Chapter 3

Singular Boundary Points of Minimal Surfaces

The first section of this chapter will be devoted to the study of minimal surfaces in the neighbourhood of boundary branch points. The fundamental tool for dealing with this problem is the *method of Hartman–Wintner* which yields asymptotic expansions for complex-valued solutions $f(w)$ of a differential inequality

$$(1) \qquad |f_{\overline{w}}(w)| \le c|w|^{-\lambda}|f(w)| \quad \text{on } B_R(0)$$

at the center $w = 0$ of a disk $B_R(0) = \{w \in \mathbb{C}\colon |w| < R\}$.

An appropriate modification of the Hartman–Wintner technique will lead to expansions for vector-valued solutions $X(w)$ of a differential inequality

$$(2) \qquad |\Delta X(w)| \le c|w|^{-\lambda}\{|X(w)| + |\nabla X(w)|\} \quad \text{on } B_R(0)$$

at $w = 0$. One of the main features of the Hartman–Wintner reasoning is that, instead of (1) and (2), one treats integral inequalities which can be considered as weak forms of the differential inequalities (1) and (2). This will enable us to deal with certain singularities at the boundary. In fact, by applying a reflection argument, it will in certain situations be possible to treat boundary singularities as interior singularities of solutions to suitably extended equations. However, to make this artifice valid, it will be indispensable to work with integral versions of (1) and (2) because they require less regularity of their solutions. We refer the reader to Section 2.10 (in particular, Theorems 1 and 2) where we have discussed the behaviour of minimal surfaces at boundary branch points in detail. We emphasize again that the Hartman–Wintner device is the essential tool in proving the asymptotic relation (7) in Section 2.10.

In Section 3.1 we shall describe an extended version of the Hartman–Wintner technique as well as some important generalizations due to Dziuk.

In Section 3.2 we shall study the asymptotic behaviour of the gradient of a minimal surface near a corner on the boundary. We shall discuss corners on a Jordan curve as well as corners between curves and supporting surfaces since

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they occur in partially free boundary problems. The results of Section 3.2 provide the *initial regularity* indispensable for the methods of Sections 3.3 and 3.4 to work. In these sections a precise discussion of the geometric behaviour of a minimal surface at corners will be given. Section 3.3 deals with the Plateau problem for piecewise smooth contours, whereas in Section 3.4 free boundary problems are investigated.

3.1 The Method of Hartman and Wintner, and Asymptotic Expansions at Boundary Branch Points

This section deals with the asymptotic behaviour of solutions to certain differential and integral inequalities at *interior singularities*. In certain situations, boundary singularities can be made into inner singularities by extending a solution, for example, by reflection.

First we shall consider complex-valued or even vector-valued solutions $f(w)$ of the differential inequality

$$|f_{\overline{w}}(w)| \le c|w|^{-\lambda}|f(w)| \tag{1}$$

in a disk $B_R(0)$, where λ and c are real constants with $0 \le \lambda < 1$ and $c > 0$, and f is of class $C^1(B_R(0), \mathbb{C}^N), N \ge 1$. As usual we write

$$g_w = \frac{\partial g}{\partial w} = \frac{1}{2}(g_u - ig_v), \quad g_{\overline{w}} = \frac{\partial g}{\partial \overline{w}} = \frac{1}{2}(g_u + ig_v).$$

Secondly, we consider vector-valued solutions $X(w) = (X^1(w), X^2(w), \dots, X^N(w))$ of

$$|\Delta X(w)| \le c|w|^{-\lambda}\{|X(w)| + |\nabla X(w)|\} \tag{2}$$

in $B_R(0), c > 0, 0 \le \lambda < 1$, which are of class C^1 of $B_R(0)$. If the right-hand side of (2) would not contain X but only ∇X, (2) could be considered as a special case of (1) by setting $f(w) := X_w(w)$.

Both (1) and (2) can be transformed into integral inequalities which require less regularity of their solutions.

For instance, let $f(w)$ be a solution of (1) in a domain $\Omega \subset \mathbb{C}$ which is of class C^1, and let $\mathcal{D} \Subset \Omega$ be an arbitrary subdomain of Ω with piecewise smooth boundary $\partial\mathcal{D}$. Choose an arbitrary function $\phi \in C^1(\Omega, \mathbb{C})$ and apply Green's formula

$$\int_{\partial\mathcal{D}} g(w)\, dw = 2i \int\int_{\mathcal{D}} \frac{\partial}{\partial \overline{w}} g(w)\, du\, dv$$

to $g(w) = \phi(w) \cdot f(w)$.

Differing from our usual notation, we denote double integrals by two integral signs.

The integral $\int_{\partial\mathcal{D}} g(w)\,dw$ stands for the complex line integral of the function g over the boundary $\partial\mathcal{D}$ which is assumed to be positively oriented with respect to $\mathcal{D}$. Then we obtain

$$\int_{\partial\mathcal{D}} \phi(w)\cdot f(w)\,dw = 2i\iint_{\mathcal{D}} \left[\phi_{\overline{w}}(w)\cdot f(w) + \phi(w)\cdot f_{\overline{w}}(w)\right]\,du\,dv,$$

and (1) yields

$$\left|\int_{\partial\mathcal{D}} \phi(w)\cdot f(w)\,dw\right| \le 2\iint_{\mathcal{D}} [|\phi_{\overline{w}}(w)| + c|w|^{-\lambda}|\phi(w)|]|f(w)|\,du\,dv. \tag{3}$$

This is the integral inequality associated with (1).

Similarly, we have

$$\int_{\partial\mathcal{D}} \phi\cdot X_w\,dw = 2i\iint_{\mathcal{D}} [\phi_{\overline{w}}\cdot X_w + \phi\cdot X_{w\overline{w}}]\,du\,dv,$$

and we derive from (2) the inequality

$$\begin{aligned}\left|\int_{\partial\mathcal{D}} \phi(w)\cdot X_w(w)\,dw\right| \le 2\iint_{\mathcal{D}} \{&|\phi_{\overline{w}}(w)||X_w(w)| \\ &+ c|w|^{-\lambda}|\phi(w)|[|X(w)| + |X_w(w)|]\}\,du\,dv.\end{aligned} \tag{4}$$

Here c is one quarter of the constant c in (2) because of $\Delta X = 4X_{w\overline{w}}$.

In the following we shall work with inequalities (3) and (4) rather than with (1) or (2) respectively. Note that (3) makes sense even for continuous $f(w)$, and (4) can even be considered for functions X of class C^1. Hence we give the following

Definition 1. *A mapping $f(w) = (f^1(w), \dots, f^N(w)), w \in B_R(0)$, is said to satisfy* **Assumption (A1)** *on $B_R(0)$ if it is of class $C^0(B_R(0)\setminus\{0\}, \mathbb{C}^N)$ and fulfils (3) for every $\phi \in C^1(B_R(0), \mathbb{C})$ and for every $\mathcal{D} \Subset B_R(0)$ with piecewise smooth boundary $\partial\mathcal{D}$.*

Similarly, $X(w) = (X^1(w), \dots, X^N(w)), w \in B_R(0)$, is said to fulfil **Assumption (A2)** *on $B_R(0)$ if it is of class $C^1(B_R(0), \mathbb{R}^N)$ and satisfies (4) for every $\phi \in C^1(B_R(0), \mathbb{C})$ and for each $\mathcal{D} \Subset B_R(0)$ with piecewise smooth boundary.*

Then we are going to prove the following two theorems:

Theorem 1. *Let $f(w)$ satisfy (A1) on $B_R(0)$, and suppose that $f(w) \not\equiv 0$ in $B_R(0)$, and that there exists a number $\lambda' \in [0,1)$ such that*

$$f(w) = O(|w|^{-\lambda'}) \quad \text{as } w \to 0.$$

Then there is a nonnegative integer ν such that $\lim_{w\to 0} w^{-\nu} f(w)$ exists and is different from zero.

Theorem 2. *Let $X(w)$ satisfy* (A2) *on $B_R(0)$ and suppose that there is a nonnegative integer ν such that*

$$X(w) = o(|w|^\nu) \quad \text{as } w \to 0. \tag{5}$$

Then the limit $\lim_{w\to 0} X_w(w)w^{-\nu}$ exists. In addition, if $X(w) \not\equiv 0$, then there is a first nonnegative integer ν such that (5) *does not hold and, moreover, that $\lim_{w\to 0} X_w(w) \cdot w^{-\mu}$ exists for $\mu = \nu - 1$ and is different from zero.*

We shall prove both theorems simultaneously. The proof of the second theorem differs from the first one in that we have as well to estimate the additional term involving $|X(w)|$. We will return to this in detail after completing the proof of Theorem 1. Without loss of generality we may assume that f is a scalar function.

Before entering into the proofs, we first mention two interesting corollaries.

Corollary 1. *Let $f(w)$ satisfy the assumptions of Theorem 1. Then there exists a nonnegative integer ν and a complex number $a \not\equiv 0$ such that*

$$f(w) = aw^\nu + o(|w|^\nu) \quad \textit{as } w \to 0.$$

Corollary 2. *Let $X(w)$ satisfy* (A2) *on $B_R(0)$ and suppose that $X(0) = 0$ but $X(w) \not\equiv 0$ on $B_R(0)$. Then there exists a nonnegative integer μ and a nonzero complex vector A such that*

$$X_w(w) = Aw^\mu + o(|w|^\mu) \quad \textit{as } w \to 0, \tag{6}$$

and

$$X(w) = \operatorname{Re}\{Bw^{\mu+1}\} + o(|w|^{\mu+1}) \quad \textit{as } w \to 0, \tag{7}$$

where $B = 2(\mu + 1)^{-1}A$.

Proof of Corollary 2. Equation (6) is nothing but a different formulation of the second statement in Theorem 2. Relation (7) follows by a suitable integration. In fact,

$$\begin{aligned} X(w) &= \int_0^1 [uX_u(tw) + vX_v(tw)]\, dt = \int_0^1 \operatorname{Re}[2wX_w(tw)]\, dt \\ &= \int_0^1 \operatorname{Re}[2(At^\mu w^{\mu+1} + t^\mu o(|w|^{\mu+1})]\, dt \\ &= \operatorname{Re}\left[\frac{2}{\mu+1} Aw^{\mu+1}\right] + o(|w|^{\mu+1}). \end{aligned}$$

□

Now we begin with the proof of Theorem 1.

Lemma 1. *Let f satisfy* (A1) *and suppose that there exists some nonnegative integer μ such that*

$$f(w) = o(|w|^{\mu-1}) \quad \textit{as } w \to 0.$$

Then $f(w) = O(|w|^\mu|)$ as $w \to 0$.

Proof. Let $r < R, \xi \in B_r(0), \xi \neq 0, \varepsilon < \min(\frac{|\xi|}{2}, r - |\xi|)$, and put

$$\mathcal{D}_{r,\varepsilon} := B_r(0) \setminus [B_\varepsilon(0) \cup B_\varepsilon(\xi)].$$

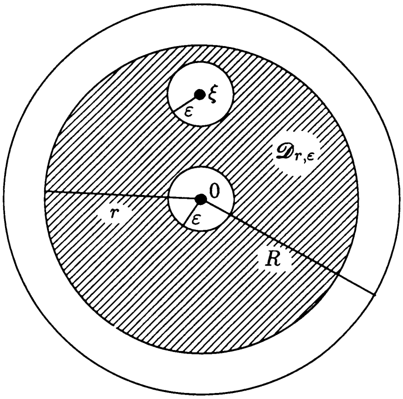

Fig. 1.

Now we test inequality (3) with the function $\phi(w) = \frac{1}{w^\mu} \frac{1}{w-\xi}$ and the domain $\mathcal{D}_{r,\varepsilon}$. This yields the estimate

$$(8) \quad \left| \int_{\partial \mathcal{D}_{r,\varepsilon}} w^{-\mu}(w-\xi)^{-1} f(w)\, dw \right| \leq 2c \iint_{\mathcal{D}_{r,\varepsilon}} |w|^{-\mu-\lambda} |w-\xi|^{-1} |f(w)|\, du\, dv.$$

The result will now follow by letting ε tend to zero. To accomplish this it will be necessary to consider the boundary integrals on the left-hand side of (6) separately. Firstly, we have

$$\int_{|w-\xi|=\varepsilon} w^{-\mu}(w-\xi)^{-1} f(w)\, dw = i \int_0^{2\pi} (\xi + \varepsilon e^{i\varphi})^{-\mu} f(\xi + \varepsilon e^{i\varphi})\, d\varphi$$

whence

$$(9) \qquad \begin{aligned} \lim_{\varepsilon \to 0} \int_{|w-\xi|=\varepsilon} w^{-\mu}(w-\xi)^{-1} f(w)\, dw &= i \int_0^{2\pi} \xi^{-\mu} f(\xi)\, d\varphi \\ &= 2\pi i f(\xi) \xi^{-\mu}. \end{aligned}$$

Furthermore we obtain

$$\begin{aligned} \left| \int_{|w|=\varepsilon} w^{-\mu}(w-\xi)^{-1} f(w)\, dw \right| &\leq \int_{|w|=\varepsilon} \left| \frac{f(w)}{w^{\mu-1}} \right| |w(w-\xi)|^{-1} |dw| \\ &\leq \frac{2}{|\xi|} \int_0^{2\pi} \frac{|f(\varepsilon e^{i\varphi})|}{\varepsilon^{\mu-1}}\, d\varphi, \end{aligned}$$

where we have used that $|w-\xi| > \frac{|\xi|}{2}$ for $w \in \partial B_\varepsilon(0)$. Hence

$$\lim_{\varepsilon\to 0}\left|\int_{|w|=\varepsilon} w^{-\mu}(w-\xi)^{-1} f(w)\,dw\right| = 0, \tag{10}$$

taking $f(w) = o(|w|^{\mu-1})$ into account.

Now we conclude from (8) the inequality

$$\begin{aligned}\left|2\pi i f(\xi)\xi^{-\mu} - \int_{|w|=r} w^{-\mu}(w-\xi)^{-1} f(w)\,dw\right| \\ \le 2c\iint_{|w|\le r} |w|^{-\mu-\lambda}|w-\xi|^{-1}|f(w)|\,du\,dv.\end{aligned} \tag{11}$$

Define J_1 and J_2 by the formulas

$$J_1(\xi) := \int_{|w|=r} |w|^{-\mu}|w-\xi|^{-1}|f(w)||dw|$$

and

$$J_2(\xi) := \iint_{|w|\le r} |w|^{-\mu-\lambda}|w-\xi|^{-1}|f(w)|\,du\,dv;$$

then (11) implies the inequality

$$2\pi|f(\xi)\xi^{-\mu}| \le J_1(\xi) + 2cJ_2(\xi). \tag{12}$$

It follows from (12) that the lemma is proved if we find uniform bounds for $J_1(\xi)$ and $J_2(\xi)$ and all $\xi \in B_{r_0}(0)$ for some $0 < r_0 < r$. The uniform boundedness of $J_1(\xi)$ is obvious since the singularities of the integrand, 0 and ξ, have positive distance from the circle $|w| = r$. Now we show that $J_1(\xi)$ provides an upper bound for $J_2(\xi)$. To this end we multiply (12) by $|\xi|^{-\lambda}|\xi - w_0|^{-1}$ where $w_0 \in B_r(0)$ and integrate over the disk $B_r(0)$; it follows that

$$\begin{aligned}&2\pi J_2(w_0) \\ &\quad = 2\pi\iint_{|\xi|<r} |\xi|^{-\mu-\lambda}|\xi-w_0|^{-1}|f(\xi)|\,d\xi_1\,d\xi_2 \le I_1(w_0) + I_2(w_0),\end{aligned} \tag{13}$$

where $\xi = \xi_1 + i\xi_2$ and

$$\begin{aligned}I_1(w_0) &:= \iint_{|\xi|<r} |\xi|^{-\lambda}|\xi-w_0|^{-1}J_1(\xi)\,d\xi_1\,d\xi_2,\\ I_2(w_0) &:= 2c\iint_{|\xi|<r} |\xi|^{-\lambda}|\xi-w_0|^{-1}J_2(\xi)\,d\xi_1\,d\xi_2.\end{aligned}$$

Using the identity

$$(w-\xi)^{-1}(\xi-w_0)^{-1} = (w-w_0)^{-1}[(w-\xi)^{-1} + (\xi-w_0)^{-1}]$$

and interchanging the order of integration we obtain

$$\begin{aligned} I_1(w_0) &\le \iint_{|\xi|<r} \left\{ \int_{|w|=r} [|w-\xi|^{-1} + |\xi-w_0|^{-1}]|\xi|^{-\lambda}|w-w_0|^{-1} \right. \\ &\qquad \left. \cdot |w|^{-\mu}|f(w)||dw| \right\} d\xi_1\, d\xi_2 \\ &\le Mr^{1-\lambda} \int_{|w|=r} |w|^{-\mu}|w-w_0|^{-1}|f(w)||dw| \\ &= Mr^{1-\lambda} J_1(w_0), \end{aligned}$$

where we have used the inequality

$$(*) \qquad \iint_{|\xi|<r} \{|w-\xi|^{-1} + |\xi-w_0|^{-1}\}|\xi|^{-\lambda}\, d\xi_1\, d\xi_2 \le Mr^{1-\lambda}$$

which will be proved later. Similarly we obtain the estimate

$$\begin{aligned} I_2(w_0) &\le 2c \iint_{|\xi|<r} \left\{ \iint_{|w|<r} [|w-\xi|^{-1} + |\xi-w_0|^{-1}]|\xi|^{-\lambda} \right. \\ &\qquad \left. \cdot |w-w_0|^{-1}|w|^{-\mu-\lambda}|f(w)|du\, dv \right\} d\xi_1\, d\xi_2 \\ &\le 2Mr^{1-\lambda} cJ_2(w_0) \end{aligned}$$

with the same constant M.

Finally we infer from (13) the inequality

$$2\pi J_2(w_0) \le Mr^{1-\lambda}[J_1(w_0) + 2cJ_2(w_0)]$$

which implies

$$(14) \qquad 2(M^{-1}\pi r^{\lambda-1} - c)J_2(w_0) \le J_1(w_0).$$

If we now choose $r_0 < (\frac{\pi}{cM})^{1/(1-\lambda)}$, then the inequality

$$J_2(w_0) \le \tfrac{1}{2}(M^{-1}\pi r^{\lambda-1} - c)^{-1} J_1(w_0)$$

holds for all $w_0 \in B_{r_0}(0)$, and Lemma 1 is proved. □

Now we have to add a proof of inequality $(*)$. In fact we shall prove a slightly more general result known as

E. *Schmidt's inequality* (see Vekua [1], p. 39). Suppose that $w_1, w_2 \in B_r(0)$, and that $\alpha, \beta < 2$ are positive real constants. Then

$$(15)\qquad \iint_{B_r(0)} |\xi - w_1|^{-\alpha} |\xi - w_2|^{-\beta}\, d\xi_1\, d\xi_2$$
$$\le \begin{cases} M_1 |w_1 - w_2|^{2-\alpha-\beta} & \text{if } a + \beta > 2, \\ M_2 + 8\pi |\log |w_1 - w_2|| & \text{if } a + \beta = 2, \\ M_3 r^{2-\alpha-\beta} & \text{if } \alpha + \beta < 2, \end{cases}$$

where $\xi = \xi_1 + i\xi_2$, and M_1, M_2, M_3 are constants depending only on α and β.

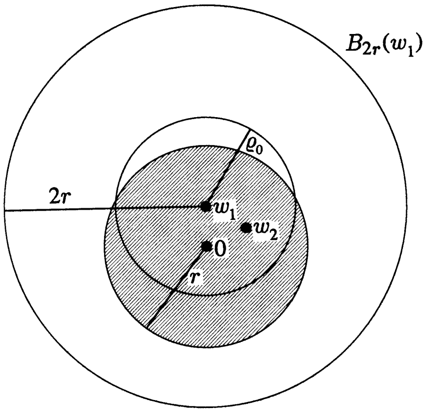

Fig. 2.

Proof of (15). We replace $B_r(0)$ by the larger domain $B_{2r}(w_1) \supset B_r(0)$. If we put $\rho_0 = 2|w_1 - w_2|$, we have for all $\xi \in B_{2r}(w_1) \setminus B_{\rho_0}(w_1)$ that $2|\xi - w_2| \ge |\xi - w_1|$ which yields

$$(16)\quad \iint_{B_{2r}(w_1)\setminus B_{\rho_0(w_1)}} |\xi - w_1|^{-\alpha} |\xi - w_2|^{-\beta}\, d\xi_1\, d\xi_2 \le 2^{1+\beta}\pi \int_{\rho_0}^{2r} \rho^{1-\alpha-\beta}\, d\rho$$
$$\le \begin{cases} 2^{1+\beta}\pi \frac{|w_1 - w_2|^{2-\alpha-\beta}}{\alpha+\beta-2} & \text{if } \alpha + \beta > 2, \\ 2^{1+\beta}\pi \log \frac{r}{|w_1 - w_2|} & \text{if } \alpha + \beta = 2, \\ \frac{2^{3-\alpha}}{2-\alpha-\beta} r^{2-\alpha-\beta} & \text{if } \alpha + \beta < 2. \end{cases}$$

Applying the linear transformation $\xi^* = \frac{\xi - w_1}{|w_2 - w_1|}$ which maps $B_{\rho_0}(w_1)$ onto $B_2(0)$, we conclude from the change-of-variables formula that

$$(17)\qquad \iint_{B_{\rho_0}(0)} |\xi - w_1|^{-\alpha} |\xi - w_2|^{-\beta}\, d\xi_1\, d\xi_2$$
$$= |w_1 - w_2|^{2-\alpha-\beta} \iint_{B_2(0)} |\xi^*|^{-\alpha} \left| \xi^* - \frac{w_2 - w_1}{|w_2 - w_1|} \right|^{-\beta} d\xi_1^*\, d\xi_2^*.$$

By virtue of $\alpha, \beta < 2$, the integral on the right-hand side can be estimated by a *finite* constant $M(\alpha, \beta)$. Inequality (15) follows by combining the above estimates. $\square$

Lemma 2. *Suppose that* f *satisfies assumption* (A1) *and that* $f(w) = o(|w|^{\mu-1})$ *as* $w \to 0$ *for some nonnegative integer* μ. *Then the limit* $\lim_{w\to 0} f(w)w^{-\mu}$ *exists.*

Proof. Let $g(w) := f(w)w^{-\mu}$ and

$$F_r(\xi) = (2\pi i)^{-1} \int_{|w|=r} g(w)(w-\xi)^{-1}\, dw, \quad r \in (0, R).$$

Then $F_r(\xi)$ is holomorphic on $B_r(0)$, and from inequality (11) we infer for all $\xi \in B_r(0) \setminus \{0\}$ the relation

$$\begin{aligned} |g(\xi) - F_r(\xi)| &\le \frac{c}{\pi} \iint_{B_r(0)} |w|^{-\mu-\lambda} |w-\xi|^{-1} |f(w)|\, du\, dv \\ &\le c_1 \iint_{B_r(0)} |w|^{-\lambda} |w-\xi|^{-1}\, du\, dv, \end{aligned}$$

where we have used Lemma 1. We infer from inequality (15) the estimate

$$|g(\xi) - F_r(\xi)| \le c_2 r^{1-\lambda}$$

for all $\xi \in B_r(0) \setminus \{0\}$. Again from the boundedness of $g(w)$ we conclude the existence of a sequence $\{w_n\}_{n\in\mathbb{N}}$ tending to zero such that

$$a := \lim_{n\to\infty} g(w_n) \in \mathbb{C},$$

whence

$$|a - F_r(0)| \le c_2 r^{1-\lambda},$$

and since $\lambda < 1$, we obtain

$$\lim_{r\to 0} F_r(0) = a.$$

Finally we conclude from

$$|g(\xi) - a| \le |g(\xi) - F_r(\xi)| + |F_r(\xi) - F_r(0)| + |F_r(0) - a| \le c_3 r^{1-\lambda}$$

the relation

$$\lim_{\xi\to 0} g(\xi) = a. \qquad \square$$

Proof of Theorem 1. The theorem will be proved if we can find an integer $\nu \ge -1$ with the properties $f(w) = o(|w|^{\nu})$ but $f(w) \ne o|w|^{\nu+1}$ as $w \to 0$, taking Lemma 2 into account. Let us assume on the contrary that for all nonnegative ν the relation $f(w) = o(|w|^\nu)$ holds true. We will then show that $f \equiv 0$ on $B_R(0)$.

To accomplish this, we recall inequality (12) with $w_0 = 0$:

$$2(M^{-1}\pi r^{\lambda-1} - c)J_2(0) \leq J_1(0),$$

where

$$J_1(0) = \int_{|w|=r} |w|^{-\nu-1}|f(w)||dw|,$$

and

$$J_2(0) = \int_{|w|=r} |w|^{-\nu-\lambda-1}|f(w)|\, du\, dv.$$

We select $r < (\frac{\pi}{cM})^{1/(1-\lambda)}$ and suppose that there exists some $\xi_0 \in B_r(0)$ with $f(\xi_0) \neq 0$. Clearly there exist numbers $0 < \delta_1 \leq \delta_2, \varepsilon > 0$, such that $B_\varepsilon(\xi_0) \Subset B_r(0)$ and

$$2(M^{-1}\pi r^{\lambda-1} - c)\delta_1[|\xi_0| + \varepsilon]^{-\nu-\lambda-1} \leq \delta_2 r^{-\nu-1}.$$

Therefore there exists some constant c_1 independent of ν such that

$$0 < c_1 \leq \left(\frac{\varepsilon + |\xi_0|}{r}\right)^{\nu+1}.$$

This relation, however, cannot hold for all $\nu \in \mathbb{Z}$ since $|\xi_0| + \varepsilon < r$. In conclusion we have shown that $f = 0$ on $B_r(0)$ for some sufficiently small r, and *a* continuation argument implies $f = 0$ on $B_R(0)$. This completes the proof of Theorem 1. $\square$

Next we are going to prove Theorem 2.

Lemma 3. *Suppose $X(w) \in C^1(B_r(0), \mathbb{R}^N)$ satisfies $X(0) = 0$ and $X_w(w) = o(|w|^{\mu-1})$ as $w \to 0$. Then $X(w) = o(|w|^\mu)$.*

Proof. Fix $w \in B_r(0)$. Then a simple integration yields

$$\begin{aligned} \frac{X(w)}{w^\mu} &= \int_0^1 \left\{\frac{u}{w^\mu}X_u(tw) + \frac{v}{w^\nu}X_v(tw)\right\} dt \\ &= \int_0^1 \frac{2}{w^\mu}\mathrm{Re}(wX_w(tw))\, dt \\ &= 2\int_0^1 t^{\mu-1}\left\{\frac{1}{(tw)^\mu}\mathrm{Re}(twX_w(tw))\right\} dt \\ &= \frac{2}{\mu}\frac{1}{(t_0w)^\mu}\mathrm{Re}(t_0wX_w(t_0w)) \end{aligned} \tag{18}$$

for some $t_0 \in (0,1)$. Consequently,

$$\lim_{w\to 0}\frac{X(w)}{w^\mu} = 0.$$

$\square$

The following auxiliary result provides a counterpart to Lemma 1.

Lemma 4. *Let X satisfy Assumption* (A2)*, and suppose that there exists some nonnegative integer μ such that $X_w(w) = o(|w|^{\mu-1})$ as $w \to 0$. Then $X_w(w) = O(|w|^{\mu})$ as $w \to 0$.*

Proof. As in the proof of Lemma 1 we put

$$\mathcal{D}_{r,\varepsilon} = B_r(0) \setminus [B_\varepsilon(0) \cup B_\varepsilon(\xi)]$$

and

$$\phi(w) = \frac{1}{w^\mu} \cdot \frac{1}{w-\xi}.$$

Then (4) yields the inequality

$$\left| \int_{\partial \mathcal{D}_{r,\varepsilon}} w^{-\mu}(w-\xi)^{-1} X_w(w)\, dw \right| \tag{19}$$
$$\leq 2c \iint_{\mathcal{D}_{r,\varepsilon}} |w|^{-\mu-\lambda}|w-\xi|^{-1}[|X(w)| + |X_w(w)|]\, du\, dv,$$

taking $\phi_{\overline{w}} = 0$ on $\mathcal{D}_{r,\varepsilon}$ into account. Note that Lemma 3 and the inequality $0 \leq \lambda < 1$ imply the boundedness of the integral

$$J_3(\xi) := \iint_{B_r(0)} |w|^{-\mu-\lambda}|w-\xi|^{-1}|X(w)|\, du\, dv.$$

Now we can proceed as in the proof of Lemma 1, i.e. we let $\varepsilon \to 0$ and obtain the estimate

$$2\pi |X_w(\xi)\xi^{-\mu}| \leq J_1(\xi) + 2c[J_2(\xi) + J_3(\xi)], \tag{20}$$

where f has to be replaced by X_w in the formulas for J_1 and J_2 respectively, i.e.,

$$J_1(\xi) := \int_{|w|=r} |w|^{-\mu}|w-\xi|^{-1}|X_w(w)||dw|,$$

and

$$J_2(\xi) := \iint_{|w|\leq r} |w|^{-\mu-\lambda}|w-\xi|^{-1}|X_w(w)|\, du\, dv.$$

Since the boundedness of J_1 is obvious for small ξ, we only show the boundedness of J_2. To this end we multiply (18) by $|\xi|^{-\lambda}|\xi - w_0|^{-1}, w_0 \in B_r(0)$, and integrate over $B_r(0)$. Then we obtain

$$2\pi J_2(w_0) \leq I_1(w_0) + I_2(w_0) + I_3(w_0), \tag{21}$$

where

$$I_1(w_0) := \iint_{|\xi|<r} |\xi|^{-\lambda}|\xi - w_0|^{-1} J_1(\xi)\, d\xi_1\, d\xi_2,$$

$$I_2(w_0) := 2c \iint_{|\xi|<r} |\xi|^{-\lambda}|\xi - w_0|^{-1} J_2(\xi)\, d\xi_1\, d\xi_2,$$

$$I_3(w_0) := 2c \iint_{|\xi|<r} |\xi|^{-\lambda}|\xi - w_0|^{-1} J_3(\xi)\, d\xi_1\, d\xi_2,$$

and $\xi = \xi_1 + i\xi_2$.

As in the proof of Lemma 1 we conclude

$$I_1(w_0) \le Mr^{1-\lambda} J_1(w_0),$$

$$I_2(w_0) \le 2Mr^{1-\lambda} cJ_2(w_0).$$

Similarly we infer from (15) and

$$(w-\xi)^{-1}(\xi - w_0)^{-1} = (w - w_0)^{-1}[(w-\xi)^{-1} + (\xi - w_0)^{-1}]$$

the estimate

$$I_3(w_0) \le 2M_3^2 c_1 r^{2(1-\lambda)} \quad \text{for some constant } c_1.$$

Finally, the boundedness of J_2 follows from (21) and the above estimates if we choose $r > 0$ suitably small. The assertion of the lemma follows from relation (20) since the right-hand side of (20) remains bounded as $\xi \to 0$. □

Lemma 5. *Let $X(w)$ satisfy assumption* (A2) *and suppose that for some non-negative integer μ we have*

$$X_w(w) = o(|w|^{\mu-1}) \quad \textit{as } w \to 0.$$

Then the limit $\lim_{w\to 0} X_w(w)w^{-\mu}$ exists.

Proof. We put $g(w) := X_w(w)w^{-\mu}$ and

$$F_r(\xi) := (2\pi i)^{-1} \int_{|w|=r} g(w)(w-\xi)^{-1}\, dw.$$

In the relation (19) we let ε tend to zero (cf. the proof of Lemma 1) and obtain the inequality

$$|g(\xi) - F_r(\xi)| \le \frac{c}{\pi} \iint_{B_r(0)} |w|^{-\mu-\lambda}|w-\xi|^{-1}[|X(w)| + |X_w(w)|]\, du\, dv,$$

holding for all $\xi \in B_r(0) \setminus \{0\}$. Now Lemmata 3 and 4 imply

$$|g(\xi) - F_r(\xi)| \le c_1 \iint_{B_r(0)} |w|^{-\lambda}|w-\xi|^{-1}\, du\, dv$$

for all $\xi \in B_r(0) \setminus \{0\}$ and some constant c_1. From here on we can proceed exactly as in the proof of Lemma 2. □

Proof of Theorem 2. Recall that $X(w)$ satisfies assumption (A2) and that for some nonnegative $\nu \in \mathbb{Z}$ we have

$$X(w) = o(|w|^\nu) \quad \text{as } w \to 0. \tag{22}$$

We first show that the limit $\lim_{w\to 0} X_w(w)w^{-\nu}$ exists. Since X is supposed to be differentiable, this clearly holds when $\nu = 0$. On the other hand, if $\nu = 1$ we infer from (22) that

$$X_w(w) = o(1) \quad \text{as } w \to 0,$$

and an application of Lemma 5 implies the existence of

$$\lim_{w\to 0} X_w(w)w^{-1}.$$

In order to prove the general case $\nu > 1$, we shall inductively show that

$$X_w(w) = o(|w|^{\mu-1}) \quad \text{as } w \to 0 \tag{23}$$

holds for all $\mu \in [1, \nu]$. (The result will then follow by a further application of Lemma 5.)

Assume the validity of (23) for some $\mu < \nu$; then there exists some number $a \in \mathbb{C}$ such that

$$\lim_{w\to 0} X_w(w)w^{-\mu} = a. \tag{24}$$

We show that $a = 0$. To this end, observe that we can write

$$\begin{aligned} \frac{X(u,0)}{u^{\mu+1}} &= \int_0^1 \frac{u}{u^{\mu+1}} X_u(tu, 0)\, dt \\ &= \int_0^1 t^\mu \frac{X_u(tu,0)}{(tu)^\mu}\, dt = \frac{X_u(t_0u,0)}{(t_0u)^\mu} \int_0^t t^\mu\, dt \\ &= \frac{1}{\mu+1} \frac{X_u(t_0u,0)}{(t_0u)^\mu} \quad \text{for some } t_0 \in (0,1). \end{aligned} \tag{25}$$

On the other hand we infer from (24) that the function $g(w) := X_w(w)w^{-\mu}$ is continuous at $w = 0$, and again (24) implies

$$a = \lim_{n\to\infty} \frac{X_w(u_n, 0)}{u_n^\mu} = \lim_{n\to\infty} \frac{X_u(u_n,0) - iX_v(u_n,0)}{2u_n^\mu}$$

whence

$$\operatorname{Re} a = \lim_{n\to\infty} \frac{X_u(u_n,0)}{2u_n^\mu} \quad \text{for every sequence } u_n \to 0. \tag{26}$$

Now (25), (26), the assumption $X(w) = o(|w|^\nu)$ as $w \to 0$, and the continuity of $g(w)$ yield the relation

$$\operatorname{Re} a = 0.$$

Furthermore, we infer from (24) that

$$a = \lim_{n\to\infty} \frac{X_u(0,v_n) - iX_v(0,v_n)}{2i^\mu v_n^\mu},$$

and in particular, if μ is even,

$$\pm \operatorname{Im} a = \lim_{n\to\infty} \frac{X_v(0,v_n)}{2v_n^\mu}, \quad \text{where } v_n \to 0.$$

Hence the same argument yields that $\operatorname{Im} a = 0$, provided that μ is even. If μ is odd we consider the function $Y(w) = wX(w)$. Then $Y_w(w) = X(w) + wX_w(w)$, and therefore

$$\lim_{w\to 0} \frac{Y_w(w)}{w^{\mu+1}} = \lim_{w\to 0} \frac{X(w)}{w^{\mu+1}} + \lim_{w\to 0} \frac{X_w(w)}{w^\mu} = a.$$

Also $Y(w) = o(|w|^{\mu+2})$ as $w \to 0$, whence

$$a = \lim_{n\to\infty} \frac{Y_u(0,v_n) - iY_v(0,v_n)}{2i^{\mu+1} v_n^{\mu+1}},$$

whenever $v_n \to 0$ with $n \to \infty$. Thus we obtain that $\operatorname{Im} a = 0$ if we repeat the argument above. This proves the first part of Theorem 2. To establish the second statement we assume on the contrary that, for all nonnegative $\mu \in \mathbb{Z}$, we have $X(w) = o(|w|^\mu)$. It will then be shown that $X_w \equiv 0$ in $B_R(0)$ contradicting the assumption that $X(w) \not\equiv 0$ on $B_R(0)$.

Note that, by the first part of Theorem 2, we obtain the relation

$$X_w(w) = O(|w|^\mu) \quad \text{for all } \mu,$$

and in particular

$$X_w(w) = o(|w|^{\mu-1}) \quad \text{as } w \to 0$$

and for all nonnegative $\mu \in \mathbb{Z}$. We are thus in a position to repeat the argument given in the proof of Lemma 4. Inequality (21) with $w_0 = 0$ now reads as

$$2\pi J_2(0) \le I_1(0) + I_2(0) + I_3(0), \tag{27}$$

where

$$\begin{aligned} I_1(0) &= \iint_{|\xi|<r} |\xi|^{-\lambda-1} J_1(\xi)\, d\xi_1\, d\xi_2, \\ I_2(0) &= 2c \iint_{|\xi|<r} |\xi|^{-\lambda-1} J_2(\xi)\, d\xi_1\, d\xi_2, \\ I_3(0) &= 2c \iint_{|\xi|<r} |\xi|^{-\lambda-1} J_3(\xi)\, d\xi_1\, d\xi_2, \quad \xi = \xi_1 + i\xi_2, \end{aligned}$$

and

$$
\begin{aligned}
J_1(\xi) &= \int_{|w|=r} |w|^{-\mu}|w-\xi|^{-1}|X_w(w)||dw|, \\
J_2(\xi) &= \iint_{|w|\le r} |w|^{-\mu-\lambda}|w-\xi|^{-1}|X_w(w)|\,du\,dv, \\
J_3(\xi) &= \iint_{|w|<r} |w|^{-\mu-\lambda}|w-\xi|^{-1}|X_w(w)|\,du\,dv, \quad w=u+iv.
\end{aligned}
$$

As in the proof of Lemma 1, i.e. using the Schmidt's inequality and the identity

$$
(w-\xi)^{-1}\xi^{-1} = w^{-1}[(w-\xi)^{-1}+\xi^{-1}], \tag{28}
$$

we obtain the estimates

$$
I_1(0) \le Mr^{1-\lambda}J_1(0), \tag{29}
$$

$$
I_2(0) \le 2Mr^{1-\lambda}cJ_2(0). \tag{30}
$$

Again we infer from (28) and (15) that

$$
\begin{aligned}
I_3(0) &= 2c\iint_{|\xi|<r} d\xi_1\,d\xi_2|\xi|^{-\lambda}|\xi|^{-1}\iint_{|w|<r}|w|^{-\mu-\lambda}|w-\xi|^{-1}|X_w(w)|\,du\,dv \\
&= 2c\iint_{|w-\xi|<r} du\,dv|w|^{-\mu-\lambda}|w|^{-1}|X_w(w)| \\
&\quad \cdot\iint_{|\xi|<r}|\xi|^{-\lambda}[(w-\xi)^{-1}+\xi^{-1}]\,d\xi_1\,d\xi_2 \\
&\le 4cM_3r^{1-\lambda}\iint_{|w|<r}|w|^{-\mu-\lambda-1}\left[\int_0^1\left|\frac{d}{dt}X(tw)\right|\,dt\right]\,du\,dv \\
&\le 4cM_3r^{1-\lambda}\int_0^1 dt\left[\iint_{|w|<r}|w|^{-\mu-\lambda}|X_w(tw)|\,du\,dv\right].
\end{aligned}
$$

Now we put $z := tw$, $z = z_1 + iz_2$, and employ the change-of-variables formula. This yields

$$
\begin{aligned}
I_3(0) &\le 4cM_3r^{1-\lambda}\int_0^1 dt\; t^{\mu+\lambda-2}\iint_{|z|<tr}|z|^{-\mu-\lambda}|X_w(z)|\,dz_1\,dz_2 \\
&\le \frac{4cM_3r^{1-\lambda}}{\mu+\lambda-1}\iint_{|z|<r}|z|^{-\mu-\lambda}|X_w(z)|\,dz_1\,dz_2,
\end{aligned}
$$

where we have assumed that $\mu+\lambda\ge 2$.

In the following estimates we let $r<1$. Then

$$
I_3(0) \le 4cM_3r^{1-\lambda}\iint_{|w|<r}|w|^{-\mu-\lambda-1}|X_w(w)|\,du\,dv,
$$

or equivalently

$$I_3(0) \le 4cM_3 r^{1-\lambda} J_2(0). \tag{31}$$

The estimates (27), (29), (30), and (31) now imply that

$$2\pi J_2(0) \le Mr^{1-\lambda} J_1(0) + 2Mr^{1-\lambda} cJ_2(0) + 4cM_3 r^{1-\lambda} J_2(0),$$

whence we obtain for small $r > 0$ and some $\delta > 0$ independent of μ that

$$\delta J_2(0) \le J_1(0),$$

or, more explicitly

$$\delta \iint_{|w|<r} |w|^{-\mu-\lambda-1} |X_w(w)| \, du \, dv \le \iint_{|w|=r} |w|^{-\mu-1} |X_w(w)| |dw|$$

for all nonnegative μ. If we now assume the existence of some w_0 such that $X_w(w_0) \ne 0$, we are led to a contradiction exactly as in the proof of Theorem 1.

We have shown that there exists some finite integer ν with

$$\begin{aligned} X(w) &= o(|w|^{\nu-1}), \\ X(w) &\ne o(|w|^{\nu}) \quad \text{as } w \to 0. \end{aligned} \tag{32}$$

By the first part of Theorem 2 we conclude the existence of the limit

$$\lim_{w\to 0} X_w(w) w^{-\nu+1} = A.$$

If $A = 0$, we could infer from Lemma 3 that $X(w) = o(|w|^\nu)$ contradicting (32), and Theorem 2 is proved. □

Now we shall consider a further generalization of Theorem 1 which will enable us to treat certain systems of differential inequalities as well. This will be of importance in Sections 3.3 and 3.4.

Definition 2. *Two complex-valued functions $F(w), G(w)$ are said to satisfy* **Assumption (A3)** *if they are of class $C^{0,1}(B'_\delta, \mathbb{C}), B'_\delta = \{0 < |w| < \delta\}$, and if there are numbers $\alpha, \beta, \nu \in (0,1), \alpha + \beta = 1$ such that the relations*

$$\begin{cases} |F(w)| = O(|w|^{\nu-\alpha}) \\ |G(w)| = O(|w|^{\nu-\beta}) \end{cases} \quad \text{as } w \to 0, \tag{33}$$

and the inequalities

$$\begin{cases} |F_{\overline{w}}(w)| \le c[|w|^{-\beta}|F(w)|^2 + |w|^{\beta-2\alpha}|G(w)|^2], \\ |G_{\overline{w}}(w)| \le c[|w|^{\alpha-2\beta}|F(w)|^2 + |w|^{-\alpha}|G(w)|^2] \end{cases} \tag{34}$$

hold true almost everywhere on $B'_\delta = B_\delta \setminus \{0\}$ for some constant $c > 0$.

Here and in the following we shall work with the concept of generalized complex derivatives which are defined analogously to generalized real (or weak) derivatives, and we refer the interested reader to the monograph of Vekua [1,2] for more detailed background information. Note that by a theorem of Rademacher (see e.g. Federer [1]) every Lipschitz-continuous function has a weak derivative which is bounded.

Theorem 3. *Suppose that F and G satisfy assumption* (A3) *on B'_δ. Then there exists a nonnegative integer m such that the functions*

$$f^m(w) := w^{-m}F(w), \quad g^m(w) := w^{-m}G(w)$$

satisfy one of the following two conditions (i) *or* (ii)*:*

(i) $$f^m(w) \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \textit{for all } \mu < \min(1, m+\alpha),$$

$$\begin{aligned} &f^m(0) \neq 0, \\ &|f^m_{\overline{w}}(w)| = O(|w|^{m-\beta}) \quad \textit{as } w \to 0, \\ &|g^m_{\overline{w}}(w)| = O(|w|^{m+\alpha-2\beta}) \quad \textit{as } w \to 0; \end{aligned}$$

(ii) $$g^m(w) \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \textit{for all } \mu < \min(1, m+\beta),$$

$$\begin{aligned} &g^m(0) \neq 0, \\ &|f^m_{\overline{w}}(w)| = O(|w|^{m+\beta-2\alpha}) \quad \textit{as } w \to 0, \\ &|g^m_{\overline{w}}(w)| = O(|w|^{m-\alpha}) \quad \textit{as } w \to 0. \end{aligned}$$

For the proof of Theorem 3 we shall need the following auxiliary results.

Lemma 6. *Suppose that $f \in C^{0,1}(B'_\delta, \mathbb{C})$ satisfies*

(35) $$|f(w)| = o(|w|^{-1}) \quad \textit{as } w \to 0$$

and

(36) $$|f_{\overline{w}}(w)| = O(|w|^\lambda) \quad \textit{as } w \to 0$$

with some exponent $\lambda > -2$. Then we have:

$$f \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \textit{for all } \mu < \min(1, 1+\lambda), \ \textit{if } \ \lambda > -1,$$

or

$$|f(w)| = O(|w|^{-\varepsilon}), \quad (w \to 0), \quad \textit{for all } \varepsilon > 0, \ \textit{if } \ \lambda = -1,$$

or

$$|f(w)| = O(|w|^{1+\lambda}), \quad (w \to 0), \ \textit{if } \ \lambda < -1,$$

Proof. Since $f_{\overline{w}}(w) \in L_1(B_\delta(0), \mathbb{C})$, we can apply Theorem 1.16 in Vekua [1] which implies that the sum

$$f(w) + \frac{1}{\pi} \iint_{B_\delta(0)} f_{\overline{w}}(\xi)(\xi - w)^{-1}\, d\xi_1\, d\xi_2$$

is holomorphic in $B_\delta(0)$. Hence it is sufficient to prove that the above alternative holds for the function

$$g(w) = \frac{1}{\pi} \iint_{B_\delta(0)} f_{\overline{w}}(\xi)(\xi - w)^{-1}\, d\xi_1\, d\xi_2.$$

If $\lambda > -1$, we conclude for $w_1, w_2 \in B'_\delta = B_\delta \setminus \{0\}$ the inequality

$$\begin{aligned} |g(w_1) - g(w_2)| &= \frac{1}{\pi} \left| \iint_{B_\delta(0)} f_{\overline{w}}(\xi) \frac{(w_1 - w_2)}{(w_1 - \xi)(w_2 - \xi)}\, d\xi_1\, d\xi_2 \right| \\ &\le \text{const}\ |w_1 - w_2| \iint_{B_\delta(0)} \frac{|\xi|^\lambda}{|w_1 - \xi||w_2 - \xi|}\, d\xi_1\, d\xi_2. \end{aligned}$$

Using Hölder's inequality, we obtain for each $\mu \in (0, 1 + \lambda)$ the estimate

$$\begin{aligned} |g(w_1) - g(w_2)| \le c|w_1 - w_2| &\left[\iint_{B_\delta(0)} |\xi|^{2\lambda/(1-\mu)}\, d\xi_1\, d\xi_2 \right]^{(1-\mu)/2} \\ &\cdot \left[\iint_{B_\delta(0)} (|w_1 - \xi||w_2 - \xi|)^{-2/(1+\mu)}\, d\xi_1\, d\xi_2 \right]^{(1+\mu)/2}. \end{aligned}$$

Now inequality (15) implies

$$|g(w_1) - g(w_2)| \le c|w_1 - w_2|^{1 + (2 - (4/(1+\mu)))(1+\mu)/2} = c|w_1 - w_2|^\mu.$$

If $\lambda < -1$, we infer again from (15) that

$$|g(w)| \le c \iint_{B_\delta(0)} |\xi|^\lambda |\xi - w|^{-1}\, d\xi_1\, d\xi_2 \le c|w|^{1+\lambda}$$

for some suitable constant c. Finally, if $\lambda = -1$, it follows that

$$|g(w)| \le c_1 + c_2 |\log|w||. \qquad \square$$

In the discussion to follow we shall always assume that $0 < \alpha \le \frac{1}{2} \le \beta < 1$. Note that this is without loss of generality since $\alpha + \beta = 1$ and because of the symmetry of the following assertions both in α and β and with respect to F and G. Observe also that we can (and will, if necessary) decrease the decay exponent ν in the relation (33).

Lemma 7. *Suppose that F and G satisfy assumption* (A3) *on $B'_\delta = B_\delta \setminus \{0\}$ with $\alpha \le \frac{1}{2}$. Then $F \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu \in (0, \alpha)$ and, furthermore, the relations*

$$\left.\begin{aligned} |F_{\overline{w}}(w)| &= O(|w|^{-\beta}) \\ |G_{\overline{w}}(w)| &= O(|w|^{\alpha-2\beta}) \end{aligned}\right. \qquad \text{as } w \to 0$$

hold true almost everywhere on B_δ.

Proof. The proof is based on an iteration argument where one has to use Lemma 6 in each step. To start, let us assume that $\nu < \alpha$, whence for some $k_0 \in \mathbb{N} \cup \{0\}$ we have that $2^{k_0}\nu \le \alpha < 2^{k_0+1}\nu$. Now assume that for some $k \in \mathbb{N} \cup \{0\}, k \le k_0$, we have

$$|F(w)| = O(|w|^{2^k\nu-\alpha}), \quad |G(w)| = O(|w|^{2^k\nu-\beta}).$$

Then (34) implies

$$|F_{\overline{w}}(w)| = O(|w|^{2^{k+1}\nu-\alpha-1}), \quad |G_{\overline{w}}(w)| = O(|w|^{2^{k+1}\nu-\beta-1}).$$

From Lemma 6 we infer

$$|F(w)| = O(|w|^{2^{k+1}\nu-\alpha}) \quad \text{as } w \to 0 \text{ if } k < k_0$$

or

$$F(w) \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \text{for all } \mu < 2^{k_0+1}\nu - \alpha \text{ if } k = k_0.$$

Also,

$$|G(w)| = O(1 + |w|^{2^{k+1}\nu-\beta}) \quad \text{if } k \le k_0.$$

By virtue of (33) we can start the iteration by putting $k = 0$. In conclusion we obtain that

$$F \in C^{0,\mu} \quad \text{for all } \mu < 2^{k_0+1}\nu - \alpha$$

and in particular

$$|F(w)| = O(1), \quad |G(w)| = O(1 + |w|^{2^{k_0+1}\nu-\beta}).$$

Again we infer from (34) that

$$|F_{\overline{w}}(w)| = O(|w|^{-\beta}) = O(|w|^{\alpha-1}),$$

since $2^{k_0+2}\nu > 2\alpha$, and

$$|G_{\overline{w}}(w)| = O(|w|^{\alpha-2\beta}) = O(|w|^{1-3\beta}).$$

Finally we infer from Lemma 6 that $F \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $0 < \mu < \alpha$. $\square$

In the next lemma we improve the regularity of G provided we know that $F(0) = 0$.

Lemma 8. *Suppose F and G satisfy* (A3) *on $B'_\delta = B_\delta \setminus \{0\}$ with $\alpha \le \frac{1}{2}$, and that $F(0) = 0$. Then we have $G \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu \in (0, \beta)$ and*

$$|F_{\overline{w}}(w)| = O(|w|^{1-3\alpha}), \quad |G_{\overline{w}}(w)| = O(|w|^{-\alpha}) \quad \textit{as } w \to 0.$$

Proof. Since $F(0) = 0$, we infer from Lemma 7 that $|F(w)| = O(|w|^\mu)$ for all $\mu < \alpha$. Hence the function $f(w) := \frac{F(w)}{w}$ satisfies

$$|f(w)| = O(|w|^{\mu-1}) \quad \text{as } w \to 0, \text{ for all } 0 < \mu < \alpha.$$

By Lemma 7 we have $|G_{\overline{w}}(w)| = O(|w|^{\alpha-2\beta})$ as $w \to 0$, and Lemma 6 yields

$$|G(w)| = O(1 + |w|^{\alpha-2\beta+1}) \quad \text{if } \alpha - 2\beta \ne -1,$$

that is,

$$|G(w)| = O(|w|^{-\varepsilon}) \quad \text{for all } \varepsilon > 0, \text{ if } \alpha - 2\beta = -1 \left(\text{i.e. } \alpha = \frac{1}{3}\right).$$

Using inequalities (34) we obtain the system

$$\begin{cases} |f_{\overline{w}}(w)| \le c_1[|w|^\alpha |f|^2 + |w|^{-3\alpha}|G|^2], \\ |G_{\overline{w}}(w)| \le c_2[|w|^{3\alpha}|f|^2 + |w|^{-\alpha}|G|^2], \end{cases} \tag{37}$$

which holds true for almost all $w \in B_\delta$.

If $\alpha - 2\beta = -1$ (or equivalently $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$), we infer from (37) the relations

$$\left.\begin{cases} |f_{\overline{w}}(w)| = O(|w|^{-1-\varepsilon}) & \text{as } w \to 0, \\ |G_{\overline{w}}(w)| = O(|w|^{-1/3-\varepsilon}) & \text{as } w \to 0, \end{cases}\right. \quad \text{for all } \varepsilon > 0 \tag{38}$$

whence in particular

$$G(w) \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \text{for all } \mu < \frac{2}{3} = \beta$$

and

$$|G(w)| = O(1), \quad |f(w)| = O(|w|^{-\varepsilon}), \tag{39}$$

for all $\varepsilon > 0$. Inserting (39) into (37) we obtain

$$\begin{aligned} |f_{\overline{w}}(w)| &= O(|w|^{-1}) = O(|w|^{-3\alpha}), \\ |G_{\overline{w}}(w)| &= O(1) = O(|w|^{1-3\alpha}), \end{aligned}$$

and therefore

$$|F_{\overline{w}}(w)| = O(1) = O(|w|^{1-3\alpha}),$$

because of $F_{\overline{w}} = w f_{\overline{w}}$.

Now we deal with the case $\alpha - 2\beta > -1$ (or equivalently $\beta < \frac{2}{3}, \alpha > \frac{1}{3}$): Inserting the relations

$$|G(w)| = O(1) \quad \text{and} \quad |f(w)| = O(|w|^{\mu-1}), \quad \mu < \alpha,$$

in (37), we obtain

$$\left.\begin{aligned} |f_{\overline{w}}(w)| &= O(|w|^{-3\alpha}) \\ |G_{\overline{w}}(w)| &= O(|w|^{-\alpha}) \end{aligned}\right. \quad \text{as } w \to 0.$$

Now Lemma 6 implies that

$$|F_{\overline{w}}(w)| = O(|w|^{1-3\alpha}),$$

$$G(w) \in C^{0,\mu}(B_\delta, \mathbb{C}) \quad \text{for all } \mu < 1 - \alpha = \beta.$$

Finally, we have to treat the case $\alpha - 2\beta < -1$ (or $\beta > \frac{2}{3}$ and $\alpha < \frac{1}{3}$): To this end we fix some $\mu < \alpha$ and select some $k_0 \in \mathbb{N} \cup \{0\}$ with the property $2^{k_0}(\mu + \alpha) < 1 - \alpha < 2^{k_0+1}(\mu + \alpha)$. Assume that for some $k \le k_0$ the relations

$$(40_k) \qquad \begin{aligned} |f(w)| &= O(|w|^{2^k(\mu+\alpha)-\alpha-1}) \quad \text{as } w \to 0, \\ |G(w)| &= O(|w|^{2^k(\mu+\alpha)+\alpha-1}) \quad \text{as } w \to 0, \end{aligned}$$

hold true. Then it follows from (37) that

$$|f_{\overline{w}}(w)| = O(|w|^{2^{k+1}(\mu+\alpha)-\alpha-2})$$

and

$$|G_{\overline{w}}(w)| = O(|w|^{2^{k+1}(\mu+\alpha)+\alpha-2}).$$

If $k < k_0$, then Lemma 6 applies and we arrive at the relations

$$|f(w)| = O(|w|^{2^{k+1}(\mu+\alpha)-\alpha-1}),$$

$$|G(w)| = O(|w|^{2^{k+1}(\mu+\alpha)+\alpha-1});$$

in other words, the validity of (40_k) implies the validity of (40_{k+1}). On the other hand, for $k = k_0$ we obtain

$$(41) \qquad \begin{aligned} |f(w)| &= O(1 + |w|^{2^{k_0+1}(\mu+\alpha)-\alpha-1}), \\ |G(w)| &= O(1). \end{aligned}$$

We can start the induction because (40_k) holds with $k = 0$ taking $\mu < \alpha$ into account.

We insert (41) into (37) and get $|f_{\overline{w}}(w)| = O(|w|^{-3\alpha})$ as $w \to 0$ and $|G_{\overline{w}}(w)| = O(|w|^{-\alpha})$ whence we infer by means of Lemma 6 that $G \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu < 1 - \alpha = \beta$ and also $|F_{\overline{w}}(w)| = O(|w|^{1-3\alpha})$. $\square$

Next we suppose that both F and G vanish at zero. Then on account of the Lemmata 8 and 6 we conclude that the functions

$$f(w) := w^{-1}F(w) \quad \text{and} \quad g(w) := w^{-1}G(w)$$

fulfil the relations

$$|f(w)| = \begin{cases} O(1+|w|^{1-3\alpha}) & \text{if } \alpha \neq \frac{1}{3}, \\ O(|w|^{-\varepsilon}) & \text{for all } \varepsilon > 0, \text{ if } \alpha = \frac{1}{3} \end{cases}$$

and

$$|g(w)| = O(|w|^{-\alpha}).$$

Therefore there exists some number $\lambda' \in (0,1)$ such that the mapping

$$h(w) := (f(w), g(w))$$

satisfies the relation

$$|h(w)| = O(|w|^{-\lambda'}) \quad \text{as } w \to 0.$$

From (34) we easily infer an estimate of the type

$$|h_{\overline{w}}(w)| \leq c|w|^{-\lambda}|h(w)|$$

holding almost everywhere on B_δ with some constants c and $\lambda \in (0,1)$. Thus we are in a position to apply Corollary 1 of this section to the function h and obtain the existence of some positive integer m and of a complex vector $A \in \mathbb{C}^2 \setminus \{0\}$ such that

$$h(w) = Aw^{m-1} + o(|w|^{m-1}) \quad \text{as } w \to 0 \tag{42}$$

holds true on B_δ.

Now we come to the *proof of Theorem* 3.

Without loss of generality we only consider the case $\alpha \leq \frac{1}{2}$. We distinguish between the following alternatives (which clearly exhaust all possibilities!):

$$\begin{aligned} &(\alpha)\ F(0) \neq 0, \quad G(0) \neq 0, \quad (\beta)\ F(0) \neq 0, \quad G(0) = 0, \\ &(\gamma)\ F(0) = 0, \quad G(0) \neq 0, \quad (\delta)\ F(0) = 0, \quad G(0) = 0. \end{aligned}$$

If (α) or (β) hold true, then Lemma 5 yields that (i) must be satisfied with $m = 0$. In view of Lemma 8 we obtain (ii) with $m = 0$ provided that (γ) holds true. Finally, let us assume that $F(0) = G(0) = 0$. Then (42) is equivalent to

$$\left.\begin{aligned} F(w) &= aw^m + o(|w|^m) \\ G(w) &= bw^m + o(|w|^m) \end{aligned}\right. \quad \text{as } w \to 0$$

with complex numbers a, b which are not both equal to zero, and we obtain

$$\begin{cases} f^m(w) = a + o(1) \\ g^m(w) = b + o(1) \end{cases} \quad \text{as } w \to 0. \tag{43}$$

On the other hand, we easily derive from (34) the inequalities

$$\begin{cases} |f^m_{\overline{w}}(w)| \le c[|w|^{m-\beta}|f^m(w)|^2 + |w|^{m+\beta-2\alpha}|g^m(w)|^2], \\ |g^m_{\overline{w}}(w)| \le c[|w|^{m+\alpha-2\beta}|f^m(w)|^2 + |w|^{m-\alpha}|g^m(w)|^2], \end{cases} \tag{44}$$

and, together with (41), this yields

$$|f^m_{\overline{w}}(w)| = O(|w|^{m-\beta}) \quad \text{and} \quad |g^m_{\overline{w}}(w)| = O(|w|^{m+\alpha-2\beta}) \quad \text{as } w \to 0.$$

But then Lemma 6 can be applied which proves that $f^m \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu < 1$. Assuming that $f^m(0) \ne 0$ we have thus shown that (i) holds true.

So let us assume that $f^m(0) = a = 0$ (whence $b = g^m(0) \ne 0$). Then clearly $|f^m(w)| = O(|w|^\mu)$ as $w \to 0$ for all $\mu < 1$, and (44) implies

$$|f^m_{\overline{w}}(w)| = O(|w|^{m+\beta-2\alpha})$$

and

$$|g^m_{\overline{w}}(w)| = O(|w|^{m-\alpha}).$$

Again, by Lemma 6 it follows that $g^m \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu < 1$, and hence (ii) holds true; thus Theorem 3 is proved. □

3.2 A Gradient Estimate at Singularities Corresponding to Corners of the Boundary

In this section we consider solutions $X = X(u, v)$ of the Plateau problem $\mathcal{P}(\Gamma)$ for a Jordan curve Γ consisting of two regular pieces Γ^+ and Γ^- of class $C^{2,\mu}$ which enclose a positive angle $\beta < \pi$ at a common point $P \in \Gamma^+ \cap \Gamma^-$. We are then interested in the behaviour of X near the corner point P and, in particular, in asymptotic expansions for the gradient $\nabla X(u, v)$ near the point $w_0 \in \partial B$ which corresponds to P. More generally, let $X \in \mathcal{C}(\Gamma, S)$ be a solution to the free boundary problem $\mathcal{P}(\Gamma, S)$ and suppose that the configuration $\langle \Gamma, S\rangle$ satisfies some chord-arc condition (see Section 2.5). Then we conclude from Theorem 2 of Section 2.5 that X is globally Hölder continuous on the closure of the semi-disk $B = \{(u, v)\colon u^2 + v^2 < 1, v > 0\}$, i.e.,

$$X \in C^{0,\alpha}(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$$

for some $\alpha > 0$. Assuming the usual three-point condition, the points (1, 0) and $(-1, 0)$ are mapped onto the corner points $P_1, P_2 \in \Gamma \cap S$ respectively. Hence our interest is concentrated on the behaviour of $\nabla X(w)$ when $w \to \pm 1$ respectively.

We first mention a (local) result concerning the Plateau problem.

Theorem 1. *Let $\Gamma^+, \Gamma^- \subset \mathbb{R}^3$ be pieces of regular Jordan arcs of class $C^{2,\mu}$ which meet at a point $P \in \mathbb{R}^3$ forming a positive angle $\beta < \pi$. Suppose that*

$$X \in C^{0,\alpha}(\overline{B}_\delta^+, \mathbb{R}^3) \cap C^2(\overline{B}_\delta^+ \setminus \{0\}, \mathbb{R}^3),$$

where $B_\delta^+ := \{w = (u,v)\colon |w| < \delta, v > 0\}$ is a minimal surface which satisfies the boundary conditions $X\colon I_\delta^\pm \to \Gamma^\pm$ with $I_\delta^\pm := \{(u,0)\colon 0 < \pm u < \delta\}$ and $X(0) = P$. Then we obtain the asymptotic relation

$$|\nabla X(w)| = O(|w|^{\alpha-1}) \quad \text{as } w \to 0.$$

For the free boundary problem we shall prove

Theorem 2. *Let Γ be a regular Jordan curve of class $C^{2,\mu}$ which has only its two endpoints P_1, P_2 in common with a regular closed surface S of class C^3. Suppose that $X \in \mathcal{C}(\Gamma, S)$ solves the partially free minimum problem $\mathcal{P}(\Gamma, S)$ and that Γ, S satisfy some chord-arc condition. Then $X(u,v)$ is of class $C^{0,\alpha}(\overline{B}, \mathbb{R}^3) \cap C^{2,\alpha}(\overline{B} \setminus \{1,-1\})$ for some $\alpha > 0$ where $B = \{(u,v)\colon u^2 + v^2 < 1, v > 0)$, and there holds the expansion*

$$|\nabla X(w)| = O(|w \mp 1|^{\alpha-1}) \quad \text{as } w \to \pm 1. \tag{1}$$

We shall only prove Theorem 2 since the proof of the first theorem is similar. Note that we only have to show the asymptotic relation (1) since the asserted regularity properties of X were already proved in Chapter 2. Also, it will be convenient to replace the semi-disk B by the upper half-plane

$$H = \{(u,v) \in \mathbb{R}^2\colon v > 0\}.$$

We may further assume that the point $(u,v) = (0,0)$ is mapped into the corner point $P_1 \in \Gamma \cap S$. Observe that this simplification is without loss of generality since the conformal map

$$w = w(z) = -\left[\frac{1-z}{1+z}\right]^2$$

maps the semi-disk $B = \{(u,v)\colon u^2 + v^2 < 1, v > 0\}$ conformally onto H, and the point $(1,0)$ into $(0,0)$. (Note that $w(z)$ is not conformal at the boundary point $z = 1$.) Furthermore, if X is of class $C^{0,\alpha}(\overline{B}) \cap C^2(B)$, then $Y(w) := X(z(w))$ is of class $C^{0,\alpha/2}(\overline{H})$, and if Y satisfies an asymptotic relation of the type

$$|\nabla Y(w)| = O(|w|^{\alpha/2-1}) \quad \text{as } w \to 0,$$

then also

$$\begin{aligned} |\nabla X(z)| &= O\left(|\nabla Y(w)| \left|\frac{dw}{dz}\right|\right) = O(|1-z|^{\alpha-2} \cdot |1-z|) \\ &= O(|1-z|^{\alpha-1}) \quad \text{as } z \to 1, z \in B. \end{aligned}$$

Since we only deal with local properties of X we may throughout this section require the following Assumption A to be satisfied by the minimal surface X.

Assumption A. *Let $\delta > 0$ be some positive number and put*

$$B_\delta^+ := \{w = (u,v) \in \mathbb{R}^2 \colon |w| < \delta, v > 0\},$$

$$I_\delta^+ := \{w = (u,0) \colon 0 < u < \delta\},$$

$$I_\delta^- := \{w = (u,0) \colon -\delta < u < 0\}.$$

Suppose that the minimal surface $X = X(u,v)$ is of class $C^{0,\alpha}(\overline{B}_\delta^+, \mathbb{R}^3) \cap C^{2,\alpha}(\overline{B}_\delta^+ \setminus \{0\})$ and satisfies the following boundary conditions:
(i) $X \colon I_\delta^- \to \Gamma$ *is weakly monotonic;*
(ii) $X(I_\delta^+) \subset S, X(0) = 0 = P_1 \in \Gamma \cap S$;
(iii) $X_v|_{I_\delta^+}$ *is orthogonal to S along the free trace $X|_{I_\delta^+}$.*

Then Theorem 2 follows from

Proposition 1. *Let $X \in C^{2,\alpha}(\overline{B}_\delta^+ \setminus \{0\}) \cap C^{0,\alpha}(\overline{B}_\delta^+)$ be a minimal surface which fulfills assumption (A). Then the gradient ∇X satisfies*

$$|\nabla X(w)| = O(|w|^{\alpha-1}) \quad \textit{as } w \to 0. \tag{2}$$

The proof of Proposition 1 rests on a further investigation of solutions $\tilde{X}(w)$ of the differential inequality

$$|\Delta \tilde{X}(u,v)| \le a|\nabla \tilde{X}(u,v)|^2 \tag{3}$$

which was already considered in Section 2.2. We recall Proposition 1 of Section 2.2.

Proposition A. *There is a continuous function $\kappa(t), 0 \le t < 1$, with the following properties: For any solution $\tilde{X} \in C^2(B_R(w_0), \mathbb{R}^N)$ of the differential inequality (3) satisfying*

$$|\tilde{X}(w)| \le M, \quad w \in B_R(w_0) \tag{4}$$

for some M with $aM < 1$, the estimates

$$|\nabla \tilde{X}(w_0)| \le \kappa(aM)\frac{M}{R} \quad \textit{and} \tag{5}$$

$$|\nabla \tilde{X}(w_0)| \le \frac{\kappa(aM)}{R} \sup_{w \in B_R(w_0)} |\tilde{X}(w) - \tilde{X}(w_0)| \tag{6}$$

hold true.

Lemma 1. *Let $D \subset B_1(0)$ be a domain such that $\overline{D}$ contains the origin. Suppose that $\tilde{X} \in C^2(D, \mathbb{R}^N) \cap C^0(\overline{D}, \mathbb{R}^N)$ satisfies inequality (3). Then there exists some $\delta > 0$ such that the estimate*

$$|\nabla \tilde{X}(w_0)| \le \varepsilon^{-1} \cdot \mathrm{const} \sup_{B_\varepsilon(w_0)} |\tilde{X}(w) - \tilde{X}(w_0)| \tag{7}$$

holds true for all $w_0 \in D \cap B_\delta(0)$ and for all $\varepsilon > 0$ with $B_\varepsilon(w_0) \subset D \cap B_\delta(0)$.

Proof. We put $Y(w) = \frac{1}{2a}[\tilde{X}(w) - \tilde{X}(0)], w \in D$, and choose $\delta > 0$ so small that $\sup_{D\cap B_\delta(0)} |Y(w)| < 1$. Then Y satisfies (3) on $D \cap B_\delta(0)$ with $a = \frac{1}{2}$. Applying Proposition A to the function $Y \in C^2(B_\varepsilon(w_0))$ and to $M = 1, a = \frac{1}{2}$, we get the estimate

$$|\nabla Y(w_0)| \le \frac{\kappa(1/2)}{\varepsilon} \sup_{B_\varepsilon(w_0)} |Y(w) - Y(w_0)|,$$

i.e.,

$$|\nabla \tilde{X}(w_0)| \le \frac{\kappa(1/2)}{\varepsilon} \sup_{B_\varepsilon(w_0)} |\tilde{X}(w) - \tilde{X}(w_0)|$$

as required. □

In order to state our results in a convenient way, we make the following

Assumption B. *For some fixed angle $\pi \ge \gamma > 0$ we denote by D_ρ the domain*

$$D_\rho := \{w = re^{i\varphi} \colon 0 < \varphi < \gamma, r < \rho\}$$

where r, φ denote polar coordinates about the origin. Let

$$\tilde{X}(w) = (\tilde{X}^1(w), \dots, \tilde{X}^N(w)), \quad w = (u, v) \in D_\rho,$$

be a mapping of class $C^0(\overline{D}_\rho, \mathbb{R}^N) \cap C^2(D_\rho, \mathbb{R}^N)$ which satisfies

$$|\Delta \tilde{X}(w)| \le a |\nabla \tilde{X}(w)|^2 \quad \text{on } D_\rho \tag{3}$$

and

$$|\tilde{X}(w)| \le c_1 |w|^\alpha \quad \text{on } D_\rho \tag{8}$$

with numbers $a, c_1 > 0$ and $0 < \alpha < 1$.

For arbitrary fixed $\theta \in (0, \gamma/2)$ we put

$$\begin{aligned} D_{\rho,\theta} &:= \{w = re^{i\varphi} \colon \theta < \varphi < \gamma - \theta, 0 < r < \rho\}, \\ D^1_{\rho,\theta} &:= \{w = re^{i\varphi} \colon 0 < \varphi < \theta, 0 < r < \rho\}, \\ D^2_{\rho,\theta} &:= \{w = re^{i\varphi} \colon \gamma - \theta < \varphi < \gamma, 0 < r < \rho\}. \end{aligned}$$

Then we have

Lemma 2. *Suppose $\tilde{X}$ satisfies Assumption B on D_ρ. Then, for every $\theta \in (0, \frac{\gamma}{2})$, there exists a constant $c_2 = c_2(\theta, a, c_1)$ such that the inequality*

$$|\nabla \tilde{X}(w_0)| \le c_2 |w_0|^{\alpha - 1} \tag{9}$$

holds true for all $w_0 \in D_{\delta_1, \theta}$ and for some $\delta_1 \in (0, \rho)$.

Proof. Let $\delta > 0$ denote the number determined in Lemma 1. We take $\delta_1 := \frac{1}{2}\min(\delta, \rho)$ and put $\varepsilon := \frac{1}{2}|w_0|\sin\theta$. Then $B_\varepsilon(w_0) \subset D_\rho \cap B_\rho(0)$ for all $w_0 \in D_{\delta_1,\theta}$, and Lemma 1 implies the estimate

$$\begin{aligned}
|\nabla \tilde{X}(w_0)| &\le \text{const } \varepsilon^{-1} \sup_{B_\varepsilon(w_0)} |\tilde{X}(w) - \tilde{X}(w_0)| \\
&\le \text{const } c_1 \varepsilon^{-1} [|w_0|^\alpha + (|w_0| + \varepsilon)^\alpha] \\
&\le c_2(\theta, a, c_1)|w_0|^{\alpha-1}.
\end{aligned}$$

The estimate (9) controls the behaviour of the gradient on the segments $D_{\delta,\theta}$. To obtain also some information on the remaining parts $D^1_{\delta,\theta}$ or $D^2_{\delta,\theta}$, we have to make additional assumptions.

Lemma 3. *Suppose that $\tilde{X}$ satisfies Assumption B, and let $\theta \in (0, \min\{\frac{\pi}{16}, \frac{\gamma}{4}\})$. In addition, assume that $\tilde{X}(re^{i\varphi}) = 0$ on $0 < r < \rho$ and $\varphi = 0$ or $\varphi = \gamma$, respectively. Then for small $\delta > 0$ we obtain the estimate*

$$|\nabla \tilde{X}(w_0)| \le \text{const } |w_0|^{\alpha - 1} \tag{10}$$

on $D^1_{\delta,\theta}$ or $D^2_{\delta,\theta}$ respectively.

Proof. It is sufficient to prove (10) for $w_0 \in D^1_{\delta,\theta}$. To this end we select some $\delta < \min(\rho, 1)$ such that

$$aM < 1, \tag{11}$$

where

$$M := \sup_{D_\delta} |\tilde{X}(w)|$$

and where a denotes the constant in (3). Applying Proposition A, we derive the gradient bound

$$|\nabla \tilde{X}(w_0)| \le c\varepsilon^{-1} \sup_{B_\varepsilon(w_0)} |\tilde{X}(w) - \tilde{X}(w_0)| \tag{12}$$

holding for some constant c independent of ε and for all $\varepsilon > 0$ satisfying $0 < \varepsilon < \text{dist}(w_0, \partial D_\delta)$.

Now we restrict w_0 further so that $|w_0| < \frac{\delta}{2}$. Put $u_0 = \operatorname{Re} w_0$, $R_\theta := 2u_0 \sin\theta$, $w_1 = (u_0, 0)$ and $B^+_{R_\theta}(w_1) := B_{R_\theta}(w_1) \cap \{(u,v)\colon v > 0\}$.

Then we find

$$B_\varepsilon(w_0) \subset B^+_{R_\theta}(w_1) \quad \text{for all } \varepsilon < \operatorname{dist}(w_0, \partial D_\delta)$$

and

$$B^+_{2R_\theta}(w_1) \subset D_\delta$$

taking the smallness of θ into account. We define harmonic functions $\varphi(w) = (\varphi^1(w), \dots, \varphi^N(w))$ and $\psi(w)$ by

$$\Delta\varphi = 0 \quad \text{on } B^+_{2R_\theta}(w_1), \quad \varphi(w) = \tilde{X}(w) \quad \text{on } \partial B^+_{2R_\theta}(w_1),$$

and

$$\Delta\psi = 0 \quad \text{on } B^+_{2R_\theta}(w_1), \quad \psi(w) = |\tilde{X}(w)|^2 \quad \text{on } \partial B^+_{2R_\theta}(w_1).$$

Consider the function

$$K(w) := \langle \tilde{X}(w) - \varphi(w), e\rangle + \frac{a}{2(1-aM)}\{\psi(w) - |\tilde{X}(w)|^2\},$$

$w \in B^+_{2R_\theta}(w_1)$, where $e \in \mathbb{R}^N$ is an arbitrary unit vector. Then

$$\begin{aligned}\Delta K(w) &= \langle \Delta\tilde{X}, e\rangle - \frac{a}{1-aM}\{|\nabla\tilde{X}|^2 + \langle\Delta\tilde{X}, \tilde{X}\rangle\} \\ &\le |\Delta\tilde{X}| - \frac{a}{1-aM}|\nabla\tilde{X}|^2 + \frac{a}{1-aM}|\Delta\tilde{X}||\tilde{X}| \\ &\le a|\nabla\tilde{X}|^2 - \frac{a}{1-aM}|\nabla\tilde{X}|^2 + \frac{a^2M}{1-aM}|\nabla\tilde{X}|^2 = 0\end{aligned}$$

for $w \in B^+_{2R_\theta}(w_1)$. Furthermore we have $K(w) = 0$ along $\partial B^+_{2R_\theta}(w_1)$; hence we conclude from the maximum principle that $K(w) \ge 0$ on $B^+_{2R_\theta}(w_1)$. In other words,

$$\langle \varphi(w) - \tilde{X}(w), e\rangle \le \frac{a}{2(1-aM)}\psi(w) - \frac{a}{2(1-aM)}|\tilde{X}(w)|^2.$$

Since e is an arbitrary unit vector, this implies the estimate

$$|\varphi(w) - \tilde{X}(w)| \le \frac{a}{2(1-aM)}\{\psi(w) - |\tilde{X}(w)|^2\},$$

in particular

$$|\tilde{X}(w)| \le |\varphi(w)| + \frac{a}{2(1-aM)}|\psi(w)| \quad \text{for } w \in B^+_{2R_\theta}(w_1). \tag{13}$$

On the other hand, we infer from (8) the inequality

$$\begin{aligned}|\tilde{X}(w)| &\le c_1\{|w_1| + 2R_\theta\}^\alpha \\ &\le c_1\{1 + 4\sin\theta\}^\alpha |w_0|^\alpha\end{aligned}$$

for all $w \in \partial B^+_{2R_\theta}(w_1)$, whence

$$
\begin{aligned}
&|\varphi(w)| \le c_2(\theta)|w_0|^\alpha, \quad w \in B^+_{2R_\theta}(w_1), \\
&|\psi(w)| \le c_2^2(\theta)|w_0|^{2\alpha} \le c_2^2(\theta)|w_0|^\alpha, \quad w \in B^+_{2R_\theta}(w_1),
\end{aligned} \tag{14}
$$

since $|w_0| < \delta < 1$. Employing the reflection principle for harmonic functions, it is possible to extend φ and ψ harmonically onto the disk $B_{2R_\theta}(w_1)$, taking account of the fact that φ, ψ vanish along the line $\{(u,0)\colon u_0 - 2R_\theta < u < u_0 + 2R_\theta\}$. Denoting the reflected functions again by φ and ψ, we see that (14) continues to hold. The mean value theorem yields the relations

$$
\begin{aligned}
|\nabla\varphi(w)| &\le \frac{1}{R_\theta} \sup_{B_{R_\theta}(w_1)} |\varphi|, \quad w \in B_{R_\theta}(w_1), \\
|\nabla\psi(w)| &\le \frac{1}{R_\theta} \sup_{B_{R_\theta}(w_1)} |\psi|, \quad w \in B_{R_\theta}(w_1).
\end{aligned}
$$

Together with (14) this implies

$$
\begin{aligned}
|\nabla\varphi(w)| &\le c_3(\theta)|w_0|^{\alpha-1}, \\
|\nabla\psi(w)| &\le c_4(\theta)|w_0|^{\alpha-1}
\end{aligned}
$$

for all $w \in B_{R_\theta}(w_1)$.

Finally we conclude from (13) and from the mean value theorem that

$$
\begin{aligned}
|\tilde{X}(w)| &\le |\varphi(w) - \varphi(w_1)| + \frac{a}{2(1-aM)}|\psi(w) - \psi(w_1)| \\
&\le c_5(a,M,\theta)|w_0|^{\alpha-1}|w - w_1| \\
&\le c_5(a,M,\theta)|w_0|^{\alpha-1} 2\ \mathrm{dist}(w_0, \partial D_\delta),
\end{aligned} \tag{15}
$$

for all $w \in B_{\mathrm{dist}(w_0,\partial D_\delta)}(w_0)$. The desired result than follows from (15) and (12) taking $\varepsilon = \frac{1}{2}\mathrm{dist}(w_0, \partial D_\delta)$. $\square$

Lemmata 2 and 3 imply the following

Proposition 2. *Suppose that $\tilde{X}$ satisfies Assumption B and that $\tilde{X}(re^{i\theta}) = 0$ for $0 < r < \rho, \varphi = 0$ or $\varphi = \gamma$. Then the asymptotic relation*

$$
|\nabla\tilde{X}(w)| = O(|w|^{\alpha-1}) \quad \textit{as } w \to 0
$$

holds true.

Now we turn to the

Proof of Proposition 1. (and hence of Theorem 2). Since we have assumed that $X(0) = P_1 = 0$, we infer from the Hölder continuity the estimate

$$
|X(w)| \le c_1|w|^\alpha \quad \text{as } w \to 0.
$$

Let us fix some $\theta \in (\theta, \frac{\pi}{16})$ and take $\gamma = \pi$ (see Assumption B). It follows that the minimal surface $X(w)$ satisfies Assumption B with $\rho = \delta, a = 0$, and from Lemma 2 we infer the estimate

$$(16) \qquad |\nabla X(w)| \le c_2 |w|^{\alpha - 1} \quad \text{for all } w \in D_{\delta_1, \theta}$$

and some $\delta_1 \in (0, \rho)$.

Next we prove (16) on

$$D^2_{\delta_3, \theta} = \{ w = re^{i\varphi} \colon 0 < r < \delta, \pi - \theta < \varphi < \pi \}$$

for some $\delta_3 \le \delta$. Recall that the segment $I^-_\delta = \{ re^{i\varphi} \colon 0 < r < \delta, \varphi = \pi \}$ is mapped onto Γ. Employing a suitable orthogonal transformation of $\mathbb{R}^3$ we assume that Γ is locally described by two differentiable functions $x = h_1(z), y = h_2(z), z \in [0, \varepsilon)$, with the properties $h_1(0) = h_2(0) = h'_1(0) = h'_2(0) = 0$ and

$$(17) \qquad |h'_i(z)| < \frac{1}{4}, \quad i = 1, 2, \; z \in [0, \varepsilon).$$

We extend the functions h_1, h_2 as even functions to the interval $(-\varepsilon, \varepsilon)$ and define

$$(18) \qquad \begin{aligned} \tilde{x}(w) &:= x(w) - h_1(z(w)) \\ \tilde{y}(w) &:= y(w) - h_2(z(w)) \end{aligned} \quad \text{for } w \in D_{\delta_2},$$

where we have chosen δ_2 so as to satisfy $z(D_{\delta_2}) \subset (-\varepsilon, \varepsilon)$. Consider the mapping $\tilde{X}(w) := (\tilde{x}(w), \tilde{y}(w)), w \in D_{\delta_2}$, which fulfils

$$(19) \qquad \tilde{X}(w) = (0, 0) \quad \text{on } I^-_{\delta_2}.$$

Furthermore, since $X(w) = (x(w), y(w), z(w))$ is harmonic we obtain

$$\Delta \tilde{x}(w) = -h''_1(z(w)) |\nabla z(w)|^2, \quad w \in D_{\delta_2},$$

$$\Delta \tilde{y}(w) = -h''_2(z(w)) |\nabla z(w)|^2, \quad w \in D_{\delta_2},$$

whence

$$(20) \qquad |\Delta \tilde{X}(w)| \le c |\nabla z(w)|^2, \quad w \in D_{\delta_2},$$

with a suitable constant c. Relations (17) and (18) imply the estimate

$$(21) \qquad \begin{aligned} |\nabla x(w)| &\le \frac{1}{4} |\nabla z(w)| + |\nabla \tilde{x}(w)|, \\ |\nabla y(w)| &\le \frac{1}{4} |\nabla z(w)| + |\nabla \tilde{y}(w)|, \end{aligned} \quad w \in D_{\delta_2}.$$

From the conformality condition we conclude

$$\begin{aligned}|\nabla z(w)|^2 &\le |\nabla x(w)|^2 + |\nabla y(w)|^2\\ &\le \frac{1}{16}|\nabla z|^2 + \frac{1}{2}|\nabla z||\nabla \tilde{x}| + |\nabla \tilde{x}|^2\\ &\quad + \frac{1}{16}|\nabla z|^2 + \frac{1}{2}|\nabla z||\nabla \tilde{y}| + |\nabla \tilde{y}|^2\\ &\le \frac{5}{8}|\nabla z|^2 + \frac{5}{4}\{|\nabla \tilde{x}|^2 + |\nabla \tilde{y}|^2\} \quad \text{on } D_{\delta_2},\end{aligned}$$

thus

$$|\nabla z(w)|^2 \le \frac{10}{3}|\nabla \tilde{X}(w)|^2, \quad w \in D_{\delta_2}. \tag{22}$$

Inequality (20) now yields $|\Delta \tilde{X}(w)| \le a|\nabla \tilde{X}(w)|^2, w \in D_{\delta_2}$, for some constant a. By virtue of the relation (19) we are in a position to apply Lemma 3 to the function $\tilde{X}$, and we obtain the estimate

$$|\nabla \tilde{X}(w)| \le c|w|^{\alpha-1} \quad \text{on } D^2_{\delta_3,\theta} \tag{23}$$

for some number $\delta_3 \le \delta_2$. Finally it follows from (21) and (22) that X itself satisfies (23), i.e. $|\nabla X| \le c|w|^{\alpha-1}$ on $D^2_{\delta_3,\theta}$.

Now we have to verify (23) on the set

$$D^1_{\delta_6,\theta} = \{w = re^{i\varphi}\colon 0 < r < \delta_6, 0 < \varphi < \theta\}$$

with $\delta_6 > 0$ chosen appropriately. Performing a suitable rotation in $\mathbb{R}^3$ we can assume that S is locally given by

$$z = f(x, y)$$

with some differentiable function f defined in a neighbourhood of zero such that

$$f(0,0) = 0, \quad \nabla f(0,0) = 0.$$

Define

$$\begin{aligned}\tilde{z}(w) &:= z(w) - f(x(w), y(w)),\\ \tilde{x}(w) &:= x(w) + \tilde{z}(w) f_x(x(w), y(w)) n(w),\\ \tilde{y}(w) &:= y(w) + \tilde{z}(w) f_y(x(w), y(w)) n(w),\end{aligned}$$

where

$$n(w) := [1 + f_x^2(x(w), y(w)) + f_y^2(x(w), y(w))]^{-1}$$

and $w \in D_{\delta_2}$ with δ_2 so small that $(x(w), y(w))$ is contained in a neighbourhood of zero where f is defined. We remark that $\tilde{z}(w) = 0$ on $I^+_{\delta_2}$ and secondly, because of

$$\begin{aligned}\tilde{x}_v(u,v) &= x_v(u,v) + \tilde{z}_v(u,v) f_x(x(u,v), y(u,v)) n(u,v)\\ &\quad + \tilde{z}(u,v)[f_x(x(u,v), y(u,v)) n(u,v)]_v,\end{aligned}$$

we have for $w \in I^+_{\delta_2}$ the equality

$$\begin{aligned}\tilde{x}_v(u,v) &= x_v(u,v) + \{z_v(u,v) - f_x(x(u,v),y(u,v))x_v(u,v)\\ &\qquad - f_y(x(u,v),y(u,v))y_v(u,v)\} f_x(x(u,v),y(u,v))n(u,v).\end{aligned}$$

Equivalently, for $w \in I^+_{\delta_2}$,

$$\tilde{x}_v(u,v) = x_v(u,v) - \langle X_v(u,v), N_S(X(u,v))\rangle n^1(X(u,v)),$$

where

$$N_s(X(u,v)) = (n^1(X(u,v)), n^2(X(u,v)), n^3(X(u,v)))$$

denotes the upward unit normal of S at $X(u,v)$. However, X intersects S orthogonally along $I^+_{\delta_2}$; thus

$$\tilde{x}_v(u,v) = 0 \quad \text{on } I^+_{\delta_2}.$$

Analogously we find

$$\tilde{y}_v(u,v) = 0 \quad \text{on } I^+_{\delta_2},$$

whence the function $\tilde{X}(u,v) := (\tilde{x}(u,v), \tilde{y}(u,v))$, $(u,v) \in D_{\delta_2}$, satisfies

$$\tilde{X}_v(u,v) = 0 \quad \text{on } I^+_{\delta_2}. \tag{24}$$

Furthermore we infer from the definition of $\tilde{x}, \tilde{y}, \tilde{z}$, from $f(0,0) = 0$, $\nabla f(0,0) = 0$, $X(0,0) = 0$, as well as from the continuity of X the relation

$$|\tilde{x}_v(w)|^2 \geq \text{const}\{|x_v(w)|^2 - \varepsilon[|y_v(w)|^2 + |z_v(w)|^2]\},$$

which holds true for $w \in D_{\delta_3}, \delta_3 = \delta_3(\varepsilon) \leq \delta_2$, and for arbitrary fixed $\varepsilon > 0$. We observe that similar relations hold for $\tilde{x}_u, \tilde{y}_v$ and $\tilde{y}_u$.

From the conformality condition we first obtain that $|\nabla z|^2 \leq |\nabla x|^2 + |\nabla y|^2$ and hence

$$|\nabla X(u,v)|^2 \leq \text{const } |\nabla \tilde{X}(u,v)|^2 \quad \text{on } D_{\delta_3}. \tag{25}$$

Similar arguments show that for some $\delta_4 \leq \delta_3$ the estimate

$$|\Delta \tilde{X}(u,v)| \leq \text{const } |\nabla \tilde{X}(u,v)|^2, \quad (u,v) \in D_{\delta_4}, \tag{26}$$

holds true. We reflect $\tilde{X}(u,v)$ so as to obtain a function $\overline{X}(u,v)$ given by

$$\overline{X}(u,v) = \begin{cases} \tilde{X}(u,v), & (u,v) \in \overline{D}_{\delta_4},\\ \tilde{X}(u,-v), & (u,-v) \in D_{\delta_4}.\end{cases}$$

By virtue of (24) we obtain for each function $\Phi \in C^1_c(B_{\delta_4}(0) \setminus \overline{I^-_{\delta_4}}, \mathbb{R}^2)$ the equalities

$$
\begin{aligned}
&\int_{B_{\delta_4}(0)} \nabla\overline{X}(u,-v)\cdot\nabla\Phi(u,v)\,du\,dv \\
&\quad = \int_{D_{\delta_4}} \nabla\tilde{X}(u,v)\cdot\nabla\Phi(u,v)\,du\,dv + \int_{B_{\delta_4}\setminus D_{\delta_4}} \nabla\tilde{X}(u,-v)\cdot\nabla\Phi(u,v)\,du\,dv \\
&\quad = -\int_{D_{\delta_4}} \Delta\tilde{X}\cdot\Phi\,du\,dv - \int_{I^+_{\delta_4}} \tilde{X}_v(u,0)\cdot\Phi(u,0)\,du \\
&\qquad - \int_{B_{\delta_4}\setminus\overline{D}_{\delta_4}} \Delta\tilde{X}\cdot\Phi\,du\,dv + \int_{I^+_{\delta_4}} \tilde{X}_v(u,0)\cdot\Phi(u,0)\,du \\
&\quad = \int_{B_{\delta_4}} \overline{F}(u,v,\overline{X}(u,v),\nabla\overline{X}(u,v))\cdot\Phi(u,v)\,du\,dv
\end{aligned}
$$

for some function $\overline{F}$ which grows quadratically in $|\nabla\overline{X}|$ (compare with inequality (26)). By construction, the function $\overline{X}(u,v)$ is of class $C^0(\overline{B_{\delta_4}(0)},\mathbb{R}^2)\cap C^1(B_{\delta_4}(0)\setminus\overline{I^-_{\delta_4}})$, and the preceding discussion shows that it is a weak solution of the two-dimensional system

$$
\Delta\overline{X} = \overline{F}(u,v,\overline{X},\nabla\overline{X}) \quad \text{in } B_{\delta_4}(0)\setminus\overline{I^-_{\delta_4}}. \tag{27}
$$

Standard regularity theory (see, for instance, Section 2.1, and Morrey [8], Gilbarg-Trudinger [1]) implies that $\overline{X}$ is in fact of class $C^2(B_{\delta_4}(0)\setminus\overline{I}_{\delta_4})$ and satisfies (27) classically on all $B_{\delta_4}(0)\setminus\overline{I^-_{\delta_4}}$. Finally we apply Lemma 1 to the domain $D = B_{\delta_4}\setminus\overline{I^-_{\delta_4}}$ and to the function $\overline{X}$; the resulting inequality is

$$
|\nabla\overline{X}(w_0)| \le \text{const}\ \varepsilon^{-1}\sup_{B_\varepsilon(w_0)}|\overline{X}(w)-\overline{X}(w_0)|
$$

for all w_0 and ε with the property $B_\varepsilon(w_0)\subset B_{\delta_5}\setminus\overline{I^-_{\delta_5}}$, where $\delta_5\le\delta_4$ is the constant determined in Lemma 1. If w_0 is restricted to lie in $D^1_{\delta_6,\theta}$, $\delta_6 := \frac{1}{2}\delta_5$, then a suitable choice of ε would be $\varepsilon = \frac{1}{2}|w_0|$. Hence

$$
\begin{aligned}
|\nabla\overline{X}(w_0)| &\le \text{const}\ \varepsilon^{-1}[|w_0|^\alpha + (|w_0|+\varepsilon)^\alpha] \\
&\le \text{const}\ |w_0|^{\alpha-1},
\end{aligned}
$$

that is, (23) holds true on $D^1_{\delta_6,\theta}$. Because of (25) we finally arrive at relation (2). □

3.3 Minimal Surfaces with Piecewise Smooth Boundary Curves and Their Asymptotic Behaviour at Corners

In the previous section we proved an asymptotic estimate for the gradient of a minimal surface X at a corner P of a given piecewise smooth boundary arc $\Gamma^+ \cup \Gamma^-$. It is the purpose of this section to obtain some more precise information on the asymptotic behaviour of X_w near the corner P. To give an idea what might happen we start with a simple but characteristic *example*:

Let $\alpha \in (0, 1)$ and $k \in \mathbb{N} \cup \{0\}$ be given and define

$$X(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in \overline{B} = \{u^2 + v^2 \leq 1, v \geq 0\}$$

by

$$\begin{aligned} x(u, v) &= \operatorname{Re}(w^{\alpha+2k}), \\ y(u, v) &= \operatorname{Im}(w^{\alpha+2k}), \quad w = u + iv \in \overline{B}, \\ z(u, v) &\equiv 0. \end{aligned}$$

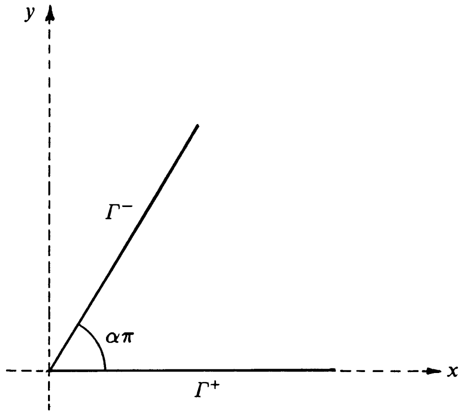

Fig. 1.

Then $X(u, v)$ is a minimal surface (i.e. $\Delta X = 0, \langle X_w, X_w \rangle = 0$) which maps the intervals $I^+ = (0, 1), I^- = (-1, 0)$ onto the straight arcs

$$\Gamma^+ = \{(x, y, z) \in \mathbb{R}^3 \colon z = 0, \arg(x + iy) = 0, 0 < x^2 + y^2 < 1\}$$

and

$$\Gamma^- = \{(x, y, z) \in \mathbb{R}^3 \colon z = 0, \arg(x + iy) = \pi\alpha, \ 0 < x^2 + y^2 < 1\}$$

respectively, and the point $w = 0$ into the origin of $\mathbb{R}^3$. Note that Γ^+, Γ^- form an angle $\beta = \alpha\pi$ at zero and that X has a branch point at zero if $k \geq 1$ whence X winds around zero k-times. However, there is another possible solution to the Plateau problem determined by Γ^+ and Γ^-, namely the surface

$$X_1(u, v) = (x_1(u, v), y_1(u, v), z_1(u, v))$$

the components of which are defined by

$$x_1(u,v) = \mathrm{Re}(\overline{w}^{2-\alpha+2k}), \quad y_1(u,v) = \mathrm{Im}(\overline{w}^{2-\alpha+2k}), \quad z_1(u,v) = 0,$$

with $w = u + iv \in \overline{B}$ and $\overline{w} = u - iv$. Here the semi-disk $\overline{B}$ is mapped into the great angle $(2-\alpha)\pi$ which is formed by Γ^+, Γ^- at zero. Again it is possible that branch points occur and that the surface winds about the origin. In Theorem 1 of this section we shall show that this behaviour is typical of a minimal surface X which is bounded by two Jordan arcs forming a positive angle $\alpha\pi$ at a corner P where Γ^+ and Γ^- are tangent to the x,y-plane.

Before we can formulate the main theorem of this section, we have to state the basic assumptions describing the geometric situation which is to be considered.

Assumption A. *Γ^+, Γ^- are regular arcs of class $C^{2,\mu}, \mu \in (0,1)$, which intersect at the origin, thereby enclosing an angle of $\pi\alpha, \alpha \in (0,1)$. The sets $B_\delta^+, I_\delta^+, I_\delta^-$ are defined by*

$$\begin{aligned} B_\delta^+ &:= \{w = (u,v) \in \mathbb{R}^2 \colon |w| < \delta, v > 0\}, \\ I_\delta^+ &:= \{w = (u,0) \in \mathbb{R}^2 \colon 0 < u < \delta\}, \\ I_\delta^- &:= \{w = (u,0) \in \mathbb{R}^2 \colon -\delta < u < 0\}, \end{aligned}$$

(and, as usual, we will identify $w = u + iv \in \mathbb{C}$ with $w = (u,v) \in \mathbb{R}^2$ and I_δ^+ with $(0,\delta) \subset \mathbb{R}$, etc.).

Let X be a minimal surface which is of class $C^{0,\nu}(\overline{B_{\delta+}}, \mathbb{R}^3) \cap C^2(\overline{B_\delta^+} \setminus \{0\}, \mathbb{R}^3)$ for some $\nu \in (0,1)$ and $\delta > 0$, and satisfies the boundary conditions $X \colon I_\delta^\pm \to \Gamma^\pm$ and $X(0) = 0$. Moreover, we assume that there exist functions $h_1^\pm, h_2^\pm \in C^{2,\mu}(\overline{I_\varepsilon^\pm}, \mathbb{R}), \varepsilon > 0$, such that

$$\Gamma^+ = \{(t, h_1^+(t), h_2^+(t)) \colon t \in \overline{I_\varepsilon^+}\} \quad \textit{and} \quad \Gamma^- = \{(t, h_1^-(t), h_2^-(t)) \colon t \in \overline{I_\varepsilon^-}\},$$

and that furthermore

$$h_j^\pm(0) = 0, \quad j = 1,2, \quad \textit{and} \quad h_1^{\pm\prime}(0) = \pm\cot\left(\frac{\alpha\pi}{2}\right), \quad h_2^{\pm\prime}(0) = 0$$

hold true.

We note that Assumption A is quite natural and not restrictive since, by performing suitable translations and rotations, we can achieve that any pair of piecewise smooth boundary curves Γ^+, Γ^- will satisfy this assumption. Also, by the results of Chapter 2, any minimal surface bounded by Γ^+, Γ^- has the desired regularity properties.

The main result of this section will be

Theorem 1. *Suppose that the Assumption A holds. Then there exist Hölder continuous complex valued functions Φ_1 and Φ_2 defined on the closure of some semidisk $B_\delta^+, \delta > 0$, such that the following assertions hold true:*

$$\Phi_1(0) \neq 0, \quad \Phi_2(0) \neq 0, \quad \Phi_1^2(0) + \Phi_2^2(0) = 0, \tag{1}$$

$$x_w(w) = w^\gamma \Phi_1(w), \quad y_w(w) = w^\gamma \Phi_2(w), \quad \textit{and} \quad |z_w(w)| = O(|w|^\lambda), \tag{2}$$

where $\gamma = \alpha - 1 + 2k$ *or* $\gamma = 1 - \alpha + 2k$ *for some* $k \in \mathbb{N} \cup \{0\}$ *and* $\lambda > \gamma$. *Furthermore there exists some* $c \in \mathbb{C} \setminus \{0\}$ *such that*

$$x(w) + iy(w) = \begin{cases} w^{\alpha+2k}[c + o(1)] \\ or \\ \overline{w}^{2-\alpha+2k}[c + o(1)] \end{cases} \quad \textit{as } w \to 0, \tag{3}$$

and

$$z(w) = O(|w|^{\lambda+1}) \quad \textit{as } w \to 0.$$

Finally, the unit normal $N(w) = \frac{(X_u \wedge X_v)(w)}{|(X_u \wedge X_v)(w)|}$ *tends to a limit as* $w \to 0$:

$$\lim_{w \to 0} N(w) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}. \tag{4}$$

Remark 1. Theorem 1 extends without essential changes to conformal solutions $X(w)$ of the system $\Delta X = f(X, \nabla X)$, where the right-hand-side grows quadratically in $|\nabla X|$. Also two-dimensional surfaces in $\mathbb{R}^n$, $n \geq 3$, can be treated.

In the case of polygonal boundaries we can say more:

Theorem 2. *Suppose that* Assumption A *holds where* Γ^+, Γ^- *are straight lines. Then there exist holomorphic functions* H_j *and* $\hat{H}_j$, $j = 1, 2, 3$, *which are defined on a disk* B_δ *for some* $\delta > 0$, *such that the following holds true*:

$$wH_1^2(w) + 4H_2(w)H_3(w) = 0, \tag{5}$$

$$X_w(w) = w^{\alpha-1} H_2(w) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + w^{-\alpha} H_3(w) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + H_1(w) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6}$$

$$\begin{aligned} x(w) + iy(w) &= w^\alpha \hat{H}_2(w) + \overline{w}^{1-\alpha} \hat{H}_3(\overline{w}), \\ z(w) &= \operatorname{Re}(w \hat{H}_1(w)), \end{aligned} \tag{7}$$

where $w \in \overline{B}_\delta^+ \setminus \{0\}$. *Furthermore,* (1) *holds true as well.*

The idea of the proof of Theorems 1 and 2 is to eventually apply Theorem 3 of Section 3.1 to a certain set of functions involving the gradient X_w. Here it is necessary and convenient to use first a reflection procedure followed by a smoothing argument. The new function of interest is then defined on a neighbourhood $B_\delta \setminus \{0\}$ of zero, and it turns out that Theorem 3 of Section 3.1 can be employed. A further essential ingredient is Theorem 1 of Section 3.2 which provides the starting regularity and thus makes our argument work.

Proof of Theorem 1. Because of the continuity of X it is possible to select $\delta > 0$ so small that $x(\overline{B}_\delta^+) \subset [-\varepsilon, \varepsilon]$; this will henceforth be assumed. By Assumption A we have on $I_\delta^\pm$ the equality

$$X(u,0) = (x(u,0), h_1^\pm(x(u,0)), h_2^\pm(x(u,0))),$$

whence

$$X_u(u,0) = (1, h_1^{\pm\prime}(x(u,0)), h_2^{\pm\prime}(x(u,0)))x_u(u,0).$$

The conformality conditions (which also hold on $I_\delta^\pm$) imply that

$$\langle X_v(u,0), (1, h_1^{\pm\prime}(x(u,0)), h_2^{\pm\prime}(x(u,0)))\rangle = 0 \quad \text{on } I_\delta^\pm. \tag{8}$$

Now we put

$$a^\pm(t) := [1 + h_1^{\pm\prime}(t)^2 + h_2^{\pm\prime}(t)^2]^{-1/2}(1, h_1^{\pm\prime}(t), h_2^{\pm\prime}(t)), \quad t \in [-\varepsilon, \varepsilon],$$

and consider the linear mappings

$$S^\pm(t)y := 2\langle a^\pm(t), y\rangle a^\pm(t) - y$$

which are defined for $t \in [-\varepsilon, \varepsilon]$ and $y \in \mathbb{R}^3$. Then, using (8), we infer

$$S^\pm(x(u,0))X_u(u,0) = X_u(u,0),$$

$$S^\pm(x(u,0))X_v(u,0) = -X_v(u,0), \quad \text{where } (u,0) \in I_\delta^\pm.$$

This may be rewritten as

$$S^\pm(x(w))X_w(w) = X_{\overline{w}}(w) \quad \text{for all } w \in I_\delta^\pm. \tag{9}$$

Since $S^\pm(t), t \in [-\varepsilon, \varepsilon]$, is a family of reflections, there exist orthogonal matrices $O^\pm(t)$ such that

$$S^\pm(t) = O^\pm(t)\,\mathrm{Diag}[-1,-1,1]O^\pm(t)^t,$$

where we have used the notation

$$\mathrm{Diag}[\alpha, \beta, \gamma] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

and A^t denotes the transpose of the matrix A. Furthermore we define

$$T^\pm(t) := O^\pm(0)O^\pm(t)^t$$

with

$$O^+(0) = \lim_{t\to 0^+} O^+(t) \quad \text{and} \quad O^-(0) = \lim_{t \to 0^-} O^-(t).$$

Now put

$$T(t) := \begin{cases} T^+(t) & \text{if } 0 \le t \le \varepsilon, \\ T^-(t) & \text{if } -\varepsilon \le t < 0. \end{cases}$$

It follows that the matrix function T is of class $C^0[(-\varepsilon, \varepsilon], \mathbb{R}^9)$ since

$$\lim_{t\to 0^+} T(t) = T^+(0) = \mathrm{Id} = T^-(0) = \lim_{t\to 0^-} T(t),$$

and, because of the assumptions on $h_1^\pm, h_2^\pm$, the matrix T is even of the class

$$C^{0,1}([-c, c], \mathbb{R}^9)$$

(although $S^\pm(t)$ is not even continuous at zero).

Next we consider the complex valued function $g(w)$ defined by

$$g(w) := T(x(w)) \cdot X_w(w) \quad \text{for all } w \in B_\delta^+. \tag{10}$$

We claim that g has the reflection property

$$S^\pm(0) g(w) = \overline{g(w)} \quad \text{for all } w \in I_\delta^\pm. \tag{11}$$

In fact, it follows from (9) that

$$\begin{aligned} S^\pm(0) g(w) &= S^\pm(0) T^\pm(x(w)) X_w(w) \\ &= S^\pm(0) O^\pm(0) O^\pm(x(w))^t X_w(w) \\ &= O^\pm(0) \,\mathrm{Diag}[-1,-1,1] O^\pm(x(w))^t X_w(w) \\ &= T^\pm(x(w)) S^\pm(x(w)) X_w(w) \\ &= T^\pm(x(w)) X_{\overline{w}}(w) = \overline{g(w)}. \end{aligned}$$

We now reflect g across the u-axis by

$$G(w) = \begin{cases} g(w) & \text{if } w \in \overline{B}_\delta^+ \setminus \{0\}, \\ S^+(0)\overline{g}(\overline{w}) & \text{if } \overline{w} \in B_\delta^+. \end{cases} \tag{12}$$

Then we have

Lemma 1. *The function G is of class $C^{0,1}(B_\delta \setminus \overline{I_\delta^-}, \mathbb{C}^3)$, and there exists some constant $c > 0$ such that the estimate*

$$|G_{\overline{w}}(w)| \le c |G(w)|^2 \tag{13}$$

holds true almost everywhere on $B_\delta \setminus \{0\}$. Furthermore $G(w)$ satisfies

$$G_1^2(w) + G_2^2(w) + G_3^2(w) = 0, \quad w \in B_\delta \setminus \overline{I_\delta^-}, \tag{14}$$

$$|G(w)| = O(|w|^{\nu - 1}) \quad \text{as } w \to 0, \tag{15}$$

where ν denotes the Hölder exponent of X. Finally there holds the jump relation

$$\lim_{v\to 0^+} G(u, v) = S^-(0) S^+(0) \lim_{v \to 0^-} G(u, v) \tag{16}$$

for all $u \in I_\delta^-$.

Proof. Since T is Lipschitz continuous and $X \in C^2(\overline{B}^+_\delta \setminus \{0\}, \mathbb{R}^3)$ we also have $g \in C^{0,1}(\overline{B}^+_\delta \setminus \{0\}, \mathbb{C}^3)$, and because of (11) we obtain $G \in C^{0,1}(B_\delta \setminus \overline{I^-_\delta}, \mathbb{C}^3)$. To establish (13), we remark that almost everywhere on B_δ we find

$$G_{\overline{w}}(w) = \begin{cases} g_{\overline{w}} = [T'(x(w))x_{\overline{w}}(w)]X_w(w) & \text{if } w \in B^+_\delta, \\ S^+(0)\overline{g}_{\overline{w}}(\overline{w}) = S^+(0)[T'(x(\overline{w}))\overline{x}_{\overline{w}}(\overline{w})]X_{\overline{w}}(\overline{w}) & \text{if } \overline{w} \in B^+_\delta \end{cases}$$

whence

$$\begin{aligned} |G_{\overline{w}}(w)| &\le c_1|T'(x(w))||x_w||X_w| \le c_2|X_w(w)|^2 \\ &\le c_3|T^{-1}(x(w))g(w)|^2 \le c_4|g(w)|^2 \le c_5|G(w)|^2 \end{aligned}$$

for suitable constants $c_1, \dots, c_5$. From the conformality condition $\langle X_w, X_w\rangle = 0$ we easily conclude (14), taking the orthogonality of the matrices $T^\pm$ into account.

The relation (15) follows from the estimate $|G(w)| \le c_6|\nabla X|$ and from Theorem 1 of Section 3.2. Finally, to prove (16), we calculate by means of (11) that

$$\begin{aligned} \lim_{v\to 0^+} G(u,v) &= g(u,0) = S^-(0)\overline{g}(u,0) \\ &= S^-(0)S^+(0)S^+(0)\overline{g}(u,0) = S^-(0)S^+(0)\lim_{v\to 0^-} G(u,v), \end{aligned}$$

where we have used that $S^+(0)S^+(0) = \mathrm{Id}$. $\square$

The function $G(w)$ itself is not yet accessible to the methods which were developed at the end of Section 3.1, because of the jump relation (16). To overcome this difficulty, we have to smooth the function G, which will be carried out in what follows. Recall that the jump of G at I^-_δ is given by

$$S^-(0)S^+(0) = \begin{pmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha & 0 \\ \sin 2\pi\alpha & \cos 2\pi\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can diagonalize $S^-(0)S^+(0)$ using the unitary matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -i & i \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

and obtain

$$S^-(0)S^+(0) = U\,\mathrm{Diag}[1, e^{2\pi i(\alpha-1)}, e^{-2\pi i\alpha}]U^*,$$

where $U^* = \overline{U}^t$. We define a new function $F(w), w \in B_\delta \setminus \overline{I^+_\delta}$, by

$$F(w) := \mathrm{Diag}[1, w^{1-\alpha}, w^\alpha]U^*G(w) \tag{17}$$

or, more explicitly,

$$F(w) = \begin{pmatrix} F_1(w) \\ F_2(w) \\ F_3(w) \end{pmatrix} = \begin{pmatrix} G_3(w) \\ \frac{1}{\sqrt{2}} w^{1-\alpha}[G_1(w) + iG_2(w)] \\ \frac{1}{\sqrt{2}} w^{\alpha}[G_1(w) - iG_2(w)] \end{pmatrix}.$$

We claim that F is continuous on the punctured disk $B_\delta(0) \setminus \{0\}$. In fact, we infer from (16) the relation

$$\begin{aligned}
\lim_{v\to 0^+} F(u,v) &= \mathrm{Diag}[1, u^{1-\alpha}e^{i\pi(1-\alpha)}, u^{\alpha}e^{i\pi\alpha}]U^* \lim_{v\to 0^+} G(u,v) \\
&= \mathrm{Diag}[1, u^{1-\alpha}e^{i\pi(1-\alpha)}, u^{\alpha}e^{i\pi\alpha}]U^* S^-(0)S^+(0) \lim_{v\to 0^-} G(u,v) \\
&= \mathrm{Diag}[1, u^{1-\alpha}e^{i\pi(1-\alpha)}, u^{\alpha}e^{i\pi\alpha}]U^* U \\
&\quad \cdot \mathrm{Diag}[1, e^{2\pi i(\alpha-1)}, e^{-2\pi i\alpha}]U^* \lim_{v\to 0^-} G(u,v)) \\
&= \mathrm{Diag}[1, u^{1-\alpha}e^{-i\pi(1-\alpha)}, u^{\alpha}e^{-i\pi\alpha}]U^* \lim_{v\to 0^-} G(u,v) \\
&= \lim_{v\to 0^-} F(u,v).
\end{aligned}$$

Since $G \in C^{0,1}(B_\delta \setminus \overline{I_\delta^-}, \mathbb{C}^3)$, and by Assumption A, it follows that F is even Lipschitz continuous on the punctured disk $B_\delta \setminus \{0\}$.

Lemma 2. *The function $F(w) = (F_1(w), F_2(w), F_3(w))$ defined by (17) belongs to the class $C^{0,1}(B_\delta(0) \setminus \{0\}, \mathbb{C}^3)$ and satisfies*

$$\begin{aligned}
&F_1^2(w)w + 2F_2(w)F_3(w) = 0 \quad \textit{for } w \in B_\delta \setminus \{0\}, \\
&|F_1(w)| = O(|w|^{\nu-1}) \quad \textit{as } w \to 0,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
|F_2(w)| &= O(|w|^{\nu-\alpha}) \quad \textit{as } w \to 0, \\
|F_3(w)| &= O(|w|^{\nu-\beta}) \quad \textit{as } w \to 0,
\end{aligned} \tag{19}$$

where ν denotes the Hölder exponent of X, and $\beta = 1 - \alpha$. Furthermore, the following differential inequalities hold true:

$$\begin{aligned}
|F_{1\overline{w}}(w)| &\le c\{|w|^{-2\beta}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2\}, \\
|F_{2\overline{w}}(w)| &\le c\{|w|^{-\beta}|F_2(w)|^2 + |w|^{\beta-2\alpha}|F_3(w)|^2\}, \\
|F_{3\overline{w}}(w)| &\le c\{|w|^{\alpha-2\beta}|F_2(w)|^2 + |w|^{-\alpha}|F_3(w)|^2\}
\end{aligned} \tag{20}$$

almost everywhere on $B_\delta \setminus \{0\}$ for some constant $c > 0$.

Proof. We conclude from (14) and (17) that

$$\begin{aligned} 0 &= G_1^2(w) + G_2^2(w) + G_3^2(w) = \tfrac{1}{2}[w^{\alpha-1}F_2(w) + w^{-\alpha}F_3(w)]^2 \\ &\quad - \tfrac{1}{2}[w^{\alpha-1}F_2(w) - w^{-\alpha}F_3(w)]^2 + F_1^2(w) \\ &= 2w^{-1}F_2(w)F_3(w) + F_1^2(w), \end{aligned}$$

whence (18) follows. From the definition of $F(w) = (F_1(w), F_2(w), F_3(w))$ and from (15) we infer the relations (19). To prove the inequalities (20) we first note that

$$|G(w)|^2 = |F_1(w)|^2 + |w|^{-2\beta}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2,$$

whence we obtain from (13) and (17) the inequalities

$$\begin{aligned} |F_{1\overline{w}}(w)| &\le c[|F_1(w)|^2 + |w|^{-2\beta}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2], \\ |F_{2\overline{w}}(w)| &\le c[|w|^{\beta}|F_1(w)|^2 + |w|^{-\beta}|F_2(w)|^2 + |w|^{\beta-2\alpha}|F_3(w)|^2], \\ |F_{3\overline{w}}(w)| &\le c[|w|^{\alpha}|F_1(w)|^2 + |w|^{\alpha-2\beta}|F_2(w)|^2 + |w|^{-\alpha}|F_3(w)|^2]. \end{aligned}$$

On the other hand, relation (18) yields the estimate

$$\begin{aligned} |F_1(w)|^2 &\le |w|^{-1}|w|^{2\alpha-1}|F_2(w)|^2 + |w|^{-1}|w|^{1-2\alpha}|F_3(w)|^2 \\ &= |w|^{2(\alpha-1)}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2 \\ &= |w|^{-2\beta}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2. \end{aligned}$$

Together with the above inequalities we finally obtain (20). This finishes the proof of Lemma 2. □

Now we are in a position to apply Theorem 3 of Section 3.1. We can assume without loss of generality that $0 < \alpha \le \beta < 1$.

Lemma 3. *There exists a nonnegative integer m such that the functions $f_i^m(w) := w^{-m}F_i(w), i = 1, 2, 3$, do not vanish simultaneously at zero and that one of the following conditions holds true*:

(i) $\quad f_2^m \in C^{0,\mu}(B_\delta, \mathbb{C})$ *for all* $\mu < \min(1, m + \alpha)$, $\quad f_2^m(0) \neq 0$,

$$\begin{aligned} |f_{1\overline{w}}^m(w)| &= O(|w|^{m-2\beta}) \quad \text{as } w \to 0, \\ |f_{2\overline{w}}^m(w)| &= O(|w|^{m-\beta}) \quad \text{as } w \to 0, \\ |f_{3\overline{w}}^m(w)| &= O(|w|^{m+\alpha-2\beta}) \quad \text{as } w \to 0, \end{aligned}$$

a.e. on $B_\delta \setminus \{0\}$.

(ii) $\quad f_3^m \in C^{0,\mu}(B_\delta, \mathbb{C})$ *for all* $\mu < \min(1, m + \beta)$, $\quad f_3^m(0) \neq 0$

$$\begin{aligned} |f^m_{1\overline{w}}(w)| &= O(|w|^{m-2\alpha}) \quad \textit{as } w \to 0, \\ |f^m_{2\overline{w}}(w)| &= O(|w|^{m+\beta-2\alpha}) \quad \textit{as } w \to 0, \\ |f^m_{3\overline{w}}(w)| &= O(|w|^{m-\alpha}) \quad \textit{as } w \to 0, \end{aligned}$$

a.e. on $B_\delta \setminus \{0\}$.

In addition, if $m \geq 1$*, then in both cases*

(21) $$f^m_2(0) f^m_3(0) = 0.$$

Proof of Lemma 3. From Theorem 3 of Section 3.1 we infer that (i) or (ii) has to hold, except for the assertions concerning f^m_1. We recall the cases $(\alpha), (\beta), (\gamma)$, and (δ) which occurred in the proof of Theorem 3 in Section 3.1. Let us treat these cases separately.

(α) $F_2(0) \neq 0, F_3(0) \neq 0$: Then (i) of Theorem 3 in Section 3.1 holds with $m = 0$. In particular, $|F_2(w)| = O(1)$ as $w \to 0$, and $|F_{3\overline{w}}(w)| = O(|w|^{\alpha-2\beta})$ as $w \to 0$. But then Lemma 6 of Section 3.1 implies that

$$|F_3(w)| = \begin{cases} O(1 + |w|^{\alpha-2\beta+1}) & \text{if } \alpha - 2\beta \neq -1, \\ O(|w|^{-\varepsilon}) \quad \text{for all } \varepsilon > 0 & \text{if } \alpha - 2\beta = -1. \end{cases}$$

Now relation (20_1) yields

$$|F_{1\overline{w}}(w)| = O(|w|^{-2\beta}| \quad \text{as } w \to 0,$$

which is the desired assertion.

(β) $F_2(0) \neq 0, F_3(0) = 0$: Here we obtain (i) of Section 3.1, Theorem 3 with $m = 0$. Thus we can proceed as in case (α).

(γ) $F_2(0) = 0, F_3(0) \neq 0$: In this case we obtain (ii) of Theorem 3 in Section 3.1 with $m = 0$. In particular,

$$|F_3(w)| = O(1) \quad \text{as } w \to 0,$$

$$|F_{2\overline{w}}(w)| = O(|w|^{\beta-2\alpha}) \quad \text{as } w \to 0.$$

But $0 < \alpha \leq \frac{1}{2} \leq \beta < 1$ and $\beta - 2\alpha = 1 - 3\alpha \geq -\frac{1}{2}$; therefore we conclude from Lemma 6 of Section 3.1 that $F_2 \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu < \min(1, 1 + \beta - 2\alpha)$. Since $F_2(0) = 0$ we have that $|F_2(w)| = O(|w|^\mu), w \to 0, \mu < \min(1, 1+\beta-2\alpha)$, and relation (20_1) implies

$$|F_{1\overline{w}}(w)| = O(|w|^{-2\beta+2\mu} + |w|^{-2\alpha}) = O(|w|^{-2\alpha}) \quad \text{as } w \to 0,$$

if we choose μ in such a way that $2\mu - 2\beta \geq 0$.

(δ) $F_2(0) = F_3(0) = 0$: In this case we find

$$(22)\qquad \begin{aligned} F_2(w) &= aw^m + o(|w|^m) \quad \text{as } w \to 0, \\ F_3(w) &= bw^m + o(|w|^m) \quad \text{as } w \to 0, \end{aligned}$$

with $a, b \in \mathbb{C}$ not both equal to zero and $m \geq 1$. A direct consequence of (20) is the following system:

$$(23)\qquad \begin{aligned} |f^m_{1\overline{w}}(w)| &\leq c[|w|^{m-2\beta}|f_2^m(w)|^2 + |w|^{m-2\alpha}|f_3^m(w)|^2], \\ |f^m_{2\overline{w}}(w)| &\leq c[|w|^{m-\beta}|f_2^m(w)|^2 + |w|^{m+\beta-2\alpha}|f_3^m(w)|^2], \\ |f^m_{3\overline{w}}(w)| &\leq c[|w|^{m+\alpha-2\beta}|f_2^m(w)|^2 + |w|^{m-\alpha}|f_3^m(w)|^2], \end{aligned}$$

while (18) yields

$$(24)\qquad w[f_1^m(w)]^2 + 2f_2^m(w)f_3^m(w) = 0 \quad \text{in } B_\delta \setminus \{0\}.$$

The relations (22) and (24) imply that

$$|f_2^m(w)|, |f_3^m(w)| = O(1) \quad \text{as } w \to 0$$

and

$$|f_1^m(w)| = o(|w|^{-1}) \quad \text{as } w \to 0.$$

Now (23_1) yields $|f^m_{1\overline{w}}(w)| = O(|w|^{m-2\beta})$, and by Lemma 6 of Section 3.1 we find that $f_1^m \in C^{0,\mu}(B_\delta, \mathbb{C})$ for all $\mu < \min(1, m - 2\beta + 1)$. By letting $w \to 0$ in relation (24) we conclude (21): $ab = f_2^m(0)f_3^m(0) = 0$.

First subcase: $a \neq 0, b = 0$. Then case (i) of Theorem 3, Section 3.1, holds with $m \geq 1$, and this implies (i) of Lemma 3 since we have already shown that

$$|f^m_{1\overline{w}}(w)| = O(|w|^{m-2\beta}) \quad \text{as } w \to 0.$$

Second subcase: $a = 0, b \neq 0$. Here case (ii) of Theorem 3, Section 3.1, holds with $m \geq 1$. In particular,

$$\begin{aligned} |f^m_{2\overline{w}}(w)| &= O(|w|^{m+\beta-2\alpha}) \quad \text{as } w \to 0, \\ |f^m_{3\overline{w}}(w)| &= O(|w|^{m-\alpha}) \quad \text{as } w \to 0. \end{aligned}$$

By virtue of $a = f_2^m(0) = 0$ and Lemma 6 of Section 3.1 we find

$$|f_2^m(w)| = O(|w|^\mu) \quad \text{for all } \mu < \min(1, m + \beta - 2\alpha + 1) = 1.$$

Finally we obtain from (23_1)

$$|f^m_{1\overline{w}}(w)| = O(|w|^{m-2\beta}|w|^{2\mu} + |w|^{m-2\alpha}) = O(|w|^{m-2\alpha}) \quad \text{as } w \to 0.$$

Thus Lemma 3 is proved. □

Now we finish the *proof of Theorem* 1.
From (10), (12) and (17) we infer

$$\begin{aligned} X_w(w) &= T(x(w))^* U \ \mathrm{Diag}[1, w^{\alpha-1}, w^{-\alpha}] F(w) \\ &= w^{\alpha-1} T(x(w))^* U \ \mathrm{Diag}[w^{1-\alpha}, 1, w^{1-2\alpha}] F(w). \end{aligned}$$

Let us assume that (i) of Lemma 3 holds true whence in particular

$$f_2^m \in C^{0,\mu}(B_\delta, \mathbb{C}), \quad f_2^m(0) \neq 0.$$

Now we define $\psi = (\psi_1, \psi_2, \psi_3)$ by

$$\begin{aligned} \psi(w) &:= w^{-m} U \,\mathrm{Diag}\,[w^{1-\alpha}, 1, w^{1-2\alpha}] F(w) \\ &= \frac{1}{\sqrt{2}} f_2^m(w) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} w^{1-2\alpha} f_3^m(w) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ &\quad + w^{1-\alpha} f_1^m(w) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{25}$$

and claim that ψ is Hölder continuous in $\overline{B}_\delta^+$ for $0 < \delta \ll 1$ and satisfies

$$\psi_1(0) \neq 0, \quad \psi_2(0) \neq 0, \quad \psi_3(0) = 0. \tag{+}$$

First we note that $\psi_3(w) = w^{1-\alpha} f_1^m(w)$ is Hölder continuous in $\overline{B}_\delta^+$ with $\psi_3(0) = 0$. In fact, if $m \geq 1$ then f_1^m is Hölder continuous according to part (δ) in the proof of Lemma 3. If, however, $m = 0$, then $f_1^0 = F_1$, and so we have according to Lemma 3, (i), and formula (18) that

$$|F_1(w)| = O(|w|^{\nu-1}), |F_{1,\overline{w}}(w)| = O(|w|^{-2\beta}) \quad \text{for } w \to 0. \tag{++}$$

We distinguish two cases:

(i) $2\alpha > 1$: Then we have $-2\beta > -1$, and by (++) and Lemma 6 in Section 3.1, the function f_1^0 is Hölder continuous in $\overline{B}_\delta^+$. Thus also ψ_3 is Hölder continuous in $\overline{B}_\delta^+$, and $\psi_3(0) = 0$ since $\alpha < 1$.

(ii) $2\alpha < 1$: By (++) and Lemma 6 in Section 3.1 the function $w f_1^0(w)$ is of class $C^\mu(B_\delta)$ for all $\delta < 2\alpha$, and $w f_1^0(w) \to 0$ as $w \to 0$. Therefore,

$$|w^{1-\alpha} f_1^0(w)| = O(|w|^{\alpha-\epsilon}) \quad \text{for } w \to 0 \text{ and } 0 < \epsilon \ll 1;$$

consequently ψ_3 is continuous in $\overline{B}_\delta^+$ with $\psi_3(0) = 0$. Now we estimate $|\psi_3(w_1) - \psi_3(w_2)|$ for any $w_1, w_2 \in \overline{B}_\delta^+ \setminus \{0\}$; w.l.o.g. we assume $|w_1| \leq |w_2|$, whence $|w_2| \geq \frac{1}{2}|w_2 - w_1|$. Then

$$\begin{aligned}
&|\psi_3(w_1) - \psi_3(w_2)| \\
&\quad \le |w_1^{-\alpha} - w_2^{-\alpha}||w_1 f_1^0(w_1)| + |w_2|^{-\alpha}|w_1 f_1^0(w_1) - w_2 f_2^0(w_2)| \\
&\quad \le c\{|w_1|^{-\alpha}|w_2|^{-\alpha}|w_1 - w_2|^{\alpha}|w_1|^{2\alpha-\epsilon} + |w_2|^{-\alpha}|w_1 - w_2|^{2\alpha-\epsilon}\} \\
&\quad \le c\{|w_2|^{-\alpha}|w_1 - w_2|^{\alpha}|w_1|^{\alpha-\epsilon} + |w_1 - w_2|^{\alpha-\epsilon}\} \le c|w_1 - w_2|^{\alpha-\epsilon},
\end{aligned}$$

which shows that ψ_3 is Hölder continuous in $\overline{B_\delta^+}$ also in case (ii).

Now we infer from (18), (24) the identity

$$w^{1-2\alpha} f_3^m(w) = -\frac{[w^{1-\alpha} f_1^m(w)]^2}{2 f_2^m(w)}.$$

Since the right-hand side is Hölder continuous in B_δ^+ (note that $f_2^m(0) \neq 0$), the same holds for $w^{1-2\alpha} f_2^m(w)$; thus we arrive at $w^{1-2\alpha} f_3^m(w) \to 0$ for $w \to 0$. Therefore, ψ is Hölder continuous and satisfies (+). Because of $T(0) = \mathrm{Id}$, also the function $\Phi(w) := T(x(w))^* \psi(w)$ satisfies $\Phi_1(0) \neq 0, \Phi_2(0) \neq 0$ and $\Phi_3(0) = 0$. Since T is Lipschitz continuous and ψ and X are Hölder continuous, also Φ is of class $C^{0,\nu_1}(\overline{B_\delta^+}, \mathbb{C}^3)$ where $\nu_1 := \min(\nu, \mu)$. Because of (25) we have

$$X_w(w) = w^{\alpha-1+m}\Phi(w) = w^{\alpha-1+m} T(x(w))^* \psi(w),$$

that is,

$$\begin{aligned}
x_w(w) &= w^{\alpha-1+m}\Phi_1(w), \quad \Phi_1(0) \neq 0, \\
y_w(w) &= w^{\alpha-1+m}\Phi_2(w), \quad \Phi_2(0) \neq 0, \\
|z_w(w)| &= |w^{\alpha-1+m}||\Phi_3(w)| = O(|w|^{\lambda}),
\end{aligned} \tag{26}$$

as $w \to 0$, with $\lambda > \alpha - 1 + m$. This proves relation (2).

On the other hand, let us assume that (ii) of Lemma 3 occurs; then we argue with the function

$$\tilde{\psi}(w) := \frac{1}{\sqrt{2}} f_3^m(w) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} w^{2\alpha-1} f_2^m(w) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + w^{\alpha} f_1^m(w) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

instead of ψ, and similar arguments show that $\tilde{\psi}(w)$ is Hölder continuous on $\overline{B_\delta^+}$ and that $\tilde{\psi}_1(0) \neq 0, \tilde{\psi}_2(0) \neq 0, \tilde{\psi}_3(0) = 0$. In this case we have $\gamma = m - \alpha$ because of

$$w^{m-\alpha}\tilde{\psi}(w) = U\,\mathrm{Diag}[1, w^{\alpha-1}, w^{-\alpha}]F(w)$$

and

$$X_w(w) = w^{m-\alpha} T(x(w))^* \tilde{\psi}(w);$$

thus (2) holds with $\gamma = m - \alpha$.

From the conformality condition

$$x_w^2(w) + y_w^2(w) + z_w^2(w) = 0, \quad w \in I_\delta^- \cup I_\delta^+,$$

we infer, using (2), that

$$0 = w^{2\gamma}[\Phi_1^2(w) + \Phi_2^2(w)] + O(|w|^{2\lambda}) \quad \text{as } w \to 0,$$

where $\gamma = \alpha - 1 + m$ or $m - \alpha$. Letting $w \to 0$, we obtain

$$0 = \Phi_1^2(0) + \Phi_2^2(0)$$

which proves (1).

Relation (3) follows by integrating formula (2), using the fact that

$$X(w) = 2 \text{ Re}\left[\int_0^r X_w(\underline{r}e^{i\varphi})e^{i\varphi}\, d\underline{r}\right].$$

Thus

$$x(w) + iy(w) = \begin{cases} w^{\alpha+m}[c + o(1)] \\ \overline{w}^{m-\alpha+1}[c + o(1)] \end{cases} \quad \text{as } w \to 0 \tag{27}$$

in the two cases respectively. Relation (27) and the boundary conditions imply that $m = 2k$ in the first and $m = 2k + 1$ in the second case.

Finally we have to consider the normal

$$N(w) = (N_1, N_2, N_3) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}.$$

Since

$$|X_u \wedge X_v| = 2|X_w|^2 = 2|w|^{2\gamma}[|\Phi_1(w)|^2 + |\Phi_2(w)|^2] + O(|w|^{2\lambda}), \quad \lambda > \gamma.$$

and

$$X_u \wedge X_v = 2(\text{Im}(y_w z_{\overline{w}}), -\text{Im}(x_w z_{\overline{w}}), \text{Im}(x_w y_{\overline{w}}))$$

we find by means of (2) that

$$\lim_{w\to 0} N_1(w) = \lim_{w\to 0}\left[\text{const}\frac{|w|^{\gamma+\lambda}}{|w|^{2\gamma}}\right] = 0 \quad \text{since } \lambda > \gamma,$$

and

$$\lim_{w\to 0} N_2(w) = 0.$$

Finally

$$\lim_{w\to 0} N_3(w) = 2\lim_{w\to 0}\frac{\text{Im}(x_w(w)y_{\overline{w}}(w))}{|X_w(w)|^2} = 2\frac{\text{Im}(\Phi_1(0)\overline{\Phi_2(0)})}{|\Phi_1(0)|^2 + |\Phi_2(0)|^2} = \pm 1$$

since $\Phi_1(0) = \pm i\Phi_2(0)$, and Theorem 1 is proved. □

Proof of Theorem 2. If Γ^+ and Γ^- are straight lines, then the matrix T is the identity $\mathrm{Id}|_{\mathbb{R}}^3$, whence $g(w) = X_w(w)$ and $\frac{dG}{d\overline{w}}(w) = X_{w\overline{w}}(w) = 0$ almost everywhere in $B_\delta - \{0\}$. According to Theorem 1.15 in Vekua [1] (or Satz 1.17 in Vekua [2]) we see that G and hence F are holomorphic on $B_\delta(0)$. By the definition of F we obtain

$$\begin{aligned} X_w(w) &= G(w) = U \ \mathrm{Diag}[1, w^{\alpha-1}, w^{-\alpha}]F(w) \\ &= \frac{w^{\alpha-1}F_2(w)}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + \frac{w^{-\alpha}F_3(w)}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + F_1(w) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Putting $H_1 := F_1, H_2 := (\sqrt{2})^{-1}F_2$ and $H_3 := (\sqrt{2})^{-1}F_3$ we obtain representation (6). Finally (7) follows by integration, and (5) is a consequence of (18). Thus Theorem 2 is proved. □

3.4 An Asymptotic Expansion for Solutions of the Partially Free Boundary Problem

The aim of this section is to prove an analogue of Theorem 1 in Section 3.3 for minimal surfaces with partially free boundaries. Here the point of interest is the intersection point of the boundary arc Γ with the supporting surface S. Let us again start with an instructive *example*:

Let S be the coordinate plane $\{z = 0\}$ and

$$\Gamma = \{(x, y, z) \colon z = x \ \tan(\alpha\pi), y = 0, 0 \le x \le 1\},$$

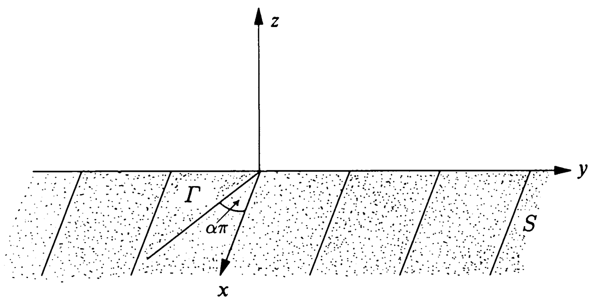

Fig. 1.

where $\alpha \in (0, \frac{1}{2})$. For each $k \in \mathbb{N} \cup \{0\}$ we consider the functions

$$f_1(w) = w^{\alpha+2k}, \quad f_2(w) = \overline{w}^{2-\alpha+2k},$$

$$f_3(w) = -w^{\alpha+1+2k}, \quad f_4(w) = -\overline{w}^{1-\alpha+2k},$$

and the associated minimal surfaces

$$X_j(u,v) = (x_j(u,v), y_j(u,v), z_j(u,v)), \quad j \in \{1,2,3,4\};$$

given by

$$\begin{aligned} &x_j(u,v) = \operatorname{Re} f_j(w), \quad y_j(u,v) = 0, \quad z_j(u,v) = \operatorname{Im} f_j(w), \\ &w \in B = \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1, v > 0\}, \quad w = u + iv. \end{aligned}$$

Then each $X_j, j = 1,2,3,4$, is a minimal surface which maps the interval $[-1,0]$ onto Γ and $[0,1]$ into S while $X(0,0) = 0$. Also X_j meets the surface S orthogonally along its trace $X_j|_{[0,1]}$, and hence it is a stationary solution of a free boundary problem determined by Γ and S. We shall prove that any minimal surface with a free boundary behaves near the corner point like one of the four solutions constructed above. More precisely, it will be shown that

$$X_w(w) = w^\gamma \Phi(w) \quad \text{as } w \to 0, \tag{1}$$

where $\gamma > -1$, and $\Phi(w) = (\Phi_1(w), \Phi_2(w), \Phi_3(w))$ denotes some Hölder continuous complex valued function with $\Phi_1(0) \neq 0, \Phi_3(0) \neq 0$, and $\Phi_2(0) = 0$ if $\alpha \neq \frac{1}{2}$. From the representation (1) we deduce that the surface normal tends to a limiting position as $w \to 0$. If in particular $\alpha \neq \frac{1}{2}$, then the tangent space of X at the corner $P \in \Gamma \cap S$ is spanned by the normal to S at P and the tangent to Γ at P. Thus the solution surface X must meet the point P at one of the angles $\alpha\pi, (2-\alpha)\pi, (1-\alpha)\pi$ and $(\alpha+1)\pi$ depending on whether X behaves like f_1, f_2, f_3, or f_4, respectively. In each of these cases X may penetrate S and can wrap P k-times.

Let us recall some notation. We define the sets I_δ^-, I_δ^+ as in Sections 3.2 and 3.3, and we formulate Assumption A similar as in Section 3.2:

Assumption A. *Let S be a regular surface of class C^3, and Γ be a regular arc of class $C^{2,\mu}$ which meets S in a common point P at an angle $\alpha\pi$ with $0 < \alpha \leq \frac{1}{2}$. We assume that P is the origin O, that the x,y-plane is tangent to S at O, and that the tangent vector to Γ at O lies in the x,z-plane. Moreover, let $X(u,v)$ be a minimal surface of class $C^{0,v}(\overline{B_\delta^+}, \mathbb{R}^3) \cap C^2(\overline{B_\delta^+} \setminus \{0\}, \delta > 0$, which satisfies the boundary conditions*

$$X \colon I_\delta^- \to \Gamma, \quad X \colon I_\delta^+ \to S, \quad X(0) = P. \tag{2}$$

We also suppose that X intersects S orthogonally along its free trace $X|_{I_\delta^+}$.

The main result of this section is

Theorem 1. *Suppose that Assumption A holds. Then there exists an $R > 0$ and a Hölder continuous function $\Phi(w) = (\Phi_1(w), \Phi_2(w), \Phi_3(w))$ defined on $\overline{B_R^+}$ such that*

$$X_w(w) = w^\gamma \Phi(w) \tag{3}$$

holds true on $\overline{B_R^+} \setminus \{0\}$ with either $\gamma = \alpha - 1 + m$ or $\gamma = -\alpha + m$ for some integer $m \geq 0$. Moreover, we have $\Phi_1(0), \Phi_2(0), i\Phi_3(0) \in \mathbb{R}$ and

$$\Phi_1(0) = \pm i\Phi_3(0) \neq 0, \quad \Phi_2(0) = 0 \quad \text{if } \alpha \neq \tfrac{1}{2}, \tag{4}$$

that is,

$$\Phi_1^2(0) + \Phi_2^2(0) + \Phi_3^2(0) = 0 \tag{5}$$

and at least two $\Phi_j(0) \neq 0$ if $\alpha = \frac{1}{2}$. The unit normal vector

$$N(w) = (N_1(w), N_2(w), N_3(w)) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(w)$$

satisfies

(6)

$$\begin{cases} \lim_{w\to 0} N(w) = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix} & \text{if } \alpha \neq \tfrac{1}{2}, \\ \lim_{w\to 0} N(w) = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}, & \text{if } \alpha = \tfrac{1}{2}, \text{ where } c_1, c_2 \in \mathbb{R} \text{ and } c_1^2 + c_2^2 = 1. \end{cases}$$

For the trace $X(u, 0), u \in \overline{I_R^+}$, we find

$$X(u, 0) = u^{\gamma+1}\psi(u) \tag{7}$$

with some Hölder continuous function ψ such that $\psi(0) = (\Phi_1(0), \Phi_2(0), 0)$. Furthermore, the oriented tangent vector $t(u) = \frac{X_u(u,0)}{|X_u(u,0)|}, u \in I_R^+$, satisfies

$$\lim_{w\to 0^+} t(u) = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{if } \alpha \neq \frac{1}{2}, \tag{8}$$

and

$$\lim_{u\to 0^+} t(u) = \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} \quad \text{if } \alpha = \frac{1}{2}, \text{ where } d_1^2 + d_2^2 = 1. \tag{9}$$

If, in addition, S is a plane and if Γ is a straight line segment, then there exist functions H_1, H_2, H_3, holomorphic on $B_R(0)$, such that

$$(10)\quad X_w(w) = w^{\alpha-1}H_1(w)\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} + w^{-\alpha}H_3(w)\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} + w^{-1/2}H_2(w)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

holds true on $\overline{B_R^+} \setminus \{0\}$ *and*

$$(11)\qquad H_2^2(w) + 4H_1(w)H_3(w) = 0 \quad on\ B_R(0).$$

Corollary 1. *If* $\alpha \neq \frac{1}{2}$, *then there exist some* $c \in \mathbb{C} \setminus \{0\}$ *and some integer* $k \geq 0$ *such that one of the following four expansions holds true:*

$$(12)\qquad (x+iz)(w) = \begin{cases} w^{\alpha+2k}[c+o(1)], & w \to 0, \\ \overline{w}^{2-\alpha+2k}[c+o(1)], & w \to 0, \\ w^{\alpha+1+2k}[c+o(1)], & w \to 0, \\ \overline{w}^{1-\alpha+2k}[c+o(1)], & w \to 0. \end{cases}$$

Moreover

$$|y(w)| = O(|w|^{\lambda+1}) \quad as\ w \to 0,\ for\ some\ \lambda > \gamma$$

where γ *is the exponent in the expansion* $(x+iz)(w) = w^{\gamma}[c+o(1)]$ *stated in* (12).

The *proof of Theorem* 1 consists in an adaptation of the method which was developed in Section 3.3 for the proof of the corresponding result, see Theorem 1 in Section 3.3. So from time to time our presentation will be sketchy and leave the details to the reader as an instructive exercise. We begin the proof of Theorem 1 with a description of a *reflection* and a *smoothing procedure*. To this end let us henceforth assume that S is locally described by

$$z = f(x,y), (x,y) \in B_\varepsilon(0) = \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 < \varepsilon\},$$

where $f \in C^3(B_\varepsilon(0), \mathbb{R})$, and $f(0,0) = 0, \nabla f(0,0) = 0$. Also, Γ may locally be described by two functions $h_1(t)$ and $h_2(t)$ of class $C^{2,\mu}([0,\varepsilon], \mathbb{R})$ such that $(h_1(t), h_2(t), t) \in \Gamma$ for $t \in [0,\varepsilon]$, and $h_1(0) = h_2(0) = h_2'(0) = 0$ while $h_1'(0) = \cot\,\alpha\pi$. Thus it follows that the unit tangent vector of Γ at zero is then given by $(\cos\alpha\pi, 0, \sin\alpha\pi)$. Because of the continuity of X we can select a number $R > 0$ such that

$$X(\overline{B_\delta^+}) \subset \mathcal{K}_\varepsilon(0) = \{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 < \varepsilon\}.$$

We define the unit vector $a(t), t \in [0,\varepsilon]$, by

$$a(t) := [h_1'(t)^2 + h_2'(t)^2 + 1]^{-1/2}\begin{pmatrix} h_1'(t) \\ h_2'(t) \\ 1 \end{pmatrix}$$

and the reflection across Γ by

$$R_\Gamma(t)Q := 2\langle a(t), Q\rangle a(t) - Q$$

for $Q \in \mathbb{R}^3, t \in [0, \varepsilon]$. Similarly, we define reflections across S by

$$R_S(x,y)Q := Q - 2\langle N_S(x,y), Q\rangle N_S(x,y),$$

for all $Q \in \mathbb{R}^3$ and $(x,y) \in B_\varepsilon(0) \subset \mathbb{R}^2$, where

$$N_S(x,y) = [1 + f_x^2(x,y) + f_y^2(x,y)]^{-1/2} \begin{pmatrix} -f_x(x,y) \\ -f_y(x,y) \\ 1 \end{pmatrix}$$

is the unit normal of S at the point $(x, y, f(x, y))$. Identifying the reflections R_Γ and R_S with their respective matrices $R_\Gamma(t)$ and $R_S(x,y)$, we may construct orthogonal matrices $O_\Gamma(t)$ and $O_S(x,y)$ with the properties[1]

$$R_\Gamma(t) = O_\Gamma(t)\,\mathrm{Diag}[-1,-1,1]O_\Gamma^*(t),$$

$$R_S(x,y) = O_S(x,y)\,\mathrm{Diag}[1,1,-1]O_S^*(x,y).$$

We put

$$T_\Gamma(t) := O_\Gamma(0)O_\Gamma^*(t)$$

and

$$T_S(x,y) := O_S(0,0)O_S^*(x,y).$$

Thus we have obtained matrices R_S and T_S which are of class $C^2(B_\varepsilon(0), \mathbb{R}^9)$, $B_\varepsilon(0) \subset \mathbb{R}^2$, while R_Γ and T_Γ are of class $C^{1,\mu}([0,\varepsilon], \mathbb{R}^9)$. If we extend $a(t), t \in [0,\varepsilon]$ by $\tilde{a}(t) = a(-t)$ for $t \in [-\varepsilon, 0]$ and call the extended functions again a, R_Γ and T_Γ, then also $a, R_\Gamma, T_\Gamma \in C^{1,\mu}([-\varepsilon,\varepsilon])$. Now let K_τ denote the cone with vertex 0 and opening angle τ whose axis is given by $x = z \cot \alpha\pi, z \geq 0, y = 0$. We assume that τ is so small that the vertex 0 is the only point of $K_{2\tau} \cap S$ in the ball $\mathcal{K}_\varepsilon(0)$. Next we choose a real valued differentiable function η defined on the punctured ball $\mathcal{K}_\varepsilon(0) \setminus \{0\} = \{0 < x^2 + y^2 + z^2 < \varepsilon\}$ which satisfies

$$\eta(x,y,z) = \begin{cases} 1 & \text{on } K_\tau \cap [\mathcal{K}_\varepsilon(0) \setminus \{0\}], \\ 0 & \text{on } \mathcal{K}_\varepsilon(0) \setminus \{0\} \setminus K_{2\tau}, \end{cases}$$

and

$$|\nabla\eta(x,y,z)| \leq \mathrm{const}[x^2 + y^2 + z^2]^{-1/2} \quad \text{on } \mathcal{K}_\varepsilon \setminus \{0\}.$$

We extend η (noncontinuously) by defining $\eta(0,0,0) = 0$, and denote by $T = T(x,y,z), (x,y,z) \in \mathcal{K}_\varepsilon(0)$, the matrix-valued function

$$T(x,y,z) := \eta(x,y,z)[T_\Gamma(z) - T_S(x,y)] + T_S(x,y).$$

[1] As the symbols t and T are used otherwise, we presently denote the transpose of a matrix A by A^*.

Then T is continuous at zero because $\lim_{(x,y,z)\to 0} T(x,y,z)$ exists and is equal to $\mathrm{Id}_{\mathbb{R}^3}$. In fact, T is even Lipschitz continuous on $\mathcal{K}_\varepsilon(0) \subset \mathbb{R}^3$ because of

$$|T_\Gamma(z) - T_S(x,y)| \le \mathrm{const}[x^2 + y^2 + z^2]^{1/2}$$

and hence $|\nabla T(x,y,z)|$ stays bounded as $(x,y,z) \to 0$. Defining

$$g(w) := T(X(w))X_w(w) \quad \text{for } w \in \overline{B_R^+} \setminus \{0\}$$

we then obtain

Lemma 1. *The function $g(w)$ is of class $C^{0,1}(\overline{B_R^+} \setminus \{0\}, \mathbb{C}^3)$ and has the following properties:*

$$(13) \qquad R_\Gamma(0)g(w) = \overline{g(w)} \quad \textit{for all } w \in I_R^-,$$

and

$$(14) \qquad R_S(0)g(w) = \overline{g(w)} \quad \textit{for all } w \in I_R^+,$$

where $R_S(0) := R_S(0,0)$.

Proof. The Lipschitz continuity of $g(w)$ is an immediate consequence of the Lipschitz continuity of T and of the regularity properties of X. Relation (13) follows similarly as equation (11) in Section 3.3 using the fact that $T(X(w)) = T_\Gamma(z(w))$ if $w \in I_R^-$. To prove (14), we let $w \in I_R^+$; then

$$X_u(w) = (x_u(w), y_u(w), f_x(x,y)x_u(w) + f_y(x,y)y_u(w))$$

and

$$\langle X_u(w), N_S(x(w), y(w))\rangle = 0.$$

From the transversality condition we infer that

$$X_v(w) = \langle X_v(w), N_S(x(w), y(w))\rangle N_S(x(w), y(w)),$$

for all $w \in I_R^+$ whence

$$R_S(x(w), y(w))X_u(w) = X_u(w),$$

and

$$R_S(x(w), y(w))X_v(w) = -X_v(w)$$

or equivalently

$$(15) \qquad R_S(x(w), y(w))X_w(w) = X_{\overline{w}}(w), \quad w \in I_R^+.$$

Now, using (15) and the definition of T, we obtain

$$\begin{aligned}
\overline{g(w)} &= T(X(w))X_{\overline{w}}(w) = T_S(x(w), y(w))X_{\overline{w}}(w) \\
&= T_S(x, y)R_S(x, y)X_w \\
&= O_S(0)O_S^*(x, y)O_S(x, y)\,\mathrm{Diag}[1, 1, -1]O_S^*(x, y)X_w \\
&= O_S(0)\,\mathrm{Diag}[1, 1, -1]O_S^*(0)O_S(0)O_S^*(x, y)X_w \\
&= R_S(0)T_S(x, y)X_w = R_S(0)T(X(w))X_w(w) \\
&= R_S(0)g(w),
\end{aligned}$$

where the argument of X, x, y is always $w \in I_R^+$. □

We now reflect $g(w)$ so as to obtain a function $G(w)$,

$$G(w) := \begin{cases} g(w) & \text{if } w \in \overline{B_R^+} \setminus \{0\}, \\ R_S(0)\overline{g(\overline{w})} & \text{if } \overline{w} \in B_R^+; \end{cases} \tag{16}$$

then $G \in C^{0,1}(B_R(0) \setminus \overline{I_R^-}, \mathbb{C}^3)$ and $\lim_{v \to 0^+} G(w) = R_\Gamma(0)R_S(0) \times \lim_{v \to 0^-} G(w)$ for all $w = (u, v)$ with $u \in I_R^-$. Furthermore, G satisfies

$$|G_{\overline{w}}(w)| \le c|G(w)|^2 \tag{17}$$

almost everywhere in B_R, and we infer from Proposition 1 in Section 3.2 that

$$|G(w)| \le c|w|^{\nu - 1}, \quad w \in B_R \setminus \{0\}, \tag{18}$$

with some constant c, where ν denotes the Hölder exponent of X.

Next we are going to smoothen the jump of G on the interval I_R^- by multiplication with a singular matrix function which is related to the eigenvalues of the matrix $R_\Gamma(0)R_S(0)$. It follows easily that

$$R_\Gamma(0)R_S(0) = \begin{pmatrix} \cos 2\pi\alpha & 0 & -\sin 2\pi\alpha \\ 0 & -1 & 0 \\ \sin 2\pi\alpha & 0 & \cos 2\pi\alpha \end{pmatrix}$$

and

$$R_\Gamma(0)R_S(0) = U\ \mathrm{Diag}[e^{i2\pi(\alpha-1)}, e^{-i\pi}, e^{-i2\pi\alpha}]U^*,$$

where U^* is the unitary matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & -i \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The smoothed function $F(w) = (F_1(w), F_2(w), F_3(w))$ is now defined by

$$F(w) := \mathrm{Diag}[w^{1-\alpha}, w^{1/2}, w^\alpha]U^*G(w), \quad \text{for all } w \in B_R(0) \setminus \{0\}. \tag{19}$$

Equation (19) is equivalent to

(20) $$X_w(w) = T(X(w))^{-1} U \,\mathrm{Diag}[w^{\alpha-1}, w^{-1/2}, w^{-\alpha}] F(w) \quad \text{for all } w \in B_R(0) \setminus \{0\}.$$

It is easily seen that F is continuous; in particular, we have

$$\lim_{v \to +0} F(u,v) = \lim_{v \to -0} F(u,v) \quad \text{for all } u \in I_R^-.$$

In fact we find

Lemma 2. *The function $F(w)$ is of class $C^{0,1}(B_R(0)) \setminus \{0\}, \mathbb{C}^3)$ and satisfies the relations*

(21) $$\begin{aligned} |F_1(w)| &= O(|w|^{\nu-\alpha}) \\ |F_2(w)| &= O(|w|^{\nu-1/2}) \quad \text{as } w \to 0 \\ |F_3(w)| &= O(|w|^{\nu-\beta}) \end{aligned}$$

and $\beta = 1 - \alpha$. Furthermore the following differential system holds almost everywhere on $B_R(0)$:

(22) $$\begin{aligned} |F_{1\overline{w}}| &\le c[|w|^{\alpha-1}|F_1|^2 + |w|^{1-3\alpha}|F_3|^2], \\ |F_{2\overline{w}}| &\le c[|w|^{(1/2)-2\beta}|F_1|^2 + |w|^{(1/2)-2\alpha}|F_3|^2], \\ |F_{3\overline{w}}| &\le c[|w|^{\alpha-2\beta}|F_1|^2 + |w|^{-\alpha}|F_3|^2], \end{aligned}$$

where we have dropped the argument w. Moreover, there exist complex-valued functions χ_1, χ_2, χ_3 which are Hölder continuous on $B_R(0)$ such that

(23) $$\begin{aligned} &F_2^2(w)\chi_1(w) + 2F_1(w)F_3(w)\chi_2(w) \\ &\quad = [w^{2\alpha-1}F_1^2(w) + w^{1-2\alpha}F_3^2(w)](1 - \chi_3(w)), \\ &\quad \text{and} \quad \chi_j(0) = 1 \quad \text{for } j = 1,2,3. \end{aligned}$$

Proof. Relations (21) follow from the definition of F and from (18). The Lipschitz continuity of F on the punctured disk is a consequence of the Lipschitz continuity of G and of the continuity of F at I_R^-. The conformality condition $\langle X_w, X_w \rangle = 0$, the definition of G and the relation $T(0) = \mathrm{Id}$ imply the existence of Hölder continuous functions $a_1(w), a_2(w), a_3(w)$ such that

$$a_1(w)G_1^2(w) + a_2(w)G_2^2(w) + a_3(w)G_3^2(w) = 0 \quad \text{in } B_R(0) \setminus \{0\},$$

and

$$a_1(0) = a_2(0) = a_3(0) = 1.$$

Then (23) follows with

$$\chi_1(w) = a_2(w), \quad \chi_2(w) = \tfrac{1}{2}(a_1(w) + a_3(w)),$$

$$\chi_3(w) = 1 + \tfrac{1}{2}(a_1(w) - a_2(w)).$$

From the definition of G we derive

$$|G(w)|^2 = |w|^{-2\beta}|F_1(w)|^2 + |w|^{-1}|F_2(w)|^2 + |w|^{-2\alpha}|F_3(w)|^2,$$

and inequality (17) together with (19) yields

$$\begin{aligned}
|F_{1\overline{w}}(w)| &\le c|w|^{1-\alpha}|G(w)|^2,\\
|F_{2\overline{w}}(w)| &\le c|w|^{1/2}|G(w)|^2,\\
|F_{3\overline{w}}(w)| &\le c|w|^{\alpha}|G(w)|^2,
\end{aligned}$$

whence

$$\begin{aligned}
|F_{1\overline{w}}(w)| &\le c[|w|^{-\beta}|F_1|^2 + |w|^{-\alpha}|F_2|^2 + |w|^{\beta-2\alpha}|F_3|^2],\\
|F_{2\overline{w}}(w)| &\le c[|w|^{(1/2)-2\beta}|F_1|^2 + |w|^{-1/2}|F_2|^2 + |w|^{(1/2)-2\alpha}|F_3|^2],\\
|F_{3\overline{w}}(w)| &\le c[|w|^{\alpha-2\beta}|F_1|^2 + |w|^{-\beta}|F_2|^2 + |w|^{-\alpha}|F_3|^2].
\end{aligned}$$

On the other hand, we deduce from (23) the inequality

$$\begin{aligned}
|F_2|^2 &\le c[|F_1||F_3| + |w|^{2\alpha-1}|F_1|^2 + |w|^{1-2\alpha}|F_3|^2]\\
&\le c[|w|^{2\alpha-1}|F_1|^2 + |w|^{1-2\alpha}|F_3|^2].
\end{aligned}$$

These inequalities imply system (22). □

Relations (21_1), (21_3) and (22_1), (22_3) are equivalent to (33) and (34) respectively stated in Section 3.1. Hence we infer from Theorem 3 in Section 3.1, similarly as in Lemma 3 of Section 3.3, the following

Lemma 3. *There exists a nonnegative integer m such that the functions $f_j^m(w) := w^{-m}F_j(w), j = 1, 2, 3$ either satisfy*
(i) $f_1^m(0) \ne 0, f_1^m \in C^{0,\mu}(B_R, \mathbb{C})$ *for all* $\mu < \min(1, m+\alpha)$, *and*

$$\begin{aligned}
|f_{1\overline{w}}^m(w)| &= O(|w|^{m-\beta})\\
|f_{2\overline{w}}^m(w)| &= O(|w|^{m+2\alpha-3/2}) \quad \text{as } w \to 0,\\
|f_{3\overline{w}}^m(w)| &= O(|w|^{m+3\alpha-2})
\end{aligned}$$

or
(ii) $f_1^m(0) = 0$ *and* $f_3^m(0) \ne 0, f_3^m \in C^{0,\mu}(B_R, \mathbb{C})$ *for every* $\mu < \min(1, m+\beta)$, *and*

$$\begin{aligned}
|f_{1\overline{w}}^m(w)| &= O(|w|^{m+\beta-2\alpha})\\
|f_{2\overline{w}}^m(w)| &= O(|w|^{m+1/2-2\alpha}) \quad \text{as } w \to 0,\\
|f_{3\overline{w}}^m(w)| &= O(|w|^{m-\alpha})
\end{aligned}$$

almost everywhere on B_R.

If $m \ge 1$, then in both cases

$$[f_2^m(0)]^2 + 2f_1^m(0)f_3^m(0) = 0. \tag{24}$$

Proof. This can be proved like the corresponding result, Lemma 3, in Section 3.3. □

Now we can continue with the *proof of Theorem* 1. Assume that case (i) of Lemma 3 holds true; then we put

$$\psi(w) := \frac{1}{\sqrt{2}} f_1^m(w) \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} w^{1-2\alpha} f_3^m(w) \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} + w^{1/2-\alpha} f_2^m(w) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\Phi(w) := T^{-1}(X(w))\psi(w), \quad w \in \overline{B_R^+}.$$

Now we claim that ψ is Hölder continuous in $\overline{B}_R^+$. On account of Lemma 3, (i) we first have $f_1^m \in C^{0,\mu}(B_R, \mathbb{C})$ for all $\mu < \min\{1, m+\alpha\}$. Then we distinguish two cases:

1.) $m \geq 1$. Then the functions $w^{1-2\alpha} f_3^m(w)$ and $w^{1/2-\alpha} f_2^m(w)$ are Hölder continuous. Indeed, we have for $w \to 0$:

$$(+) \qquad \begin{aligned} &|f_3^m(w)| = O(1), \quad \text{by construction;} \\ &|f_2^m(w)| \leq c\{|w|^{\alpha-1/2}|f_1^m(w)| + |w|^{1/2-\alpha}|f_3(w)|\} = O(|w|^{\alpha-1/2}). \end{aligned}$$

Here we have employed (23) and $\alpha \leq 1$. On account of Lemma 3, (i), we see that Lemma 6 in Section 3.1 yields the Hölder continuity of f_2^m and f_3^m, and therefore of ψ, in $\overline{B}_R^+$.

2.) $m = 0$. Now we use (21) instead of (+). By Lemma 6 in Section 3.1 and Lemma 3, (i), we see that $f_2^0 = F_2$ is Hölder continuous for $\alpha > 1/4$, and so is $f_3^0 = F_3$ for $\alpha > 1/3$.

If $\alpha \leq 1/3$, we consider the function $wF_3(w)$, which satisfies

$$|wF_3(w)| = O(|w|^{\nu+\alpha}), \quad |[wF_3(w)]_{\overline{w}}| = O(|w|^{3\alpha-1}) \quad \text{for } w \to 0.$$

Hence $wF_3(w)$ is Hölder continuous for any exponent $< 3\alpha$, and it vanishes for $w = 0$.

For arbitrary $w_1, w_2 \in \overline{B}_R^+ \setminus \{0\}$ and $0 < \epsilon \ll 1$ we estimate the expression $|w_1^{1-2\alpha}F_3(w_1) - w_2^{1-2\alpha}F_3(w_2)|$ as follows, using w.l.o.g. that $|w_1| \leq |w_2|$ whence $|w_2| \geq (1/2)|w_1 - w_2|$:

$$\begin{aligned} &|w_1^{1-2\alpha}F_3(w_1) - w_2^{1-2\alpha}F_3(w_2)| \\ &\quad \leq |w_1^{-2\alpha} - w_2^{-2\alpha}||w_1F_3(w_1)| + |w_2|^{-2\alpha}|w_1F_3(w_1) - w_2F_3(w_2)| \\ &\quad \leq c\{|w_1|^{-2\alpha}|w_2|^{-2\alpha}|w_1 - w_2|^{2\alpha}|w_1|^{3\alpha-\epsilon} + |w_2|^{-2\alpha}|w_1 - w_2|^{3\alpha-\epsilon}\} \\ &\quad \leq c|w_1 - w_2|^{\alpha-\epsilon}. \end{aligned}$$

Thus $w^{1-2\alpha}F_3(w)$ is Hölder continuous in $\overline{B}_R^+$, and similarly one shows the Hölder continuity of $w^{1/2-\alpha}F_2(w)$. Since T^{-1} is Lipschitz continuous, and $X(w)$ is Hölder continuous in $\overline{B}_R^+$, we see that $T^{-1}(X(w))$ is Hölder continuous in $\overline{B}_R^+$. This yields the Hölder continuity of $\Phi(w) = T^{-1}(X(w))\psi(w)$. On the other hand, it follows from definition (20) that

$$X_w(w) = w^{\alpha-1+m}\Phi(w), \quad w \in \overline{B_R^+} \setminus \{0\}. \tag{25}$$

Because of $T(0) = \mathrm{Id}$, we obtain for $\alpha < \frac{1}{2}$ the relations

$$\begin{gathered} \Phi_1(0) = \frac{1}{\sqrt{2}} if_1^m(0), \quad \Phi_3(0) = \frac{1}{\sqrt{2}} f_1^m(0) \neq 0 \\ |\Phi_2(w)| = O(|w|^{\mu'}) \quad \text{for some } \mu' > 0. \end{gathered} \tag{26}$$

Then (25) yields

$$|y_w(w)| = O(|w|^{\lambda'}) \quad \text{for some } \lambda' > \alpha - 1 + m,$$

and we also have $\Psi_2(0) = 0$, which is sufficient for the proof of the theorem. If the second alternative of Lemma 3 holds true, we consider instead of ψ the function $\tilde{\psi}$ given by

$$\tilde{\psi}(w) := \frac{1}{\sqrt{2}} f_3^m(w) \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} w^{2\alpha-1} f_1^m(w) \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} + w^{\alpha-1/2} f_2^m(w) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\tilde{\Phi}(w) := T^{-1}(X(w))\tilde{\psi}(w).$$

Then $\tilde{\Phi}(w)$ is Hölder continuous and we have

$$X_w(w) = w^{-\alpha+m}\tilde{\Phi}(w), \quad w \in \overline{B_R^+}(0) \setminus \{0\}, \tag{27}$$

which together with (25) proves (3) of Theorem 1. Also we find for $\alpha < \frac{1}{2}$ that

$$\Phi_1(0) = \frac{-i}{\sqrt{2}} f_3^m(0) \neq 0, \quad \Phi_3(0) = \frac{1}{\sqrt{2}} f_3^m(0) \neq 0, \quad \Phi_2(0) = 0,$$

since $(w^{2\alpha-1}f_1^m)(0) = 0$ and $(w^{\alpha-1/2}f_2^m)(0) = 0$. The last relation follows because of $f_1^m(0) = 0$, relation (24) if $m \geq 1$ or (23) for $m = 0$, Lemma 3 (ii) and Lemma 6 of Section 3.1.

If $\alpha = \frac{1}{2}$, we obtain relation (3) with

$$\Phi(w) = T^{-1}(X(w)) \left[\frac{1}{\sqrt{2}} f_3^m(w) \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} f_1^m(w) \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} + f_2^m(w) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

and

$$\begin{aligned}\Phi_1^2(0)+\Phi_2^2(0)+\Phi_3^2(0) &= -\tfrac{1}{2}[f_1^2(0)-2f_1(0)f_3(0)+f_3^2(0)]\\ &\quad +f_2^2(0)+\tfrac{1}{2}[f_1^2(0)+2f_1(0)f_3(0)+f_3^2(0)]\\ &= 2f_1(0)f_3(0)+f_2^2(0)=0\end{aligned}$$

by (23) and (24).

Then the unit normal of $X(w)$ given by $N(w)=\frac{X_u\wedge X_v}{|X_u\wedge X_v|}(w)$ satisfies by virtue of (25) or (27) and because of

$$(X_u\wedge X_v)(w)=2(\mathrm{Im}(y_w z_{\overline{w}}),-\mathrm{Im}(x_w z_{\overline{w}}),\mathrm{Im}(x_w y_{\overline{w}}))$$

the relation

$$\lim_{w\to 0} N(w)=2[|\Phi_1(0)|^2+|\Phi_2(0)|^2+|\Phi_3(0)|^2]^{-1}\begin{pmatrix}\mathrm{Im}(\Phi_2(0)\overline{\Phi_3}(0))\\ \mathrm{Im}(\Phi_3(0)\overline{\Phi_1}(0))\\ \mathrm{Im}(\Phi_1(0)\overline{\Phi_2}(0))\end{pmatrix}.$$

But now relation (15) implies $R_S(0)\Phi(0)=\overline{\Phi(0)}$, and this means that

$$\mathrm{Im}\,\Phi_1(0)=0,\quad \mathrm{Im}\,\Phi_2(0)=0,\quad \text{and}\quad \mathrm{Re}\,\Phi_3(0)=0.$$

Also, if $\alpha<\frac{1}{2}$, then $\Phi_2(0)=0$, and we arrive at

$$N_1(0)=0,\quad N_3(0)=0,\quad N_2(0)=\pm 1,$$

whereas, if $\alpha=\frac{1}{2}$, we conclude that

$$N_j(0)=\pm\,\mathrm{Re}\,\Phi_j(0)[(\mathrm{Re}\,\Phi_1(0))^2+(\mathrm{Re}\,\Phi_2(0))^2]^{-1/2},\quad j=1,2,$$

and

$$N_3(0)=0.$$

Finally, we obtain for the tangent vector $t(u)=\frac{X_u(u,0)}{|X_u(u,0)|}$, $u>0$, the asymptotic behaviour

$$\lim_{u\to 0+} t(u)=[(\mathrm{Re}\,\Phi_1(0))^2+(\mathrm{Re}\,\Phi_2(0))^2]^{-1/2}\begin{pmatrix}\mathrm{Re}\,\Phi_1(0)\\ \mathrm{Re}\,\Phi_2(0)\\ 0\end{pmatrix},$$

which proves the relations (8) and (9).

If S is a plane and Γ is a straight line, then $T=\mathrm{Id}_{\mathbb{R}^3}$ and $g=X_w$. Hence G is holomorphic on $B_R\setminus\{0\}$ and F is holomorphic on B_R. Finally (10) and (11) follows from (20) if we take

$$H_1:=\frac{1}{\sqrt{2}}F_1,\quad H_2:=F_2,\quad H_3:=\frac{1}{\sqrt{2}}F_3,$$

and Theorem 1 is proved. □

3.5 Scholia

3.5.1 References

The basic idea of this chapter, the Hartman–Wintner method, was described and developed in the paper [1] of Hartman and Wintner in 1953. Its relevance for the theory of nonlinear elliptic systems with two independent variables was emphasized by E. Heinz. In particular, he discovered the use of this method for obtaining asymptotic expansions of minimal surfaces at boundary branch points, and of H-surfaces at branch points in the interior and at the boundary.

The results of Sections 3.2–3.4 concerning minimal surfaces with non-smooth boundaries are due to Dziuk (cf. his papers [1–4]). His work is based on methods by Vekua [1,2], Heinz [5], and Jäger [1–3].

Earlier results on the behaviour of minimal surfaces at a corner were derived by H.A. Schwarz [3] and Beeson [1]. The boundary behaviour of conformal mappings at corners was first treated by Lichtenstein, and then by Warschawski [4]. The continuity of minimal surfaces in Riemannian manifolds at piecewise smooth boundaries was investigated by Jost [12].

The proofs in the paper [1] of Marx based on joint work of Marx and Shiffman concerning minimal surfaces with polygonal boundaries are somewhat sketchy and contain several large gaps. Heinz [19–24] was able to fill these gaps and to develop an interesting theory of quasi-minimal surfaces bounded by polygons, thereby generalizing classical work of Fuchs and Schlesinger on linear differential equations in complex domains that have singularities (see Schlesinger [1]). A survey of Heinz's work can be found in the Scholia of Chapter 6 of Vol. 1.

In this context we also mention the work of Sauvigny [3–6]. The papers of Garnier are also essentially concerned with minimal surfaces having polygonal boundaries, but apparently these results were rarely studied in detail and did not have much influence on the further progress. This might be both unjustified and unfortunate, see the recent thesis by L. Desideri.

3.5.2 Hölder Continuity at Intersection Points

In Theorem 1 of Section 3.4 we have derived asymptotic expansions for $X_w(w)$ and $N(w)$ at the points $w_0 = \pm 1$ if $X : B \to \mathbb{R}^3$ is a minimal surface of class $\mathcal{C}(\Gamma, S)$ with the parameter domain $B = \{w = u + iv : |w| \le 1, v > 0\}$ that is bounded by $I = \{(u, 0) : |u| < 1\}$ and $C = \{w : |w| = 1, v \ge 0\}$, and $w_0 = \pm 1$ are mapped onto the two points P_1, P_2 where the arc Γ meets the surface S. The basic assumption (cf. Assumption A) was that X is Hölder continuous on $\overline{B}$. Recently, F. Müller [4] has proved that Hölder continuity of X on $\overline{B}$ follows from the much weaker assumption that X merely be continuous on $\overline{B}$. His reasoning even applies to continuous solutions X of

$$|\varDelta X| \le a|\nabla X|^2, \tag{1}$$

satisfying also

$$|X_u|^2 = |X_v|^2, \quad \langle X_u \cdot X_v \rangle = 0, \tag{2}$$

i.e. to H-surfaces with $\sup |H| \le \text{const}$.

Let us choose the corner $w_0 = 1$ of B, and consider the 3-gon $\Omega_\delta := B \cap B_\delta(1)$ as well as the arcs $I_\delta := I \cap \partial\Omega_\delta$ and $C_\delta := C \cap \partial\Omega_\delta$. We assume that both Γ and S are of class C^3, and that Γ meets S in P_1, P_2 only. We fix $P := P_2$ which is assumed to correspond to the corner $w_0 = 1$, i.e. $X(1) = P$. Then F. Müller's result reads as follows:

Theorem 1. *Suppose that*

$$X \in C^0(\overline{\Omega}_\delta, \mathbb{R}^3) \cap H_2^1(\Omega_\delta, \mathbb{R}^3) \cap C^2(\overline{\Omega}_\delta \setminus \{1\}, \mathbb{R}^3)$$

satisfies (1) *and* (2) *in* Ω_δ *as well as*

$$X(w) \in \Gamma \quad \textit{for } w \in C_\delta, \quad X(1) = P, \tag{3}$$

$$X(w) \in S \quad \textit{and} \quad X_v(w) \perp T_{X(w)}S \quad \textit{for } w \in I_\delta. \tag{4}$$

Then we obtain $X \in C^{0,\mu}(\overline{\Omega}_{\delta'})$ *for some* $\mu \in (0,1)$ *and some* $\delta' \in (0,\delta)$.

Sketch of the Proof. 1. Let us introduce local coordinates $y = (y^1, y^2, y^3)$ about P in the same way as in Section 2.7 such that 0 corresponds to P. Suppose that x and y are related by a C^2-diffeomorphism $y \mapsto x = h(y)$ from the ball $\mathcal{K}_r(0) := \{y \in \mathbb{R}^3 : |y| < r\}$ onto a neighbourhood U of P such that $h^{-1}(S \cap U) = \mathcal{K}_r(0) \cap \{y^3 = 0\} = B_r(0) \times \{0\}$ and

$$g_{jk}(y^1, y^2, 0) = \operatorname{diag}(\mathcal{E}(y^1, y^2), \mathcal{E}(y^1, y^2), 1) \quad \text{for } (y^1, y^2) \in B_r(0), \tag{5}$$

as well as $g_{13} = g_{31} = g_{23} = g_{32} = 0$ and $g_{33} = 1$ in $\mathcal{K}_r(0)$,

$$m|\xi|^2 \le g_{jk}(y)\xi^j\xi^k \le m^{-1}|\xi|^2 \quad \text{for } y \in \mathcal{K}_r(0), \ \ \xi \in \mathbb{R}^3, \tag{6}$$

$$\left| \frac{\partial g_{jk}}{\partial y^\ell}(y) \right| \le M \quad \text{for } y \in \mathcal{K}_r(0). \tag{7}$$

2. Then there is an $\epsilon \in [0,\delta]$ such that $X(\overline{\Omega}_\epsilon) \subset U$. We may assume that $\epsilon = \delta$. Then $Y := h^{-1}(X)$ lies in the same class as X and satisfies

$$\begin{aligned} &|\Delta Y| \le b|\nabla Y|^2 \quad \text{in } \Omega_\epsilon \ \text{ for some } b \in \mathbb{R}, b < 0, \\ &g_{jk}(y) y_w^j y_w^k = 0 \quad \text{in } \Omega_\epsilon, \\ &y(w) \in \Gamma^* := h^{-1}(\Gamma \cap U) \quad \text{for } w \in C_\epsilon, \\ &y_v^1(w) = 0, \quad y_v^2(w) = 0, \quad y^3(w) = 0 \quad \text{for } w \in I_\epsilon. \end{aligned} \tag{8}$$

We can assume that $\Gamma^* \setminus \{0\} \subset \mathcal{H}^+ := \{y^3 > 0\}$. Set

$$\tilde{Y}(w) = (\tilde{y}^1(w), \tilde{y}^2(w), \tilde{y}^3(w)) := \begin{cases} Y(w) & \\ & \text{for} \\ (y^1(w), y^2(w), -y^3(w)) & \end{cases} \begin{matrix} w \in \overline{\Omega}_\epsilon, \\ \\ w \in \overline{\Omega}^*_\epsilon, \end{matrix}$$

where $\Omega^*_\epsilon := \{w \in \mathbb{C} : \overline{w} \in \Omega_\epsilon\}$.

Let $\tau : B \to \tilde{\Omega}_\epsilon := \Omega_\epsilon \cup I_\epsilon \cup \Omega^*_\epsilon$ be a conformal mapping of the unit disk B onto $\tilde{\Omega}_\epsilon$, and set $Z := \tilde{Y} \circ \tau$ and

$$\gamma_{jk} := \tilde{g}_{jk} \circ \tau \quad \text{with } \tilde{g}_{jk}(w) := \begin{cases} g_{jk}(Y(w)) & \\ & \text{for} \\ g_{jk}(Y(\overline{w})) & \end{cases} \begin{matrix} w \in \overline{\Omega}_\epsilon, \\ \\ w \in \overline{\Omega}^*_\epsilon. \end{matrix}$$

Furthermore, let

$$\Gamma^+ := \Gamma^*, \quad \Gamma^- := \{(z^1, z^2, z^3) \in \mathbb{R}^3 : (z^1, z^2, -z^3) \in \Gamma^+\}.$$

Then for some $\rho \in (0,1)$, $S_\rho(0) := B \cap B_\rho(0)$, and $I^+_\rho := [0, \rho)$, $I^-_\rho := (-\rho, 0]$ and a proper choice of τ we obtain for $Z|_{S_\rho(0)}$, which is again denoted by Z, the following relations, by employing the special form of the g_{jk}:

$$(9) \qquad \begin{aligned} &|\Delta Z| \le b|\nabla Z|^2 \quad \text{in } S_\rho(0), \\ &\gamma_{jk} z^j_w z^k_w = 0 \quad \text{in } S_\rho(0), \\ &Z(w) \in \Gamma^+ \quad \text{for } w \in I^+_\rho, \quad Z(w) \in \Gamma^- \quad \text{for } w \in I^-_\rho. \end{aligned}$$

By a suitable change of the z-coordinates we can arrange for

$$\Gamma^\pm = \{(z^1, z^2, z^3) \in \mathbb{R}^3 : z^j = h^j_\pm(z^1), 0 \le \pm z^3 \le \epsilon_0, j = 1, 2\}$$

with some $\epsilon_0 > 0$ and $h^j_- \in C^2([-\epsilon_0, 0]), h^j_+ \in C^2([0, \epsilon_0])$,

$$(10) \qquad (h^1_\pm)'(0) = \pm\text{cotg}\left(\frac{\alpha\pi}{2}\right), \quad (h^2_\pm)'(0) = 0, \quad \alpha \in [0,1].$$

Set

$$h^j(t) := \left\{ \begin{matrix} h^j_-(t) & & -\epsilon_0 \le t_0 \le 0 \\ & \text{for} & \\ h^j_+(t) & & 0 \le t_0 \le \epsilon_0 \end{matrix} \right\}, \quad j = 1, 2.$$

For $0 < \rho \ll 1$ we define $\zeta = (\zeta^1, \zeta^2)$ by

$$(11) \qquad \zeta^j = z^j - h^j(z^3), \quad j = 1, 2.$$

Then $\zeta \in C^{0,1}(\overline{S}_\rho(0) \setminus \{0\}, \mathbb{R}^2) \cap C^0(\overline{S}_\rho(0), \mathbb{R}^2) \cap H^1_2(S_\rho(0), \mathbb{R}^2)$ satisfies

$$\int_{S_\rho(0)} \sum_{j=1}^{2} \nabla\zeta^j \nabla\varphi^j \, du\, dv = \int_{S_\rho(0)} \sum_{j=1}^{2} [g^j \nabla\varphi^j + f^j \varphi^j] \, du\, dv \tag{12}$$
$$\text{for all } \varphi = (\varphi^1, \varphi^2) \in C_c^\infty(S_\rho(0), \mathbb{R}^2),$$

where we have set for $j = 1, 2$:

$$f^j := -\Delta z^j \in L_1(S_\rho(0)), \quad g^j := -(h^j)'(z^2)\nabla z^3 \in L_2(S_\rho(0), \mathbb{R}^2). \tag{13}$$

Claim. *For* $0 < \rho \ll 1$ *we have*

$$|g^1 \nabla\zeta^1| \le a_1 |\nabla\zeta^1|^2 + b_1 |\nabla\zeta^2|^2, \tag{14}$$

$$|g^1| \le a_2 |\nabla\zeta^1| + b_2 |\nabla\zeta^2|, \tag{15}$$

$$|g^2| \le a_3(\rho) |\nabla\zeta|, \tag{16}$$

$$|f^j| \le b_3 |\nabla\zeta|^2 \quad \text{for } j = 1, 2, \tag{17}$$

with positive constants $a_1, a_2 \in [0, 1), b_1, b_2, b_3$, *and a function* $a_3(t) \to +0$ *as* $t \to +0$.

Suppose that the claim is proved. Using the boundary condition

$$\zeta(w) = 0 \quad \text{for } w \in I_\rho := \{w = u \in \mathbb{R} : |u| \le \rho\} \tag{18}$$

we extend ζ to a continuous function $\tilde\zeta$ on $\overline{B}_\rho(0)$by setting

$$\tilde\zeta(w) := \begin{cases} \zeta(w) & \text{for } w \in S_\rho(0), \\ -\zeta(\overline{w}) & \text{for } \overline{w} \in S_\rho(0). \end{cases}$$

Furthermore, we have $\tilde\zeta \in H_2^1(S_\rho(0), \mathbb{R}^2)$. In addition, we reflect $g_1^j = -(h^j)'(z^3) z_u^3$ and f^j in an odd way and $g_2^j = -(h^j)'(z^3) z_v^3$ evenly across I_ρ, obtaining $\tilde g_1^j, \tilde f^j, \tilde g_2^j$. Then it follows

$$\int_{B_\rho(0)} \sum_{j=1}^{2} \nabla\tilde\zeta^j \nabla\varphi^j \, du\, dv = \int_{B_\rho(0)} \sum_{j=1}^{2} [\tilde g^j \nabla\varphi^j + \tilde f^j \varphi^j] \, du\, dv \tag{19}$$
$$\text{for all } \varphi \in \overset{\circ}{H}{}_2^1 (B_\rho(0), \mathbb{R}^2) \cap L_\infty(B_\rho(0), \mathbb{R}^2).$$

One checks that $\tilde f^j$ and $\tilde g^j$ satisfy growth conditions analogous to (14)–(17) where the ζ^j are to be replaced by $\tilde\zeta^j$, whereas $a_1, a_2, a_3(\rho), b_1, b_2, b_3$ remain the same. Now one can apply a procedure due to Dziuk [1] (cf. the proof of Satz 1 in [1]) to show that $\tilde\zeta$ satisfies a “Dirichlet growth condition” on some disk $\overline{B}_{\rho'}(0)$ with $0 < \rho' \ll 1$, and the same holds for ζ on $\overline{S}_{\rho'}(0)$. From (11) one infers that also z^1 and z^2 satisfy such a condition on $\overline{S}_{\rho'}(0)$, using also (15), (16), and

$$|z_w^j| \overset{(11)}{\leq} |\zeta_w^j| + |(h^j)' z_w^3| \overset{(15),(16)}{\leq} c|\zeta_w|, \quad j = 1, 2,$$

and

$$\gamma_{jk} z_w^j z_w^k = 0 \tag{20}$$

implies that

$$|\nabla z^3|^2 \leq \text{const}(|\nabla z^1|^2 + |\nabla z^2|^2) \quad \text{on } \overline{S}_{\rho'}(0) \quad \text{for } 0 < \rho' \ll 1.$$

Consequently, $Z = (z^1, z^2, z^3)$ satisfies a Dirichlet growth condition on $\overline{S}_{\rho'}(0)$, and therefore Z is Hölder continuous on $\overline{S}_{\rho'}(0)$. Since $\tilde{Y} = Z \circ \tau^{-1}$, it follows that $\tilde{Y}$ is Hölder continuous on the closure of $\tilde{\Omega}_{\epsilon'}$ for $0 < \epsilon' \ll 1$, and Y is Hölder continuous on $\overline{\Omega}_{\epsilon'}$ for $0 < \epsilon' \ll 1$. Since $X = h(Y)$, we finally conclude that $X \in C^{0,\mu}(\overline{\Omega}_{\delta'})$ for some $\mu \in (0, 1)$ and some $\delta' \in (0, \delta)$.

It remains to prove the Claim. We begin with (15). From (20) and the special structure of the g_{jk}, and therefore of the γ_{jk}, it follows that

$$-(z_w^3)^2 - \gamma_{11}(z_w^1)^2 = 2\gamma_{12} z_w^1 z_w^2 + \gamma_{22}(z_w^2)^2 \quad \text{in } S_\rho(0).$$

Inserting $z_w^1 = \zeta_w^1 + (h^1)'(z^3) z_w^3$ into the left-hand side we find

$$-\gamma(z_w^3 - \xi^1 \zeta_w^1)(z_w^3 - \xi_2 \zeta_w^1) = 2\gamma_{12} z_w^1 z_w^2 + \gamma_{22}(z_w^2)^2 \quad \text{in } S_\rho(0) \tag{21}$$

with

$$\xi_{1,2} := -\gamma^{-1}[\gamma_{11}(h^1)'(z^3) \pm i\sqrt{\gamma_{11}}],$$

$$\gamma := 1 + \gamma_{11}[(h^1)'(z^3)]^2.$$

We have

$$|\xi_1| = |\xi_2| = \left\{ \frac{\tilde{\gamma_{11}}}{1 + \tilde{\gamma}_{11}[(h^1)'(z^3)]^2} \right\}^{\frac{1}{2}} \quad \text{in } S_\rho(0). \tag{22}$$

If $|z_w^3| \leq |\xi_1||\zeta_w^1|$, we find

$$|g^1| \leq 2|(h^1)'(z^3)||\xi_1||\zeta_w^1| \leq a_2 |\nabla \zeta^1|,$$

and this is (15) with $a_2 < 1$ and $b_2 = 0$.

Otherwise we infer from (21)

$$\gamma(|z_w^3| - |\xi_1||\zeta_w^1|) \leq |2\gamma_{12} z_w^1 z_w^2 + \gamma_{22}(z_w^2)^2|;$$

thus,

$$\sqrt{\gamma}|z_w^3| \le \left\{ \sqrt{|\gamma_{12}|}|(h^1)'(z^3)| + \sqrt{\gamma_{22} + |\gamma_{12}|}(h^2)'(z^3) \right\} |z_w^3| + \left\{ \sqrt{|\gamma_{12}|} + \sqrt{\gamma}|\xi_1| \right\} |\zeta_w^1| + \sqrt{\gamma_{22} + |\gamma_{12}|}|\zeta_w^2|.$$

Furthermore there is a function $c(t)$ with $c(t) \to +0$ as $t \to 0$ such that

$$|\tilde{\gamma}_{12}| + |(h^2)'(z^3)| \le c(\rho) \quad \text{on } S_\rho(0), \tag{23}$$

due to (5) and (10). Thus we have for $0 < \rho \ll 1$ that

$$|z_w^3| \le [1 - \tilde{c}(\rho)]^{-1} \left\{ \left[\left(\frac{|\gamma_{12}|}{\gamma} \right)^{\frac{1}{2}} + |\xi_1| \right] |\zeta_w^1| + \left[\left(\frac{\gamma_{22} + |\gamma_{12}|}{\gamma} \right)^{\frac{1}{2}} |\zeta_w^2| \right] \right\} \quad \text{in } S_\rho(0) \tag{24}$$

with $\tilde{c}(\rho) \to +0$ as $\rho \to 0$.

Using (22) and again (23), we obtain for $0 < \rho \ll 1$ that

$$|g^1| \le 2|(h^1)'(z^3)||z_w^3| \le a_2|\nabla\zeta^1| + b_2|\nabla\zeta^2| \quad \text{in } S_\rho(0)$$

with $a_2 \in (0,1)$ and $b_2 > 0$, as claimed in (15).

The estimate (14) follows easily from (15), and (16) and (17) are derived from (10) and (24). Thus we have verified the "Dziuk estimates" of the Claim, and the proof of the theorem is complete. □

3.5.3

We also note that Dziuk [1] has proved Hölder continuity of a minimal surface $X \in \mathcal{C}(\Gamma)$ at a corner of the boundary contour Γ, assuming only $X \in C^0(\overline{B}, \mathbb{R}^3)$. This is relevant for Theorem 1 in Section 3.3 where we have assumed that $X \in C^{0,\mu}(\overline{B}_\delta(0), \mathbb{R}^3)$, which in Chapter 2 was only proved for minimizers.