th component of every point of $\mathbb{R}^3$ unchanged. Thus $\sigma_1 \cup A_2\sigma_2$ is connected. In a second step we find points

$$P_2 \in \sigma_2 \quad \text{and} \quad Q_2 \in \sigma_3 \cup \cdots \cup \sigma_p, \quad \text{say,} \quad Q_2 \in \sigma_3,$$

such that $P_2^j = Q_2^j$, and a translation $A_3 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$P \mapsto P + (P_1 - Q_1) + (P_2 - Q_2).$$

Again, A_3 leaves the j-th component of every point in $\mathbb{R}^3$ unchanged, and $\sigma_1 \cup A_2\sigma_2 \cup A_3\sigma_3$ is connected. Proceeding by induction, we find translations $A_4, \ldots, A_p$ such that $c_j := \sigma_1 \cup A_2\sigma_2 \cup \cdots \cup A_p\sigma_p$ is a connected curve.

Now let $\mathfrak{X}_1(s), \ldots, \mathfrak{X}_p(s)$ be the parametrizations of $\sigma_1, \ldots, \sigma_p$ with respect to their arc lengths, and

$$x_1(s), \quad 0 \le s \le L_1, \ldots, x_p(s), \quad 0 \le s \le L_p,$$

be their j-th components. We can assume that $\mathfrak{X}_1(0) = P_1$ and $\mathfrak{X}_2(0) = Q_1$ whence $x_1(0) = x_1(L_1) = x_2(0)$. Define

$$y_1(s) := \begin{cases} x_1(s) & \text{for } 0 \le s \le L_1, \\ x_2(s - L_1) & \text{for } L_1 \le s \le L_1 + L_2 \end{cases}$$

and

$$z_2(s) := y_1(s + s_2),$$

where s_2 is chosen in such a way that $z_2(0) = y_1(s_2) = P_2^j = Q_2^j$. Then both $y_1(s)$ and $z_2(s)$ are continuous and periodic with the period $L_1 + L_2$, and we have a.e. that $|\dot{y}_1(s)| = 1$ and $|\dot{z}_2(s)| = 1$.

In the second step we define

$$y_2(s) := \begin{cases} z_2(s) & \text{for } 0 \le s \le L_1 + L_2, \\ x_3(s - L_1 - L_2) & \text{for } L_1 + L_2 \le s \le L_1 + L_2 + L_3 \end{cases}$$

and

$$z_3(s) := y_2(s + s_3),$$

where s_3 is chosen in such a way that $z_3(0) = y_2(s_3) = P_3^j = Q_3^j$. Finally, after $p-1$ steps, we obtain a continuous function $y_p(s), 0 \le s \le L := L_1 + \cdots + L_p$, which is periodic with the period L, and $|\dot{y}_p(s)| = 1$ a.e. on $[0, L]$.

By Wirtinger's inequality we obtain

$$\text{(30)} \qquad \int_0^L |y_{p-1}(s)|^2 \, ds \le \left(\frac{L}{2\pi}\right)^2 \int_0^L |\dot{y}_{p-1}(s)|^2 \, ds,$$

as the mean value of the function y_{p-1} is zero. By construction it follows that

$$\int_0^L |y_{p-1}(s)|^2 \, ds = \int_{\partial M} |\mathfrak{X}^j(s)|^2 \, d\,\mathrm{vol}_{\partial M}$$

and that

$$\int_0^L |\dot{y}_{p-1}(s)|^2\, ds = \int_{\partial M} \left|\frac{d}{ds}\mathcal{X}^j\right|^2 d\,\mathrm{vol}_{\partial M}$$

whence

$$\int_{\partial M} |\mathcal{X}^j|^2\, d\,\mathrm{vol}_{\partial M} \le \left(\frac{L}{2\pi}\right)^2 \int_M \left|\frac{d}{ds}\mathcal{X}^j\right|^2 d\,\mathrm{vol}_{\partial M}. \tag{31}$$

As j is an arbitrary index in $\{1, 2, 3\}$, we may sum up the equations (31) for $j = 1, 2, 3$, thus obtaining

$$\int_{\partial M} |\mathcal{X}^2|\, d\,\mathrm{vol}_{\partial M} \le \left(\frac{L}{2\pi}\right)^2 \int_{\partial M} \left|\frac{d}{ds}\mathcal{X}\right|^2 d\,\mathrm{vol}_{\partial M}. \tag{32}$$

Thus Wirtinger's inequality can be generalized to weakly connected boundaries $\mathcal{X} : \partial M \to \mathbb{R}^3$ in the form (32). Now we can proceed as in case (i) to obtain the isoperimetric inequality (26).

Let us now suppose that equality holds in the isoperimetric inequality, i.e.,

$$4\pi A(\mathcal{X}) = L^2(\mathcal{X}).$$

Then, in particular, equality holds in (28) implying that

$$|\mathcal{X}| \equiv \text{const} =: R \quad \text{on } \partial M,$$

i.e., $\partial\mathcal{X}$ lies on a sphere of radius R, and $R > 0$ since $X(w) \not\equiv 0$.

Now let P be some point on the curve σ_1 which is not the image of a branch point of $\mathcal{X}$. The parametrization of the curves c_j introduced above with respect to the arc length s can now be chosen such that

$$c_j(0) = P \quad \text{for all } j$$

and, if P is suitably selected, that for some neighbourhood $(-\varepsilon, \varepsilon)$ of 0 the curve $c_j(s)$ parametrizes a part of σ_1. Now equality in the isoperimetric inequality implies equality in Wirtinger's inequality for the j-th component c_j^j of the curve c_j, thus

$$c_j^j(s) = a^j \cos\left(\frac{2\pi}{L}s\right) + b^j \sin\left(\frac{2\pi}{L}s\right)$$

for two constants a^j and b^j, $L = L(\mathcal{X})$; in particular, we have for all j that

$$\begin{aligned} c_j^j(0) &= p^j = a^j, \\ \left(\frac{d}{ds}c_j^j\right)(0) &= \frac{d\sigma_1^j}{ds}(0) = b^j\,\frac{2\pi}{L}. \end{aligned}$$

Since $\partial\mathcal{X}$ lies on a sphere, the vectors

$$a = (a^1, a^2, a^3), \quad b = (b^1, b^2, b^3)$$

are mutually perpendicular, and they satisfy

$$R = |a| = |b| = \frac{L}{2\pi}.$$

Since $R > 0$, at least one of the components of a, say a^{j_0}, does not vanish; consequently the function $c_{j_0}^{j_0}(s)$ has exactly two critical points in the interval $[0, L)$. This implies that the boundary $\partial\mathfrak{X}$ of the minimal surface $\mathfrak{X}$ under consideration has only one component. In fact, $c_{j_0}^{j_0}$ is the j_0-th component of the curve obtained by shifting the boundary components $\sigma_1, \ldots, \sigma_p$ of $\partial\mathfrak{X}$ together in a plane perpendicular to the j_0-axis, and every curve σ_j contributes at least two critical points to the function $c_{j_0}^{j_0}$, so that $c_{j_0}^{j_0}$ has at least four critical points if p is greater than one.

This proves that the functions c_j^j are simply the j-th components of the one and only boundary curve σ_1. The preceding identities show that σ_1 is a circle of radius $R = \frac{L}{2\pi}$, the boundary of a plane disk containing $\mathfrak{X}$; see the convex hull theorem in Section 4.1. □

We shall now study a minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ in the three-dimensional space defined on a compact manifold M whose boundary ∂M has exactly two components $\partial^+ M$ and $\partial^- M$. Let us see what happens if $\partial\mathfrak{X}$ is *not* weakly connected.

Denote by $\partial^+\mathfrak{X} = \mathfrak{X}(\partial^+ M)$ and $\partial^-\mathfrak{X} = \mathfrak{X}(\partial^- M)$ the components of $\partial\mathfrak{X}$. They lie in some ball $\overline{B}_R(0) \subset \mathbb{R}^3$. We claim that there is a hyperplane E_*^1 with normal $N_* \in S^2$ through a point P_* such that the components $\partial^\pm\mathfrak{X}$ of $\partial\mathfrak{X}$ lie in the two closed half spaces $H_1^\pm$ defined by E_*^1 respectively and such that $\partial^+\mathfrak{X}$ and $\partial^-\mathfrak{X}$ touch E_*^1.

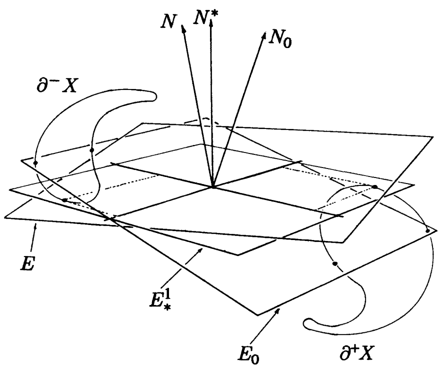

Fig. 2. Construction of E_*^1

Such a plane E_*^1 can be constructed as follows: First of all, there is a plane E_0 with normal N_0 which intersects $\partial^+\mathfrak{X}$ and $\partial^-\mathfrak{X}$. Then consider the open

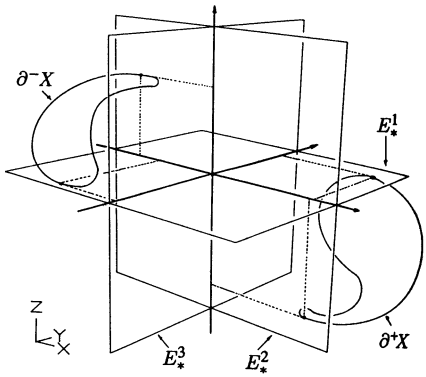

Fig. 3. The planes E^1_*, E^2_*, E^3_*

set $U \subset S^2$ of all unit vectors N for which there is a $P \in \overline{B}_R(0)$ such that the oriented plane $E(P, N)$ through P with normal N separates $\partial\mathcal{X}$, (i.e., $\partial^+\mathcal{X}$ and $\partial^-\mathcal{X}$ lie in the *open* half spaces defined by $E(P, N)$). U is not empty since by assumption $\partial\mathcal{X}$ is not weakly connected (see Fig. 2).

Now take a sequence of planes $E_n = E(P_n, N_n)$ such that $P_n \in \overline{B}_r(0)$, $N_n \in U$, and such that

$$\lim_{n\to\infty} \langle N_n, N_0\rangle = \sup\{\langle N, N_0\rangle : N \in U\};$$

this expression is positive since U is open. Passing to a subsequence we may assume that P_n converges to P_* and N_n to N_*. The plane $E^1_* = E(P_*, N_*)$ then has the desired property (cf. Fig. 2).

No plane parallel to E^1_* separates $\partial\mathcal{X}$, therefore some plane E^2 orthogonal to E^1_* separates $\partial\mathcal{X}$ since it is not weakly connected. Proceeding as above we can now construct a plane E^2_* perpendicular to E^1_* such that $\partial^+\mathcal{X}$ and $\partial^-\mathcal{X}$ lie again in the two closed half spaces defined by E^2_* and such that both components of $\partial\mathcal{X}$ touch E^2_*. Once again none of the planes parallel to E^2_* separates $\partial\mathcal{X}$, hence there has to be a third plane E^3_* orthogonal to E^1_* as well as E^2_* which separates ∂X (Fig. 3). Thus we can choose x, y, z-coordinate axes such that E^1_*, E^2_* and E^3_* correspond to the x, y-, x, z- and y, z-planes respectively and such that $\partial^\pm\mathcal{X}$ lies in the octant

$$\{(x, y, z) : x, y, z \geq 0 (\leq 0)\};$$

in particular, putting $V = \frac{1}{\sqrt{3}}(1, 1, 1)$, the components $\partial^\pm\mathcal{X}$ lie in the cones

$$C^\pm = \left\{P \in \mathbb{R}^3 : \pm\langle P, V\rangle \geq \frac{|P|}{\sqrt{3}}\right\}$$

respectively. The opening angle of this cone is 54.7356103... degrees.

As we have proved in Section 4.2, this implies that the minimal surface $\mathcal{X}$ is not connected. According to Section 3.6 of Vol. 1, there are not compact global minimal surfaces without boundary. Therefore M has exactly two components M^+ and M^- with boundaries $\partial^+ M$ and $\partial^- M$ respectively. Applying the isoperimetric inequality to both of them we obtain

$$\begin{aligned} 4\pi A(\mathcal{X}) &= 4\pi \left\{ A(\mathcal{X}^+) + A(\mathcal{X}^-) \right\} \\ &\le L^2(\partial^+ \mathcal{X}) + L^2(\partial^- \mathcal{X}) \\ &< L^2(\mathcal{X}). \end{aligned}$$

(Here $L(\partial^\pm \mathcal{X})$ denotes the length of $\partial^\pm \mathcal{X}$, and $L(\mathcal{X})$ is the length of $\partial \mathcal{X}$, i.e., $L(\mathcal{X}) = L(\partial^+ \mathcal{X} \cup \partial^- \mathcal{X})$.) Thus we have proved the following

Corollary 1. *If $\mathcal{X}$ is a global compact minimal surface of class $C^1(M, \mathbb{R}^3)$ having at most finitely many branch points and whose boundary has no more than two connected components, then we have the isoperimetric inequality*

$$4\pi A(\mathcal{X}) \le L^2(\mathcal{X}),$$

and equality holds if and only if $\mathcal{X}$ is a plane disk.

One undesirable feature of our isoperimetric inequality is that the minimal surface $\mathcal{X}$ has to be of class C^1 up to the boundary. For a minimal surface $X : B \to \mathbb{R}^3$ defined on the disk $B = \{w : |w| < 1\}$, it follows that the lengths of the boundaries of the surfaces $Z^{(r)}(w) := X(rw)$, $0 < r < 1$, and $w \in B$, tend to the length of the boundary of X, if $X \in C^0(\overline{B}, \mathbb{R}^3)$ and $X|_C$ is rectifiable.

Such a continuity property is also known for doubly connected minimal surfaces defined on annuli; cf. Feinberg [1]. Thus we obtain also

Corollary 2. *If $X : \Omega \to \mathbb{R}^3$ is a minimal surface with $X \in C^0(\overline{\Omega}, \mathbb{R}^3)$, which has a rectifiable boundary and whose parameter domain Ω is either a disk or an annulus, then we have*

$$A(X) \le \frac{1}{4\pi} L^2(X).$$

It can be seen that equality holds if and only if $X(\Omega)$ is a plane disk. Note that Corollary 2 is a generalization of Theorem 1.

4.6 Estimates for the Length of the Free Trace

In this section we want to estimate the length of the free trace of a minimal surface $X : B \to \mathbb{R}^3$ in two situations. First we assume that the image $X(I)$, $I \subset \partial B$, is contained in some part S_0 of the support surface S which can be viewed as the graph of some function $\psi : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, having a bounded gradient (that is, the Gauss image of S is compactly contained in some open hemisphere of S^2). Secondly, we shall study the case that S satisfies a (two-sided) sphere condition.

To begin with the first situation, we assume that S is an embedded regular surface of class C^1 in $\mathbb{R}^3$, and that Γ is a rectifiable Jordan arc of length $L(\Gamma)$ with endpoints P_1 and P_2 on S. We shall not exclude that Γ and S have also other points in common. Nevertheless, we can define the class $\mathcal{C}(\Gamma, S)$ of admissible surfaces $X : B \to \mathbb{R}^3$ for the semifree problem with respect to the boundary configuration $\langle \Gamma, S \rangle$ as in 4.6 of Vol. 1. For technical reasons we imagine such surfaces to be parametrized on the semidisk $B = \{w : |w| < 1, \operatorname{Im} w > 0\}$, the boundary of which consists of the interval I and the circular arc C. For any $X \in \mathcal{C}(\Gamma, S)$, the Jordan arc Γ is the weakly monotonic image of C under X, by Σ we want to denote the free trace $X : I \to \mathbb{R}^3$ of the mapping X on the support surface S. The total variation

$$L(\Sigma) := \int_I |dX|$$

will be called the *length of the free trace* Σ.

Definition 1. *We say that some orientable part S_0 of S fulfils a λ-graph condition, $\lambda > 0$, if there is a unit vector $N_0 \in \mathbb{R}^3$ such that the (suitably chosen) field $N_S(P)$ of unit normals on S satisfies the condition*

$$\langle N_0, N_S(P) \rangle \geq \lambda \quad \text{for all } P \in S_0. \tag{1}$$

Proposition 1. *Let X be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ (see Section 1.4, Definition 1) which satisfies the following two conditions:*

(i) *The free boundary curve $X(I)$ is contained in an open, orientable part S_0 of S which fulfils a λ-graph condition, $\lambda > 0$.*

(ii) *The scalar product $\langle X_v, N_S(X) \rangle$ does not change its sign on I.*

Then the length $L(\Sigma)$ of the free trace Σ, given by $X : I \to \mathbb{R}^3$, is estimated from above by

$$L(\Sigma) \leq \lambda^{-1} L(\Gamma), \tag{2}$$

and the area $A(X) = D(X)$ is bounded by

$$A(X) \leq \frac{(1+\lambda)^2}{4\pi\lambda^2} L^2(\Gamma). \tag{3}$$

Moreover, the surface X is continuous on $\overline{B}$.

Supplement. *If we drop the assumption that X* maps *C monotonically onto Γ, we obtain the estimates*

$$L(\Sigma) \leq \lambda^{-1} \int_C |dX|, \quad A(X) \leq \frac{(1+\lambda)^2}{4\pi\lambda^2} \left(\int_C |dX| \right)^2$$

instead of (2) *and* (3).

Proof of Proposition 1. We can assume that both

$$\langle X_v, N_S(X)\rangle \geq 0$$

and

(4) $$\langle N_0, N_S(X)\rangle \geq \lambda > 0$$

hold on I (we possibly have to replace N_S and N_0 by $-N_S$ and $-N_0$ respectively). As X is assumed to be stationary in $\mathcal{C}(\Gamma, S)$, we have by definition that X is of class $C^1(B \cup I, \mathbb{R}^3)$ and meets S_0 perpendicularly. Consequently we have

$$X_v = |X_v| N_S(X) \quad \text{on } I,$$

and the conformality relation $|X_u| = |X_v|$ yields

(5) $$X_v = |X_u| N_S(X) \quad \text{on } I.$$

Integration by parts implies

$$0 = \int_B \Delta X \, du \, dv = \int_{\partial B} \frac{\partial}{\partial \nu} X \, d\mathcal{H}^1,$$

where ν is the exterior normal to ∂B. Introducing polar coordinates r, φ by $u + iv = re^{i\varphi}$, we arrive at

$$\int_I X_v \, du = \int_C X_r \, d\varphi,$$

and (5) yields

$$\int_I N_S(X)|X_u| \, du = \int_C X_r \, d\varphi.$$

Multiplying this identity by N_0, we arrive at

(6) $$\lambda L(\Sigma) \leq \int_I \langle N_0, N_S(X)\rangle |dX| = \int_C \langle N_0, X_r\rangle \, d\varphi,$$

taking (4) into account, and the conformality relation

$$|X_r| = |X_\varphi| \quad \mathcal{H}^1\text{-a.e. on } C$$

yields

(7) $$\lambda L(\Sigma) \leq \int_C \cos\alpha(\varphi)|X_\varphi| \, d\varphi \leq \int_C |dX|,$$

where $\alpha(\varphi)$ denotes the angle between N_0 and the side normal $X_r(1,\varphi)$ to Γ on X at the point $X(1,\varphi)$. This implies (2), and (3) follows from the isoperimetric inequality

$$A(X) \leq \frac{1}{4\pi}\left(\int_{\partial B} |dX|\right)^2.$$

Finally, a by now standard reasoning yields $X \in C^0(\overline{B}, \mathbb{R}^3)$, taking the relations $D(X) < \infty$ and $L(\Gamma) < \infty$ into account. The "Supplement" is proved by the same reasoning. □

Remark 1. As X intersects S_0 perpendicularly along Σ, the assumption (ii) is certainly satisfied if X possesses no boundary branch points on the free boundary I. Taking the asymptotic expansion of X at boundary branch points into account (see Section 2.10), we see that there are no branch points on I if, for any $r \in (0,1)$, there is a $\delta \in (0, \sqrt{1-r^2})$ such that the part $X : \{w = u + iv : |u| < r, 0 < |v| < \delta\} \to \mathbb{R}^3$ of the minimal surface X lies "on one side of S_0". The last assumption means that, close to I, the minimal surface X does not penetrate the supporting surface S_0.

Moreover, we read off from the asymptotic expansion that $\langle X_v, N_S(X)\rangle$ does not change its sign on I close to branch points of even order. Thus condition (ii) is even fulfilled if branch points of odd order are excluded on I.

Remark 2. By exploiting (6) somewhat more carefully, we can derive an improvement of estimate (2). To this end, we introduce the representation $\{\xi(s) : 0 < s \le l\}$, $l = L(\Gamma)$, of the Jordan arc Γ with respect to its parameter s of the arc length. Then $\xi'(s)$ is defined a.e. on $[0,l]$, and $|\xi'(s)| = 1$. Let $\beta(s) \in [0, \frac{\pi}{2}]$ be the angle between N_0 and the unoriented tangent $T(s)$ of Γ at $\xi(s)$, given by $\pm\xi'(s)$. Then we obtain

$$\langle X_r, N_0\rangle \le |X_r| \cos\left(\frac{\pi}{2} - \beta\right) = |X_r| \sin\beta = |X_\varphi| \sin\beta$$

and, because of $ds = |X_\varphi|\, d\varphi$ and of the monotonicity of the mapping $X : C \to \Gamma$, we infer from (6) the following variant of (7):

$$\lambda L(\Sigma) \le \int_0^l \sin\beta(s)\, ds.$$

This yields the following sharpened version of (2):

$$L(\Sigma) \le \frac{1}{\lambda} \int_\Gamma \sin\beta(s)\, ds. \tag{8}$$

Remark 3. The estimate (2) *is* optimal in the sense that the number λ^{-1} cannot be replaced by a smaller constant. In order to see this, we consider for $0 < \gamma < \frac{\pi}{2}$ the surface

$$S := \{(x,y,z) : y = (\tan\gamma)(x+1) \text{ for } x \le 0, y = (\tan\gamma)(1-x) \text{ for } x \ge 0\}$$

and the arc

$$\Gamma := \{(x,0,0) : |x| \le 1\}$$

(cf. Fig. 1). Let

$$N_0 := (0,1,0),$$

and consider the minimal surface

$$X(w) := (\operatorname{Re}\tau(w), \operatorname{Im}\tau(w), 0),$$

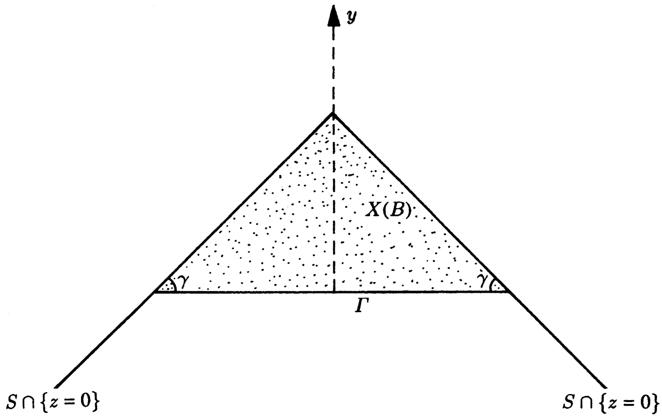

Fig. 1. Remark 3: The estimate (2) is sharp

where $\tau(\omega)$ denotes the conformal mapping of B onto the triangle $\Delta \subset \mathbb{C}$ with the vertices $-1, 1, i$ tan γ keeping ± 1 fixed and mapping i onto 0. Here we have

$$\langle N_0, N_S(P)\rangle = \cos\gamma > 0$$

and

$$L(\Sigma) = \frac{1}{\cos\gamma} L(\Gamma),$$

which shows that the estimate (2) is sharp. However the support surface S of our example does not quite match with the assumptions of Proposition 1 as it is only a Lipschitz surface. By smoothing the surface S at the edge $E := \{(0, \tan\gamma, z)\}$, we can construct a sequence of support surfaces $S_n \in C^\infty$ and a sequence of minimal surfaces $X_n \in \mathcal{C}(\Gamma, S_n)$ whose free traces Σ_n are estimated by

$$L(\Sigma_n) \le \lambda_n^{-1} L(\Gamma)$$

with numbers λ_n tending to $\lambda := \cos\gamma$. As we have

$$\inf_{P\in S_n} \langle N_0, N_{S_n}(P)\rangle = \cos\gamma$$

for all $n = 1, 2, \ldots$ if we construct S_n from S by smoothing around the edge E, it follows that (2) is also sharp in the class of C^∞-support surfaces.

Remark 4. The λ-graph condition (i) in Proposition 1 is crucial. By way of example we shall, in fact, show that one cannot bound the length $L(\Sigma)$ of the free trace Σ in terms of $L(\Gamma)$ and S alone if the λ-graph condition is dropped.

To this end we construct a regular support surface S of class C^∞ which is perpendicularly intersected by the planes

$$\Pi_n := \{(x, y, z) : x = n\}, \quad n = 1, 2, \ldots.$$

We can arrange matters in such a way that the Gauss image of S is contained in the northern hemisphere $S^2 \cap \{z \geq 0\}$ of S^2 and that every intersection curve $S \cap \Pi_n$ consists of an semi-ellipse

$$E_n := \left\{(x, y, z) : x = n, y^2 + n^{-2}z^2 = 1, z \geq 0\right\}$$

and of two rays $\{(n, \pm 1, z) : z \leq 0\}$; cf. Fig. 2. Moreover, we choose Γ_n as straight segments in Π_n connecting the endpoints of E_n,

$$\Gamma_n := \{(n, y, 0) : |y| \leq 1\}.$$

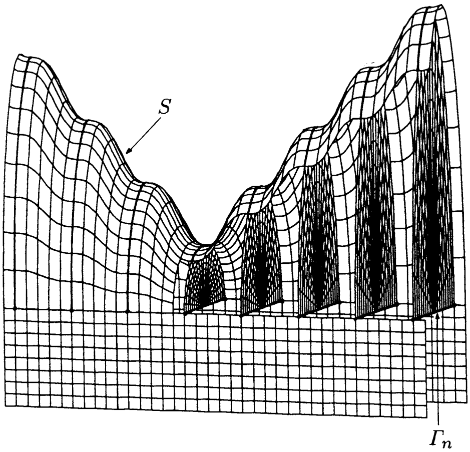

Fig. 2. A supporting surface S, Jordan curves Γ_n of length 2 with endpoints on S, and a sequence of stationary minimal surfaces for these boundary configurations whose surface areas and the lengths of whose free boundaries are unbounded, cf. Remark 4

Finally we choose conformal maps $\tau_n(w) = y_n(w) + iz_n(w)$ of B onto the solid semi-ellipse E_n^* in the y, z-plane, given by

$$E_n^* := \left\{(y, z) : y^2 + n^{-2}z^2 < 1, z > 0\right\},$$

which map C onto Γ_n. Then the minimal surfaces

$$X_n(w) := (n, \operatorname{Re} \tau_n(w), \operatorname{Im} \tau_n(w))$$

are stationary in $\mathcal{C}(\Gamma_n, S)$. Their areas $A(X_n)$ and the lengths $L(\Sigma_n)$ of their free traces tend to infinity as $n \to \infty$ whereas $L(\Gamma_n)$ is always equal to 2.

Note that the support surface S of our example satisfies a λ-graph condition with the forbidden value $\lambda = 0$ if we choose N_0 as $(0, 0, 1)$, but it does not fulfil a λ-graph condition for any $\lambda > 0$, no matter what we choose N_0 to be.

By a slight change of the previous reasoning, the reader may construct a similar example of a support surface S with only one Jordan arc Γ such that $\langle \Gamma, S\rangle$ bounds infinitely many stationary minimal surfaces $X_n \in \mathcal{C}(\Gamma, S)$, $n \in \mathbb{N}$, having the property that $A(X_n) \to \infty$ and $L(\Sigma_n) \to \infty$ as $n \to \infty$.

Remark 5. We can use Proposition 1 to derive a priori estimates for the derivatives of X up to the free boundary I. The key step is the following: Suppose that the assumptions of Proposition 1 are satisfied. Let w_0 be some point on I, $d := 1 - |w_0|$, and let r, θ be polar coordinates around w_0, that is, $w = w_0 + re^{i\theta}$. Set $S_r(w_0) := B \cap B_r(w_0)$ and

$$\varphi(r) := \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv = 2 \int_0^r \int_0^\pi |X_\theta|^2 \rho^{-1} \, d\rho \, d\theta.$$

Then we have

$$\varphi'(r) = 2r^{-1} \int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta.$$

By an obvious modification of the proof of Proposition 1 we obtain

$$\varphi(r) \le 2\lambda_1 \left\{ \int_0^\pi |X_\theta(r,\theta)| \, d\theta \right\}^2, \quad \lambda_1 := \frac{(1+\lambda)^2}{4\pi\lambda^2},$$

and Schwarz's inequality yields

$$\varphi(r) \le \pi\lambda_1 r \varphi(r) \quad \text{for } 0 < r < d$$

whence

$$\varphi(r) \le \varphi(d)(r/d)^{2\mu} \quad \text{for } 0 \le r \le d \tag{9}$$

with

$$\mu := \frac{1}{2\pi\lambda_1} = \frac{2\lambda^2}{(1+\lambda)^2}.$$

Because of

$$\varphi(d) \le 2D(X) \le 2\lambda_1 L^2(\Gamma) \tag{10}$$

we arrive at the following result:

If the assumptions of Proposition 1 are satisfied, then, for any $w_0 \in I_d := \{w \in I : |w| < 1 - d\}$, $0 < d < 1$, *we have*

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le K(r/d)^{2\mu} \quad \text{for } r \in [0, d], \tag{11}$$

where

$$\mu := \frac{2\lambda^2}{(1+\lambda)^2}, \quad K := \frac{(1+\lambda)^2}{2\pi\lambda^2} L^2(\Gamma). \tag{12}$$

By a reasoning used in the proofs of the Theorems 1 and 4 of Section 2.5 we obtain:

There is a constant K^* *depending only on* λ *and* $L(\Gamma)$ *such that, for any* $w_0 \in \overline{B}$ *satisfying* $|w_0| \le 1 - d$ *and any* $r \in [0, d], 0 < d < 1$, *we have*

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le K^*(r/d)^{2\mu}, \tag{13}$$

and Morrey's Dirichlet growth theorem yields

$$[X]_{\mu, \overline{Z}_d} \le c(\mu) \, d^{-\mu} \sqrt{K^*} \tag{14}$$

(cf. Section 2.5, Theorem 1).

Remark 6. In consideration of Remark 4 and of the observation stated at the beginning of Section 2.6 it cannot be expected that estimates of the type (13) and (14) hold with some constant K^* depending only on $L(\Gamma)$ and S if we drop assumption (i) in Proposition 1. Nevertheless one could expect such estimates to be true with numbers K^* depending solely on $L(\Gamma), S$ and $D(X)$.

This seems to be unknown in general except for the following particular case which we want to formulate as

Proposition 2. *Let* X *be a stationary minimal surface in* $\mathcal{C}(\Gamma, S)$ *which lies in the exterior of an open, convex subset* $\mathcal{K}$ *of* $\mathbb{R}^3$ *that is bounded by* S. *Suppose also that* $S = \partial\mathcal{K}$ *is a regular surface of class* C^3, *and suppose that the unit normal* N_S *of* S *pointing into the set* $\mathcal{K}$ *satisfies the following condition:*

(iii) *There exist two constants* $\rho > 0$ *and* $\lambda > 0$ *such that* $\langle N_S(P), N_2(Q)\rangle \ge \lambda$ *is fulfilled for any two points* $P, Q \in S$ *whose* S*-intrinsic distance is at most* ρ.

Then there is a constant K^* *depending only on* $L(\Gamma)$, S *and* $D(X)$ *such that the inequalities* (13) *and* (14) *hold true.*

Let us sketch the proof. We begin with the following

Lemma 1. *Suppose that the assumptions of Proposition* 2 *are satisfied. Let* r, θ *be polar coordinates about some points* $w_0 \in I$, *defined by* $w = w_0 + re^{i\theta}$, *and set*

$$\psi(r) := \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta, \quad 0 \le r \le 1 - |w_0|.$$

Then $\psi(r)$ *is a monotonically increasing function of* r *in* $[0, 1 - |w_0|]$.

Proof. By Proposition 1 of Section 2.8 it follows that $X \in C^2(B \cup I, \mathbb{R}^3)$. Set $I_r(w_0) := \{w \in I : |w - w_0| < r\}$. Then, by partial integration we obtain

$$\begin{aligned} r\psi'(r) &= \int_0^\pi \frac{\partial}{\partial r} |X_\theta(r, \theta)|^2 r \, d\theta \\ &= \int_{S_r(w_0)} \Delta |X_\theta|^2 \, du \, dv + \int_{I_r(w_0)} \frac{\partial}{\partial v} |X_\theta|^2 \, du. \end{aligned} \tag{15}$$

Let X^* be the adjoint minimal surface to X. Then the mapping $f : B \to \mathbb{C}^3$ defined by $f(w) = X(w) + iX^*(w)$ is holomorphic. Consequently also $wf'(w)$ is holomorphic, and

$$|w|^2|f'(w)|^2 = r^2|\nabla X|^2 = 2|X_\theta|^2$$

is subharmonic. Thus we arrive at

$$\Delta|X_\theta|^2 \geq 0 \quad \text{on } B. \tag{16}$$

Moreover, the conformality relations imply

$$|X_\theta|^2 = r^2|X_u|^2,$$

where $r^2 = |w - w_0| = (u - u_0)^2 + v^2$ for $w_0 = u_0 \in I$, and therefore

$$\frac{\partial}{\partial v}|X_\theta|^2 = 2v|X_u|^2 + 2r^2\langle X_u, X_{uv}\rangle.$$

Thus we obtain

$$\frac{\partial}{\partial v}|X_\theta|^2 = 2(u - u_0)^2\langle X_u, X_{uv}\rangle \quad \text{on } I.$$

Differentiating $\langle X_u, X_v = 0\rangle$ with respect to u it follows that

$$\langle X_u, X_{uv}\rangle = -\langle X_{uu}, X_v\rangle \quad \text{on } B \cup I,$$

and consequently

$$\frac{\partial}{\partial v}|X_\theta|^2 = -2(u - u_0)^2\langle X_{uu}, X_v\rangle \quad \text{on } I.$$

Note that X_v points in the direction of the exterior normal of $\mathcal{K}$ whereas X_{uu} points into the interior of $\mathcal{K}$ since $X : I \to \mathbb{R}^3$ maps I into the boundary S of the open convex set $\mathcal{K}$. Thus we have

$$\langle X_{uu}, X_v\rangle \leq 0 \quad \text{on } I$$

and therefore

$$\frac{\partial}{\partial v}|X_\theta|^2 \geq 0 \quad \text{on } I. \tag{17}$$

On account of (15)–(17) we infer that $\psi'(r) \geq 0$. □

In the same way, the following result can be established, we leave its proof to the reader.

Lemma 2. *Suppose that the assumptions of Proposition 2 are satisfied, and let $w = w_0 + re^{i\theta}$ for some point $w_0 \in I$. Then, for any $p \in [1, \infty)$, the function*

$$\psi_p(r) := \int_0^\pi |X_\theta(r, \theta)|^p \, d\theta$$

is a monotonically increasing function of $r \in [0, 1 - |w_0|]$.

Now we turn to the

Proof of Proposition 2. Fix some $d \in (0,1)$, and let w_0 be an arbitrary point on I with $|w_0| < 1-d$. By $X(r,\theta)$ we denote the representation of X in polar coordinates r, θ about w_0 (i.e. $w = w_0 + re^{i\theta}$). Set

$$\chi(r) := \frac{\sqrt{\pi}}{\lambda} \left\{ \int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta \right\}^{1/2},$$

$$\chi^*(r) := \left\{ \frac{2\pi D(X)}{\lambda^2 \log 1/r} \right\}^{1/2}.$$

By the reasoning of the Courant–Lebesgue lemma (see Section 4.4) we infer that, for any $r \in (0, d^2)$, there exists some $r' \in (r, \sqrt{r})$ such that $\chi(r') \le \chi^*(r)$ holds true. On account of Lemma 1, the function χ is increasing whence

$$\chi(r) \le \chi^*(r) \quad \text{for all } r \in (0, d^2). \tag{18}$$

Since χ^* is strictly increasing, we have

$$\chi^*(r) < \rho \quad \text{if and only if} \quad r < \lambda_2, \tag{19}$$

where the number λ_2 is defined by

$$\lambda_2 := \exp\left(-\frac{2\pi D(X)}{\lambda^2 \rho^2} \right).$$

Let us now introduce the increasing function

$$l(r) := \int_{I_r(w_0)} |dX|,$$

and set

$$\mathcal{I}(w_0) := \left\{ r \in (0, d^2) : l(r) < \rho \right\}.$$

Clearly, $\mathcal{I}(w_0)$ is an open and non-empty interval contained in $(0, d^2)$, and therefore

$$m := \sup \mathcal{I}(w_0)$$

is a positive number which is not contained in $\mathcal{I}(w_0)$. Set

$$\delta := \min\{d^2, \lambda^2\}.$$

We claim that the interval $(0, \delta)$ is contained in $\mathcal{I}(w_0)$, independently of the choice of w_0. Otherwise we had $m < \delta$, whence

$$m < d^2 \quad \text{and} \quad m < \lambda^2,$$

and therefore also

(20) $$\chi^*(m) < \rho,$$

on account of (19). For any $r \in (0, m)$ we have $r \in \mathcal{I}(w_0)$, and therefore $l(r) < \rho$. Applying assumption (iii), we infer as in Proposition 1 that

(21) $$l(r) \le \lambda^{-1} \int_0^\pi |X_\theta(r,\theta)|\, d\theta$$

and Schwarz's inequality yields

(22) $$l(r) \le \chi(r) \quad \text{for all } \ r \in (0, m).$$

From (18)–(21) we infer that

$$l(r) \le \chi(r) \le \chi^*(r) < \chi^*(m) < \rho \quad \text{for all } \ r \in (0, m)$$

whence, by $r \to m - 0$, we deduce that

$$l(m) < \rho.$$

This implies $m \in \mathcal{I}(w_0)$ which is impossible. Thus we have proved:

The interval $(0, \delta)$ *lies in* $\mathcal{I}(w_0)$, *for any* $w \in I$ *with* $|w_0| < 1 - d$.

Thus we obtain (21) for all $r \in (0, \delta)$, and the isoperimetric inequality yields

(23) $$\int_{S_r(w_0)} |\nabla X|^2 du\, dv \le 2\lambda_1 \left\{ \int_0^\pi |X_\theta(r,\theta)|\, d\theta \right\}^2$$

with $\lambda_1 = (1+\lambda)^2/(4\pi\lambda^2)$, for all $r \in (0, \delta)$ and for all $w_0 \in I$ with $|w_0| < 1-d$.

Now we can proceed as in Remark 5 in order to prove the assertion of Proposition 2.

Theorem 1. *Let* S *be an admissible*[1] *surface of class* C^3, *and assume that* X *is a critical point of Dirichlet's integral which has the following properties:*

(i) *The free trace* $X(I)$ *is contained in an open, orientable part* S_0 *of* S *that fulfils a* λ*-graph condition,* $\lambda > 0$.

(ii) *There exist no branch points of* X *on* I *which are of odd order.*

Then the length $L(\Sigma) = \int_I |dX|$ *of the free trace* Σ *given by* $X : I \to \mathbb{R}^3$ *is estimated by* (2)*:*

$$L(\Sigma) \le \lambda^{-1} L(\Gamma),$$

and the area $A(X)$ *of* X *is estimated by* (3)*:*

$$A(X) \le \lambda_1 L^2(\Gamma), \quad \lambda_1 := \frac{(1+\lambda)^2}{4\pi\lambda^2}.$$

[1] The condition of "admissibility" is essentially a uniformity condition at infinity which is formulated in Section 2.6.

Proof. By Theorem 4 of Section 2.7, the surface X is a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ which is of class $C^1(B \cup I, \mathbb{R}^3)$. Then the assertion follows from Proposition 1 and from Remark 1. $\square$

Note that the λ-graph condition imposes no bounds on the principal curvature of S_0. Thus S_0 was allowed to have arbitrarily sharp wrinkles.

The following assumption is in a sense complementary to the λ-graph condition; it implies a bound on the principal curvatures of S but does not restrict the position of the Gauss image N_S of S.

Definition 2. *We say that a surface S in $\mathbb{R}^3$ satisfies a* (two-sided) R-sphere condition, *if S is a C^2-submanifold of $\mathbb{R}^3$ which is the boundary of an open set U of $\mathbb{R}^3$, and if for every $P \in S$ the tangent balls*

$$B^{\pm}(P, R) := \{Q \in \mathbb{R}^3 : |P \pm RN_S(P) - Q| < R\} \tag{24}$$

do not contain any points of S. Here N_S denotes the exterior unit normal of S with respect to U (see Fig. 3).

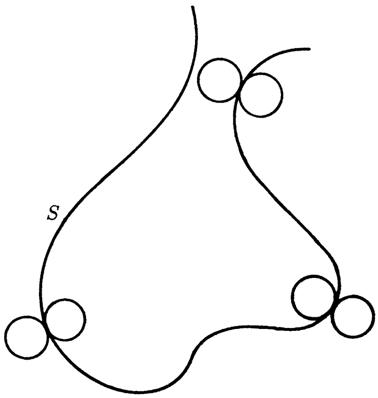

Fig. 3. The R-sphere condition

Theorem 2. *Let S be a support surface satisfying an R-sphere condition, and let Γ be a rectifiable Jordan arc with its endpoints on S. Let X be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ with the free trace Σ given by $X : I \to \mathbb{R}^3$. Then the length $L(\Sigma)$ of Σ can be estimated by*

$$L(\Sigma) \le L(\Gamma) + \frac{2}{R} D(X). \tag{25}$$

This estimate is optimal in the sense that 2 *cannot be replaced by any smaller number.*

For the proof we need the following

Lemma 3. *Suppose that the surface S satisfies an R-sphere condition. Then its principal curvatures are bounded from above by $1/R$. Moreover, any point P in the tubular neighbourhood*

$$T_R := \{Q \in \mathbb{R}^3 : \operatorname{dist}(Q,S) < R\} \tag{26}$$

has a unique representation of the form

$$P = F(P) + \rho(P) N_S(F(P)), \tag{27}$$

where $F(P) \in S$ is the unique foot of P on S, $\rho(P)$ is the oriented distance from S to P, and $N_S(Q)$ denotes the exterior normal to S at $Q \in S$ (i.e., $\rho(P) < 0$ if $P \in U$, and $\rho(P) \geq 0$ if $P \in \mathbb{R}^3 - U$). The distance function ρ is of class C^2 (and of class C^m or $C^{m,\alpha}$ if $S \in C^m$ or $C^{m,\alpha}$ respectively, $m \geq 2, 0 < \alpha < 1$), and we have

$$D\rho(P) = N_S(F(P)) \quad \textit{for all } P \in T_R. \tag{28}$$

Finally the eigenvalues of the Hessian matrix $H(P) = D^2\rho(P) = (\rho_{x^i x^k}(P))$ at any $P \in T_R$ are bounded from above by $[R - |\rho(P)|]^{-1}$, and the Hessian annihilates normal vectors, i.e.,

$$H(P) N_S(F(P)) = 0 \quad \textit{for all } P \in T_R. \tag{29}$$

(Here D denotes the three-dimensional gradient in $\mathbb{R}^3$.)

Proof. The representation formula (27) in the tubular neighbourhood T_R is fairly obvious. The other results follow from (27) by means of the implicit function theorem using the fact that the principal curvature of S at P are precisely the eigenvalues of the Hessian of a nonparametric representation of S close to P whose x, y-plane is parallel to the tangent plane of S at P. We omit the details and refer the reader to Gilbarg and Trudinger [1], Appendix (pp. 383–384), for the pertinent estimates.

Proof of Theorem 2 in the special case that X has no branch point of odd order on I. For any $\delta > 0$ we can choose a function $\varphi(t), t \in \mathbb{R}$, of class $C_C^\infty((-R,R))$ having the following properties:

$$\begin{aligned} 0 \leq \varphi \leq 1, \quad \varphi(0) = 1, \quad \varphi(t) = \varphi(-t),& \\ \varphi(t) \leq (1 - R^{-1}|t|)(1+\delta) \quad \text{for } |t| \leq R,& \\ |\varphi'(t)| \leq R^{-1}(1+\delta).& \end{aligned} \tag{30}$$

Then we define a C^1-vector field Z on $\mathbb{R}^3$ by

$$Z(P) = \begin{cases} \varphi(\rho(P)) N_S(F(P)) & \text{for } P \in T_R, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

We clearly have

$$(32)\qquad \begin{aligned} &|Z(P)| \le 1 \quad \text{for all } P \in \mathbb{R}^3, \\ &Z(Q) = N_S(Q) \quad \text{for } Q \in S, \end{aligned}$$

and we claim that also

$$(33)\qquad |\nabla Z(P) \le R^{-1}(1+\delta) \quad \text{for all } P \in \mathbb{R}^3$$

holds true. As ∇Z vanishes in the exterior of T_R, we have to prove (33) only for $P \in T_R$. Thus we fix some $P \in T_R$ and some unit vector $\nu \in \mathbb{R}^3$. Then the directional derivative $\frac{\partial Z}{\partial \nu}(P)$ is given by

$$\frac{\partial Z}{\partial \nu}(P) = \varphi'(\rho(P))\frac{\partial \rho}{\partial \nu}(P) N_S(F(P)) + \varphi(\rho(P))\frac{\partial}{\partial \nu} N_S(F(P)).$$

If $\nu = \pm N_S(F(P))$, then

$$\frac{\partial Z}{\partial \nu}(P) = \varphi'(\rho(P))\frac{\partial \rho}{\partial \nu}(P) N_S(F(P)),$$

and by (28) and (30_3) it follows that

$$(34)\qquad \left|\frac{\partial Z}{\partial \nu}(P)\right| \le \frac{1+\delta}{R}.$$

If ν is orthogonal to $N_S(F(P))$, then $\frac{\partial \rho}{\partial \nu}(P) = 0$, and Lemma 3 yields

$$\frac{\partial}{\partial \nu} N_S(F(P)) = \frac{\partial}{\partial \nu}\nabla\rho(P) = \nabla^2\rho(P)\nu$$

and

$$|\nabla^2\rho(P)\nu| \le |\nabla^2\rho(P)||\nu| \le (R - |\rho(P)|)^{-1}.$$

In conjunction with (30_2) it follows that

$$\left|\frac{\partial Z}{\partial \nu}(P)\right| \le (1 - R^{-1}|\rho(P)|)(1+\delta)(R - |\rho(P)|)^{-1} = \frac{1+\delta}{R}$$

and thus (34) holds true if $\nu \perp N_S(F(P))$. Hence (34) is satisfied for all unit vectors ν, and we have established property (33).

By means of the vector field Z on $\mathbb{R}^3$ we define a surface $Y(w), w \in B$, of class $L_\infty \cap H_2^1(B, \mathbb{R}^3)$, setting $Y(w) = Z(X(w))$.

Given any $\varepsilon \in (0,1)$, we can find two numbers $\varepsilon_1, \varepsilon_2 \in (\varepsilon^2, \varepsilon)$ such that

$$(35)\qquad \int_{\gamma_1(\varepsilon)} |dX| + \int_{\gamma_2(\varepsilon)} |dX| \le 2\left\{\frac{\pi D(X)}{\log(1/\varepsilon)}\right\}^{1/2},$$

where $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ denote the circular arcs

$$\gamma_1(\varepsilon) := \{w \in B : |w-1| = \varepsilon_1, \operatorname{Im} w > 0\},$$

$$\gamma_2(\varepsilon) := \{w \in B : |w+1| = \varepsilon_2, \operatorname{Im} w > 0\};$$

see Section 4.4 of Vol. 1.

Now we apply Green's formula to the functions X, Y and to the domain $\Omega(\varepsilon)$ which is obtained from the semidisk B by removing the parts which are contained in the disks $B_{\varepsilon_1}(1)$ or $B_{\varepsilon_2}(-1)$, respectively:

$$\Omega(\varepsilon) := B \setminus [B_{\varepsilon_1}(1) \cup B_{\varepsilon_2}(-1)].$$

Thus we obtain

$$(36)\quad \int_{\Omega(\varepsilon)} \langle \nabla X, \nabla Y\rangle \, du\, dv = -\int_{\Omega(\varepsilon)} \langle \Delta X, Y\rangle \, du\, dv + \int_{\partial\Omega(\varepsilon)} \left\langle \frac{\partial X}{\partial v}, Y \right\rangle d\mathcal{H}^1,$$

where ν denotes the exterior normal on $\partial\Omega(\varepsilon)$. Set

$$I(\varepsilon) := I \cap \partial\Omega(\varepsilon) \quad \text{and} \quad C(\varepsilon) := C \cap \partial\Omega(\varepsilon).$$

Then

$$\partial\Omega(\varepsilon) = I(\varepsilon) \cup C(\varepsilon) \cup \gamma_1(\varepsilon) \cup \gamma_2(\varepsilon).$$

On the interval $I(\varepsilon)$, we have $d\mathcal{H}^1 = du$, $\frac{\partial X}{\partial \nu} = -X_v$, $Y = N_S(X)$, and $X_v = \pm|X_v| N_S(X)$. As there exist no branch points of odd order on I, the vector X_v always points in the direction of $N_S(X)$ or in the direction of $-N_S(X)$. Thus we can assume that

$$X_v = |X_v| N_s(X) \quad \text{on } I(\varepsilon),$$

and we arrive at

$$\left\langle \frac{\partial X}{\partial \nu}, Y \right\rangle = -|X_v| = -|X_u| \quad \text{on } I(\varepsilon).$$

This implies

$$(37)\qquad \int_{I(\varepsilon)} \left\langle \frac{\partial X}{\partial \nu}, Y \right\rangle d\mathcal{H}^1 = \int_{I(\varepsilon)} |dX|.$$

Let $\frac{\partial}{\partial \tau} X$ be the tangential derivative of X along $\partial\Omega(\varepsilon)$. The conformality relations yield

$$\left|\frac{\partial X}{\partial \nu}\right| = \left|\frac{\partial X}{\partial \tau}\right|$$

and therefore

$$\left|\left\langle \frac{\partial X}{\partial \nu}, Y \right\rangle\right| \le \left|\frac{\partial X}{\partial \nu}\right| |Y| \le \left|\frac{\partial X}{\partial \tau}\right| \quad \text{along } \partial\Omega(\varepsilon).$$

Consequently, we have

$$(38)\qquad \left|\int_{C(\varepsilon)+\gamma_1(\varepsilon)+\gamma_2(\varepsilon)} \left\langle \frac{\partial X}{\partial \nu}, Y \right\rangle d\mathcal{H}^1\right| \\ \le \int_{C(\varepsilon)} |dX| + \int_{\gamma_1(\varepsilon)} |dX| + \int_{\gamma_2(\varepsilon)} |dX| \le L(\Gamma) + f(\varepsilon),$$

where the remainder term $f(\varepsilon)$ tends to zero as $\varepsilon \to +0$, by virtue of (35).

Finally we infer from $Y = Z \circ X$ and from (33) that

$$|\nabla Y| \le \frac{1+\delta}{R} |\nabla X|$$

whence

$$(39)\qquad \left|\int_{\Omega(\varepsilon)} \langle \nabla X, \nabla Y\rangle \, du\, dv\right| \le \frac{1+\delta}{R} \int_{\Omega(\varepsilon)} |\nabla X|^2 \, du\, dv.$$

Because of $\Delta X = 0$ we infer from (36) in conjunction with (37)–(39) that

$$(40)\qquad \int_{I(\varepsilon)} |dX| \le L(\Gamma) + \frac{1+\delta}{R} \int_{\Omega(\varepsilon)} |\nabla X|^2 \, du\, dv + f(\varepsilon).$$

Letting $\varepsilon \to +0$, it follows that

$$L(\Sigma) \le L(\Gamma) + \frac{1+\delta}{R} \int_B |\nabla X|^2 \, du\, dv.$$

Since we can choose $\delta > 0$ as small as we please, we arrive at the desired inequality

$$L(\Sigma) \le L(\Gamma) + 2R^{-1} D(X).$$

In order to show that the estimate (25) is optimal we consider the following *examples*. Let S be the circular cylinder of radius R given by

$$S := \left\{(x, y, z) : x^2 + y^2 = R^2\right\},$$

and let Γ be the straight arc

$$\Gamma := \left\{(x, y, z) : x = a, y^2 \le R^2 - a^2, z = 0\right\},$$

where a denotes some number with $0 < a < R$. Then it is easy to define a planar minimal surface $X : B \to \mathbb{R}^3$ which is stationary in $\mathcal{C}(\Gamma, S)$ and maps B conformally onto the planar domain $\Omega = \{(x, y, z) : z = 0, x^2 + y^2 < R^2, x < a\}$.

If a tends to R then $L(\Sigma)$ converges to $2\pi R$ and $cR^{-1}D(X)$ to $c\pi R$ whereas $L(\Gamma)$ shrinks to zero. This shows that $c = 2$ is the optimal value in the estimate

$$L(\Sigma) \le L(\Gamma) + cR^{-1}D(X),$$

and the proof of Theorem 2 is complete in the special case that there are no branch points of odd order on I. □

The proof of Theorem 2 in the general case will be based on the relation

$$|X_v| = |D_v\rho(X)| \quad \text{along } I. \tag{41}$$

This follows by differentiating the relation

$$X = F(X) + \rho(X)N_S(F(X)) \quad \text{on } B \cup I$$

which holds on $B \cup I$ close to I (cf. (27)). Hence we can express the length of the free trace Σ as

$$L(\Sigma) = \int_I |D_v\rho(X)|\, du. \tag{42}$$

If $X(B)$ were contained in the tubular neighbourhood T_R of S, we could write

$$\int_I D_v\rho(X)\, du = -\int_B \Delta\rho(X)\, du\, dv + \int_C \frac{\partial}{\partial\nu}\rho(x)\, d\mathcal{H}^1.$$

If $D_v\rho(X)$ has a uniform sign on I, we could use this identity to derive an estimate for $L(\Sigma)$. However, since both facts are not guaranteed, we shall instead construct some function $\eta(w)$ of which we can prove that

$$\eta \ge |D_\nu\rho(X)| \quad \text{on } I \tag{43}$$

holds true. Then we can estimate $L(\Sigma)$ from above by the integral $\int_I \eta_v\, du$ which is transformed into

$$-\int_B \Delta\eta\, du\, dv + \int_C \frac{\partial}{\partial\nu}\eta\, d\mathcal{H}^1,$$

and this integral will be estimated in terms of X.

In order to define η we first introduce

$$\Psi(t) := \int_0^t \varphi(s)\, ds,$$

where φ is a function of class $C_c^\infty((-R,R))$ satisfying (30). Then Ψ satisfies

$$\begin{aligned} \Psi(t) &= \Psi(R) \quad \text{for } t \ge R, \quad \Psi(t) = -\Psi(R) \quad \text{for } t \ge -R,\\ \Psi(0) &= 0, \quad \Psi'(0) = 1, \quad 0 \le \Psi' \le 1,\\ \Psi(t) &\le (1 - R^{-1}|t|)(1+\delta) \quad \text{for } |t| \le R,\\ |\Psi''(t)| &\le R^{-1}(1+\delta). \end{aligned} \tag{44}$$

Secondly we define

$$\zeta(P) := \Psi^2(\rho(P)) \quad \text{if } P \in \mathbb{R}^3,$$

$$\alpha(w) := \delta v, \quad w = u + iv.$$

Then $\eta(w)$ will be defined as

$$\eta(w) := \left\{\alpha^2(w) + \zeta(X(w))\right\}^{1/2} \quad \text{for } w \in \overline{B}.$$

The function η is of class $C^2(B) \cap C^1(B \cup I)$, and its boundary values on C are absolutely continuous. Moreover, we have

$$\nabla\eta = \left\{\alpha^2 + \zeta(X)\right\}^{1/2} \left[\alpha\nabla\alpha + \frac{1}{2}\nabla\zeta(X)\right]$$

whence it follows that

$$\eta_v = \left\{\alpha^2 + \zeta(X)\right\}^{-1/2} \left[\delta\alpha + \Psi'(\rho(X))\Psi(\rho(X))D_v\rho(X)\right].$$

Here and in the sequel we use the notation $\nabla\zeta(X)$ for $\nabla(\zeta \circ X)$, $D_v\rho(X)$ for $D_v(\rho \circ X)$, etc.

Set $w = u_0 + iv$, $v > 0$, and let $v \to +0$. Then $\Psi(\rho(X)) \to 0$, $\Psi'(\rho(X)) \to 1$, and l'Hospital's rule yields

$$\frac{\Psi(\rho(X(u_0 + iv)))}{v} \to D_v\rho(X)\Big|_{w=w_0}.$$

Hence η_v tends to

$$\frac{\delta^2 + |D_v\rho(X)|^2}{\{\delta^2 + |D_v\rho(X)|^2\}^{1/2}} = \left\{\delta^2 + |D_v\rho(X)|^2\right\}^{1/2} \geq |D_v\rho(X)|$$

whence we have established (43).

Next we want to estimate $-\Delta\eta$ from above. We have

$$\begin{aligned} \nabla\zeta(X) &= 2\Psi'(\rho(X))\Psi(\rho(X))\nabla\rho(X), \\ \Delta\zeta(X) &= 2\gamma'(\rho(X))|\nabla\rho(X)|^2 + 2\gamma(\rho(X))\Delta\rho(X), \end{aligned}$$

where we have set

$$\gamma := \Psi\Psi',$$

and

$$\begin{aligned} -\Delta\eta = &\left\{\alpha^2 + \zeta(X)\right\}^{-3/2} \left|\alpha\nabla\alpha + \frac{1}{2}\nabla\zeta(X)\right|^2 \\ &- \left\{\alpha^2 + \zeta(X)\right\}^{-1/2} \left[|\nabla\alpha|^2 + \frac{1}{2}\Delta\zeta(X)\right]. \end{aligned}$$

This implies (with $\zeta = \zeta(X)$, $\rho = \rho(X)$, $\gamma = \gamma(\rho(X))$, etc.) that

$$\begin{aligned}
-\Delta\eta &= \left\{\alpha^2+\zeta\right\}^{-3/2}|\alpha\nabla\alpha+\gamma\nabla\rho|^2\\
&\quad-\left\{\alpha^2+\zeta\right\}^{-1/2}(|\nabla\alpha|^2+\Psi'^2|\nabla\rho|^2+\Psi\Psi''|\nabla\rho|^2+\gamma\Delta\rho)\\
&= -\left\{\alpha^2+\zeta\right\}^{-3/2}|\alpha\Psi'\nabla\rho-\Psi\nabla\alpha|^2\\
&\quad-\left\{\alpha^2+\zeta\right\}^{-1/2}(\Psi\Psi''|\nabla\rho|^2+\gamma\Delta\rho)\\
&\le -\left\{\alpha^2+\zeta\right\}^{-1/2}\Psi(\Psi''|\nabla\rho|^2+\Psi'\Delta\rho)\\
&\le \left\{\alpha^2+\Psi^2\right\}^{-1/2}\Psi(|\Psi''||\nabla\rho|^2+|\Psi'||\Delta\rho|)\\
&\le |\Psi''||\nabla\rho|^2+|\Psi'||\Delta\rho|.
\end{aligned}$$

Thus we have

$$-\Delta\eta \le |\Psi''(\rho(X))||\nabla\rho(X)|^2+|\Psi'(\rho(X))||\Delta\rho(X)|. \tag{45}$$

We can restrict our attention to the set

$$B' := \{w\in B : \rho(X(w)) < R\}$$

since $\Psi'(\rho(X))$ and $\Psi''(\rho(X))$ vanish in $B\setminus B'$ whence also $\Delta\eta=0$ in $B\setminus B'$. In B' we have

$$|\Psi''(\rho(X))||\nabla\rho(X)|^2 \le \frac{1+\delta}{R}|\nabla\rho(X)|^2 \tag{46}$$

and

$$|\Psi'(\rho(X))| \le (1+\delta)(1-R^{-1}|\rho(X)|), \tag{47}$$

taking (44) into account.

Furthermore we have

$$\Delta\rho(X) = X_u H(X) X_u + X_v H(X) X_v \tag{48}$$

with $H(X) = (\rho_{x^i x^k}(X))$ = Hessian matrix of ρ composed with X. By means of Lemma 3 we infer that

$$|X_u H(X) X_u + X_v H(X) X_v| \le (R-|\rho(X)|)^{-1}\left\{|\nabla X|^2-|\nabla\rho(X)|^2\right\} \tag{49}$$

since $|X_u|^2 - D_u\rho(X)|^2$ is the square of the norm of the tangential component of X_u, and an analogous statement holds for $|X_v|^2 - D_v\rho(X)|^2$. Combining (47), (48) and (49), we arrive at

$$|\Psi'(\rho(X))||\Delta\rho(X)| \le \frac{1+\delta}{R}[|\nabla X|^2-|\nabla\rho(X)|^2]. \tag{50}$$

Then (45), (46) and (50) yield

$$-\Delta\eta \le \frac{1+\delta}{R}|\nabla X|^2 \tag{51}$$

on B', and therefore also on B.

Moreover, a straight-forward estimation yields

$$\left|\frac{\partial\eta}{\partial\nu}\right| \le \left\{\left|\frac{\partial\alpha}{\partial\nu}\right|^2 + \left|\frac{\partial}{\partial\nu}\Psi(X)\right|^2\right\}^{1/2} \le \sqrt{\delta^2 + |X_r|^2} = \sqrt{\delta^2 + |X_\theta|^2} \quad \text{on } C$$

and therefore

$$\left|\frac{\partial\eta}{\partial\nu}\right| \le \delta + |X_\theta| \quad \text{on } C. \tag{52}$$

Now choose $\Omega(\varepsilon)$ as in the proof of the special case as

$$\Omega(\varepsilon) = B \setminus [B_{\varepsilon_1}(1) \cup B_{\varepsilon_2}(-1)]$$

with

$$\partial\Omega(\varepsilon) = I(\varepsilon) \cup C(\varepsilon) \cup \gamma_1(\varepsilon) \cup \gamma_2(\varepsilon).$$

Then we obtain

$$\int_{I(\varepsilon)} D_v\eta\, du = -\int_{\Omega(\varepsilon)} \Delta\eta\, du\, dv + \int_{C(\varepsilon)+\gamma_1(\varepsilon)+\gamma_2(\varepsilon)} \frac{\partial}{\partial\nu}\eta\, d\mathcal{H}^1. \tag{53}$$

By (41) and (43) it follows that

$$\begin{aligned} \int_{I(\varepsilon)} |dX| &= \int_{I(\varepsilon)} |X_u|\, du = \int_{I(\varepsilon)} |X_v|\, dv \\ &= \int_{I(\varepsilon)} |D_v\rho(X)|\, du \le \int_{I(\varepsilon)} D_v\eta\, du, \end{aligned} \tag{54}$$

taking $|X_u| = |X_v|$ into account. Thus, by virtue of (51)–(54), we obtain that

$$\begin{aligned} \int_{I(\varepsilon)} |dX| &\le \int_{\Omega(\varepsilon)} (\Delta\eta)\, du\, dv + \int_{C(\varepsilon)+\gamma_1(\varepsilon)+\gamma_2(\varepsilon)} \left|\frac{\partial\eta}{\partial\nu}\right| d\mathcal{H}^1 \\ &\le \frac{1+\delta}{R}\int_{\Omega(\varepsilon)} |\nabla X|^2\, du\, dv + \int_{C(\varepsilon)} \{\delta + |X_\theta|\}\, d\mathcal{H}^1 \\ &\quad + \int_{\gamma_1(\varepsilon)+\gamma_2(\varepsilon)} \frac{\partial\eta}{\partial\nu}\, d\mathcal{H}^1. \end{aligned}$$

Letting first δ and then ε tend to zero, we arrive at

$$\int_I |dX| \le \frac{1}{R}\int_R |\nabla X|^2\, du\, dv + \int_C |dX|,$$

where the integral over $\gamma_1(\varepsilon)$ is dealt with in the same way as in the previous proof for the special case. □

Remark 7. There is no estimate of the form

$$L(\Sigma) \le c_1 L(\Gamma) + c_2 H_0 D(X)$$

or of the form

$$L(\Sigma) \le c_1 L(\Gamma) + c_2 K_0 D(X)$$

with absolute constants c_1 and c_2, where H_0 and K_0 denote upper bounds for $|H|$ and $|K|^{1/2}$, respectively, H and K being the mean curvature and Gauss curvature of S. In fact, the second inequality is ruled out by the cylinder example discussed before, and the first is disproved by a similar example where one replaces the cylinder surface by a suitable catenoid as supporting surface S (see Fig. 4). In other words, it is quite natural that in (25) an upper bound for the two principal curvatures κ_1 and κ_2 of S enters and not an upper bound for the mean curvature H or for the Gauss curvature K.

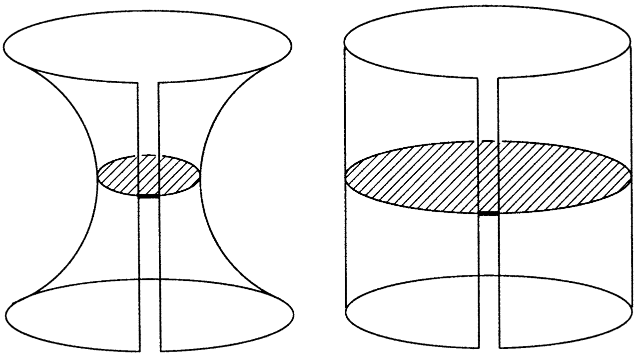

Fig. 4. The examples of Remark 7

Remark 8. Suppose that not all of Γ lies in S. Then, by choosing $\varphi(t)$ in such a way that $\varphi(t) < 1$ for all $t \ne 0$, a close inspection of the proof of Theorem 2 shows that we have in fact the strict inequality

$$L(\Sigma) < L(\Gamma) + \frac{2}{R} D(X) \tag{55}$$

instead of (25).

Remark 9. In addition to the assumptions of Theorem 2 we now assume that $X(B)$ is contained in a ball $\mathcal{K}_{R_0}(P) = \{Q : |P - Q| \le R_0\}$ of $\mathbb{R}^3$. Then the linear isoperimetric inequality of Section 6.3 implies that

$$D(X) \le \frac{R_0}{2} \{L(\Gamma) + L(\Sigma)\}.$$

If $R > R_0$, we infer in conjunction with (25)

$$L(\Sigma) \le \frac{R + R_0}{R - R_0} L(\Gamma). \tag{56}$$

This is an analogue to the inequality (2) in Proposition 1. From the example $S = \partial\mathcal{K}_{R_0}(P)$ we infer that $L(\Sigma)$ can in general not be bounded from above by $L(\Gamma)$. In this case the inequality (56) fails since we have $R = R_0$.

Remark 10. An estimate similar to (25) can be given for stationary minimal surfaces with completely free boundaries. In fact, *suppose that* $X : B \to \mathbb{R}3$ *is a stationary minimal surface in* $\mathcal{C}(S)$ *and assume that* S *satisfies an* R*-sphere condition. Then it follows that the free trace* Σ *of* X *satisfies*

$$L(\Sigma) \le 2R^{-1} D(X). \tag{57}$$

Note that we cannot prove strict inequality as equality holds for the cylinder

$$S = \left\{ (x, y, z) : x^2 + y^2 = 1 \right\},$$

where $R = 1$ and for

$$X(w) = (\mathrm{Re}(w^n), \mathrm{Im}(w^n), 0), \quad w = u + iv,\ n \in \mathbb{N}.$$

Let us conclude this section by a brief discussion of surfaces $X : B \to \mathbb{R}^3$, parametrized over the unit disk which are the class $H_2^1 \cap C^2(B, \mathbb{R}^3)$ and satisfy both

$$\Delta X = 2H X_u \wedge X_v$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

That is, the surface X has constant mean curvature H at all points w where $\nabla X(w) \ne 0$. We shall in the following assume that $X(w) \not\equiv \text{const}$. Then branch points w_0 of X are isolated, and $X_w(w)$ possesses an asymptotic expansion

$$X_w(w) = A(w - w_0)^m + o(|w - w_0|^m) \quad \text{as } w \to w_0,$$

$$A \in \mathbb{C}^3, \quad A \ne 0, \quad \langle A, A \rangle = 0, \quad m \in \mathbb{N},$$

which is completely analogous to asymptotic expansions of minimal surfaces at branch points w_0 derived in Vol. 1, Section 3.2 (see Section 3.1).

Moreover we assume that X is of class $\mathcal{C}(S)$ where the support surface S satisfies an R-sphere condition (cf. Definition 2), and that $S = \partial U$ where U is an open (nonempty) set in $\mathbb{R}^3$.

Finally we suppose that X is of class $C^1(\overline{B}, \mathbb{R}^3)$ and intersects S perpendicularly along its free trace Σ given by $X : \partial B \to \mathbb{R}^3$.

We shall call such surfaces *stationary H-surfaces in* $\mathcal{C}(S)$. Then, by the same computations as in the proof of Theorem 2, we obtain the following *analogue of* (57) *for "stationary H-surfaces in the class* $\mathcal{C}(S)$*"*:

(58) $$L(\Sigma) \le 2(|H|) + R^{-1})D(X).$$

Whenever X satisfies an isoperimetric inequality of the kind

(59) $$D(X) \le cL^2(\Sigma),$$

it follows that

$$L(\Sigma) \le 2(|H| + R^{-1})cL^2(\Sigma)$$

whence

(60) $$L(\Sigma) \ge \frac{1}{2c(|H| + R^{-1})}.$$

In particular, for stationary minimal surfaces in $\mathcal{C}(S)$ we have $H = 0$ and $c = \frac{1}{4\pi}$, whence

(61) $$L(\Sigma) \ge 2\pi R.$$

This is a remarkable *lower bound* for the length of the free trace of a stationary minimal surface in $\mathcal{C}(S)$.

One encounters stationary H-surfaces as solutions of the so-called *partition problem*. Given an open set U in $\mathbb{R}^3$ of finite volume V and with $S = \partial U$, this is the following task:

Among all surfaces Z of prescribed topological type which are contained in $\overline{U}$, have their boundaries on S, and divide U in two disconnected parts U_1 and U_2 of prescribed ratio of volumes, one is to find a surface X which assigns a minimal value or at least a stationary value to its surface area (Dirichlet integral).

One can show[2] that any solution $X : B \to \mathbb{R}^3$ of the partition problem is a surface of constant mean curvature H which is regular up to its free boundary and intersects $S = \partial U$ perpendicularly along $\Sigma = X|_{\partial B}$. That is, *any solution of the partition problem for U is a stationary H-surface in $\mathcal{C}(S)$, $S := \partial U$.*

If $\overline{U}$ is a closed convex body $\mathcal{K}$ whose boundary $S = \partial\mathcal{K}$ satisfies an R-sphere condition, and if R_* is the inradius of $\mathcal{K}$ (i.e., the radius of the largest ball contained in $\mathcal{K}$), then one can also prove the following lower bound for the length $L(\Sigma)$ of the free trace Σ of any stationary H-surface $X : B \to \mathbb{R}^3$ in $\mathcal{C}(S)$ that is parametrized on the unit disk and satisfies $X(B) \subset \mathcal{K}$:

(62) $$L(\Sigma) \ge \frac{2\pi R_*}{1 + (\operatorname{diam}\mathcal{K} - R_*)|H|}.$$

For $H = 0$ this reduces to

(63) $$L(\Sigma) \ge 2\pi R_*.$$

As we have $R_* \ge R$, this inequality is an improvement of (61).

[2] Cf. Grüter-Hildebrandt-Nitsche [2].

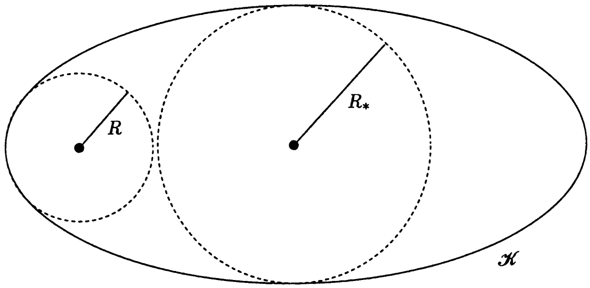

Fig. 5. The inradius R_*, and the smallest curvature radius R

Proof of estimate (62). Set $L := L(\Sigma)$, and define the parameter of the arc length of Σ by

$$s(\theta) := \int_0^\theta |X_\theta(e^{i\theta})|\, d\theta = \int_0^\theta |X_r(e^{i\theta})|\, d\theta$$

($r, \theta =$ polar coordinates about the origin $w = 0$).

Let $\theta(s)$ be the inverse function, $0 \le s \le L$, and introduce the representation

$$Z(s) := X(e^{i\theta(s)}), \quad 0 \le s \le L,$$

of Σ with respect to the parameter s. Moreover let $N_S(P)$ be the exterior unit normal of S at the point $P \in S$. As the H-surface X meets S perpendicularly along Σ, we have

$$X_r(e^{i\theta}) = |X_r(e^{i\theta})| N_S(X(e^{i\theta}))$$

and therefore

$$\int_{\partial B} X_r\, d\theta = \int_\Sigma N_S(Z)\, ds := \int_0^L N_S(Z(s))\, ds. \tag{64}$$

Secondly, a partial integration yields

$$2\int_B X_u \wedge X_v\, du\, dv = \int_{\partial B} X \wedge dX = \int_\Sigma Z \wedge dZ, \tag{65}$$

and another partial integration implies

$$\int_{\partial B} X_r\, d\theta = \int_B \Delta X\, du\, dv.$$

On account of $\Delta X = 2H X_u \wedge X_v$ we thus obtain

$$\int_{\partial B} X_r\, d\theta = 2H \int_B X_u \wedge X_v\, du\, dv. \tag{66}$$

Now we infer from (64)–(66) that

(67) $$\int_\Sigma \{N_S(Z)\,ds - HZ \wedge dZ\} = 0.$$

Set

$$\overline{Z} := \fint_0^L Z(s)\,ds.$$

Then Wirtinger's inequality (Section 6.3, Lemma 2) yields

(68) $$\int_0^L |Z - \overline{Z}|^2\,ds \le \frac{L^3}{4\pi^2}.$$

Let us now introduce the support function $\sigma(P)$ of the convex surface S by

$$\sigma(P) := \langle P, N_S(P)\rangle\,,$$

where we have identified P with the radius vector $\overrightarrow{OP}$ from the origin 0 to the point P. We can assume that 0 is the center of the in-ball $B_{R_*}(0)$ of $\mathcal{K}$. Then we obtain

$$\sigma(P) \ge R_* \quad \text{for all } P \in S.$$

Consequently we have

$$\begin{aligned} R_*L - \int_0^L \left\langle \overline{Z}, N_S(Z)\right\rangle\,ds &\le \int_0^L \langle Z, N_S(Z)\rangle\,ds - \int_0^L \left\langle \overline{Z}, N_S(Z)\right\rangle\,ds \\ &= \int_0^L \left\langle Z - \overline{Z}, N_S(Z)\right\rangle\,ds \\ &\le L^{1/2}\left\{\int_0^L |Z - \overline{Z}|^2\,ds\right\}^{1/2} \le \frac{L^2}{2\pi}, \end{aligned}$$

taking also (68) into account.

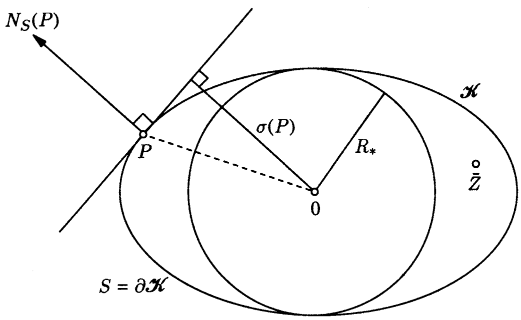

Fig. 6. Concerning the proof of formula (62)

In conjunction with (67) we arrive at

$$
\begin{aligned}
R_* L &= \left\{ R_* L - \int_0^L \langle \overline{Z}, N_S(Z) \rangle \, ds \right\} + \int_0^L \langle \overline{Z}, N_S(Z) \rangle \, ds \\
&\leq \frac{L^2}{2\pi} + H \int_0^L [\overline{Z}, Z, Z'] \, ds.
\end{aligned}
$$

Here $[A_1, A_2, A_3]$ denotes the volume form $\langle A_1, A_2 \wedge A_3 \rangle = \det(A_1, A_2, A_3)$ of three vectors A_1, A_2, A_3 of $\mathbb{R}^3$. Because of the identity

$$
[\overline{Z}, Z, Z'] = [\overline{Z}, Z - \overline{Z}, Z']
$$

we arrive at

$$
\begin{aligned}
R_* L &\leq \frac{L^2}{2\pi} + H \int_0^L [\overline{Z}, Z - \overline{Z}, Z'] \, ds \\
&\leq \frac{L^2}{2\pi} + |H| \int_0^L |\overline{Z}| |Z - \overline{Z}| |Z'| \, ds \\
&\leq \frac{L^2}{2\pi} + |H| |\overline{Z}| \sqrt{L} \left\{ \int_0^L |Z - \overline{Z}|^2 \, ds \right\}^{1/2} \\
&\leq \frac{L^2}{2\pi} (1 + |H \overline{Z}|).
\end{aligned}
$$

Moreover, an elementary estimation yields

$$
|\overline{Z}| \leq \operatorname{diam} \mathcal{K} - R_*,
$$

and therefore

$$
R_* L \leq \frac{1}{2\pi} \{1 + (\operatorname{diam} \mathcal{K} - R_*) |H|\} \, L^2.
$$

Now (62) is an obvious consequence of this inequality. □

Let us conclude this section with the remark that equality in (63) implies that X is a disk.

4.7 Obstacle Problems and Existence Results for Surfaces of Prescribed Mean Curvature

In this section we treat obstacle problems, that is, we look for surfaces of minimal area (or minimal Dirichlet integral) which are spanning a prescribed closed boundary curve Γ and avoid certain open sets (the "obstacles"). This means that the competing surfaces of the variational problem are confined to some closed set $\mathcal{K}$ which is a subset of $\mathbb{R}^3$ or, more generally a subset of a three dimensional manifold M. In Chapter 4 of Vol. 1 we have very thoroughly described the minimization procedure which leads to a solution of Plateau's problem for minimal surfaces. In addition we have outlined the extension of this argument to a more general variational integral, see Theorem in No. 6 of the Scholia to that chapter. Therefore we refrain from repeating the procedure here and refer to Chapter 4 of Vol. 1 as well as to the pertinent literature cited therein. Instead we focus on higher regularity results for obstacle problems. Note that the optimal regularity which can be expected is $C^{1,1}$-regularity of a solution. Indeed, this can already be seen by considering a thread of minimal length which is spanned between two fixed points and touches an (analytic) obstacle in a whole interval.

In a first step we prove Hölder continuity of any solution, and later in Theorem 6 we use a difference quotient technique to show $H^2_{s,\mathrm{loc}}$-regularity for any solution of the variational problem. By standard Sobolev imbedding results this implies the Hölder continuity of the first derivatives.

We also study the Plateau problem for surfaces of prescribed mean curvature in Euclidean space $\mathbb{R}^3$. Here one prescribes a real valued function H on $\mathbb{R}^3$ and asks for a surface X which is bounded by a given closed Jordan curve Γ and has prescribed mean curvature $H(X(u,v))$ at a particular point $X(u,v)$. Clearly, if $H \equiv 0$, we recover the classical Plateau problem for minimal surfaces. In this section we discuss some classical existence and also non-existence results for the general Plateau problem described above.

Set

$$B = \{w \in \mathbb{C} : |w| < 1\} \quad \text{and} \quad C := \{w \in \mathbb{C} : |w| = 1\} = \partial B$$

and let Γ denote a closed Jordan curve in $\mathbb{R}^3$ i.e. a topological image of C. Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a given function which is bounded and continuous.

Definition 1. *Given a closed Jordan curve Γ in $\mathbb{R}^3$ and a bounded continuous function $H : \mathbb{R}^3 \to \mathbb{R}$. We say that $X : \overline{B} \to \mathbb{R}^3$ is a solution of Plateau's problem determined by Γ and H (in short: an "H-surface spanned by Γ") if it fulfills the following three conditions:*

(i) $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$.

(ii) *X satisfies in B the equations*

$$\triangle X = 2H(X(u,v)) X_u \wedge X_v \tag{1}$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{2}$$

(iii) *The restriction $X_{|C}$ of X to the boundary C of the parameter domain B is a homeomorphism of C onto Γ.*

It follows from Chapter 2.5 of Vol. 1 that every H-surface X spanned by Γ has mean curvature $H = H(X(u,v))$ at each *regular* point $(u,v) \in B$.

Since, for $H \equiv 0$, each H-surface with boundary Γ provides a solution to the classical Plateau problem for minimal surfaces, it is conceivable that a similar variational approach using a more general energy functional instead of Dirichlet's integral might be successful.

Before we define a suitable energy functional we remark at the outset that Plateau's problem can certainly not be solvable for arbitrary Γ and H, in other words there are necessary conditions for existence.

To see this let us suppose that $X \in C^2(\overline{B}, \mathbb{R}^3)$ is a solution of (ii) and (iii) with $H \equiv \text{const}$.

Then, by integrating (ii) we obtain

$$\int_B \triangle X \, du \, dv = \int_B \operatorname{div} \nabla X \, du \, dv = 2H \int_B (X_u \wedge X_v) \, du \, dv$$

and Gauß' and Green's theorem yield

$$\begin{aligned}
&\int_{\partial B} \nabla X \cdot n \, ds \\
&= 2H \int_B \begin{bmatrix} y_u z_v - z_u y_v \\ -x_u z_v + x_v z_u \\ x_u y_v - x_v y_u \end{bmatrix} du \, dv \\
&= H \int_B \begin{bmatrix} (y z_v)_u - (z_u y)_v \\ (z x_v)_u - (x_u z)_v \\ (x y_v)_u - (x y_u)_v \end{bmatrix} du \, dv + H \int_B \begin{bmatrix} (z y_u)_v - (z y_v)_u \\ (x z_u)_v - (x z_v)_u \\ (y x_u)_v - (y x_v)_u \end{bmatrix} du \, dv \\
&= H \int_{\partial B} \begin{bmatrix} y z_u \, du + y z_v \, dv \\ x_u z \, du + z x_v \, dv \\ x y_u \, du + x y_v \, dv \end{bmatrix} + H \int_{\partial B} \begin{bmatrix} -z y_u \, du - z y_v \, dv \\ -x z_u \, du - x z_v \, dv \\ -y x_u \, du - y x_v \, dv \end{bmatrix}.
\end{aligned}$$

On the other hand we have

$$X \wedge X_u = \begin{pmatrix} y z_u - z y_u \\ -x z_u + z x_u \\ x y_u - y x_u \end{pmatrix} \quad \text{and} \quad X \wedge X_v = \begin{pmatrix} y z_v - z y_v \\ -x z_v + z x_v \\ x y_v - y x_v \end{pmatrix}$$

and therefore

$$\begin{aligned}
\int_{\partial B} \frac{\partial X}{\partial r} \, ds &= H \int_{\partial B} (X \wedge X_u) \, du + H \int_{\partial B} (X \wedge X_v) \, dv \\
&= H \int_{\partial B} X \wedge dX.
\end{aligned}$$

In particular this implies the relation

$$|H| \left| \int_{\partial B} X \wedge dX \right| = \left| \int_{\partial B} \frac{\partial X}{\partial r} \, ds \right|$$

from which we conclude the necessary condition

$$\begin{aligned}
|H| \left| \int_{\partial B} X \wedge dX \right| \leq \int_{\partial B} \left| \frac{\partial X}{\partial r} \right| ds = \int_{\partial B} |X_\theta(1, \theta)| \, d\theta = L(\Gamma) \\
= \text{length of the curve } \Gamma.
\end{aligned}$$

(Note that here we have used the conformality relation (ii).)

Putting $k(\Gamma) := |\int_{\partial B} X \wedge dX|$ we obtain the following *necessary condition of Heinz [12]* which we formulate as a nonexistence result.

Theorem 1. *Suppose $k(\Gamma) > 0$. Then there is no solution $X \in C^2(\overline{B}, \mathbb{R}^3)$ of Plateau's problem determined by Γ and $H \equiv$ const, if*

$$|H| > \frac{L(\Gamma)}{k(\Gamma)}.$$

This theorem also holds for solutions $X \in C^2(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ as was proved by Heinz [12] using an appropriate approximation procedure.

Example. Let Γ be a circle of radius R,

$$\Gamma = \{(R\cos\theta, R\sin\theta, 0) \in \mathbb{R}^3 : \theta \in [0, 2\pi)\}.$$

Then

$$k(\Gamma) = \left|\int_{\partial B} X \wedge dX\right| = \left|\int_{\partial B} X \wedge X_\theta d\theta\right| = \left|\int_{\partial B} R^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d\theta\right| = 2\pi R^2.$$

Hence there is no solution of Plateau's problem for a circle of radius R and constant mean curvature H if

$$|H| > \frac{2\pi R}{2\pi R^2} = \frac{1}{R}.$$

Also, if Γ is "close to" a circle of radius R, we cannot expect the existence of an H-surface bounded by Γ and constant H bigger than $\frac{1}{R}$. We will see later on in this section, that this conditions is sharp.

Recall now that every minimizer X of the Dirichlet integral within the class $\mathcal{C}(\Gamma)$ is harmonic in B.

Furthermore we have seen in Theorem 1 of Section 4.5 in Vol. 1 that the conformality conditions (2) hold if the first inner variation $\partial D(X, \lambda)$ vanishes for all vector fields $\lambda \in C^1(\overline{B}, \mathbb{R}^3)$ (which is the case for a minimizer of $D(\cdot)$). As a suitable energy functional to be considered one might therefore try an integral $\mathcal{F}$ of the type

$$\mathcal{F}(X) = D(X) + V(X)$$

consisting of the Dirichlet integral and a "volume" term

$$V(X) := \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv,$$

where $Q = (Q^1, Q^2, Q^3)$ denotes a C^1-vector field defined on $\mathbb{R}^3$ or a subset $\mathcal{K}$ of $\mathbb{R}^3$. Since $V(\cdot)$ is invariant with respect to all orientation preserving C^1-diffeomorphisms of $\overline{B}$ this term would not alter the conformality of minimizers.

Note also that $V = V(X)$ equals the algebraic volume enclosed by the surface X and the cone over the boundary Γ weighted by the factor div Q, as follows easily by applying Gauß's theorem.

We observe that the Euler equation for the functional $\mathcal{F}$ is given by the system

$$\triangle X^\ell = \operatorname{div} Q(X)(X_u \wedge X_v)_\ell \tag{3}$$

for $\ell = 1, 2, 3$. (Compare Vol. 1, Section 4.5; here we have put $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$).

If in addition to the first outer variation also the first inner variation $\partial\mathcal{F}(X,\lambda) = \partial D(X,\lambda)$ vanishes for all C^1-vector fields $\lambda = (\mu,\nu)$, then it follows that the conformality condition

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0 \tag{4}$$

hold true (almost every where) in B.

Theorem 1 of Vol. 1, Section 2.6 now states that a solution X of (3) which satisfies (4) has mean curvature

$$H(X) = \frac{1}{2}\operatorname{div} Q(X)$$

at each regular point $(u,v) \in B$ of X.

We are thus led to consider the "energy" functional

$$\mathcal{F}(X) = \frac{1}{2}\int_B |\nabla X|^2\,du\,dv + \int_B \langle Q(X), X_u \wedge X_v\rangle\,du\,dv,$$

where the vector field Q is of class $C^1(\mathbb{R}^3,\mathbb{R}^3)$ or $C^1(\mathcal{K},\mathbb{R}^3), \mathcal{K} \subset \mathbb{R}^3$, and has to be determined such that

$$\operatorname{div} Q(x) = 2H(x) \tag{5}$$

for all $x \in \mathbb{R}^3$ or $\mathcal{K}$ respectively.

In addition $\mathcal{F}(\cdot)$ has to be coercive on the set of admissible functions, i.e. there are positive numbers $m_0 \le m_1$ so that

$$m_0 D(X) \le \mathcal{F}(X) \le m_1 D(X) \tag{6}$$

holds for every admissible X.

The Lagrangian $e \equiv e(x,p_1,p_2)$ of $\mathcal{F}$ is given by

$$e(x,p_1,p_2) = \frac{1}{2}(|p_1|^2 + |p_2|^2) + \langle Q(x), p_1 \wedge p_2\rangle,$$

where $x \in \mathbb{R}^3$ or $\mathcal{K}$ and $p_1, p_2 \in \mathbb{R}^3$.

Assuming that

$$\sup_{\mathcal{K}} |Q| = |Q|_{0,\mathcal{K}} < 1 \tag{7}$$

we immediately conclude coerciveness of $\mathcal{F}(\cdot)$ since we obtain from Schwarz's inequality

$$\frac{1}{2}(1-|Q|_{0,\mathcal{K}})(|p_1|^2+|p_2|^2) \le e(x,p_1,p_2) \le \frac{1}{2}(1+|Q|_{0,\mathcal{K}})(|p_1|^2+|p_2|^2),$$

that is (6) follows with constants

$$m_0 := (1-|Q|_{0,\mathcal{K}}) > 0 \quad \text{and} \quad m_1 := (1+|Q|_{0,\mathcal{K}}).$$

In order to avoid additional difficulties which arise from the discussion of an obstacle problem it would be desirable to construct a vector field Q of class C^1 which is defined on $\mathcal{K} = \mathbb{R}^3$ and is subject to (5) and (7). However, even in the case $H = \text{const}$, a quick inspection of equation (5), using Gauß's theorem, shows that the quantity $|Q|_{0,\partial B_R}$ has to grow linearly in the radius R; in other words (7) can not hold for $\mathcal{K} = \mathbb{R}^3$, even if $H = \text{const}$.

Hence we consider the following strategy:

I) **The vector field Q:**
For given Γ and H satisfying conditions to be determined later, find a closed set $\mathcal{K} \subset \mathbb{R}^3$ such that $\Gamma \subset \mathcal{K}$ together with a vector field $Q \in C^1(\mathcal{K}, \mathbb{R}^3)$ which fulfills the conditions (5) and (7).

II) **The obstacle problem:**
Define the set of admissible functions $\mathcal{C} = \mathcal{C}(\Gamma, \mathcal{K}) := \mathcal{C}^*(\Gamma) \cap H_2^1(B, \mathcal{K})$, where $\mathcal{C}^*(\Gamma)$ denotes the class of H_2^1-surfaces spanning Γ which are normalized by a three point condition, and $H_2^1(B, \mathcal{K})$ denotes the subset of all Sobolev functions $f \in H_2^1(B, \mathbb{R}^3)$ which map almost all of B into $\mathcal{K}$. Solve the obstacle problem

$$\mathcal{P}(\Gamma, \mathcal{K}): \qquad \mathcal{F}(\cdot) \to \min \quad \text{in } \mathcal{C}(\Gamma, \mathcal{K})$$

and establish some initial regularity of the solutions assuming appropriate regularity hypotheses on $\mathcal{K}$. Instead of a variational equality $\delta\mathcal{F} = 0$, a solution X of $\mathcal{P}(\Gamma, \mathcal{K})$ in general merely satisfies a variational inequality $\delta\mathcal{F} \ge 0$. Therefore we have to apply a suitable inclusion principle.

III) **Geometric maximum principle:**
Determine conditions on H and $\mathcal{K}$ (or $\partial\mathcal{K}$ respectively) which guarantee that the "coincidence" set

$$\mathcal{T} := \{w \in B : X(w) \in \partial\mathcal{K}\}$$

is empty for a minimizer or a stationary point X of $\mathcal{F}$ in $\mathcal{C}$. In this case X maps B into the interior of $\mathcal{K}$ and hence satisfies the Euler-equation $\delta\mathcal{F} = 0$ in a weak sense. We refer to the Enclosure Theorems 2 and 3 in Section 4.4 for the pertinent results; however note that more elementary arguments suffice, when $\mathcal{K}$ is a ball or a cylinder.

IV) **Regularity:**
Show that under natural assumption on H (and Γ) a minimizer of $\mathcal{F}$ in $\mathcal{C}$ is a classical $C^{2,\alpha}$ solutions of the H-surface system (1) and (2). Note that the conformality conditions (2) are automatically satisfied, compare the discussion in Vol. 1, Section 4.5, and in No. 6 of the Scholia to Chapter 4 of Vol. 1.

Ad I) **Construction of the vector field** Q

The construction device requires $Q \in C^1(\mathcal{K}, \mathbb{R}^3)$ with the properties

$$\operatorname{div} Q(x) = 2H(x) \quad \text{for all } x \in \mathcal{K}$$

and some given $H \in C^0(\mathcal{K}, \mathbb{R})$ and, in addition,

$$|Q|_{0,\mathcal{K}} < 1, \quad \text{see (5) and (7).}$$

The simplest situation occurs, when $\mathcal{K} = \overline{B}_R(0) \subset \mathbb{R}^3$ and $H \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. The vectorfield

$$(8) \quad Q(x) := \frac{2}{3}\left(\int_0^{x^1} H(\tau, x^2, x^3)\, d\tau, \int_0^{x^2} H(x^1, \tau, x^3)\, d\tau, \int_0^{x^3} H(x^1, x^2, \tau)\, d\tau\right)$$

clearly is of class $C^1(\mathbb{R}^3, \mathbb{R}^3)$ and satisfies (5) on $\mathbb{R}^3$ (and in particular on $\mathcal{K}$). Also

$$|Q(x)| \le \frac{2}{3}|x|\, |H|_{0,\mathcal{K}} \quad \text{for all } x \in \mathcal{K},$$

whence $|Q|_{0,\mathcal{K}} \le \frac{2}{3} R |H|_{0,\mathcal{K}}$; therefore $\mathcal{F}(\cdot)$ is coercive, if we take $\mathcal{K} = B_R(0)$ and

$$(9) \qquad |H|_{0,B_R(0)} < \frac{3}{2} R^{-1}.$$

Now let $\mathcal{K} = Z_R(0)$ be the cylinder

$$Z_R := \{(x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 \le R^2\}$$

and $H \in C^1(\mathbb{R}^3)$. Instead of (8) we put

$$(10) \qquad Q(x) := \left(\int_0^{x^1} H(\tau, x^2, x^3)\, d\tau, \int_0^{x^2} H(x^1, \tau, x^3)\, d\tau, 0\right),$$

which is again of class $C^1(\mathbb{R}^3, \mathbb{R}^3)$ and fulfills relation (5) for all $x \in \mathbb{R}^3$. Furthermore

$$|Q(x)| \leq |H|_{0,\mathcal{K}}((x^1)^2 + (x^2)^2)^{1/2} \quad \text{for } x \in \mathbb{R}^3,$$

that is

$$|Q|_{0,\mathcal{K}} \leq R \cdot |H|_{0,\mathcal{K}}.$$

In particular $\mathcal{F}(\cdot)$ is coercive if $\mathcal{K} = Z_R(0)$ and

$$|H|_{0,Z_R} < \frac{1}{R}. \tag{11}$$

Finally suppose $\mathcal{K} \subset S_R$ is a slab of width $2R$,

$$S_R = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : -R \leq x_3 \leq R\}.$$

Putting

$$Q(x) := 2\left(0, 0, \int_0^{x^3} H(x^1, x^2, \tau)\, d\tau\right)$$

we then have

$$\operatorname{div} Q(x) = 2H(x) \quad \text{in } S_R$$

and

$$|Q(x)| \leq 2|H(x)| \cdot |x_3|.$$

Therefore $\mathcal{F}(\cdot)$ is coercive in this case if $\mathcal{K} \subset S_R$ and

$$|H|_{0,S_R} < \frac{1}{2R}.$$

The situation for general $\mathcal{K} \subset \mathbb{R}^3$ is more involved, although the essential idea is fairly simply, namely to consider a Dirichlet problem for the nonparametric mean curvature equation in $\mathcal{K}$. To this end suppose that $u = u(x^1, x^2, x^3) \in C^1(\mathcal{K}, \mathbb{R})$ solves the mean curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 2H \quad \text{in } \mathcal{K} \tag{12}$$

then the vector field

$$Q(x) := \frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}$$

certainly satisfies (5) and also (7) $|Q|_{0,\mathcal{K}} < 1$ holds, provided u has globally bounded gradient on $\mathcal{K}$.

For a bounded set $\mathcal{K} \subset \mathbb{R}^3$ with boundary $\partial\mathcal{K} \in C^2$ and for constant H the equation (12) with boundary condition $u = 0$ on $\partial\mathcal{K}$ is uniquely solvable with $u \in C^{2,\alpha}(\overline{\mathcal{K}})$ if and only if the inward mean curvature Λ of $\partial\mathcal{K}$ satisfies

$$(13) \qquad |H| \le \Lambda \quad \text{along } \partial\mathcal{K},$$

for a proof of this result see e.g. Gilbarg and Trudinger [1] Theorem 16.11, or Serrin [4].

To describe the condition on $\mathcal{K}$ and Λ in the case of variable H we let $\rho(x) := \text{dist}\,(x, \partial\mathcal{K})$ denote the distance of $x \in \mathcal{K}$ to the boundary $\partial\mathcal{K}$ of $\mathcal{K}$, cp. the discussion of the distance function in Section 4.4. Furthermore we extend the mean curvature function Λ from $\partial\mathcal{K}$ to $\mathcal{K}$ by putting

$$\Lambda(x) = \Lambda_{\rho(x)}(x)$$

to equal the mean curvature at x of the local surface $\mathcal{S}_{\rho(x)}$ through x which is parallel to $\partial\mathcal{K}$ at distance $\rho(x)$ in case this surface exists and is of class C^2. Otherwise we let $\Lambda_{\rho(x)}(x) = +\infty$. Condition (13) may now be replaced by

$$(14) \qquad |H(x)| \le (1 - a\rho(x))\Lambda_{\rho(x)}(x) + \frac{a}{2}$$

for $x \in \mathcal{K}$, where a denotes some number with $0 \le a \le \inf_{x\in\mathcal{K}} \rho^{-1}(x)$.

Theorem 2. *Suppose $\mathcal{K} \subset \mathbb{R}^3$ is the closure of a C^2 domain whose boundary $\partial\mathcal{K}$ has uniformly bounded principal curvatures and a global inward parallel surface at distance $\varepsilon > 0$. In addition assume that $\sup_{\mathcal{K}} \rho(x) < \infty$ and let $H \in C^1(\mathcal{K}, \mathbb{R})$ have uniformly bounded C^1-norm on $\mathcal{K}$ with (13) and (14) being fulfilled for some a, $0 \le a \le \inf_{\mathcal{K}} \rho^{-1}(x)$. Then there exists a solution $u \in C^2(\mathcal{K})$ of equation (12) with uniformly bounded gradient on $\mathcal{K}$. In particular there exists a C^1-vector field Q satisfying (5) and (7).*

The proof of Theorem 2 in case of bounded domains is due to Serrin [4]; the generalization to unbounded $\mathcal{K}$ can be found in Gulliver and Spruck [2].

Ad II) **The obstacle problem**

Let $\Gamma \in \mathbb{R}^3$ be a closed Jordan curve and $\mathcal{K} \subset \mathbb{R}^3$ a closed set which contains Γ. Also put

$$\mathcal{C} = \mathcal{C}(\Gamma, \mathcal{K}) = \mathcal{C}^*(\Gamma) \cap H_2^1(B, \mathcal{K})$$

to denote the class of $H_2^1(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$-surfaces which map ∂B weakly monotonic onto Γ, satisfy a three point condition and have an image almost everywhere in $\mathcal{K}$.

Since, in Section 4.8, we study surfaces of prescribed mean curvature in a Riemannian three-manifold we consider now somewhat more generally functionals $\mathcal{F}(\cdot)$ which are the sum of a Riemannian Dirichlet integral and a suitable volume term.

Put $\mathcal{F}(X) = E(X) + V(X)$, where

$$E(X) := \frac{1}{2} \int_B g_{ij}(X)(X_u^i X_u^j + X_v^i X_v^j)\, du\, dv$$

and

$$V(X) := \int_B \langle Q(X), X_u \wedge X_v \rangle\, du\, dv,$$

that is

$$\mathcal{F}(X) = \int_B e(X, \nabla X)\, du\, dv$$

with the Lagrangian

$$e(x,p) = \frac{1}{2} g_{ij}(x)(p_1^i p_1^j + p_2^i p_2^j) + \langle Q(x), p_1 \wedge p_2 \rangle,$$

where $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and $p = (p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$. In No. 6 of the Scholia to Vol. 1, Chapter 4, we have outlined the proof of the following

Theorem 3. *Suppose* $Q \in C^0(\mathcal{K}, \mathbb{R}^3), g_{ij} \in C^0(\mathcal{K})$ $g_{ij} = g_{ji}$ *for all* $i, j = 1, 2, 3$, *and let* $0 < m_0 \le m_1$ *be constants with the property* $m_0(|p_1|^2 + |p_2|^2) \le e(x,p) \le m_1(|p_1|^2 + |p_2|^2)$ *for all* $(x, p_1, p_2) \in \mathcal{K} \times \mathbb{R}^3 \times \mathbb{R}^3$. *Moreover assume that* $\mathcal{K}$ *is a closed set in* $\mathbb{R}^3$ *such that* $\mathcal{C} = \mathcal{C}(\Gamma, \mathcal{K})$ *is nonempty. Then the variational problem*

$$\mathcal{P} = \mathcal{P}(\Gamma, \mathcal{K}) : \qquad \mathcal{F} \to \min \quad \text{in } \mathcal{C}$$

has a solution. Every solution $X \in \mathcal{C}$ *satisfies the conformality relations*

$$g_{ij} X_u^i X_u^j = g_{ij} X_v^i X_v^j \quad \text{and} \quad g_{ij} X_u^i X_v^j = 0 \tag{15}$$

almost everywhere in B. □

In order to obtain continuity for solutions of $\mathcal{P}$ we have to assume more regularity of $\mathcal{K}$ or $\partial\mathcal{K}$ respectively. A reasonable quantitative notion is the “quasiregularity” of $\mathcal{K}$.

Definition 2. *A closed set* $\mathcal{K} \subset \mathbb{R}^3$ *is called “quasiregular”, if*

(a) $\mathcal{K}$ *is equal to the closure of its interior* $\mathring{\mathcal{K}}$;
(b) *there are positive numbers* d *and* M *such that for each point* $x_0 \in \mathcal{K}$ *there exists a compact convex set* K^* $(\mathring{K}^* \neq \emptyset)$ *and a* C^1*-diffeomorphism* g *defined on some open neighbourhood of* K^* *with* $g : K^* \to \mathcal{K} \cap \overline{B_d(x_0)}$ *with*

$$|Dg|^2_{0,K^*} \le M \quad \text{and} \quad |Dg^{-1}|^2_{0,\mathcal{K}\cap\overline{B_d}(x_0)} \le M.$$

Remarks. (i) Closed convex sets $\mathcal{K}$ with $\mathcal{K} = \overline{\overset{\circ}{\mathcal{K}}}$ are quasiregular.

(ii) If $\mathcal{K} = \overline{\overset{\circ}{\mathcal{K}}} \subset \mathbb{R}^3$ is compact with $\partial\mathcal{K} \in C^1$, then $\mathcal{K}$ is quasiregular.

(iii) Suppose $\mathcal{K} = \overline{\overset{\circ}{\mathcal{K}}}$, $\partial\mathcal{K} \in C^2$ and $\partial\mathcal{K}$ has uniformly bounded principal curvatures and a global parallel surface in $\overset{\circ}{\mathcal{K}}$, then $\mathcal{K}$ is quasiregular; for a proof see Gulliver and Spruck [2].

Theorem 4. *Let the assumption of Theorem 3 be satisfied and suppose that $\mathcal{K} \subset \mathbb{R}^3$ is quasiregular. Furthermore let X be a solution of the problem*

$$\mathcal{P}(\Gamma, \mathcal{K}): \qquad \mathcal{F} \to \min \ \text{ in } \mathcal{C}(\Gamma, \mathcal{K}).$$

Then there is a number $\mu > 0$ such that

$$\int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \le \left(\frac{r}{R}\right)^{2\mu} \int_{B_R(w_0)} |\nabla X|^2 \, du \, dv \tag{16}$$

for all $r \in (0, R]$ and $w_0 \in B$ with $0 < R \le \operatorname{dist}(w_0, \partial B)$. It follows that X is of class $C^{0,\mu}(B, \mathbb{R}^3)$. Furthermore X is continuous up to the boundary.

Proof. Let X be a minimizer of the functional $\mathcal{F}$ in $\mathcal{C}$. For an arbitrary point $w_0 \in B$ we define

$$\phi(r) = \phi(r, w_0) = \int_{B_r(w_0)} |\nabla X|^2 \, du \, dv,$$

where $0 < r \le R = \operatorname{dist}(w_0, \partial B)$.

Introducing polar coordinates (ρ, θ) around w_0 by $w = w_0 + \rho e^{i\theta}$ and writing (with a slight but convenient abuse of notation) $X(w) = X(w_0 + \rho e^{i\theta}) = X(\rho, \theta)$, we get

$$\phi(r) = \int_0^r \int_0^{2\pi} \left\{ |X_\rho|^2 + \frac{1}{\rho^2} |X_\theta|^2 \right\} \rho \, d\rho \, d\theta.$$

Furthermore, by selecting an ACM-representative of X again denoted by X, we can assume that for almost all $\theta \in [0, 2\pi]$ the restriction $X(\cdot, \theta)$ is absolutely continuous in $\rho \in [\varepsilon, R]$, $\varepsilon > 0$, and $X(\rho, \cdot)$ is absolutely continuous in $\theta \in [0, 2\pi]$ for almost all $\rho \in [0, R]$.

There is a Lebesgue null set $\mathcal{N} \subset [0, R]$ such that for $r \in [0, R] \setminus \mathcal{N}$ we have

(i) $X(r, \cdot)$ is absolutely continuous on $[0, 2\pi]$,

(ii) $\int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta < \infty$,

(iii) $\phi(r)$ is differentiable with

$$\phi'(r) = \int_0^{2\pi} \left\{ |X_\rho(r, \theta)|^2 + \frac{1}{r^2} |X_\theta(r, \theta)|^2 \right\} r \, d\theta \geq \frac{1}{r} \int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta,$$

i.e.

$$\int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta \leq r \cdot \phi'(r) \quad \text{for all } r \in [0, R]. \tag{17}$$

Take a radius $r \in [0, R] \setminus \mathcal{N}$ for which

$$\int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta < \frac{\pi^{-1}}{2} d^2, \tag{18}$$

where d denotes the constant in the definition of quasiregularity. Then for any $\theta_0, \theta_1 \in [0, 2\pi]$ we infer the estimate

$$|X(r, \theta_1) - X(r, \theta_0)| \leq \left| \int_{\theta_1}^{\theta_2} |X_\theta(r, \theta)| \, d\theta \right| \leq \sqrt{2\pi} \left\{ \int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta \right\}^{\frac{1}{2}} < d$$

and hence the image of the curve $X(r, \cdot)$ is contained in $\mathcal{K} \cap \overline{B_d(x_0)}$, where $x_0 = X(r, \theta_0)$ is an arbitrary point on that curve. According to the definition of quasiregularity there is a C^1-diffeomorphism $h = g^{-1} : \mathcal{K} \cap \overline{B_d(x_0)} \to K^*$, where K^* is a compact and convex set. Hence the curve $\zeta(\theta) := h(X(r, \theta))$ is of class $H_2^1([0, 2\pi], \mathbb{R}^3)$ with values in the convex set K^*. Now let $H = H(w)$ denote the harmonic vector function defined in $B_r(w_0)$ whose boundary values are given by $\zeta(\theta)$, i.e.

$$H(w_0 + re^{i\theta}) = \zeta(\theta) = h(X(r, \theta))$$

for $0 \leq \theta \leq 2\pi$. By the maximum principle and the convexity of K^* it follows that the image $H(B_r(w_0)) \subset K^*$ and therefore the function $g \circ H \in H_2^1(B_r(w_0), \mathcal{K})$ with boundary trace $X(r, \theta)$. Setting

$$Y(w) := \begin{cases} g \circ H(w) & \text{for } w \in B_r(w_0), \\ X(w) & \text{for } w \in B \setminus B_r(w_0) \end{cases}$$

we therefore obtain a function $Y \in \mathcal{C}(\Gamma, \mathcal{K})$. Since X is a minimizer of $\mathcal{F}$ in $\mathcal{C} = \mathcal{C}(\Gamma, \mathcal{K})$ we have

$$\mathcal{F}(X) \leq \mathcal{F}(Y)$$

and by the coercivity assumption and the quasiregularity of $\mathcal{K}$ it follows

$$\begin{aligned} m_0 \int_{B_r(w_0)} |\nabla X|^2 \, du \, dv &\le m_1 \int_{B_r(w_0)} |\nabla Y|^2 \, du \, dv \\ &= m_1 \int_{B_r(w_0)} |\nabla (g \circ H)|^2 \, du \, dv \le m_1 M \int_{B_r(w_0)} |\nabla H|^2 \, du \, dv, \end{aligned}$$

that is

$$\phi(r) \le \frac{m_1}{m_0} M \int_{B_r(w_0)} |\nabla H|^2 \, du \, dv. \tag{19}$$

On the other hand an expansion of ζ and H in Fourier series yields

$$\zeta(\theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta)),$$

and

$$H(w) = A_0 + \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n [A_n \cos(n\theta) + B_n \sin(n\theta)],$$

which yields

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv = \pi \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2),$$

and

$$\int_0^{2\pi} |\zeta_\theta|^2 \, d\theta = \pi \sum_{n=1}^{\infty} n^2(|A_n|^2 + |B_n|^2).$$

In particular we have

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv \le \int_0^{2\pi} |\zeta_\theta|^2 \, d\theta. \tag{20}$$

But from $\zeta(\theta) = h(X(r, \theta))$ we obtain, using the quasiregularity of $\mathcal{K}$ again,

$$\int_0^{2\pi} |\zeta_\theta|^2 \, d\theta \le M \int_0^{2\pi} |X_\theta|^2 \, d\theta. \tag{21}$$

Relations (19), (20), (21) and (17) now yield the estimate

$$\phi(r) \le \frac{m_1}{m_0} M^2 \int_0^{2\pi} |X_\theta|^2 \, d\theta \le \frac{m_1}{m_0} M^2 r \phi'(r)$$

for almost every $r \in [0, R]$.

On the other hand, if (18) does not hold, then we trivially have

$$\phi(r) \le \phi(R) \le \phi(R) \cdot \frac{2\pi}{d^2} \int_0^{2\pi} |X_\theta|^2 \, d\theta \le \frac{2\pi}{d^2} D(X) r \cdot \phi'(r),$$

again by using (17). Concluding we obtain in both cases the inequality $\phi(r) \le C \cdot r\phi'(r)$, where we have put $C := \max(2\pi d^{-2} D(X), \frac{m_1}{m_0} M^2)$. From this inequality we finally obtain by a simple integration

$$\phi(r) \le \left(\frac{r}{R}\right)^{2\mu} \phi(R)$$

for all $r \in [0, R]$ and $\mu := \frac{1}{2C}$.

Now $X \in C^{0,\mu}(B)$ follows from Dirichlet's growth theorem, see e.g. Gilbarg and Trudinger [1], Theorem 7.19.

To prove continuity of X up to the boundary, we apply a conformal mapping τ which maps the unit disk onto the upper half plane and the unit circle onto the real axis. Since τ maps circles onto circles, leaves the Dirichlet integral invariant and is locally bi-Lipschitz, it follows that $X \circ \tau^{-1}$ satisfies again condition (16) in a neighbourhood of any boundary point of the half plane, possibly with an additional constant factor K on the right hand side. In addition we may choose τ in such a way that an arbitrary but fixed point $e^{i\Theta}$ is mapped onto the origin. We are thus led to consider the following situation: Let Ω be the rectangle $\{w = u + iv \in \mathbb{C} : |u| < 2, 0 < v < 2\}$ and suppose $X \in H_2^1(\Omega, \mathbb{R}^3)$ possesses continuous boundary trace $\xi(u) = X(u, 0), u \in (-2, 2)$. Then we have to show that $X(w) \to \xi(0)$ as $w \to 0$. To this end we introduce the entities

$$\varepsilon(X, u, h) := \left(\int_{u-2h}^{u+2h} \int_0^{2h} |\nabla X|^2 \, du \, dv \right)^{\frac{1}{2}},$$
$$\omega(\xi, h) := \sup_{|u'-u''| \le h} |\xi(u') - \xi(u'')|$$

and let $w = u + ih$ be an arbitrary point with $|u| < 1, 0 < h < \frac{1}{2}$. Recalling Morrey's proof of Dirichlet's growth theorem (see Morrey [8], Theorem 3.5.2) we obtain by virtue of condition (16) the estimate

$$|X(u, h) - X(u', h)| \le C_0 k \varepsilon(X, u, h) |u - u'|^\mu h^{-\mu}$$

for all u' with $|u - u'| \le h < \frac{1}{2}$ with some constant c_0 depending only on μ.

Next we select a $u_1 \in [-1, 1]$, $|u - u_1| < h$ with the properties

(i) $X(u_1, \cdot) \in H_2^1([0, 2], \mathbb{R}^3)$,
(ii) $X(u_1, v) \to \xi(u_1)$ as $v \to 0^+$,
(iii) $\varepsilon^2(X, u, h) = \int_{u-2h}^{u+2h} \int_0^{2h} |\nabla X|^2 \, du \, dv \ge h \int_0^h |X_v(u_v, u)|^2 \, dv$.

Consequently

$$|X(u_1,h)-\xi(u_1)| \le \int_0^h |X_v(u_1,v)|\,dv \le \sqrt{h}\left(\int_0^h |X_v(u_1,v)|^2\,dv\right)^{\frac{1}{2}} \le \varepsilon(X,u,h)$$

by (iii). Finally we obtain for all $u \in \mathbb{R}$ with $|u| < h' < \frac{1}{2}$,

$$|X(u,h)-\xi(0)| \le |X(u,h)-X(u_1,h)| + |X(u_1,h)-\xi(u_1)| + |\xi(u_1)-\xi(u)| + |\xi(u)-\xi(0)| \le (c_0k+1)\varepsilon(X,u,h) + \omega(\xi,h)+\omega(\xi,h'),$$

whence $X(u,h)\to\xi(0)$ as $(u,h)\to(0,0)$. This proves that $X\in C^0(\overline{B},\mathbb{R}^3)$.□

By the same reasoning we can show

Proposition 1. *Let $\mathbb{F}$ be a family of functions $X \in H_2^1(B,\mathbb{R}^3)$ whose boundary values are equicontinuous on ∂B. Suppose that*

$$\int_{B_r(w_0)} |\nabla X|^2\,du\,dv \le k^2\left(\frac{r}{R}\right)^{2\mu}\int_{B_R(w_0)}|\nabla X|^2\,du\,dv$$

holds for all $r\in(0,R]$ and $w_0\in B$ with $0<R\le \operatorname{dist}(w_0,\partial B)$ and uniform constants k and μ for all $X\in\mathbb{F}$. Furthermore, assume that there exist a number $A>0$ and a function $\eta(r)$ on $0<r<\infty$ with $\lim_{r\to 0}\eta(r)=0$, all independent of $X\in\mathbb{F}$, such that $D_B(X)=\int_B|\nabla X|^2\,du\,dv\le A$, $D_{B\cap B_r(w^*)}(X)\le\eta(r)$ for $w^*\in\partial\Omega$ and $0<r<\infty$, for all $X\in\mathbb{F}$. Then the family $\mathbb{F}$ is equicontinuous on $\overline{B}$. □

In Section 4.5 we have derived a formula for the *inner variation* of a functional $\mathcal{F}$, see the formulae in Section 4.5 of Vol. 1, (15) and (20). In particular the conformality relations (15) hold if the first inner variation $\partial\mathcal{F}$ vanishes for all vector fields λ.

Now we have to consider "*outer variations*", that is variations of the type $X_\varepsilon = X+\varepsilon\varphi$.

Assumption A. *Let $\mathcal{K}\in\mathbb{R}^3$ be a closed set and $Q\in C^1(S,\mathbb{R}^3)$, $g_{ij}\in C^1(S,\mathbb{R})$ for all $i,j=1,2,3$ and some open set S containing $\mathcal{K}$. In addition suppose that Q and g_{ij} satisfy*

$$\left|\frac{\partial Q^j}{\partial x^i}\right|_{0,\mathcal{K}}<\infty,\qquad \left|\frac{\partial g_{ij}}{\partial x^k}\right|_{0,\mathcal{K}}<\infty \tag{22}$$

for all $i,j,k=1,2,3$ and suppose

$$e(x,p_1,p_2) = \frac{1}{2} g_{ij}(x) p^i_\alpha p^j_\alpha + \langle Q(x), p_1 \wedge p_2 \rangle$$

is coercive, i.e.

$$m_0\{|p_1|^2 + |p_2|^2\} \le e(x,p_1,p_2) \le m_1\{|p_1|^2 + |p_2|^2\}$$

for all $(x,p_1,p_2) \in \mathcal{K} \times \mathbb{R}^3 \times \mathbb{R}^3$ *and suitable constants* $0 < m_0 \le m_1$.

Theorem 5 (First variation formula). *Assume* $Q, g_{ij}, \mathcal{K}$ *and* $e(x,p_1,p_2)$ *fulfill Assumption A. Let* $X \in H^1_2(B,\mathcal{K})$ *and* $\varphi \in L_\infty(B,\mathbb{R}^3)$ *be functions such that* $X + \varepsilon\varphi \in H^1_2(B,\mathcal{K})$ *for all* $\varepsilon \in [0,\varepsilon_0)$ *and some* $\varepsilon_0 > 0$. *Then the first (outer) variation* $\delta\mathcal{F}(X,\varphi) = \lim_{\epsilon\to 0^+} \frac{\mathcal{F}(X+\varepsilon\varphi)-\mathcal{F}(X)}{\varepsilon}$ *exists and is given by*

$$\begin{aligned}
&\delta\mathcal{F}(X,\varphi) \\
&\quad = \int_B \left\{ g_{ij}(X) X^i_{u^\alpha} \varphi^j_{u^\alpha} + \frac{1}{2} \frac{\partial g_{ij}(X)}{\partial x^e} X^i_{u^\alpha} X^j_{u^\alpha} \varphi^\ell \right. \\
&\qquad \left. + \left\langle \frac{\partial Q}{\partial x^j}(X), X_u \wedge X_v \right\rangle \varphi^j + \langle Q(X), X_u \wedge \varphi_v + \varphi_u \wedge X_v \rangle \right\} du_1\, du_2.
\end{aligned}$$

Furthermore, if $\varphi \in \mathring{H}^1_2(B,\mathbb{R}^3) \cap L_\infty(B,\mathbb{R}^3)$ *then*

$$\begin{aligned}
\delta\mathcal{F}(X,\varphi) = \int_B &\left\{ g_{ij}(X) X^i_{u^\alpha} \varphi^j_{u^\alpha} + \frac{1}{2} \frac{\partial g_{ij}}{\partial x^\ell} X^i_{u^\alpha} X^j_{u^\alpha} \varphi^\ell \right. \\
&\left. + \operatorname{div} Q(X) \langle X_u \wedge X_v, \varphi \rangle \right\} du\, dv,
\end{aligned} \tag{23}$$

where

$$\operatorname{div} Q(X) = \frac{\partial Q^1}{\partial x^2}(X) + \frac{\partial Q^2}{\partial x^2}(X) + \frac{\partial Q^3}{\partial x^3}(X).$$

Remark. $\delta\mathcal{F}(X,\varphi)$ is called the *first (outer)* variation of $\mathcal{F}$ at X in direction φ.

We have adopted the summation convention that Latin indices have to be summed from 1 to 3 and Greek indices from 1 to 2. Also we have replaced (u,v) by (u_1,u_2).

Proof of Theorem 5. We compute

$$
\begin{aligned}
&\frac{1}{\varepsilon}[\mathcal{F}(X+\varepsilon\varphi)-\mathcal{F}(X)]-\delta\mathcal{F}(X,\varphi)\\
&\quad=\frac{1}{\varepsilon}\int_B\bigg\{\frac{1}{2}[g_{ij}(X+\varepsilon\varphi)(X^i+\varepsilon\varphi^i)_{u^\alpha}(X^j+\varepsilon\varphi^j)_{u^\alpha}-g_{ij}(X)X^i_{u^\alpha}X^j_{u^\alpha}]\\
&\qquad+\langle Q(X+\varepsilon\varphi),(X_u+\varepsilon\varphi_u)\wedge(X_v+\varepsilon\varphi_v)\rangle-\langle Q(X),X_u\wedge X_v\rangle\bigg\}\,du\,dv\\
&\qquad-\int_B\bigg\{g_{ij}(X)X^i_{u^\alpha}\varphi^j_{u^\alpha}+\frac{1}{2}\frac{\partial g_{ij}}{\partial x^e}X^i_{u^\alpha}X^j_{u^\alpha}\varphi^e\\
&\qquad+\left\langle\frac{\partial Q}{\partial x^j},X_u\wedge X_v\right\rangle\varphi^j+\langle Q(X),X_u\wedge\varphi_v+\varphi_u\wedge X_v\rangle\bigg\}\,du\,dv\\
&\quad=\int_B\bigg\{\frac{1}{2}\left[\frac{1}{\varepsilon}(g_{ij}(X+\varepsilon\varphi)-g_{ij}(X))-\frac{\partial g_{ij}}{\partial x^e}\varphi^e\right]X^i_{u^\alpha}X^j_{u^\alpha}\\
&\qquad+\left\langle\frac{1}{\varepsilon}(Q(X+\varepsilon\varphi)-Q(X))-\frac{\partial Q}{\partial x^j}(X)\varphi^j,X_u\wedge X_v\right\rangle\\
&\qquad+[g_{ij}(X+\varepsilon\varphi)-g_{ij}(X)]X^i_{u^\alpha}\varphi^j_{u^\alpha}\\
&\qquad+\langle Q(X+\varepsilon\varphi)-Q(X),X_u\wedge\varphi_v+\varphi_u\wedge X_v\rangle\\
&\qquad+\frac{\varepsilon}{2}g_{ij}(X+\varepsilon\varphi)\varphi^i_{u^\alpha}\varphi^j_{u^\alpha}+\varepsilon Q(X+\varepsilon\varphi)(\varphi_u\wedge\varphi_v)\bigg\}\,du\,dv\\
&\quad=\int_B a^\varepsilon_{ij}(w)X^i_{u^\alpha}X^j_{u^\alpha}\,du\,dv+\int_B b^\varepsilon_i(w)(X_u\wedge X_v)_i\,du\,dv\\
&\qquad+\int_B c^\varepsilon_{ij}(w)X^i_{u^\alpha}\varphi^j_{u^\alpha}\,du\,dv+\int_B d^\varepsilon_i(w)[(X_u\wedge\varphi_v)_i+(\varphi_u\wedge X_v)_i]\,du\,dv\\
&\qquad+\varepsilon\int_B[h^\varepsilon_{ij}(w)\varphi^i_{u^\alpha}\varphi^j_{u^\alpha}+f^\varepsilon_i(w)(\varphi_u\wedge\varphi_v)_i]\,du\,dv
\end{aligned}
$$

with obvious choices of bounded and measurable functions $a^\varepsilon_{ij},\dots,f^\varepsilon_i$ on B whose $L_\infty(B)$-norms are uniformly bounded with respect to ε. Furthermore

$$
a^\varepsilon_{ij}(\cdot),\ b^\varepsilon_i(\cdot),\ c^\varepsilon_{ij}(\cdot),\ d^\varepsilon_i\to 0
$$

a.e. on B as $\varepsilon\to 0$.

For any measurable set $\Omega\subset B$ we have

$$
\begin{aligned}
&\left|\int_\Omega a^\varepsilon_{ij}X^i_{u^\alpha}X^j_{u^\alpha}\,du\,dv\right|\le cD_\Omega(X)=c\int_\Omega|\nabla X|^2\,du\,dv,\\
&\left|\int_\Omega b^\varepsilon_i(X_u\wedge X_v)_i\,du\,dv\right|\le cD_\Omega(X),\\
&\left|\int_\Omega c^\varepsilon_{ij}X^i_{u^\alpha}\varphi^j_{u^\alpha}\,du\,dv\right|\le c(D_\Omega(X))^{\frac{1}{2}}(D_\Omega(\varphi))^{\frac{1}{2}},\\
&\left|\int_\Omega d^\varepsilon_i[(X_u\wedge\varphi_v)_i+(\varphi_u\wedge X_v)_i]\,du\,dv\right|\le c(D_\Omega(X))^{\frac{1}{2}}(D_\Omega(\varphi))^{\frac{1}{2}}
\end{aligned}
$$

and

$$\left| \int_\Omega b^\varepsilon_{ij} \varphi^i_{u^\alpha} \varphi^j_{u^\alpha} \, du \, dv \right| \le c D_\Omega(\varphi),$$
$$\left| \int_\Omega f^\varepsilon_i (\varphi_u \wedge \varphi_v)_i \, du \, dv \right| \le c D_\Omega(\varphi)$$

for a constant c independent of ε. This implies the uniform absolute continuity of the integrals under consideration. By virtue of Vitali's convergence theorem the first part of Theorem 5 follows. Finally formula (23) can be derived by an integration by parts using an appropriate approximation argument. □

Remarks. (i) The statements of Theorem 5 hold true without the hypotheses (22), if $X \in C^0(\overline{B}, \mathbb{R}^3)$ or even $X \in L_{\infty,\mathrm{loc}}(B, \mathbb{R}^3)$, which is – by Theorem 4 – true for solutions X of $\mathcal{P}(\Gamma, \mathcal{K})$.

(ii) The first variation formula (23) continues to hold if Q is not necessarily C^1 but $\operatorname{div} Q$ is defined (possibly in a weak sense!). For a proof and an application of this remark see the proof of Theorem 8, in particular relation (37).

A consequence of Theorem 5 is the Euler equation for the functional $\mathcal{F} = E + V$ (see also Theorem 7), namely

$$\triangle X^\ell + \Gamma^\ell_{jk}(X^j_u X^k_u + X_v X^k_v) = \operatorname{div} Q(X) g^{\ell m}(X_u \wedge X_v)_m, \quad \ell = 1, 2, 3, \tag{24}$$

where the Christoffel symbols $\Gamma_{jk\ell}$ and Γ^ℓ_{jk} are given by (cp. Vol. 1, Chapter 1)

$$\Gamma_{jk\ell} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^\ell} - \frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} \right), \qquad \Gamma^\ell_{jk} = g^{\ell m} \Gamma_{jmk}.$$

Indeed, (24) follows from the first variation formula (23) on testing with $\varphi = (\varphi^1, \varphi^2, \varphi^3)$, where $\varphi^j = g^{jk}(X)\psi^k$ with $\psi = (\psi^1, \psi^2, \psi^3) \in C^\infty_0(B, \mathbb{R}^3)$, and the fundamental lemma of the calculus of variations.

A major step in the regularity theory for obstacle problems is the following

Theorem 6. *Suppose* $Q \in C^2(S, \mathbb{R}^3)$, $g_{ij} \in C^2(S, \mathbb{R})$, $i, j = 1, 2, 3$ *and* $e(x, p_1, p_2)$ *satisfy Assumption A (possibly without relation* (22)*), where* $\mathcal{K}$ *is quasiregular and of class* C^3 *and* $S \subset \mathbb{R}^3$ *is open with* $\mathcal{K} \subset S$. *Then each solution* $X \in \mathcal{C}(\Gamma, \mathcal{K})$ *of the obstacle problem*

$$\mathcal{P}(\Gamma, \mathcal{K}): \qquad \mathcal{F} \to \min \ \text{in } \mathcal{C}(\Gamma, \mathcal{K})$$

is of class $H^2_s(B', \mathbb{R}^3) \cap C^{1,\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ *for all* $B' \subset\subset B$ *and all* $s, \alpha \in \mathbb{R}$ *with* $0 \le s < \infty$ *and* $1 < \alpha < 1$.

This result holds under somewhat weaker regularity hypothesis on Q, see the Remark at the end of the proof of Theorem 6.

The key argument of the proof of Theorem 6 is given in the following Lemma 1 where the L_2-estimates of the second derivatives are established.

Definition 3. *Let $\Omega' \subset \mathbb{R}^2$ be a bounded open set, $K \subset \mathbb{R}^3$ a closed set and $S \subset \mathbb{R}^3$ some open set containing K. Consider functions $A = A(w,z,p) = (A_j^\alpha)(w,z,p), j = 1,2,3, \alpha = 1,2$ and $B = B(w,z,p) = B_j(w,z,p), j = 1,2,3$ of class C^1 on $\Omega' \times S \times \mathbb{R}^6$ such that the inequalities*

$$m_2|\eta|^2 \le A^\alpha_{jp^k_\beta}(\xi)\eta^j_\alpha\eta^k_\beta,$$

$$\left|A^\alpha_{jp^k_\beta}(\xi)\right| \le m_3$$

and

$$|A(\xi)|^2 + |A_w(\xi)|^2 + |A_z(\xi)|^2 + |B(\xi)| + |B_w(\xi)| + |B_z(\xi)| + |B_p(\xi)|^2 \\ \le m_4(1 + |p|^2)$$

hold for all $\xi = (w,z,p) \in \Omega' \times K \times \mathbb{R}^6$ and for all $\eta = (\eta_1, \eta_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ with positive constants $m_2, m_3, m_4 \in \mathbb{R}$ independent of ξ.

Lemma 1. *Suppose A, B and Ω' satisfy Definition 3 with $K = B_1^+(0) := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1, z \ge 0\}$. Moreover let $z = z(w) \in H_2^1(\Omega', B_1^+)$ have the following properties*

(a) *There are positive numbers M_0 and μ such that*

$$\int_{\Omega' \cap B_\rho(\zeta)} |\nabla z|^2 \, du \, dv \le M_0 \rho^{2\mu} \quad \text{for all disks } B_\rho(\zeta) \subset \mathbb{R}^2, \tag{25}$$

(b) *For all $\varphi \in C_c^0(\Omega', \mathbb{R}^3) \cap H_2^1(\Omega', \mathbb{R}^3)$ with $z^3 - \varepsilon\varphi^3 \ge 0$ for $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0(\varphi) > 0$, the variational inequality*

$$\int_{\Omega'} \{A_j^\alpha(w,z,\nabla z)\varphi^j_{u^\alpha} + B_j(w,z,\nabla z)\varphi^j\} \, du \, dv \le 0 \tag{26}$$

is satisfied.

Then we have $z \in H_2^2(\Omega'', \mathbb{R}^3) \cap H_s^1(\Omega'', \mathbb{R}^3)$ for $\Omega'' \Subset \Omega'$ and all $s \in [1, \infty)$.

Proof. Pick any $\zeta_0 \in \Omega'$ and consider a disk $B_{3R_0}(\zeta_0) \Subset \Omega'$, $0 < R_0 < 1$ and choose $R \in (0, R_0)$. Then there exists a function $\eta \in C_c^\infty(B_{2R}(\zeta_0))$ satisfying $0 \le \eta \le 1$, $|\nabla \eta| \le \frac{2}{R}$ and $\eta(w) = 1$ for $w \in B_R(\zeta_0)$. Moreover let us denote by $\triangle_h z$ the difference quotient

$$\triangle_h z = \frac{1}{h}[z(w + h\zeta) - z(w)], \quad h \ne 0,$$

in the direction of a unit vector $\zeta \in \mathbb{R}^2$. Then we have the relation

$$z(w) + \epsilon\triangle_{-h}[\eta^2(w)\triangle_h z(w)] = \frac{\epsilon}{h^2}\eta^2(w)z(w + h\zeta) \\ + \left\{1 - \frac{\epsilon}{h^2}[\eta^2(w) + \eta^2(w - h\zeta)]\right\} z(w) + \frac{\epsilon}{h^2}\eta^2(w - h\zeta)z(w - h\zeta).$$

Therefore $\varphi = -\triangle_{-h}[\eta^2 \triangle_h z]$ is of class $C_c^{0,\mu}(\Omega', \mathbb{R}^3) \cap H_2^1(\Omega', \mathbb{R}^3)$ for $0 < |h| < R$ and satisfies $z^3 - \epsilon\varphi^3 \geq 0$ provided $0 \leq \epsilon < \epsilon_0 = \frac{h^2}{2}$. Thus φ is admissible in (26) and we obtain

$$(27) \quad \int \left\{ \triangle_h A(w, z, \nabla z) \nabla(\eta^2 \triangle_h z) + \triangle_h B(w, z, \nabla z)(\eta^2 \triangle_h z) \right\} du\, dv \leq 0,$$

where we have for simplicity omitted the domain of integration Ω'. Now we use the identity

$$(28) \quad \triangle_h A(w, z(w), \nabla z(w)) = \int_0^1 A_w(\xi(t))\, dt \cdot \zeta + \int_0^1 A_z(\xi(t))\, dt \cdot \triangle_h z(w) + \int_0^1 A_p(\xi(t))\, dt \cdot \nabla \triangle_h z(w),$$

where $\xi(t) = (w + th\zeta, z(w) + th\triangle_h z(w), \nabla z(w) + th\nabla\triangle_h z(w))$ and analogous expressions holding for $\triangle_h B(w, z(w), \nabla z(w))$. Observe that the set $B_{3R}(\zeta_0) \times B_1^+ \times \mathbb{R}^6$ is convex and $z : \Omega' \to B_1^+$; hence $\xi(t) \in B_{3R}(\zeta_0) \times B_1^+ \times \mathbb{R}^6$ for all $t \in [0,1], |h| < R$ and $w \in B_{2R}(\zeta_0) \Supset \operatorname{supp} \eta$.

By virtue of Definition 3

$$(29) \quad \begin{aligned} &|\triangle_h A(w, z, \nabla z)| \leq m_5\{(1 + |\nabla z| + |\nabla z_h|) \cdot (1 + |\triangle_h z|) + |\nabla \triangle_h z|\}, \\ &|\triangle_h A(w, z, \nabla z) - \int_0^1 A_p(\xi(t))\, dt \nabla \triangle_h z(w)| \\ &\quad \leq m_5(1 + |\nabla z| + |\nabla z_h|)(1 + |\triangle_h z|), \\ &|\triangle_h B(w, z, \nabla z| \leq m_6\{(1 + |\nabla z|^2 + |\nabla z_h|^2)(1 + |\triangle_h z|) \\ &\qquad + (1 + |\nabla z| + |\nabla z_h|)|\nabla \triangle_h z|\} \end{aligned}$$

with suitable constants m_5, m_6 and $z_h(w) := z(w + h\zeta)$. Again from Definition 3 we infer

$$(30) \quad m_2 \int_{\Omega'} |\eta \nabla \triangle_h z|^2\, du\, dv \leq \int_{\Omega'} \eta^2 \int_0^1 A_p(\xi(t))\, dt \nabla \triangle_h z \nabla \triangle_h z\, du\, dv.$$

Now we use the variational inequality (27) and relation (28) together with $\nabla(\eta^2 \triangle_h z) = 2\eta \nabla \eta \triangle_h z + \eta^2 \nabla \triangle_h z$ and infer

$$\begin{aligned} &\int_{\Omega'} \int_0^1 A_p\, dt \nabla \triangle_h z \nabla \triangle_h z \eta^2\, du\, dv \\ &\quad \leq -\int_{\Omega'} \int_0^1 A_p\, dt \nabla \triangle_h z \nabla \eta \triangle_h z \cdot 2\eta\, du\, dv \\ &\qquad - \int_{\Omega'} \int_0^1 A_w\, dt \zeta [2\eta \nabla \eta \triangle_h z + \eta^2 \nabla \triangle_h z]\, du\, dv \\ &\qquad - \int_{\Omega'} \int_0^1 A_z\, dt \triangle_h z [2\eta \nabla \eta \triangle_h z + \eta^2 \nabla \triangle_h z]\, du\, dv \\ &\qquad - \int_{\Omega'} \triangle_h B \eta^2 \triangle_h z\, du\, dv. \end{aligned}$$

Inequality (30) implies the estimate

$$
\begin{aligned}
m_2 \int_{\Omega'} |\mu \nabla \triangle_h z|^2 \, du \, dv &\le c \int_{\Omega'} \eta |\nabla \eta| |\triangle_h z| |\nabla \triangle_h z| \, du \, dv \\
&+ c \int_{\Omega'} \int_0^1 |A_w| \, dt \; \eta |\nabla \eta| |\triangle_h z| \, du \, dv + c \int_{\Omega'} \int_0^1 |A_w| \, dt \; \eta^2 |\nabla \triangle_h z| \, du \, dv \\
&+ c \int_{\Omega'} \int_0^1 |A_z| \, dt |\triangle_h z|^2 \eta |\nabla \eta| \, du \, dv \\
&+ c \int_{\Omega'} \int_0^1 |A_z| \, dt \; \eta^2 |\triangle_h z| |\nabla \triangle_h z| \, du \, dv \\
&+ c \int_{\Omega'} |\triangle_h B| \eta^2 |\triangle_h z| \, du \, dv,
\end{aligned}
$$

where here and in the following c denotes some constant independent of h and R (and only depending on $m_2, \dots, m_6$).

Definition 3 yields the estimates

$$
\begin{aligned}
|A_w| &\le c(1 + |\nabla z| + |\nabla z_h|), \\
|A_z| &\le c(1 + |\nabla z| + |\nabla z_h|)
\end{aligned}
$$

and together with (29) and the previous inequality we get

$$
\begin{aligned}
\int_{\Omega'} |\eta \nabla \triangle_h z|^2 \, du \, dv &\le c \int_{\Omega'} \eta |\nabla \eta| |\triangle_h z| |\nabla \triangle_h z| \, du \, dv \\
&+ c \int_{\Omega'} \eta |\nabla \eta| |\triangle_h z| \{1 + |\nabla z| + |\nabla z_h|\} \, du \, dv \\
&+ c \int_{\Omega'} \eta^2 |\nabla \triangle_h z| \{1 + |\nabla z| + |\nabla z_h|\} \, du \, dv \\
&+ c \int_{\Omega'} \eta |\nabla \eta| |\triangle_h z|^2 \{1 + |\nabla z| + |\nabla z_h|\} \, du \, dv \\
&+ c \int_{\Omega'} \eta^2 |\triangle_h z| |\nabla \triangle_h z| \{1 + |\nabla z| + |\nabla z_h|\} \, du \, dv \\
&+ c \int_{\Omega'} \eta^2 |\triangle_h z| \{(1 + |\nabla z|^2 + |\nabla z_h|^2)(1 + |\triangle_h z|) \\
&+ (1 + |\nabla z| + |\nabla z_h|) |\nabla \triangle_h z|\} \, du \, dv.
\end{aligned}
$$

Taking the elementary inequality $2ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$ for $\epsilon > 0$, into account we can estimate the different integrands as follows

$$\begin{aligned}
&\eta|\nabla\eta||\triangle_h z||\nabla\triangle_h z| \le \epsilon\eta^2|\nabla\triangle_h z|^2 + \frac{1}{\epsilon}|\nabla\eta|^2|\triangle_h z|^2,\\
&\eta|\nabla\eta||\triangle_h z|^2 \le \eta^2|\triangle_h z|^2 + |\nabla\eta|^2|\triangle_h z|^2,\\
&\eta|\nabla\eta||\triangle_h z||\nabla z| \le \eta^2|\nabla z|^2 + |\nabla\eta|^2|\triangle_h z|^2,\\
&\eta|\nabla\eta||\triangle_h z|^2|\nabla z| \le \eta^2|\triangle_h z|^2|\nabla z|^2 + |\nabla\eta|^2|\triangle_h z|^2,\\
&\eta^2|\triangle_h z||\nabla z||\nabla\triangle_h z| \le \epsilon\eta^2|\nabla\triangle_h z|^2 + \frac{1}{\epsilon}\eta^2|\nabla z|^2|\triangle_h z|^2,
\end{aligned}$$

and the other terms are treated similarly. In this way we get for $\epsilon > 0$ arbitrary

$$\begin{aligned}
(31) \qquad \int_{\Omega'} |\eta\nabla\triangle_h z|^2\,du\,dv &\le \epsilon\int_{\Omega'} |\eta\nabla\triangle_h z|^2\,du\,dv\\
&+ c\left(1+\frac{1}{\epsilon}\right)\int_{\Omega'} \eta^2(|\nabla z|^2 + |\nabla z_h|^2)|\triangle_h z|^2\,du\,dv\\
&+ c\left(1+\frac{1}{\epsilon}\right)\\
&\times \int_{\Omega'} \{\eta^2(1+|\triangle_h z|^2 + |\nabla z|^2 + |\nabla z_h|^2 + |\nabla\eta|^2|\triangle_h z|^2\}\,du\,dv.
\end{aligned}$$

For some constant c depending on $m_2,\dots,m_6$ but not on h, R or ϵ. We observe that for $|h| < R$ we have (see e.g. Lemma 7.23 in Gilbarg and Trudinger [1])

$$\int_{B_{2R}(\zeta_0)} |\triangle_h z|^2\,du\,dv \le \int_{B_{3R}(\zeta_0)} |\nabla z|^2\,du\,dv,$$

and therefore

$$\begin{aligned}
(32) \qquad &\int_{\Omega'} \{\eta^2(1+|\triangle_h z|^2 + |\nabla z|^2 + |\nabla z_h|^2) + |\nabla\eta|^2|\triangle_h z|^2\}\,du\,dv\\
&\le \frac{c}{R^2}\int_{\Omega'} |\nabla z|^2\,du\,dv + cR^2.
\end{aligned}$$

Next we apply the Dirichlet-growth condition (25) which yields

$$\int_{B_{2R}(\zeta_0)\cap B_\rho(\xi)} (|\nabla z|^2 + |\nabla z_h|^2)\,du\,dv \le 2M_0\rho^{2\mu}$$

for all disks $B_\rho(\xi) \subset \mathbb{R}^2$. Now Lemma 2 in Section 2.7 applied to the functions $q(w) := |\nabla z(w)|^2 + |\nabla z_h(w)|^2 \in L_1(B_{2R}(\zeta_0))$ and to $\phi(w) := \eta\triangle_h z \in \overset{\circ}{H}{}^1_2(B_{2R}(\zeta_0), \mathbb{R}^3)$ gives the estimate

(33)
$$\begin{cases} \int_{B_{2R}(\zeta_0)} \eta^2(|\nabla z|^2 + |\nabla z_h|^2)|\triangle_h z|^2 \, du \, dv \\ \quad \le \overline{C}(M_0, \mu) R^{2\mu} \int_{B_{2R}(\zeta_0)} |\nabla(\eta \triangle_h z)^2| \, du \, dv \\ \quad \le C(M_0, \mu) R^{2\mu} \left\{ \int_{B_{2R}(\zeta_0)} \eta^2 |\nabla \triangle_h z|^2 \, du \, dv + R^{-2} \int_{\Omega'} |\nabla z|^2 \, du \, dv \right\} \end{cases}$$

for $|h| < R$, since

$$|\nabla(\eta \triangle_h z)|^2 \le (|\nabla \eta||\triangle_h z| + \eta |\nabla \triangle_h z|)^2 \le 2\eta^2 |\nabla \triangle_h z|^2 + \frac{8}{R^2} |\triangle_h z|^2$$

and with constants $C(M_0, \mu)$ indepent of h and R. The formulae (31), (32) and (33) yield

$$\begin{aligned} &\int_{\Omega'} |\eta \nabla \triangle_h z|^2 \, du \, dv \\ &\le \left[\epsilon + c \left(1 + \frac{1}{\epsilon} \right) C(M_0, \mu) R^{2\mu} \right] \int_{B_{2R}} \eta^2 |\nabla \triangle_h z|^2 \, du \, dv \\ &\quad + c \left(1 + \frac{1}{\epsilon} \right) C(M_0, \mu) R^{2\mu - 2} \int_{\Omega'} |\nabla z|^2 \, du \, dv \\ &\quad + c \left(1 + \frac{1}{\epsilon} \right) \left\{ \frac{c}{R^2} \int_{\Omega'} |\nabla z|^2 \, du \, dv + cR^2 \right\}. \end{aligned}$$

By an appropriate choice of $\epsilon > 0$ and $R \in (0, R_0)$ the coefficient $[\ldots]$ can be made arbitrary small, for instance $[\ldots] < \frac{1}{2}$.

Hence the term $[\ldots] \int_{B_{2R}} \eta^2 |\nabla \triangle_h z|^2 \, du \, dv$ can be absorbed by the left hand side and we obtain an estimate of the type

$$\int_{B_{2R}(\zeta_0)} \eta^2 |\nabla \triangle_h z|^2 \, du \, dv \le \text{const} \quad \text{for all } |h| < R$$

and some constant depending on $M_0, \mu, m_2, \ldots, m_6$ and the Dirichlet integral of z, but not on h. We conclude that the weak derivatives $D_i D_j z, \quad i, j = 1, 2$ exist and that $z \in H_2^2(B_R(\xi_0), \mathbb{R}^3)$, since $\eta \equiv 1$ on $B_R(\xi_0)$ (see e.g. Lemma 7.24 in Gilbarg and Trudinger [1]). Then a covering argument yields that $z \in H_2^2(\Omega'', \mathbb{R}^3)$ for all $\Omega'' \Subset \Omega'$, and by the Sobolev imbedding theorem we finally obtain $z \in H_s^1(\Omega'', \mathbb{R}^3)$ for any subset $\Omega'' \Subset \Omega'$ and all $s \in [1, \infty)$. □

Now we turn to the

Proof of Theorem 6. **Step I: L_2-estimates of the second derivatives.** By virtue of Theorem 4 we have $X \in C^{0,\mu}(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3)$ for some $\mu > 0$. For some arbitrary point $\zeta_0 \in B$ either $X(\zeta_0) \in \partial\mathcal{K}$ or $X(\zeta_0) \in \operatorname{int}\mathcal{K}$. We treat the first case by reducing it to Lemma 1; the second case can be handled similarly. Since $\mathcal{K}$ is of class C^3 there exists a neighbourhood U of $X(\zeta_0)$ and a C^3-diffeomorphim $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ with inverse χ which maps $U \cap \mathcal{K}$ onto $B_1^+(0)$, $U \cap \partial\mathcal{K}$ onto $B_1^+(0) \cap \{x_3 = 0\}$ and $X(\zeta_0)$ onto 0. For sufficiently small $\rho_0 > 0$ and $\Omega' := B_{\rho_0}(\zeta_0)$ we have $\Omega' \Subset B$ and $z := \psi \circ X \in H_2^1(\Omega', B_{1/2}^+(0)) \cap C^{0,\mu}(\Omega',\mathbb{R}^3)$. Consider any $\varphi \in C_c^0(\Omega',\mathbb{R}^3) \cap H_2^1(\Omega',\mathbb{R}^3)$ with the property $z^3(w) - \epsilon\varphi^3(w) \ge 0$ for all $w \in \Omega'$ and sufficiently small $\epsilon \ge 0$. Then the mapping $X_\epsilon := \chi(z - \epsilon\varphi) \in \mathcal{C}(\Gamma,\mathcal{K})$ for $\epsilon \in [0,\epsilon_0)$, $\epsilon_0 = \epsilon_0(\varphi)$, while clearly $X_0 = X$. By the minimum property of X we have $\mathcal{F}(X) \le \mathcal{F}(X_\epsilon)$ for all $\epsilon \in [0,\epsilon_0)$. Introduce the integral $\tilde{\mathcal{F}}(Y) := \int_B \tilde{e}(Y,\nabla Y)\,du\,dv$, whose integrand is defined by

$$\tilde{e}(y,q) := e(\chi(y),\chi_y(y)q),$$

for $(y,q) \in \mathcal{K}^* \times \mathbb{R}^6$, $\mathcal{K}^* := \psi(\mathcal{K})$ and where $\chi_y = D\chi : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the Jacobian of χ while $e(x,p) = \frac{1}{2}g_{ij}(x)[p_1^i p_1^j + p_2^i p_2^j] + \langle Q(x), p_1 \wedge p_2\rangle$, $p = (p_1,p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Since $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^3-diffeomorphism it is not difficult (but somewhat tedious) to prove that the functions defined by

$$A_j^\alpha(y,q) := \tilde{e}_{q_\alpha^j}(y,q)$$

and

$$B_j(y,q) := \tilde{e}_{y^j}(y,q) \quad \text{for } \alpha = 1,2 \text{ and } j = 1,2,3,$$

satisfy the growth and coercivity conditions of Definition 3.

Furthermore, arguments similar to those used in the proof of the variation formula Theorem 5 show that the first variation

$$\delta\tilde{\mathcal{F}}(z,\varphi) = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}(\tilde{\mathcal{F}}(z+\epsilon\varphi) - \tilde{\mathcal{F}}(z))$$

exists for functions φ considered above and is given by

$$\delta\tilde{\mathcal{F}}(z,\varphi) = \int_{\Omega'} \{A_j^\alpha(z,\nabla z)\varphi_{u^\alpha}^j + B_j(z,\nabla z)\varphi^j\}\,du\,dv.$$

By the minimality of X we infer that $\delta\tilde{\mathcal{F}}(z,\varphi) \le 0$ is satisfied for all $\varphi \in C_c^0(\Omega',\mathbb{R}^3) \cap H_2^1(\Omega',\mathbb{R}^3)$ with $z^3 - \epsilon\varphi^3 \ge 0$ on Ω' and $0 \le \epsilon < \epsilon_0(\varphi)$.

By Theorem 4 X satisfies a Dirichlet growth condition of the type (16), whence also $z = \psi \circ X$ fulfills the estimate

$$\int_{\Omega' \cap B_\rho(\xi)} |\nabla z|^2\,du\,dv \le M_o \rho^{2\mu}$$

for every ball $B_\rho(\xi) \subset \mathbb{R}^2$ with constant $M_0 = \lambda(R-\rho_0)^{-2\mu} D_B(X)$ where $R = \mathrm{dist}(\xi_0, \partial B)$ and $\lambda := |\mathcal{G} \circ \chi|_{0,B_1^+}$, $\mathcal{G} := \psi_x^t \psi_x$; here we have used $\nabla z = \psi_x \circ X$ and $|\nabla z(w)|^2 = \nabla X(w) \mathcal{G}(X(w)) \nabla X(w) \le \lambda |\nabla X(w)|^2$ for all $w \in \Omega' = B_{\rho_0}(\zeta_0)$.

Now we can apply Lemma 1 and obtain $z \in H_2^2(\Omega'', \mathbb{R}^3) \cap H_s^1(\Omega'', \mathbb{R}^3)$ for all $1 \le s < \infty$ and domains $\Omega'' \Subset \Omega'$. Taking $X = \chi \circ z$ on $B_{\rho_0}(\zeta_0)$ into account, we see by a covering argument that $X \in H_2^2(\Omega', \mathbb{R}^3) \cap H_s^1(\Omega', \mathbb{R}^3)$ for all subsets $\Omega' \Subset B$ and all numbers $s \in [1, \infty)$.

Step II. L_s-estimates of the second derivatives.

Case 1. $X(\zeta_0) \in \mathrm{int}\, \mathcal{K}$.

Since X is continuous also $X(B_{R_0}(\zeta_0)) \subset \mathrm{int}\, \mathcal{K}$ for some $0 < R_0 \ll 1$, whence we obtain $\delta \mathcal{F}(X, \varphi) = 0$, i.e. by Theorem 5

$$\int_B \{ g_{jk}(X) X^j_{u^\alpha} \varphi^k_{u^\alpha} + \frac{1}{2} \frac{\partial g_{jk}}{\partial x^e} X^j_{u^\alpha} X^k_{u^\alpha} \varphi^\ell + \mathrm{div}\, Q(X) \langle X_u \wedge X_v, \varphi \rangle \} \, du\, dv = 0,$$

for every φ of class $\overset{\circ}{H}{}_2^1(B_{R_0}(\zeta_0), \mathbb{R}^3) \cap L_\infty(B_{R_0}(\zeta_0), \mathbb{R}^3)$. Therefore, since $X \in H^2_{2,\mathrm{loc}}(B, \mathbb{R}^3)$, the Euler equations

$$\triangle X^\ell + \Gamma^\ell_{jk} X^j_{u^\alpha} X^k_{u^\alpha} = \mathrm{div}\, Q(X) g^{\ell m} (X_u \wedge X_v)_m$$

hold almost everywhere on $B_{R_0}(\zeta_0)$, whence we have the estimate

$$|\triangle X(w)| \le C |\nabla X(w)|^2$$

a.e. on $B_{R_0}(\zeta_0)$ for some constant $c > 0$. Since $\nabla X \in L_{2s,\mathrm{loc}}$ on B for all $s \in [1, \infty)$ we get $\triangle X \in L_s$ on $B_{R_0}(\zeta_0)$ and therefore conclude by standard L_p-theory (e.g. Gilbarg and Trudinger [1]) that $X \in H_s^2(B_{R_0}(\zeta_0), \mathbb{R}^3)$.

Case 2. $X(\zeta_0) \in \partial \mathcal{K}$.

Since $\mathcal{K}$ is of class C^3 there exists a neighbourhood U of $X(\zeta_0)$ and a C^3-diffeomorphism ψ of $\mathbb{R}^3$ onto itself which maps $U \cap \mathcal{K}$ onto B_1^+ and $U \cap \partial \mathcal{K}$ onto $B_1^0 := B_1^+ \cap \{x^3 = 0\}$ and $\psi(X(\zeta_0)) = 0$, $\det \psi_x > 0$.

For sufficiently small $R_0 > 0$ and $\Omega' = B_{R_0}(\zeta_0) \Subset B$ we have $z := \psi \circ X \in H_2^1(\Omega', B^+_{1/2})$. Pick any $\varphi \in C_c^0(\Omega', \mathbb{R}^3) \cap H_2^1(\Omega', \mathbb{R}^3)$ with the property that $z^3(w) - \epsilon \varphi^3(w) \ge 0$ for all $w \in \Omega'$, provided that $\epsilon > 0$ is sufficiently small. As in Step I consider the functional

$$\tilde{\mathcal{F}}(Y) = \int_B \tilde{e}(Y, \nabla Y) \, du\, dv,$$

where $\tilde{e}(y, q) := e(\chi(y), \chi_y(y) q)$ and $q = (q_1, q_2) \in \mathbb{R}^3 \times \mathbb{R}^3$. A simple calculation shows that we have

$$\tilde{e}(y, q) = \tilde{g}_{\ell m}(y) q^\ell_\alpha q^m_\alpha + \langle \tilde{Q}(y), q_1 \wedge q_2 \rangle,$$

where

$$\tilde g_{\ell m} = (g_{jk}\circ\chi)\chi^j_{y^\ell}\chi^k_{y^m} \quad \text{and}$$
$$\tilde Q = (\det\chi_y)\chi_y^{-1}(Q\circ\chi).$$

In other words, $\tilde{\mathcal{F}}$ is of the same structure as $\mathcal{F}$ and we can apply Theorem 5. Also we have $\delta\tilde{\mathcal{F}}(z,\varphi) \le 0$ for all $\varphi \in C^0_c(\Omega',\mathbb{R}^3)\cap H^1_2(\Omega',\mathbb{R}^3)$ with the property that

$$z^3(w) - \epsilon\varphi^3(w) \ge 0 \quad \text{for all } w\in\Omega'$$

and $0\le\epsilon\le\epsilon_0=\epsilon_0(\varphi)$. In particular we are free to make arbitrary "tangential" variations, in other words

$$\delta\tilde{\mathcal{F}}(z,\varphi)=0 \quad \text{for all } \varphi=(\varphi^1,\varphi^2,0)\in C^0_c(\Omega',\mathbb{R}^3)\cap H^1_2(\Omega',\mathbb{R}^3).$$

Theorem 5 now implies

$$\begin{aligned} \tilde g_{1j}(z)\triangle z^j + \tilde\Gamma_{j1k} z^j_{u^\alpha} z^k_{u^\alpha} &= \operatorname{div}\tilde Q(z)(z_u\wedge z_v)_1,\\ \tilde g_{2j}(z)\triangle z^j + \tilde\Gamma_{j2k} z^j_{u^\alpha} z^k_{u^\alpha} &= \operatorname{div}\tilde Q(z)(z_u\wedge z_v)_2 \end{aligned} \tag{34}$$

a.e. on Ω', where $\tilde\Gamma_{j\ell k}$ are the Christoffel symbols of the first kind corresponding to $\tilde g_{ij}$. Introduce the "coincidence" set

$$\begin{aligned}\mathcal{T}_z &:= \{w\in\Omega' = B_{R_0}(\zeta_0) : z^3(w)=0\}\\ &= \{w\in\Omega' : X(w)\in\partial\mathcal{K}\}.\end{aligned}$$

By a well known property of Sobolev functions we get $\nabla z^3(w)=0, \nabla^2 z^3(w) = 0$ a.e. on $\mathcal{T}_z$. Hence, on account of (34)

$$\begin{aligned} \tilde g_{11}(z)\triangle z^1 + \tilde g_{12}\triangle z^2 &= \ell_1(z,\nabla z),\\ \tilde g_{21}(z)\triangle z^1 + \tilde g_{22}\triangle z^2 &= \ell_2(z,\nabla z),\\ \triangle z^3 &= 0 \end{aligned} \tag{35}$$

a.e. on $\mathcal{T}_z$, where the right hand side grows quadratically in $|\nabla z|$, i.e.

$$|\ell_1(z,\nabla z)| + |\ell_2(z,\nabla z)| \le c|\nabla z|^2 \quad \text{on } \Omega'$$

for some constant c.

The coercivity of $e(x,p)$ (cf. Assumption A) implies that

$$\tilde m_0|\xi|^2 \le \tilde g_{jk}(z)\xi^j\xi^k \le \tilde m_1|\xi|^2$$

for all $(z,\xi)\in\mathcal{K}^{**}\times\mathbb{R}^3, \mathcal{K}^{**}\subset\mathcal{K}^* = \psi(\mathcal{K})$, where $\tilde m_0\le\tilde m_1$ are positive numbers. Therefore we infer from equation (35)

$$|\triangle z| \le c^*|\nabla z|^2 \quad \text{a.e. on } \mathcal{T}_z$$

for some number c^*.

On the other hand we get as in case 1

$$\triangle z^\ell + \tilde{\Gamma}^\ell_{jk}(z) z^j_{u^\alpha} z^k_{u^\alpha} = \operatorname{div} \tilde{Q}(z) \tilde{g}^{\ell m}(z)(z_u \wedge z_v)_m$$

for $\ell = 1, 2, 3$ a.e. on the (open) set $\Omega' \setminus \mathcal{T}_z$; whence also

$$|\triangle z| \le c^{**} |\nabla z|^2 \quad \text{a.e. on } \Omega' \setminus \mathcal{T}_z.$$

Concluding we have

$$|\triangle z| \le c |\nabla z|^2 \quad \text{a.e. on } \Omega'$$

with $c := \max(c^*, c^{**})$. Now we can proceed as in case 1 and obtain $z \in H^2_s(B_R(\zeta_0), \mathbb{R}^3)$ for any $R \in (0, R_0)$ and any $s \in [1, \infty)$. This implies that $X \in H^2_s(\Omega', \mathbb{R}^3)$ for all $\Omega' \Subset B$ and all $s \in [1, \infty)$. Finally, by Sobolev imbedding theorem we infer that also $X \in C^{1,\alpha}(\Omega', \mathbb{R}^3)$ for all $\Omega' \Subset B$ and all $\alpha \in [0, 1)$. This completes the proof of Theorem 6. □

Remark. The assertion of Theorem 6 still holds true if the condition $Q \in C^2(S, \mathbb{R}^3)$ is replaced by the weaker assumption $Q \in C^1(\mathcal{K})$ and $\operatorname{div} Q \in C^1(\mathcal{K})$. This observation is of importance for the solution of Plateau's problem for H-surfaces in the set $\mathcal{K}$.

Proof of the Remark. A careful scrutinizing of the steps in the proof of Theorem 6 shows that Step II (L_p-estimates of second derivatives) only requires $Q \in C^1(\mathcal{K})$. Returning to Step I we consider the functional $\tilde{\mathcal{F}}(Y) = \int_B \tilde{e}(Y, \nabla Y)\, du\, dv$, where $\tilde{e}(y, q_1, q_2) = \tilde{g}_{\ell m}(y) q^\ell_\alpha q^m_\alpha + \langle \tilde{Q}(y), q_1 \wedge q_2 \rangle$ with $\tilde{g}_{\ell m}(y) = g_{jk}(\chi(y)) \chi^j_{y^\ell} \chi^k_{y^m}$ and $\tilde{Q}(y) = (\det \chi_y(y))[\chi_y(y)]^{-1} Q(\chi(y))$. By Theorem 5 the first variation $\delta \tilde{\mathcal{F}}(z, \varphi)$ for

$$z \in C^0_c(\Omega', B^+_{\frac{1}{2}}(0)) \cap H^1_2(\Omega', \mathbb{R}^3) \quad \text{and} \quad \varphi \in C^0_c(\Omega', \mathbb{R}^3) \cap H^1_2(\Omega', \mathbb{R}^3)$$

is given by

$$\begin{aligned} &\delta \tilde{\mathcal{F}}(z, \varphi) \\ &\quad = \int_{\Omega'} \left\{ \tilde{g}_{ij}(z) z^i_{u^\alpha} z^j_{u^\alpha} + \frac{1}{2} \frac{\partial \tilde{g}_{ij}}{\partial y^\ell} z^i_{u^\alpha} z^j_{u^\alpha} \varphi^\ell + \operatorname{div} \tilde{Q}(z) \langle z_u \wedge z_v, \varphi \rangle \right\} du\, dv, \end{aligned}$$

where

$$\operatorname{div} \tilde{Q}(z) = \frac{\partial \tilde{Q}^1}{\partial y^1}(z) + \frac{\partial \tilde{Q}^2}{\partial y^2}(z) + \frac{\partial \tilde{Q}^3}{\partial y^3}(z).$$

Therefore, in order to apply Lemma 1 and the same arguments as in Step I in the proof of Theorem 6, it is sufficient to show that still we have $\operatorname{div} \tilde{Q}(y) \in C^1(K)$ under the weaker assumption Q, $\operatorname{div} Q \in C^1$. To this end we put

$\tilde{Q}(y) = (\det \chi_y(y))Q^*(y)$ with $Q^*(y) := [\chi_y(y)]^{-1}Q(\chi(y))$ and observe that it remains to show $\operatorname{div} Q^* \in C^1$, since χ is of class C^3. Let

$$\chi_y(y) = \begin{bmatrix} \dfrac{\partial \chi^1}{\partial y^1} & \dfrac{\partial \chi^1}{\partial y^2} & \dfrac{\partial \chi^1}{\partial y^3} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial \chi^3}{\partial y^1} & \dfrac{\partial \chi^3}{\partial y^2} & \dfrac{\partial \chi^3}{\partial y^3} \end{bmatrix},$$

then, since $\psi(\chi(y)) = y$ we have $\psi_x(\chi(y)) \cdot \chi_y(y) = \mathrm{Id}$ and

$$\chi_y^{-1}(y) = \begin{bmatrix} \dfrac{\partial \psi^1}{\partial x^1} & \cdots & \dfrac{\partial \psi^1}{\partial x^3} \\ \vdots & & \vdots \\ \dfrac{\partial \psi^3}{\partial x^1} & \cdots & \dfrac{\partial \psi^3}{\partial x^3} \end{bmatrix} (\chi(y)).$$

In particular we have $\frac{\partial \psi^k}{\partial x^i}(\chi(y)) \cdot \frac{\partial \chi^j}{\partial y^k}(y) = \delta_i^j$ for $i, j = 1, 2, 3$. Next we compute

$$\begin{aligned} \frac{\partial Q^{*k}}{\partial y^i} &= \frac{\partial}{\partial y^i}\left[\frac{\partial \psi^k}{\partial x^j}(\chi(y))Q^j(\chi(y))\right] \\ &= \frac{\partial}{\partial y^i}\left[\frac{\partial \psi^k}{\partial x^j}(\chi(y))\right] Q^j(\chi(y)) + \frac{\partial \psi^k}{\partial x^j}(\chi(y))\frac{\partial Q^j}{\partial y^i}(\chi(y)) \\ &= \frac{\partial}{\partial y^i}\left[\frac{\partial \psi^k}{\partial x^j}(\chi(y))\right] Q^j(\chi(y)) + \frac{\partial \psi^k}{\partial x^j}(\chi(y))\frac{\partial Q^j}{\partial x^\ell}(\chi(y))\frac{\partial \chi^\ell}{\partial y^i}, \end{aligned}$$

i.e.

$$\begin{aligned} \operatorname{div} Q^*(y) &= \frac{\partial}{\partial y^k}\left[\frac{\partial \psi^k}{\partial x^j}(\chi(y))\right] Q^j(\chi(y)) + \frac{\partial \psi^k}{\partial x^j}(\chi(y))\frac{\partial Q^j}{\partial x^\ell}(\chi(y))\frac{\partial \chi^\ell}{\partial y^k} \\ &= \frac{\partial}{\partial y^k}\left[\frac{\partial \psi^k}{\partial x^j}(\chi(y))\right] Q^j(\chi(y)) + \operatorname{div} Q(\chi(y)) \end{aligned}$$

which is of class $C^1(K)$. Now Lemma 1 can be applied and the proof can be completed as in Theorem 6. □

Theorem 7 (Regularity off the coincidence set). *Suppose that Assumption A is satisfied (possibly without condition (22)), $\mathcal{K}$ is quasiregular and $g_{ij} \in C^{1,\beta}(\mathcal{K})$, $\operatorname{div} Q \in C^{0,\beta}(\mathcal{K})$ for $0 < \beta < 1$ and $i, j = 1, 2, 3$. Let X be a solution for $\mathcal{P}(\Gamma, \mathcal{K})$ in $\mathcal{C}(\Gamma, \mathcal{K})$ and put $\Omega := \{w \in B : X(w) \in \partial\mathcal{K}\}$ to denote the coincidence set. Then $X \in C^{2,\beta}(B \setminus \Omega, \mathbb{R}^3)$ and satisfies the Euler equation (24) classically on $B \setminus \Omega$.*

Proof. By Theorem 4, $X \in C^0(\overline{B}, \mathbb{R}^3)$; therefore $B \setminus \Omega$ is an open set and for each $w_0 \in B \setminus \Omega$ there is a disk $B_\rho(w_0)$ which is contained in $B \setminus \Omega$. Consequently for any testfunction $\varphi \in C_c^\infty(B_\rho(w_0),\mathbb{R}^3)$ we have $X+\epsilon\varphi \in \mathcal{C}(\Gamma,\mathcal{K})$ for all $\epsilon \in (-\epsilon_0,\epsilon_0)$, $\epsilon_0=\epsilon_0(\varphi)>0$ sufficiently small, and the minimizing property of X implies

$$\mathcal{F}(X) \le \mathcal{F}(X+\epsilon\varphi) \quad \text{for all } \epsilon \in (-\epsilon_0,\epsilon_0).$$

Whence $\delta\mathcal{F}(X,\varphi)=0$ and by Theorem 5 we obtain

$$\int_B \left\{ g_{jk}(X) X^j_{u^\alpha}\varphi^k_{u^\alpha} + \frac{1}{2}\frac{\partial g_{jk}}{\partial x^\ell} X^j_{u^\alpha}X^k_{u^\alpha}\varphi^\ell + \operatorname{div} Q(X)\langle X_u \wedge X_v, \varphi\rangle \right\} du\, dv = 0.$$

Put $\varphi^j := g^{jk}(X)\psi^k$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}) and $\psi=(\psi^1,\psi^2,\psi^3)\in C_c^\infty(B_\rho(w_0),\mathbb{R}^3)$ is arbitrary. A simple calculation yields

$$\int_B \left\{ X^\ell_{u^\alpha}\psi^\ell_{u^\alpha} - \Gamma^\ell_{jk}(X) X^j_{u^\alpha}X^k_{u^\alpha}\psi^\ell + \operatorname{div} Q(X) g^{\ell m}(X)(X_u\wedge X_v)_m\psi^\ell \right\} du\, dv = 0$$

for all $\psi \in C_c^\infty(B\setminus\Omega,\mathbb{R}^3)$ applying appropriate partitions of unity. The fundamental lemma in the calculus of variations shows that (24) is the Euler equation of $\mathcal{F}$. A regularity theorem of Tomi [1] (for a similar reasoning due to Heinz see also Section 2.1 and 2.2) now implies that $X \in C^{1,\mu}(B\setminus\Omega,\mathbb{R}^3)$ for all $\mu \in (0,1)$. Alternatively, we might also apply Theorem 6 assuming the somewhat stronger hypotheses $g_{ij}\in C^2(S)$ and $Q \in C^2(S,\mathbb{R}^3)$, where S denotes an open set containing $\mathcal{K}$. Finally classical results from potential theory yields that $X \in C^{2,\beta}(B\setminus\Omega,\mathbb{R}^3)$. □

Now we solve the Plateau problem for surfaces of prescribed mean curvature H. We start with Jordan curves Γ which are contained in a closed ball $\overline{B_R(P_0)}\subset\mathbb{R}^3$.

Theorem 8. *Let $\mathcal{K}$ be the closed ball $\overline{B_R(P_0)}$ of radius R and center P_0 and denote by H a function of class $C^{0,\beta}(\mathcal{K}), 0<\beta<1$, satisfying*

$$|H|_{0,\mathcal{K}} < \frac{3}{2}R^{-1} \quad \text{and} \quad |H|_{0,\partial\mathcal{K}} \le R^{-1}.$$

Suppose $\Gamma\subset\mathcal{K}$ is a closed Jordan curve such that $\mathcal{C}(\Gamma,\mathcal{K})$ is nonempty. Then there exists a surface X of class $\mathcal{C}(\Gamma,\mathcal{K})\cap C^{2,\beta}(B,\mathbb{R}^3)\cap C^0(\overline{B},\mathbb{R}^3)$, which maps ∂B homeomorphically onto Γ and satisfies

$$\triangle X = 2H(X) X_u\wedge X_v \quad \text{in } B,$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0 \quad \text{in } B.$$

Proof. Without loss of generality we take $P_0 = 0 \in \mathbb{R}^3$ and extend H to some ball $\overline{B}_{R+r_0}(0)$ such that $|H|_{0,B_{R+r_0}} < \frac{3}{2}(R+r_0)^{-1}$ and $|H(x)|\,|x| \le 1$ for all $x \in \overline{B}_{R+r_0} - B_R$ and some $r_0 > 0$. We remark here that the first variation formula (23) of Theorem 5 extends to cases where Q is not necessarily of class C^1 but $\operatorname{div} Q$ is defined (possibly in a weak sense). Here we define the vectorfield

$$Q(x) = \frac{2}{3}\left(\int_0^{x^1} H(\tau, x^2, x^3)\,d\tau, \int_0^{x^2} H(x^1, \tau, x^3)\,d\tau, \int_0^{x^3} H(x^1, x^2, \tau)\,d\tau\right)$$

which, although not necessarily of class $C^1(B_{R+r_0}, \mathbb{R}^3)$, satisfies $\operatorname{div} Q = 2H$. We claim that $\delta\mathcal{F}_Q(X, \varphi)$ exists for all $X \in H_2^1(B, \overline{B}_{R+r_0})$, $\varphi \in \overset{\circ}{H}{}_2^1(B, \mathbb{R}^3) \cap L_\infty(B, \mathbb{R}^3)$ and is given by (23) i.e.

$$\delta\mathcal{F}_Q(X, \varphi) = \int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\,du\,dv.$$

Note that here we have written $\mathcal{F}_Q$ to indicate the dependence of $\mathcal{F}$ on Q. Now, to see that (23) holds in this case we take a sequence $H_n \in C^1(\overline{B}_{R+r_0})$ s.t. $|H_n - H|_{0,B_{R+r_0}} \to 0$, $n \to \infty$ and define $Q_n \in C^1(\overline{B}_{R+r_0}, \mathbb{R}^3)$ by

$$Q_n(x) = \frac{2}{3}\left(\int_0^{x^1} H_n(\tau, x^2, x^3)\,d\tau, \int_0^{x^2} H_n(x^1, \tau, x^3)\,d\tau, \int_0^{x^3} H_n(x^1, x^2, \tau)\,d\tau\right)$$

and

$$\mathcal{F}_{Q_n}(X) = \frac{1}{2}\int_B |\nabla X|^2\,du\,dv + \int_B \langle Q_n(X), X_u \wedge X_v\rangle\,du\,dv.$$

Relation (23) of Theorem 5 implies

$$\begin{aligned}\delta\mathcal{F}_{Q_n}(X, \varphi) &= \int_B \{\langle \nabla X, \nabla\varphi\rangle + \operatorname{div} Q_n(X)\langle X_u \wedge X_v, \varphi\rangle\}\,du\,dv\\ &= \int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H_n(X)\langle X_u \wedge X_v, \varphi\rangle\}\,du\,dv,\end{aligned}$$

whence, as $n \to \infty$

$$\delta\mathcal{F}_{Q_n}(X, \varphi) \to \int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\,du\,dv. \tag{36}$$

On the other hand we have

$$\frac{\mathcal{F}_Q(X+\epsilon\varphi)-\mathcal{F}_Q(X)}{\epsilon} = \frac{\mathcal{F}_{Q_n}(X+\epsilon\varphi)-\mathcal{F}_{Q_n}(X)}{\epsilon}$$
$$+\frac{1}{\epsilon}\{\mathcal{F}_Q(X+\epsilon\varphi)-\mathcal{F}_Q(X)-\mathcal{F}_{Q_n}(X+\epsilon\varphi)+\mathcal{F}_{Q_n}(X)\}$$
$$= \frac{\mathcal{F}_{Q_n}(X+\epsilon\varphi)-\mathcal{F}_{Q_n}(X)}{\epsilon}$$
$$+\frac{1}{\epsilon}\left\{\int_B \langle Q-Q_n,(X_u+\epsilon\varphi_u)\wedge(X_v+\epsilon\varphi_v)\rangle\,du\,dv\right.$$
$$\left.+\int_B \langle Q-Q_n, X_u\wedge X_v\rangle\,du\,dv\right\} = \frac{\mathcal{F}_{Q_n}(X+\epsilon\varphi)-\mathcal{F}_{Q_n}(X)}{\epsilon}$$
$$+\int_B \langle Q-Q_n, X_u\wedge\varphi_v+\varphi_u\wedge X_v\rangle\,du\,dv$$
$$+\epsilon\int_B \langle Q-Q_n,\varphi_u\wedge\varphi_v\rangle\,du\,dv.$$

Letting $\epsilon\to 0$ we find that $\delta\mathcal{F}_Q(X,\varphi)$ exists and is given by

$$\delta\mathcal{F}_Q(X,\varphi)=\delta\mathcal{F}_{Q_n}(X,\varphi)+\int_B \langle Q-Q_n, X_u\wedge\varphi_v+\varphi_u\wedge X_v\rangle\,du\,dv.$$

Since $|Q-Q_n|_{0,B_{R+r_0}}\le \text{const}|H-H_n|_{0,B_{R+r_0}}\to 0$ as $n\to\infty$ we conclude, by letting $n\to\infty$ and using (36), the first variation formula

$$\delta\mathcal{F}_Q(X,\varphi)=\int_B\{\langle\nabla X,\nabla\varphi\rangle+2H(X)\langle X_u\wedge X_v,\varphi\rangle\}\,du\,dv. \tag{37}$$

Next we observe that for every $x\in\overline{B_{R+r_0}}(0)$ we have $|Q(x)|\le\frac{2}{3}|x|\ |H|_{0,B_{R+r_0}}$, whence $|Q|_{0,B_{R+r_0}}<1$. By the discussion following Theorem 1 and by virtue of Theorems 3 and 4 we can find a solution $X\in\mathcal{C}(\Gamma,\overline{B_{R+r_0}(0)})$ of the variational problem $\mathcal{F}_Q(X)\to\min$ in the class $\mathcal{C}(\Gamma,\overline{B_{R+r_0}(0)})$, which in addition belongs to the spaces $C^{0,\alpha}(B,\mathbb{R}^3)\cap C^0(\overline{B},\mathbb{R}^3)$. Consider the function $\varphi(w):=\max(|X(w)|^2-R^2,0)\cdot X$ which is of class $\mathring{H}^1_2(B,\mathbb{R}^3)\cap L_\infty(B,\mathbb{R}^3)$ and satisfies $X-\epsilon\varphi\in\mathcal{C}(\Gamma,\overline{B_{R+r_0}(0)})$ for all $\epsilon\in[0,\epsilon_0)$, provided ϵ_0 is sufficiently small. Since X is a minimizer in that class we have $\mathcal{F}_Q(x)\le\mathcal{F}_Q(x-\epsilon\varphi)$ for all $\epsilon\in[0,\epsilon_0)$ and therefore $\delta\mathcal{F}_Q(X,\varphi)\le 0$. On the other hand we compute, using well known properties to Sobolev functions

$$\nabla\varphi=(\varphi_u,\varphi_v)=\begin{cases}2\langle X,\nabla X\rangle X+(|X|^2-R^2)\nabla X, & \text{on } \{w:|X(w)|>R\},\\ 0, & \text{on } \{w:|X(w)|\le R\}.\end{cases}$$

From the first variation formula (37) and the variational inequality $\delta\mathcal{F}_Q(X,\varphi)\le 0$ we derive

$$\int_{B\cap\{|X(w)|>R\}}\{2\langle X,X_u\rangle^2+2\langle X,X_v\rangle^2+(|X|^2-R^2)(|X_u|^2+|X_v|^2)$$
$$+2H(X)\langle X,X_u\wedge X_v\rangle(|X|^2-R^2)\}\,du\,dv\le 0. \tag{38}$$

But on the set $\{w : |X(w)| > R\}$ we have

$$
\begin{aligned}
|2H(X)(|X|^2 - R^2)\langle X, X_u \wedge X_v\rangle| &\le (|X|^2 - R^2)|H(X)|\,|X|(|X_u|^2 + |X_v|^2) \\
&\le (|X|^2 - R^2)(|X_u|^2 + |X_v|^2),
\end{aligned}
$$

whence by (38) it follows that $\langle X, X_u\rangle = \langle X, X_v\rangle = 0$ a.e. on $\{w : |X(w)| > R\}$. This implies that the function $\eta(w) := \max(|X(w)|^2 - R^2, 0)$ belongs to $H_2^1(B) \cap C^0(\overline{B})$ whose derivative is

$$
\nabla\eta = \begin{cases} \langle X, \nabla X\rangle & \text{on } \{|X(w)|^2 > R^2\}, \\ 0 & \text{on } \{|X(w)|^2 \le R^2\} \end{cases}
$$

must vanish identically on B, since $\eta = 0$ on ∂B. Therefore $|X(w)| \le R$ on B and the coincidence set $\Omega = \{w \in B : X(w) \in \partial B_{R+r_0}\}$ is empty. Now observe that Theorem 7 is applicable here, since we have already proved the variational formula (37) to also hold in this case; furthermore we have by assumption $\operatorname{div} Q = 2H \in C^{0,\beta}(\mathcal{K})$. By Theorem 7 we get $X \in C^{2,\beta}(B, \mathbb{R}^3)$ and the system

$$
\begin{gathered}
\triangle X = 2H(X) X_u \wedge X_v, \\
|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0 \quad \text{in } B
\end{gathered}
$$

is satisfied in a classical sense.

The topological character of the boundary mapping $X_{|\partial B} : \partial B \to \Gamma$ is proved similarly as in Theorem 3 of Chapter 4.5 in Vol. 1. Indeed in some neighbourhood of a boundary branch point $w_0 \in \partial B$ we have the asymptotic expansion $X_w(w) = a(w - w_0)^\nu + o(|w - w_0|^\nu)$ for some integer $\nu \ge 1$ and some $a \in \mathbb{C}^3 \setminus \{0\}$, provided X is of class C^1 in a neighbourhood $U_0 \subset \overline{B}$ of w_0 (cf. Section 2.10). Therefore $|\nabla X(w)| > 0$ for $w \in \partial B$ with $0 < |w - w_0| < \epsilon$. We conclude that $X(w)$ cannot be constant on any open arc $\Gamma_0 \subset \partial B$, because this would imply $X \in C^1(B \cup \Gamma_0, \mathbb{R}^3)$ and, because of the conformality relations, $\nabla X = 0$ on Γ_0, an obvious contradiction. □

Remark. The proof of Theorem 8 also shows the existence of a conformal weak solution $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{1,\alpha}(B, \mathbb{R}^3)$ of the system $\triangle X = 2H(X) X_u \wedge X_v$, if H is only of class $C^0(\mathcal{K})$; also X maps ∂B homeomorphically onto Γ.

By Theorem 1 the sharpness of the existence result Theorem 8 follows if all closed curves $\Gamma \subset B_R(p_0)$ are considered. However, for certain shapes one expects better results for geometric reasons. Consider for instance a long and "thin" Jordan curve Γ, say a slightly perturbed rectangle of sidelengths ϵ and ϵ^{-1} respectively where $\epsilon > 0$ is small. Then Theorem 8 asserts the existence of a solution if $|H| < \epsilon$. However, a much better result holds in this situation.

Theorem 9. *Suppose $\mathcal{K} \subset \mathbb{R}^3$ is a closed circular cylinder $\overline{C}_R$ of radius $R > 0$ and $\Gamma \subset \overline{C}_R$ is a closed Jordan curve such that $\mathcal{C}(\Gamma, \mathcal{K})$ is nonempty. Denote* *by H a function of class $C^{0,\beta}(\mathcal{K}), 0 < \beta < 1$, satisfying $|H|_{0,\partial\mathcal{K}} \le \frac{1}{2R}$ and $|H|_{0,\mathcal{K}} < \frac{1}{R}$. Then the Plateau problem determined by H and Γ is solvable, i.e. there exists a surface $X \in \mathcal{C}(\Gamma, \mathcal{K}) \cap C^{2,\beta}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ with*

$$\triangle X = 2H(X) X_u \wedge X_v \quad \text{in } B, \quad \text{and} \quad |X_u|^2 = |X_v|^2, \langle X_u, X_v\rangle = 0 \quad \text{in } B,$$

which maps ∂B homeomorphically onto Γ.

Proof. The proof is similar to the one of Theorem 8. Without loss of generality, we assume at the outset that

$$\mathcal{K} = \overline{C}_R = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 \le R^2\},$$

and $H \in C^0(\overline{C}_{R_0})$ for some $R_0 > R$, satisfies

$$|H|_{0,R_0} < \frac{1}{R_0}, \quad |y|\ |H(x)| \le \frac{1}{2} \tag{39}$$

for all $x = (x^1, x^2, x^3) \in \overline{C}_{R_0} \setminus C_R$ and $y := (x^1, x^2, 0)$. As vector field Q we choose

$$Q(x) := \left(\int_0^{x^1} H(\tau, x^2, x^3)\, d\tau,\ \int_0^{x^2} H(x^1, \tau, x^3)\, d\tau, 0 \right),$$

which again satisfies

$$\operatorname{div} Q(x) = 2H(X) \quad \text{in } C_{R_0}$$

and

$$|Q(x)| = \{(Q^1(x))^2 + (Q^2(x))^2\}^{\frac{1}{2}} \le |H|_{0,C_{R_0}} \{(x^1)^2 + (x^2)^2\}^{\frac{1}{2}} = |H|_{0,C_{R_0}} |y|.$$

Whence, by (39) it follows that $|Q|_{0,C_{R_0}} < 1$. Therefore the variational problem

$$(\mathcal{P}): \qquad \mathcal{F}(X) = \frac{1}{2} \int_B |\nabla X|^2\, du\, dv + \int_B \langle Q(X), X_u \wedge X_v\rangle\, du\, dv \to \min$$

in $\mathcal{C}(\Gamma, \overline{C}_{R_0})$ is solvable; let $X \in \mathcal{C}(\Gamma, \overline{C}_{R_0}) \cap C^{0,\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ be a conformally parametrized solution (cf. Theorems 3 and 4). Denote by $Y(w) := (x^1(w), x^2(w), 0)$ the projection of $X(w)$ onto the plane $x^3 = 0$ and consider the $\mathring{H}_2^1(B, \mathbb{R}^3) \cap L_\infty(B, \mathbb{R}^3)$ function $\varphi(w) := \max(|Y(w)|^2 - R^2, 0) \cdot Y(w)$. We have $X - \epsilon\varphi \in \mathcal{C}(\Gamma, \overline{C}_{R_0}) \cap H_2^1(B, \overline{C}_{R_0})$ for all $\epsilon \in [0, \epsilon_0)$, provided $\epsilon_0 > 0$ is sufficiently small. Whence, by the minimality of X,

$$\mathcal{F}(X) \le \mathcal{F}(X - \epsilon\varphi) \quad \text{for all } \epsilon \in [0, \epsilon_0). \tag{40}$$

By the same reasoning as in the proof of Theorem 8 we see that the first variation $\delta\mathcal{F}(X, \varphi)$ exists and is given by (see relation (37))

$$\delta\mathcal{F}(X,\varphi) = \int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\, du\, dv,$$

whence by (40) we arrive at the variational inequality

$$\int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\, du\, dv \le 0.$$

Now, since

$$\nabla\varphi = (\varphi_u, \varphi_v) = \begin{cases} 2\langle Y, \nabla Y\rangle Y + (|Y|^2 - R^2)\nabla Y & \text{on } \{|Y(w)| > R\}, \\ 0 & \text{on } \{|Y(w)| \le R\} \end{cases}$$

we infer

$$(41) \qquad \delta\mathcal{F}(X,\varphi) = \int_{B\cap\{|Y|>R\}} \{2\langle Y, Y_u\rangle^2 + 2\langle Y, Y_v\rangle^2 + (|Y|^2 - R^2)(|Y_u|^2 + |Y_v|^2) + 2H(X)\langle X_u \wedge X_v, Y\rangle(|Y|^2 - R^2)\}\, du\, dv \le 0.$$

By virtue of the conformality relation $|X_u|^2 = |X_v|^2, \langle X_u, X_v\rangle = 0$ a.e. on B, we obtain as in the proof of Theorem 2 in Section 4.1 the inequality

$$(42) \qquad |\nabla x^3|^2 \le |\nabla x^1|^2 + |\nabla x^2|^2 = |\nabla Y|^2.$$

Whence

$$\begin{aligned} 2|H(X)\langle X_u \wedge X_v, Y\rangle| &\le 2|H(X)| \cdot |Y| \cdot \{|x_u^2 x_v^3 - x_u^3 x_v^2|^2 + |x_u^3 x_v^1 - x_u^1 x_v^3|^2\}^{\frac{1}{2}} \\ &\le 2|H(X)|\, |Y| \{|\nabla x^2|^2 |\nabla x^3|^2 + |\nabla x^1|^2 |\nabla x^3|^2\}^{\frac{1}{2}} = 2|H(X)||Y|\, |\nabla x^3|\, |\nabla Y| \\ &\le 2|H(X)|\, |Y|\, |\nabla Y|^2 \le |\nabla Y|^2 = |Y_u|^2 + |Y_v|^2 \quad \text{a.e. on } \{w : |Y(w)| > R\}, \end{aligned}$$

where we have used (42) and (39). By virtue of (41) this now implies that $\langle Y, Y_u\rangle = \langle Y, Y_v\rangle = 0$ a.e. on $\{w : |Y(w)| > R\}$. In other words, the H_2^1-function $\eta(w) := \max(|Y(w)|^2 - R^2, 0)$ has vanishing derivative a.e. in B and hence vanishes identically. This means that the coincidence set $\Omega := \{w \in B : X(w) \in \partial C_{R_0}\}$ is empty and by Theorem 7 we conclude that $X \in C^{2,\beta}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ satisfies the Euler equation

$$\triangle X = 2H(X) X_u \wedge X_v \quad \text{in } B$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0$$

in the classical sense. The rest of the proof is the same as in Theorem 8. □

Now we consider Plateau's problem for surfaces of prescribed mean curvature H and boundary Γ which are confined to arbitrary sets $\mathcal{K}$. In particular it is desirable to describe geometric conditions on H and $\mathcal{K}$ or $\partial\mathcal{K}$ respectively, which guarantee the existence of a solution to this problem. In this respect Theorem 2 and Enclosure Theorems 2 and 3 of Section 4.4 are of crucial importance. We recall the definition of the "mean curvature" function $\Lambda_\rho(x)$ for $x \in \mathcal{K}$ to denote the mean curvature at x of the surface $\mathcal{S}_{\rho(x)}$ through x which is parallel to $\partial\mathcal{K}$ at distance $\rho = \rho(x)$, if this is defined and is equal to infinity otherwise.

Theorem 10. *Suppose $\mathcal{K} \subset \mathbb{R}^3$ is the closure of a C^3 domain whose boundary $\partial\mathcal{K}$ has uniformly bounded principal curvatures and a global inward parallel surface at distance $\epsilon > 0$. Assume also that $\sup_{\mathcal{K}} \rho(x) < \infty$ and $H \in C^1(\mathcal{K})$ has uniformly bounded C^1-norm on $\mathcal{K}$ with*

$$|H(x)| \le \Lambda(x) \quad \textit{for all } x \in \partial\mathcal{K}, \tag{43}$$

and

$$|H(x)| \le (1 - a\rho(x))\Lambda_\rho(x) + \frac{a}{2} \quad \textit{for all } x \in \mathcal{K} \tag{44}$$

and some number a, $0 \le a \le \inf_{\mathcal{K}} \rho^{-1}(x)$. Finally let $\Gamma \subset \mathcal{K}$ denote a closed Jordan curve such that $\mathcal{C}(\Gamma, \mathcal{K}) \neq \emptyset$. Then there exists a solution $X \in C^{2,\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathcal{K})$ of the Plateau problem which is determined by H and Γ. Furthermore X satisfies the H-surface system 1) *and* 2) *classically in B and maps the boundary of B homeomorphically onto Γ. Moreover, if in addition*

$$|H(x)| \le \Lambda_\rho(x) \tag{45}$$

holds for all x in a small strip in $\mathcal{K}$ near $\partial\mathcal{K}$ and $\Gamma \cap$ int $\mathcal{K} \neq \emptyset$, then every solution X maps B into the interior of $\mathcal{K}$. Finally, if for some point $x_0 \in \partial\mathcal{K}$ we have

$$|H(x_0)| < \Lambda(x_0), \tag{46}$$

then there is a neighbourhood $U(x_0) \subset \mathbb{R}^3$ such that no $w_0 \in B$ is mapped into $U(x_0)$. In particular if (46) *holds true for all $x_0 \in \mathcal{K}$, then $X(B) \subset \operatorname{int}\mathcal{K}$. (Clearly,* (45) *follows from* (44)*, if $a = 0$.)*

Proof. First we remark that $\mathcal{K}$ is quasiregular; for a proof see Lemma 2.4 in Gulliver and Spruck [2]. Furthermore by Theorem 2 there is a vector field $Q \in C^1(\mathcal{K}, \mathbb{R}^3)$ which satisfies

$$\operatorname{div} Q(x) = 2H(x) \quad \text{for all } x \in \mathcal{K}$$

and $|Q|_{0,\mathcal{K}} < 1$. Now Theorems 3, 4 and 6, in particular the Remark at the end of the proof of Theorem 6 imply the existence of a conformally parametrized solution $X \in \mathcal{C}(\Gamma, \mathcal{K}) \cap H^2_{s,\mathrm{loc}}(B, \mathbb{R}^3) \cap C^{1,\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$, for all $s < \infty$, and $0 < \alpha < 1$, of the variational problem

$$\mathcal{P}(\Gamma, \mathcal{K}): \quad \mathcal{F}(X) = \frac{1}{2}\int_B |\nabla X|^2\, du\, dv + \int_B \langle Q(X), X_u \wedge X_v\rangle\, du\, dv \to \min \quad \text{in } \mathcal{C}(\Gamma, \mathcal{K}).$$

By Theorem 5 the first variation $\delta\mathcal{F}(X, \varphi)$ exists, is given by

$$\delta\mathcal{F}(X, \varphi) = \int_B \{\langle \nabla X, \nabla\varphi\rangle + 2H(X)\langle X_u \wedge X_v, \varphi\rangle\}\, du\, dv$$

and satisfies – since X is a minimum of $\mathcal{F}$ in $\mathcal{C}(\Gamma, \mathcal{K})$ – the relation

$$\delta\mathcal{F}(X, \varphi) \geq 0$$

for all $\varphi \in \overset{\circ}{H}{}^1_2(B, \mathbb{R}^3) \cap L_\infty(B)$ such that $(X + \epsilon\varphi) \in \mathcal{C}(\Gamma, \mathcal{K})$. Assumption (43) together with Enclosure Theorem 3 of Section 4.4 yield that $X \in H^2_{s,\mathrm{loc}}(B, \mathbb{R}^3) \cap C^{1,\alpha}(B, \mathbb{R}^3)$ satisfies the system

$$\triangle X = 2H(X) X_u \wedge X_v$$

almost everywhere in B. Since the right hand side is Hölder continuous it follows from Schauder theory that X is of class $C^{2,\alpha}(B, \mathbb{R}^3)$ and satisfies the H-surface system in a classical sense.

By Enclosure Theorem 2 of Section 4.4 and since $X \in C^0(\overline{B}, \mathbb{R}^3)$, we see that $X(B) \subset \operatorname{int}\mathcal{K}$, if (45) holds and $\Gamma \cap \operatorname{int}\mathcal{K} \neq \emptyset$. The rest of the assertion is a consequence of Corollary 3 in Section 4.4. That the boundary mapping $X_{|\partial B} : \partial B \to \Gamma$ is a homeomorphism follows in a standard manner. Theorem 10 is completely proved. □

Let us close this section with a simple example when $\mathcal{K} = \{\xi \in \mathbb{R}^3 : |\xi| \leq R\}$ is the closed ball of radius R and center zero. Formula (44) then is equivalent to

$$|H(x)| \leq (1 - a(R - |x|))\frac{1}{|x|} + \frac{a}{2},$$

where $0 \leq a \leq R^{-1}$; or

$$|H(x)| \leq \frac{1}{|x|}(1 - aR) + \frac{3a}{2}$$

for all $x \in \mathcal{K}$. For $a = R^{-1}$ we recover the result of Theorem 8, while a new existence result is obtained when $a = 0$. In this case the condition requires $|H(x)| \leq \frac{1}{|x|}$ for all $x \in \mathcal{K}$, whence we obtain the existence of an H-surface in $\mathcal{K}$ which lies strictly interior to $\mathcal{K}$ if $\Gamma \cap \operatorname{int}\mathcal{K} \neq \emptyset$.

4.8 Surfaces of Prescribed Mean Curvature in a Riemannian Manifold

In this section we shall extend the methods which we have introduced in Section 4.7 to surfaces of prescribed mean curvature in a three-dimensional Riemannian manifold. We assume the reader's acquaintance with basic Riemannian geometry; however we repeat some of the underlying concepts and calculations when assumed necessary. In particular we discuss in this section estimates for Jacobi fields. As standard reference on differential geometry we refer to the monographs by Gromoll, Klingenberg, and Meyer [1], do Carmo [3], Jost [18], and Kühnel [2], and we also refer to Chapter 1 of Vol. 1, where most of the formulas needed later can also be found. In what follows we shall assume, unless stated otherwise, that M is a three-dimensional, connected, orientable, and complete Riemannian manifold of class C^4 with scalar product $\langle X, Y\rangle$ and norm $\|X\| = \langle X, X\rangle^{\frac{1}{2}}$ for $X, Y \in T_pM$, $p \in M$, where T_pM denotes the tangent space of M at p. Observe that this notation contrasts with the one in the last section, where $\langle\cdot,\cdot\rangle$ has denoted the Euclidean scalar product, which in this chapter will simply be written as $X \cdot Y$.

If $\varphi : U \to \mathbb{R}^3$, $U \subset M$ an open set, denotes a chart we let $x = (x^1, x^2, x^3) = \varphi(p)$ stand for the local coordinates and $\partial_k = \frac{\partial}{\partial x^k} = X_k$ denote their basis fields. We put

$$\begin{aligned} g_{ij}(x) &= \langle \partial_i, \partial_j\rangle = \langle X_i, X_j\rangle, \quad g(x) = \det(g_{ij}(x)),\\ (g^{ij})_{i,j} &= (g_{ij})^{-1}_{i,j}, \quad D_{\partial_i}\partial_j = \Gamma^\ell_{ij}\partial_\ell = D_{X_i}X_j = \Gamma^\ell_{ij}X_\ell \end{aligned}$$

and $\Gamma_{ijk} = \langle D_{\partial_i}\partial_k, \partial_j\rangle$, compare the formulas in Vol. 1, Section 1.5. Here D denotes covariant differentiation on M, g_{ij} is the metric and Γ_{ijk}, Γ^k_{ij} stand for the Christoffel symbols. From Chapter 1 we recall the relation

$$\Gamma^k_{ij} = g^{km}\Gamma_{imj} \quad \text{and} \quad \Gamma_{ijk} = \frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k}\right\}.$$

A mapping $f : B \to M$ of the unit disk B into M represents a surface of (prescribed) mean curvature H in M, if it is of class C^2 and any local representation $X(w) = \varphi \circ f(w)$ satisfies in B (or a suitable subset of B) the system

$$\triangle X^\ell + \Gamma^\ell_{jk}X^j_{u^\alpha}X^k_{u^\alpha} = 2H(X)\sqrt{g(x)}g^{\ell m}(X)(X_u \wedge X_v)_m$$

for $\ell = 1, 2, 3$, and the conformality condition

$$g_{ij}X^i_uX^j_u = g_{ij}X^i_vX^j_v, \quad g_{ij}X^i_uX^j_v = 0.$$

We shall confine ourselves to surfaces which are contained in a "Riemann normal chart" (φ, U) with center $p \in M$. Here (φ, U) is called a Riemann normal chart with center p, if $U \subset M$ is an open set with $p \in U$ and $\varphi : U \to$

$\mathbb{R}^3$ is of the form $\varphi = j \circ \exp_p^{-1}$, where $\exp_p : T_pM \to M$ is the exponential map with center p and $j : T_pM \to \mathbb{R}^3$ is a linear isometry. Recall that the map $\exp_p : T_pM \to M$ is defined by $\exp_p(v) = c(1)$ for $v \in T_pM$, where $c = c(t)$ is the geodesic in M with $c(0) = p$ and $\dot{c}(0) = v$. Hence every point $q \in U$ can be connected with p by exactly one shortest geodesic which is the image of a straight line through 0 in T_pM under the exponential map $\exp_p$.

Since we want to solve the Plateau problem for surfaces of prescribed mean curvature in M via a minimization procedure of the functional $\mathcal{F}(X)$ which we have investigated in Section 4.7, it is of crucial importance to have a quantitative control of the metric tensor and the Christoffel symbols in terms of the curvature of the underlying manifold M. This will be established by invoking estimates for Jacobi fields along geodesics. These estimates are of independent interest and will be of importance later in Subsection 4.8.3.

4.8.1 Estimates for Jacobi Fields

Throughout this subsection we assume that M is a complete m-dimensional Riemannian manifold of class C^4 with covariant derivative D and Riemann curvature tensor $R(X,Y)Z$ (for a definition and properties of R, see e.g. Vol. 1, Sections 1.3 and 1.5). A geodesic $c(t)$ starting for $t = 0$ at $p \in M$ is then defined for all times $t \geq 0$.

A vector field J along a geodesic $c : [0, \infty) \to M$ with $\dot{c}(0) \neq 0$ is said to be a *Jacobi field* along c if it satisfies

$$\frac{D}{dt}\frac{D}{dt}J + R(J, \dot{c})\dot{c} = 0. \tag{1}$$

If no misunderstanding is possible, we shall abbreviate both the ordinary derivation $\frac{d}{dt}$ and the covariant derivation $\frac{D}{dt}$ with a superscript dot. Then (1) takes the form

$$\ddot{J} + R(J, \dot{c})\dot{c} = 0. \tag{1'}$$

Here $R(X,Y)Z$ denotes the Riemann curvature tensor of M. The linear equation (1), the so-called *Jacobi equation* of the geodesic c, is nothing but the Euler equation of the second variation of the Dirichlet integral $\int \langle \dot{c}, \dot{c} \rangle \, dt$ at c. In local coordinates, the Jacobi equation is equivalent to the system of m linear ordinary differential equations of second order

$$\ddot{\eta}^k + R^k_{\ell rs}(c)\eta^\ell \dot{c}^r \dot{c}^s = 0$$

for the unknown functions $\eta^k(t), k = 1, \ldots, m$. Thus the Jacobi fields along a geodesic c span a $2m$-dimensional linear space over $\mathbb{R}$ which we denote by J_c. In particular, the tangent vector $\dot{c}$ of a geodesic c is a Jacobi field of constant length $\|\dot{c}(0)\|$ along c, since

$$\frac{D}{dt}\dot{c} = 0, \quad R(\dot{c}, \dot{c})\dot{c} = 0$$

and

$$\frac{d}{dt}\|\dot{c}\|^2 = 2\left\langle \dot{c}, \frac{D}{dt}\dot{c}\right\rangle = 0.$$

Moreover, if J and $J^* \in J_c$, then

$$\begin{aligned}\frac{d}{dt}\left\{\langle \dot{J}, J^*\rangle - \langle J, \dot{J}^*\rangle\right\} &= \langle \ddot{J}, J^*\rangle - \langle J, \ddot{J}^*\rangle \\ &= -\langle R(J,\dot{c})\dot{c}, J^*\rangle + \langle R(J^*,\dot{c})\dot{c}, J\rangle = 0.\end{aligned}$$

We therefore obtain

$$\langle \dot{J}, J^*\rangle - \langle J, J^*\rangle = \text{const} \quad \text{for all } J, J^* \in J_c$$

and in particular, for $J^* = \dot{c}$, we arrive at

$$(2) \qquad \langle \dot{J}, \dot{c}\rangle = \text{const} \quad \text{for all } J \in J_c.$$

Suppose now that $c : [0,\infty) \to M$ is a geodesic normalized by the condition $\|\dot{c}\| = 1$. Then, by setting

$$J^T = \alpha\dot{c}, \quad \alpha = \langle J, \dot{c}\rangle, \quad J^\perp = J - J^T,$$

we can decompose each Jacobi field $J \in J_c$ into a tangential component J^T and a normal component $J^\perp$:

$$J = J^T + J^\perp.$$

We claim that both J^T and $J^\perp$ are Jacobi fields. In fact, equation (2) implies $\ddot{\alpha} = 0$, and therefore $(J^T)^{\cdot\cdot} + R(J^T,\dot{c})\dot{c} = (J^T)^{\ddot{}} = (\alpha\dot{c})^{\dot{}} = (\dot{\alpha}\dot{c})^{\dot{}} = \ddot{\alpha}\dot{c} = 0$ if we take $\ddot{c} = 0$ into account.

The tangential part J^T is of the form

$$(3) \qquad J^T(t) = \{at + b\}\dot{c}(t),$$

where

$$(3') \qquad a = \langle \dot{J}(0), \dot{c}(0)\rangle, \quad b = \langle J(0), \dot{c}(0)\rangle.$$

Thus the growth of the tangential part $J^T(t)$ can easily be determined from the initial values $J(0)$ and $\dot{J}(0)$.

Hence we can control the growth of all Jacobi fields if we can estimate the normal Jacobi fields. These are the elements of J_c orthogonal to $\dot{c}$ which, by (3), span a $(2m-2)$-dimensional subspace of J_c that is denoted by $J_c^\perp$.

Unfortunately, there is no simple way to compute the normal Jacobi fields, yet they can fairly well be estimated in terms of upper and lower bounds on the sectional curvature of M. To see this, we consider the solutions of the scalar differential equation

$$\ddot{f} + \kappa f = 0, \quad \kappa \in \mathbb{R},$$

which also satisfy

$$-\left(\frac{\dot{f}}{f}\right)^{\cdot} = \kappa + \left(\frac{\dot{f}}{f}\right)^2,$$

wherever f does not vanish. In particular the solutions s_κ and c_κ of the initial value problems

$$\boxed{\begin{array}{l} \ddot{s}_\kappa + \kappa s_\kappa = 0 \\ s_\kappa(0) = 0, \ \dot{s}_\kappa(0) = 1 \end{array}} \quad \text{and} \quad \boxed{\begin{array}{l} \ddot{c}_\kappa + \kappa c_\kappa = 0 \\ c_\kappa(0) = 1, \ \dot{c}_\kappa(0) = 0 \end{array}}$$

We have

$$\begin{array}{lll} s_\kappa(t) = t, & c_\kappa(t) = 1 & \text{if } \kappa = 0, \\[2ex] s_\kappa(t) = \dfrac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t, & c_\kappa(t) = \cos \sqrt{\kappa} t & \text{if } \kappa > 0, \\[2ex] s_\kappa(t) = \dfrac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} t, & c_\kappa(t) = \cosh \sqrt{-\kappa} t & \text{if } \kappa < 0. \end{array}$$

Put

$$t_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\[2ex] +\infty & \text{if } \kappa \le 0 \end{cases}$$

that is, t_κ is the first positive zero of $s_\kappa(t)$.

Lemma 1. *Let $c : [0, \infty) \to M$ be a geodesic with $\|\dot{c}\| = 1$, and suppose that some $J \in J_c^\perp$ satisfies $\|J\| > 0$ on $(0, t^*)$. Finally we assume that, for some number κ, the sectional curvature K of M is bounded on $\Gamma_{t^*} = \{c(t) : 0 \le t \le t^*\}$ by the inequality $K \le \kappa$. Then $\|J\|$ satisfies the differential inequality*

$$\frac{d^2}{dt^2} \|J\| + \kappa \|J\| \ge 0 \quad \text{on } (0, t^*). \tag{4}$$

Proof. We first obtain

$$\frac{d}{dt} \|J\| = \|J\|^{-1} \langle J, \dot{J} \rangle, \tag{5}$$

whence

$$\begin{aligned} \frac{d^2}{dt^2} \|J\| &= \|J\|^{-1} \langle J, \ddot{J} \rangle + \|J\|^{-1} \|\dot{J}\|^2 - \|J\|^{-3} \langle J, \dot{J} \rangle^2 \\ &= \|J\|^{-1} \langle J, \ddot{J} \rangle + \|J\|^{-3} \left\{ \|J\|^2 \|\dot{J}\|^2 - \langle J, \dot{J} \rangle^2 \right\}, \end{aligned}$$

and, by Schwarz's inequality, we arrive at

$$\frac{d^2}{dt^2}\|J\| \geq \|J\|^{-1}\langle J, \ddot{J}\rangle. \tag{6}$$

The Jacobi equation (1′), on the other hand, implies

$$\langle J, \ddot{J}\rangle = -\langle R(J, \dot{c})\dot{c}, J\rangle.$$

The term on the right hand side is nothing but $-K\|J\|^2$, where $K = K(t)$ denotes the sectional curvature of M at $c(t)$ with respect to the two-plane spanned by $J(t)$ and $\dot{c}(t)$. Thus we find

$$\langle J, \ddot{J}\rangle\|J\|^{-1} = -K\|J\| \geq -\kappa\|J\|. \tag{7}$$

Finally, (4) follows from (6) and (7).

Lemma 2. *Let the assumption of Lemma 1 be satisfied. If, moreover, we assume that $J(0) = 0$ and $t^* \leq t_\kappa$, then*

$$\frac{d}{dt}\left\{\frac{\|J\|}{s_\kappa}\right\} \geq 0 \quad \text{on } (0, t^*). \tag{8}$$

Proof. Set

$$Z = \|J\|^{\cdot} s_\kappa - \|J\|\dot{s}_\kappa.$$

Then, for $0 < t < t^*$, we obtain $Z(t) \geq 0$ since

$$\dot{Z} = \|J\|^{\cdot\cdot} s_\kappa - \|J\|\ddot{s}_\kappa = s_\kappa\left\{\|J\|^{\cdot\cdot} + \kappa\|J\|\right\} \geq 0,$$

if we take (4) into account. Hence, for any $t_0 \in (0, t^*)$, we infer that

$$Z(t) \geq Z(t_0) \quad \text{for all } t \in (t_0, t^*).$$

Moreover, (5) yields

$$\|J\|^{\cdot} \leq \|\dot{J}\|,$$

and therefore

$$|Z| \leq \|\dot{J}\|s_\kappa + \|J\|\ |\dot{s}_\kappa| \quad \text{on } (0, t^*).$$

As $t_0 \to +0$, we have $s_\kappa(t_0) \to 0$ and $\|J(t_0)\| \to 0$, whence $Z(t_0) \to 0$ and $Z \geq 0$ on $(0, t^*)$. Then the desired inequality (8) follows from

$$\frac{d}{dt}\left\{\frac{\|J\|}{s_\kappa}\right\} = \frac{Z}{s_\kappa^2}.$$

Theorem 1. *Let $c : [0,\infty) \to M$ be a geodesic with $\|\dot{c}\| = 1$, and let J be a normal Jacobi field along c which satisfies $J(0) = 0$. We moreover suppose that the sectional curvature K of M has an upper bound κ on $\Gamma_{t_\kappa} = \{c(t) : 0 \le t \le t_\kappa\}$. Then*

$$\|\dot{J}(0)\| s_\kappa(t) \le \|J(t)\| \quad \textit{for all } t \in [0, t_\kappa). \tag{9}$$

Proof. If $\dot{J}(0) = 0$, (9) obviously is correct. We therefore may assume that $\|\dot{J}(0)\| > 0$, whereas $J(0) = 0$. Then there is a number $t^* \in (0, t_\kappa)$ such that $\|J\| > 0$ on $(0, t^*)$, and Lemma 2 implies

$$\frac{\|J\|}{s_\kappa}(t_0) \le \frac{\|J\|}{s_\kappa}(t) \quad \text{for } 0 < t_0 \le t < t^*.$$

As t_0 tends to $+0$, the quotient on the left hand side is an expression of the kind $\frac{0}{0}$ which, according to L'Hospital's rule, is determined by

$$\lim_{t_0 \to +0} \frac{\|J\|^2}{s_\kappa^2} = \lim_{t_0 \to +0} \frac{\frac{d}{dt}\|J\|^2}{\frac{d}{dt} s_\kappa^2} = \lim_{t_0 \to +0} \frac{\frac{d^2}{dt^2}\|J\|^2}{\frac{d^2}{dt^2} s_\kappa^2} = \|\dot{J}(0)\|^2,$$

since

$$\begin{aligned}
&\frac{d}{dt} s_\kappa^2(t_0) \to 0, \quad \frac{d^2}{dt^2} s_\kappa^2(t_0) \to 2,\\
&\frac{d}{dt}\|J\|^2(t_0) = 2\langle J, \dot{J}\rangle(t_0) \to 0,\\
&\frac{d^2}{dt^2}\|J\|^2(t_0) = 2\left\{\|\dot{J}\|^2 + \langle \ddot{J}, J\rangle\right\}(t_0)\\
&\qquad\qquad = 2\left\{\|\dot{J}\|^2 - \langle R(J,\dot{c})\dot{c}, J\rangle\right\}(t_0) \to 2\|\dot{J}(0)\|^2,
\end{aligned}$$

and (9) is proved for $0 \le t \le t^*$. We then conclude that $J(t)$ cannot vanish before t_κ, and thus (9) must hold for all $t \in [0, t_\kappa)$.

By the same reasoning, we can prove

Theorem 1′. *Let $c : [0,\infty) \to M$ be a geodesic with $\|\dot{c}\| = 1$, and let $J \in J_c^\perp$. Suppose also that the sectional curvature K satisfies $K \le \kappa$ on $\Gamma_{\tau_\kappa} = \{c(t) : 0 \le t \le \tau_\kappa\}$ where τ_κ is the first positive zero of*

$$\varphi(t) = \|J(0)\| c_\kappa(t) + \|J\|^{\cdot}(0) s_\kappa(t),$$

and $\|J\|^{\cdot}(0) = \|J\|^{-1}\langle J, \dot{J}\rangle(0)$. We then obtain

$$\varphi(t) \le \|J(t)\| \quad \textit{for } 0 \le t < \tau_\kappa \tag{10}$$

and

$$\|J(t)\| \le \frac{\|J(t^*)\|}{\varphi(t^*)}\varphi(t) \quad \textit{for all } t \in [0, t^*], \tag{11}$$

where $0 < t^* < \tau_\kappa$.

Remark. Here we have assumed that $J(0) \neq 0$; the case $J(0) = 0$ is handled by a limit consideration.

We now turn to another class of Jacobi field estimates derived from a lower bound on the sectional curvature of M.

To this end, let $c : [0, \infty) \to M$ again be a unit speed geodesic, and let $X_1, X_2, \ldots, X_m$ be m parallel vector fields along c which, at every point $c(t)$ of the geodesic, yield on orthogonal frame of the tangent space $T_{c(t)}M$. In other words, we have

$$\dot{X}_k = 0 \quad \text{and} \quad \langle X_k, X_\ell \rangle = \delta_{k\ell}.$$

Then every vector field U along c can be written as

$$U(t) = u^k(t) X_k(t).$$

If we identify $\mathbb{R}^m$ with $T_{c(0)}M$ and introduce the vector function $u : [0, \infty) \to \mathbb{R}^m$ by

$$u(t) = \big(u^1(t), \ldots, u^m(t)\big)^T$$

we obtain a 1-1-correspondence between the vector functions $u : [0, \infty) \to \mathbb{R}^m$ and the vector fields U along c given by parallel translation.

To any $m \times m$-matrix function $B(t) = (b^\ell_k(t))$ which acts on vector functions $u(t)$ according to $(B(t)u(t))^\ell = b^\ell_k(t) u^k(t)$, we can associate an operator, again called B, acting on vector fields $U = u^k X_k$ by the rule

$$(BU)(t) = (B(t)u(t))^\ell X_\ell(t)$$

if the vectorfield U is identified with the function u.

We, in particular, can associate with every Jacobi field $J = J^k X_k$ a vector function $I = (J^1, \ldots, J^m)$ which satisfies

$$\ddot{I} + R_c I = 0, \tag{12}$$

where the matrix function $R_c(t) = (R^s_k(t))$ is defined by

$$R^s_k = R^s_{k\ell r} \dot{c}^\ell \dot{c}^r,$$

where

$$\dot{c} = \dot{c}^k X_k \quad \text{and} \quad R(J, \dot{c})\dot{c} = R^s_{k\ell r} J^k \dot{c}^\ell \dot{c}^r X_s.$$

The well known symmetry relation

$$\langle R(U, \dot{c})\dot{c}, V \rangle = \langle R(V, \dot{c})\dot{c}, U \rangle$$

implies the symmetry of R_c.

Next we choose a basis $J_1, \dots, J_m$ of the m-dimensional subspace $\mathring{J}_c := \{J \in J_c : J(0) = 0\}$ of J_c with $\dot{J}_k(0) = X_k(0)$. By Theorem 1 and by (3), the tangent vectors $J_1(t), \dots, J_m(t)$ are linearly independent for all $t \in (0, t_\kappa)$ if we assume $K \le \kappa$. Let now I_k be the vector functions corresponding to the Jacobi vectors J_k. Then the matrix $A(t)$, defined by

$$A = (I_1, I_2, \dots, I_m),$$

is invertible and satisfies

$$\ddot{A} + R_c A = 0, \quad A(0) = 0, \quad \dot{A}(0) = 1, \tag{13}$$

where 1 denotes the unit matrix (δ_k^ℓ). We therefore can define the matrix function

$$S(t) = -\dot{A}(t)A^{-1}(t) \quad \text{for } t \in (0, t_\kappa),$$

which satisfies the Riccati equation

$$\dot{S} = R_c + S^2, \tag{14}$$

since the differentiation of $AA^{-1} = 1$ and $S = -\dot{A}A^{-1}$ yields $(A^{-1})^{\cdot} = -A^{-1}\dot{A}A^{-1}$ and $\dot{S} = -\ddot{A}A^{-1} - \dot{A}(A^{-1})^{\cdot} = -\ddot{A}A^{-1} + (\dot{A}A^{-1})^2$, and from (13) we infer $\ddot{A}A^{-1} = -R_c$. Moreover,

$$S(t) = -t^{-1} \cdot 1 + 0(1) \quad \text{as } t \to +0 \tag{15}$$

since $A(t) = t \cdot 1 + \cdots$ and $\dot{A}(t) = 1 + \cdots$.

We also claim that $S(t)$ is a symmetric operator on $T_{c(t)}M$, i.e. we must prove that

$$\langle S(t_0)U_0, V_0 \rangle = \langle U_0, S(t_0)V_0 \rangle$$

holds for every $t_0 \in (0, t_\kappa)$ and for each pair of tangent vectors $U_0 = u_0^k X_k(t_0)$, $V_0 = v_0^k X_k(t_0) \in T_{c(t_0)}M$.

But, if we introduce the two parallel vector fields $U(t) = u^k X_k(t)$ and $V(t) = v^k X_k(t)$ with $u = A^{-1}(t_0)u_0$ and $v = A^{-1}(t_0)v_0$, this is equivalent to saying that the function

$$\phi = \langle \dot{A}U, AV \rangle - \langle AU, \dot{A}V \rangle$$

vanishes for $t = t_0$, which is proved by showing that ϕ identically vanishes on $(0, t_\kappa)$. In fact, we infer from the definition of ϕ that

$$\lim_{t \to +0} \phi(t) = 0,$$

and, on the other hand, ϕ is constant because of

$$\begin{aligned}\dot{\phi} &= \langle \ddot{A}U, AV\rangle - \langle AU, \dot{A}V\rangle \\ &= -\langle R_c AU, AV\rangle + \langle AU, R_c AV\rangle = 0.\end{aligned}$$

Let now J be an arbitrary normal Jacobi field in $\mathring{\mathcal{J}}_c$, and let I be the associated vector function. Then we infer from $\dot{A} = -SA$ that

$$\dot{I} = -SI \quad \text{or} \quad \dot{J} = -SJ \tag{16}$$

holds on $(0, t_\kappa)$. We fix some $t_0 \in (0, t_\kappa)$ and set $U_0 = u_0^k X_k(t_0) = \|J(t_0)\|^{-1} J(t_0)$. Moreover, we define a parallel vector field U along c with $U(t_0) = U_0$ by setting $U(t) = u_0^k X_k(t)$. Then we claim that the function

$$k(t) = \langle SU, U\rangle(t)$$

satisfies

$$-k \le \frac{\dot{s}_\omega}{s_\omega} \quad \text{on } (0, t_\kappa) \tag{16'}$$

provided that $\omega \le K \le \kappa$ is assumed. We also note that $\omega \le \kappa$ implies $t_\kappa \le t_\omega$.

From (16′) we infer that

$$-\frac{\langle SJ, J\rangle}{\|J\|^2}(t) \le \frac{\dot{s}_\omega}{s_\omega}(t)$$

holds for $t = t_0$. Since t_0 was arbitrary, this inequality is true for all $t \in (0, t_\kappa)$, and, together with (16), we arrive at

$$\frac{\|J\|^{\cdot}}{\|J\|} = \frac{\langle J, \dot{J}\rangle}{\|J\|^2} = -\frac{\langle J, SJ\rangle}{\|J\|^2} \le \frac{\dot{s}_\omega}{s_\omega}$$

which is to hold on $(0, t_\kappa)$.

On the other hand, by repeating the proof of Lemma 2 and by taking Theorem 1 into account, we obtain

$$Z = \|J\|^{\cdot} s_\kappa - \|J\| \dot{s}_\kappa \ge 0 \quad \text{on } (0, t_\kappa).$$

Hence we have

Theorem 2. *Let J be a normal Jacobi field with $J(0) = 0$ along a unit speed geodesic $c : [0, \infty) \to M$, and suppose that $\omega \le K \le \kappa$ holds on the set $\{c(t) : t \in (0, t_\kappa)\}$. Then we may conclude that*

$$\frac{\dot{s}_\kappa}{s_\kappa} \le \frac{\langle J, \dot{J}\rangle}{\|J\|^2} \le \frac{\dot{s}_\omega}{s_\omega} \quad \textit{on } (0, t_\kappa). \tag{17}$$

It remains to prove (16′). We first note that $\|U\| = 1$ and $\langle U, \dot{c}\rangle = 0$ hold on $[0, \infty)$, since these relations are true for $t = t_0$, and $U, \dot{c}$ are parallel.

Thus we get

$$\omega \le \langle R(U, \dot c)\dot c, U\rangle,$$

and

$$\langle SU, U\rangle^2 \le \|SU\|^2 = \langle S^2 U, U\rangle.$$

Furthermore, (14) yields

$$\begin{aligned}\frac{d}{dt}\langle SU, U\rangle &= \langle R_c U, U\rangle + \langle S^2 U, U\rangle \\ &= \langle R(U, \dot c)\dot c, U\rangle + \langle S^2 U, U\rangle,\end{aligned}$$

and therefore

$$\dot k \ge \omega + k^2 \quad \text{on } (0, t_\kappa). \tag{18}$$

Consider the function

$$h = s_\omega k + \dot s_\omega$$

which then satisfies

$$\dot h \ge hk, \tag{19}$$

as we see from

$$\dot h = \dot s_\omega k + s_\omega \dot k + \ddot s_\omega \ge \dot s_\omega k + s_\omega k^2 + (\ddot s_\omega + \omega s_\omega)$$

if we take (18) and $\ddot s_\omega + \omega s_\omega = 0$ into account. By differentiating, one checks the identity

$$h(t)\exp\left(-\int_\epsilon^t k(s)\,ds\right) = h(\epsilon) + \int_\epsilon^t (\dot h - hk)(s)\exp\left(-\int_\epsilon^s k(\tau)\,d\tau\right)\,ds,$$

$0 < \epsilon < t < t_\kappa$, and thus by (19):

$$h(t) \ge h(\epsilon)\exp\left(\int_\epsilon^t k(s)\,ds\right).$$

As ϵ tends to $+0$, (15) yields $k(\epsilon) = -\frac{1}{\epsilon} + 0(1)$, whence $h(\epsilon) \to 0$ and $k(\epsilon) \to -\infty$. We infer

$$h(t) \ge 0 \quad \text{for } t \in (0, t_\kappa),$$

which is equivalent to (16′), and thus Theorem 2 is proved.

From Theorem 2 we infer that

$$\frac{d}{dt}\left\{\frac{\|J\|}{s_\omega}\right\} = \frac{\|J\|^{\cdot} s_\omega - \|J\|\dot{s}_\omega}{s_\omega^2} = \frac{\|J\|}{s_\omega}\left\{\frac{\|J\|^{\cdot}}{\|J\|} - \frac{\dot{s}_\omega}{s_\omega}\right\} = \frac{\|J\|}{s_\omega}\left\{\frac{\langle J, \dot{J}\rangle}{\|J\|^2} - \frac{\dot{s}_\omega}{s_\omega}\right\}$$
$$\leq 0,$$

i.e., the function $\|J\|/s_\omega$ is decreasing on $(0, t_\omega)$ and then the same reasoning as in the proof of Theorem 1 yields $\|\dot{J}(0)\|s_\omega(t) \geq \|J(t)\|$ for $t \in (0, t_\kappa)$.

Thus we have proved

Theorem 3. *Let J be a normal Jacobi field with $J(0) = 0$ along a unit speed geodesic $c : [0, \infty) \to M$, and suppose that the sectional curvature K of M satisfies $\omega \leq K \leq \kappa$ on the set $\{c(t) : t \in (0, t_\kappa)\}$. Then the function $\frac{\|J\|}{s_\omega}$ is decreasing in $(0, t_\kappa)$, and we have*

$$\|J(t)\| \leq \|\dot{J}(0)\|s_\omega(t) \quad \textit{for all } t \in (0, t_\kappa). \tag{20}$$

Remarks. 1. We first note that the completeness of M was not really needed. It was only used to insure the existence of $c(t)$ for all $t \in (0, t_\kappa)$. If we instead assume that $c(t)$ is defined for $0 \leq t \leq R$, the estimates (9), (17) and (20) will hold for $0 < t < \min(t_\kappa, R)$.

2. From $\omega \leq K \leq \kappa$ and $\langle R(J, \dot{c})\dot{c}, J\rangle = K(t)\|J^\perp\|^2$ we conclude that

$$\omega\|J^\perp\|^2 \leq \langle R(J, \dot{c})\dot{c}, J\rangle \leq \kappa\|J^\perp\|^2,$$

and therefore

$$\omega\|J\|^2 \leq \langle R(J, \dot{c})\dot{c}, J\rangle \leq \kappa\|J\|^2, \tag{21}$$

if we also assume that $\omega \leq 0 \leq \kappa$. The inequality (21) was all we needed to derive the statements of the Theorems 1–3, and the assumption $\langle J, \dot{c}\rangle = 0$ was nowhere else used. Thus these statements remain true for all Jacobi fields J along c with $J(0) = 0$.

3. Let us once again assume that $\omega \leq K \leq \kappa$ and $\omega \leq 0 \leq \kappa$, and suppose that $J \in \mathring{J}_c$, but not necessarily $\|\dot{c}\| = 1$. Then we define $r = \|\dot{c}\|$, $\underline{c}(\tau) = c(\tau/r)$, $\underline{J}(\tau) = J(\tau/r)$, and note that $\underline{J} \in \mathring{J}_{\underline{c}}$ and $\|\dot{\underline{c}}\| = 1$, whence, by (17),

$$\sqrt{\kappa}\,\mathrm{ctg}\,\sqrt{\kappa}\tau \leq \frac{\langle \underline{J}, \dot{\underline{J}}\rangle}{\|\underline{J}\|^2}(\tau) \leq \sqrt{-\omega}\,\mathrm{ctgh}\,\sqrt{-\omega}\tau,$$

and therefore

$$r\sqrt{\kappa}\,\mathrm{ctg}\,\sqrt{\kappa}rt \leq \frac{\langle J, \dot{J}\rangle}{\|J\|^2}(t) \leq r\sqrt{-\omega}\,\mathrm{ctgh}\,\sqrt{-\omega}rt.$$

If we introduce the functions

$$a_\kappa(t) = t\sqrt{\kappa}\,\mathrm{ctg}\,\sqrt{\kappa}t \qquad \text{for } 0 \le t < \pi/\sqrt{\kappa},$$
$$a_\omega(t) = t\sqrt{-\omega}\,\mathrm{ctgh}\,\sqrt{-\omega}t \quad \text{for } 0 \le t < \infty,$$

we arrive at

$$a_\kappa(r)\|J(1)\|^2 \le \langle J(1), \dot{J}(1)\rangle \le a_\omega(r)\|J(1)\|^2 \tag{22}$$

and

$$\{a_\kappa(r) - 1\}\|J(1)\|^2 \le \langle \dot{J} - J, J\rangle(1) \le \{a_\omega(r) - 1\}\|J(1)\|^2 \tag{23}$$

provided that $\sqrt{\kappa}r < \pi$.

By the same scaling argument, we derive from (9) and (20) the inequalities

$$\|\dot{J}(0)\|^2 r^{-2} s_\kappa^2(rt) \le \|J(t)\|^2 \le \|\dot{J}(0)\|^2 r^{-2} s_\omega^2(rt) \quad \text{if } 0 < rt < \pi/\sqrt{\kappa}.$$

By setting

$$b_\kappa(t) = \frac{\sin\sqrt{\kappa}t}{\sqrt{\kappa}t} \quad \text{and} \quad b_\omega(t) = \frac{\sinh\sqrt{-\omega}t}{\sqrt{-\omega}t},$$

we arrive at

$$\|\dot{J}(0)\|^2 b_\kappa^2(r) \le \|J(1)\|^2 \le \|\dot{J}(0)\|^2 b_\omega^2(r) \tag{24}$$

provided that $\sqrt{\kappa}r < \pi$.

Let us collect these results in the following

Theorem 4. *Let J be a Jacobi field with $J(0) = 0$ along a geodesic $c : [0,1] \to M$ with $r = \|\dot{c}(0)\|$, and suppose that the sectional curvature K of M satisfies $\omega \le K \le \kappa$ on the arc c. Then, if $\omega \le 0 \le \kappa$ and $r\sqrt{\kappa} < \pi$, the estimates (22)–(24) hold.*

Remark. We observe that $a_\omega, b_\omega \ge 1$ and $a_\kappa, b_\kappa \le 1$, in particular $a_\omega(0) = a_\kappa(0) = b_\omega(0) = b_\kappa(0) = 1$.

4.8.2 Riemann Normal Coordinates

Let $\psi(t,\alpha)$ be a mapping $\psi : [0,R] \times [-\alpha_0, \alpha_0] \to M$ such that, for every $\alpha \in [-\alpha_0, \alpha_0]$, $\alpha_0 > 0$, the curve $c(t) = \psi(t,\alpha)$ is a geodesic in M. Then $J(t) = \frac{\partial\psi}{\partial\alpha}(t,\alpha)$ is a Jacobi field along c. This follows from the identities

$$\frac{D}{\partial t}\frac{\partial\psi}{\partial\alpha} - \frac{D}{\partial\alpha}\frac{\partial\psi}{\partial t} = 0$$

and

$$\frac{D}{\partial t}\frac{D}{\partial\alpha}Z - \frac{D}{\partial\alpha}\frac{D}{\partial t}Z = R\left(\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial\alpha}\right)Z,$$

where Z denotes an arbitrary vector field along ψ. In fact, we have

$$\begin{aligned}\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial\psi}{\partial\alpha} &= \frac{D}{\partial t}\frac{D}{\partial\alpha}\frac{\partial\psi}{\partial t} = \frac{D}{\partial\alpha}\frac{D}{\partial t}\frac{\partial\psi}{\partial t} + R\left(\frac{\partial\psi}{\partial t},\frac{\partial\psi}{\partial\alpha}\right)\frac{\partial\psi}{\partial t} \\ &= 0 - R\left(\frac{\partial\psi}{\partial\alpha},\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial t}\end{aligned}$$

or

$$\ddot{J} + R(J,\dot{c})\dot{c} = 0.$$

This idea to construct Jacobi fields will be used in the following.

In what follows we identify the tangent space of T_pM at $v \in T_pM$ with T_pM itself and write $T_v(T_pM) \equiv T_pM$. The exponential map $\exp_p\colon\ T_pM \to M$ with center p is defined by $\exp_p(v) = c(1)$ for $v \in T_pM$, where c is the geodesic with $c(0) = p$, $\dot{c}(0) = v$.

Let $q = \exp_p v$. Then, by *Gauss's lemma*, the differential $(d\exp_p)_v$: $T_v(T_pM) = T_pM \to T_qM$ satisfies

$$\text{(25)} \qquad \langle\xi,\eta\rangle_p = \langle\tilde{\xi},\tilde{\eta}\rangle_q,$$

where $\eta \in T_v(T_pM) \hat{=} T_pM$ is the radial vector parallel to v (i.e. $\eta = v$ after identification of T_pM and $T_v(T_pM)$) and $\tilde{\xi},\tilde{\eta}$ are defined by

$$\text{(25')} \qquad \tilde{\xi} = (d\exp_p)_v(\xi), \quad \tilde{\eta} = (d\exp_p)_v(\eta).$$

A "normal chart" (φ, U) with center $p \in M$ is given by an open set $U \subset M$ with $p \in U$, and by a mapping $\varphi : U \to \mathbb{R}^m$ of the form $\varphi = j \cdot \exp_p^{-1}$, where $j : T_pM \to \mathbb{R}^m$ is a linear isometry, and $\exp_p^{-1}$ is supposed to be existing on U.

Let $e_1,\dots,e_m$ be the orthogonal base of T_pM which under j corresponds to the standard base $(1,0,\dots,0),\dots,(0,\dots,0,1)$ of the Euclidean space $\mathbb{R}^m$. Since T_pM is identified with $T_v(T_pM)$ for all $v \in T_pM$, we may consider $e_1,\dots,e_m$ as m orthogonal vector fields on T_pM, and the base vector fields $X_1,\dots,X_m$ of the normal chart (φ, U) are given by

$$X_i(q) = (d\exp_p)e_i,$$

where $q = \exp_p v$.

Let c be the geodesic with $c(0) = p$ and $\dot{c}(0) = v$ for some $v \in T_pM$, and let $\xi = \xi^k e_k$ be an arbitrary vector in T_pM. Then $c(t) = \exp_p(tv)$, and, for each $\alpha, \psi(t,\alpha) = \exp_p\{t(v+\alpha\xi)\}$ defines a geodesic $\psi(\cdot,\alpha) : [0,\infty) \to M$ with $\psi(0,\alpha) = p$. By our previous remarks, $J(t) = \frac{\partial\psi}{\partial\alpha}(t,0)$ therefore is a Jacobi field along c and, moreover,

$$\frac{\partial\psi}{\partial\alpha}(t,0) = (d\exp_p)_{tv}(t\xi) = t\xi^k(d\exp_p)_{tv}e_k = t\xi^k X_k(c(t)).$$

Thus we have proved:

Lemma 3. *If $c : [0, \infty) \to M$ is a geodesic with $c(0) = p$ and $\dot{c}(0) = v \in T_pM$, then, for every $\xi = \xi^k e_k$, $J(t) = t\xi^k X_k(c(t))$ defines a Jacobi field J along c with $J(0) = 0, \dot{J}(0) = \xi^k X_k(p)$ and, if $q = c(1)$, with*

$$J(1) = \xi^k X_k(q), \quad \dot{J}(1) = \left\{\xi^\ell + \Gamma^\ell_{ik}(q)\xi^i \dot{c}^k(1)\right\} X_\ell(q). \tag{26}$$

For each normal chart (ψ, U) with center p, we may introduce Riemann normal coordinates by

$$x = \varphi(q)$$

for all $q \in U$. Let $X_1, \dots, X_m$ be the base vector fields on U corresponding to the chart (φ, U). Then $q_{k\ell}(q) = \langle X_k(q), X_\ell(q)\rangle_q$ are the components of the fundamental tensor on U, and $\Gamma_{ik\ell}(q)$ and $\Gamma^\ell_{ik}(q)$ denote the Christoffel symbols of the first and second kind. For the sake of brevity, we set

$$g_{k\ell}(x) := g_{k\ell}\left(\varphi^{-1}(x)\right), \quad \Gamma_{ik\ell}(x) := \Gamma_{ik\ell}\left(\varphi^{-1}(x)\right), \quad \text{etc.}$$

without using different notation.

We obviously have

$$\varphi(p) = 0.$$

Moreover, $(d\exp_p)_0$ is the identical map, whence $X_i(p) = e_i$, and therefore

$$g_{k\ell}(p) = \delta_{k\ell} \quad \text{or} \quad g_{k\ell}(0) = \delta_{k\ell}.$$

Let $c(t) = \exp_p tv$, where $v = x^k e_k$ and $j(v) = x = (x^1, \dots, x^m)$. Then $\eta(t) := \varphi(c(t))$ satisfies

$$\ddot{\eta}^\ell + \Gamma^\ell_{ik}(\eta)\dot{\eta}^i\dot{\eta}^k = 0.$$

On the other hand, the definition of φ implies $\eta(t) = tx$ and therefore $c(t) = \varphi^{-1}(tx)$ and $\Gamma^\ell_{ik}(tx)x^i x^k = 0$, in particular, $\Gamma^\ell_{ik}(0)x^i x^k = 0$ for all $x \in \mathbb{R}^m$. Therefore,

$$\Gamma^\ell_{ik}(0) = \Gamma_{ik\ell}(0) = 0 \quad \text{or} \quad \Gamma^\ell_{ik}(p) = \Gamma_{ik\ell}(p) = 0$$

since $\Gamma^\ell_{ik} = \Gamma^\ell_{ki}$.

Let $\xi = e_k$, and $\eta = x^\ell e_\ell$ be a radial vector that coincides with $v = \dot{c}(0)$. Then

$$\langle \xi, \eta\rangle_v = \langle e_k, x^\ell e_\ell\rangle_v = x^\ell \delta_{k\ell} = x^k.$$

Since $X_i(q) = (d\exp_p)_v e_i$, we infer from Gauss's lemma (25), (25′) that

$$\begin{aligned} x^k = \langle \xi, \eta\rangle_v = \langle (d\exp_p)_v \xi, (d\exp_p)_v \eta\rangle &= \langle X_k(q), x^\ell X_\ell(q)\rangle_q = x^\ell g_{k\ell}(q) \\ &= x^\ell g_{k\ell}(x). \end{aligned}$$

Thus we have

$$x^k = x^\ell g_{k\ell}(x) \quad \text{and also} \quad x^k = x^\ell g^{k\ell}(x).$$

Moreover, one also infers from Gauss's lemma that the distance $d(p,q)$ of the two points $p, q \in U$ with $p = c(0)$, $q = c(1) = \exp_p v$ is given by

$$d(p,q) = \|\dot{c}\| = \|v\| = |x|,$$

where $|x| = \sqrt{\delta_{k\ell} x^k x^\ell}$ denotes the Euclidian length of the vector $x \in \mathbb{R}^m$. Hence we have proved:

Lemma 4. *If $x = \varphi(q)$ are Riemann normal coordinates with center p on the set $U \subset M$, then*

$$g_{ik}(0) = \delta_{ik}, \quad \Gamma_{ik\ell}(0) = 0, \quad \Gamma^\ell_{ik}(0) = 0, \tag{27}$$

$$x^k = g_{k\ell}(x) x^\ell, \quad x^k = g^{k\ell}(x) x^\ell, \tag{28}$$

$$d(p,q) = |x|. \tag{29}$$

Moreover, if $v = x^m e_m \in T_pM, x = (x^1, \dots, x^m) \in \mathbb{R}^m$, and if $c(t)$ denotes the geodesic $\exp_p tv$ with $c(0) = p$ and $\dot{c}(0) = v$, then $\varphi(c(t)) = tx$. □

For some real-valued function $f(x)$, we write

$$f_\ell(x) = \frac{\partial f}{\partial x^\ell}(x).$$

Then the following holds:

Lemma 5. *If $x = \varphi(q)$ are Riemann normal coordinates, then*

$$x^k g_{ik,\ell}(x) = \delta_{i\ell} - g_{i\ell}(x), \quad x^k g^{ik}_\ell(x) = \delta^{i\ell} - g^{i\ell}(x), \tag{30}$$

$$x^i x^k g_{ik,\ell}(x) = x^i x^\ell g_{ik,\ell}(x) = x^k x^\ell g_{ik,\ell}(x) = 0, \tag{31}$$

$$x^i x^k g^{ik}_{,\ell}(x) = x^i x^\ell g^{ik}_{,\ell}(x) = x^k x^\ell g^{ik}_{,\ell}(x) = 0,$$

$$x^\ell \left\{ \Gamma_{i\ell k}(x) + \Gamma_{ik\ell}(x) \right\} = \delta_{ik} - g_{ik}(x), \tag{32}$$

$$x^\ell \Gamma^\ell_{ik}(x) = x^\ell \Gamma_{i\ell k}(x), \tag{33}$$

$$x^i x^k \Gamma_{ik\ell}(x) = x^i x^\ell \Gamma_{ik\ell}(x) = x^i x^k \Gamma^k_{i\ell}(x) = x^i x^\ell \Gamma^k_{i\ell}(x) = 0. \tag{34}$$

Proof. By differentiating the formulas (28), we obtain (30), and (31) is a consequence of (28) and (30). The identity $\Gamma_{ik\ell}+\Gamma_{i\ell k}=g_{k\ell,i}$ together with (30) yields (32). Finally, if we take $\Gamma_{i\ell k}=g_{\ell j}\Gamma^j_{ik}$ into account, (28) implies (33), and (34) follows from (31). □

Let us now return to the formulas (26) of Lemma 3. If $c(t)=\exp_p tv$ and $v=x^k e_k$, then we infer from Lemma 4 that $x=\varphi(q)$ with $q=c(1)$, and $\dot{c}^k(1)=x^k$ if $\dot{c}(t)=\dot{c}^k(t)X_k(c(t))$. Hence, the Jacobi field $J(t)=t\xi^k X_k(c(t))$ fulfills

$$\dot{J}(1)=\left\{\xi^\ell+\Gamma^\ell_{ik}(x)\xi^i x^k\right\}X_\ell(q). \tag{35}$$

Thus we obtain the relations

$$\|\dot{J}(0)\|^2=\delta_{k\ell}\xi^k\xi^\ell,\quad \|J(1)\|^2=g_{k\ell}(x)\xi^k\xi^\ell, \tag{36}$$

and

$$\begin{aligned}\langle\dot{J}(1)-J(1),J(1)\rangle &= \Gamma^\ell_{ik}(x)\xi^i x^k g_{\ell j}(x)\xi^j\\ &= \Gamma_{ijk}(x)\xi^i\xi^j x^k.\end{aligned} \tag{37}$$

We also note that $r:=d(p,q)=|x|=\|\dot{c}\|$.

For any $p_0\in M$, the interior $\mathring{S}$ of the set $\{V\in T_{p_0}M:\|V\|=d(p_0,\exp_{p_0}V)\}$ is an open, starshaped neighbourhood of 0 in $T_{p_0}M$. If we denote the cut locus of p_0 in M by $C(p_0)=\exp_{p_0}(\partial\mathring{S})\subset M$ then the exponential map $\exp_{p_0}:\mathring{S}\to M$ is a C^2-diffeomorphism onto $S(p_0):=\exp_{p_0}(\mathring{S})$ and we can define Riemann normal coordinates $x=\varphi(q)$ for $q\in U=S(p_0)$. In addition, if K denotes the sectional curvature of M we define the numbers

$$\kappa(A):=\max\Big\{0,\sup_A K\Big\},$$

$$\omega(A):=\min\Big\{0,\inf_A K\Big\}\quad\text{for } A\subset M,$$

and

$$\begin{aligned}\kappa(x)&:=\kappa([0,x])=\kappa([p_0,p]),\\ \omega(x)&:=\omega([0,x])=\omega([p_0,p]),\end{aligned}$$

where $[p_0,p]$ is the geodesic segment between p_0 and p which in normal coordinates is just the segment $[0,x]$ on the ray from the origin 0 through x. Recall that we also use the notation

$$a_\kappa(t)=t\sqrt{\kappa}\,\mathrm{ctg}\,\sqrt{\kappa}t\quad\text{for } 0\le t<\pi/\sqrt{\kappa},$$

$$a_\omega(t) = t\sqrt{-\omega}\mathrm{ctgh}\sqrt{-\omega}t \quad \text{for } 0 \le t < \infty$$

and

$$b_\kappa(t) = \frac{\sin\sqrt{\kappa}t}{\sqrt{\kappa}t}, \quad b_\omega(t) = \frac{\sinh\sqrt{-\omega}t}{\sqrt{-\omega}t}.$$

Theorem 5. *Let M be a complete Riemannian manifold and $x = \varphi(q)$ denote Riemann normal coordinates for $q \in S(p_0)$. With respect to those coordinates the following estimates are true:*

$$\{a_{\kappa(x)}(|x|) - 1\}g_{i\kappa}(x)\xi^i\xi^k \le \Gamma_{ik\ell}(x)x^i\xi^k\xi^\ell \le \{a_{\omega(x)}(|x|) - 1\}g_{ik}\xi^i\xi^k, \tag{38}$$

$$b^2_{\kappa(x)}(|x|)\xi^i\xi^i \le g_{ik}\xi^i\xi^k \le b^2_{\omega(x)}(|x|)\xi^i\xi^i, \tag{39}$$

$$b^2_{\kappa(x)}(|x|) \le \sqrt{g}(x) \le b^2_{\omega(x)}(|x|) \tag{40}$$

for all $\xi \in \mathbb{R}^m$ and all $x \in \varphi(S(p_0))$ with $|x| \cdot \kappa(x) < \pi$.

Proof. The inequalities (38) and (39) readily follow from the estimates (23) and (24) of Theorem 4 and from (36) and (37). Finally relation (28) implies that $\lambda = 1$ is one of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix $g_{k\ell}(x)$ and by virtue of (39) we have $b^2_{\kappa(x)}(|x|) \le \lambda_k \le b^2_{\omega(x)}(|x|)$ for $k = 1, 2, 3$. This yields estimate (40). □

Theorem 6. *Let the assumptions of Theorem 5 be satisfied, and set $f(q) = \frac{1}{2}d^2(p,q)$. Then we have*

$$a_{\kappa(q)}(r)\|\xi\|^2 \le (D^2 f)_q(\xi,\xi) \le a_{\omega(q)}(r)\|\xi\|^2 \tag{41}$$

for all $q \in M$ with $r = d(p,q) \le R$ and for $\xi \in T_qM$, where $(D^2f)_q(\xi,\xi)$ denotes the Hessian form of f at q (cp. Section 1.5 *of Vol.* 1*, equation* (28)*).*

Proof. Let $c(t) = \exp_p tv,\ q = c(1)$, and $\xi = \xi^k X_k(q) \in T_qM$. Then $J(t) = t\xi^k X_k(c(t))$ forms a Jacobi field J along c with $J(1) = \xi$ and $\|J(1)\|^2 = \|\xi\|^2$. Consider normal coordinates $x = \varphi(q)$ with center at p, and set $F(x) = f(q)$. Then

$$(D^2 f)_q(\xi,\xi) = F_{,ik}(x)\xi^i\xi^k - \Gamma^\ell_{ik}(x)F_{,\ell}(x)\xi^i\xi^k.$$

Since $F(x) = \frac{1}{2}|x|^2$, we get

$$(D^2 f)_q(\xi,\xi) = \delta_{ik}\xi^i\xi^k - \Gamma^\ell_{ik}(x)x^\ell\xi^i\xi^k = \Gamma_{ik\ell}(x)x^\ell\xi^i\xi^k + g_{ik}(x)\xi^i\xi^k$$

by virtue of (32) and (33). We then derive from (26) that

$$\langle \dot{J}(1), J(1)\rangle = (D^2 f)_q(\xi,\xi)$$

and thus (41) follows from (22). □

Theorem 7. *Let M be a complete Riemannian manifold, the sectional curvature K of which is bounded from above by*

$$K \le \kappa, \quad \kappa \ge 0,$$

on some ball $B_R(p)$ that does not meet the cut locus of its center p. Moreover, let $R\sqrt{\kappa} < \pi/2$. Then any two points q_1, q_2 of $B_R(p)$ can be connected by a geodesic arc contained in $B_R(p)$. This arc does not contain any pairs of conjugate points, and it is shortest among all arcs in $B_R(p)$ that join q_1 and q_2.

This result was proved by Jost [19].

4.8.3 Surfaces of Prescribed Mean Curvature in a Riemannian Manifold

In the following we consider a complete three-dimensional Riemannian manifold of class C^4. Since we restrict our considerations to surfaces in a normal chart (φ, U) with center p_0, we shall identify any point $q \in U \subset M$ with its normal coordinates $x = \varphi(q) \in \mathbb{R}^3$. Correspondingly any subset $\mathcal{K} \subset U$ is identified with $\varphi(\mathcal{K})$ and any surface $f : B \to U$ is identified with $X = X(w) = \varphi \circ f(w)$. In this way we obtain a natural definition of the Sobolev classes $H_s^1(B, U)$ as subsets of $H_s^1(B, \mathbb{R}^3)$. We recall the definition of a normal neighbourhood $U = S(p_0) = \exp_{p_0} \mathring{S}$, where $\mathring{S}$ is equal to the interior of $\{V \in T_{p_0}M : \|V\| = d(p_0, \exp_{p_0} V)\}$. Define the (Riemannian) cross product of two vector fields $Y = Y^k(x)X_k$, $Z = Z^\ell(x)X_\ell$ with respect to a chart x by $Y \times Z := \sqrt{g}g^{jk}(Y \wedge Z)_k$, where $(Y \wedge Z)_1 = Y^2Z^3 - Y^3Z^2$ etc. We then obtain the relation $\langle Y_1, Y_2 \times Y_3 \rangle = \sqrt{g}Y_1 \cdot (Y_2 \wedge Y_3)$, where the dot denotes the Euclidean scalar product.

Lemma 6. *For any $H \in C^1(U, \mathbb{R})$ we define the vector potentials*

$$Q(x) = \mu(x)x \quad \textit{and} \quad Q^*(x) = \frac{1}{\sqrt{g(x)}}Q(x), \tag{42}$$

where $x \in U$ and

$$\mu(x) = 2\int_0^1 t^2\sqrt{g(tx)}H(tx)\,dt.$$

(i) *Q and Q^* are of class $C^1(U, \mathbb{R}^3)$ and*

$$\operatorname{div} Q = 2\sqrt{g}H, \quad \operatorname{Div} Q^* = 2H, \tag{43}$$

where $\operatorname{div} Q$ *denotes the (noninvariantly defined) expression $\sum_{k=1}^3 \frac{\partial Q^k}{\partial x^k}$, while* $\operatorname{Div} Q^*$ *stands for the divergence on M, i.e. we have*

$$\operatorname{div} Q^* + \Gamma^j_{jk}Q^{*k} = \frac{1}{\sqrt{g}}\operatorname{div}(\sqrt{g}Q^*) \quad \textit{(see Chapter 1.5 of Vol. 1).}$$

(ii) *Suppose* $\mathcal{K} \subset S(p_0)$ *is starshaped with respect to* p_0 *and let*

$$\rho_{\mathcal{K}}^+ := \sup_{x\in\mathcal{K}}(|x|\sqrt{\kappa(x)}) < \pi, \quad \rho_{\mathcal{K}}^- := \sup_{x\in\mathcal{K}}(|x|\sqrt{-\omega(x)}) < \infty, \tag{44}$$

and

$$\begin{cases} b_+(\tau) := \dfrac{\sin\tau}{\tau} & \text{for } 0 \le \tau \le \pi, \\[2ex] b_-(\tau) := \dfrac{\sinh\tau}{\tau} & \text{for } \tau \ge 0. \end{cases} \tag{45}$$

Then we have

$$q_{\mathcal{K}}^* := \sup_{\mathcal{K}} \|Q^*(x)\| \le \frac{2}{3}\frac{b_-^2(\rho_\kappa^-)}{b_+^2(\rho_\kappa^+)}|H|_{0,\mathcal{K}} \cdot \sup_{\mathcal{K}}|x|. \tag{46}$$

Moreover, if in addition $\mathcal{K}$ *is compact and* $q_{\mathcal{K}}^* < 1$*, then for this* $\mathcal{K}$

$$\begin{aligned} e(x,\eta) &:= \frac{1}{2}g_{jk}(x)\eta_\alpha^j\eta_\alpha^k + Q(x)\cdot(\eta_1 \wedge \eta_2) \\ &= \|\eta_1\|^2 + \|\eta_2\|^2 + \langle Q^*(x), \eta_1 \times \eta_2\rangle \end{aligned} \tag{47}$$

satisfies Assumption A of Section 4.7 with

$$m_0 := (1 - q_{\mathcal{K}}^*)b_+^2(\rho_{\mathcal{K}}^+) \quad \textit{and} \quad m_1 := (1 + q_{\mathcal{K}}^*)b^2(\rho_{\mathcal{K}}^-).$$

Proof. (i) The function $\mu(x)$ is well defined for $x \in S(p_0)$ because this set is starshaped with respect to p_0 and the differentiability of Q and Q^* is obvious. Equation (43) follows by using an integration by parts.

(ii) The estimate (46) is obtained from (40), the definition of Q^* and the monotonicity properties of the functions b_- and b_+^{-1}. Furthermore, if $\eta = (\eta_1, \eta_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ then

$$\begin{aligned} Q\cdot(\eta_1 \wedge \eta_2) = \langle Q^*, \eta_1 \times \eta_2\rangle &\le \|Q^*\|\,\|\eta_1 \times \eta_2\| \le \|Q^*\|\,\|\eta_1\|\,\|\eta_2\| \\ &\le \frac{1}{2}\|Q^*\|\{\|\eta_1\|^2 + \|\eta_2\|^2\} \end{aligned}$$

and in view of (29) we have

$$b_+^2(\rho_{\mathcal{K}}^+)|\xi|^2 \le \|\xi\|^2 \le b_-^2(\rho_{\mathcal{K}}^-)|\xi|^2.$$

Combining these estimates we obtain (47),

$$\frac{1}{2}(1 - q_{\mathcal{K}}^*)b_+^2(\rho_{\mathcal{K}}^+)|\xi|^2 \le e(x,\eta) \le \frac{1}{2}(1 + q_{\mathcal{K}}^*)b_-^2(\rho_{\mathcal{K}}^-)|\xi|^2. \qquad \square$$

Definition 1. *A subset $\mathcal{K}$ of $S(p_0)$ is called a "gauge ball" in M with center p_0 if there exists an open neighbourhood $U \subset S(p_0)$ of p_0 which is starshaped with respect to p_0, a function $k \in C^2(U, \mathbb{R})$ and a real number $R > 0$ such that*

(i) $\mathcal{K} = \mathcal{K}_R(p_0) := \{x \in U : k(x) \le R^2\}$,

(ii) $k(0) = 0,\ Dk(0) = 0$,

(iii) $\gamma := \inf_{x\in\mathcal{K}} \gamma_k(x) > 0$, *where for $x \in U$,*

$$\gamma_k(x) := \inf\{D^2k(x; \xi, \xi) : \xi \in T_xM, \|\xi\| = 1\}$$

and $D^2k(x;\xi,\eta) = D^2k_q(\xi,\eta)$, $q = \varphi(x)$, stands for the Hessian form $\langle D_\xi Dk, \eta\rangle_q$. A function k with these properties is called a gauge function.

Remark 1. In local coordinates the coefficients of the Hessian form $D^2k(x;\xi,\eta)$ are given by

$$D^2k(x; X_j, X_\ell) = \frac{\partial^2 k(x)}{\partial x^j \partial x^\ell} - \Gamma^m_{j\ell}(x)\frac{\partial k(x)}{\partial x^m}. \tag{48}$$

Remark 2. Since we are dealing with Riemann normal coordinates, Lemma 5, (34) is applicable; in particular it follows that $x^i x^\ell \Gamma^j_{i\ell}(x) = 0$ for all $x \in S(p_0)$, $j = 1, 2, 3$. This yields by virtue of (48), (ii), (iii) and Taylor's formula the following estimates

$$x^j \frac{\partial k}{\partial x^j}(x) \ge \gamma |x|^2 \quad \text{for all } x \in \mathcal{K}, \quad \text{and} \quad k(x) \ge \frac{1}{2}\gamma|x|^2 \quad \text{for all } x \in \mathcal{K}. \tag{49}$$

Therefore, each gauge ball $\mathcal{K}_R(p_0)$ in M is bounded and hence also relatively compact in M, according to the Theorem of Hopf and Rinow. Also, every gauge ball is starshaped with respect to $p_0 = 0$. Indeed, (49) implies that the function $g(t) = k(tx)$ is strictly increasing in $t \in [0, 1]$ for any $x \in \mathcal{K}$, $x \ne 0$, which yields the assertion.

The most important example of a gauge function on M is furnished by the square of the distance function (cp. Lemma 4)

$$k_0(x) := |x|^2 = d^2(p_0, p) \quad \text{on } U = S(p_0),$$

where x denotes normal coordinates around $p_0 = 0$. Using relation (28) in Lemma 4 we find

$$\frac{\partial k_0}{\partial x^j} = 2x^j = 2g_{j\ell}x^\ell$$

and

$$\begin{aligned}\frac{\partial^2 k_0(x)}{\partial x^j \partial x^\ell} - \Gamma^m_{j\ell}(x)\frac{\partial k_0(x)}{\partial x^m} &= 2\frac{\partial}{\partial x^\ell}[g_{jk}x^k] - 2\Gamma^m_{j\ell}g_{mk}x^k \\ &= 2g_{jk,\ell}x^k + 2g_{jk}\delta^k_\ell - 2g^{mn}\Gamma_{jn\ell}g_{mk}x^k \\ &= 2g_{jk,\ell}x^k + 2g_{j\ell} - 2\Gamma_{jk\ell}x^k.\end{aligned}$$

By virtue of (30) and (32) in Lemma 5 we can compute the coefficients of the Hessian form

$$(50)\quad \begin{aligned} D^2 k_0(x; X_j, X_\ell) &= \frac{\partial^2 k_0}{\partial x^j \partial x^\ell} - \Gamma^m_{j\ell} \frac{\partial k_0}{\partial x^m} = 2\delta_{j\ell} - 2g_{j\ell} \\ &\quad + 2g_{j\ell} + 2\Gamma_{j\ell k} x^k - 2\delta_{j\ell} + 2g_{j\ell} = 2[g_{j\ell} + \Gamma_{j\ell k}] x^k, \end{aligned}$$

and

$$(51)\qquad \|Dk_0(x)\| = 2|x|.$$

Lemma 7. (i) *Suppose that the sectional curvature of M is bounded from above, i.e. $\kappa(M) < \infty$. Then for $\mathcal{K} = \{x \in S(p_0) : |x| \le R\}$ and $R < \frac{\pi}{2\sqrt{\kappa(M)}}$ we have*

$$\inf_{\mathcal{K}} \gamma_{k_0}(x) \ge 2a_{\kappa(M)}(R) > 0,$$

and

$$\frac{\gamma_{k_0}(x)}{\|Dk_0(x)\|} \ge \frac{a_{\kappa(M)}}{R} > 0 \quad \textit{for } x \in \mathcal{K} \setminus \{0\}.$$

(ii) *If only $\rho^+_{\mathcal{K}} = \sup_{x\in\mathcal{K}}(|x|\sqrt{\kappa(x)}) < \frac{\pi}{2}$ holds, then we obtain instead $\inf_{x\in\mathcal{K}} \gamma_{k_0}(x) \ge 2a_+(\rho^+_{\mathcal{K}}) > 0$, and*

$$\frac{\gamma_{k_0}(x)}{\|Dk_0(x)\|} \ge \frac{a_+(\rho^+_{\mathcal{K}})}{R} > 0 \quad \textit{for } x \in \mathcal{K} \setminus \{0\};$$

here we have put $a_+(t) := t\,\mathrm{ctg}(t)$.

Proof. (i) and (ii) follow from the definition of γ_{k_0}, relation (50), (51) and (38) of Theorem 5 and the monotonicity of the functions a_κ and a_+ respectively. □

Lemma 8 (Inclusion Principle). *Let $\mathcal{K} = \mathcal{K}_R(p_0)$ be a compact gauge ball and consider the Lagrangian (47) and the corresponding variational integral*

$$\begin{aligned} \mathcal{F}(X) &= \int_B e(X, \nabla X)\, du\, dv \\ &= \int_B \left\{ \frac{1}{2} g_{ij}(X) X^i_{u^\alpha} X^j_{u^\alpha} + Q(X) \cdot (X_u \wedge X_v) \right\} du\, dv. \end{aligned}$$

Suppose that $Q \in C^1(S(p_0), \mathbb{R}^3)$ satisfies

$$(52)\qquad \operatorname{div} Q = 2\sqrt{g} H \quad \textit{on } S(p_0)$$

and

$$(53) \qquad |H(x)| \le \frac{\gamma_k(x)}{\|Dk(x)\|} \quad \textit{for all } x \in \mathcal{K} \setminus \{p_0\}.$$

Moreover, denote by X *a function of class* $H_2^1(B, \mathcal{K}) \cap C^0(\overline{B}, \mathbb{R}^3)$ *satisfying*

$$(54) \qquad \delta\mathcal{F}(X, \phi) \ge 0 \quad \textit{for every } \phi \in L_{\infty,c}(B, \mathbb{R}^3)$$

such that $X + \epsilon\phi \in H_2^1(B, \mathcal{K})$ *for sufficiently small* $\epsilon > 0$. *Then* $X(\overline{B}) \subset \mathcal{K}_r$ *provided* $X(\partial B) \subset \mathcal{K}_r$ *for some* $r \le R$.

Proof. Define $\phi = (\phi^1, \phi^2, \phi^3)$ by

$$\phi^\ell(w) = \eta(w) g^{\ell m}(X(w)) \frac{\partial k}{\partial x^m}(X(w)),$$

where $\eta \in C_c^1(B, \mathbb{R})$ satisfies $0 \le \eta \le 1$ and X is a solution of (54). Since $\mathcal{K} \Subset U, X \in C^0(\overline{B}, \mathcal{K})$ and $\phi \in C_c^0(B, \mathbb{R}^3)$ there is a $\mathcal{K}' \Subset U$ such that $X - \epsilon\phi \in H_2^1(B, \mathcal{K}')$ for sufficiently small $|\epsilon|$. Hence $k(X(w) - \epsilon\phi(w))$ is defined for all $w \in B$, provided $|\epsilon|$ is small. Furthermore we have

$$\begin{aligned}
(55) \quad & k(X - \epsilon\phi) \\
&= k(X) - \epsilon k_{x^j}(X)\phi^j + \epsilon^2 \int_0^1 (1-t) k_{x^j x^\ell}(X - \epsilon t\phi)\phi^j \phi^\ell \, dt \\
&= k(X) - \epsilon\eta g^{\ell m}(X) k_{x^\ell}(X) k_{x^m}(X) \\
&\quad + \epsilon^2\eta^2 \int_0^1 (1-t) k_{x^j x^\ell}(X - \epsilon t\phi) g^{jm}(X) g^{\ell n}(X) k_{x^m}(X) k_{x^n}(X) \, dt.
\end{aligned}$$

Since (g^{jk}) is a positive definite matrix and $\mathcal{K}$ is compact, there is a constant $c > 0$ such that everywhere on B

$$g^{jk}(X(w))\xi^j \xi^k \ge c|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3.$$

Also since $(X - \epsilon\phi)(w) \in \mathcal{K}' \Subset U$ for every $w \in B$ there is a constant $c' > 0$ such that the integral in (55) can be estimated in absolute value by $c' k_{x^j}(X) k_{x^j}(X)$ for all $w \in B$.

Thus we obtain

$$k(X - \epsilon\phi) \le k(X) - \epsilon\eta c k_{x^j}(X) k_{x^j}(X) + \epsilon^2\eta^2 c' k_{x^j}(X) k_{x^j}(X)$$

which implies that

$$k(X - \epsilon\phi) \le R \quad \text{for all } w \in B \text{ and } 0 < \epsilon < \epsilon_0 := \frac{c}{c'}.$$

Therefore the function $-\phi = (-\phi^1, -\phi^2, -\phi^3)$ is admissible in (54) and by Theorem 5 in Section 4.7 in particular (23), we have

$$\begin{aligned}
\delta\mathcal{F}(X, \phi) = \int_B \{ & g_{j\ell}(X) X^j_{u^\alpha} [\eta g^{\ell m}(X) k_{x^m}(X)]_{u^\alpha} \\
& + \frac{1}{2} \frac{\partial g_{j\ell}}{\partial x^n} X^j_{u^\alpha} X^\ell_{u^\alpha} \eta g^{mn} k_{x^m}(X) \\
& + \eta \operatorname{div} Q(X)(X_u \wedge X_v)^j \cdot g^{jm} k_{x^m}(X)\} \, du \, dv \le 0.
\end{aligned}$$

Using the expression

$$D^2k(X; X_{u^\alpha}, X_{u^\alpha}) = \frac{\partial^2 k(X)}{\partial x^j \partial x^\ell} X^j_{u^\alpha} X^\ell_{u^\alpha} - \Gamma^m_{j\ell} \frac{\partial k(X)}{\partial x^m} X^j_{u^\alpha} X^\ell_{u^\alpha}$$

for the Hessian form of k and

$$\begin{aligned} 2H(X)\langle X_u \times X_v, Dk(X)\rangle &= 2H(X)\sqrt{g}Dk \cdot (X_u \wedge X_v) \\ &= \operatorname{div} Q g^{j\ell} k_{x^\ell}(X)(X_u \wedge X_v)^j \end{aligned}$$

we obtain the inequality

$$\begin{aligned} 0 &\geq \int_B \eta_{u^\alpha} [k(X)]_{u^\alpha} \, du \, dv \\ &\quad + \int_B \left\{ \eta D^2 k(X; X_{u^\alpha}, X_{u^\alpha}) + 2H(X)\langle X_u \times X_v, Dk(X)\rangle \right\} du \, dv \\ &\geq \int_B \eta_{u^\alpha} [k(X)]_{u^\alpha} \, du \, dv \\ &\quad + \int_B \eta \{ \gamma_k(X)(\|X_u\|^2 + \|X_v\|^2) \\ &\quad - 2|H(X)| \, \|X_u\| \, \|X_v\| \, \|Dk(X)\| \} \, du \, dv. \end{aligned}$$

By assumption (53) $|H(X)| \leq \frac{\gamma_k(X)}{\|Dk(X)\|}$ and because of $\|X_u\| \; \|X_v\| \leq \frac{1}{2}(\|X_u\|^2 + \|X_v\|^2)$ it follows that $\{\ldots\} \geq 0$ on B, whence

$$\int_B \eta_{u^\alpha} [k(X)]_{u^\alpha} \, du \, dv \leq 0$$

for all $\eta \in C^1_c(B)$ with $0 \leq \eta \leq 1$. Therefore $k(X(u,v)) \in C^0(\overline{B}) \cap H^1_2(B)$ is subharmonic in B and the assertion follows from the maximum principle. □

Note that by the strong maximum principle (cp. Gilbarg and Trudinger, Theorem 8.19) we may even conclude $X(B) \subset \operatorname{int} \mathcal{K}_r$ or $X(B) \subset \partial \mathcal{K}_r$.

Now we can prove the main result of this section.

Theorem 8. *Let $\mathcal{K} = \mathcal{K}_R(p_0)$ be a compact gauge ball in M. Suppose that the restriction $\rho^+_{\mathcal{K}} < \pi$ on the sectional curvature of M is satisfied and that Γ is a closed Jordan curve in $\mathcal{K}$ such that $\mathcal{C}(\Gamma, \mathcal{K}) \neq \emptyset$. Finally let H be a function of class $C^{0,\beta}(\mathcal{K}), 0 < \beta < 1$, satisfying the conditions*

$$(56) \qquad |H|_{0,\mathcal{K}} < \frac{3}{2} \frac{1}{\sup_{p \in \mathcal{K}} d(p_0, p)} \frac{b^2_+(\rho^+_{\mathcal{K}})}{b^2_-(\rho^-_{\mathcal{K}})},$$

and

$$(57) \qquad |H(x)| \leq \frac{\gamma_k(x)}{\|Dk(x)\|} \quad \text{for all } x \in \mathcal{K} \setminus \{p_0\}.$$

Then there exists an X *of class* $\mathcal{C}(\Gamma, \mathcal{K}) \cap C^{2,\beta}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ *with*

$$\triangle X^\ell + \Gamma^\ell_{ij} X^i_{u^\alpha} X^j_{u^\alpha} = 2H(X)\sqrt{g(X)}g^{\ell m}(X_u \wedge X_v)_m \tag{58}$$

in B *for* $\ell = 1, 2, 3$, *and such that the conformality relations*

$$g_{ij} X^i_u X^j_u = g_{ij} X^i_v X^j_v, \quad g_{ij} X^i_u X^j_v = 0$$

hold everywhere in B. *Furthermore* X *maps* ∂B *homeomorphically onto* Γ.

In other words, we have determined a surface X in the Riemannian manifold M which has mean curvature $H(X)$ in B (except, possibly at isolated branch points) and which is spanned by the Jordan arc Γ.

Proof of Theorem 8. We extend H continuously to some compact gauge ball $\mathcal{K}_{R+\epsilon}(p_0), \epsilon > 0$, such that (56) and (57) continue to hold for $\mathcal{K} = \mathcal{K}_{R+\epsilon}(p_0)$. Consider the variational problem

$$\mathcal{F}(X) = \int_B \left\{ \frac{1}{2} g_{ij}(X) X^i_{u^\alpha} X^j_{u^\alpha} + Q(X) \cdot (X_u \wedge X_v) \right\} du\, dv \to \min$$

in $\mathcal{C}(\Gamma, \mathcal{K}_{R+\epsilon})$ where $Q(x) = \mu(x) \cdot x$, $\mu(x) = 2\int_0^1 t^2 \sqrt{g(tx)} H(tx)\, dt$ as in Lemma 6. Relation (46) and assumption (56) imply that $\mathcal{F}(\cdot)$ is coercive; also $\mathcal{K}_{R+\epsilon}$ is quasiregular. Hence we may apply Theorems 3 and 4 in Section 4.7 and obtain the existence of a conformally parametrized solution $X \in \mathcal{C}(\Gamma, \mathcal{K}_{R+\epsilon}) \cap C^0(\overline{B}, \mathcal{K}_{R+\epsilon}) \cap C^{0,\alpha}(B, \mathbb{R}^3)$. By a reasoning analogous to the one in the proof of Theorem 8 in Section 4.7 one can see that the first variation formula

$$\begin{aligned} &\delta\mathcal{F}(X, \phi) \\ &\quad = \int_B \left\{ g_{j\ell} X^j_{u^\alpha} X^\ell_{u^\alpha} + \frac{1}{2} \frac{\partial g_{j\ell}}{\partial x^n} X^j_{u^\alpha} X^\ell_{u^\alpha} \phi^n + 2H\sqrt{g}(X_u \wedge X_v)^j \phi^j \right\} du\, dv \end{aligned}$$

holds for all $\phi \in \overset{\circ}{H}{}^1_2(B, \mathbb{R}^3) \cap L_\infty(B, \mathbb{R}^3)$, cp. Theorem 5 in Section 4.7. Moreover, it follows from the minimum property of X that the variational inequality

$$\delta\mathcal{F}(X, \phi) \ge 0$$

holds for all $\phi \subset \overset{\circ}{H}{}^1_2(B, \mathbb{R}^3) \cap L_\infty(B, \mathbb{R}^3)$ with $X + \epsilon\phi \in H^1_2(B, \mathcal{K}_{R+\epsilon})$. The inclusion principle Lemma 8 now implies that the coincidence set $\Omega = \{w \in B : X(w) \in \partial\mathcal{K}_{R+\epsilon}\}$ must be empty. Finally Theorem 7 in Section 4.7 shows that $X \in C^{2,\beta}(B, \mathcal{K}_R) \cap C^0(\overline{B}, \mathbb{R}^3)$ is a conformal solution of the system (58).

The topological character of the boundary mapping follows in a standard way. Theorem 8 is completely proved. □

We finally consider the special case $k = k_0$.

Theorem 9. *Let $\mathcal{K}_R = \{x \in M : |x| \le R\} \cap S(p_0)$ be a compact gauge ball, where $k_0(x) = |x| = d(p_0, p)$. Suppose that*

$$R < \frac{\pi}{2\sqrt{\kappa(M)}} \quad \text{and} \quad \omega(M) > -\infty.$$

Let $\Gamma \subset \mathcal{K}$ be a closed Jordan curve such that $\mathcal{C}(\Gamma, \mathcal{K})$ is nonempty and suppose that H is a function of class $C^{0,\beta}(\mathcal{K}, \mathbb{R})$, $0 < \beta < 1$, for which

$$|H|_{0,\mathcal{K}} < \min\left\{ \frac{a_{\kappa(M)}(R)}{R}, \frac{3b_+^2(R\sqrt{\kappa(M)})}{2b_-^2(R\sqrt{-\omega(M)})} \right\}.$$

Then the assertion of Theorem 8 holds. □

4.9 Scholia

4.9.1 Enclosure Theorems and Nonexistence

The observation that a connected minimal surface lies in the convex hull of its boundary (cf. Theorem 1 of Section 4.1) has been made a long time ago and was, for instance, known to T. Radó (see e.g. [21]). Apparently S. Hildebrandt [11] was the first to observe that also certain nonconvex sets can be used for enclosing minimal surfaces and H-surfaces, and to apply this fact for proving nonexistence of connected minimal surfaces whose boundaries are "too far apart", cf. Theorem 2.3 of Section 4.1. Earlier, J.C.C. Nitsche [13,15] had proved various results about the "extension" of minimal surfaces with two boundary curves, thereby obtaining nonexistence results; cf. also Nitsche [28], pp. 474–498. The results by Hildebrandt [11] were improved and generalized in several directions; a survey of this work is presented in Sections 4.1–4.4, based on papers by Osserman and Schiffer [1], Böhme, Hildebrandt, and Tausch [1], Gulliver and Spruck [1,2], Hildebrandt [8,11], Hildebrandt and Kaul [1], U. Dierkes [1–4,6,11], Dierkes and Huisken [1,2], and Dierkes and Schwab [1]. We particularly mention the geometric maximum principle in Dierkes [6] which is based on a pull–back version of the standard monotonicity formula from geometric measure theory due to M. Grüter [2] (see also Section 2.6 of this volume as well as Böhme, Hildebrandt, and Tausch [1] for a related technique). Furthermore we refer to the maximum principles proved in Gulliver, Osserman, and Royden [1], R. Gulliver [7], and particularly we mention the work of K. Steffen [6] and of Duzaar and Steffen [5–7] where geometric maximum principles of an optimal form are derived. Theorems 3–6 in Section 4.3 are due to Dierkes [11] and Dierkes and Schwab [1]. It is interesting to note that – despite its simplicity – the argument used here is of considerable generality and is applicable to a number of important situations. For example, K. Ecker [2,3] could give a very simple proof of the "neck-pinching" phenomenon for mean curvature flow by using a parabolic version of the polynomial

$$p_j = \sum_{i=1}^{n+k-j} |x^i|^2 - \frac{(n-j)}{j} \sum_{i=n+k-j+1}^{n+k} |x^i|^2 \quad (\text{for } k = 1).$$

Furthermore, U. Clarenz [1,2] applied the same argument to $\mathcal{F}$-*minimal immersions* which arise as extremals of parametric integrals of the type

$$\int_M \mathcal{F}(X, N)\, dA$$

for suitable homogenous integrands $\mathcal{F}$ depending on the position X and the normal of an immersion. Again, general necessary conditions for $\mathcal{F}$-minimal surfaces are obtained, and the method can be generalized to corresponding parabolic flow problems as well. For details in this direction see Winklmann [1].

Apparently the first *barrier-principle* for minimal immersions with arbitrary codimension is due to Jorge and Tomi [1]; however, see also the geometric inclusion principle for energy minimizers obtained earlier by R. Gulliver [1]. The barrier principle for submanifolds with arbitrary codimension and bounded mean curvature, formulated in Theorem 1 of Section 4.4, is due to Dierkes and Schwab [1].

Geometric inclusion principles valid for conformal H_2^1-solutions of the variational inequality (9), Section 4.4, were found by Steffen [6], cp. also Duzaar and Steffen [5–7]. The versions presented in Theorem 2 and 3 require a priori $C^1 \cap H^2_{2,\mathrm{loc}}$-regularity of the solution, which is, however, always satisfied in the application we have in mind later in Section 4.7, due to certain regularity results for obstacle problems, cp. Section 4.8. Our proof of Theorem 2 in Chapter 4.4 is self-contained and independent of the argument in Duzaar and Steffen [5–7]; it cannot be extended to H_2^1-subsolutions. The proof of Theorem 3 in Chapter 4.4 is reminiscent to Proposition 2.4 in Duzaar and Steffen [7] and uses the same type of test function argument. We also mention the geometric inclusion principle of Gulliver and Spruck [2] which uses strict energy minimality of the solution considered. In fact, pushing in a surface under an assumption on the boundary curvature similar to those in Theorems 2 and 3 of Section 4.4 saves energy, and hence energy minimizers cannot touch the boundary of the inclusion domain.

The following terminology due to P. Levy has become customary (see Nitsche [28], pp. 364, 671–672, [37], pp. 354, 373): A closed set $\mathcal{K}$ in $\mathbb{R}^3$ is said to be *H-convex* if for every point $P \in \partial\mathcal{K}$ there is a locally supporting minimal surface $\mathcal{M}$, i.e.: For any $P \in \mathcal{M}$ there is an $\epsilon > 0$ such that $\mathcal{K} \cap B_\epsilon(P)$ lies on one side of $\mathcal{M} \cap B_\epsilon(P)$.

If $\partial\mathcal{K}$ is a regular C^2-surface then H-convexity of $\mathcal{K}$ means that the mean curvature Λ of $\partial\mathcal{K}$ with respect to the inward normal is nonnegative.

4.9.2 The Isoperimetric Problem. Historical Remarks and References to the Literature

Among all closed curves of a given length, the circle encloses a domain of maximal area. This is the classical *isoperimetric property of the circle* which was already known in antiquity. The first transmitted proof of this property is due to Zenodorus who lived between 200 B.C. and 100 A.D. Concerning the history of the isoperimetric problem we refer to Gericke [1]. Of the later proofs we mention that of Galilei [1], pp. 57–60 who prompts Sagredo to say at the end of the discussion:

"Mà dove siamo trascorsi à ingolfarci nella Geometria ..."[3]

The problem became again popular through the work of Steiner who contributed many beautiful ideas to this and to related questions. Yet all of his proofs were imperfect as they only showed that no other curve than the circle can enclose maximal area. It remained open whether there is a curve of given perimeter whose interior maximizes area. The first rigorous proof of the isoperimetric property of the circle was given by Weierstrass in his lectures, and his student H.A. Schwarz established the isoperimetric property of the sphere, a much more difficult question. A beautiful discussion of the isoperimetric problem can be found in Blaschke's classic [3]: *Kreis und Kugel* (with a historical survey in §14).

Fig. 1. Rügen, an island in the Baltic Sea, furnishes an example of a planar domain whose area A is far less than $L^2/4\pi$, L being the length of its circumference. It shows how bold it is to draw conclusions about the area of a domain from the time it takes to sail around it

[3] It seems that Galileo was enthusiastic by rights as his reasoning (according to an oral communication by E. Giusti) can be turned into a proof that is correct by our standards.

Concerning references to the modern literature we refer to Nitsche [28], pp. 290–292, and particularly to Osserman's survey paper [19] that provides a thorough discussion of all pertaining results as well as a report on related questions.

The isoperimetric inequality for minimal surfaces of the type of the disk was first proved by Carleman [3] in 1921.

Beckenbach and Radó [1] proved in 1933: Let S be a surface in $\mathbb{R}^3$ with Gauss curvature K. *Then the inequality* $4\pi A \le L^2$ *holds for all simply connected domains* Ω *in* S *(*A *= area* S*,* L *= length* ∂S*) if and only if* K *is nonpositive.*

The simple connectivity of Ω is crucial as one immediately realizes by looking a long cylinders. Moreover, in the Beckenbach–Radó theorem it is essential that S is a regular surface, whereas in Carleman's theorem the minimal surface may have branch points. Note that in Theorems 1 and 2 of Section 4.5 the minimal surface is allowed to have arbitrarily many branch points.

It is still an open question whether the sharp isoperimetric inequality

$$A(\mathfrak{X}) \le \frac{1}{4\pi} L^2(\mathfrak{X}) \tag{1}$$

holds for any compact minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ with boundary, or if additional assumptions on $\mathfrak{X}$ are truly necessary for (1) to be true. It is, however, known that certain extra-assumptions suffice to ensure the validity of (1). For instance, Osserman and Schiffer [1] proved (1) for minimal surfaces $\mathfrak{X} : M \to \mathbb{R}^3$ defined on an annulus M, and Feinberg [1] showed that (1) also holds for annulus-type surfaces $\mathfrak{X} : M \to \mathbb{R}^n, n \ge 2$. The Osserman–Schiffer result implies that the sharp isoperimetric inequality also holds for minimal surfaces of the topological type of the Möbius strip, see Osserman [18]. The beautiful result of Theorem 3 of Section 4.5 was found by Li, Schoen, and Yau [1]. Amazingly it is strong enough to (essentially) imply the Osserman–Schiffer result. Other interesting conditions guaranteeing (1) were discovered by Alexander-Hoffmann-Osserman [1] and by Osserman [17].

A variant of the *linear isoperimetric inequality (21)* in Section 4.5 using the oscillation of a minimal surface X was pointed out by Nitsche [28]. Küster [3] showed that the radius of the smallest ball containing $X(B)$ leads to the optimal version of the inequality for which equality holds precisely for plane disks.

Concerning generalizations of the isoperimetric inequality to H-surfaces we refer e.g. to papers by Heinz and Hildebrandt [2], Heinz [11], and Kaul [2,3]. A survey of the entire field of geometric inequalities can be found in the treatise of Burago and Zalgaller [1]. B. White [3] showed that, for each integer $n > 1$, there is a smooth Jordan curve Γ in $\mathbb{R}^4$ such that $(1/n)\alpha(n\Gamma) < (1/k)(\alpha(k\Gamma)$ for $1 \le k < n$. Here $\alpha(k\Gamma)$ denotes the least area (counting multiplicities) of any oriented surface with boundary $k\Gamma$ (= k-fold multiple of Γ). In a different way, examples of this kind were somewhat earlier constructed by F. Morgan.

4.9.3 Experimental Proof of the Isoperimetric Inequality

There are two simple soap film experiments by means of which one can demonstrate the isoperimetric property of the circle. For instance, take a wire that has the shape of a plane curve, attach a handle to it, and dip it into a soap solution. On removing it from the liquid, a soap film spanning the wire will be formed. Then place a thin loop of thread onto the film and break the part of the soap film inside of the loop with a blunt tool. As the soap film wants to reduce its area, it will pull the thread tight into the shape of a circle (see Fig. 2). The soap film has minimal energy and therefore minimal area; hence the interior of the strained loop is maximizing area, and since the thread apparently has the form of a circle, we have an "experimental proof" of its isoperimetric property. Further experiments and results with soap films and threads will be described in Chapter 5.

Another experimental proof will be obtained by blowing a soap bubble between two parallel wetted glass plates. Let us begin with a bubble in the form of a hemisphere sitting on one of the plates. By blowing more air into the bubble, it will enlarge until it touches the other plate, whereupon it changes into a circular cylinder that meets both plates perpendicularly in circles (see Fig. 3). The cylinder has minimal area among all surfaces enclosing a fixed volume which touch both plates (a discussion of related mathematical questions can be found in papers by Athanassenas [1,2] and Vogel [1]), whence one concludes that the circle has minimal length among all closed curves bounding the same amount of area. But this "dual property" is equivalent to the isoperimetric property of the circle. This second experiment was apparently first described by Courant (see Courant and Robbins [1]).

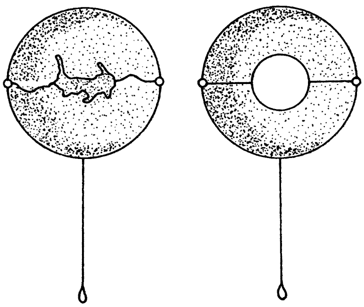

Fig. 2. Experimental proof of the isoperimetric inequality

4.9.4 Estimates for the Length of the Free Trace

The first estimate of this kind was derived by Hildebrandt and Nitsche [4]; an improved version of their result with the optimal constant 2 is due to

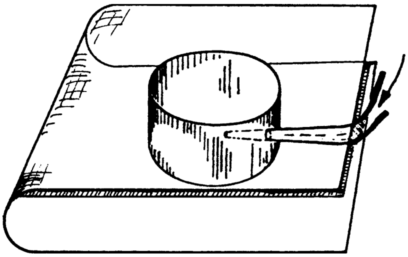

Fig. 3. Another experimental demonstration of this isoperimetric property of the circle

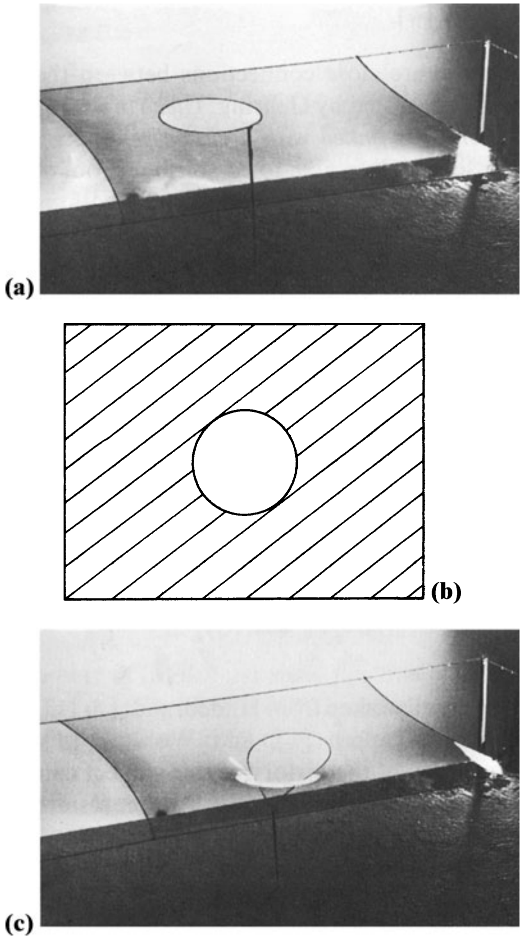

Fig. 4. (**a**), (**b**) Experimental proof of the isoperimetric property of the circle. (**c**) If the thread is pulled down, one obtains a curve of constant curvature (see Chapter 5). (**a**), (**c**) courtesy of Institut für Leichte Flächentragwerke, Stuttgart – Archive

Küster [2]. Finally Dziuk [8] removed the assumption that the minimal surface be free of branch points of odd order on the free boundary. We have presented this result as Theorem 2.

Using the idea of Hildebrandt and Nitsche, Ye [2] has stated estimates of the length of the free trace and of the area of a minimal surface with a partially free boundary in terms of the length of the fixed boundary, in case that the supporting surface is a strict graph (= λ-graph). Ye also provided the example described in Remark 4 which shows that the estimates of Section 4.6, Proposition 1 and Theorem 1, are in a sense optimal. Küster [2] contributed Remark 7, which shows that neither a bound on the Gauss curvature of the support surface S nor a bound on its mean curvature will imply an estimate such as stated in Theorem 2 of Section 4.6; instead, one needs bounds on both principal curvatures of S. Hence the *R-sphere condition* is really adequate in Theorem 2 and by no means artificial.

The partition problem was treated in the paper [2] of Grüter, Hildebrandt, and Nitsche. These authors derived boundary regularity for arbitrary stationary solutions as well as bounds on the length of the free trace such as stated in formulas (58)–(63) of Section 4.6.

We finally mention that an approach to estimates on the length of the free trace for area-minimizing solutions of free boundary problems can already be found in the fundamental work of H. Lewy [4].

Osserman [18] pointed out that there are close connections between the isoperimetric inequality and an inequality suggested by Gehring: *Given in* $\mathbb{R}^3$ *any closed Jordan curve* Γ *of length* $L(\Gamma)$ *which is linked with a closed set* Σ *such that* dist $(\Gamma, \Sigma) \geq r$, *then* $L(\Gamma) \geq 2\pi r$. Osserman was able to establish a proof of Gehring's inequality by means of the isoperimetric inequality. Generalizations to higher dimensions ($n > 3$) follow from work of White [1] and Almgren [7]. Other proofs and generalizations were given by Bombieri and Simon [1], Gage [1], and Gromov [1].

4.9.5 The Plateau Problem for H-Surfaces

In Sections 4.7 and 4.8 we have discussed the Plateau problem for H-surfaces in Euclidean space and in Riemannian manifolds. For $H = \text{const}$ this problem was first treated by E. Heinz [2], H. Werner [1,2], and S. Hildebrandt [4,7], and for variable H by Hildebrandt [5,6]. The Riemannian case was first studied by Hildebrandt and Kaul [1] and R. Gulliver [3]. Further pioneering work in this field is due to H. Wente [1–4,6–8], K. Steffen [1–6], Brezis and Coron [1,3], and M. Struwe [5,7]. The optimal results are due to K. Steffen [6] and Duzaar and Steffen [6].

We particularly mention the solution of Rellich's problem by the work of Brezis and Coron [1,3], M. Struwe [5,7], and K. Steffen [6].

In Section 4.7 (Theorems 3–6) we have outlined several regularity results for variational problems with obstacles due to S. Hildebrandt [12,13]; for similar results see Tomi [4]. We have added some important remarks to make these results accessible to applications for the existence procedure for H-surfaces, cp. Theorems 8 and 9. The existence result for H-surfaces in a closed ball is due to Hildebrandt [5,6]. Our proof given here is a slight modification of his argument. Theorem 9 was found independently and almost simultaneously by Gulliver and Spruck [1] and Hildebrandt [10].

A slight improvement of Gulliver and Spruck's [2] existence theorem for H-surfaces contained in arbitrary closed sets K with suitably curved boundaries is presented in Theorem 10. We have replaced their *pushing in* argument for minimizers by the geometric maximum principles *Enclosure Theorem* 2 and 3 of Section 4.4.

H. Wente [1–4] and K. Steffen [1–6] have initiated a completely different approach to prove existence theorems for H-surfaces by invoking the isoperimetric inequality in a suitable way. In his pioneering work, Wente [1] considered the energy functional for constant H,

$$E_H(x) = D(x) + 2HV,$$

where

$$V(x) = \frac{1}{3}\int_B X \cdot (X_u \wedge X_v)\, du\, dv$$

is the volume enclosed by the surface X and the cone over the boundary trace of X. Using the isoperimetric inequality in $\mathbb{R}^3$ he was able to prove lower semicontinuity of $E_H(\cdot)$ in a class of surfaces with suitably small Dirichlet integral. In a mayor achievement, Steffen [1–6] generalized and improved these results to variable H.

The following result holds:

Theorem 1. (Wente, Steffen). *Suppose that*

$$\sup_{\mathbb{R}^3} |H| \le c\sqrt{\frac{\pi}{A_\Gamma}},$$

where A_Γ is the infimum of area of all surfaces spanned by Γ and $c = \sqrt{2/3}$. Then there is an H-surface X bounded by Γ.

Clearly this theorem gives better existence results than the Theorems 6–9 in Section 4.7 for curves Γ which are of the shape of a curled and knotted rectangle of side lengths ϵ and $\frac{1}{\epsilon}$ spread over a large region of $\mathbb{R}^3$. Probably the optimal constant c in Theorem 1 is $c = 1$. According to Heinz [12] (see Section 4.7, Theorem 1), c cannot be larger than one, and a result by Struwe for constant H indicates that $c = 1$ is the best possible value. Using concepts from geometric measure theory, Steffen [3,4] introduced his notion of an H-volume, replacing the volume term V_H above, thereby obtaining several striking existence theorems under very natural conditions on the prescribed curvature H. A typical result is the following

Theorem 2. (Steffen [4]). *Suppose* $H : \mathbb{R}^3 \to \mathbb{R}$ *satisfies*

$$\int_{\mathbb{R}^3} |H|^3 \, dx < \frac{9\pi}{2}.$$

Then there is an H*-surface* X *bounded by* Γ *which is as regular as* H *(and* Γ*) permit. In particular, if* H *is continuous then* X *is of class* $C^{1,\alpha}(B, \mathbb{R}^3)$ *for every* $0 < \alpha < 1$*, and if* H *is locally Hölder continuous on* $\mathbb{R}^3$ *then* X *is of class* $C^{2,\alpha}$ *and solves the* H*-surface system in the classical sense.*

It may surprise that no condition on the boundary curve Γ is needed here. We remark that all results mentioned above possess suitable analogs for H-surfaces in three-dimensional manifolds M, see Hildebrandt and Kaul [1], Gulliver [3], Steffen [6], and Duzaar and Steffen [6,7]. Corresponding results hold also for H-surfaces which are restricted to lie in given sets K of $\mathbb{R}^3$, see Steffen [4] and Dierkes [2]. We mention in particular the survey articles by Steffen [6] and Duzaar and Steffen [6,7] for a thorough account of existence results for H-surfaces in three-manifolds which are not restricted to a coordinate patch.

Surfaces with prescribed mean curvature vector in manifolds of arbitrary dimensions were found by R. Gulliver [1].

The differential geometric background of Section 4.8 is taken from papers by Hildebrandt and Kaul [1], H. Karcher [6], and S. Hildebrandt [17].