Chapter 2

The Boundary Behaviour of Minimal Surfaces

In this chapter we deal with the boundary behaviour of minimal surfaces, with particular emphasis on the behaviour of stationary surfaces at their free boundaries. This and the following chapter will be the most technical and least geometric parts of our lectures. They can be viewed as a section of the regularity theory for nonlinear elliptic systems of partial differential equations. Yet these results are crucial for a rigorous treatment of many geometrical questions, and thus they will again illustrate what role the study of partial differential equations plays in differential geometry.

The first part of this chapter, comprising Sections 2.1–2.3, deals with the boundary behaviour of minimal surfaces at a fixed boundary. Consider for example a minimal surface $X: B \to \mathbb{R}^3$ which is continuous on $\bar{B}$ and maps ∂B onto some closed Jordan curve Γ. Then we shall prove that X is as smooth on $\bar{B}$ as Γ, more precisely, that X is of class $C^\infty(\bar{B}, \mathbb{R}^3)$ (or $X \in C^\omega(\bar{B}, \mathbb{R}^3)$, or $X \in C^{m,\alpha}(\bar{B}, \mathbb{R}^3)$) if Γ is of class C^∞ (or $\Gamma \in C^\omega$, or $\Gamma \in C^{m,\alpha}$, respectively). These results are worked out in Section 2.3. In Section 2.1 we shall supply some results from potential theory that will be needed, and in Section 2.2 we shall derive various regularity results and estimates for vector-valued solutions X of differential inequalities of the kind

$$|\Delta X| \le a|\nabla X|^2$$

which will be crucial for our considerations in Section 2.3.

The central part of this chapter consists of Sections 2.4–2.9 where we prove analogous regularity results for minimal surfaces with free boundaries on a support surface S. If the boundary ∂S of S is empty, the reasoning is considerably simpler than for $\partial S \neq \emptyset$; in fact this second case has to be viewed as a Signorini problem (or else, as a thin obstacle problem). For a survey of the results on the boundary behaviour of minimal surfaces with free boundaries we refer the reader to Section 2.4.

Finally, in Section 2.10, we shall derive an asymptotic expansion for any minimal surface at a boundary branch point which is analogous to the expan-

U. Dierkes, S. Hildebrandt, A.J. Tromba, *Regularity of Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 340,
DOI 10.1007/978-3-642-11700-8_2,

sion at an interior branch point that was obtained in Section 3.2 of Vol. 1. The results of Section 2.10 are based on the discussion in Chapter 3.

2.1 Potential-Theoretic Preparations

In this section we want to supply some results from potential theory which will be needed in Section 2.3 for investigating the boundary behaviour of minimal surfaces which are bounded by smooth Jordan arcs. The reader who is well acquainted with Schauder estimates may skip this part at a first reading. Although a large part of the material can be found in the treatise of Gilbarg and Trudinger [1], a brief presentation may be welcome because it will enable the reader to study the essential results of Section 2.2 on solutions of differential inequalities without consulting additional sources.

In what follows we shall use the following notation: We write $w = u + iv$, $\zeta = \xi + i\eta$, $d\zeta = d\xi + i\,d\eta$, and $d^2\zeta = d\xi\,d\eta$ denotes the two-dimensional area element. Moreover, we set

$$\begin{aligned}
\frac{\partial}{\partial w} &= \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right), \\
\Delta &= \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4\frac{\partial}{\partial w}\frac{\partial}{\partial \overline{w}}, \\
B_R &= B_R(0) = \{w \in \mathbb{C}\colon |w| < R\}, \quad B := B_1(0).
\end{aligned}$$

Green's function $G_R(w,\zeta)$ for the disk B_R is given by

(1) $$G_R(w,\zeta) = \frac{1}{2\pi}\log\left|\frac{R^2 - \overline{w}\zeta}{R(\zeta - w)}\right|,$$

and the Poisson kernel $\mathcal{P}_R(w,\varphi) = \mathcal{P}^*_R(w,\zeta), w = re^{i\theta}, \zeta = \mathrm{Re}^{i\varphi} \in \partial B_R$, is defined[1] by

(2) $$\begin{aligned}
\mathcal{P}_R(w,\varphi) &= \frac{1}{2\pi}\frac{R^2 - r^2}{R^2 - 2rR\ \cos(\theta - \varphi) + r^2} = \frac{1}{2\pi}\ \mathrm{Re}\frac{R + re^{i(\theta-\varphi)}}{R - re^{i(\theta-\varphi)}} \\
&= \frac{1}{2\pi}\ \mathrm{Re}\frac{\zeta + w}{\zeta - w} = \frac{1}{2\pi}\frac{R^2 - |w|^2}{|\zeta - w|^2} = -R\frac{\partial}{\partial \nu_\zeta}G_R(w,\zeta),
\end{aligned}$$

where ν_ζ denotes the exterior normal to ∂B_R at ζ. One computes that

(3) $$\frac{\partial}{\partial w}G_R(w,\zeta) = \frac{1}{4\pi}\left(\frac{1}{\zeta - w} - \frac{\overline{\zeta}}{R^2 - w\overline{\zeta}}\right),$$

[1] Note that often the expression $\frac{1}{R}\mathcal{P}_R(w,\varphi) = -\frac{\partial}{\partial\nu_\zeta}G_R(w,\zeta)$ is called *Poisson kernel*; cf. for instance Gilbarg and Trudinger [1], formula (2.29).

whence it follows that

$$\frac{\partial^s}{\partial w^s} G_R(w,\zeta) = \frac{(s-1)!}{4\pi}\left[\frac{1}{(\zeta-w)^s} - \frac{\overline{\zeta}^s}{(R^2-w\overline{\zeta})^s}\right]. \tag{4}$$

A straight-forward estimation shows that

$$R|\zeta - w| \le |R^2 - w\overline{\zeta}| \quad \text{for all } w,\zeta \in \overline{B}_R,$$

which implies

$$\left|\frac{\partial^s}{\partial w^s} G_R(w,\zeta)\right| \le \frac{(s-1)!}{2\pi}\frac{1}{|\zeta-w|^s} \quad \text{for all } \zeta, w \in \overline{B}_R \text{ with } w \ne \zeta. \tag{5}$$

The following results is a direct consequence of Green's formula and can be found in any textbook on partial differential equations.[2]

Proposition 1. *Any function $x \in C^0(\overline{B}_R) \cap C^2(B_R)$ with $q := \Delta x \in L_\infty(B_R)$ and $\boldsymbol{x}(\varphi) := x(Re^{i\varphi})$ can be written in the form*

$$x(w) = h(w) - \int_{B_R} G_R(w,\zeta) q(\zeta)\, d^2\zeta, \tag{6}$$

where

$$h(w) := \int_0^{2\pi} \mathcal{P}_R(w,\varphi)\boldsymbol{x}(\varphi)\, d\varphi \tag{7}$$

denotes the harmonic function in B_R which is continuous on $\overline{B}_R$ and satisfies $h = x$ on ∂B_R.

Proposition 2. *Suppose that $\boldsymbol{x}(\varphi)$ is of class $C^2(\mathbb{R})$ and periodic with the period 2π, and let $q(w)$ be of class $L_\infty(B)$. Assume also that*

$$\sup_B |q| \le \alpha, \quad \sup_{\mathbb{R}} |\boldsymbol{x}''| \le \beta$$

holds for some numbers α, β. Then the function $x(w), w \in B$, defined by (6) and (7) for $R = 1$, can be extended to $\overline{B}$ as a function which is of class $C^{1,\mu}(\overline{B})$ for any $\mu \in (0,1)$ and satisfies $x(e^{i\varphi}) = \boldsymbol{x}(\varphi)$. For suitable numbers $c_1(\alpha,\beta)$ and $c_2(\alpha,\beta,\mu)$ depending only on the indicated parameters and not on q and $\boldsymbol{x}$, we have

$$|\nabla x|_{0,\overline{B}} \le c_1(\alpha,\beta), \quad [\nabla x]_{\mu,\overline{B}} \le c_2(\alpha,\beta,\mu). \tag{8}$$

If $q \in C^{0,\sigma}(B)$ holds for some $\sigma \in (0,1)$, then we have $x \in C^{2,\sigma}(B)$, and the equation $\Delta x = q$ is satisfied on B. Moreover, for any R, R' with $0 < R' < R \le 1$ the function $y(w)$ defined by

[2] Cf. for instance Gilbarg and Trudinger [1], p. 18; John [1], p. 96.

(9) $$y(w) := \int_{B_R} G_R(w,\zeta) q(\zeta)\, d^2\zeta$$

is of class $C^{2,\sigma}(B_R)$ *and satisfies*

(10) $$|y|_{2+\sigma, B_{R'}} \le c(R, R', \sigma)|q|_{0+\sigma, B_R}.$$

Here and in the following we use the notation

$$|x|_{0,B} = \sup_B |x|, \quad [x]_{\mu,B} = \sup\left\{ \frac{|x(w) - x(w')|}{|w - w'|^\mu} : w, w' \in B, w \ne w' \right\},$$

$$|x|_{s,B} = \sum_{k=0}^{s} |\nabla^k x|_{0,B}, \quad |x|_{s+\mu,B} + |x|_{s,B} + [\nabla^s x]_{\mu,B}.$$

Moreover, we shall use the notation $\nabla_w = (\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ in order to distinguish the real gradient $\nabla_w f = (f_u, f_v)$ of a function $f(u,v)$ from its complex derivative $f_w = \frac{1}{2}(f_u - i f_v)$.

The reader will find more complete results on Schauder estimates in Gilbarg and Trudinger [1], Chapters 2–4 and 6; Morrey [8], Chapters 2 and 6; Stein [1]; Agmon, Douglis, and Nirenberg [1,2]. We shall use some of these refined results later on. For the present the reader might welcome to see how one can obtain Schauder estimates in the simple situation at hand. Proposition 2 and related results will be proved by a sequence of auxiliary results.

Lemma 1. *Let* $H(w,\zeta)$ *be a* C^2*-kernel on the set* $\{w, \zeta \in \overline{B}_R : w \ne \zeta\}$ *such that*

(11) $$|H(w,\zeta)| \le b\left|\log\frac{1}{r}\right|, \quad |\nabla_w H(w,\zeta)| \le \frac{b}{r}, \quad |\nabla_w^2 H(w,\zeta)| \le \frac{b}{r^2}$$

holds for $r = |w - \zeta|$ *and some constant* $b > 0$*. In addition we assume that* $q \in L_\infty(B_R)$*. Then the function* $y(w)$ *defined by*

(12) $$y(w) = \int_{B_R} H(w,\zeta) q(\zeta)\, d^2\zeta, \quad w \in B_R,$$

can be extended to a function of class $C^{1,\mu}(\overline{B}_R)$ *satisfying*

(13) $$|y|_{1,\overline{B}_R} \le c_1(b, R)|q|_{0,\overline{B}_R},$$

(14) $$[\nabla y]_{\mu,\overline{B}_R} \le c_2(b, R, \mu)|q|_{0,\overline{B}_R},$$

with constants c_1, c_2 *depending on the parameters* b, R *and* b, R, μ*, respectively. Moreover, we have*

(15) $$\nabla_w y(w) = \int_{B_R} \nabla_w H(w,\zeta) q(\zeta)\, d^2\zeta \quad \text{for } w \in B_R.$$

Proof. As $\frac{1}{r} \in L_1(B_R)$ and $q \in L_\infty(B_R)$, the integrals (12) and (15) are well defined. We choose a cut-off function $\eta_h \in C^\infty(\mathbb{R})$ with $0 \le \eta_h \le 1, \eta_h(r) = 0$ for $r \le h, \eta_h(r) = 1$ for $r \ge 2h$, and $\eta_h'(r) \le \frac{2}{h}$. Then we set

$$H_h(w,\zeta) := \eta_h(r) H(w,\zeta) \quad \text{with } r = |w-\zeta|.$$

Then we have $|H_h| \le |H|$ and $H = H_h$ for $r \ge 2h$. The function

$$y_h(w) := \int_{B_R} H_h(w,\zeta) q(\zeta)\, d^2\zeta, \quad w \in B_R, \tag{16}$$

is of class C^2 and, setting $a := |q|_{0,B_R}$, we obtain

$$\begin{aligned} |y(w) - y_h(w)| &\le a \int_{B_R \cap B_{2h}(w)} \{|H| + |H_h|\}\, d^2\zeta \le 2a \int_{B_R \cap B_{2h}(w)} |H|\, d^2\zeta \\ &\le \text{const} \cdot h \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Thus we infer that $y \in C^0(B_R)$.

Now we define

$$z(w) := \int_{B_R} \nabla_w H(w,\zeta) q(\zeta)\, d^2\zeta, \quad w \in B_R.$$

We want to show that $y \in C^1(B_R)$ and $\nabla_w y = z$. In fact, we have

$$\nabla y_h(w) = I_1^h(w) + I_2^h(w)$$

with

$$\begin{aligned} I_1^h(w) &:= \int_{B_R} \eta_h(r) \nabla_w H(w,\zeta) q(\zeta)\, d^2\zeta, \\ I_2^h(w) &:= \int_{B_R} \nabla_w \eta_h(r) H(w,\zeta) q(\zeta)\, d^2\zeta. \end{aligned}$$

As before we show

$$|z(w) - I_1^h(w)| \le \text{ const } \cdot h \to 0 \quad \text{as } h \to 0,$$

and a straight-forward estimate yields

$$|I_2^h(w)| \le \text{const } h^{-1} h^{2-\alpha} \quad \text{for any } \alpha > 0$$

whence we infer that ∇y_h tends uniformly to z on every $\Omega \Subset B_R$. Together with the uniform convergence of y_h to y on $\Omega \Subset B_R$ as $h \to 0$ we infer that $y \in C^1(B_R)$ and $\nabla y(w) = z(w)$ for any $w \in B_R$. Consequently

$$\begin{aligned} |y(w)| + |\nabla_w y(w)| &\le \int_{B_R} \{|H(w,\zeta)| + |\nabla_w H(w,\zeta)|\} |q(\zeta)|\, d^2\zeta \\ &\le c(b,R) a \quad \text{for all } w \in B_R; \end{aligned}$$

thus (13) is also verified.

Now let $w_1, w_2 \in B_R$, and set $\rho := |w_1 - w_2|$. Then we infer from (15) that

$$\begin{aligned}
\left|\frac{\partial y}{\partial u}(w_1) - \frac{\partial y}{\partial u}(w_2)\right| &= \left|\int_{B_R} \left\{\frac{\partial H}{\partial u}(w_1,\zeta) - \frac{\partial H}{\partial u}(w_2,\zeta)\right\} q(\zeta)\, d^2\zeta\right| \\
&\le a \int_{B_R\cap B_{2\rho}(w_1)} \left|\frac{\partial H}{\partial u}(w_1,\zeta)\right| d^2\zeta + a\int_{B_R\cap B_{2\rho}(w_1)} \left|\frac{\partial H}{\partial u}(w_2,\zeta)\right| d^2\zeta \\
&\quad + a\int_{B_R\setminus B_{2\rho}(w_1)} \left|\frac{\partial H}{\partial u}(w_1,\zeta) - \frac{\partial H}{\partial u}(w_2,\zeta)\right| d^2\zeta.
\end{aligned}$$

Note that

$$\left|\frac{\partial H}{\partial u}(w_1,\zeta)\right| \le b|w_1-\zeta|^{-1}, \quad \left|\frac{\partial H}{\partial u}(w_2,\zeta)\right| \le b|w_2-\zeta|^{-1},$$

and the mean value theorem implies

$$\left|\frac{\partial H}{\partial u}(w_1,\zeta) - \frac{\partial H}{\partial u}(w_2,\zeta)\right| \le \frac{2b\rho}{|w^*-\zeta|^2}$$

for some $w^* = (1-t)w_1 + tw_2, 0 < t < 1$. If $|\zeta - w_1| \ge 2\rho$, we infer that

$$|\zeta - w^*| \ge |\zeta - w_1| - |w_1 - w^*| \ge \frac{1}{2}|\zeta - w_1|,$$

and therefore

$$\left|\frac{\partial H}{\partial u}(w_1,\zeta) - \frac{\partial H}{\partial u}(w_2,\zeta)\right| \le \frac{8b\rho}{|\zeta-w_1|^2} \quad \text{for } |\zeta - w_1| \ge 2\rho.$$

Thus we arrive at

$$\begin{aligned}
&\left|\frac{\partial y}{\partial u}(w_1) - \frac{\partial y}{\partial u}(w_2)\right| \\
&\quad \le ab\bigg[\int_{B_{2\rho}(w_1)} |w_1-\zeta|^{-1}\, d^2\zeta \\
&\qquad + \int_{B_{3\rho}(w_2)} |w_2-\zeta|^{-1}\, d^2\zeta + 8\rho\int_{B_R\setminus B_{2\rho}(w_1)} |w_1-\zeta|^{-2}\, d^2\zeta\bigg] \\
&\quad \le ab\left[4\pi\rho + 6\pi\rho + 16\pi\rho \log\frac{R}{\rho}\right] \le ac(b,R,\mu)\rho^\mu
\end{aligned}$$

for any $\mu \in (0,1)$ and $\rho = |w_1 - w_2|$, and (14) is proved. The estimates (13) and (14) imply that y can be extended to $\overline{B}_R$ as a function of class $C^{1,\mu}(\overline{B}_R)$ for any $\mu \in (0,1)$. □

Lemma 2. *Let $H(w,\zeta)$ be a kernel of the form*

$$H(w,\zeta) = K(w-\zeta) = K(u-\xi, v-\eta)$$

for some function $K(\zeta)$ *which is of class* C^2 *on* $\{\zeta \neq 0\}$*, and suppose that* $H(w,\zeta)$ *satisfies the growth condition* (11)*. Furthermore we assume that* $q(w)$ *is of class* $C^{0,\mu}(\overline{B}_R), 0 < \mu < 1$*. Then the function* $y(w)$ *defined by* (12) *is of class* $C^2(B_R)$*, and we have*

$$|\nabla^2 y|_{0,B_{R'}} \le c_1 |q|_{\mu,B_R}, \quad |y|_{2,B_{R'}} \le c_2 |q|_{\mu,B_R} \tag{17}$$

for $0 < R' < R$*. Here* c_1 *and* c_2 *denote constants depending solely on* b, μ, R *and* R'*.*

Moreover, if $K(\zeta)$ *is of class* C^3 *for* $\zeta \neq 0$ *and if also*

$$|\nabla_w^3 H(w,\zeta)| \le b|w-\zeta|^{-3}, \tag{11*}$$

then $y(w)$ *is of class* $C^{2,\mu}(B_R)$ *and satisfies*

$$|\nabla^2 y|_{\mu,B_{R'}} \le c_3 |q|_{\mu,B_R}, \quad |y|_{2+\mu,B_{R'}} \le c_4 |q|_{\mu,B_R} \tag{18}$$

for $0 < R' < R$*. Here the numbers* c_3 *and* c_4 *only depend on* b, μ, R'*, and* R*.*

Proof. We set again $H_h = \eta_h H$ where η_h is chosen as in the proof of Lemma 1; but in addition we arrange that $|\eta_h''(r)| \le \gamma h^{-2}$ for some constant $\gamma > 0$. Then

$$z_h(w) := \int_{B_R} \nabla_w H_h(w,\zeta) q(\zeta)\, d^2\zeta$$

is of class $C^1(\overline{B}_R, \mathbb{R}^2)$, and for $D = \frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$ we can write

$$\begin{aligned} Dz_h(w) &= \int_{B_R} D\nabla_w H_h(w,\zeta) q(\zeta)\, d^2\zeta \\ &= \int_{B_R} D\nabla_w H_h(w,\zeta)[q(\zeta)-q(w)]\, d^2\zeta + q(w) \int_{B_R} D\nabla_w H_h(w,\zeta)\, d^2\zeta. \end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned} \int_{B_R} D_u \nabla_w H_h(w,\zeta)\, d^2\zeta &= -\int_{B_R} D_\xi \nabla_w H_h(w,\zeta)\, d^2\zeta \\ &= -\int_{\partial B_R} \nabla_w H_h(w,\zeta) \cos\alpha\, ds(\zeta), \end{aligned}$$

where ds is the line element on ∂B_R and $\cos\alpha = \frac{\xi}{|\zeta|}$. If $2h < |w-\zeta|$ we obtain

$$Dz_h(w) = \phi_h(w) - q(w) \int_{\partial B_R} \nabla_w H(w,\zeta) \cos\alpha\, ds(\zeta)$$

with $\cos\alpha = \frac{\xi}{|\zeta|}$ or $= \frac{\eta}{|\zeta|}$ and

$$\phi_h(w) := \int_{B_R} D\nabla_w H_h(w,\zeta)[q(\zeta)-q(w)]\, d^2\zeta.$$

Similarly we set

(19) $$\phi(w) := \int_{B_R} D\nabla_w H(w,\zeta)[q(\zeta) - q(w)]\, d^2\zeta.$$

For $r = |w - \zeta|$ and $a := |q|_{\mu, B_R}$ we have

$$|D\nabla_w H(w,\zeta)| \le br^{-2} \quad \text{and} \quad |q(\zeta) - q(w)| \le ar^\mu,$$

whence

(20) $$|\phi(w)|, |\phi_h(w)| \le 2\pi ab\mu^{-1} R^\mu.$$

By a similar reasoning we obtain ($\nabla = \nabla_w$ and $0 < h \ll 1$):

$$\begin{aligned} |\phi_h(w) - \phi(w)| &\le \int_{B_R \cap B_{2h}(w)} |\eta_h(w) - 1||D\nabla H(w,\zeta)||q(\zeta) - q(w)|\, d^2\zeta \\ &\quad + \int_{B_R} \{|\nabla^2 \eta_h||H| + 2|\nabla \eta_h||\nabla H|\}|q(\zeta) - q(w)|\, d^2\zeta \\ &\le \text{const} \cdot h^\mu \left(1 + \log \frac{1}{h}\right) \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Thus $Dz_h(w)$ tends uniformly to

$$\phi(w) - q(w) \int_{\partial B_R} \nabla_w H(w,\zeta) \cos\alpha(\zeta)\, ds(\zeta)$$

as $h \to 0$, for $w \in B_{R'}$ and $0 < R' < R$. On the other hand, if y_h is defined by (16), we know that

$$z_h = \nabla y_h, \quad Dz_h = D\nabla y_h, \quad z_h \in C^1,$$

and, as shown in the proof of Lemma 1, we also have

$$\lim_{h \to 0} |y - y_h|_{1, B_{R'}} = 0 \quad \text{for } 0 < R' < R.$$

Consequently we have $y \in C^2(B_R)$ and

(21) $$D\nabla y(w) = \phi(w) - q(w) \int_{\partial B_R} \nabla_w H(w,\zeta)\, \cos\alpha(\zeta)\, ds(\zeta) \quad \text{for } |w| < R.$$

Now inequalities (17) follow from (20) and (21).

Finally, taking assumption (11*) into account, we derive from the representation formulas (19) and (21) that y is of class $C^{2,\mu}(B_R)$ and in conjunction with (17) that the estimates (18) are satisfied. Since we may proceed in the same way as in the last part of the proof of Lemma 1, we shall skip this part of the proof. □

Proposition 3. *Suppose that $x \in C^0(\overline{B}_R) \cap C^2(B_R)$ and that $\nabla x \in L_2(B_R)$ and $\Delta x \in L_\infty(B_R)$. Then we obtain the following representation formulas which are satisfied for $w \in B_R$:*

$$x(w) = \frac{1}{2\pi} \int_{\partial B_R} \left(\mathrm{Re} \frac{\zeta + w}{\zeta - w} \right) x(\zeta) \frac{d\zeta}{i\zeta} - \int_{B_R} G_R(w, \zeta) \Delta x(\zeta)\, d^2\zeta, \tag{22}$$

$$\frac{\partial}{\partial w} x(w) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{x(\zeta)}{(\zeta - w)^2}\, d\zeta - \int_{B_R} \frac{\partial}{\partial w} G_R(w, \zeta) \Delta x(\zeta)\, d^2\zeta, \tag{23}$$

$$x_u(0) = \frac{1}{\pi R^2} \int_{B_R} x_u(u, v)\, du\, dv - \frac{1}{2\pi} \int_{B_R} u \left[\frac{1}{r^2} - \frac{1}{R^2} \right] \Delta x(u, v)\, du\, dv, \tag{24}$$

$$x_v(0) = \frac{1}{\pi R^2} \int_{B_R} x_v(u, v)\, du\, dv - \frac{1}{2\pi} \int_{B_R} v \left[\frac{1}{r^2} - \frac{1}{R^2} \right] \Delta x(u, v)\, du\, dv, \tag{25}$$

$$r = |w| = \sqrt{u^2 + v^2}.$$

Proof. Formula (22) is merely a reformulation of (6) and (7). Differentiating (22), it follows in conjunction with Lemma 2 (in particular, with (15)) that (23) holds if we take

$$\frac{2\zeta}{(\zeta - w)^2} = \frac{\partial}{\partial w} \frac{\zeta + w}{\zeta - w} = 2 \frac{\partial}{\partial w} \mathrm{Re} \frac{\zeta + w}{\zeta - w} = 2 \frac{\partial}{\partial w} \frac{R^2 - |w|^2}{|\zeta - w|^2}$$

for $\zeta \in \partial B_R$ into account.

By applying (23) to $w = 0$ and noting that

$$\frac{\partial}{\partial w} G_R(0, \zeta) = \frac{1}{4\pi} \left(\frac{1}{\zeta} - \frac{\overline{\zeta}}{R^2} \right)$$

we infer that

$$x_w(0) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{x(\zeta)}{\zeta^2}\, d\zeta - \int_{B_R} \frac{\overline{\zeta}}{4\pi} \left(\frac{1}{|\zeta|^2} - \frac{1}{R^2} \right) \Delta x(\zeta)\, d^2\zeta.$$

Because of $\zeta^{-2}\, d\zeta = -R^{-2}\, d\overline{\zeta}$, it follows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial B_R} \zeta^{-2} x(\zeta)\, d\zeta &= -\frac{1}{2\pi i R^2} \int_{\partial B_R} x(\zeta)\, d\overline{\zeta} \\ &= -\frac{1}{2\pi i R^2} \int_{\partial B_R} x(\zeta)(d\xi - i\, d\eta) \\ &= -\frac{1}{2\pi i R^2} \int_{B_R} (-i x_\xi - x_\eta)\, d\xi\, d\eta \\ &= \frac{1}{\pi R^2} \int_{B_R} x_\zeta\, d^2\zeta. \end{aligned}$$

Replacing ζ by w, we arrive at (24) and (25) by separating the real and imaginary parts. Actually we first prove (24) and (25) for $B_{R'}$, $R' < R$, instead for B_R, and then we let $R' \to R$. □

Now we prove Schwarz's result concerning the boundary continuity of Poisson's integral.

Lemma 3. *Let $\boldsymbol{x}(\varphi)$ be a continuous, 2π-periodic function on $\mathbb{R}$, and let $h(w) := \int_0^{2\pi} \mathcal{P}_R(w,\varphi)\boldsymbol{x}(\varphi)\,d\varphi$ be the corresponding Poisson integral, which is a harmonic function of $w \in B_R$. Then we obtain $h(w) \to \boldsymbol{x}(\varphi)$ as $w \to Re^{i\varphi}$. Thus $h(w)$ can be extended to a continuous function on $\overline{B}_R$ such that $h(Re^{i\varphi}) = \boldsymbol{x}(\varphi)$ for all $\varphi \in \mathbb{R}$.*

Proof. It suffices to treat the case $R = 1$. Then we have to prove $\lim_{r\to 1-0} h(re^{i\theta}) = \boldsymbol{x}(\theta)$ uniformly in $\theta \in \mathbb{R}$. We can write

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-r|^2}\boldsymbol{x}(\theta+\varphi)\,d\varphi.$$

Because of the identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-r|^2}\,d\varphi = 1$$

it follows that

$$h(re^{i\theta}) - \boldsymbol{x}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-r|^2}[\boldsymbol{x}(\theta+\varphi)-\boldsymbol{x}(\theta)]\,d\varphi$$

whence

$$\begin{aligned} |h(re^{i\theta}) - \boldsymbol{x}(\theta)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{|e^{i\varphi}-r|^2}|\boldsymbol{x}(\theta+\varphi)-\boldsymbol{x}(\theta)|\,d\varphi \\ &\le \frac{1}{2\pi}\int_{-\pi}^{-\delta}\cdots + \frac{1}{2\pi}\int_{-\delta}^{\delta}\cdots + \frac{1}{2\pi}\int_{\delta}^{\pi}\cdots = I_1 + I_2 + I_3 \end{aligned}$$

for any $\delta \in (0, \frac{\pi}{2})$. Fix some $\varepsilon > 0$ and choose $\delta > 0$ so small that $|\boldsymbol{x}(\varphi) - \boldsymbol{x}(\theta)| < \varepsilon$ for all φ and θ with $|\varphi - \theta| < \delta$. Then we obtain

$$I_2 \le \varepsilon \cdot \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{1-r^2}{|e^{i\varphi}-r|^2}\,d\varphi \le \varepsilon.$$

Moreover, by setting $M := \max_{\mathbb{R}} |\boldsymbol{x}|$ we obtain

$$I_1, I_3 \le \frac{1}{2\pi}(\pi-\delta)(1-r)(1+r)\frac{2M}{\sin^2\delta} \le \frac{2M}{\sin^2\delta}(1-r)$$

since $|e^{i\varphi} - r|^2 \ge \sin^2\delta$ for $\delta \le |\varphi| \le \pi$. Thus we arrive at

$$|h(re^{i\theta}) - \boldsymbol{x}(\theta)| \le \varepsilon + \frac{4M}{\sin^2 \delta(\varepsilon)}(1-r) \quad \text{for } r \in (0,1)$$

and therefore

$$\lim_{r\to 1-0} h(re^{i\theta}) = \boldsymbol{x}(\theta) \quad \text{uniformly in } \theta. \qquad \square$$

As a by-product of this proof we have found:

Lemma 4. *Let $h \in C^0(\overline{B}) \cap C^2(B)$ be harmonic in B and suppose that $|h(e^{i\varphi}) - h(e^{i\theta})| < \varepsilon$ holds for all φ with $|\varphi - \theta| \le \delta, \delta \in \left(0, \frac{\pi}{2}\right)$. Then it follows that*

$$|h(re^{i\theta}) - h(e^{i\theta})| \le \varepsilon + \frac{4|h|_{0,\partial B}}{\sin^2 \delta}(1-r) \tag{26}$$

holds for all $r \in (0,1)$.

Lemma 5. *Let $h \in C^0(\overline{B}_R) \cap C^2(B_R)$ be harmonic in B_R, and suppose that the boundary values $\boldsymbol{x}(\varphi)$ of h defined by $\boldsymbol{x}(\varphi) := h(Re^{i\theta})$ are of class $C^2(\mathbb{R})$ and satisfy $|\boldsymbol{x}''(\varphi)| \le k$ for all $\varphi \in \mathbb{R}$. Then we obtain*

$$|\nabla h(w)| \le cR^{-1}k \quad \textit{for all } w \in B_R, \tag{27}$$

where c is an absolute constant independent of h and R.

Proof. By virtue of an obvious scaling argument we can restrict our attention to the case $R = 1$. Then we have to prove

$$|\nabla h(w)| \le \text{const } k \quad \text{for } w \in B.$$

Let h^* be the conjugate harmonic function to h. Then $f(w) := h(w) + ih^*(w)$ is a holomorphic function of $w = u + iv$, and we have the convergent power series expansion

$$f(w) = \sum_{l=0}^{\infty} c_l w^l \quad \text{for } |w| < 1.$$

Set $c_0 = \frac{1}{2}(a_0 - ib_0), c_l = a_l - ib_l$ if $l \ge 1, a_l, b_l \in \mathbb{R}$. Then we have for $w = re^{i\varphi}$ that

$$h(w) = \frac{a_0}{2} + \sum_{l=1}^{\infty} r^l (a_l \cos l\varphi + b_l \sin l\varphi)$$

whence

$$\begin{aligned}
\boldsymbol{x}(\varphi) &= \frac{a_0}{2} + \sum_{l=1}^{\infty} (a_l \cos l\varphi + b_l \sin l\varphi), \\
a_l &= \frac{1}{\pi} \int_0^{2\pi} \boldsymbol{x}(\varphi) \cos l\varphi \, d\varphi = -\frac{1}{\pi l^2} \int_0^{2\pi} \boldsymbol{x}''(\varphi) \cos l\varphi \, d\varphi, \\
b_l &= \frac{1}{\pi} \int_0^{2\pi} \boldsymbol{x}(\varphi) \sin l\varphi \, d\varphi = -\frac{1}{\pi l^2} \int_0^{2\pi} \boldsymbol{x}''(\varphi) \sin l\varphi \, d\varphi.
\end{aligned}$$

Because of $f' = h_u + ih_u^* = h_u - ih_v$ we infer that

$$\begin{aligned}|\nabla h(w)| &= |f'(w)| = \left|\sum_{l=1}^{\infty} l(a_l - ib_l)w^{l-1}\right| \\ &\le \sum_{l=1}^{\infty} l\sqrt{a_l^2 + b_l^2} \le \left\{\sum_{l=1}^{\infty} \frac{1}{\pi l^2}\right\}^{1/2} \left[\int_0^{2\pi} |\boldsymbol{x}''(\varphi)|^2\, d\varphi\right]^{1/2} \\ &\le \frac{\pi}{\sqrt{3}} k \quad \text{if } |w| < 1,\end{aligned}$$

taking Schwarz's inequality into consideration as well as Parseval's relation for the Fourier series expansion of $\boldsymbol{x}''$. □

The next result is known in the literature as *Theorem of Korn and Privalov.*

Lemma 6. *Let $f(w) = h(w) + ih^*(w)$ be holomorphic in $B_R, h = \operatorname{Re} f$, $h^* = \operatorname{Im} f$, and suppose that $h^* \in C^0(\overline{B}_R) \cap C^{0,\mu}(\partial B_R)$ holds for some $\mu \in (0,1)$. Then f is of class $C^{0,\mu}(\overline{B}_R)$ and we have*

$$[f]_{\mu,\overline{B}_R} \le c(\mu)[h^*]_{\mu,\partial B_R}. \tag{28}$$

Proof. We can assume that $R = 1$ applying a scaling argument. Set $H := [h^*]_{\mu,\partial B}$. Then we have

$$|h^*(e^{i\theta}) - h^*(e^{i\varphi})| \le H|e^{i\theta} - e^{i\varphi}|^\mu \tag{29}$$

for all $\theta, \varphi \in \mathbb{R}$. Fix some $\varphi \in [0, 2\pi)$ and consider the function

$$\psi(w) = \operatorname{Re}(1 - we^{-i\varphi})^\mu$$

which can be viewed as a univalent harmonic function of $w \in B$. Introducing the angle α between the rays $\{te^{i\varphi} : t \ge 0\}$ and $\{t(e^{i\varphi} - w) : t \ge 0\}$, we obtain

$$\psi(w) = |w - e^{i\varphi}|^\mu \cos(\mu\alpha(w)),$$

where $|\alpha(w)| \le \frac{\pi}{2}$. Thus we infer from (29) that

$$-\frac{H\psi(w)}{\cos\frac{\mu\pi}{2}} \le h^*(w) - h^*(e^{i\varphi}) \le \frac{H\psi(w)}{\cos\frac{\mu\pi}{2}} \tag{30}$$

holds for all $w \in \partial B$. Applying the maximum principle, we obtain that (31) holds for all $w \in \overline{B}$ and in particular for all $w \in B_{1-r}(re^{i\varphi})$ if $0 < r < 1$. Hence we infer that

$$|h^*(w) - h^*(e^{i\varphi})| \le \frac{2^\mu H}{\cos\frac{\mu\pi}{2}}(1 - r)^\mu \quad \text{for } |w - re^{i\varphi}| < 1 - r \tag{31}$$

is satisfied.

Now we set $w_0 := re^{i\varphi}, 0 < r < 1$, and $h_0^* := h^*(w_0)$. Applying Gauss's mean value theorem to the harmonic function h_u^* and to the ball $B_\rho(w_0)$ with some radius $\rho \in (0, 1-r)$, we obtain

$$h_u^*(w_0) = \frac{1}{\pi\rho^2}\int_{B_\rho(w_0)} h_u^*(w)\,d^2w = \frac{1}{\pi\rho^2}\int_{B_\rho(w_0)} (h^* - h_0^*)_u\,d^2w,$$

and an integration by parts yields

$$h_u^*(w_0) = \frac{1}{\pi\rho^2}\int_{\partial B_\rho(w_0)} \frac{u - u_0}{\rho}(h^* - h_0^*)\,ds.$$

Analogously,

$$h_v^*(w_0) = \frac{1}{\pi\rho^2}\int_{\partial B_\rho(w_0)} \frac{v - v_0}{\rho}(h^* - h_0^*)\,ds.$$

Thus we obtain

$$(32) \qquad |\nabla h^*(w_0)| \le 2\rho^{-1}|h^* - h_0^*|_{0,\partial B_\rho(w_0)}.$$

If we let $\rho \to 1 - r$ and combine the resulting inequality with (31), it follows that

$$|\nabla h^*(w_0)| \le \frac{4H}{\cos\frac{\mu\pi}{2}}(1-r)^{-1+\mu}.$$

Since we can choose φ arbitrarily, we obtain that

$$(33) \qquad |f'(w)| \le c(\mu)H(1-|w|)^{-1+\mu} \quad \text{for all } w \in B,$$

if we set $c(\mu) := 4(\cos\frac{\mu\pi}{2})^{-1}$. Then it follows from (33) for $0 \le r < 1$ that

$$(7.33') \qquad |f'(w)| \le c(\mu)H(r-|w|)^{-1+\mu} \quad \text{for all } w \in B_r.$$

For any r and r' with $0 \le r < r' < 1$ we now conclude

$$|f(r'e^{i\theta}) - f(re^{i\theta})| = \left|\int_{re^{i\theta}}^{r'e^{i\theta}} f'(w)\,dw\right| \le c(\mu)H\int_r^{r'} (r'-\rho)^{-1+\mu}\,d\rho$$

whence

$$(34) \qquad |f(r'e^{i\theta}) - f(re^{i\theta})| \le c(\mu)\mu^{-1}H(r'-r)^\mu \quad \text{for } 0 \le r < r' < 1.$$

We infer that $\lim_{r\to 1-0} f(re^{i\theta})$ exists for any $\theta \in \mathbb{R}$. Setting $\xi(\varphi) := \lim_{r\to 1-0} f(re^{i\varphi})$ we extend $f(w)$ from B to $\overline{B}$ by defining $f(e^{i\varphi}) := \xi(\varphi)$. We now want to show that $f \in C^{0,\mu}(\overline{B})$. In fact, setting $c^*(\mu) := \mu^{-1}c(\mu)$ we obtain from (34) that

$$(34') \qquad |f(r'e^{i\theta}) - f(re^{i\theta})| \le c^*(\mu)H(r'-r)^\mu \quad \text{for } 0 \le r \le r' \le 1, \theta \in \mathbb{R},$$

and in particular

$$|\xi(\theta) - f(re^{i\theta})| \le c^*(\mu)H(1-r)^\mu \quad \text{for } 0 < r < 1 \text{ and } \theta \in \mathbb{R}.$$

Then it follows for $\theta_1 < \theta_2$ that

$$\begin{aligned} |\xi(\theta_1) - \xi(\theta_2)| &\le |\xi(\theta_1) - f(re^{i\theta_1})| + |\xi(\theta_2) - f(re^{i\theta_1})| \\ &\le |\xi(\theta_1) - f(re^{i\theta_1})| + |\xi(\theta_2) - f(re^{i\theta_2})| + |f(re^{i\theta_1}) - f(re^{i\theta_2})| \\ &\le 2c^*(\mu)H(1-r)^\mu + \int_{\theta_1}^{\theta_2} |f'(re^{i\theta})| r \, d\theta. \end{aligned}$$

Moreover, we derive from (33) that

$$\int_{\theta_1}^{\theta_2} |f'(re^{i\theta})| r \, d\theta \le Hc(\mu) r(1-r)^{-1+\mu}(\theta_2 - \theta_1).$$

Suppose that $0 < \theta_2 - \theta_1 < 1$, and choose $r = 1 - (\theta_2 - \theta_1)$. Then it follows that

$$|\xi(\theta_1) - \xi(\theta_2)| \le \{2c^*(\mu) + c(\mu)\}H|\theta_2 - \theta_1|^\mu$$

if $|\theta_1 - \theta_2| \le 1$. Renaming $8c^*(\mu) + 4c(\mu)$ by $c(\mu)$, we arrive at

$$(35) \qquad |\xi(\theta_1) - \xi(\theta_2)| \le c(\mu)H|\theta_1 - \theta_2|^\mu \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R}.$$

Applying the maximum principle to the modulus of the holomorphic mapping $f(e^{i\alpha}w) - f(w), w \in B$, we see that

$$\max_{w \in \overline{B}} |f(e^{i\alpha}w) - f(w)| \le \max_{w \in \partial B} |f(e^{i\alpha}w) - f(w)|$$

holds for all $\alpha \in \mathbb{R}$, and in view of (35) we obtain

$$|f(e^{i\alpha}w) - f(w)| \le c(\mu)H|\alpha|^\mu \quad \text{for } w \in \overline{B} \text{ and } \alpha \in \mathbb{R}.$$

This estimate is equivalent to

$$(35') \quad |f(re^{i\theta_2}) - f(re^{i\theta_1})| \le c(\mu)H|\theta_1 - \theta_2|^\mu \quad \text{for } 0 \le r \le 1,\ \theta_1, \theta_2 \in \mathbb{R}.$$

Combining the estimates (34′) and (35′) we arrive at

$$\begin{aligned} |f(r_1e^{i\theta_1}) - f(r_2e^{i\theta_2})| &\le |f(r_1e^{i\theta_1}) - f(r_2e^{i\theta_1})| + |f(r_2e^{i\theta_1}) - f(r_2e^{i\theta_2})| \\ &\le c^*(\mu)H|r_1 - r_2|^\mu + c(\mu)H|\theta_1 - \theta_2|^\mu \end{aligned}$$

for arbitrary $r_1, r_2 \in [0,1]$ and $\theta_1, \theta_2 \in \mathbb{R}$.

If $w_1 = r_1e^{i\theta_1}, w_2 = r_2e^{i\theta_2}, \frac{1}{2} \le r_1, r_2 \le 1, |\theta_2 - \theta_1| \le \pi$, then there is a constant K such that

$$|r_1 - r_2|^\mu + |\theta_1 - \theta_2|^\mu \le K|w_1 - w_2|^\mu.$$

Consequently we have

$$|f(w_1) - f(w_2)| \le c(\mu)H|w_1 - w_2|^\mu$$

for all $w_1, w_2 \in \overline{B} \setminus B_{1/2}$ and for some constant $c(\mu)$, and because of (33) the same estimate holds for any $w_1, w_2 \in \overline{B}_{1/2}$. Then we easily infer that

$$|f(w_1) - f(w_2)| \le c(\mu)H|w_1 - w_2|^\mu \quad \text{for all } w_1, w_2 \in \overline{B}$$

holds true. □

Lemma 7. *Suppose that $h \in C^0(\overline{B}_R) \cap C^2(B_R)$ is harmonic in B_R and that its boundary values $\boldsymbol{x}(\varphi) := h(Re^{i\varphi})$ satisfy $\boldsymbol{x} \in C^2(\mathbb{R})$ and $|\boldsymbol{x}''(\varphi)| \le k$ for all $\varphi \in \mathbb{R}$. Then we obtain $h \in C^{1,\mu}(\overline{B}_R)$ for every $\mu \in (0,1)$ and*

$$[\nabla h]_{\mu,\overline{B}_R} \le c(\mu)R^{-1-\mu}k, \tag{36}$$

where the number $c(\mu)$ only depends on μ.

Proof. It is sufficient to prove the result for $R = 1$. Let us introduce the tangential difference quotient

$$(T_\theta h)(re^{i\varphi}) := \frac{1}{\theta}[h(re^{i(\varphi+\theta)}) - h(re^{i\varphi})]$$

and note that $(T_\theta h)(w)$ is a harmonic function of $w \in B$ which is continuous on $\overline{B}$ and has the boundary values

$$(\tau_\theta \boldsymbol{x})(\varphi) := \frac{1}{\theta}[\boldsymbol{x}(\varphi + \theta) - \boldsymbol{x}(\varphi)].$$

By assumption the boundary values $(\tau_\theta \boldsymbol{x})(\varphi)$ tend uniformly to $\boldsymbol{x}'(\varphi)$ as $\theta \to 0$. Then, on account of Harnack's first convergence theorem, we easily infer that the functions $(T_\theta h)(w)$ tend uniformly on $\overline{B}$ to the harmonic function $h_\varphi(w)$ with the boundary values $\boldsymbol{x}'(\varphi) = \frac{\partial}{\partial\varphi}h(e^{i\varphi})$ on ∂B which, by assumption, are Hölder continuous for any exponent $\mu < 1$, and

$$[\boldsymbol{x}']_{\mu,\mathbb{R}} \le 2\pi k. \tag{37}$$

Consider a holomorphic function $f(w)$ on B with $f = h + ih^*$, that is, $h = \operatorname{Re} f, h^* = \operatorname{Im} f$. Then $g(w) := iwf'(w) = \frac{\partial f}{\partial\varphi}(w), w = re^{i\varphi}$, is another holomorphic function on B with $\frac{\partial h}{\partial\varphi} = \operatorname{Re} g$ and $\boldsymbol{x}'(\varphi) = \frac{\partial h}{\partial\varphi}(e^{i\varphi}), \boldsymbol{x}' \in C^{0,\mu}(\mathbb{R})$ for any $\mu \in (0,1)$. Hence we can apply Lemma 6 to the holomorphic function $ig(w) = -wf'(w), w \in B$, and we obtain that $ig(w)$ is of class $C^{0,\mu}(\overline{B})$. This implies

$$f' \in C^{0,\mu}(\overline{B} \setminus B_{1/2}),$$

and inequalities (28) and (37) yield

(38) $$[f']_{\mu,B\setminus B_{1/2}} \le \text{const} \cdot k.$$

Moreover, (27) implies

$$|f'|_{0,B} \le \text{const} \cdot k,$$

and Cauchy's estimate for holomorphic functions then gives

$$|f''|_{0,B_{1/2}} \le \text{const} \cdot k,$$

whence

$$[f']_{\mu,B_{1/2}} \le \text{const} \cdot k.$$

Combining this estimate with (38), we arrive at the desired inequality

$$[f']_{\mu,B} \le \text{const} \cdot k.$$

If we now recall that $f' = h_u - ih_v$, we find that the lemma is proved. □

Proof of Proposition 2. We now see that Proposition 2 is a direct consequence of Lemmata 1–7 in conjunction with Proposition 1 and with formulas (1)–(5). □

Remark. We have formulated the estimates and the regularity results of Proposition 1 in a global way. Analogous local results can be derived by similar methods, but certain changes will be necessary to obtain local estimates at the boundary. A very simple approach to local $C^{1,\mu}$-estimates is based on a reflection method: it will be described in the next section.

2.2 Solutions of Differential Inequalities

In this section we want to derive a priori estimates for solutions $X(u,v) = X(w) = (x^1(w), x^2(w), \dots, x^N(w))$ of differential inequalities

(1) $$|\Delta X| \le a|\nabla X|^2,$$

which can equivalently be written as

(1′) $$|X_{w\overline{w}}| \le a|X_w|^2.$$

Here a denotes a fixed nonnegative constant.

Lemma 1. *Let $X \in C^2(\Omega, \mathbb{R}^N)$ be a solution of (1) in the open set Ω of $\mathbb{R}^2$ which satisfies $|X|_{0,\Omega} \le M$. Then we obtain*

(2) $$\Delta|X|^2 \ge 2(1 - aM)|\nabla X|^2$$

in Ω. In particular, if $aM < 1$, then $|X|^2$ is subharmonic in Ω.

Proof. Because of

$$|\langle X, \Delta X\rangle| \le |X||\Delta X| \le aM|\nabla X|^2,$$

the inequality (2) is an immediate consequence of the identity

$$\Delta |X|^2 = 2|\nabla X|^2 + 2\langle X, \Delta X\rangle \tag{3}$$

which holds for every mapping X of class C^2. □

Lemma 2. *Suppose that $X \in C^0(\overline{B}_R(w_0), \mathbb{R}^N) \cap C^2(B_R(w_0), \mathbb{R}^N)$ satisfies (1) in $B_R(w_0)$. Assume also that $|X(w)| \le M$ for $w \in \overline{B}_R(w_0)$ and $aM < 1$ are satisfied. Then for any $\rho \in (0, R)$ we have*

$$\int_{B_\rho(w_0)} |\nabla X|^2 \, du \, dv \le \frac{1}{\log \frac{R}{\rho}} \frac{2\pi M}{1 - aM} \max_{w \in \partial B_R(w_0)} |X(w) - X(w_0)| \tag{4}$$

and

$$\int_{B_\rho(w_0)} |\nabla X|^2 \, du \, dv \le \frac{1}{\log \frac{R}{\rho}} \frac{4\pi M^2}{1 - aM}. \tag{5}$$

Proof. Choose some $\rho \in (0, R)$ and apply Proposition 1 of Section 2.1 to the function $x(w) := |X(w)|^2$ and to the domain $B_R(w_0)$ instead of $B_R = B_R(0)$, assuming in addition that X is of class C^2 on $\overline{B}_R(w_0)$. Then formula (6) of Section 2.2 yields

$$\frac{1}{2\pi} \int_0^{2\pi} [x(w_0 + Re^{i\varphi}) - x(w_0)] \, d\varphi = \frac{1}{2\pi} \int_{B_R(w_0)} \log \frac{R}{|w - w_0|} \Delta x \, d^2 w.$$

Because of

$$|x(w) - x(w_0)| \le 2M |X(w) - X(w_0)|$$

we infer that

$$\frac{1}{2\pi} \int_{B_R(w_0)} \log \frac{R}{|w - w_0|} \Delta x \, d^2 w \le 2M \max_{w \in \partial B_R(w_0)} |X(w) - X(w_0)|.$$

On the other hand, Lemma 1 gives

$$2(1 - aM)|\nabla X|^2 \le \Delta x,$$

whence

$$(1 - aM)\pi^{-1} \int_{B_R(w_0)} \log \frac{R}{|w - w_0|} |\nabla X|^2 \, d^2 w \le 2M \max_{w \in B_R(w_0)} |X(w) - X(w_0)|.$$

Moreover,

$$\log \frac{R}{\rho} \int_{B_\rho(w_0)} |\nabla X|^2 \, d^2 w \le \int_{B_R(w_0)} \log \frac{R}{|w - w_0|} |\nabla X|^2 \, d^2 w$$

if $0 < \rho < R$, and (4) is proved. The additional hypothesis can be removed if we first apply the reasoning to ρ and R' with $0 < \rho < R' < R$, and then let $R' \to R - 0$. Inequality (5) is a direct consequence of (4). □

Proposition 1. *There is a continuous function $\kappa(t), 0 \le t < 1$, with the following property: For any solution $X \in C^2(B_R(w_0), \mathbb{R}^N)$ of the differential inequality (1) in $B_R(w_0)$ satisfying*

$$(6) \qquad |X(w)| \le M \quad \text{for } w \in B_R(w_0)$$

and for some constant M with $aM < 1$, the estimates

$$(7) \qquad |\nabla X(w_0)| \le \kappa(aM)\frac{M}{R}$$

and

$$(8) \qquad |\nabla X(w_0)| \le \frac{\kappa(aM)}{R} \sup_{w \in B_R(w_0)} |X(w) - X(w_0)|$$

hold true.

Proof. Fix any $R' \in (0, R)$, and consider the nonnegative function

$$f(w) := (R' - |w - w_0|)|\nabla X(w)|$$

on $\overline{B}_{R'}(w_0)$ which vanishes on $\partial B_{R'}(w_0)$. Then there is some point $w_1 \in B_{R'}(w_0)$ where $f(w)$ assumes its maximum K, i.e.,

$$f(w_1) = K := \max\,\{f(w) \colon w \in \overline{B}_{R'}(w_0)\}.$$

Set $r = |w - w_1|$ for $w \in \overline{B}_{R'}(w_0)$ and $\rho := R' - |w_1 - w_0|$. Clearly, we have $0 < \rho < R'$.

By formulas (24) and (25) of Section 2.1, we obtain for any $\theta \in (0, 1)$ that

$$\begin{aligned} X_u(w_1) = &\frac{1}{\pi\rho^2\theta^2}\int_{B_{\rho\theta}(w_1)} X_u\,du\,dv \\ &-\frac{1}{2\pi}\int_{B_{\rho\theta}(w_1)} (u - u_1)\left(\frac{1}{r^2} - \frac{1}{\rho^2\theta^2}\right)\Delta X\,du\,dv, \end{aligned}$$

and an analogous formula holds for $X_v(w_1)$. By means of Schwarz's inequality we infer that

$$|\nabla X(w_1)| \le \frac{1}{\sqrt{\pi}\rho\theta}\left\{\int_{B_{\rho\theta}(w_1)} |\nabla X|^2\,du\,dv\right\}^{1/2} + \frac{1}{2\pi}\int_{B_{\rho\theta}(w_1)} \frac{|\Delta X|}{r}\,du\,dv.$$

Applying Lemma 2, (5) to $\rho\theta$ and ρ instead of ρ and R, we also obtain

$$\int_{B_{\rho\theta}(w_1)} |\nabla X|^2\,du\,dv \le \frac{c(a, M)}{\log\frac{1}{\theta}}.$$

Taking $|\Delta X| \le a|\nabla X|^2$ into account, we arrive at

$$|\nabla X(w_1)| \le \frac{\sqrt{c}}{\sqrt{\pi}\rho\theta\left(\log\frac{1}{\theta}\right)^{1/2}} + \frac{a}{2\pi}\int_{B_{\rho\theta}(w_1)} \frac{1}{r}|\nabla X|^2\,du\,dv$$

and

$$\frac{a}{2\pi}\int_{B_{\rho\theta}(w_1)} \frac{1}{r}|\nabla X|^2\,du\,dv \le a\rho\theta \sup_{B_{\rho\theta}(w_1)} |\nabla X|^2.$$

On account of

$$K = f(w_1) = \rho|\nabla X(w_1)|$$

we obtain

$$|\nabla X(w_1)| = K/\rho.$$

Moreover, if $r = |w - w_1| < \rho\theta$, it follows that

$$R' - |w - w_0| \ge R' - |w_0 - w_1| - |w - w_1| = \rho - r > (1-\theta)\rho.$$

Thus we infer from

$$|\nabla X(w)|(R' - |w - w_0|) \le K \quad \text{for all } w \in B_{R'}(w_0)$$

that

$$|\nabla X(w)| \le \frac{K}{(1-\theta)\rho} \quad \text{for all } w \in B_{\rho\theta}(w_1)$$

holds true, and we conclude that

$$\frac{a}{2\pi}\int_{B_{\rho\theta}(w_1)} \frac{1}{r}|\nabla X|^2\,du\,dv \le \frac{a\theta K^2}{(1-\theta)^2\rho}$$

whence

$$\frac{K}{\rho} \le \frac{\sqrt{c}}{\sqrt{\pi}\rho\theta\left(\log\frac{1}{\theta}\right)^{1/2}} + \frac{a\theta K^2}{(1-\theta)^2\rho},$$

and finally

$$K \le \frac{\sqrt{c/\pi}}{\theta\left(\log\frac{1}{\theta}\right)^{1/2}} + \frac{a\theta K^2}{(1-\theta)^2}.$$

Set

$$\alpha(\theta) := \frac{a\theta}{(1-\theta)^2}, \quad \beta(\theta) := \frac{\sqrt{c/\pi}}{\theta\left(\log\frac{1}{\theta}\right)^{1/2}}.$$

Then we have

$$\alpha K^2 - K + \beta \ge 0,$$

or equivalently

$$\left(K - \frac{1}{2\alpha}\right)^2 \ge \frac{1 - 4\alpha\beta}{4\alpha^2}.$$

Note that

$$\alpha(\theta)\beta(\theta) = \frac{a\sqrt{c/\pi}}{(1-\theta)^2\left(\log\frac{1}{\theta}\right)^{1/2}}, \quad c = c(a, M).$$

Hence there exists a number $\theta_0(a, M) \in (0,1)$ such that

$$4\alpha(\theta)\beta(\theta) \le \frac{3}{4} \quad \text{if } 0 < \theta \le \theta_0,$$

that is,

$$\sqrt{1 - 4\alpha(\theta)\beta(\theta)} \ge \frac{1}{2} \quad \text{if } 0 < \theta \le \theta_0.$$

Set

$$m^-(\theta) := \frac{1 - \sqrt{1 - 4\alpha(\theta)\beta(\theta)}}{2\alpha(\theta)}, \quad m^+(\theta) := \frac{1 + \sqrt{1 - 4\alpha(\theta)\beta(\theta)}}{2\alpha(\theta)}.$$

Then we infer for any $\theta \in (0, \theta_0]$ that either

$$\text{(i) } K \le m^-(\theta), \quad \text{or} \quad \text{(ii) } K \ge m^+(\theta)$$

holds true.

Moreover, the functions $m^-(\theta)$ and $m^+(\theta)$ are continuous on $(0, \theta_0]$ and satisfy

$$m^-(\theta) < m^+(\theta) \quad \text{for } 0 < \theta \le \theta_0$$

and

$$\lim_{\theta \to +0} m^+(\theta) = \infty.$$

The last relation yields that case (ii) cannot occur for θ close to zero; hence we have $K \le m^-(\theta)$ for θ near zero, and a continuity argument then implies

$$K \le m^-(\theta) \quad \text{for all } \theta \in (0, \theta_0],$$

in particular, $K \le m^-(\theta_0)$. Finally, for $\theta \in (0, \theta_0]$, we also obtain that

$$\begin{aligned} m^-(\theta) &= \frac{1 - \sqrt{1 - 4\alpha(\theta)\beta(\theta)}}{2\alpha(\theta)} = \frac{1}{2\alpha(\theta)} \frac{1 - (1 - 4\alpha(\theta)\beta(\theta))}{1 + \sqrt{1 - 4\alpha(\theta)\beta(\theta)}} \\ &= \frac{2\beta(\theta)}{1 + \sqrt{1 - 4\alpha(\theta)\beta(\theta)}} \le \frac{2\beta(\theta)}{1 + 1/2} = \frac{4}{3}\beta(\theta) \end{aligned}$$

whence

$$m^-(\theta_0) < \tfrac{4}{3}\beta(\theta_0).$$

Consequently,

$$K \le \frac{4}{3}\beta(\theta_0) = \frac{4\sqrt{c/\pi}}{3\theta_0\big(\log \frac{1}{\theta_0}\big)^{1/2}} := c^*(a, M).$$

Because of

$$R'|\nabla X(w_0)| = f(w_0) \le f(w_1) = K \le c^*(a, M)$$

we arrive at

$$|\nabla X(w_0)| \le c^*(a, M)/R' \quad \text{for any } R' \in (0, R)$$

whence

$$(7^*) \qquad |\nabla X(w_0)| \le c^*(a, M)/R.$$

This estimate is close to (7). We now introduce the function $\kappa(t) := c^*(t, 1)$. A close inspection of the previous computations shows that $\kappa(t)$ can assumed to be an increasing and continuous function on the interval $[0, 1)$.

In order to prove (7) we assume that $M > 0$ because that inequality trivially holds true if $M = 0$. Then $Z(w) := M^{-1}X(w)$ satisfies both $|Z(w)| \le 1$ and

$$|\Delta Z| \le aM|\nabla Z|^2.$$

Applying the estimate (7^*) to Z, we arrive at

$$|\nabla Z(w_0)| \le \kappa(aM)/R.$$

Multiplying this inequality by M, we obtain (7).

Estimate (8) is now an easy consequence of (7). To see this we introduce the quantity

$$m := \sup\,\{|X(w) - X(w_0)| \colon w \in B_R(w_0)\}.$$

If $m = 0$ or $m = \infty$, the estimate (8) is true for trivial reasons. If $M \le m < \infty$, (8) follows directly from (7). If $0 < m < M$, we introduce $Z := m^{-1}[X - X(w_0)]$ and obtain as before

$$|\nabla Z(w_0)| \le \kappa(am)/R \le \kappa(aM)/R,$$

and this implies (8). □

Corollary 1. *Suppose that $X \in C^2(B_R(w_0), \mathbb{R}^N)$ is a solution of (1) in $B_R(w_0)$ satisfying*

$$|X(w)| \le M \quad \textit{for } w \in B_R(w_0)$$

and for some constant M with $aM < 1$. Then we have

$$(9) \qquad |\nabla X(w)| \le \kappa(aM)\frac{M}{\rho} \quad \textit{for all } w \in B_{R-\rho}(w_0),\ 0 < \rho < R.$$

Proposition 2. *Let* $X \in C^0(\overline{B}_R(w_0), \mathbb{R}^N) \cap C^2(B_R(w_0), \mathbb{R}^N)$ *be a solution of the differential inequality* (1) *in* $B_R(w_0)$*, and suppose that* $|X(w)| \le M$ *holds for all* $w \in \overline{B}_R(w_0)$ *and for some number* M *with* $2aM < 1$*. Moreover, set* $x(w) := |X(w)|^2$*, and let* $H \in C^0(\overline{B}_R(w_0), \mathbb{R}^N) \cap C^2(B_R(w_0), \mathbb{R}^N)$ *and* $h \in C^0(\overline{B}_R(w_0)) \cap C^2(B_R(w_0))$ *be the solutions of the boundary value problems*

$$\Delta H = 0 \quad \text{in } B_R(w_0), \qquad H = X \quad \text{on } \partial B_R(w_0), \tag{10}$$

$$\Delta h = 0 \quad \text{in } B_R(w_0), \qquad h = x \quad \text{on } \partial B_R(w_0). \tag{11}$$

Then for any $w \in B_R(w_0)$ *and* $w^* \in \partial B_R(w_0)$ *we have the inequality*

$$\begin{aligned} |X(w) - X(w^*)| \le \frac{a}{2(1-2aM)} |h(w) - h(w^*)| \\ + \frac{1-aM}{1-2aM} |H(w) - H(w^*)|. \end{aligned} \tag{12}$$

Proof. Inequality (2) implies

$$|\nabla X|^2 \le \frac{1}{2(1-aM)} \Delta x,$$

which in conjunction with

$$|\Delta X| \le a |\nabla X|^2$$

yields

$$|\Delta X| \le \frac{a}{2(1-aM)} \Delta x.$$

Pick some constant vector $E \in \mathbb{R}^N$ with $|E| = 1$ and consider the auxiliary function $z \in C^0(\overline{B}_R(w_0)) \cap C^2(B_R(w_0))$ which is defined by

$$z(w) := \frac{a}{2(1-aM)} [x(w) - h(w)] + \langle H(w) - X(w), E \rangle$$

and vanishes on $\partial B_R(w_0)$. Because of

$$\begin{aligned} \Delta z &= \frac{a}{2(1-aM)} \Delta x - \langle \Delta X, E \rangle \\ &\ge \frac{a}{2(1-aM)} \Delta x - |\Delta X| \ge 0, \end{aligned}$$

we see that z is subharmonic on $B_R(w_0)$. Then the maximum principle yields

$$\langle H - X, E \rangle \le \frac{a}{2(1-aM)} [h - x] \quad \text{on } \overline{B}_R(w_0)$$

for any unit vector E of $\mathbb{R}^N$, and we conclude that

$$|H - X| \le \frac{a}{2(1-aM)}[h - x] \tag{13}$$

holds on $\overline{B}_R(w_0)$. Moreover, the inequality $|X(w)| \le M$ in conjunction with the maximum principle gives

$$|H(w)| \le M \quad \text{for all } w \in \overline{B}_R(w_0).$$

Then we obtain

$$\begin{aligned} h - x &= (|H|^2 - |X|^2) + (h - |H|^2) \\ &\le 2M(|H| - |X|) + (h - |H|^2), \end{aligned}$$

whence

$$|H - X| \le \frac{aM}{1-aM}|H - X| + \frac{a}{2(1-aM)}(h - |H|^2).$$

Since

$$0 < 1 - \frac{aM}{1-aM} = \frac{1-2aM}{1-aM} < 1,$$

it follows that

$$|H - X| \le \frac{a}{2(1-2aM)}(h - |H|^2) \quad \text{on } \overline{B}_R(w_0). \tag{14}$$

For $w^* \in \partial B_R(w_0)$ we have

$$|X(w^*)|^2 = x(w^*) = |H(w^*)|^2 = h(w^*),$$

and therefore

$$\begin{aligned} |X(w) - X(w^*)| &\le |X(w) - H(w)| + |H(w) - H(w^*)| \\ &\le \frac{a}{2(1-2aM)}(h(w) - |H(w)|^2) + |H(w) - H(w^*)|. \end{aligned}$$

Because of

$$\begin{aligned} h(w) - |H(w)|^2 &= h(w) - h(w^*) + |H(w^*)|^2 - |H(w)|^2 \\ &\le |h(w) - h(w^*)| + 2M|H(w) - H(w^*)|, \end{aligned}$$

we may now conclude that (12) holds for any $w \in \overline{B}_R(w_0)$ and for any $w^* \in \partial B_R(w_0)$. □

Remark 1. Note that the differential inequality (1) remains invariant with respect to conformal transformations of the parameter domain. Thus we can carry over Proposition 2 from $B_R(w_0)$ to any bounded domain Ω in $\mathbb{C}$ which is of the conformal type of the disk and has a closed Jordan curve as its boundary.

Proposition 3. *For any $a \geq 0, R > 0, M \geq 0$, and $k \geq 0$ with $2aM < 1$, there is a number $c = c(a, R, M, k) \geq 0$ having the following property:*

Let $X \in C^0(\overline{B}_R(w_0), \mathbb{R}^N) \cap C^2(B_R(w_0), \mathbb{R}^N)$ be a solution of (1) in $B_R(w_0)$ satisfying $|X(w)| \leq M$ for all $w \in \overline{B}_R(w_0)$. Suppose also that the boundary values $\mathcal{X}(\varphi) := X(w_0 + Re^{i\varphi})$ are of class $C^2(\mathbb{R})$ and satisfy $|\mathcal{X}''(\varphi)| \leq k$ for all $\varphi \in \mathbb{R}$. Then we have

$$|\nabla X(w)| \leq c(a, R, M, k) \quad \text{for all } w \in B_R(w_0). \tag{15}$$

Proof. It suffices to treat the case $w_0 = 0$ and $R = 1$, that is, we consider the parameter domain $B = B_1(0)$. Let $w = re^{i\theta}, 0 < r < 1$, be an arbitrary point of B. By formula (8) of Proposition 1 we have

$$|\nabla X(w)| \leq \frac{c(a, M)}{1 - r} \sup\{|X(w') - X(w)| : w' \in B_{1-r}(w)\}. \tag{16}$$

Moreover, for $w, w' \in B$ and $w^* \in \partial B$ it follows from Proposition 2 that

$$\begin{aligned} |X(w) - X(w')| &\leq |X(w) - X(w^*)| + |X(w') - X(w^*)| \\ &\leq \frac{a}{2(1 - 2aM)}\{|h(w) - h(w^*)| + |h(w') - h(w^*)|\} \\ &\quad + \frac{1 - aM}{1 - 2aM}[|H(w) - H(w^*)| + |H(w') - H(w^*)|] \end{aligned} \tag{17}$$

holds true where H and h are harmonic in B and have the boundary values X and $x := |X|^2$ respectively on ∂B. By Lemma 5 of Section 2.1 we obtain that

$$|\nabla H(w)| \leq ck \quad \text{for all } w \in B$$

whence

$$|H(w_1) - H(w_2)| \leq ck|w_1 - w_2| \quad \text{for all } w_1, w_2 \in \overline{B}.$$

Therefore we have

$$|H(w) - H(w^*)| + |H(w') - H(w^*)| \leq 3ck(1 - r) \tag{18}$$

for $w = re^{i\theta}, w^* = e^{i\theta}, w' \in B_{1-r}(w)$.

Furthermore, the boundary values $\eta(\varphi) := |\mathcal{X}(\varphi)|^2$ of $x(e^{i\varphi})$ satisfy $\eta'' = 2|\mathcal{X}'|^2 + 2\langle \mathcal{X}, \mathcal{X}'' \rangle$, hence

$$|\eta''| \leq 2|\mathcal{X}'|^2 + 2|\langle \mathcal{X}, \mathcal{X}'' \rangle| \leq 2|\mathcal{X}'|^2 + 2Mk.$$

Let $E \in \mathbb{R}^N$ be a constant unit vector. Then we have

$$\int_0^{2\pi} \langle E, \mathcal{X}'(\varphi) \rangle \, d\varphi = 0.$$

Consequently, there is some $\varphi_0 \in [0, 2\pi]$ such that

$$\langle E, \mathfrak{X}'(\varphi_0)\rangle = 0$$

and therefore

$$\langle E, \mathfrak{X}'(\varphi)\rangle = \int_{\varphi_0}^{\varphi} \langle E, \mathfrak{X}''(\varphi)\rangle \, d\varphi.$$

Hence we obtain

$$|\langle E, \mathfrak{X}'(\varphi)\rangle| \le 2\pi k \quad \text{for all } \varphi \in [0, 2\pi].$$

Since E can be chosen as an arbitrary vector of $\mathbb{R}^N$, we conclude that

$$|\mathfrak{X}'(\varphi)| \le 2\pi k \quad \text{for all } \varphi \in \mathbb{R},$$

and therefore

$$|\eta''| \le 8\pi^2 k^2 + 2Mk.$$

Then we infer from Lemma 5 of Section 2.1 that

$$|\nabla h(w)| \le c^*(1 + k^2) \quad \text{for all } w \in B$$

whence

$$|h(w_1) - h(w_2)| \le c^*(1 + k^2)|w_1 - w_2| \quad \text{for all } w_1, w_2 \in \overline{B}$$

and consequently

$$|h(w) - h(w^*)| + |h(w') - h(w^*)| \le 3c^*(1 + k^2)(1 - r) \tag{19}$$

for $w = re^{i\theta}, w^* = e^{i\theta}, w' \in B_{1-r}(w)$.

Combining (17), (18), and (19), we arrive at

$$|X(w) - X(w')| \le c(a, M, k)(1 - r) \quad \text{for } w = re^{i\theta}, 0 < r < 1, \text{ and } w' \in B_{1-r}(w),$$

and this implies

$$|\nabla X(w)| \le c(a, M, k) \quad \text{for all } w \in B,$$

taking (16) into account. □

Theorem 1. *Suppose that the assumptions of Proposition 3 are satisfied. Then X is of class $C^{1,\mu}(\overline{B}_R(w_0), \mathbb{R}^N)$ for all $\mu \in (0, 1)$ and we have*

$$[\nabla X]_{\mu, \overline{B}_R(w_0)} \le c(a, R, M, k, \mu). \tag{20}$$

Proof. This result is an immediate consequence of Proposition 2 of Section 2.1 in conjunction with Proposition 3 that we have just proved. □

Now we come to the proof of the most important result with regard to the next section.

Theorem 2. *For $w_0 \in \partial B$, we introduce the set $S_\rho(w_0) := B \cap B_\rho(w_0)$. Assume that, for some $\rho \in (0,1)$, $X \in C^0(\overline{S}_\rho(w_0), \mathbb{R}^N) \cap C^2(S_\rho(w_0), \mathbb{R}^N)$ is a solution of the differential inequality (1) in $S_\rho(w_0)$ that vanishes on $\partial S_\rho(w_0) \cap \partial B$. Then we obtain $X \in C^{1,\mu}(\overline{S}_{\rho'}(w_0), \mathbb{R}^N)$ for every $\mu \in (0,1)$ and every $\rho' \in (0,\rho)$.*

Proof. It suffices to show that for any $w^* = e^{i\theta} \in \partial S_\rho(w_0) \cap \partial B$ there is a $\delta > 0$ such that $X \in C^{1,\mu}(\overline{S}_\delta(w^*), \mathbb{R}^N)$, where $S_\delta(w^*)$ denotes the circular two-gon $B \cap B_\delta(w^*)$. We may also assume that $a > 0$.

Thus, having fixed an arbitrary $w^* = e^{i\theta} \in \partial B \cap \partial S_\rho(w_0)$, we first choose an $\varepsilon > 0$ such that $S_{3\varepsilon}(w^*) \subset S_\rho(w_0)$ holds and that

$$\sup\{|X(w)|\colon w \in S_{3\varepsilon}(w^*)\} \le \frac{1}{4a}.$$

Then the mapping

$$Z(w) := 4aX, \quad w \in S_{3\varepsilon}(w^*),$$

satisfies the inequalities

$$|Z| \le 1 \quad \text{and} \quad |\Delta Z| \le \tfrac{1}{4}|\nabla Z|^2$$

in $S_{3\varepsilon}(w^*)$.

We now consider the functions $H(w)$ and $h(w)$ which are harmonic in $S_{3\varepsilon}(w^*)$ and which have the boundary values X and $|X|^2$ respectively on $\partial S_{3\varepsilon}$. As h and H vanish on the circular arc

$$C := \partial B \cap \partial S_{3\varepsilon}(w^*)$$

we can extend h and H to harmonic functions in $B_{3\varepsilon}(w^*)$ by reflection at C, applying Schwarz's reflection principle. Hence there is a number $c(\varepsilon)$ such that

$$|\nabla H(w)| + |\nabla h(w)| \le c(\varepsilon) \quad \text{for all } w \in B_{2\varepsilon}(w^*)$$

whence

$$\text{(21)} \qquad |H(w_1) - H(w_2)| + |h(w_1) - h(w_2)| \le c(\varepsilon)|w_1 - w_2| \quad \text{for all } w_1, w_2 \in B_{2\varepsilon}(w^*).$$

Fix some $w = re^{i\varphi} \in S_\varepsilon(w^*)$. Then we have $|w| = r > 1-\varepsilon$ and, for $|w-w'| < 1-r$, we have $|w' - w^*| \le |w' - w| + |w - w^*| < \varepsilon + \varepsilon = 2\varepsilon$, and consequently

$$B_{1-r}(w) \subset S_{2\varepsilon}(w^*) \subset S_{3\varepsilon}(w^*) \subset S_\rho(w_0).$$

By Proposition 2 and the subsequent Remark 1 we obtain for $\Omega := S_{3\varepsilon}(w^*), w' \in \Omega$ and $e^{i\varphi} \in \partial B \cap \partial S_{3\varepsilon}(w^*)$ that

$$|Z(w')| = |Z(w') - Z(e^{i\varphi})| \le \tfrac{1}{4}|h(w') - h(e^{i\varphi})| + \tfrac{3}{2}|H(w') - H(e^{i\varphi})|.$$

In connection with (21) we infer for any $w' \in B_{1-r}(w)$ that

$$|Z(w')| \le \tfrac{7}{4}c(\varepsilon)|w' - e^{i\varphi}| \le \tfrac{7}{2}c(\varepsilon)(1-r) < 4c(\varepsilon)(1-r).$$

In other words, we have

$$\sup\,\{|Z(w')|\colon w' \in B_{1-r}(w)\} \le 4c(\varepsilon)(1-r)$$

for any $w \in S_\varepsilon(w^*)$ with $|w| = r$.

Moreover, we infer from Proposition 1, (8) that

$$|\nabla Z(w)| \le \frac{2\kappa(1/4)}{1-r}\sup\{|Z(w')|\colon w' \in B_{1-r}(w)\}$$

for any $w \in S_\varepsilon(w^*)$ with $|w| = r$.

This implies

$$|\nabla Z(w)| \le c^*(\varepsilon) \quad \text{for all } w \in S_\varepsilon(w^*).$$

Since $X = \frac{1}{4a}Z$, we conclude that

$$|\Delta X|_{0,S_\varepsilon(w^*)} \le \text{const}, \quad \text{and} \quad |\nabla X|_{0,S_\varepsilon(w^*)} \le \text{const}.$$

Now we choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^2)$ with $\eta(w) = 1$ for $w \in B_\delta(w^*), \delta := \frac{\varepsilon}{2}$, and with $\eta(w) = 0$ for $|w - w^*| \ge \frac{3}{4}\varepsilon$. Then the mapping $Y := \eta X$ on $S_\varepsilon(w^*)$ satisfies

$$\Delta Y = \eta\Delta X + 2\nabla\eta\cdot\nabla X + \Delta\eta X$$

and therefore

$$|\Delta Y(w)| \le \text{const} \quad \text{for all } w \in S_\varepsilon(w^*), \tag{22}$$

$$Y(w) = 0 \quad \text{on } \partial S_\varepsilon(w^*), \tag{23}$$

$$Y(w) = 0 \quad \text{for all } w \in S_\varepsilon(w^*) \text{ with } \tfrac{3}{4}\varepsilon < |w - w^*| < \varepsilon. \tag{24}$$

Consider a conformal mapping τ of the unit disk B onto the two-gon $S_\varepsilon(w^*)$. We can extend τ to a homeomorphism of $\overline{B}$ onto $\overline{S}_\varepsilon(w^*)$, and it can be assumed that $\zeta = \pm 1$ are mapped onto the two vertices of the two-gon. By the reflection principle the mapping $\tau(\zeta)$ is holomorphic on $\overline{B}\setminus\{-1,1\}$. Then it follows from (22) and (24) that the mapping $Y^*(\zeta) := Y(\tau(\zeta))$ is of class $C^0(\overline{B},\mathbb{R}^N)\cap C^2(B,\mathbb{R}^N)$ and satisfies

$$|\Delta Y^*(\zeta)| \le \text{const} \quad \text{for all } \zeta \in B.$$

Moreover, we infer from (23) that

$$Y^*(\zeta) = 0 \quad \text{for all } \zeta \in \partial B.$$

Thus we can apply Proposition 2 of Section 2.1 and obtain that $Y^* \in C^{1,\mu}(\overline{B},\mathbb{R}^N)$ for any $\mu \in (0,1)$. It follows that $Y \in C^{1,\mu}(\overline{S}_\varepsilon(w^*),\mathbb{R}^N)$, and therefore $X \in C^{1,\mu}(\overline{S}_\delta(w^*),\mathbb{R}^N)$ as $Y(w) = X(w)$ for all $w \in \overline{S}_\delta(w^*)$, $\delta = \frac{\varepsilon}{2}$. □

2.3 The Boundary Regularity of Minimal Surfaces Bounded by Jordan Arcs

In this section we want to investigate the boundary behaviour of minimal surfaces at smooth Jordan arcs. The results can be applied to minimal surfaces bounded by one or several Jordan curves, to solutions of the partially free boundary problem, and to solutions of the thread problem (see Chapters 1 and 5 as well as Vol. 3, Chapters 1, 2).

As all results are of local nature, it suffices to formulate them on simply connected boundary domains, say, for minimal surfaces $X\colon B \to \mathbb{R}^3$ defined on the unit disk $B = \{w \in \mathbb{C}\colon |w| < 1\}$. The same results can be carried over without any problem to minimal surfaces $X\colon B \to \mathbb{R}^N, N \geq 2$. At the end of this section we shall sketch analogous results for minimal surfaces $X\colon B \to \mathcal{M}$ in an n-dimensional Riemannian manifold $\mathcal{M}$.

The main theorem of this section is the following result.

Theorem 1. *Consider a minimal surface $X\colon B \to \mathbb{R}^3$ of class $C^0(B\cup\gamma, \mathbb{R}^3)\cap C^2(B, \mathbb{R}^3)$ which maps an open subarc γ of ∂B into an open Jordan arc Γ of $\mathbb{R}^3$ which is a regular curve of class $C^{m,\mu}$ for some integer $m \geq 1$ and some $\mu \in (0, 1)$. Then X is of class $C^{m,\mu}(B\cup\gamma, \mathbb{R}^3)$. Moreover, if Γ is a regular real analytic Jordan arc, then X can be extended as a minimal surface across γ.*

In fact, we shall only prove a slightly weaker result. We want to show that the statement of the theorem holds under the assumption $\Gamma \in C^{m,\mu}$ with $m \geq 2$ and $0 < \mu < 1$. It remains to verify that the assumption $\Gamma \in C^{1,\mu}$ implies $X \in C^{1,\mu}(B\cup\gamma, \mathbb{R}^3)$. This can be carried out by employing a reflection method combined with refined potential-theoretic estimates. A version of this reasoning was invented by W. Jäger [3]. Other methods to prove this initial step can be found in Nitsche [16,20] and [28] (see Kapitel V, 2.1), Kinderlehrer [1], and Warschawski [5].

It will turn out that the method to be described also covers the boundary behaviour of surfaces of prescribed mean curvature at a smooth arc. Thus we shall deal with this more general result.

Theorem 2. *Let $X \in C^0(B\cup\gamma, \mathbb{R}^3)\cap C^2(B, \mathbb{R}^3)$ be a solution of the equations*

$$\Delta X = 2H(X)X_u \wedge X_v, \tag{1}$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0 \tag{2}$$

in B which maps an open subarc $\gamma = \{e^{i\theta}\colon \theta_1 < \theta < \theta_2\}$ of ∂B into some open regular Jordan arc Γ of $\mathbb{R}^3$, i.e. $X(w) \in \Gamma$ for all $w \in \gamma$. Then the following holds:

(i) *If $\mathcal{H}(w) := H(X(w))$ is of class $L_\infty(B)$, and if $\Gamma \in C^2$, then we obtain that $X \in C^{1,\mu}(B \cup \gamma, \mathbb{R}^3)$ for any $\mu \in (0, 1)$.*

(ii) *If H is of class $C^{0,\mu}$ on $\mathbb{R}^3$, and if $\Gamma \in C^{2,\mu}, 0 < \mu < 1$, then $X(w)$ is of class $C^{2,\mu}(B \cup \gamma, \mathbb{R}^3)$.*

Proof. (i) It suffices to show that for any $w_0 \in \gamma$ there is some $\delta > 0$ such that $X \in C^{1,\mu}(\overline{S}_\delta(w_0), \mathbb{R}^3), 0 < \mu < 1$, provided that $\mathcal{H}(w) := H(X(w))$ is of class $L_\infty(B)$ and that $\Gamma \in C^2$. Here $S_\delta(w_0)$ denotes as usual the two-gon $B \cap B_\delta(w_0)$.

Thus we fix some $w_0 \in \gamma$. Without loss of generality we may assume that $X(w_0) = 0$. For sufficiently small $\rho > 0$ we can represent $\Gamma \cap \mathcal{K}_\rho(0)$ in the form

$$(3) \qquad x^1 = g^1(t), \quad x^2 = g^2(t), \quad x^3 = t, \quad |t| < 2t_0,$$

where the functions $g^1(t)$ and $g^2(t)$ are of class C^2, and by a suitable motion in $\mathbb{R}^3$ we can arrange that

$$(4) \qquad g^k(0) = 0, \quad \dot{g}^k(0) = 0, \quad k = 1, 2,$$

choosing the parameter t appropriately.

We may also assume that $w_0 = 1$ and that $\gamma = \{w \in \partial B \colon |w - 1| < R_0\}$ for some $R_0 \in (0, 1)$. Choosing $t_0 > 0$ and $R \in (0, R_0]$ sufficiently small we can achieve that

$$(5) \qquad |\dot{g}^1(t)|^2 + |\dot{g}^2(t)|^2 \le \tfrac{1}{8} \quad \text{for } |t| < t_0$$

and

$$(6) \qquad |x^3(w)| < t_0 \quad \text{for } w \in \overline{S}_R(1).$$

Consider the auxiliary function $Y(w) = (y^1(w), y^2(w))$ which is defined by

$$(7) \qquad y^k(w) := x^k(w) - g^k(x^3(w)), \quad k = 1, 2, \ w \in \overline{S}_R(1),$$

where $X(w) = (x^1(w), x^2(w), x^3(w))$. Clearly, we have $Y \in C^0(\overline{S}_R(1), \mathbb{R}^2) \cap C^2(S_R(1), \mathbb{R}^2)$ and $Y(w) = 0$ for $w \in \partial B \cap \partial S_R(1)$. Moreover, we infer from (1) and from the relations

$$(8) \qquad \Delta y^k = \Delta x^k - \dot{g}^k(x^3)\Delta x^3 - \ddot{g}^k(x^3)|\nabla x^3|^2, \quad k = 1, 2,$$

that

$$(8^*) \qquad |\Delta Y| \le \alpha |\nabla X|^2$$

holds for some constant $\alpha > 0$.

In addition, we have

$$(9) \qquad x^k_w = y^k_w + \dot{g}^k(x^3)x^3_w, \quad k = 1, 2,$$

and therefore

$$(10) \qquad \begin{aligned} |x^1_w|^2 + |x^2_w|^2 &\le 2|y^1_w|^2 + 2|y^2_w|^2 + 2|x^3_w|^2 \sum_{k=1}^{2} |\dot{g}^k(x^3)|^2 \\ &\le 2|y^1_w|^2 + 2|y^2_w|^2 + \tfrac{1}{4}|x^3_w|^2 \end{aligned}$$

since (5) and (6) imply $\sum_{k=1}^{2} |\dot{g}^k(x^3)|^2 \le \frac{1}{8}$.

Now we write the conformality relations (2) as

$$0 = \langle X_w, X_w \rangle = (x_w^1)^2 + (x_w^2)^2 + (x_w^3)^2 \tag{11}$$

whence

$$|x_w^3|^2 \le |x_w^1|^2 + |x_w^2|^2$$

and therefore

$$\tfrac{1}{2}|X_w|^2 \le |x_w^1|^2 + |x_w^2|^2. \tag{12}$$

From (10) and (11) we infer

$$\tfrac{1}{4}|X_w|^2 \le 2|Y_w|^2$$

whence

$$|\nabla X|^2 \le 8|\nabla Y|^2. \tag{13}$$

From (8*) and (12) we derive the differential inequality

$$|\Delta Y| \le 8\alpha |\nabla Y|^2 \quad \text{on } S_R(1), \tag{14}$$

and we know already that

$$Y = 0 \quad \text{on } \partial B \cap \partial S_R(1). \tag{15}$$

Thus we can apply Theorem 2 of Section 2.2 to $Y\colon \overline{S}_R(1) \to \mathbb{R}^2$, and we obtain $Y \in C^{1,\mu}(\overline{S}_\varepsilon(1), \mathbb{R}^2)$ for any $\varepsilon \in (0, R)$ and any $\mu \in (0,1)$.

Combining (9) and (11) it follows that

$$0 = \sum_{k=1}^{2} (y_w^k)^2 + 2\sum_{k=1}^{2} \dot{g}^k(x^3) y_w^k x_w^3 + \left\{ 1 + \sum_{k=1}^{2} |\dot{g}^k(x^3)|^2 \right\} (x_w^3)^2. \tag{16}$$

If we introduce

$$p^k(t) := \frac{\dot{g}^k(t)}{q(t)}, \quad q(t) := 1 + \sum_{k=1}^{2} |\dot{g}^k(t)|^2, \tag{17}$$

this relation can be rewritten as

$$\left[x_w^3 + \sum_{k=1}^{2} p^k(x^3) y_w^k \right]^2 = \left\{ \sum_{k=1}^{2} p^k(x^3) y_w^k \right\}^2 - \frac{1}{q(x^3)} \sum_{k=1}^{2} (y_w^k)^2. \tag{18}$$

As the right-hand side of (18) is continuous in $\overline{S}_\varepsilon(1)$, it follows that [...] and therefore also x_w^3 are continuous. Thus we arrive at $X \in C^1(\overline{S}_\varepsilon(1), \mathbb{R}^3)$ for any $\varepsilon \in (0, R)$.

Multiplying (18) by $-w^2$, we obtain

$$(19)\quad \left[iwx_w^3 + \sum_{k=1}^{2} p^k(x^3) iwy_w^k\right]^2 = \frac{1}{q(x^3)} \sum_{k=1}^{2} (wy_w^k)^2 - \left\{\sum_{k=1}^{2} p^k(x^3) wy_w^k\right\}^2 .$$

Introducing polar coordinates r, φ with $w = re^{i\varphi}$, we find that

$$\begin{array}{ll} wx_w^l = \frac{1}{2}(rx_r^l - ix_\varphi^l), & wy_w^k = \frac{1}{2}(ry_r^k - iy_\varphi^k) \\ (l = 1,2,3) & (k = 1,2). \end{array}$$

For $w \in \gamma' := \partial B \cap \partial S_\varepsilon(1)$, we infer from (15) that the right-hand side of (19) is equal to

$$\frac{1}{4}|q(x^3)|^{-2} \left\{ \sum_{k=1}^{2} q(x^3)|y_r^k|^2 - \left(\sum_{k=1}^{2} \dot{g}^k(x^3) y_r^k \right)^2 \right\}$$

and this expression is real and nonnegative on account of (5) and (6). The left-hand side of (19) is of the form

$$(a+ib)^2 = (a^2 - b^2) + 2iab$$

with

$$a := \frac{r}{2}\left(x_\varphi^3 + \sum_{k=1}^{2} p^k(x^3) y_\varphi^k\right), \quad b := \frac{1}{2}\left(x_r^3 + \sum_{k=1}^{2} p^k(x^3) y_r^k\right).$$

From the relations

$$a^2 - b^2 \geq 0 \quad \text{and} \quad ab = 0$$

we infer that $b = 0$, that is,

$$(20)\qquad x_r^3 + \sum_{k=1}^{2} p^k(x^3) y_r^k = 0 \quad \text{on } \gamma'.$$

Thus we have found:

$$(21)\qquad |\Delta x^3| + |\nabla x^3| \leq \text{const} \quad \text{on } S_\varepsilon(1), \quad \frac{\partial}{\partial r} x^3 \in C^{0,\mu}(\gamma').$$

Now we choose a cut-off function $\eta \in C_c^\infty$ which is rotationally symmetric with respect to the pole $w = 1$ and satisfies $\eta(w) = 1$ for $w \in B_\delta(1), \delta := \frac{\varepsilon}{2}, \eta(w) = 0$ for $|w - 1| \geq \frac{3}{4}\varepsilon$. Set

$$(22)\qquad y(w) := \eta(w) x^3(w), \quad w \in \overline{S}_\varepsilon(1).$$

We have $y_r = \eta x_r^3 + \eta_r x^3$, and therefore

$$y_r \in C^{0,\mu}(\gamma'). \tag{23}$$

From the identity

$$\Delta y = \eta \Delta x^3 + x^3 \Delta \eta + 2\nabla \eta \cdot \nabla x^3$$

and from (21) we infer that

$$|\Delta y| \le \text{const} \quad \text{on } S_\varepsilon(1). \tag{24}$$

Finally we have

$$y(w) = 0 \quad \text{for all } w \in S_\varepsilon(1) \text{ with } \tfrac{3}{4}\varepsilon < |w-1| < \varepsilon. \tag{25}$$

Consider a conformal mapping τ of the unit disk B onto the two-gon $S_\varepsilon(1)$. We can extend τ to a homeomorphism of $\overline{B}$ onto $\overline{S}_\varepsilon(1)$, and it can be assumed that $\zeta = \pm 1$ are mapped onto the two vertices of the two-gon. By the reflection principle the mapping $\tau(\zeta)$ is holomorphic on $\overline{B} \setminus \{-1, 1\}$. Then it follows from (23) and (25) that the function $y^*(\zeta) := y(\tau(\zeta)), \zeta \in \overline{B}$, is of class $C^1(\overline{B}) \cap C^2(B)$ and satisfies

$$|\Delta y^*| \le \text{const} \quad \text{on } B,$$

$$\frac{\partial y^*}{\partial r} \in C^{0,\mu}(\partial B);$$

here the radial derivative y_r^* is the normal derivative of y^* on ∂B.

According to Section 2.1, Proposition 2, the solution $p(\zeta)$ of the boundary value problem

$$\Delta p = \Delta y^* \quad \text{in } B, \quad p = 0 \quad \text{on } \partial B$$

is of class $C^{1,\mu}(\overline{B})$ for any $\mu \in (0,1)$. Hence $h := y^* - p$ is of class $C^1(\overline{B}) \cap C^2(B)$, harmonic in B, and h_r is of class $C^{0,\mu}$ on ∂B. Therefore the conjugate harmonic function h^* with respect to h is of class $C^1(\overline{B})$ too, and the equation $h_r = h_\varphi^*$ on ∂B implies that $h^*|_{\partial B}$ is of class $C^{1,\mu}$. Applying the Korn–Privalov theorem (see Section 2.1, Lemmata 6 and 7) we infer that $h \in C^{1,\mu}(\overline{B})$, and therefore also $y^* \in C^{1,\mu}(\overline{B})$. Returning to $y = y^* \circ \tau^{-1}$ it follows that $y \in C^{1,\mu}(\overline{S}_\varepsilon(1))$. Since $y(w) = x^3(w)$ holds true for $w \in \overline{S}_\delta(1)$, $\delta = \frac{\varepsilon}{2}$, we finally arrive at $x^3 \in C^{1,\mu}(\overline{S}_\delta(1))$, and therefore $X \in C^{1,\mu}(\overline{S}_\delta(1), \mathbb{R}^3)$ for any $\mu \in (0,1)$. This concludes the proof of the first part of the theorem.

(ii) The initial step (i) is the crucial part of our investigation whereas the further proof is essentially potential-theoretic routine. However, as our estimates in Section 2.1 are not quite complete, we only want to indicate how one can proceed. The reader should use the Schauder estimates (described for example in Gilbarg and Trudinger [1]) to derive higher regularity by bootstrapping.

Thus let us assume that $H \in C^{0,\mu}(\mathbb{R}^3)$ and that $\Gamma \in C^{2,\mu}$ for some $\mu \in (0,1)$. Then $\mathcal{H}(w) := H(X(w))$ is of class $C^{0,\mu}(B \cup \gamma)$ and, using the notation of (i), the functions $g^1(t)$ and $g^2(t), |t| < 2t_0$, are of class $C^{2,\mu}$. On account of (8) and (2) the mapping Y satisfies

$$\begin{aligned} \Delta Y &= Q \quad \text{in } S_R(1), \\ Y &= 0 \quad \text{on } \partial B \cap \partial S_R(1) \end{aligned} \tag{26}$$

with $Q \in C^{0,\mu}(\overline{S}_R(1), \mathbb{R}^2)$.

Then a potential-theoretic reasoning yields $Y \in C^{2,\mu}(\overline{S}_\varepsilon(1), \mathbb{R}^2)$ for $0 < \varepsilon < R$. Now we use the equations (cf. (1) and (20))

$$\begin{aligned} \Delta x^3 &= 2H(X)(x_u^1 x_v^2 - x_v^1 x_u^2) \quad \text{in } S_R(1), \\ x_r^3 &= -\sum_{k=1}^{2} p^k(x^3) y_r^k \quad \text{on } \partial B \cap \partial S_R(1) \end{aligned} \tag{27}$$

to prove by a potential-theoretic argument that $x^3 \in C^{2,\mu}(\overline{S}_\varepsilon(1))$ for $0 < \varepsilon < R$. We only have to note that $p^1(x^3)y_r^1 + p^2(x^3)y_r^2$ is of class $C^{1,\mu}(\gamma')$ for $\gamma' = \partial B \cap \partial S_R(1)$ because of the result for Y that we obtained before. By virtue of (7) we then infer $X \in C^{2,\mu}(\overline{S}_\varepsilon(1), \mathbb{R}^3)$, and therefore $X \in C^{2,\mu}(B \cup \gamma, \mathbb{R}^3)$. □

Remark 1. Similarly one proves $X \in C^{m,\mu}(B \cup \gamma, \mathbb{R}^3)$ as claimed in Theorem 1 if $\Gamma \in C^{m,\mu}$ and $H \in C^{m-2,\mu}(\mathbb{R}^3)$. The proof is carried out by a bootstrap reasoning, considering the boundary value problems alternatingly. Since a similar idea is developed in detail in the following sections, we want to omit the proof of higher boundary regularity of X except for proving analyticity in the case that Γ is a real analytic, regular arc. This will be done next for a minimal surface. We shall present *H. Lewy's regularity theorem.*

In the following we shall suppose B to be the semidisk $\{w = u + iv \colon |w| < 1, v > 0\}$, and I will denote the straight segment $\{u \in \mathbb{R} \colon |u| < 1\}$ on the boundary of B.

Theorem 3. *Let $X \in C^0(B \cup I, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ be a minimal surface which maps I into a real-analytic and regular Jordan arc Γ in $\mathbb{R}^3$. Then X can be extended analytically across I as a minimal surface.*

Proof. Let $X^*(w)$ be an adjoint minimal surface to $X(w) = (x^1(w), x^2(w), x^3(w))$ which is assumed to satisfy $\Delta X = 0$ and (2) in B, and let

$$f(w) = X(w) + iX^*(w) = (f^1(w), f^2(w), f^3(w)) \tag{28}$$

be the holomorphic curve in $\mathbb{C}^3$ with $X = \operatorname{Re} f$ and $X^* = \operatorname{Im} f$ satisfying

$$\langle f'(w), f'(w) \rangle = 0. \tag{29}$$

By Theorem 2 we know already that $X \in C^2(B \cup I, \mathbb{R}^3)$ holds true. We have to show that for any $u_0 \in I$ there is some $\delta > 0$ such that $f(w)$ can be extended as a holomorphic curve from $S_\delta(u_0) := B \cap B_\delta(u_0)$ to $B_\delta(u_0)$. Without loss of generality we can assume that $u_0 = 0$. Set $B_\delta := B_\delta(0)$ and $S_\delta = S_\delta(0) = B \cap B_\delta$. As in the proof of Theorem 2 we can arrange for the following:

$$x^1(0) = x^2(0) = x^3(0) = 0, \quad \text{i.e.} \quad X(0) = 0.$$

For sufficiently small $\rho > 0$, we can represent $\Gamma \cap \mathcal{K}_\rho(0)$ in the form

$$x^1 = g^1(t), \quad x^2 = g^2(t), \quad x^3 = t, \quad t \in I_{2R_0} \tag{30}$$

where $I_\delta := \{t \in \mathbb{R} \colon |t| < \delta\}$, and $g^1(t)$ and $g^2(t)$ are real analytic functions on $I_{2R_0}, R_0 > 0$. Hence, choosing R_0 sufficiently small, we can assume that $g^1(\zeta)$ and $g^2(\zeta)$ are holomorphic functions of $\zeta \in B_{2R_0}$; hence

$$g(\zeta) := (g^1(\zeta), g^2(\zeta), \zeta), \quad |\zeta| < 2R_0$$

is a holomorphic curve on B_{2R_0}. In addition, we may (in accordance with $X(0) = 0$) assume that

$$g^1(0) = g^2(0) = \dot{g}^1(0) = \dot{g}^2(0) = 0$$

and

$$\left|\frac{dg^1}{d\zeta}(\zeta)\right|^2 + \left|\frac{dg^2}{d\zeta}(\zeta)\right|^2 \le \frac{1}{2} \quad \text{for } |\zeta| < 2R_0$$

are satisfied, and that

$$|f^3(w)| < R_0 \quad \text{holds for } w \in \overline{S}_R,$$

where R is a sufficiently small positive number.

Consider now the holomorphic function

$$F(w, \zeta) := \frac{\langle g'(\zeta), f'(w)\rangle}{\langle g'(\zeta), g'(\zeta)\rangle} \tag{31}$$

of $(w, \zeta) \in S_R \times B_{R_0}$, and note that F is of class C^1 on $\overline{S}_R \times \overline{B}_{R_0}$ (of course, differentiability in the second statement is real differentiability).

We claim that the differential equation

$$x^3_u(u) = F(u, x^3(u)) \quad \text{for } u \in I_R \tag{32}$$

holds true. In fact, the boundary condition $X(I) \subset \Gamma$ together with the above normalization implies that

$$X(u) = g(x^3(u)) \quad \text{for } u \in I_R, \tag{33}$$

hence

$$X_u(u) = g'(x^3(u))x^3_u(u) \quad \text{for } u \in I_R. \tag{34}$$

Therefore $x_u^3(u) = 0$ for some $u \in I_R$ yields $X_u(u) = 0$, and (2) gives $X_v(u) = 0$ or $X_u^*(u) = 0$ and consequently $f'(u) = 0$; therefore we also have $F(u, x^3(u)) = 0$, and (32) is trivially satisfied. Thus we may now assume that $x_u^3(u) \neq 0$. Because of (34) and (2), we obtain on I_R:

$$\begin{aligned} x_u^3 \langle g'(x^3), f' \rangle &= \langle g'(x^3) x_u^3, f' \rangle = \langle X_u, f' \rangle \\ &= \langle X_u, X_u + iX_u^* \rangle = |X_u|^2 - i\langle X_u, X_v \rangle = |X_u|^2 \\ &= \langle g'(x^3), g'(x^3) \rangle (x_u^3)^2 \end{aligned}$$

whence

$$x_u^3(u) = \frac{\langle g'(x^3), f' \rangle}{\langle g'(x^3), g'(x^3) \rangle}(u) = F(u, x^3(u)),$$

and (32) is verified. Thus $\zeta(u) = x^3(u)$ is a solution of the integral equation

$$\zeta(u) = \int_0^u F(\underline{u}, \zeta(\underline{u}))\, d\underline{u}. \tag{35}$$

It can easily be shown that there is some constant $M > 0$ such that

$$|F(w, \zeta) - F(w, \zeta')| \leq M |\zeta - \zeta'|$$

holds for all $w \in \overline{S}_R$ and $\zeta, \zeta' \in \overline{B}_{R_0}$. Then it follows from a standard fixed point argument that there is a number $\delta \in (0, R)$ such that the integral equation

$$z(w) = \int_0^w F(\underline{w}, z(\underline{w}))\, d\underline{w}, \quad w \in \overline{S}_\delta, \tag{36}$$

has exactly one solution $z(w), w \in \overline{S}_\delta$, in the Banach space $\mathcal{A}(\overline{S}_\delta)$ of functions $z \colon \overline{S}_\delta \to \mathbb{C}$ which are holomorphic in S_δ and continuous on $\overline{S}_\delta$. (As usual, the proof of this fact can easily be carried out by Picard's iteration method.[3]) Similarly one sees that the real integral equation (35) has (for $u \in \overline{I}_\delta$) exactly one solution $\zeta(u), u \in \overline{I}_\delta$, whence we obtain $\zeta(u) = x^3(u)$ and $\zeta(u) = z(u)$ for $|u| \leq \delta$, that is,

$$z(u) = x^3(u) \quad \text{for } u \in \overline{I}_\delta. \tag{37}$$

Consequently $z(w)$ is real-valued on I_δ, and by Schwarz's reflection principle we can extend $z(w)$ to a holomorphic function B_δ.

Now we consider the mapping $\phi \colon \overline{S}_\delta \to \mathbb{C}^3$, defined by

$$\phi(w) := f(w) - g(z(w)), \tag{38}$$

which is continuous on $\overline{S}_\delta$, holomorphic in S_δ, and purely imaginary on I_δ, since we have

[3] The integral in (36) is a complex line integral independent of the path from 0 to w within $\overline{S}_\delta$.

(39) $$\phi(u) = f(u) - g(z(u)) = X(u) + iX^*(u) - X(u) = iX^*(u)$$

on account of (33). Applying the reflection principle once again, we can extend $\phi(w)$ to a holomorphic function on B_δ, and therefore also

(40) $$f(w) = \phi(w) + g(z(w))$$

is extended to a holomorphic mapping on B_δ. □

We conclude this section by sketching the proof of a generalization of Theorem 1, employing the method of the proof of Theorem 2.

Theorem 4. *Let $\mathcal{M}$ be a Riemannian manifold of class C^2, and let Γ be an open regular Jordan arc in $\mathcal{M}$ which is of class C^2. Moreover let $X \in C^2(B, \mathcal{M}), B = \{w \in \mathbb{C}\colon |w| < 1\}$ be a minimal surface in $\mathcal{M}$. Finally we assume that γ is an open subarc of ∂B such that $X \in C^0(B \cup \gamma, \mathcal{M})$ and that $X(\gamma) \subset \Gamma$. Then we have:*

(i) *$X \in C^{1,\mu}(B \cup \gamma, \mathcal{M})$ for any $\mu \in (0, 1)$.*

(ii) *If $\mathcal{M}$ and Γ are of class $C^{m,\mu}, m \geq 2, 0 < \mu < 1$, then $X \in C^{m,\mu}(B \cup \gamma, \mathcal{M})$.*

(iii) *If $\mathcal{M}$ and Γ are real analytic, then X is real analytic in $B \cup \gamma$ and can be extended as a minimal surface across γ.*

Proof. We shall sketch a proof of (i). The results of (ii) can be derived from (i) by employing a bootstrap reasoning together with potential-theoretic estimates, as described in the proof of Theorem 2 and in Remark 1. The proof of (iii) now follows from a general theorem by Morrey [8] (cf. Theorem 6.8.2, pp. 278–279). We refer the reader to Hildebrandt [3], p. 80, for an indication how Morrey's result can be used to prove (iii). Another proof (in the spirit of H. Lewy) can be obtained by the method of F. Müller [1–3].

Let us now turn to step (i). We fix some point $w_0 \in \gamma$. Then there is some $R > 0$ such that X maps $\overline{S}_R(w_0) := \overline{B \cap B_R(w_0)}$ into some coordinate patch on the manifold $\mathcal{M}$ since X is continuous on $B \cup \gamma$. Introducing local coordinates $(x^1, x^2, \ldots, x^n)$ on this patch, we can represent X in the form

$$X(w) = (x^1(w), x^2(w), \ldots, x^n(w)) \quad \text{for } w \in S_R(w_0)$$

with $X \in C^0(\overline{S}_R(w_0) \cup \gamma, \mathbb{R}^n) \cap C^1(S_R(w_0), \mathbb{R}^n)$.

Suppose that the line element ds of $\mathcal{M}$ on the patch is given by

(41) $$ds^2 = g_{kl}(x)\, dx^k\, dx^l,$$

where repeated Latin indices are to be summed from 1 to n, and let

(42) $$\Gamma^l_{jk} = \tfrac{1}{2} g^{rl}(g_{jr,k} + g_{rk,j} - g_{jk,r})$$

be the Christoffel symbols corresponding to g_{kl}, where $(g^{rl}) = (g_{jk})^{-1}$. Then we have the equations

(43) $$\Delta x^l + \Gamma^l_{jk}(X)\{x^j_u x^k_u + x^j_v x^k_v\} = 0, \quad 1 \le l \le n,$$

and

(44) $$g_{kl}(X)x^k_u x^l_u = g_{kl}(X)x^k_v x^l_v, \quad g_{kl}(X)x^k_u x^l_v = 0.$$

(Equations (43) replace the equations $\Delta x^l = 0$ holding in the Euclidean case, and equations (44) are the Riemannian substitute of the conformality relations (2).)

Without loss of generality we may assume that $w_0 = 1$, and we set $S_R := S_R(1), 0 < R < 1$, and $\gamma' = \partial B \cap \partial S_R$. We can also assume that the coordinate patch containing $X(\overline{S}_R)$ is described by $\{x \in \mathbb{R}^n : |x| < 1\}$ and that $X(1) = 0$. Furthermore, we can assume that Γ in $\{|x| < 1\}$ is described by $x^1 = x^2 = \cdots = x^{n-1} = 0$, and that $g_{kl} \in C^1, g_{kl}(0) = \delta_{kl}$. Thus we have

$$|X(w)| < 1 \quad \text{for } w \in \overline{S}_R$$

and

$$x^\alpha(w) = 0 \quad \text{for } \alpha = 1, \dots, n-1 \text{ and } w \in \gamma'.$$

We write (44) as

$$g_{kl}(X)x^k_w x^l_w = 0, \quad w \in \overline{S}_R,$$

which can be transformed into

(45) $$\left(x^n_w + \frac{g_{\alpha n}(X)}{g_{nn}(X)} x^\alpha_w\right)^2 = \left(\frac{g_{\alpha n}(X)}{g_{nn}(X)} x^\alpha_w\right)^2 - \frac{g_{\alpha\beta}(X)}{g_{nn}(X)} x^\alpha_w x^\beta_w$$

(summation with respect to repeated Greek indices is supposed to run from 1 to $n-1$).

The definiteness of the matrix (g_{kl}) implies

$$m_1 \le g_{nn}(x) \quad \text{and} \quad |g_{kl}(x)| \le m_2 \quad \text{for } |x| < 1$$

where m_1 and m_2 denote two positive constants. Then we obtain from (45) that there is some constant $m_3 > 0$ such that

$$|\nabla x^n|^2 \le m_3 \sum_{\alpha=1}^{n-1} |\nabla x^\alpha|^2 \quad \text{on } \overline{S}_R.$$

As in the proof of Theorem 1 we infer from (43) and (45) that the mapping

$$Y(w) := (x^1(w), x^2(w), \dots, x^{n-1}(w))$$

is of class $C^0(\overline{S}_R) \cap C^2(S_R)$ and satisfies the relations

(46) $$\begin{aligned} |\Delta Y| &\le m_4 |\nabla Y|^2 \quad \text{on } S_R, \\ Y &= 0 \quad \text{on } \gamma'. \end{aligned}$$

Now we proceed as in the proof of Theorem 2. In fact, from (46) we infer that $Y \in C^{1,\nu}(\overline{S}_\varepsilon)$ for any $\nu \in (0,1)$ and $\varepsilon \in (0,R)$, whence (45) implies that x_w^n is of class $C^0(\overline{S}_\varepsilon)$. Therefore we obtain $X \in C^1(\overline{S}_\varepsilon)$.

Moreover, from (45) and (46_2) it follows that

$$\left(iwx_w^n + \frac{g_{\alpha n}}{g_{nn}} iwx_w^\alpha\right)^2 = \left\{\frac{i}{2}\left(x_r^n + \frac{g_{\alpha n}}{g_{nn}} x_r^\alpha\right) + \frac{1}{2}\left(x_\varphi^n + \frac{g_{\alpha n}}{g_{nn}} x_\varphi^\alpha\right)\right\}^2$$
$$= \frac{g_{\alpha\beta}}{g_{nn}}(wx_w^\alpha)(wx_w^\beta) - \left(\frac{g_{\alpha n}}{g_{nn}} wx_w^\alpha\right)^2 \geq 0$$

on $\gamma'' := \gamma' \cap \partial S_\varepsilon$ (cf. the computations leading to (20)). Hence we have

$$x_r^n(e^{i\varphi}) = -\frac{g_{\alpha n}(0,\dots,0,x^n(e^{i\varphi}))}{g_{nn}(0,\dots,0,x^n(e^{i\varphi}))} x_r^\alpha(e^{i\varphi}) \quad \text{on } \gamma''. \tag{47}$$

Setting

$$p = -\Gamma_{kl}^n(X)\{x_u^k x_u^l + x_v^k x_v^l\}, \tag{48}$$

it follows that

$$\Delta x^n = p \quad \text{in } S_\varepsilon, \quad x_r^n = f \quad \text{on } \gamma'', \tag{49}$$

where x^n is of class C^1 on $\overline{S}_\varepsilon$, of class C^2 on $S_\varepsilon, p \in L_\infty(S_\varepsilon), f \in C^{0,\nu}(\gamma'')$. Then a potential-theoretic reasoning yields $x^n \in C^{1,\nu}(\overline{S}_{\varepsilon'})$ for $0 < \varepsilon' < \varepsilon$ and therefore $X \in C^{1,\nu}(\overline{S}_{\varepsilon'})$.

Alternating between (49) and

$$\Delta Y = Q \quad \text{in } S_\varepsilon, \quad Y = 0 \quad \text{on } \gamma'', \tag{50}$$

where

$$Q^\alpha := -\Gamma_{kl}^\alpha(X)(x_u^k x_u^l + x_v^k x_v^l),$$

we obtain higher regularity of X at the boundary part γ. This completes the sketch of the proof. □

2.4 The Boundary Behaviour of Minimal Surfaces at Their Free Boundary: A Survey of the Results and an Outline of Their Proofs

The boundary behaviour of minimal surfaces with free boundaries is somewhat more difficult to treat than that of solutions of Plateau's problem. In fact, Courant [9,15] has exhibited a number of examples indicating that the trace of a minimal surface with a free boundary on a continuous support surface S need not be continuous. One of his examples even shows that the trace curve can be unbounded although S is smooth (but not compact). Unfortunately Courant's examples are not rigorous as their construction is based on a heuristic principle, the *bridge theorem*, which has not yet been established for solutions of free boundary problems, and therefore we shall describe Courant's idea only in the Scholia. However, one of Courant's constructions is not based on the bridge theorem and has been made perfectly rigorous by Cheung [1].

We consider here a modification of Cheung's example. The supporting surface S (see Fig. 1) in our example will be defined as follows. Let us define sets $B_1, B_2, C, E_\pm, G$ and curves $\gamma_\pm, \beta_\pm$ by

$$\begin{aligned}
B_1 &:= \{(x,y,z)\colon x = 0, -1 \le y \le 1, -3 \le z \le 0\},\\
B_2 &:= \{(x,y,z)\colon x = 0, -1 \le y \le 1, -5 \le z \le -3\},\\
C &:= \{(x,y,z)\colon z = 0, x \ge 0, -e^{-x} \le y \le e^{-x}\},\\
E_\pm &:= \{(x,y,z)\colon x \ge 0, y = \pm 1, -5 \le z \le -3\},\\
G &:= \{(x,y,z)\colon x \ge 0, -1 \le y \le 1, z = -5\},\\
\gamma_\pm &:= \{(x,y,0)\colon x \ge 0, y = \pm e^{-x}\},\\
\beta_\pm &:= \{(x,y,-3)\colon x \ge 0, y = \pm 1\}.
\end{aligned}$$

Now we connect each point $(x, \pm e^{-x}, 0)$ on $\gamma_\pm$ by straight segments with the corresponding point $(x, \pm 1, -3)$ on $\beta_\pm$, thus obtaining two ruled surfaces $F_\pm$.

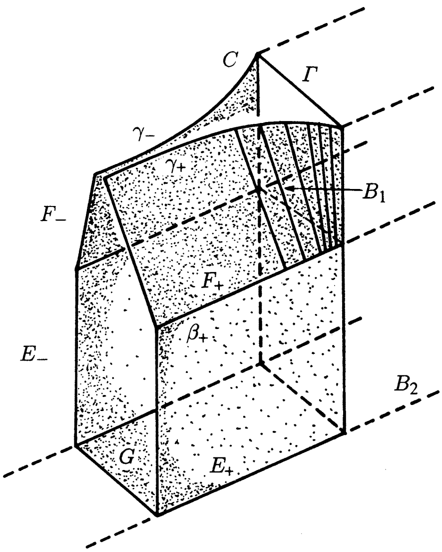

Fig. 1. A noncompact, Lipschitz continuous, nonclosed supporting surface S which satisfies no chord-arc condition. The configuration $\langle \Gamma, S\rangle$ bounds an unbounded minimal surface of the type of the disk

Let

$$S_1 : = E_+ \cup E_- \cup F_+ \cup F_- \cup G$$

and denote by S_1^* the reflection of S_1 at the plane $\{x = 0\}$. Then we define

$$S := S_1 \cup S_1^*$$

and

$$\Gamma := \{(x, y, z) \colon x = 0, z = 0, -1 \le y \le 1\}.$$

Claim. *Every solution $Y \in \mathcal{C}(\Gamma, S)$ of the corresponding free boundary problem $\mathcal{P}(\Gamma, S)$ has an unbounded trace on S; in particular Y is discontinuous along the interval I.*

In fact, suppose that $Y \in \mathcal{C}(\Gamma, S)$ is a solution of $\mathcal{P}(\Gamma, S)$ which is continuous on $B \cup \overline{I}$. Then the trace $Y(\overline{I})$ is compact and has to pass G continuously as $Y(\overline{I}) \subset S$. By a projection argument we infer that the area of the part of $Y(B)$ below the plane $\{z = -3\}$ is greater than or equal to the area of B_2 which is 4, and thus it is larger than the area of C which is 2. Thus each solution Y of $\mathcal{P}(\Gamma, S)$ must have a discontinuous trace $Y|_{\overline{I}}$. In fact, $Y|_{\overline{I}}$ cannot be contained in the subregion $S_R := \mathcal{K}_R(0) \cap S$ for any $R > 0$. (Otherwise, by Theorem 2 of Section 2.5, we would obtain $Y \in C^{0,\mu}(\overline{B}, \mathbb{R}^3)$ for some $\mu > 0$.) Hence it follows that the trace $Y|_{\overline{I}}$ is unbounded, and a projection argument shows that Y has to be a parametrization of C or of its reflection C^* at $\{x = 0\}$. In other words, if there is a solution Y of $\mathcal{P}(\Gamma, S)$, it will be given either by C or by C^*. As the existence theory of Vol. 1, Chapter 4 yields the existence of a solution of $\mathcal{P}(\Gamma, S)$, we infer that C *and* C^* *are the two solutions of* $\mathcal{P}(\Gamma, S)$ *and that there is no other solution of this minimum problem.*

By reflecting S at the plane $\{z = 0\}$ we can extend it to a Lipschitz continuous noncompact surface $\tilde{S}$ without boundary. Furthermore, by rounding off the edges of S and of $\tilde{S}$, we can even construct examples of smooth supporting surfaces, with or without boundary, having the desired property that there is no solution $Y \in C^0(B \cup \overline{I}, \mathbb{R}^3)$ of the minimum problem $\mathcal{P}(\Gamma, S)$.

Thus we have an example of a boundary configuration $\langle \Gamma, S \rangle$ consisting of a smooth arc Γ and a smooth support surface S for which the minima of area in $\mathcal{C}(\Gamma, S)$ have a discontinuous (and even unbounded) trace on S. However, the reader will note that the surface S in the Courant example does not satisfy a uniform extrinsic Lipschitz condition in $\mathbb{R}^3$, i.e., the quotient of the distance of two points on S divided by their air distance is unbounded. We say that S does not satisfy a *chord-arc condition* (the precise definition of this condition will be given in Section 2.5).

Surprisingly, the chord-arc condition suffices to enforce that all minima of a free or partially free boundary problem have a continuous free trace. In fact, we shall prove:

(i) *Suppose that X minimizes Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$ and that $D(X) > 0$. Assume also that the support surface S satisfies a chord-arc condition. Then X is of class $C^{0,\mu}(B \cup I, \mathbb{R}^3)$ for some $\mu \in (0,1)$.*

This result is the main statement of Theorem 1 in Section 2.5. The proof is based on an adaptation of Morrey's idea to compare any minimizer locally with a suitable harmonic mapping. To make this idea effective one constructs such a mapping by exploiting the chord-arc condition in order to set up its boundary values on S.

Several variants of the assertion (i) are given in Theorems 2–4 of Section 2.5. In particular, Theorem 4 of Section 2.5 provides a regularity theorem analogous to (i), holding for minimizers of a completely free boundary problem.

In Section 2.6 we shall prove regularity of stationary points of Dirichlet's integral at their free boundaries. At present it is not known whether the free trace Σ of any such surface X is a continuous curve provided that the support surface S satisfies merely a chord-arc condition. However, assuming that S is of class C^2 we obtain the desired result. More precisely, we have:

(ii) *Let S be an admissible support surface of class C^2, and suppose that X is a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Then there is some $\alpha \in (0,1)$ such that $X \in C^{0,\alpha}(B \cup I, \mathbb{R}^3)$.*

This result is the content of Theorem 2 in Section 2.6; a similar statement can be obtained for solutions of completely free boundary problems (cf. Section 2.6, Remark 2).

The proof of (ii) is quite different from that of (i). Whereas in (i) we shall proceed by deriving a priori estimates for X, the approach in (ii) is indirect. Using the finiteness of Dirichlet integral of X we shall first derive suitable *monotonicity results* for functionals that are closely related to Dirichlet's integral. Combining these results we shall infer that X has to be continuous on $B \cup I$ if $D(X) < \infty$.

Once the boundary values $X|_I$ are shown to be continuous, we can apply suitable techniques from the theory of nonlinear elliptic equations to obtain $X \in C^{0,\alpha}(B \cup I, \mathbb{R}^3), \alpha \in (0,1)$. For instance, Widman's hole-filling method (cf. Lemma 5 in Section 2.6) yields a direct way to this result; the details are carried out in the proof of Theorem 2 in Section 2.6.

Note that in all these cases the support surface S may have a nonempty boundary. If ∂S is void, we can say much more on the free trace $\Sigma = \{X(w) \colon w \in I\}$ of X on S. Roughly speaking Σ will turn out to be as good as the support surface S. We shall, in fact, obtain:

(iii) *Let S be an admissible support surface with $\partial S = \emptyset$ which is of class $C^{m,\beta}, m \geq 3, \beta \in (0,1)$. Then any stationary point X of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is of class $C^{m,\beta}(B \cup I, \mathbb{R}^3)$. If S is real analytic, then X is real analytic on $B \cup I$ and can be continued analytically across I.*

This result is formulated in Theorems 1 and 2 of Section 2.8. Starting from (ii), we shall first verify that X is contained in $C^1(B \cup I, \mathbb{R}^3)$. This can either be achieved by transforming the boundary problem for X locally into an interior regularity question for some weak solution Z of an elliptic system

$$\Delta Z = F(w)|\nabla Z|^2,$$

which is derived by a reflection argument, and then applying Tomi's regularity theorem, or by playing the full regularity machinery for nonlinear elliptic boundary value problems. The first possibility is sketched in Remark 1 of Section 2.8, whereas the second approach is discussed in Section 2.7 in great detail and in a wider context (see in particular Theorem 4 of Section 2.7).

Having proved that X is of class $C^1(B \cup I, \mathbb{R}^3)$, we use classical results from potential theory to derive $X \in C^{m,\beta}(B \cup I, \mathbb{R}^3)$ by employing a suitable bootstrap argument. The reader can find this reasoning in the proof of Proposition 1 in Section 2.8.

In Theorem 2 of Section 2.8 we show that X can be continued analytically across its free boundary if the support surface S is real analytic. To this end, we set up a Volterra integral equation

$$Z(w) = \int_0^w F(\omega, Z(\omega))\, d\omega$$

which has exactly one solution Z in the space $\mathcal{A}(\overline{S}_\delta)$ of mappings $Z\colon \overline{S}_\delta \to \mathbb{C}^3$ which are continuous on $\overline{S}_\delta$ and holomorphic in $S_\delta := \{w\colon |w| < \delta < 1, \operatorname{Im} w > 0\}$, and F is constructed in such a way that

$$Z(u) = X(u) \quad \text{for } u \in \mathbb{R} \text{ with } |u| < \delta$$

(assuming a suitable normalization of X).

Let X^* be an adjoint surface of X and $f = X + iX^*$. Then both f and $g := f - Z$ are of class $\mathcal{A}(\overline{S}_\delta)$, and we have

$$\operatorname{Im} Z = 0 \quad \text{and} \quad \operatorname{Re} g = 0 \quad \text{on } I_\delta.$$

By Schwarz's reflection principle, we can continue both Z and g across I_δ as holomorphic functions, whence also $f = g + Z$ and $X = \operatorname{Re} f$ are continued analytically across I_δ.

This approach to analyticity at the boundary, due to H. Lewy, is by far the easiest, but it cannot be carried over to H-surfaces or to minimal surfaces in a Riemannian manifold as it uses the holomorphic function $f = X + iX^*$. This tool is, however, not available in those other cases. Here one can apply a general regularity theorem due to Morrey [5] (cf. also Morrey and Nirenberg [1] and Morrey [8]), or the work of Frank Müller which extends Lewy's method to more general situations.

Let us now turn to the case when the support surface S has a nonempty boundary. Then we shall establish the following result (cf. Section 2.7, Theorem 1):

(iv) *Let S be an admissible support surface of class C^4* (by definition, this implies $\partial S \in C^4$; cf. Section 2.6, Definitions 1 and 2). *Moreover, let X be a stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$. Then X is of class $C^{1,1/2}(B \cup I, \mathbb{R}^3)$.*

According to Remark 1 in Section 1.8 of Vol. 3, this is the best possible result which can, in general, be expected. This follows from the asymptotic expansions (1) and (2) in Section 2.10 around points u_1 and u_2 on I where the free trace $X|_I$ of X on S lifts off the boundary ∂S of the support surface S. We could interpret (iv) as a regularity result for a *Signorini problem* (or else for a *thin obstacle problem*). The proof of (iv) will be carried out in three steps. First, by applying Nirenberg's difference quotient technique, we shall derive L_2-estimates for the second derivatives $\nabla^2 X$ up to the free boundary. For this purpose we need the Hölder continuity of X on $B \cup I$, established in (ii), as well as an important calculus inequality due to Morrey (cf. Section 2.7, Lemma 2) which implies that the Morrey seminorm is reproducible.

As a second step it will be shown that X is of class $C^1(B \cup I, \mathbb{R}^3)$. This follows from L_p-estimates for solutions of the Poisson equation. In order to apply these estimates we introduce suitable local coordinates $\{\mathcal{U}, g\}$ on S such that the boundary conditions for $Y(w) = (y^1(w), y^2(w), y^3(w)) = g(X(w))$ become uncoupled. For y^2 we derive a Neumann condition and for y^3 a Dirichlet condition. Then we apply the L_p-estimates to y^2 and y^3, thus obtaining Hölder continuity of ∇y^2 and ∇y^3 up to the free boundary. Finally, the continuity of y^1 up to the free boundary will be derived from the conformality relations.

As a third step we devise an iteration scheme, which allows us to attain $X \in C^{1,1/2}(B \cup I, \mathbb{R}^3)$, by exploiting once again the conformality relations.

The use of L_p-estimates can be circumvented by a method which is developed in Section 2.9. Here one derives directly that $\nabla^2 X$ satisfies a Dirichlet growth condition (i.e., has a finite Morrey seminorm) up to the free boundary by alternatively applying one of two possible Poincaré inequalities. This method is nothing but a skilful improvement of the estimates derived in step 1.

Note that the regularity results (i)–(iv) are not directly meaningful for differential geometry, as the free boundary I may contain branch points. This can, at least partially, be remedied in the following way. First, by applying a technique due to Hartman and Wintner, we show that, for every branch point $w_0 \in I$, we have an asymptotic expansion

$$X_w(w) = A(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0$$

with some $\nu \in \mathbb{N}$ and $A \in \mathbb{C}^3, A \neq 0, \langle A, A \rangle = 0$.

This implies that there exists a limit tangent plane of X as $w \to w_0$ with the normal $N_0 = \lim_{w \to w_0} N(w)$, where

$$N(w) = |X_u|^{-2}(X_u \wedge X_v),$$

and that the oriented tangent

$$t(u) := |X_u(u)|^{-1} X_v(u) \quad \textit{as } u \to w_0 = u_0 \in I$$

of the free trace $X|_I$ *is either continuous or jumps by 180 degrees; the first case occurs if the order* ν *of the branch point* $w_0 = u_0 \in I$ *is even, the second case, if* ν *is odd.*

Hence, if ν is even, the representation $\boldsymbol{x}(s)$ of the trace $\Sigma = X|_I$ with respect to its arc length

$$s = \int_{u_0}^{u} |X_u(u)|\, du$$

is of class C^1, and therefore the trace Σ can be viewed as a regular C^1-curve in the neighbourhood of $x_0 := X(u_0)$. If ν is odd, then Σ has a *cusp at* x_0, and only the unoriented tangent is continuous at x_0.

We sketch the derivation of this result in Section 2.10; the details of the Hartman–Wintner technique are given in Chapter 3. In the first two chapters of Vol. 3 we shall study cases where boundary branch points can entirely be excluded.

Most of our results will be stated and proved merely for stationary points of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$, that is, for solutions of a partially free boundary problem. Similar results hold mutatis mutandis for minimal surfaces with completely free boundaries, or for minimal surfaces of higher topological type spanned in a general boundary configuration $\langle \Gamma_1, \dots, \Gamma_l, S_1, \dots, S_m \rangle$, and their proofs can be carried out in essentially the same way. In fact, the considerations in Sections 2.7–2.10 are strictly local and require only changes in notation, and the reasoning of Sections 2.5 and 2.6 can be adjusted without major difficulties. We leave it as an exercise to the reader to carry out the details.

2.5 Hölder Continuity for Minima

Courant's examples indicate that one cannot expect a solution of a free or a semifree boundary problem to be regular at its free boundary, even if the support surface S is of class C^∞. On the other hand, we shall see that a minimal surface is continuous up to its free boundary if it is minimizing and if S satisfies a kind of uniform (local) Lipschitz condition. Such a condition on S will be called a *chord-arc* condition.

Definition. *A set* S *in* $\mathbb{R}^3$ *is said to fulfil a chord-arc condition with constants* M *and* $\delta, M \geq 1$ *and* $\delta > 0$, *if it is closed and if any two points* P *and* Q *of* S *whose distance* $|P - Q|$ *is less than or equal to* δ *can be connected in* S *by a rectifiable arc* Γ^* *whose length* $L(\Gamma^*)$ *satisfies*

$$L(\Gamma^*) \leq M|P - Q|.$$

For example, every compact regular C^1-surface S without boundary satisfies a chord-arc condition, and the same holds true if the boundary ∂S is nonempty but smooth.

Let us first deal with the semifree problem. We now denote by B the parameter domain

$$B = \{w = u + iv\colon |w| < 1, v > 0\}$$

the boundary of which consists of the circular arc

$$C = \{w = u + iv\colon |w| = 1, v \geq 0\}$$

and of the interval

$$I = \{u \in \mathbb{R}\colon |u| < 1\}$$

on the real axis.

Consider a boundary configuration $\langle \Gamma, S\rangle$ consisting of a closed set S in $\mathbb{R}^3$ satisfying a chord-arc condition and of a Jordan curve Γ in $\mathbb{R}^3$ whose endpoints P_1 and P_2 lie on $S, P_1 \neq P_2$. As in Section 4.6 of Vol. 1 we define the class of admissible surfaces for the semifree problem as the set $\mathcal{C}(\Gamma, S)$ of mappings $X \in H_2^1(B, \mathbb{R}^3)$ satisfying

(i) $X(w) \in S$ $\mathcal{H}^1$-a.e. on I;

(ii) $X\colon C \to \Gamma$ is a continuous, weakly monotonic mapping of C onto Γ such that $X(1) = P_1, X(-1) = P_2$.

Let us also introduce the sets

$$\begin{aligned} Z_d &:= \{w \in B\colon |w| < 1 - d\} = \{w \in B\colon \operatorname{dist}(w, C) > d\}, \quad (0 < d < 1),\\ S_r(w_0) &:= B \cap B_r(w_0). \end{aligned}$$

Then we can prove:

Theorem 1. *Suppose that $X \in \mathcal{C}(\Gamma, S)$ minimizes the Dirichlet integral $D(X)$ within the class $\mathcal{C}(\Gamma, S)$, and let $e = e(\Gamma, S) := \inf\{D(Y)\colon Y \in \mathcal{C}(\Gamma, S)\}$ be positive. Moreover, assume that S satisfies a chord-arc condition with constants M and δ. Then, for any $d \in (0,1)$, and $w_0 \in \overline{Z}_d$, and for any $r > 0$, we have*

$$\int_{S_r(w_0)} |\nabla X|^2\, du\, dv \leq \left(\frac{2r}{d}\right)^{2\mu} \int_B |\nabla X|^2\, du\, dv, \tag{1}$$

where

$$\mu := \min\,\{(1 + M^2)^{-1}, \delta^2/(2e\pi)\}. \tag{2}$$

It follows that X is of class $C^{0,\mu}(\overline{Z}_d, \mathbb{R}^3)$ and that

$$[X]_{\mu, \overline{Z}_d} \leq c(\mu)(1 - d)^{-\mu}\sqrt{D(X)} = c(\mu)(1 - d)^{-\mu}\sqrt{e(\Gamma, S)} \tag{3}$$

holds true for some constant $c(\mu) > 0$.

Proof. Let X be a minimizer of the Dirichlet integral in $\mathcal{C}(\Gamma, S)$. Then X is harmonic in B, satisfies the conformality relations

$$|X_u| = |X_v|, \quad \langle X_u, X_v \rangle = 0, \tag{4}$$

and

$$D(X) = e.$$

For any point $w_0 \in \overline{B}$ we define

$$\Phi(r, w_0) := \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv. \tag{5}$$

We begin by proving that for any $d \in (0,1)$ and for any $w_0 \in I$ with $|w_0| \le 1-d$ the inequality

$$\Phi(r, w_0) \le (r/d)^{2\mu} \Phi(d, w_0) \tag{6}$$

holds true for all $r \in (0, d]$. To this end we fix $w_0 \in I$ with $|w_0| \le 1 - d$ and set $B_r := B_r(w_0), S_r := S_r(w_0)$, and $\Phi(r) := \Phi(r, w_0)$. Introducing polar coordinates ρ, θ around w_0 by $w = w_0 + \rho e^{i\theta}$ and writing somewhat sloppily

$$X(w) = X(w_0 + \rho e^{i\theta}) = X(\rho, \theta),$$

we obtain

$$\Phi(r) = \int_0^r \int_0^\pi \{|X_\rho(\rho,\theta)|^2 + \rho^{-2}|X_\theta(\rho,\theta)|^2\} \rho \, d\theta \, d\rho. \tag{7}$$

From (4) we infer

$$|X_\rho|^2 = \rho^{-2}|X_\theta|^2, \quad \langle X_\rho, X_\theta \rangle = 0, \tag{8}$$

hence

$$\Phi(r) = 2\int_0^r \rho^{-1} \int_0^\pi |X_\theta(\rho,\theta)|^2 \, d\theta \, d\rho. \tag{9}$$

There is a set $\mathcal{N} \subset [0, d]$ of 1-dimensional measure zero such that

$$\int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta < \infty \quad \text{for } r \in (0,d) \setminus \mathcal{N} \tag{10}$$

and that the absolutely continuous function $\Phi(r)$ is differentiable at the values $r \in (0, d) \setminus \mathcal{N}$ and satisfies

$$\Phi'(r) = 2r^{-1} \int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta. \tag{11}$$

We can therefore assume that, for $r \in (0, d) \setminus \mathcal{N}$, the function $X(r, \theta)$ is an absolutely continuous function of $\theta \in [0, \pi]$; in particular, the limits

$$Q_1(r) := \lim_{\theta \to \pi - 0} X(r, \theta), \quad Q_2(r) := \lim_{\theta \to +0} X(r, \theta)$$

exist for $r \in (0, d) \setminus \mathcal{N}$.

Consider now any $r \in (0, d) \setminus \mathcal{N}$ for which

$$\int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta \leq \pi^{-1} \delta^2 \tag{12}$$

holds true. Then we infer from

$$|Q_1(r) - Q_2(r)| \leq \int_0^\pi |X_\theta(r, \theta)| \, d\theta \leq \sqrt{\pi} \left\{ \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta \right\}^{1/2} \tag{13}$$

the inequality

$$|Q_1(r) - Q_2(r)| \leq \delta.$$

Since S satisfies a chord-arc condition with constants M and δ, there exists a rectifiable arc

$$\Gamma^* = \{\xi(s) \colon 0 \leq s \leq l^*\}$$

of length $l^* = L(\Gamma^*)$ on S which connects the points $Q_1(r)$ and $Q_2(r)$, and whose length $L(\Gamma^*)$ satisfies

$$l^* = L(\Gamma^*) \leq M |Q_1(r) - Q_2(r)|. \tag{14}$$

We assume s to be chosen as parameter of the arc length on Γ^*. Then it follows that $|\xi'(s)| = 1$ a.e. on $[0, l^*]$. Introducing the reparametrization $\zeta(\theta), \pi \leq \theta \leq 2\pi$, of Γ^* which is defined by

$$\zeta(\theta) := \xi(\pi^{-1}(\theta - \pi) l^*),$$

we obtain

$$|\zeta_\theta(\theta)| = \text{const} = l^*/\pi \quad \text{a.e. on } [\pi, 2\pi]$$

and

$$l^* = \int_\pi^{2\pi} |\zeta_\theta| \, d\theta;$$

therefore also

$$\pi \int_\pi^{2\pi} |\zeta_\theta|^2 \, d\theta = \left(\int_\pi^{2\pi} |\zeta_\theta| \, d\theta \right)^2 = l^{*2}. \tag{15}$$

From (13)–(15) we conclude that

$$\int_\pi^{2\pi} |\zeta_\theta|^2 \, d\theta \leq M^2 \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta. \tag{16}$$

Consider now the harmonic vector function $H(w)$ in B_r whose boundary values $\eta(\theta) = H(w_0 + re^{i\theta})$ are defined by

$$\eta(\theta) := \begin{cases} X(r,\theta) & \text{for} \quad 0 \le \theta \le \pi \\ \zeta(\theta) & \phantom{\text{for}} \quad \pi \le \theta \le 2\pi \end{cases}.$$

Because of (16), we have

$$\int_0^{2\pi} |\eta_\theta|^2 \, d\theta \le (1 + M^2) \int_0^{\pi} |X_\theta(r,\theta)|^2 \, d\theta. \tag{17}$$

Expanding H and ζ in Fourier series we obtain

$$H(w) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n (A_n \cos n\theta + B_n \sin n\theta)$$

and

$$\eta(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta).$$

From these expressions, we derive

$$\int_{B_r} |\nabla H|^2 \, du \, dv = \pi \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2),$$

$$\int_0^{2\pi} |\eta_\theta|^2 \, d\theta = \pi \sum_{n=1}^{\infty} n^2(|A_n|^2 + |B_n|^2),$$

and therefore

$$\int_{B_r} |\nabla H|^2 \, du \, dv \le \int_0^{2\pi} |\eta_\theta|^2 \, d\theta. \tag{18}$$

Relations (11), (17) and (18) imply that

$$\int_{B_r} |\nabla H|^2 \, du \, dv \le \frac{1}{2}(1 + M^2) r \Phi'(r). \tag{19}$$

Next we consider the mapping $Y(w)$ on $B \cup B_r$ which is defined as

$$Y(w) := \begin{cases} H(w) & \text{for} \quad w \in B_r \\ X(w) & \phantom{\text{for}} \quad w \in B \setminus B_r \end{cases}.$$

Clearly Y is continuous and of class H_2^1 on $B \cup B_r$. Let τ be the homeomorphism of $\overline{B}$ onto $\overline{B \cup B_r}$ which maps B conformally onto $B \cup B_r$, keeping the points $1, -1, i$ fixed. Then the mapping $Z := Y \circ \tau$ is contained in $\mathcal{C}(\Gamma, S)$, and the minimum property of X implies

$$\int_B |\nabla X|^2 \, du \, dv \le \int_B |\nabla Z|^2 \, du \, dv.$$

On account of the conformal invariance of the Dirichlet integral we have

$$\int_B |\nabla X|^2 \, du \, dv \le \int_{B \cup B_r} |\nabla Y|^2 \, du \, dv,$$

and the definition of Y now implies

$$\int_{S_r} |\nabla X|^2 \, du \, dv \le \int_{B_r} |\nabla H|^2 \, du \, dv. \tag{20}$$

By virtue of (5), (19), and (20) we obtain the relation

$$\Phi(r) \le \tfrac{1}{2}(1+M^2) r \Phi'(r) \tag{21}$$

for every $r \in (0,d) \setminus \mathcal{N}$ satisfying equation (12).

On the other hand, if the equation

$$\int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta > \pi^{-1}\delta^2$$

holds for some $r \in (0,d) \setminus \mathcal{N}$, then we trivially have

$$\Phi(r) \le 2D(X) = 2e < 2e\pi\delta^{-2} \int_0^\pi |X_\theta(r,\theta)|^2 \, d\theta,$$

and the identity (11) yields

$$\Phi(r) \le \pi e \delta^{-2} r \Phi'(r). \tag{22}$$

Defining the number $\mu \in (0,1)$ as in (2), it follows that

$$2\mu\Phi(r) \le r\Phi'(r) \quad \text{for all } r \text{ in } (0,d) \setminus \mathcal{N}, \tag{23}$$

and an integration yields

$$\Phi(r) \le (r/d)^{2\mu} \Phi(d) \quad \text{for all } r \in [0,d].$$

Thus we have established (6) for any $d \in (0,1), w_0 \in I$ with $|w_0| < 1-d$, and $r \in [0,d]$.

Consider any w_0 with $|w_0| \le 1-R$ and $\operatorname{Im} w_0 \ge R$ for some $R \in (0,1)$. Then we have $B_r(w_0) \subset B$ for any $r \in (0,R)$, and analogously to (18) we obtain

$$\int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \le \int_0^{2\pi} |X_\theta(r,\theta)|^2 \, d\theta$$

for almost all $r \in (0,R)$. By (5) and (11) we therefore infer

$$\Phi(r,w_0) \le \frac{1}{2} r \frac{d}{dr} \Phi(r,w_0)$$

for a.a. $r \in (0, R)$, and an integration yields

$$\Phi(r, w_0) \le (r/R)^2 \Phi(R, w_0) \quad \text{for all } r \in [0, R]. \tag{24}$$

Finally we fix some $d \in (0,1)$ and choose an arbitrary point $w_0 \in \overline{Z}_d = \overline{B} \cap \{|w| \le 1 - d\}$. Set $u_0 = \operatorname{Re} w_0$ and $v_0 = \operatorname{Im} w_0$. We distinguish three cases:

(i) $v_0 \ge d/2$.
Choosing $R = d/2$ we infer from (24) that

$$\Phi(r, w_0) \le (2r/d)^2 \int_B |\nabla X|^2 \, du \, dv \quad \text{for } 0 \le r \le d/2 \tag{25}$$

holds true.

(ii) $0 \le v_0 \le d/2$ and $v_0 \le r \le d/2$.
Then we have $B_r(w_0) \subset B_{2r}(u_0)$, and it follows that

$$\Phi(r, w_0) \le \Phi(2r, u_0).$$

Applying (6) we have also

$$\Phi(2r, u_0) \le (2r/d)^{2\mu} \Phi(d, u_0),$$

and therefore

$$\Phi(r, w_0) \le (2r/d)^{2\mu} \int_B |\nabla X|^2 \, du \, dv. \tag{26}$$

In particular we have

$$\Phi(v_0, w_0) \le (2v_0/d)^{2\mu} \int_B |\nabla X|^2 \, du \, dv \quad \text{for any } v_0 \in [0, d/2]. \tag{27}$$

(iii) $0 \le v_0 \le d/2$ and $0 \le r \le v_0$.
Applying (24) to the case $R = v_0$ we obtain

$$\Phi(r, w_0) \le (r/v_0)^2 \Phi(v_0, w_0).$$

Combining this inequality with (27) it follows that

$$\begin{aligned} \Phi(r, w_0) &\le (r/v_0)^2 (2v_0/d)^{2\mu} \int_B |\nabla X|^2 \, du \, dv \\ &\le (2r/d)^{2\mu} \int_B |\nabla X|^2 \, du \, dv. \end{aligned} \tag{28}$$

On account of (25), (26) and (28) inequality (1) holds true for any $r \in [0, d/2]$, and for $r > d/2$ estimate (1) is satisfied for trivial reasons. The bound (3) and $X \in C^{0,\mu}(\overline{Z}_d, \mathbb{R}^3)$ now follow from Morrey's Dirichlet growth theorem (see Morrey [8], p. 79). □

Remark. Note that the assumptions of Theorem 1 do not require S to be a regular surface. In fact, S is allowed to degenerate to a rectifiable arc. Thus several variants of Theorem 1 can be proved. For instance we get:

Theorem 2. *Suppose that $S \cup \Gamma$ satisfies a chord-arc condition with constants M and δ, and let $X \in \mathcal{C}(\Gamma, S)$ be a minimizer of the Dirichlet integral in the class $\mathcal{C}(\Gamma, S)$, that is, a solution of the minimum problem $\mathcal{P}(\Gamma, S)$ considered in Section 4.6 of Vol. 1, which satisfies $D(X) > 0$. Then X is of class $C^{0,\mu}(\overline{B}, \mathbb{R}^3)$ for some $\mu \in (0,1)$.*

Fixing a third point $P_3 \in \Gamma$ and requiring $X(i) = P_3$ we can even derive an a priori estimate for $[X]_{\mu,\overline{B}}$ analogous to (3).

In particular, the chord-arc condition for $S \cup \Gamma$ implies the Hölder continuity of any minimizer X in the corners $w = \pm 1$ which are mapped by X on the points P_1 and P_2 where the arc Γ is attached to S.

If we consider minimal surfaces bounded by a preassigned closed Jordan curve Γ of finite length, we can even drop the minimizing property of X since we then can avoid the detour via the comparison surface $Z = Y \circ \tau$ obtained from X and H. Instead we derive an inequality of the type (21) directly by applying the isoperimetric inequality to the part $X|_{S_r(w_0)}$ of the minimal surfaces. Leaving a detailed discussion to the reader we just formulate the final result:

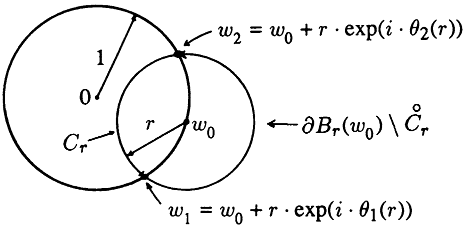

Fig. 1.

Theorem 3. *Let Γ be a closed rectifiable Jordan arc in $\mathbb{R}^3$ of the length $L(\Gamma)$ satisfying a chord-arc condition with constants M and δ. Denote by $\mathfrak{F}(\Gamma)$ a family of minimal surfaces $Y \in \mathcal{C}(\Gamma)$ bounded by Γ which maps three fixed points on $C = \partial B$ onto three fixed points on Γ. Then there exists a number $R > 0$ such that for all $X \in \mathfrak{F}(\Gamma)$ we have*

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le (r/R)^{2\mu} D(X) \quad \text{for all } r > 0 \tag{29}$$

with the exponent $\mu = (1+M)^{-2}$, and

$$[X]_{0+\mu,\overline{B}} \le cL(\Gamma), \tag{30}$$

where the constant c only depends on M, δ and on the chosen three-point condition of the family $\mathfrak{F}(\Gamma)$.[4]

Similar results hold for solutions of minimum problems with a completely free boundary, i.e., for the minimizers of the Dirichlet integral within one of the classes $\mathcal{C}(\sigma, S)$, $\mathcal{C}^+(S)$, and $\mathcal{C}(\Pi, S)$ introduced in Sections 1.1 and 1.2. As in Chapter 1 we now choose the parameter domain B as the unit disk in $\mathbb{C}$,

$$B = \{w \in \mathbb{C}\colon |w| < 1\},$$

and

$$C = \partial B = \{w \in \mathbb{C}\colon |w| = 1\}.$$

Moreover, we set

$$S_r(w_0) := B \cap B_r(w_0), \quad C_r(w_0) := \overline{B} \cap \partial B_r(w_0).$$

Theorem 4. *Let S be a closed, nonempty, proper subset of $\mathbb{R}^3$ satisfying a chord-arc condition with constants M, δ. Moreover assume that for some $\mu > 0$ the inclusion $S \to T_\mu$ of S in T_μ induces a bijection of the corresponding homotopy classes: $\tilde{\pi}_1(S) \leftrightarrow \tilde{\pi}_1(T_\mu)$.*[5] *Finally, suppose that $\mathcal{C}$ denotes one of the classes $\mathcal{C}(\sigma, S)$, $\mathcal{C}^+(S)$, $\mathcal{C}(\Pi, S)$. Then for every minimizer X of the Dirichlet integral in the class $\mathcal{C}$ there is a constant c such that*

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le c r^{2\nu} \tag{31}$$

holds for any $w_0 \in \overline{B}$ and any $r > 0$, where

$$\nu = (1 + M^2)^{-1}. \tag{32}$$

In particular, we have $X \in C^{0,\nu}(\overline{B}, \mathbb{R}^3)$ and

$$\lim_{w \to w_0} \operatorname{dist}(X(w), S) = 0 \quad \textit{for all } w_0 \in \partial B. \tag{33}$$

Sketch of the proof. Set $\delta_0 := \frac{1}{4}\pi\mu^2$; this constant is nothing but the number δ which appears in Theorem 2 of Section 1.1. Then there is a number $R_0 \in (0, 1)$ such that

$$\int_{\Omega_0} |\nabla X|^2 \, du \, dv < \delta_0 \tag{34}$$

holds true for the annular domain $\Omega_0 := \{w \in \mathbb{C}\colon 1 - R_0 < |w| < 1\}$.

For any point $w_0 \in \overline{B}$ we define

---

[4] See Hildebrandt [3], pp. 55–59, for a sketch of the proof.

[5] This is Assumption (A) of Section 1.1.

(35) $$\Phi(r) = \Phi(r, w_0) = \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv.$$

Introducing polar coordinates ρ, θ around w_0 by $w = w_0 + \rho e^{i\theta}$ and writing $X(w) = X(w_0 + \rho e^{i\theta}) = X(\rho, \theta)$, we obtain analogously to (9) that

(35′) $$\Phi(r) = 2 \int_0^r \int_{\theta_1(\rho)}^{\theta_2(\rho)} \rho^{-1} |X_\theta(\rho, \theta)|^2 \, d\theta \, d\rho$$

holds for two angles θ_1, θ_2 with $0 \le \theta_2(\rho) - \theta_1(\rho) \le 2\pi$. Consequently the absolutely continuous function $\Phi(r)$ satisfies

(36) $$\int_{\theta_1(r)}^{\theta_2(r)} |X_\theta(r, \theta)|^2 \, d\theta = \tfrac{1}{2} r \Phi'(r)$$

for all $r \in (0, \infty) \setminus \mathcal{N}$ where $\mathcal{N}$ is a one-dimensional null set.

Let $w_0 \in C$ and consider some positive number β which will be specified later. Moreover, let $r \in (0, R_0) \setminus \mathcal{N}$.

Case 1.
$$\int_{\theta_1(r)}^{\theta_2(r)} |X_\theta(r, \theta)|^2 \, d\theta \ge \pi^{-1} \beta^2.$$

Then we obtain the trivial inequality

(37) $$\Phi(r) \le 2\pi\beta^{-2} D(X) \int_{\theta_1(r)}^{\theta_2(r)} |X_\theta(r, \theta)|^2 \, d\theta = \pi\beta^{-2} D(X) r \Phi'(r).$$

Case 2.
$$\int_{\theta_1(r)}^{\theta_2(r)} |X_\theta(r, \theta)|^2 \, d\theta < \pi^{-1} \beta^2.$$

Then for any two points $P := X(r, \theta)$ and $P' := X(r, \theta')$ on $X(C_r(w_0))$ we have

$$|P' - P| \le \int_\theta^{\theta'} |X_\theta(r, \theta)| \, d\theta \le |\theta' - \theta|^{1/2} \left\{ \int_\theta^{\theta'} |X_\theta(r, \theta)|^2 \, d\theta \right\}^{1/2}$$

whence

(38) $$|P' - P| \le \int_\theta^{\theta'} |X_\theta(r, \theta)| \, d\theta \le \beta.$$

In particular, this estimate holds true for the two endpoints $Q_1(r)$ and $Q_2(r)$ of the arc $X \colon C_r(w_0) \to \mathbb{R}^3$ which lie on S. Choosing β less than or equal to δ (M and δ being the constants of the chord-arc condition of S), we have

$$|Q_1(r) - Q_2(r)| \le \delta.$$

Thus there is a rectifiable arc $\Gamma^* = \{\xi(s)\colon 0 \le s \le l^*\}$ of length $l^* = L(\Gamma^*)$ which connects the points $Q_1(r)$ and $Q_2(r)$, and whose length satisfies

$$l^* = L(\Gamma^*) < M|Q_1(r) - Q_2(r)|. \tag{39}$$

Consider now the harmonic vector function $H(w)$ in $B_r = B_r(w_0)$, the boundary values $\eta(\theta) = H(w_0 + re^{i\theta})$ of which are defined by

$$\eta(\theta) := \begin{cases} X(r,\theta) & \theta_1(r) \le \theta \le \theta_2(r) \\ & \text{for} \\ \zeta(\theta) & \theta \in [0, 2\pi] \setminus [\theta_1(r), \theta_2(\theta)] \end{cases}$$

where $\zeta(\theta)$ is a suitable reparametrization of Γ^* proportional to the arc length. Then analogously to (19) we obtain

$$\int_{B_r} |\nabla H|^2 \, du \, dv \le \frac{1}{2}(1 + M^2) r \Phi'(r). \tag{40}$$

We now define the mapping $Y(w)$ on $B \cup B_r$ by

$$Y(w) := \begin{cases} H(w) & w \in B_r \\ & \text{for} \\ X(w) & w \in B \setminus B_r \end{cases}$$

and set

$$Z := Y \circ \tau,$$

where τ is a homeomorphism of $\overline{B}$ onto $\overline{B \cup B_r}$ which maps B conformally onto $B \cup B_r$.

Claim. *The mapping Z is an admissible comparison surface, i.e. $Z \in \mathcal{C}$, if we choose β as*

$$\beta := \min\{\delta, \mu, [(1 + M^2)^{-1} \pi \delta_0]^{1/2}\}. \tag{41}$$

Then analogously to (21) we arrive at

$$\Phi(r) \le \tfrac{1}{2}(1 + M^2) r \Phi'(r). \tag{42}$$

Combining the discussion of the cases 1 and 2 we infer from (37) and (40) that

$$\Phi(r) \le \tfrac{1}{2} c r \Phi'(r) \quad \text{a.e. on } (0, R_0), \tag{43}$$

where

(44) $$c := \max\{1 + M^2, 2\pi\beta^{-2}D(X)\}.$$

Now we can proceed as in the proof of Theorem 1, and we obtain the Dirichlet growth condition (31) with $\nu = 1/c$. As this implies $X \in C^{0,\nu}(\overline{B}, \mathbb{R}^3)$, we can repeat the previous discussion in such a way that case 1 becomes void. For this purpose we only have to choose $R_0 > 0$ so small that

$$|Q_1(r) - Q_2(r)| < \delta \quad \text{for } r \in (0, R_0) \setminus \mathcal{N}.$$

Then we obtain condition (31) with the desired exponent $\nu = (1 + M^2)^{-1}$.

It remains to verify the claim.

First of all we choose a radius $\rho \in (0, 1)$ so close to 1 that the closed curve $Z\colon \partial B_\rho(0) \to \mathbb{R}^3$ is completely contained in $T_{\mu/2}$ and represents the boundary class $[Z_{|\partial B}]$. We shall show that this curve is homotopic in T_μ to some curve $X\colon \partial B_{\rho'}(0) \to \mathbb{R}^3$ which represents the boundary class of X.

Let $Q_1(r) = X(w_1), Q_2(r) = X(w_2), w_1, w_2 \in \partial B$. Since τ is a conformal mapping of B onto $B \cup B_r(w_0)$, the tangent of the curve $\tilde{C}_r(w_0) := \tau^{-1}(C_r(w_0))$ tends to a limit as w tends along $C_r(w_0)$ to one of the endpoints w_1 and w_2 of $C_r(w_0)$, and this limit is different from the tangent of ∂B. This can either be seen by an explicit computation of τ or from a general theorem of the theory of conformal mappings (cf. Carathéodory [4], p. 91). Therefore the above number ρ can be selected in such a way that $\partial B_\rho(0)$ intersects $\tilde{C}_r(w_0)$ in exactly two points z_3 and z_4, and that the curve $\tau(\partial B_\rho(0))$ is completely contained in $\Omega_\varepsilon := B_r(w_0) \cup B \setminus B_{1-\varepsilon}(0)$ where ε is chosen to satisfy $0 < \varepsilon < r \leq R_0$. Because of (34), it follows that

(45) $$2D_{\Omega_\varepsilon}(X) < \delta_0.$$

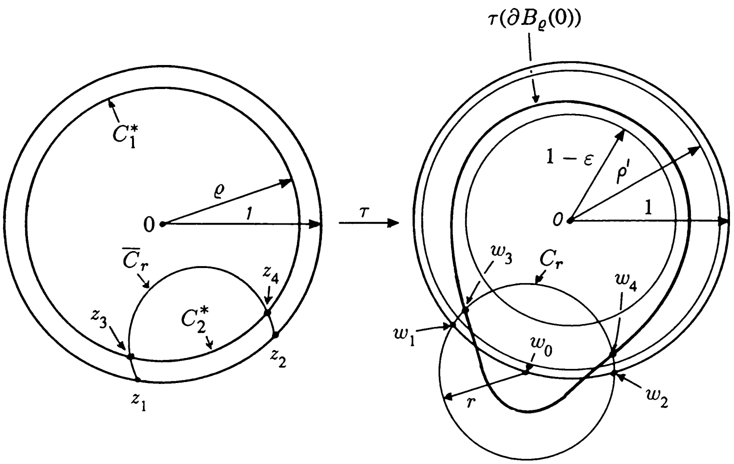

Fig. 2. This sketch illustrates the proof that the comparison surface used in the regularity proof is admissible

We can find a number $\rho' \in (1-\varepsilon, 1)$ such that the trace of the curve

$$X \colon \partial B_{\rho'}(0) \to \mathbb{R}^3$$

is completely contained in $T_{\mu/2}$ and represents the boundary class $[X|_{\partial B}]$ of X. We can also achieve that $\partial B_{\rho'}(0) \setminus B_r(w_0)$ lies between ∂B and $\tau(C_1^*)$, where C_1^* denotes that part of $\partial B_\rho(0)$ which is mapped by τ into $B \setminus B_r(w_0)$. Set $C_2^* := \partial B_\rho(0) \setminus C_1^*$. Moreover, note that the curve $X \colon C_r(w_0) \to \mathbb{R}^3$ remains completely in $T_{\mu/2}$ since its endpoints lie on S and its length is less than or equal to β (cf. (38)), and $\beta \le \mu$ on account of (41).

Finally we infer from (36) and (40) that

$$\int_{B_r(w_0)} |\nabla Y|^2 \, du \, dv \le (1+M^2) \int_{\theta_1(r)}^{\theta_2(r)} |X_\theta(r,\theta)|^2 \, d\theta,$$

and the right-hand side of this inequality is bounded from above by

$$(1+M^2)\pi^{-1}\beta^2.$$

By virtue of (41) we arrive at

$$\int_{B_r(w_0)} |\nabla Y|^2 \, du \, dv \le \delta_0. \tag{46}$$

We infer from (34), (45) and (46) as well as from Theorem 2 of Section 1.1 that all curves to be considered in the following are contained in $T_{\mu/2}$, and that we obtain the following homotopies ($\simeq$). Here C_r' will denote the subarc of $C_r(w_0)$ which connects the intersection points w_3 and w_4 of $C_r(w_0)$ with $\tau(\partial B_\rho(0))$:

$$\begin{aligned} X|_{\partial B_{\rho'}(0)} &\simeq X|_{\partial B_{\rho'}(0) \setminus B_r(w_0)} \cdot X|_{C_r(w_0) \cap \overline{B}_{\rho'}(0)} \\ &\simeq X|_{\tau(C_1^*)} \cdot X|_{C_r'} \\ &\simeq Y \circ \tau|_{C_1^*} \cdot Y|_{C_r'} \\ &\simeq Y \circ \tau|_{C_1^*} \cdot Y \circ \tau|_{C_2^*} = Z|_{\partial B_\rho(0)}. \end{aligned}$$

This completes the proof of the *claim* and thus of the theorem. □

Remark. An Inspection of the proof of Theorem 4 shows that the constant c in (31) will depend on the number R_0 which in turn depends on X. Hence (31) does not yield an a priori estimate of the Morrey seminorm or of the Hölder seminorm of X.

2.6 Hölder Continuity for Stationary Surfaces

In the previous section we have proved that minimizers of the Dirichlet integral in various classes of admissible surfaces corresponding to free boundary problems are Hölder continuous up to their free boundary. The proof has made essential use of the minimum property of the solution of the free boundary problem. In case of partially free problems we have even derived a priori estimates for the Hölder seminorm up to the free boundary. Now we want to establish Hölder continuity of stationary minimal surfaces up to the free boundary. However, we shall have to use a completely different approach in this case as we are not able to derive a priori estimates for the Hölder seminorm or even for the modulus of continuity. In fact, such estimates do not exist, as an inspection of the Schwarz examples discussed in Section 1.9 will show. Consider, for instance, the boundary configuration $\langle \Gamma, S\rangle$ depicted in Fig. 1 which consists of a cylinder surface S and of a polygon Γ with its endpoints on S. For this particular configuration the corresponding semi-free boundary problem possesses infinitely many stationary solutions, all of which are simply connected parts of helicoids, and it is fairly obvious that there is neither an upper bound for their areas (Dirichlet integrals), nor for the length of their free traces, nor for their moduli of continuity.

For this reason we shall not approach the regularity problem by deriving estimates. Instead we want to use an indirect reasoning, first proving continuity up to the boundary by a contradiction argument. We shall constrain our attention to stationary surfaces in the class $\mathcal{C}(\Gamma, S)$ defined for semi-free problems. Similar results can be obtained for stationary solutions of completely free problems without any essential alterations.

We begin by defining Assumption (B) and the notion of admissible support surfaces.

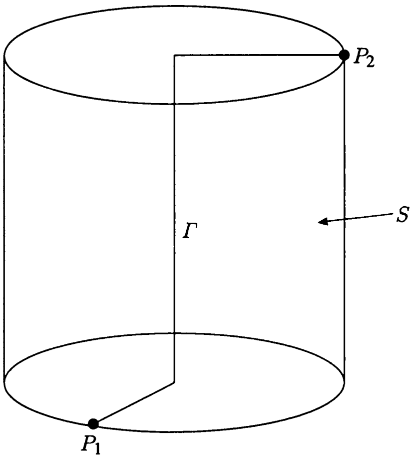

Fig. 1. The stationary solutions of the boundary value problem for the configuration $\langle \Gamma, S\rangle$ cannot be estimated a priori

Definition 1. *An* admissible support surface S *of class* $C^m, m \geq 2$, *(or of class* $C^{m,\beta}$ *with* $0 < \beta \leq 1$*) is a two-dimensional manifold of class* C^m *(or of* *class $C^{m,\beta}$) embedded in $\mathbb{R}^3$, with or without boundary, which has the following two properties:*

(i) *The boundary ∂S of the manifold S is a regular one-dimensional submanifold of class C^m (or $C^{m,\beta}$) which can be empty.*

(ii) *Assumption (B) is fulfilled.*

Assumption (B), a uniformity condition at infinity, is defined next. We write $x = (x^1, x^2, x^3), y = (y^1, y^2, y^3), \ldots$ for points $x, y, \ldots$ in $\mathbb{R}^3$.

Definition 2. *A support surface S is said to fulfil* **Assumption (B)** *if the following holds true: For each $x_0 \in S$ there exist a neighbourhood $\mathcal{U}$ of x_0 in $\mathbb{R}^3$ and a C^2-diffeomorphism h of $\mathbb{R}^3$ onto itself such that h and its inverse $g = h^{-1}$ satisfy:*

(i) *The inverse g maps $\mathcal{U}$ onto some open ball $B_R(0) = \{y \in \mathbb{R}^3 : |y| < R\}$ such that $g(x_0) = 0; 0 < R < 1$.*

(ii) *If ∂S is empty, then*

$$g(S \cap \mathcal{U}) = \{y \in B_R(0) \colon y^3 = 0\}.$$

If ∂S is nonvoid, then there exists some number $\sigma = \sigma(x_0) \in [-1, 0]$ such that

$$\begin{aligned} g(S \cap \mathcal{U}) &= \{y \in B_R(0) \colon y^3 = 0, y^1 \geq \sigma\}, \\ g(\partial S \cap \mathcal{U}) &= \{y \in B_R(0) \colon y^3 = 0, y^1 = \sigma\} \end{aligned}$$

holds true. If $x_0 \in \partial S$, then $\sigma = 0$, and $\sigma \leq -R$ if $\partial S \cap \mathcal{U}$ is empty.

(iii) *There are numbers m_1 and m_2 with $0 < m_1 \leq m_2$ such that the components*

$$g_{ik}(y) = h^l_{y^i}(y) h^l_{y^k}(y)$$

of the fundamental tensor of $\mathbb{R}^3$ with respect to the curvilinear coordinates y satisfies

$$m_1 |\xi|^2 \leq g_{ik}(y) \xi^i \xi^k \leq m_2 |\xi|^2 \quad \textit{for all } y, \xi \in \mathbb{R}^3.$$

(iv) *There exists a number $K > 0$ such that*

$$\left| \frac{\partial g_{ik}}{\partial y^l}(y) \right| \leq K$$

is satisfied on $\mathbb{R}^3$ for $i, k, l = 1, 2, 3$.

We call the pair $\{\mathcal{U}, g\}$ an **admissible boundary coordinate system centered at** *x_0.*

Let us recall the standard notation used for semifree problems and for the definition of $\mathcal{C}(\Gamma, S)$: The parameter domain B is the semidisk

$$B = \{w = u + iv \colon |w| < 1, v > 0\},$$

the boundary of which consists of the circular arc

$$C = \{w = u + iv \colon |w| = 1, v \geq 0\}$$

and of the segment

$$I = \{w \in \mathbb{R} \colon |w| < 1\}.$$

Moreover, we set

$$Z_d = \{w = u + iv \colon |w| < 1 - d, v > 0\}, \quad d \in (0,1),$$

$$S_r(w_0) = B \cap B_r(w_0), \quad I_r(w_0) = I \cap B_r(w_0),$$
$$C_r(w_0) = \overline{B} \cap \partial B_r(w_0).$$

Next we introduce some terminology with respect to a fixed *admissible boundary coordinate system* $\{\mathfrak{U}, g\}$. Given a minimal surface $X \colon B \to \mathbb{R}^3$, we use the diffeomorphism $g \colon \mathbb{R}^3 \to \mathbb{R}^3$ to define a new mapping $Y \in C^3(B, \mathbb{R}^3)$ by

(1) $$Y(u,v) := g(X(u,v)),$$

whence also

(1′) $$X(u,v) = h(Y(u,v)).$$

In other words, we have

$$Y = g \circ X \quad \text{and} \quad X = h \circ Y.$$

Quite often we use the following *normalization:*

(2) $$\begin{cases} \textit{Let } w_0 \in I, \textit{ and set } x_0 := X(w_0). \textit{ Suppose that } \{\mathfrak{U}, g\} \textit{ is an} \\ \textit{admissible boundary coordinate system for } S \textit{ centered at } x_0. \\ \textit{Then } Y(w_0) = 0. \end{cases}$$

In case of this normalization, the following holds true:

(2′) $$\begin{cases} \textit{Let } \Omega \textit{ be a subset of } B \textit{ such that } X(\Omega) \subset \mathfrak{U}. \textit{ Then we have} \\ |Y(w)| < R. \textit{ If } w_0 \in I,\ d = 1 - |w_0|,\ r < d,\ X \in C^0(\overline{S}_r(w_0), \mathbb{R}^3), \\ \textit{and } X \colon I_r(w_0) \to S, \textit{ then we have } y^3(w) = 0 \textit{ for } w \in I_r(w_0). \\ \textit{If } \partial S \textit{ is nonempty and } x_0 \in \partial S, \textit{ then we have } y^1(w) \geq \sigma \\ \textit{for all } w \in I_r(w_0). \end{cases}$$

For any $Z = (z^1, z^2, z^3) \in H_2^1(\Omega, \mathbb{R}^3), \Omega \subset \mathbb{C}$, we define the *transformed Dirichlet integral* (or: *energy functional*) $E_\Omega(Z)$ by

(3) $$E_\Omega(Z) := \frac{1}{2} \int_\Omega g_{ik}(Z)[z_u^i z_u^k + z_v^i z_v^k]\, du\, dv$$

and we set

$$(3') \qquad E(Z) := E_B(Z).$$

We note that

$$(4) \qquad E_\Omega(Z) = D_\Omega(h \circ Z) \quad \text{for all } Z \in H_2^1(\Omega, \mathbb{R}^3),$$

whence, by (1′),

$$(5) \qquad E_\Omega(Y) = D_\Omega(X), \quad E(Y) = D(X).$$

For every $\phi = (\varphi^1, \varphi^2, \varphi^3) \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$ and for $X_\varepsilon := h(Y + \varepsilon\phi)$ we have

$$\lim_{\varepsilon \to +0} \frac{1}{\varepsilon}\{E(Y + \varepsilon\phi) - E(Y)\} = \lim_{\varepsilon \to +0} \frac{1}{\varepsilon}\{D(X_\varepsilon) - D(X)\}.$$

The left-hand side is equal to the first variation $\delta E(Y, \phi)$ of E at Y in direction of ϕ, and a straightforward computation yields

$$(6) \qquad \begin{aligned} \delta E(Y, \phi) \;=\; & \int_B g_{ik}(Y)\{y_u^i \varphi_u^k + y_v^i \varphi_v^k\}\, du\, dv \\ & + \int_B \frac{1}{2} g_{ik,l}(Y)\{y_u^i y_u^k + y_v^i y_v^k\}\varphi^l\, du\, dv \end{aligned}$$

while the right-hand side tends to

$$(7) \qquad \delta D(X, \Psi_0) = \int_B \langle \nabla X, \nabla \Psi_0 \rangle\, du\, dv, \quad \Psi_0 := h_y(Y)\phi$$

because of

$$X = h(Y), \quad X_\varepsilon = h(Y + \varepsilon\phi) = h(Y) + \varepsilon\Psi(\cdot, \varepsilon) = X + \varepsilon\Psi(\cdot, \varepsilon)$$

with

$$\Psi_0 := \Psi(\cdot, 0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{X_\varepsilon - X\} = h_y(Y)\phi.$$

Thus we have

$$(8) \qquad \delta E(Y, \phi) = \delta D(X, \Psi_0).$$

Now we can reformulate the conditions which define stationary points X of the Dirichlet integral in terms of the transformed surfaces $Y = g(X)$. Recall Definition 2 in Section 1.4:

If X is a stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ and if $X_\varepsilon = X + \varepsilon\Psi(\cdot, \varepsilon)$ is an outer variation (type II*) of X with $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ for $0 \le \varepsilon < \varepsilon_0$, we have*

$$\lim_{\varepsilon \to +0} \frac{1}{\varepsilon}\{D(X_\varepsilon) - D(X)\} = \int_B \langle \nabla X, \nabla \Psi_0 \rangle\, du\, dv \ge 0$$

for $\Psi_0 := \Psi(\cdot, 0)$, *and this is equivalent to*

$$\delta E(Y, \phi) \geq 0. \tag{9}$$

This holds true in particular for every $\phi \in C_c^\infty(B, \mathbb{R}^3)$ and thus we have both

$$\delta E(Y, \phi) \geq 0 \quad \text{and} \quad \delta E(Y, -\phi) \geq 0$$

whence

$$\delta E(Y, \phi) = 0 \quad \text{for all } \phi \in C_c^\infty(B, \mathbb{R}^3). \tag{10}$$

An integration by parts yields

$$\begin{aligned} -\int_B & g_{il}(Y)\{y_u^i \varphi_u^l + y_v^i \varphi_v^l\}\, du\, dv \\ &= \int_B [g_{il}(Y) \nabla y^i \varphi^l + g_{il,k}(Y)(y_u^i y_u^k + y_v^i y_v^k)\varphi^l]\, du\, dv \end{aligned}$$

for any $\phi \in C_c^\infty(B, \mathbb{R}^3)$, and we infer from (6) and (10) that

$$\begin{aligned} \int_B [g_{il}(Y)\Delta y^i + \{g_{il,k}(Y) - \tfrac{1}{2} g_{ik,l}(Y)\}(y_u^i y_u^k + y_v^i y_v^k)]\varphi^l\, du\, dv = 0 \\ \text{for all } \phi \in C_c^\infty(B, \mathbb{R}^3). \end{aligned} \tag{11}$$

Then the fundamental lemma of the calculus of variations yields

$$g_{il}(Y)\Delta y^i + \{g_{il,k}(Y) - \tfrac{1}{2} g_{ik,l}(Y)\}(y_u^i y_u^k + y_v^i y_v^k) = 0. \tag{12}$$

Introducing the Christoffel symbols of the first kind,

$$\Gamma_{ilk} = \tfrac{1}{2}\{g_{lk,i} - g_{ik,l} + g_{il,k}\}$$

we can rewrite (12) in the form

$$g_{ik}(Y)\Delta y^i + \Gamma_{ilk}(Y)(y_u^i y_u^k + y_v^i y_v^k) = 0 \tag{13}$$

using the symmetry relation $\Gamma_{ilk} = \Gamma_{kli}$, and this implies

$$\Delta y^l + \Gamma_{jk}^l(Y)(y_u^j y_u^k + y_v^j y_v^k) = 0, \quad l = 1, 2, 3, \tag{14}$$

if, as usual, $\Gamma_{jk}^l = g^{lm}\Gamma_{jmk}$ and $(g^{lm}) = (g_{jk})^{-1}$. As one can reverse the previous computations, we have found:

The equation $\Delta X = 0$ *is equivalent to the system* (14).

Moreover, we infer by a straight-forward computation from (1′) and from $g_{ik} = h_{y^i}^l h_{y^k}^l$:

The conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

are equivalent to

$$(15) \qquad g_{jk}(Y) y_u^j y_u^k = g_{jk}(Y) y_v^j y_v^k, \quad g_{jk}(Y) y_u^j y_v^k = 0.$$

The advantage of the new coordinate representation $Y(w)$ over the old representation $X(w)$ is that we have transformed the nonlinear boundary condition $X(I) \subset S$ into linear conditions as described in (2′). We pay, however, by having to replace the linear Euler equation $\Delta X = 0$ by the nonlinear system (14). The *variational inequality* (9) will be the key to all regularity results. Together with the conformality relations (15) it expresses the fact that $X = h \circ Y$ is a stationary point of the Dirichlet integral in the class $\mathcal{C}(\Gamma, S)$. (Here Γ can even be empty if X is a stationary point for a completely free boundary configuration; however, to have a clear-cut situation, we restrict our attention to partially free problems.)

The two main steps of this section are:

(i) First we prove continuity in $B \cup I$, that is, up to the free boundary I, using an indirect reasoning. The corresponding result will be formulated as Theorem 1.

(ii) In the second step we establish Hölder continuity on $B \cup I$ employing the hole-filling technique. The corresponding result is stated as Theorem 2.

Let us begin with the *first step* by formulating

Theorem 1. *Let S be an admissible support surface of class C^2, and suppose that $X(w)$ is a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Then $X(w)$ is continuous on $B \cup I$.*

The proof of this result will be based on four lemmata which we are now going to discuss.

Lemma 1. *Let $X: B \to \mathbb{R}^3$ be a minimal surface. For any point $w^* \in B$ we introduce $x^* := X(w^*)$ and the set*

$$K_\rho(x^*) := \{w \in B \colon |X(w) - x^*| < \rho\}.$$

Then, for each open subset Ω of B with $w^* \in \Omega$, we obtain

$$\limsup_{\rho \to +0} \frac{1}{\pi \rho^2} \int_{\Omega \cap K_\rho(x^*)} |\nabla X|^2 \, du \, dv \geq 2.$$

Proof. Fix some $w^* \in B$ and some Ω in B with $w^* \in \Omega$. We can assume that $x^* = X(w^*) = 0$. Then we introduce the set

$$\mathcal{U}_\rho := \{w \colon w = w^* + te^{i\theta}, t \geq 0, \theta \in \mathbb{R}, |X(w^* + re^{i\theta})| < \rho \text{ for all } r \in [0, t]\}.$$

Clearly $\mathcal{U}_\rho$ is an open set with $w^* \in \mathcal{U}_\rho$, and we have

$$\mathcal{U}_\rho \Subset \Omega \quad \text{for } 0 < \rho \ll 1$$

and therefore

$$\mathcal{U}_\rho \Subset \Omega \cap K_\rho(x^*) \quad \text{for } 0 < \rho \ll 1.$$

Hence it suffices to prove

$$\limsup_{\rho \to +0} \frac{1}{\pi \rho^2} \int_{\mathcal{U}_\rho} |\nabla X|^2 \, du \, dv \geq 2.$$

This relation is, however, an immediate consequence of Proposition 2 in Section 3.2 of Vol. 1. □

Lemma 2. *For each $X \in C^1(B, \mathbb{R}^3)$, every $w_0 \in I$, and each $\rho \in (0, 1 - |w_0|)$, there is a number r with $\rho/2 \leq r \leq \rho$ such that*

$$\operatorname{osc}_{C_r(w_0)} X \leq (\pi / \log 2)^{1/2} \left\{ \int_{S_\rho(w_0)} |\nabla X|^2 \, du \, dv \right\}^{1/2}.$$

Proof. Let us introduce polar coordinates r, θ about w_0 setting $w = w_0 + re^{i\theta}$ and $X(r, \theta) = X(w)$. Then, for $0 \leq \theta_1 \leq \theta_2 \leq \pi$, we obtain

$$|X(r, \theta_2) - X(r, \theta_1)| \leq \int_{\theta_1}^{\theta_2} |X_\theta(r, \theta)| \, d\theta \leq \sqrt{\pi p(r)}$$

where we have set

$$p(r) := \int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta.$$

If $\rho/2 \leq r \leq \rho$, it follows that

$$\int_{\rho/2}^{\rho} p(r) \frac{dr}{r} \leq \int_{S_\rho(w_0)} |\nabla X|^2 \, du \, dv.$$

Consequently, there is a number $r \in [\rho/2, \rho]$ such that

$$\left(\int_{\rho/2}^{\rho} \frac{dt}{r} \right) p(r) \leq \int_{S_\rho(w_0)} |\nabla X|^2 \, du \, dv$$

or

$$p(r) \leq \frac{1}{\log 2} \int_{S_\rho(w_0)} |\nabla X|^2 \, du \, dv,$$

and the assertion is proved. □

Lemma 3. *Let $w_0 \in I, r \in (0, 1 - |w_0|)$, and $X \in C^1(B, \mathbb{R}^3)$. Assume also that there are positive numbers α_1 and α_2 such that*

$$\operatorname{osc}_{C_r(w_0)} X \leq \alpha_1$$

and

$$\sup_{w^* \in S_r(w_0)} \inf_{w \in C_r(w_0)} |X(w) - X(w^*)| \le \alpha_2.$$

Then we obtain

$$\operatorname{osc}_{S_r(w_0)} X \le 2\alpha_1 + 2\alpha_2.$$

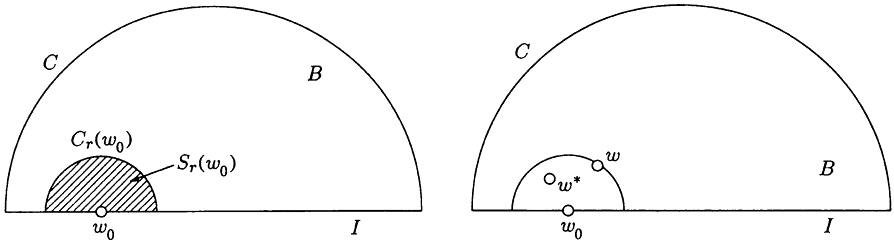

Fig. 2. A domain used in Lemma 3

Proof. Let $w \in S_r(w_0)$ and $w', w'' \in C_r(w_0)$. Then we infer from

$$|X(w) - X(w')| \le |X(w) - X(w'')| + |X(w'') - X(w')|$$

that

$$|X(w) - X(w')| \le \inf_{w'' \in C_r(w_0)} |X(w) - X(w'')| + \operatorname{osc}_{C_r(w_0)} X.$$

Thus we have

$$|X(w) - X(w')| \le \alpha_1 + \alpha_2 \quad \text{for all } w \in S_r(w_0) \text{ and } w' \in C_r(w_0).$$

This yields for arbitrary $w_1, w_2 \in S_r(w_0)$ and $w' \in C_r(w_0)$ the inequalities

$$|X(w_1) - X(w_2)| \le |X(w_1) - X(w')| + |X(w_2) - X(w')| \le 2\alpha_1 + 2\alpha_2,$$

and the assertion is proved. □

Lemma 4. *Let X be a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Suppose also that the support surface S is of class C^2, and let R, K, m_1, m_2 be the constants appearing in Assumption (B) that is to be satisfied by S. Then, for $R_1 := R\sqrt{m_2}$ and for some number $c > 0$ depending only on R, K, m_1, m_2, we have: If for some $r \in (0, 1 - |w_0|)$ and for some number $R_2 \in (0, R_1)$ the inequality*

$$\left[\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \right]^{1/2} < R_2/c$$

holds true, then it follows that

$$\sup_{w^* \in S_r(w_0)} \inf_{w \in C_r(w_0)} |X(w) - X(w^*)| \le R_2.$$

Before we come to the proof of Lemma 4 which is the main result in step 1 of our discussion, let us turn to the *Proof of Theorem 1*. Since $\int_B |\nabla X|^2\, du\, dv < \infty$, we have

$$\lim_{r\to +0} \int_{S_r(w_0)} |\nabla X|^2\, du\, dv = 0$$

for every $w_0 \in I$. Then Lemmata 2, 3, and 4 immediately imply that

$$\lim_{r\to 0} \operatorname{osc}_{S_r(w_0)} X = 0$$

for $w_0 \in I$. In conjunction with $X \in C^0(B, \mathbb{R}^3)$ we then infer that X is continuous on $B \cup I$. □

Proof of Lemma 4. Let $w_0 \in I$ and $0 < r < 1 - |w_0|$. Then we have to prove the following statement:

There is a number $c = c(R, K, m_1, m_2)$ *with the property that for any* R_2 *with* $0 < R_2 < R_1$ *and for any* $w^* \in S_r(w_0)$ *with*

$$\inf_{w\in C_r(w_0)} |X(w) - X(w^*)| > R_2 \tag{16}$$

the inequality

$$R_2 \le c \left[\int_{S_r(w_0)} |\nabla X|^2\, du\, dv \right]^{1/2} \tag{17}$$

holds true.

Thus let us consider some $w^* \in S_r(w_0), w_0 \in I, 0 < r < 1 - |w_0|$, and set

$$x^* := X(w^*), \quad \delta(x^*) := \operatorname{dist}(x^*, S).$$

We shall distinguish between two cases, $\delta(x^*) > 0$ and $\delta(x^*) = 0$.

Case (i): $\delta(x^*) > 0$.

Then we proceed as follows: Choose some function $\lambda \in C^1(\mathbb{R})$ with $\lambda' \ge 0$ and with $\lambda(t) = 0$ for $t \le 0$, and introduce the real valued function

$$\varphi(\rho) := \frac{1}{2} \int_{S_r(w_0)} \lambda(\rho - |X - x^*|) |\nabla X|^2\, du\, dv,$$

for $0 < \rho < \min\{\delta(x^*), d^2 R_2\}$, where R_2 is some number with $0 < R_2 < R_1 := R\sqrt{m_2}$, and where we have set

$$d := \frac{1}{2}\sqrt{\frac{m_1}{m_2}}, \quad 0 < d \le 1/2.$$

Define a test function $\eta(w)$ as

$$\eta(w) := \begin{cases} \lambda(\rho - |X(w) - x^*|)[X(w) - x^*] & \text{for } w \in \overline{S_r}(w_0), \\ 0 & \text{for } w \in B \setminus \overline{S_r}(w_0). \end{cases}$$

We use η to define a family $\{X_\varepsilon\}_{0 \le \varepsilon < \varepsilon_0}$ of outer variations

$$X_\varepsilon(w) := X(w) - \varepsilon\eta(w).$$

On account of

$$|X(w) - x^*| \ge R_2 > \rho \quad \text{for } w \in C_r(w_0)$$

we find that X_ε is of class $H_2^1(B, \mathbb{R}^3)$. Furthermore, we obtain

$$X_\varepsilon(w) = X(w) \quad \text{for } w \in B \setminus S_r(w_0).$$

Hence X and X_ε have the same boundary values on C. Moreover, for $\mathcal{L}^1$-almost all $w \in I$, we have $X(w) \in S$ and therefore $|X(w) - x^*| \ge \delta(x^*)$ whence $\rho - |X(w) - x^*| < 0$. This implies $\eta(w) = 0$ for $\mathcal{L}^1$-a.a. $w \in I$. Consequently we obtain $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ for $0 \le \varepsilon < \varepsilon_0$ and for any $\varepsilon_0 > 0$. As $\eta \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$, we conclude that X_ε is an admissible variation of type II in the sense of Definition 2, Section 1.4. By Section 1.4, (3) and (7), it follows that

$$\int_{S_r(w_0)} \langle \nabla X, \nabla \eta \rangle \, du \, dv \le 0$$

(in fact, even the equality sign holds true since we are allowed to take $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$), and therefore

$$\begin{aligned} &\int_{S_r(w_0)} |\nabla X|^2 \lambda(\rho - |X(w) - x^*|) \, du \, dv \\ &\quad \le \int_{S_r(w_0)} \lambda'(\rho - |X - x^*|)|X - x^*|^{-1} \\ &\qquad \cdot \{\langle X_u, X - x^* \rangle^2 + \langle X_v, X - x^* \rangle^2\} \, du \, dv. \end{aligned}$$

By virtue of the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

we have

$$\{\ldots\} \le \tfrac{1}{2} |\nabla X|^2 |X - x^*|^2,$$

and the factor 1/2 will be essential for the following reasoning.

It follows that

$$\begin{aligned} (18) \qquad &\int_{S_r(w_0)} |\nabla X|^2 \lambda(\rho - |X - x^*|) \, du \, dv \\ &\quad - \frac{1}{2} \int_{S_r(w_0)} \lambda'(\rho - |X - x^*|)|\nabla X|^2 |X - x^*| \, du \, dv \le 0. \end{aligned}$$

Since

$$\lambda'(\rho - |X - x^*|) = 0 \quad \text{if } |X - x^*| \geq \rho$$

it follows that

$$|X - x^*|\lambda'(\rho - |X - x^*|) \leq \rho\lambda'(\rho - |X - x^*|),$$

and (18) yields

$$2\varphi(\rho) - \rho\varphi'(\rho) \leq 0. \tag{18'}$$

Thus,

$$\frac{d}{d\rho}\{\rho^{-2}\varphi(\rho)\} \geq 0,$$

and it follows that

$$\rho^{-2}\varphi(\rho) \leq (\rho')^{-2}\varphi(\rho') \quad \text{for } 0 < \rho \leq \rho' < R^*, \tag{19}$$

where we have set

$$R^* := \min\{\delta(x^*), d^2 R_2\}.$$

Now we choose λ in such a way that it also satisfies

$$\lambda(t) = 1 \quad \text{for any } t \geq \varepsilon,$$

where ε denotes some positive number (in other words, we consider a family $\{\lambda_\varepsilon\}$ of cut-off functions $\lambda_\varepsilon(t)$ with the parameter ε).

Then we obtain

$$\frac{1}{2}\rho^{-2}\int_{S_r(w_0)\cap K_{\rho-\varepsilon}(x^*)} |\nabla X|^2\, du\, dv \leq \rho^{-2}\varphi(\rho),$$

where we have set

$$K_\tau(x^*) := \{w \in B\colon |X(w) - x^*| < \tau\}.$$

Letting $\varepsilon \to +0$ and then $\rho' \to R^* - 0$, we find that

$$\rho^{-2}\int_{S_r(w_0)\cap K_\rho(x^*)} |\nabla X|^2\, du\, dv \leq (R^*)^{-2}\int_{S_r(w_0)\cap K_{R^*}(x^*)} |\nabla X|^2\, du\, dv$$

taking $\lambda(t) \leq 1$ and (19) into account. Now let $\rho \to +0$. Then it follows from Lemma 1 that

$$2\pi R^{*2} \leq \int_{S_r(w_0)\cap K_{R^*}(x^*)} |\nabla X|^2\, du\, dv. \tag{20}$$

In case that $d^2 R_2 \leq \delta(x^*)$, we have by definition of R^* that $R^* = d^2 R_2$, and (20) implies

(21) $$R_2 \le \left\{ \frac{1}{2\pi d^4} \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \right\}^{1/2} \quad \text{if } d^2 R_2 \le \delta(x^*).$$

Now we treat the opposite case $\delta(x^*) < d^2 R_2$ where we have $R^* = \delta(x^*)$. Because of (20), we have already proved that

(22) $$2\pi \delta^2(x^*) \le \int_{S_r(w_0) \cap K_{\delta(x^*)}(x^*)} |\nabla X|^2 \, du \, dv.$$

(The still missing case $\delta(x^*) = 0$ is formally included and will be treated at the end of our discussion.)

First we choose some point $f \in S$ which satisfies

$$|f - x^*| = \text{dist}(x^*, S) = \delta(x^*) < d^2 R_2 \le \tfrac{1}{4} R_2.$$

Then we choose an admissible boundary coordinate system $\{\mathcal{U}, g\}$ for S centered at $x_0 := f$ as described in Definition 2, with the diffeomorphisms g and $h = g^{-1}$. As before we define by $g_{jk}(y)$ the components of the fundamental tensor:

$$g_{jk}(y) := \frac{\partial h^l}{\partial y^j}(y) \frac{\partial h^l}{\partial y^k}(y).$$

Let us introduce the transformed surface $Y(w)$ by

$$Y(w) := g(X(w)) = (y^1(w), y^2(w), y^3(w)),$$

and set

$$\|Y(w)\| := \{g_{jk}(Y(w)) y^j(w) y^k(w)\}^{1/2}.$$

For ρ with $d^{-1}\delta(x^*) < \rho < dR_2$, we define

$$\eta(w) := \begin{cases} \lambda(\rho - \|Y(w)\|) Y(w) & \text{for } w \in \overline{S}_r(w_0), \\ 0 & \text{if } w \in \overline{B} \setminus \overline{S}_r(w_0). \end{cases}$$

Firstly we prove that $\eta \in H_2^1(B, \mathbb{R}^3)$. For this it suffices to show that η vanishes on $C_r(w_0)$. For this purpose, let w be an arbitrary point on $C_r(w_0)$. By assumption (16) we have

$$R_2 \le |X(w) - x^*|,$$

whence

$$R_2 \le \delta(x^*) + |X(w) - f| \le R_2/4 + |X(w) - f|,$$

and this implies

$$R_2/2 \le |X(w) - f| \quad \text{for all } w \in C_r(w_0).$$

On the other hand, since $h(0) = f$, we obtain

$$|X(w) - f| = \left|\int_0^1 h_{y^k}(tY(w))y^k(w)\,dt\right| \leq \sqrt{m_2}|Y(w)| \leq (m_2/m_1)^{1/2}\|Y(w)\|.$$

Thus

$$\|Y(w)\| \geq (1/2)(m_1/m_2)^{1/2}R_2 = dR_2 > \rho,$$

and therefore

$$\eta(w) = 0 \quad \text{for } w \in C_r(w_0).$$

For $0 \leq \varepsilon < 1/2$ we consider the family X_ε of surfaces which are defined by

$$X_\varepsilon(w) := h(Y(w) - \varepsilon\eta(w)).$$

We want to show that X_ε is an admissible variation of X which is of type II. In fact, we have $X_\varepsilon \in H_2^1(B, \mathbb{R}^3)$ and $X_\varepsilon(w) = X(w)$ for all $w \in C$ since $\eta(w) = 0$ for $w \in C$. Now we want to show that X_ε maps $\mathcal{L}^1$-almost all points of I into S. To this end, we pick some $w \in I$ with $X(w) \in S$. If $\eta(w) = 0$, then $X_\varepsilon(w) = X(w)$, and therefore $X_\varepsilon(w) \in S$. On the other hand, if $\eta(w) \neq 0$, we have $\|Y(w)\| < \rho$ and therefore

$$|Y(w)| < \rho/\sqrt{m_1} < dR_2/\sqrt{m_1} = \frac{dR_2}{2\sqrt{m_2}}\left(\frac{1}{2}\sqrt{m_1/m_2}\right)^{-1} = \frac{R_2}{2\sqrt{m_2}} < \frac{R_1}{2\sqrt{m_2}} = \frac{R\sqrt{m_2}}{2\sqrt{m_2}} = R/2.$$

Since $X(w) \in S$, this estimate yields $y^3(w) = 0$ (see (2′)), whence

$$[Y(w) - \varepsilon\eta(w)]^3 = 0.$$

Taking the inequalities

$$|Y(w) - \varepsilon\eta(w)| \leq 2|Y(w)| < R$$

into account, we infer that

$$X_\varepsilon = h(Y - \varepsilon\eta) \in \mathcal{C}(\Gamma, S)$$

provided that $\partial S = \emptyset$. This inclusion holds as well if ∂S is nonvoid, since $y^1(w) \geq \sigma$ and $-1 \leq \sigma \leq 0$ implies

$$y^1(w) - \varepsilon\eta^1(w) = y^1(w)\{1 - \varepsilon\lambda(w)\} \geq \sigma\{1 - \varepsilon\lambda(w)\} \geq \sigma.$$

Now we define

$$\Psi(\varepsilon, w) := \begin{cases} \varepsilon^{-1}[h(Y(w) - \varepsilon\eta(w)) - h(Y(w))] & \text{for } \varepsilon > 0, \\ -\frac{\partial h}{\partial y^k}(Y(w))\eta^k(w) & \text{for } \varepsilon = 0. \end{cases}$$

Then we have

$$X_\varepsilon := h(Y - \varepsilon\eta) = X + \varepsilon\Psi(\cdot, \varepsilon) \quad \text{for } 0 \le \varepsilon < 1/2,$$

and Taylor's formula yields

$$\Psi(\varepsilon, w) = \Psi_0(w) + o(\varepsilon)$$

with

$$\Psi_0 := -h_{y^k}(Y)\eta^k \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$$

and

$$\Psi(\varepsilon, w) \to \Psi_0(w) \quad \text{a.e. on } B \text{ as } \varepsilon \to 0.$$

Moreover, the reader readily checks that

$$|\nabla\Psi(\varepsilon, \cdot)|_{L_2(B)} \le \text{const}$$

holds for some constant independent of $\varepsilon \in [0, 1/2)$. Hence the variations $\{X_\varepsilon\}_{0\le\varepsilon<1/2}$ of X are admissible, and we infer from (9) that

$$\delta E(Y, -\eta) \ge 0,$$

or

$$\delta E(Y, \eta) \le 0,$$

which implies

$$\int_{S_r(w_0)} \left[g_{jk}(Y) D_\alpha y^j D_\alpha \eta^k + \frac{1}{2} g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k \eta^l \right] du\, dv \le 0,$$

where we have set

$$u^1 = u, \quad u^2 = v, \quad D_1 = \frac{\partial}{\partial u}, \quad D_2 = \frac{\partial}{\partial v}$$

(summation with respect to Greek indices from 1 to 2, and with respect to Latin indices from 1 to 3).

Then it follows that

$$\begin{aligned}
&\int_{S_r(w_0)} g_{jk}(Y) D_\alpha y^j D_\alpha y^k \lambda(\rho - \|Y\|)\, du\, dv \\
&\qquad - \int_{S_r(w_0)} \lambda'(\rho - \|Y\|) g_{mn}(Y)(D_\alpha y^m) y^n \frac{1}{2}\|Y\|^{-1} \\
&\qquad \cdot \{2g_{jk}(Y)(D_\alpha y^j) y^k + g_{jk,l}(y)(D_\alpha y^l) y^j y^k\}\, du\, dv \\
&\quad \le -\frac{1}{2} \int_{S_r(w_0)} g_{jk,l}(y) D_\alpha y^j D_\alpha y^k y^l \lambda(\rho - \|Y\|)\, du\, dv.
\end{aligned}$$

This is equivalent to

$$
\begin{aligned}
(23)\quad &\int_{S_r(w_0)} g_{jk}(Y) D_\alpha y^j D_\alpha y^k \lambda(\rho - \|Y\|)\, du\, dv \\
&\quad - \int_{S_r(w_0)} \|Y\| \lambda'(\rho - \|Y\|) \left\{ \left[g_{jk}(Y) y_u^j \frac{y^k}{\|Y\|} \right]^2 \right. \\
&\qquad\qquad \left. + \left[g_{jk}(Y) y_v^j \frac{y^k}{\|Y\|} \right]^2 \right\} du\, dv \\
&\le -\frac{1}{2} \int_{S_r(w_0)} g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k y^l \lambda(\rho - \|Y\|)\, du\, dv \\
&\quad + \frac{1}{2} \int_{S_r(w_0)} \lambda'(\rho - \|Y\|) g_{jk}(Y) D_\alpha y^j \frac{y^k}{\|Y\|} g_{mn,l}(Y) y^m y^n D_\alpha y^l\, du\, dv.
\end{aligned}
$$

Now we set

$$\psi(\rho) := \int_{S_r(w_0)} g_{jk}(Y) D_\alpha y^j D_\alpha y^k \lambda(\rho - \|Y\|)\, du\, dv.$$

Then, by virtue of the conformality relations (15), we obtain the estimate

$$\left[g_{jk}(Y) y_u^j \frac{y^k}{\|Y\|} \right]^2 + \left[g_{jk}(Y) y_v^j \frac{y^k}{\|Y\|} \right]^2 \le \frac{1}{2} \|\nabla Y\|^2,$$

where we have set

$$\|\nabla Y\|^2 := g_{jk}(Y) D_\alpha y^j D_\alpha y^k.$$

Moreover we have

$$
\begin{aligned}
\|Y\| \lambda(\rho - \|Y\|) &\le \rho \lambda(\rho - \|Y\|), \\
\|Y\| \lambda'(\rho - \|Y\|) &\le \rho \lambda'(\rho - \|Y\|).
\end{aligned}
$$

Hence the left-hand side of (23) can be estimated from below by

$$\psi(\rho) - \tfrac{1}{2} \rho \psi'(\rho);$$

compare (18) and (18′) for an analogous computation.

The first term on the right-hand side of (23) can be estimated from above by

$$
\begin{aligned}
c(n)K &\int_{S_r(w_0)} |\nabla Y|^2 m_1^{-1/2} \|Y\| \lambda(\rho - \|Y\|)\, du\, dv \\
&\le c(n) K m_1^{-3/2} \int_{S_r(w_0)} \rho \|\nabla Y\|^2 \lambda(\rho - \|Y\|)\, du\, dv \\
&\le \tilde{M} \rho \psi(\rho),
\end{aligned}
$$

where we have set

$$\tilde{M} := c(n)Km_1^{-3/2},$$

and where $c(n)$ denotes a constant depending on the space dimension (in our case: $n = 3$).

Analogously, the second term is bounded from above by

$$\begin{aligned}
c(n)K &\int_{S_r(w_0)} \lambda'(\rho - \|Y\|)\|\nabla Y\||\nabla Y||Y|^2 \, du \, dv \\
&\le c(n)Km_1^{-3/2} \int_{S_r(w_0)} \|Y\|^2 \lambda'(\rho - \|Y\|)\|\nabla Y\|^2 \, du \, dv \\
&\le \tilde{M}\rho^2\psi'(\rho).
\end{aligned}$$

Thus we have derived the following differential inequality

$$\psi(\rho) - \tfrac{1}{2}\rho\psi'(\rho) \le \tilde{M}[\rho\psi(\rho) + \rho^2\psi'(\rho)]$$

which is equivalent to

$$-\frac{d}{d\rho}[\rho^{-2}\psi(\rho)] \le 2M\rho^{-2}\psi(\rho) + M\frac{d}{d\rho}[\rho^{-1}\psi(\rho)]$$

with

$$M := 2\tilde{M}.$$

Multiplying by $e^{2M\rho}$, we obtain

$$0 \le \frac{d}{d\rho}[e^{2M\rho}\rho^{-2}\psi(\rho)] + Me^{2M\rho}\frac{d}{d\rho}[\rho^{-1}\psi(\rho)].$$

Then by integrating between the limits ρ and $\rho', \rho < \rho'$, and by applying an integration by parts, we infer that

$$\begin{aligned}
0 &\le [e^{2M\rho}\rho^{-2}\psi(\rho)]_\rho^{\rho'} + \int_\rho^{\rho'} Me^{2M\rho}\frac{d}{d\rho}[\rho^{-1}\psi(\rho)] \, d\rho \\
&= [e^{2M\rho}\rho^{-2}\psi(\rho)]_\rho^{\rho'} + [Me^{2M\rho}\rho^{-1}\psi(\rho)]_\rho^{\rho'} - \int_\rho^{\rho'} 2M^2e^{2M\rho}\rho^{-1}\psi(\rho) \, d\rho.
\end{aligned}$$

Therefore,

$$0 \le [e^{2M\rho}\rho^{-2}\psi(\rho) + Me^{2M\rho}\rho^{-1}\psi(\rho)]_\rho^{\rho'}$$

whence

$$\rho^{-2}\psi(\rho) \le \frac{e^{2M\rho'} + \rho'Me^{2M\rho'}}{e^{2M\rho} + \rho Me^{2M\rho}}(\rho')^{-2}\psi(\rho').$$

Applying once again the reasoning which led to (20) (that is, choosing $\lambda = \lambda_\varepsilon$, and letting first $\varepsilon \to +0$ and then $\rho' \to dR_2 - 0$) and setting

$$C(R_2) := (1 + dR_2M)e^{2MdR_2},$$

we arrive at

$$(24)\qquad \rho^{-2}\int_{S_r(w_0)\cap\{w:\ \|Y(w)\|<\rho\}}\|\nabla Y\|^2\,du\,dv \leq C(R_2)(dR_2)^{-2}\int_{S_r(w_0)}\|\nabla Y\|^2\,du\,dv.$$

Furthermore,

$$S_r(w_0)\cap\{w\colon |X(w)-f|<2\delta(x^*)\}\subset S_r(w_0)\cap\{w\colon \|Y(w)\|<\rho\}$$

since

$$Y(w)=\int_0^1 g_{x^j}(tX+(1-t)f)(x^j-f^j)\,dt$$

implies

$$\|Y(w)\|\leq (m_2/m_1)^{1/2}|X(w)-f|<2(m_2/m_1)^{1/2}\delta(x^*)=d^{-1}\delta(x^*)<\rho.$$

For $\delta(x^*)>0$ and $\rho\to d^{-1}\delta(x^*)+0$, we then infer from (24) that

$$\frac{d^2}{\delta^2(x^*)}\int_{S_r(w_0)\cap\{w:\ |X(w)-f|<2\delta(x^*)\}}\|\nabla Y\|^2\,du\,dv \leq C(R_2)d^{-2}R_2^{-2}\int_{S_r(w_0)}\|\nabla Y\|^2\,du\,dv,$$

and this inequality can be rewritten in the form

$$(25)\qquad \delta(x^*)^{-2}\int_{S_r(w_0)\cap K_{2\delta(x^*)}(f)}|\nabla X|^2\,du\,dv \leq C(R_2)d^{-4}R_2^{-2}\int_{S_r(w_0)}|\nabla X|^2\,du\,dv.$$

By virtue of

$$|X(w)-f|\leq|X(w)-x^*|+|f-x^*|$$

we obtain

$$K_{\delta(x^*)}(x^*)\subset K_{2\delta(x^*)}(f),$$

and therefore

$$R_2^2\delta(x^*)^{-2}\int_{S_r(w_0)\cap K_{\delta(x^*)}(x^*)}|\nabla X|^2\,du\,dv\leq C(R_2)d^{-4}\int_{S_r(w_0)}|\nabla X|^2\,du\,dv.$$

By virtue of (22), the left-hand side is bounded from below by $2\pi R_2^2$. Thus we obtain

$$(26)\qquad R_2\leq\left\{\frac{C(R_2)}{2\pi d^4}\int_{S_r(w_0)}|\nabla X|^2\,du\,dv\right\}^{1/2}\quad \text{if } 0<\delta(x^*)<d^2R_2.$$

Combining (21) and (26), we obtain from $C(R_2) \le C(R_1), R_1 = R\sqrt{m_2}$ and $d^{-2} = 4(m_2/m_1)$ that

$$R_2 \le c \left\{ \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \right\}^{1/2} \quad \text{in Case (i): } \delta(x^*) > 0, \tag{27}$$

if we set

$$c := (2^3 \pi^{-1} (m_2/m_1)^2 C(R_1))^{1/2} = c(R, K, m_1, m_2).$$

Case (ii): $\delta(x^*) = 0$.

Here we take $f = x^*$ as the center of an admissible boundary coordinate system $\{\mathcal{U}, g\}$ for S. Then we obtain (24) for any $\rho \in (0, dR_2)$. Setting $\rho' := \rho\sqrt{m_1/m_2}$, it follows that

$$\rho^{-2} \int_{S_r(w_0) \cap K_{\rho'}(x^*)} |\nabla X|^2 \, du \, dv \le C(R_2) d^{-2} R_2^{-2} \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv.$$

Now let $\rho' \to +0$; then another application of Lemma 1 yields

$$\begin{aligned} 2\pi &\le \limsup_{\rho' \to +0} \; (\rho')^{-2} \int_{S_r(w_0) \cap K_{\rho'}(x^*)} |\nabla X|^2 \, du \, dv \\ &\le \frac{C(R_2)}{4 d^4 R_2^2} \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \end{aligned}$$

whence we obtain

$$R_2 \le c \left\{ \int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \right\}^{1/2} \quad \text{in Case (ii): } \delta(x^*) = 0. \tag{27'}$$

Combining (27) and (27′), we arrive at (17). □

Now we turn to the *second step* with the aim to prove

Theorem 2. *Let S be an admissible support surface of class C^2, and suppose that $X(w)$ is a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Then there exists a constant $\alpha \in (0,1)$ such that the following holds true:*

For every $d \in (0,1)$, there exists a constant $c > 0$ such that

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le c r^{2\alpha} \tag{28}$$

holds true for every $w_0 \in \overline{Z}_d$ and for all $r > 0$. In particular, X is of class $C^{0,\alpha}(B \cup I, \mathbb{R}^3)$.

We shall use the following simple but quite effective

Lemma 5. *Let $\varphi(r), 0 < r \le 2R$ be a nondecreasing and nonnegative function satisfying*

$$\varphi(r) \le \theta\varphi(2r) \tag{29}$$

for some $\theta \in (0,1)$ and for all $r \in (0,R]$. Then, for

$$\alpha := {}_2\log\frac{1}{\theta},$$

we have

$$\varphi(r) \le 2^\alpha \varphi(R)(r/R)^\alpha \quad \textit{for all } r \in (0,R]. \tag{30}$$

Proof. For any $r \in (0,R]$ and for any $\nu = 0,1,2,\dots$, we have

$$\varphi(2^{-\nu}r) \le \theta\varphi(2^{-\nu+1}r).$$

Iterating these inequalities, we obtain

$$\varphi(2^{-\nu}r) \le \theta^\nu\varphi(r) \quad \text{for } 0 < r \le R.$$

Fix some $r \in (0,R]$. Then there exists some integer $\nu \ge 0$ such that

$$2^{-\nu-1} < r/R \le 2^{-\nu}.$$

Since $\theta = 2^{-\alpha}$ and $\varphi(r)$ is nondecreasing, we see that

$$\varphi(r) \le \varphi(2^{-\nu}R) \le \theta^\nu\varphi(R) \le 2^{-\nu\alpha}\varphi(R) \le 2^\alpha\varphi(R)(r/R)^\alpha. \qquad \square$$

For later use we note a generalization of Lemma 5.

Lemma 6. *Let $\varphi(r), 0 < r \le 2R$, by a nondecreasing and nonnegative function satisfying*

$$\varphi(r) \le \theta\{\varphi(2r) + r^\sigma\} \tag{31}$$

for some $\theta \in (0,1)$, $\sigma > 1$, $0 < R < 1$, and for all $r \in (0,R]$. Then, for $\varepsilon \in (0, \sigma - 1)$ and for

$$\theta^* := \max\{\theta, 2^{\varepsilon-\sigma}(\theta R^\varepsilon + 1)\}, \quad \alpha := {}_2\log\frac{1}{\theta^*}, \tag{32}$$

we have

$$\varphi(r) \le 2^\alpha\{\varphi(R) + R^{\sigma-\varepsilon}\}(r/R)^\alpha \quad \textit{for all } r \in (0,R]. \tag{33}$$

Proof. Since $R^\varepsilon\theta < 1$ and $2^{\varepsilon-\sigma} < 1/2$, we infer that

$$2^{\varepsilon-\sigma}(\theta R^\varepsilon + 1) < 1$$

and therefore $0 < \theta^* < 1$. Set

$$\varphi^*(r) := \varphi(r) + r^{\sigma-\varepsilon}.$$

Then, for any $r \in (0, R]$, it follows that

$$\begin{aligned}\varphi^*(r) &\le \theta\varphi(2r) + \theta r^\sigma + r^{\sigma-\varepsilon} = \theta\varphi(2r) + r^{\sigma-\varepsilon}(\theta r^\varepsilon + 1)\\ &\le \theta\varphi(2r) + (2r)^{\sigma-\varepsilon}2^{\varepsilon-\sigma}(\theta r^\varepsilon + 1)\\ &\le \theta^*\{\varphi(2r) + (2r)^{\sigma-\varepsilon}\} = \theta^*\varphi^*(2r).\end{aligned}$$

Applying Lemma 5, we infer

$$\varphi^*(r) \le 2^\alpha \varphi^*(R)(r/R)^\alpha \quad \text{for all } r \in (0, R] \text{ and } \alpha := {}_2\log\frac{1}{\theta^*}$$

whence

$$\varphi(r) \le \varphi(r) + r^{\sigma-\varepsilon} \le 2^\alpha\{\varphi(R) + R^{\sigma-\varepsilon}\}(r/R)^\alpha \quad \text{for } 0 < r \le R. \qquad \square$$

Proof of Theorem 2. We want to show that the growth estimate (28) is satisfied for any $w_0 \in I$. Let us first assume that ∂S is empty. We introduce an admissible boundary coordinate system $\{\mathcal{U}, g\}$ for S centered at $x_0 := X(w_0)$ with the inverse mapping $h = g^{-1}$, and we set $Y := g(X)$. Then we have $Y \in C^0(B \cup I, \mathbb{R}^3)$ and $Y(w_0) = 0$, and we can find some number $\rho_0 \in (0, 1 - |w_0|)$ such that

$$|Y(w)| \le R/2 \quad \text{for } w \in \overline{S}_{\rho_0}(w_0), \qquad Y^3(w) = 0 \quad \text{for } w \in I \cap \overline{S}_{\rho_0}(w)$$

(cf. Definition 2 for the meaning of R, as well as the discussion following Definition 2).

Suppose that $X_\varepsilon := h(Y - \varepsilon\phi), |\varepsilon| < \varepsilon_0(\phi), \phi = (\varphi^1, \varphi^2, \varphi^3)$, is a family of admissible variations with $X_\varepsilon \in \mathcal{C}(\Gamma, S)$. Then we have

$$(34) \qquad \int_B g_{jk}(Y) D_\alpha y^j D_\alpha \varphi^k \, du\, dv \le -\frac{1}{2}\int_B g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k \varphi^l \, du\, dv.$$

(In fact, equality holds true.) Now let $r \in (0, \rho_0/2]$, and choose some cut-off function $\xi \in C_c^\infty(B_{2r}(w_0))$ with $\xi(w) \equiv 1$ on $B_r(w_0)$ and $0 \le \xi \le 1, |\nabla\xi| \le 2/r$.

Set $T_{2r} := S_{2r}(w_0) \setminus S_r(w_0)$,

$$\omega^1 := \fint_{T_{2r}} y^1 \, du\, dv, \quad \omega^2 := \fint_{T_{2r}} y^2 \, du\, dv, \quad \omega^3 := 0,$$

where

$$\fint_\Omega \cdots \quad \text{stands for} \quad \frac{1}{\text{meas}\,\Omega}\int_\Omega \cdots$$

$\phi = (\varphi^1, \varphi^2, \varphi^3), \varphi^k(w) := (y^k(w) - \omega^k)\xi^2(w)$ for $w \in B \cup I$. Then the test vector ϕ is admissible in (34), and we obtain

$$\int_B g_{jk}(Y) D_\alpha y^j D_\alpha y^k \xi^2 \, du \, dv + \frac{1}{2} \int_B g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k (y^l - \omega^l) \xi^2 \, du \, dv$$
$$\leq -2 \int_B g_{jk}(Y) D_\alpha y^j (y^k - \omega^k) \xi D_\alpha \xi \, du \, dv.$$

Hence, for any $\varepsilon > 0$ and some constant $K_1(\varepsilon) > 0$, we find the inequality

(35) $$m_1 \int_B |\nabla Y|^2 \xi^2 \, du \, dv - \frac{1}{2} \int_B |g_{jk,l}(Y)| |D_\alpha y^j| |D_\alpha y^k| |y^l - \omega^l| \xi^2 \, du \, dv$$
$$\leq \varepsilon \int_B |\xi|^2 \|\nabla Y\|^2 \, du \, dv + K_1(\varepsilon) \int_B \|Y - \omega\|^2 |\nabla \xi|^2 \, du \, dv,$$

where $\omega = (\omega^1, \omega^2, \omega^3)$. Since

$$\|\nabla Y\|^2 \leq m_2 |\nabla Y|^2$$

we can absorb the term

$$\varepsilon \int_B \xi^2 \|\nabla Y\|^2 \, du \, dv$$

by the first term on the left-hand side, if we choose

$$\varepsilon = \frac{m_1}{2m_2}.$$

Moreover, the absolute value of the second term of the left-hand side of (35) can be bounded from above by

$$\frac{m_1}{4} \int_B |\nabla Y|^2 \xi^2 \, du \, dv,$$

if we choose $r \in (0, \rho_1)$, where $\rho_1 \in (0, \rho_0/2)$ is a sufficiently small number depending on the modulus of continuity of X. Hence there is a number $K_2 > 0$ such that

(36) $$\int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \leq \int_{S_{2r}(w_0)} \xi^2 |\nabla Y|^2 \, du \, dv$$
$$\leq K_2 r^{-2} \int_{T_{2r}} |Y - \omega|^2 \, du \, dv$$

holds for all $r \in (0, \rho_1)$.

By Poincaré's inequality, there is a constant $K_3 > 0$ such that

(37) $$\int_{T_{2r}} |Y - \omega|^2 \, du \, dv \leq K_3 r^2 \int_{T_{2r}} |\nabla Y|^2 \, du \, dv$$

is satisfied for $0 < r < \rho_1$. Consequently, there is a constant K_4 such that

$$\int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \le K_4 \int_{S_r(w_0) \setminus S_r(w_0)} |\nabla Y|^2 \, du \, dv$$

for all $r \in (0, \rho_1)$.

Now we fill the hole $S_r(w_0)$ by adding the term $K_4 \int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv$ to both sides. Then we arrive at

$$(1 + K_4) \int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \le K_4 \int_{S_{2r}(w_0)} |\nabla Y|^2 \, du \, dv$$

whence, setting

$$\theta := \frac{K_4}{1 + K_4},$$

we attain

$$\int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \le \theta \int_{S_{2r}(w_0)} |\nabla Y|^2 \, du \, dv$$

for every $r \in (0, \rho_1)$. As $0 < \theta < 1$, we can apply Lemma 5 to $R = \rho_1$ and to $\varphi(r) := \int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv$, thus obtaining

$$\int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \le 2^{2\alpha} \int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv \left(\frac{r}{\rho_1} \right)^{2\alpha}$$

for $0 < r < \rho_1$, if we set $\alpha := \frac{1}{2} \, {}_2\log \theta$. For

$$K_5 := 2^{2\alpha}(m_2/m_1), \quad (K_5 > 1),$$

and by virtue of

$$\|\nabla Y\|^2 = |\nabla X|^2, \quad m_1 |\nabla Y|^2 \le \|\nabla Y\|^2 \le m_2 |\nabla Y|^2,$$

we obtain

$$\int_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le K_5 D(X) (r/\rho_1)^{2\alpha} \tag{38}$$

for all $r \in (0, \rho_1)$, and consequently for all $r > 0$.

Combining (38) in a suitable way with interior estimates for X, we arrive at (28). We can omit this reasoning since it would be a mere repetition of the arguments used in the second part of the proof of Theorem 1 in Section 2.5.

Finally, Morrey's Dirichlet growth theorem yields $X \in C^{0,\alpha}(B \cup I, \mathbb{R}^3)$. Thus we have proved Theorem 2 in the case that ∂S is empty.

The general case where ∂S is not necessarily empty can be settled by a slight modification of our previous reasoning.

First we note the that test function

$$\phi = (0, \varphi^2, \varphi^3), \quad \varphi^k = (y^k - \omega^k)\xi^2, \quad k = 2, 3$$

is admissible in (34), where ω^k and ξ are chosen as before. Then we obtain an inequality which coincides with (35) except for the term

$$m_1 \int_B |\nabla Y|^2 \xi^2 \, du \, dv,$$

which is to be replaced by

$$m_1 \int_B (|\nabla y^2|^2 + |\nabla y^3|^2) \xi^2 \, du \, dv.$$

However, this expression can be estimated from below by

$$\frac{m_1}{1+K^*} \int_B |\nabla Y|^2 \xi^2 \, du \, dv$$

since there is a constant $K^* > 0$ such that

$$|\nabla y^1|^2 \le K^*(|\nabla y^2|^2 + |\nabla y^3|^2)$$

holds true, and this inequality is an immediate consequence of the conformality relations (15), written in the complex form

$$\langle\langle Y_w, Y_w \rangle\rangle = 0,$$

where we have set

$$\langle\langle \xi, \eta \rangle\rangle := g_{jk} \xi^j \eta^k$$

(cf. Section 2.3, proof of Theorem 2, part (i)).

Thus we arrive again at an inequality of the type (36) from where we can proceed as before. This completes the proof of the theorem. □

Remark 1. A close inspection of the proof of Theorem 2 shows that we would obtain a priori estimates for the α-Hölder seminorm in the case that we had bounds on the modulus of continuity of X. Hence only the approach used in the proof of Theorem 1 is indirect.

Remark 2. Without any essential change we can replace the class $\mathcal{C}(\Gamma, S)$ in the previous reasoning by $\mathcal{C}(S)$. In other words, we have analogues to Theorems 1 and 2 for stationary points of Dirichlet's integral in the free boundary class $\mathcal{C}(S)$.

2.7 $C^{1,1/2}$-Regularity

In this section we want to prove $C^{1,1/2}$-regularity of a stationary point X of Dirichlet's integral up to its free boundary. As we have seen in Section 2.4, this regularity result is optimal, that is, we can in general not prove $X \in$

$C^{1,\alpha}(B \cup I, \mathbb{R}^3)$, I being the free boundary, for some $\alpha > 1/2$, if the boundary of the support surface S is nonvoid. On the other hand, if ∂S is empty or if $X|_I$ does not touch ∂S, then one might be able to achieve higher regularity as we shall see in the next section.

As in Section 2.6 we shall restrict our considerations to minimal surfaces with partially free boundaries or, more precisely, to stationary points of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$; stationary points with completely free boundaries can be treated in exactly the same way, and perfectly analogous results hold true.

Consequently we can use the same notation as in Section 2.6. Our main result will be the following

Theorem 1. *Let S be an admissible support surface of class C^4, and suppose that $X(w)$ is a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Then X is of class $C^{1,1/2}(B \cup I, \mathbb{R}^3)$.*

The proof of this result is quite involved; it will be carried out in three steps. In the first step we prove that $X \in H_2^2(Z_d, \mathbb{R}^3)$ for any $d \in (0,1)$, using Nirenberg's difference quotient technique to derive L_2-estimates for $\nabla^2 X$. Secondly, using ideas related to those of Section 2.3, it will be shown that $X \in C^1(B \cup I, \mathbb{R}^3)$. In the third part of our investigation we shall see how the boundary regularity can be pushed up to $X \in C^{1,1/2}(B \cup I, \mathbb{R}^3)$ by applying an appropriate iteration procedure.

Let us note that, assuming $X \in C^0(B \cup I, \mathbb{R}^3)$, all regularity results will be proved directly by establishing a priori estimates. Thus the only indirect proof entering into our discussion is that of Theorem 1 of Section 2.6.

Step 1. L_2-estimates for $\nabla^2 X$ up to the free boundary. Let us begin with a few remarks on difference quotients which either are well known (cf. Nirenberg [1], Gilbarg and Trudinger [1]) or can easily be derived.

We consider some function $Y \in H_2^s(Z_{d_0}, \mathbb{R}^m)$ with $0 < d_0 < 1$ and $m \geq 1$, $s \geq 1$. For $w \in Z_d$ and t with $|t| < d_0 - d$, we define the tangential shift Y_t by

$$Y_t(u,v) := Y(u+t, v)$$

and the tangential difference quotient $\Delta_t Y$ by

$$\Delta_t Y(u,v) = \frac{1}{t}[Y(u+t,v) - Y(u,v)],$$

that is,

$$\Delta_t Y(w) = \frac{1}{t}[Y_t(w) - Y(w)], \quad w = u + iv.$$

Moreover, let $D_u = \frac{\partial}{\partial u}$ be the tangential derivative with respect to the free boundary I. Then we have:

Lemma 1. (i) *Let $Y \in H_2^s(Z_{d_0}, \mathbb{R}^m), s \geq 1, m \geq 1, d_0 \in (0,1), d \in (0, d_0)$, $|t| \leq d_0 - d$. Then $Y_t, \Delta_t Y \in H_2^s(Z_d, \mathbb{R}^m)$, and*

$$\int_{Z_d} |\Delta_t Y|^2 \, du \, dv \le \int_{Z_{d_0}} |D_u Y|^2 \, du \, dv, \quad \lim_{t \to 0} \int_{Z_d} |D_u Y - \Delta_t Y|^2 \, du \, dv = 0.$$

The operators ∇ *and* Δ_t *commute; more precisely,*

$$(\Delta_t \nabla Y)(w) = (\nabla \Delta_t Y)(w) \quad \text{for } w \in Z_d,$$

and similarly

$$(\nabla Y)_t(w) = (\nabla Y_t)(w) \quad \text{for } w \in Z_d.$$

Moreover, we have the product rule

$$\Delta_t(\varphi Y) = (\Delta_t \varphi) Y_t + \varphi \Delta_t Y = (\Delta_t \varphi) Y + \varphi_t \Delta_t Y$$

on Z_d *for scalar functions* φ, *and*

$$\int_B \varphi \Delta_{-t} \psi \, du \, dv = -\int_B (\Delta_t \varphi) \psi \, du \, dv \quad \text{for } 0 < |t| \ll 1$$

if either φ *or* ψ *has compact support in* $B \cup I$.

(ii) *Similarly, if* Y *and* $D_u Y \in L_q(Z_{d_0}, \mathbb{R}^m), q \ge 1$, *then*

$$\int_{Z_d} |\Delta_t Y|^q \, du \, dv \le \int_{Z_{d_0}} |D_u Y|^q \, du \, dv, \quad \lim_{t \to 0} \int_{Z_d} |D_u Y - \Delta_t Y|^q \, du \, dv = 0.$$

(iii) *Finally, if* $Y \in H_2^s(Z_{d_0}, \mathbb{R}^m)$, *then*

$$(\nabla^p Y)_t = \nabla^p Y_t,$$

$$\int_\Omega |\nabla^p Y_t|^2 \, du \, dv = \int_{\Omega_t} |\nabla^p Y|^2 \, du \, dv, \quad 0 < |t| \ll 1,$$

for $0 \le p \le s$ *and* $\Omega_t := \{w + t\colon w \in \Omega\}$, *for any open set* $\Omega \Subset Z_{d_0} \cup I$.

Now we turn to the derivation of L_2-estimates for the second derivatives of X. We begin by linearizing the boundary conditions on X. This will be achieved by introducing suitable new coordinates on $\mathbb{R}^3$. Thus let w_0 be an arbitrary point on I, and set $x_0 := X(w_0)$. Then we choose an admissible boundary coordinate system $\{\mathcal{U}, g\}$, centered at x_0, as defined in Section 2.6. Let $h = g^{-1}$ and $Y = g \circ X$, i.e., $X = h \circ Y$. Then we can use the discussion at the beginning of Section 2.6; in particular we can employ the formulas (1)–(15) of Section 2.6.

By Theorem 1 of Section 2.6, we know that X and Y are continuous on $B \cup I$, and $Y(w_0) = 0$. Hence there is some number $\rho > 0$ such that

$$|Y(w)| < R \quad \text{for all } w \in \overline{S}_{2\rho}(w_0)$$

and therefore

$$y^3(w) = 0 \quad \text{for } w \in I_{2\rho}(w_0),$$

and, if ∂S is nonempty, we have

$$y^1(w) \geq \sigma \quad \text{for } w \in I_{2\rho}(w_0).$$

Let r be some number with $0 < r < \rho$ which is to be fixed later, and let $\eta(w)$ be some cut-off function of class $C_c^\infty(B_{2r}(w_0))$ with $\eta(w) \equiv 1$ on $B_r(w_0), 0 \leq \eta \leq 1, |\nabla\eta| \leq 2/r$, and $\eta(u,v) = \eta(u,-v)$.

Now we set

$$\phi := \Delta_{-t}\{\eta^2 \Delta_t Y\}. \tag{1}$$

We claim that

$$X_\varepsilon := h(Y + \varepsilon\phi), \quad 0 \leq \varepsilon < \varepsilon_0(\phi), \tag{2}$$

is an admissible variation of X in $\mathcal{C}(\Gamma, S)$ of type II (see Definition 2 of Section 1.4) for some sufficiently small $\varepsilon_0(\phi) > 0$. In fact, we have $\phi \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$, and

$$\begin{aligned} Y(w) + \varepsilon\phi(w) &= Y(w) + \varepsilon\Delta_{-t}\{\eta^2\Delta_t Y\}(w) \\ &= \lambda_1 Y_t(w) + \lambda_2 Y_{-t}(w) + (1 - \lambda_1 - \lambda_2)Y(w), \end{aligned}$$

where

$$\lambda_1 := \varepsilon t^{-2}\eta^2(w), \quad \lambda_2 := \varepsilon t^{-2}\eta_{-t}^2(w), \quad 0 < |t| \ll 1.$$

Thus $Y(w) + \varepsilon\phi(w), 0 \leq \varepsilon \leq t^2/2$, is a convex combination of the three points $Y(w), Y_t(w)$, and $Y_{-t}(w)$.

Since $\eta(w) = 0$ for $|w - w_0| \geq 2r$, we obtain

$$\lambda_1(w) = 0, \quad \lambda_2(w) = 0 \quad \text{if } |w - w_0| \geq 2r + |t|, \; w \in \overline{B}.$$

Therefore we have

$$Y(w) + \varepsilon\phi(w) = Y(w) \quad \text{for } |w - w_0| \geq 2r + |t|.$$

On the other hand, if $|w - w_0| < 2r + |t|, w \in \overline{B}$, then we have

$$|w \pm t - w_0| \leq 2r + 2|t|$$

and therefore

$$w, w \pm t \in \overline{S}_{2\rho}(w_0), \quad \text{provided that } |t| < \rho - r.$$

Hence, for $w \in I_{2r+|t|}(w_0)$, the points $Y(w), Y_t(w), Y_{-t}(w)$ are contained in the convex set

$$C_R' := \{y \in \mathbb{R}^3 \colon y^3 = 0, |y| < R\} \quad \text{if } \partial S = \emptyset$$

or in

$$C''_R := \{y \in \mathbb{R}^3 : y^3 = 0, y^1 \geq \sigma, |y| < R\} \quad \text{if } \partial S \neq \emptyset$$

respectively, and we have

$$S \cap \mathcal{U} = \begin{cases} h(C'_R) & \text{if } \partial S = \emptyset, \\ h(C''_R) & \text{if } \partial S \neq \emptyset. \end{cases}$$

Thus we obtain

$$X_\varepsilon(w) = h(Y(w) + \varepsilon\phi(w)) \in S \quad \text{for all } w \in I,$$

provided that $0 \leq \varepsilon < t^2/2$ and $|t| < \rho - r$, and clearly

$$X_\varepsilon(w) = X(w) \quad \text{for } w \in C = \partial B \setminus I$$

since $\phi(w) = 0$ on C. Consequently, we have

$$X_\varepsilon = h(Y + \varepsilon\phi) \in \mathcal{C}(\Gamma, S) \quad \text{for } 0 \leq \varepsilon < t^2/2 \text{ and } |t| < \rho - r,$$

and it follows from Section 2.6, (9) that

$$\delta E(Y, \phi) \geq 0.$$

Inserting the expression (1) into this inequality, we obtain

$$\int_B g_{jk}(Y) D_\alpha y^j D_\alpha \{\Delta_{-t}(\eta^2 \Delta_t y^k)\}\, du\, dv \\ \geq -\frac{1}{2} \int_B \Delta_{-t}\{\eta^2 \Delta_t y^l\} g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k\, du\, dv,$$

where $D_1 = \frac{\partial}{\partial u}$, $D_2 = \frac{\partial}{\partial v}$, $u^1 = u$, $u^2 = v$, and an integration by parts yields

$$\int_B \Delta_t[g_{jk}(Y) D_\alpha y^j] D_\alpha(\eta^2 \Delta_t y^k)\, du\, dv \\ \leq -\frac{1}{2} \int_B \eta^2 \Delta_t y^l \Delta_t[g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k]\, du\, dv;$$

see Lemma 1. Since

$$\Delta_t[g_{jk}(Y) D_\alpha y^j] = g_{jk}(Y) D_\alpha \Delta_t y^j + D_\alpha y^j_t \Delta_t g_{jk}(Y)$$

and

$$D_\alpha(\eta^2 \Delta_t y^k) = D_\alpha \eta^2 \Delta_t y^k + \eta^2 D_\alpha \Delta_t y^k,$$

we arrive at

$$\text{(3)} \quad \int_B \eta^2 g_{jk}(Y) D_\alpha \Delta_t y^j D_\alpha \Delta_t y^k \, du \, dv$$
$$\leq - \int_B 2\eta D_\alpha \eta \Delta_t y^k [g_{jk}(Y) D_\alpha \Delta_t y^j + D_\alpha y_t^j \Delta_t g_{jk}(Y)] \, du \, dv$$
$$- \int_B \eta^2 D_\alpha \Delta_t y^k D_\alpha y_t^j \Delta_t g_{jk}(Y) \, du \, dv$$
$$- 1/2 \int_B \eta^2 \Delta_t y^l \Delta_t [g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k] \, du \, dv.$$

The ellipticity condition for (g_{jk}) yields

$$\text{(4)} \quad m_1 \int_B \eta^2 |\nabla \Delta_t Y|^2 \, du \, dv \leq \int_B \eta^2 g_{jk}(Y) D_\alpha \Delta_t y^j D_\alpha \Delta_t y^k \, du \, dv.$$

Moreover, Lemma 1 implies

$$\text{(5)} \quad \Delta_t [g_{jk,l}(Y) D_\alpha y^j D_\alpha y^k]$$
$$= (\Delta_t g_{jk,l}(Y)) D_\alpha y^j D_\alpha y^k + g_{jk,l}(Y_t)(\Delta_t D_\alpha y^j) D_\alpha y^k$$
$$+ g_{jk,l}(Y_t) D_\alpha y_t^j \Delta_t D_\alpha y^k.$$

Furthermore, there is a constant $K^* > 0$ such that

$$\text{(6)} \quad |\Delta_t g_{jk}(Y)| + |\Delta_t g_{jk,l}(Y)| \leq K^* |\Delta_t Y|.$$

On account of (3)–(6), there is a number $c = c(m_2, K, K^*)$ independent of t such that

$$m_1 \int_B \eta^2 |\nabla \Delta_t Y|^2 \, du \, dv$$
$$\leq c \left\{ \int_B r^{-1} \eta |\Delta_t Y| (|\nabla \Delta_t Y| + |\nabla Y_t| |\Delta_t Y|) \, du \, dv \right.$$
$$+ \int_B \eta^2 |\nabla \Delta_t Y| |\nabla Y_t| |\Delta_t Y| \, du \, dv$$
$$\left. + \int_B \eta^2 |\Delta_t Y| (|\Delta_t Y| |\nabla Y|^2 + |\nabla \Delta_t Y| |\nabla Y| + |\nabla \Delta_t Y| |\nabla Y_t|) \, du \, dv \right\}.$$

By means of the elementary inequality

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} \, b^2$$

for any $\varepsilon > 0$, we obtain the estimate

$$m_1 \int_B \eta^2 |\nabla \Delta_t Y|^2 \, du \, dv$$
$$\leq \varepsilon \int_B \eta^2 |\nabla \Delta_t Y|^2 \, du \, dv + \frac{c^*}{\varepsilon} \left[r^{-2} \int_{S_{2r}(w_0)} |\Delta_t Y|^2 \, du \, dv \right.$$
$$\left. + \int_B \eta^2 |\Delta_t Y|^2 |\nabla Y|^2 du \, dv + \int_B \eta^2 |\Delta_t Y|^2 |\nabla Y_t|^2 \, du \, dv \right].$$

Choosing $\varepsilon := m_1/2$, we can absorb the first integral on the right-hand side by the positive term on the left-hand side, and secondly, we have

$$\begin{aligned}\int_{S_{2r}(w_0)} |\Delta_t Y|^2\,du\,dv &\le \int_B |D_u Y|^2\,du\,dv \le \int_B |\nabla Y|^2\,du\,dv\\ &\le m_1^{-1}\int_B \|\nabla Y\|^2\,du\,dv = m_1^{-1}\int_B |\nabla X|^2\,du\,dv\\ &= 2m_1^{-1}D(X).\end{aligned}$$

Thus we arrive at

$$\begin{aligned}(7)\qquad &\int_{S_{2r}(w_0)} \eta^2|\nabla\Delta_t Y|^2\,du\,dv\\ &\quad\le c^{**}\left[r^{-2}D(X)+\int_{S_{2r}(w_0)}\eta^2|\Delta_t Y|^2(|\nabla Y|^2+|\nabla Y_t|^2)\,du\,dv\right].\end{aligned}$$

Moreover, we claim that the estimate (28) in Theorem 2 of Section 2.6 implies the existence of some number c_0 independent of r and t such that

$$\begin{aligned}(8)\qquad &\int_{S_{2r}(w_0)} \eta^2|\Delta_t Y|^2(|\nabla Y|^2+|\nabla Y_t|^2)\,du\,dv\\ &\quad\le c_0 r^{2\alpha}\left\{\int_{S_{2r}(w_0)}\eta^2|\nabla\Delta_t Y|^2\,du\,dv + r^{-2}D(X)\right\}.\end{aligned}$$

Let us defer the proof of the inequality (8) until we have finished the derivation of the L_2-estimates of $\nabla^2 X$. Then we can proceed as follows:

We choose $r\in(0,\rho)$ so small that $c^{**}c_0 r^{2\alpha} < 1/2$. Then we infer from (7) and (8) the existence of a number c_1 independent of t such that

$$(9)\qquad \int_{S_{2r}(w_0)} \eta^2|\nabla\Delta_t Y|^2\,du\,dv \le c_1 D(X)$$

holds true for all t with $0<|t|<\rho-r$. If we let t tend to zero, this inequality yields

$$(10)\qquad \int_{S_{2r}(w_0)} \eta^2|\nabla D_u Y|^2\,du\,dv \le c_1 D(X)$$

since $Y = g\circ X$ is of class $C^3(B,\mathbb{R}^3)$, and from (8) and (9) we infer

$$(11)\qquad \int_{S_{2r}(w_0)} \eta^2|D_u Y|^2|\nabla Y|^2\,du\,dv \le c_2 D(X).$$

Moreover, the conformality relation

$$\|D_u Y\|^2 = \|D_v Y\|^2$$

implies that

$$|D_v Y|^2 \le (m_2/m_1)|D_u Y|^2,$$

whence we obtain

$$\int_{S_{2r}(w_0)} \eta^2 |\nabla Y|^4 \, du \, dv \le c_3 D(X), \tag{12}$$

taking (11) into account.

Moreover, by formula (14) of Section 2.6 we have

$$\Delta y^l + \Gamma^l_{jk}(Y)(y^j_u y^k_u + y^j_v y^k_v) = 0 \quad \text{in } B,$$

whence

$$|D_v^2 Y|^2 \le c_4(|D_u^2 Y|^2 + |\nabla Y|^4) \quad \text{in } B.$$

Combining the last relation with (10) and (12), we arrive at

$$\int_{S_{2r}(w_0)} \eta^2 |\nabla^2 Y|^2 \, du \, dv \le c_5 D(X) \tag{13}$$

whence

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv + \int_{S_r(w_0)} |\nabla Y|^4 \, du \, dv \le c_6 D(X). \tag{14}$$

Moreover, from $X = h(Y)$ we obtain

$$\nabla^2 X = h_{yy}(Y)\nabla Y \nabla Y + h_y(Y)\nabla^2 Y,$$

and therefore

$$|\nabla^2 X|^2 \le c_7(|\nabla Y|^4 + |\nabla^2 Y|^2).$$

By virtue of (14) it follows that

$$\int_{S_r(w_0)} |\nabla^2 X|^2 \, du \, dv \le c_8 D(X). \tag{15}$$

This is the desired estimate of $\nabla^2 X$.

Before we summarize the results of our investigation, we want to prove the estimate (8) which, so far, has remained open. We shall see how c_0 depends on X, and this will inform us about the dependence of the numbers $c_1, \dots, c_8$ on X.

The estimate (8) will be derived from Theorem 2 of Section 2.6 and from the following calculus inequality:

Lemma 2. *Let Ω be an open set in $\mathbb{C}$ of finite measure, and define $d \ge 0$ by the relation $\pi d^2 = \operatorname{meas} \Omega$. Suppose also that $q \in L_1(\Omega)$ is a function such that*

$$\int_{\Omega \cap B_r(w_0)} |q(w)| \, du \, dv \le Q r^{2\alpha} \tag{16}$$

holds for some number $Q \ge 0$, for some exponent $\alpha > 0$ and for all disks $B_r(w_0)$ in $\mathbb{C}$. Then, for any $\nu \in (0, \alpha)$, there is a number $M(\alpha, \nu) > 0$, depending only on α and ν, such that

$$\int_{\Omega \cap B_r(w_0)} |q(w)| |\phi(w)|^2 \, du \, dv \le M Q D_\Omega(\phi) d^\nu r^{2\alpha - \nu} \tag{17}$$

holds true for all $w_0 \in \mathbb{C}$, for all $r > 0$, and for any function $\phi \in \mathring{H}_2^1(\Omega, \mathbb{R}^m)$, $m \ge 1$.

Proof. As the set $C_c^\infty(\Omega, \mathbb{R}^m)$ is dense in $\mathring{H}_2^1(\Omega, \mathbb{R}^m)$, it is sufficient to prove (17) for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^m)$, taking Fatou's lemma into account.

Thus let $\phi \in C_c^\infty(\Omega, \mathbb{R}^m)$, $w = u^1 + iu^2$, $\zeta = \xi^1 + i\xi^2$, $d^2 w = du^1 \, du^2$, $d^2\zeta = d\xi^1 \, d\xi^2$. From Green's formula, we infer that

$$\phi(w) = -\frac{1}{2\pi} \int_\Omega |w - \zeta|^{-2} (\xi^\alpha - u^\alpha) D_\alpha \phi(\zeta) \, d^2\zeta$$

is satisfied for any $w \in \Omega$. Set $\Omega_r := \Omega \cap B_r(w_0)$; then we obtain

$$\begin{aligned}
&\int_{\Omega_r} |q(w)| |\phi(w)| \, d^2 w \\
&\le \frac{1}{2\pi} \int_{\Omega_r} \int_\Omega |q(w)| |w - \zeta|^{-1} |\nabla \phi(\zeta)| \, d^2\zeta \, d^2 w \\
&= \frac{1}{2\pi} \int_{\Omega_r} \int_\Omega |q(w)|^{1/2} |w - \zeta|^{-1+\nu} |q(w)|^{1/2} |w - \zeta|^{-\nu} |\nabla \phi(\zeta)| \, d^2\zeta \, d^2 w \\
&\le \frac{1}{2\pi} \left[\int_{\Omega_r} \int_\Omega |q(w)| |w - \zeta|^{2\nu - 2} \, d^2\zeta \, d^2 w \right]^{1/2} \\
&\quad \cdot \left[\int_\Omega \int_{\Omega_r} |q(w)| |w - \zeta|^{-2\nu} |\nabla \phi(\zeta)|^2 \, d^2 w \, d^2\zeta \right]^{1/2} .
\end{aligned} \tag{18}$$

By an inequality of E. Schmidt, we have

$$\int_\Omega |w - \zeta|^{2\nu - 2} \, d^2\zeta \le (\pi/\nu) d^{2\nu};$$

the simple proof of this fact is left to the reader. Then, by (16), we obtain

$$\int_{\Omega_r} \int_\Omega |q(w)| |w - \zeta|^{2\nu - 2} \, d^2\zeta \, d^2 w \le \frac{\pi}{\nu} d^{2\nu} Q r^{2\alpha}. \tag{19}$$

For $s > 0$ and $\zeta \in \mathbb{C}$ we introduce the function

$$\psi(s,\zeta) := \int_{\Omega_r \cap B_s(\zeta)} |q(w)|\, d^2w.$$

By (16), we have

$$0 \le \psi(s,\zeta) \le Q s^{2\alpha} \quad \text{for all } s > 0 \text{ and } \zeta \in \mathbb{C}$$

as well as

$$0 \le \psi(s,\zeta) \le Q r^{2\alpha} \quad \text{for all } \zeta \in \mathbb{C}.$$

Introducing polar coordinates ρ, θ about ζ by $w = \zeta + \rho e^{i\theta}$ we have

$$\psi(s,\zeta) = \int_0^s \left(\int_{\Sigma_\rho} |q(\zeta + \rho e^{i\theta})|\, d\theta \right) \rho\, d\rho,$$

where

$$\Sigma_\rho := \{\theta \colon 0 \le \theta \le 2\pi, \zeta + \rho e^{i\theta} \in \Omega_r \cap B_s(\zeta)\}.$$

It follows that

$$\frac{d}{ds}\psi(s,\zeta) = s \int_{\Sigma_s} |q(\zeta + s e^{i\theta})|\, d\theta.$$

Case 1. Let $\zeta \in \overline{B}_r(w_0)$. Then we have $|w - \zeta| \le 2r$ for any $w \in \Omega_r$. Accordingly,

$$\begin{aligned}
\int_{\Omega_r} |w-\zeta|^{-2\nu} |q(w)|\, d^2w &\le \int_0^{2r} \int_{\Sigma_s} s^{-2\nu} |q(\zeta + s e^{i\theta})| s\, d\theta\, ds \\
&= \int_0^{2r} s^{-2\nu} \frac{d}{ds}\psi(s,\zeta)\, ds = \lim_{\varepsilon \to +0} \int_\varepsilon^{2r} s^{-2\nu} \frac{d}{ds}\psi(s,\zeta)\, ds \\
&= \lim_{\varepsilon \to +0} [s^{-2\nu}\psi(s,\zeta)]_\varepsilon^{2r} + 2\nu \lim_{\varepsilon \to +0} \int_\varepsilon^{2r} s^{-2\nu-1}\psi(s,\zeta)\, ds \\
&\le Q(2r)^{2\alpha-2\nu} + 2\nu Q \frac{1}{2(\alpha-\nu)}(2r)^{2\alpha-2\nu} = c(\alpha,\nu) Q r^{2\alpha-2\nu}.
\end{aligned}$$

Case 2. If $\zeta \in \Omega \setminus \overline{B}_r(w_0)$, then we have

$$\zeta_0 := w_0 + \frac{r}{|\zeta - w_0|}(\zeta - w_0) \in \overline{B}_r(w_0).$$

Moreover, for all $w \in \Omega_r$, it follows by a simple geometric consideration (cf. Fig. 1) that

$$|\zeta - w| \ge |\zeta_0 - w|.$$

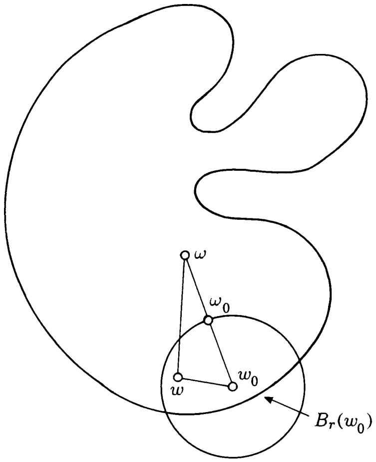

Fig. 1.

Consequently,

$$\int_{\Omega_r} |q(w)||\zeta - w|^{-2\nu}\, d^2w \le \int_{\Omega_r} |q(w)||\zeta_0 - w|^{-2\nu}\, d^2w,$$

and, by case 1,

$$\int_{\Omega_r} |q(w)||\zeta_0 - w|^{-2\nu}\, d^2w \le c(\alpha,\nu)\, Q r^{2\alpha-2\nu}.$$

Thus we have found that

$$\int_{\Omega_r} |q(w)||\zeta - w|^{-2\nu}\, d^2w \le c(\alpha,\nu)\, Q r^{2\alpha-2\nu} \quad \text{for all } \zeta \in \Omega.$$

Consequently,

$$\begin{aligned}(20) \qquad &\int_{\Omega}\int_{\Omega_r} |q(w)||w-\zeta|^{-2\nu}|\nabla\phi(\zeta)|^2\, d^2w\, d^2\zeta \\ &\le c(\alpha,\nu)\, Q r^{2\alpha-2\nu} \int_{\Omega} |\nabla\phi(\zeta)|^2\, d^2\zeta.\end{aligned}$$

From (18), (19), and (20) we infer that

$$(21) \qquad \int_{\Omega_r} |q(w)||\phi(w)|\, d^2w \le c^*(\alpha,\nu)\, Q d^{\nu} D_{\Omega}^{1/2}(\phi) r^{2\alpha-\nu}.$$

In other words, the function $q^* := q\phi$ satisfies

$$(21') \qquad \int_{\Omega\cap B_r(w_0)} |q^*(w)|\, d^2w \le Q^* r^{2\alpha^*} \quad \text{for all disks } B_r(w_0),$$

where

$$Q^* := c^*(\alpha, \nu) Q d^\nu D_\Omega^{1/2}(\phi), \quad 2\alpha^* := 2\alpha - \nu.$$

Now let ν and μ be two positive numbers such that $\nu + \mu < \alpha$. Then it follows that

$$\alpha^* - \mu = \alpha - \frac{\nu}{2} - \mu > \alpha - (\nu + \mu) > 0.$$

Hence we can apply the estimate (21′) to q^*, Q^*, α^*, μ instead of q, Q, α, ν, and thus we obtain

$$\int_{\Omega_r} |q^*(w)||\phi(w)|\, d^2w \le c^*(\alpha, \mu) Q^* d^\mu D_\Omega^{1/2}(\phi) r^{2\alpha^* - \mu}$$

or equivalently

$$\int_{\Omega \cap B_r(w_0)} |q(w)||\phi(w)|^2\, d^2w \le c^*(\alpha, \nu) c^*(\alpha, \mu) Q d^{\nu+\mu} D_\Omega(\phi) r^{2\alpha - (\nu+\mu)}.$$

Replacing $\nu + \mu$ by ν and $c^*(\alpha, \nu)c^*(\alpha, \mu)$ by $M(\alpha, \nu)$, we arrive at the desired inequality (17). □

Now we come to the *proof of formula* (8). From $Y = g(X)$ it follows that

$$|\nabla Y| \le \sqrt{m_2} |\nabla X|$$

whence by Section 2.6, Theorem 2 (and, in particular, Section 2.6, (28)) we obtain that

$$\int_{B \cap B_\tau(\zeta_0)} |\nabla Y|^2\, du\, dv \le Q \tau^{2\alpha} \tag{22}$$

holds for some constant $Q > 0$, some $\alpha \in (0, 1)$, and for all disks $B_\tau(\zeta_0)$. Therefore,

$$\int_{S_{2r}(w_0) \cap B_\tau(\zeta_0)} (|\nabla Y|^2 + |\nabla Y_t|^2)\, du\, dv \le 2Q\tau^{2\alpha}$$

for some $Q > 0, \alpha \in (0, 1)$, and for all disks $B_\tau(\zeta_0)$ and all t with $|t| < t_0$ and $0 < t_0 \ll 1$.

Let $\Omega := B_{2r}(w_0), w_0 \in I$, and set

$$q(w) := |\nabla Y(w)|^2 + |\nabla Y_t(w)|^2, \quad \phi(w) := \eta(w) \Delta_t Y(w) \quad \text{for } w \in S_{2r}(w_0)$$

and

$$q(u, v) := q(u, -v),\ \phi(u, v) := \phi(u, -v) \quad \text{for } w = u + iv \in B_{2r}(w_0) \text{ and } v < 0.$$

Applying Lemma 2, we can infer that

$$\int_{\Omega \cap B_\tau(\zeta_0)} |q(w)||\phi(w)|^2\, du\, dv \le 2MQD_\Omega(\phi)(2r)^\nu \tau^{2\alpha - \nu}.$$

In particular, for $\zeta_0 = w_0$ and $\tau = 2r$, we have $\Omega = B_\tau(\zeta_0)$ and therefore

$$\int_{B_{2r}(w_0)} |q(w)||\phi(w)|^2 \, du \, dv \le 4MQD_{B_{2r}(w_0)}(\phi)(2r)^{2\alpha}$$

whence, for reasons of symmetry,

$$\begin{aligned}&\int_{S_{2r}(w_0)} \eta^2 |\Delta_t Y|^2 (|\nabla Y|^2 + |\nabla Y_t|^2) \, du \, dv \\ &\qquad \le 4MQ 2^{2\alpha} r^{2\alpha} \int_{S_{2r}(w_0)} |\nabla(\eta \Delta_t Y)|^2 \, du \, dv.\end{aligned}$$

Moreover,

$$\begin{aligned}|\nabla(\eta\Delta_t Y)|^2 &= |\nabla\eta\Delta_t Y + \eta\nabla\Delta_t Y|^2 \\ &\le 2\eta^2|\nabla\Delta_t Y|^2 + 8r^{-2}|\Delta_t Y|^2.\end{aligned}$$

Setting

$$c_0 := 2^{5+2\alpha} MQ \max\{1, m_1^{-1}\}, \tag{23}$$

we arrive at formula (8). From (38) in Section 2.6 it follows that Q is of the form

$$Q = cD(X), \tag{24}$$

where c depends on the diffeomorphism g and on the modulus of continuity of X on $B \cup I$. Hence also the constants $c_1, \ldots, c_6$ are of the form $cD(X)$ with c depending on g and on the modulus of continuity of X.

Let us summarize the results (9)–(15), (22)–(24).

Theorem 2. *Let S be an admissible support surface of class C^3, and suppose that X is a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Then, for any $d \in (0,1)$, there is a constant $c > 0$ depending only on $d, |g|_3, D(X)$, and the modulus of continuity of X such that*

$$\int_{Z_d} (|\nabla^2 X|^2 + |\nabla X|^4) \, du \, dv \le c \tag{25}$$

holds true.

Applying Sobolev's embedding theorem (see Gilbarg and Trudinger [1]), we derive the following result from Theorem 2:

Theorem 3. *Let S be an admissible support surface of class C^3, and suppose that X is a stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$. Then, for any $d \in (0,1)$ and for any p with $2 < p < \infty$, there is a constant $c > 0$ depending only on $d, p, |g|_3, D(X)$, and the modulus of continuity of X such that*

$$\int_{Z_d} |\nabla X|^p \, du \, dv < c \tag{26}$$

holds true. Moreover both X_u *and* X_v *have an* L_2*-trace on every compact subinterval of* I*.*

In brief, we have shown that any stationary minimal surface X in $\mathcal{C}(\Gamma, S)$ is of class $H_2^2 \cap H_p^1(Z_d, \mathbb{R}^3)$ for any $d \in (0,1)$ and any p with $2 < p < \infty$, and $X_u, X_v \in L_2(I', \mathbb{R}^3)$ for every $I' \subset\subset I$.

Step 2. Continuity of the first derivatives at the free boundary. The aim of this step is the proof of the following

Theorem 4. *Let* S *be an admissible support surface of class* C^3*. Then any stationary point* X *of Dirichlet's integral in* $\mathcal{C}(\Gamma, S)$ *is of class* $C^1(B \cup I, \mathbb{R}^3)$*.*

Proof. We choose $w_0 \in I, x_0 = X(w_0), \rho > 0$, and a boundary coordinate system $\{\mathcal{U}, g\}$ centered at x_0 as before, and we set $Y = g \circ X = (y^1, y^2, y^3)$. Then we have

$$\Delta y^l + \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k = 0. \tag{27}$$

Hence, for any $\phi = (\varphi^1, \varphi^2, \varphi^3) \in C_c^\infty(S_{2\rho}(w_0) \cup I_{2\rho}(w_0), \mathbb{R}^3)$, the equation

$$\delta E(Y, \phi) = 0 \tag{28}$$

is equivalent to

$$\int_{I_{2\rho}(w_0)} g_{jk}(Y) y_v^j \varphi^k \, du = 0. \tag{29}$$

Case 1. ∂S is empty.

Then ϕ is admissible for (28) if $\varphi^3(w) = 0$ on $I_{2\rho}(w_0)$. We conclude from (29) that

$$\begin{aligned} g_{j1}(Y) y_v^j &= 0 \quad \text{a.e. on } I_{2\rho}(w_0), \\ g_{j2}(Y) y_v^j &= 0 \quad \text{a.e. on } I_{2\rho}(w_0), \\ y^3 &= 0 \qquad\quad \text{on } I_{2\rho}(w_0) \end{aligned} \tag{30}$$

since Theorem 3 implies that both Y_u and Y_v are of class $L_2(I', \mathbb{R}^3)$ for every $I' \Subset I$.

Case 2. ∂S is nonempty.

Then ϕ is admissible for (28) if $\varphi^3(w) = 0$ on $I_{2\rho}(w_0)$ and if $\varphi^1|_{I_{2\rho}(w_0)}$ has its support in $I^+_{2\rho}(w_0) := I_{2\rho}(w_0) \cap \{y^1(w) > \sigma\}$. We conclude from (29) that

$$\begin{aligned} g_{j1}(Y) y_v^j &= 0 \quad \text{a.e. on } I^+_{2\rho}(w_0), \\ g_{j2}(Y) y_v^j &= 0 \quad \text{a.e. on } I_{2\rho}(w_0), \\ y^3 &= 0 \qquad\quad \text{on } I_{2\rho}(w_0). \end{aligned} \tag{31}$$

Now we claim that, for $S \in C^3$ and $S \in C^4$, we can find an admissible coordinate system $\{\mathcal{U}, g\}$ for S centered at x_0 which is of class C^2 or C^3 respectively, and satisfies

$$(g_{jk}(y^1, y^2, 0)) = \begin{bmatrix} g_{11}(y^1, y^2, 0) & 0 & 0 \\ 0 & g_{22}(y^1, y^2, 0) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{32}$$

for all $(y^1, y^2, 0) \in B_R(0)$ if $\partial S = \emptyset$, or for all $(y^1, y^2, 0) \in B_R(0) \cap \{y^1 \geq \sigma\}$ if $\partial S \neq \emptyset$. Note, however, that we lose an order of differentiability if we pass from S to g in case of the particular coordinate system $\{\mathcal{U}, g\}$ with property (32). Let us postpone the construction of this coordinate system; first we want to exploit (32) to derive regularity.

The special form of the metric tensor (g_{jk}) simplifies the equations (30) to

$$\begin{aligned} y^1_v &= 0 \\ y^2_v &= 0 \quad \text{a.e. on } I_{2\rho}(w_0) \text{ if } \partial S = \emptyset, \\ y^3 &= 0 \end{aligned} \tag{33}$$

and (31) takes the special form

$$\begin{aligned} y^1_v &= 0 \quad \text{a.e. on } I^+_{2\rho}(w_0) \\ y^2_v &= 0 \quad \text{a.e. on } I_{2\rho}(w_0) \quad \text{if } \partial S \neq \emptyset. \\ y^3 &= 0 \qquad\quad \text{on } I_{2\rho}(w_0). \end{aligned} \tag{34}$$

Furthermore, we infer from Theorem 3 that

$$\Delta Y \in L_p(S_{2\rho}(w_0), \mathbb{R}^3) \quad \text{for any } p \in (1, \infty), \tag{35}$$

provided that S is of class C^3 which implies $h \in C^2$ and $\Gamma^l_{jk} \in C^0$. In *case* 1, we infer from (33) and (35) by means of classical results from potential theory that $Y \in H^2_p(S_{2r}(w_0), \mathbb{R}^3)$ for any $p \in (1, \infty)$, any $w_0 \in I$, and any $r \in (0, \rho)$; cf. Morrey [8], Theorem 6.3.7, or Agmon, Douglis, and Nirenberg [1, 2], for the pertinent L_p-estimates.

Then we obtain $Y \in C^{1,\beta}(S_{2r}(w_0), \mathbb{R}^3)$ for all $\beta \in (0, 1)$, taking a Sobolev embedding theorem into account; cf. Gilbarg and Trudinger [1], Chapter 7, or Morrey [8], Theorem 3.6.6.

If $S \in C^4$, then $h \in C^3$ and $\Gamma^l_{jk} \in C^1$, and consequently $\Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k \in C^{0,\beta}(S_{2r}(w_0)), l = 1, 2, 3$. On account of (27) and (33), we then obtain

$$\Delta Y \in C^{0,\beta}(S_{2r}(w_0), \mathbb{R}^3) \quad \text{for any } \beta \in (0, 1). \tag{36}$$

By classical potential-theoretic results of Korn–Lichtenstein–Schauder, we infer from (33) and (36) that $Y \in C^{2,\beta}(S_{2r}(w_0), \mathbb{R}^3)$ holds for any $\beta \in (0, 1)$ and any $r \in (0, \rho)$. A simple proof can be derived from the Korn–Privalov theorem; see Section 2.1, Lemma 6. Since $h \in C^3$ and $X = h \circ Y$, it follows that

$$X \in C^{2,\beta}(B \cup I, \mathbb{R}^3).$$

Note that the same result can be derived under the weaker assumption $S \in C^3$ if we do not work with the special coordinate system (32) where one derivative is lost. Then we have to use (36) and the more complicated boundary conditions (30). Applying Morrey's results (see [8], Chapter 6), together with a strengthening by F.P. Harth [2], we obtain the desired result.

Thus, if ∂S is empty, we have proved a result which is much stronger than Theorem 4:

Theorem 5. *Let S be an admissible support surface of class C^3, and suppose that ∂S is empty. Then any stationary point X of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is of class $C^{2,\beta}(B \cup I, \mathbb{R}^3)$ for any $\beta \in (0,1)$.*

Remark 1. If ∂S is nonempty, the same holds true if $X|_I$ does not touch ∂S.

Remark 2. In addition to Theorem 5, the Schauder–Lichtenstein estimates together with our previous bounds (see Theorem 3) imply that there exists a number c depending only on $d \in (0,1)$, $\beta \in (0,1)$, $|g|_3$, $D(X)$, and the modulus of continuity of X such that

$$|X|_{2+\beta, Z_d} \le c \tag{37}$$

holds true for any $d \in (0,1)$ and any $\beta \in (0,1)$.

These remarks complete our discussion in the case that ∂S is empty.

Now we turn to case 2, i.e. $\partial S \neq \emptyset$. As before, we have (35), and therefore in particular

$$\Delta y^2, \Delta y^3 \in L_p(S_{2\rho}(w_0)) \quad \text{for any } p \in (1, \infty),$$

and the second and third equation of (35) yield

$$y_v^2 = 0 \quad \text{and} \quad y^3 = 0 \quad \text{a.e. on } I_{2\rho}(w_0).$$

By the same reasoning as in case 1 we first obtain $y^2, y^3 \in H_p^2(S_{2r}(w_0))$ for $p \in (1, \infty)$ and $r \in (0, \rho)$, and then

$$y^2, y^3 \in C^{1,\beta}(S_{2r}(w_0)) \quad \text{for any } \beta \in (0,1) \text{ and } r \in (0, \rho). \tag{38}$$

(For this result, we only use $S \in C^3$, whence $h \in C^2$ and $\Gamma_{jk}^l \in C^0$.)

The function $y^1(w)$ satisfies

$$\begin{aligned} &\Delta y^1 \in L_p(S_{2\rho}(w_0)) \quad \text{for any } p \in (1, \infty), \\ &y_v^1 = 0 \quad \text{a.e. on } I_{2\rho}^+(w_0), \quad y^1 = \sigma \quad \text{on } I_{2\rho}(w_0) \setminus I_{2\rho}^+(w_0). \end{aligned} \tag{39}$$

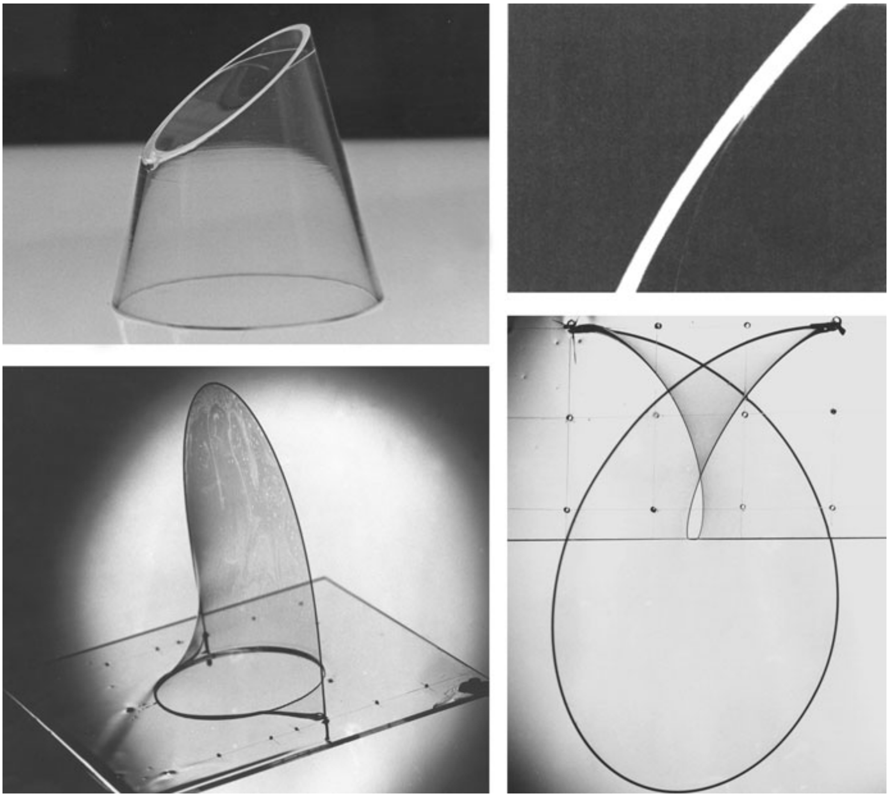

Fig. 2. Soap films attaching smoothly to the boundary of the support surface. Courtesy of E. Pitts (above) and Institut für Leichte Flächentragwerke, Stuttgart – Archive (below)

It is not at all clear how to exploit (39) (cf., however, Section 2.9). Therefore we shall instead use the conformality relations in complex notation,

$$g_{jk}(Y)y_w^j y_w^k = 0, \tag{40}$$

in order to show that $y^1 \in C^1(\overline{S}_{2r}(w_0))$ for some $r \in (0, \rho)$. Our reasoning will be similar as in the proof of Theorem 2 in Section 2.3 (see formulas (16)–(20) of Section 2.3). Since (g_{jk}) is a positive definite matrix, there is some $\gamma > 0$ such that

$$g_{11}(Y(w)) \geq \gamma \quad \text{for all } w \in \overline{S}_{2\rho}(w_0).$$

Hence we can rewrite (40) as

$$\left\{ y_w^1 + \frac{g_{1L}(Y)}{g_{11}(Y)} y_w^L \right\}^2 = \left[\frac{g_{1L}(Y)}{g_{11}(Y)} y_w^L \right]^2 - \frac{g_{LM}(Y)}{g_{11}(Y)} y_w^L y_w^M, \tag{41}$$

where repeated indices L, M are to be summed from 2 to 3. If we introduce the complex-valued function $f(w)$ by

(42) $$f(w) := y_w^1 + \frac{g_{1L}(Y)}{g_{11}(Y)} y_w^L, \quad w \in \overline{S}_{2\rho}(w_0),$$

we infer from (41) and (38) as well as from $Y \in C^{0,\beta}(B \cup I, \mathbb{R}^3)$ that

(43) $$f^2 \in C^{0,\beta}(\overline{S}_{2r}(w_0)) \quad \text{for } 0 < \beta < 1,$$

whence

$$f^2 \in C^0(\overline{S}_{2r}(w_0)).$$

In addition, we have $f \in C^0(S_{2r}(w_0))$. By the following lemma it will be seen that $f(w)$ is continuous on $\overline{S}_{2r}(w_0)$.

Lemma 3. *Let $f(w)$ be a complex-valued continuous function on an open connected set Ω in $\mathbb{C}$ such that its square $f^2(w)$ has a continuous extension to $\overline{\Omega}$. Suppose also that $\partial\Omega$ is non-degenerate in the sense that, for every $w_0 \in \partial\Omega$, there exists a $\delta > 0$ such that $\Omega_\delta(w_0) := \Omega \cap B_\delta(w_0)$ is connected. Then $f(w)$ can continuously be extended to $\overline{\Omega}$.*

Proof. Let w_0 be an arbitrary point on $\partial\Omega$. Then there exists a complex number z such that $f^2(w) \to z$ as $w \to w_0$, $w \in \Omega$. If $z = 0$, then $|f(w)|^2 \to 0$, and therefore $f(w) \to 0$ as $w \to w_0$. If $z \neq 0$, then we choose some $\zeta \neq 0$, $\zeta \in \mathbb{C}$, such that $z = \zeta^2$. We pick an $\varepsilon > 0$ such that $0 < \varepsilon < |\zeta|$. Then there exists a number $\delta > 0$ such that $\Omega_\delta(w_0)$ is connected, and that f maps $\Omega_\delta(w_0)$ into the disconnected set $B_\varepsilon(\beta) \cup B_\varepsilon(-\beta)$. Since $f\colon \Omega \to \mathbb{C}$ is continuous, the image $f(\Omega_\delta(w_0))$ is connected, and therefore already contained in one of the disks $B_\varepsilon(\beta), B_\varepsilon(-\beta)$. Thus $\lim_{w\to w_0} f(w)$ exists and is equal to β or $-\beta$. Set

$$F(w) := \begin{cases} f(w) & \text{for } w \in \Omega \\ \lim_{\tilde{w}\to w} f(\tilde{w}) & \text{for } w \in \partial\Omega \end{cases}.$$

Clearly this function is a continuous extension of f to $\overline{\Omega}$, and the lemma is proved. □

Thus we have found that the function $f(w)$, defined by (42), is continuous on $\overline{S}_{2r}(w_0)$ for some $r \in (0, \rho)$. Since

(42′) $$g(w) := \frac{g_{1L}(w)}{g_{11}(w)} y_w^L$$

is Hölder continuous on $\overline{S}_{2r}(w_0)$, we infer that

$$y_w^1 = f(w) - g(w)$$

is continuous on $\overline{S}_{2r}(w_0)$, and therefore $Y \in C^1(\overline{S}_{2r}(w_0), \mathbb{R}^3)$. This implies $X \in C^1(B \cup I, \mathbb{R}^3)$, and Theorem 4 is proved.

However, we still have to verify that we can find a coordinate system $\{\mathcal{U}, g\}$ centered at x_0 which satisfies (32). To this end, we choose a neighbourhood $\mathcal{U}$ of the point $x_0 \in \mathcal{U}$ and an orthogonal parameter representation $x = t(y^1, y^2)$, $(y^1, y^2) \in P$, of $S \cap \mathcal{U}$ with $x_0 = t(0, 0)$. In other words, we have $\mathcal{F} = 0$, where

$$\mathcal{E} := |t_{y^1}|^2, \quad \mathcal{F} := \langle t_{y^1}, t_{y^2} \rangle, \quad \mathcal{G} := |t_{y^2}|^2$$

are the coefficients of the first fundamental form of S. Moreover, set

$$\mathcal{W} := |t_{y^1} \wedge t_{y^2}| = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{\mathcal{E}\mathcal{G}}$$

and let

$$n := \frac{1}{\mathcal{W}}(t_{y^1} \wedge t_{y^2})$$

be the surface normal of S. If $\partial S = \emptyset$, we can assume that the parameter domain P is given by $P = K_R$ where

$$K_R := \{(y^1, y^2) \colon |y^1|^2 + |y^2|^2 < R^2\}, \quad 0 < R < 1.$$

If $\partial S \neq \emptyset$, we can assume that S is part of a larger surface S_0 such that $S_0 \cap \mathcal{U}$ is represented on K_R in the form $x = t(y^1, y^2), (y^1, y^2) \in K_R$, and that $S \cap \mathcal{U}$ is given by $x = t(y^1, y^2), (y^1, y^2) \in P = K_R \cap \{y^1 \geq \sigma\}, \sigma \in [-1, 0]$. We can also suppose that $\partial S \cap \mathcal{U}$ is represented by t on $K_R \cap \{y^1 = \sigma\}$. Choosing $R \in (0, 1)$ sufficiently small, we can in addition assume that

$$h(y) := t(y^1, y^2) + y^3 n(y^1, y^2), \quad y = (y^1, y^2, y^3) \in B_R(0), \tag{44}$$

provides a diffeomorphism of $B_R(0) = \{y \in \mathbb{R}^3 \colon |y| < R\}$ onto some neighbourhood of x_0 which will again be denoted by $\mathcal{U}$. Then h maps C'_R or C''_R onto $S \cap \mathcal{U}$ if ∂S is void or nonvoid respectively, where

$$\begin{aligned} C'_R &= \{y \in \mathbb{R}^3 \colon y^3 = 0, |y| < R\}, \\ C''_R &= \{y \in \mathbb{R}^3 \colon y^3 = 0, y^1 \geq \sigma, |y| < R\}. \end{aligned}$$

Moreover, we may assume that h can be extended to a diffeomorphism of $\mathbb{R}^3$ onto itself; let g be its inverse. Then $\{\mathcal{U}, g\}$ is an admissible boundary coordinate system for S centered at x_0, which is of class C^2 or C^3 if S is of class C^3 or C^4, respectively (because of the special form (44) of h involving the surface normal n of S, we unfortunately lose one derivate).

The components $g_{jk} = h^l_{y^j} h^l_{y^k}$ of the metric tensor are computed as

$$\begin{aligned} g_{11} &= |h_{y^1}|^2 = \mathcal{E} - 2y^3\mathcal{L} + (y^3)^2|n_{y^1}|^2, \\ g_{22} &= |h_{y^2}|^2 = \mathcal{G} - 2y^3\mathcal{N} + (y^3)^2|n_{y^2}|^2, \\ g_{33} &= |h_{y^3}|^2 = |n|^2 = 1, \\ g_{12} &= g_{21} = \langle h_{y^1}, h_{y^2} \rangle = \mathcal{F} - 2y^3\mathcal{M} + (y^3)^2\langle n_{y^1}, n_{y^2} \rangle, \\ g_{13} &= g_{31} = \langle h_{y^1}, h_{y^3} \rangle = \langle t_{y^1}, n \rangle + y^3\langle n_{y^1}, n \rangle = 0, \\ g_{23} &= g_{32} = \langle h_{y^2}, h_{y^3} \rangle = \langle t_{y^2}, n \rangle + y^3\langle n_{y^2}, n \rangle = 0 \end{aligned}$$

for $y \in B_R(0)$. Hence we obtain

$$(g_{jk}(y^1, y^2, 0)) = \begin{bmatrix} \mathcal{E}(y^1, y^2) & 0 & 0 \\ 0 & \mathcal{G}(y^1, y^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for $y \in C'_R$ or C''_R respectively, and the proof of Theorem 4 is complete. □

Step 3. Regularity of class $C^{1,1/2}$ at the free boundary.

Now we turn to the final part of our discussion. We are going to prove Theorem 1. Our main tool will be

Lemma 4. *Let $f(u)$ be a complex-valued continuous function of the real variable u on a closed subinterval I' of I, and set $a(u) = \operatorname{Re} f(u), b(u) = \operatorname{Im} f(u)$. We suppose that $a(u)b(u) \equiv 0$ on I', and that there are positive numbers α and c such that $\alpha \le 1$ and*

(45) $$|f^2(u_1) - f^2(u_2)| \le c^2 |u_1 - u_2|^{2\alpha} \quad \textit{for all } u_1, u_2 \in I'.$$

Then it follows that

(46) $$|f(u_1) - f(u_2)| \le 2c|u_1 - u_2|^{\alpha} \quad \textit{for all } u_1, u_2 \in I'.$$

Proof. Let $u_1, u_2 \in I', u_1 \ne u_2$, and set $f_1 := f(u_1), f_2 := f(u_2)$.

(i) Let $c|u_1 - u_2|^\alpha \le |f_1 + f_2|$. Then we obtain

$$\begin{aligned} c|u_1 - u_2|^\alpha |f_1 - f_2| &\le |f_1 + f_2||f_1 - f_2| \\ &= |f_1^2 - f_2^2| \le c^2 |u_1 - u_2|^{2\alpha} \end{aligned}$$

and consequently

$$|f_1 - f_2| \le c|u_1 - u_2|^\alpha.$$

(ii) If $|f_1 + f_2| < c|u_1 - u_2|^\alpha$ and $\operatorname{Re} f_1 = \operatorname{Im} f_2 = 0$, or else $\operatorname{Re} f_2 = \operatorname{Im} f_1 = 0$, then $|f_1 - f_2| = |f_1 + f_2|$, and consequently

$$|f_1 - f_2| < c|u_1 - u_2|^\alpha.$$

(iii) If $|f_1 + f_2| < c|u_1 - u_2|^\alpha$ and $\operatorname{Im} f_1 = \operatorname{Im} f_2 = 0$, then either $|f_1 - f_2| \le |f_1 + f_2|$, and therefore

$$|f_1 - f_2| < c|u_1 - u_2|^\alpha,$$

or else $|f_1 - f_2| > |f_1 + f_2|$, whence

$$|a_1 + a_2| < |a_1 - a_2| \quad \text{for } a_1 := \operatorname{Re} f_1, a_2 := \operatorname{Re} f_2.$$

Since $a(u)$ is continuous on I', there is a number u_0 between u_1 and u_2 such that $a(u_0) := \operatorname{Re} f_0 = 0$, where we have set $f_0 := f(u_0)$. Thus each of the pairs

$\{f_1, f_0\}$ and $\{f_2, f_0\}$ is either in case (i) or in case (ii), and by the previous conclusions we obtain

$$|f_1 - f_0| \le c|u_1 - u_0|^\alpha \quad \text{and} \quad |f_2 - f_0| \le c|u_2 - u_0|^\alpha$$

whence

$$|f_1 - f_2| \le 2c|u_1 - u_2|^\alpha.$$

(iv) If $|f_1 + f_2| < c|u_1 - u_2|^\alpha$ and $\operatorname{Re} f_1 = \operatorname{Re} f_2 = 0$, then we obtain by a reasoning analogous to (iii) that

$$|f_1 - f_2| \le 2c|u_1 - u_2|^\alpha.$$

Because of $a(u)b(u) \equiv 0$ on I', we have exhausted all possible cases and the lemma is proved. □

Proof of Theorem 1. We choose $w_0 \in I, x_0 := X(w_0), \rho > 0, r \in (0, \rho)$, and a boundary coordinate system $\{\mathcal{U}, g\}$ with (32) as before, and set again $Y = g \circ X$. As we have now assumed that $S \in C^4$, we have $g, h \in C^3$, and therefore $\Gamma^l_{jk} \in C^1$, $\Gamma^l_{jk}(Y) \in C^1(S_{2r}(w_0))$.

Since we have already treated the case $\partial S = \emptyset$, we can concentrate our attention on the case $\partial S \neq \emptyset$ where we have the boundary conditions (34).

Let $f(w)$ be the complex-valued function defined by (42). Then, by (32), it follows that

$$f(w) = y^1_w(w) \quad \text{for all } w \in I_{2r}(w_0). \tag{47}$$

Furthermore, the equations (32) and (41) imply

$$f^2 = -\frac{g_{22}(Y)}{g_{11}(Y)}(y^2_w)^2 - \frac{1}{g_{11}(Y)}(y^3_w)^2 \quad \text{on } I_{2r}(w_0). \tag{48}$$

Since $y^1_w = \frac{1}{2}(y^1_u - iy^1_v)$ and $y^2, y^3 \in C^{1,\beta}(\overline{S}_{2r}(w_0))$ for any $\beta \in (0,1)$ and $Y \in C^1(S_{2r}(w_0), \mathbb{R}^3)$, we infer that $f(u)$ with $u \in I' := \overline{I}_{2r}(w_0)$ satisfies the assumptions of Lemma 4 for all $\alpha \in (0, 1/2)$. Consequently y^1 is of class $C^{1,\alpha}(\overline{I}_{2r}(w_0))$ for all $\alpha \in (0, 1/2)$. Moreover, the Euler equation

$$\Delta y^1 = -\Gamma^1_{jk}(Y)(y^j_u y^k_u + y^j_v y^k_v)$$

can be written in the form

$$\Delta y^1 + a y^1_u + b y^1_v = p + q|\nabla y^1|^2$$

with functions $a, b, p, q \in C^{0,\beta}(\overline{S}_{2r}(w_0))$. Then an appropriate modification of potential-theoretic estimates (see Gilbarg and Trudinger [1], Widman [1,2]) yields $y^1 \in C^{1,\alpha}(\overline{S}_{2r}(w_0))$ for all $\alpha \in (0, 1/2)$.

Next we use the Euler equations

$$\begin{aligned} \Delta y^2 &= -\Gamma^2_{jk}(Y)(y^j_u y^k_u + y^j_v y^k_v) \\ \Delta y^3 &= -\Gamma^3_{jk}(Y)(y^j_u y^k_u + y^j_v y^k_v) \end{aligned} \quad \text{in } S_{2r}(w_0)$$

and the boundary conditions

$$y^2_v = 0, \quad y^3_v = 0 \quad \text{in } I_{2r}(w_0)$$

for any $r \in (0, \rho)$, as well as $Y \in C^{1,\alpha}(\overline{S}_{2r}(w_0), \mathbb{R}^3)$ to conclude that both y^2 and y^3 are of class $C^{2,\alpha}(\overline{S}_{2r}(w_0, \mathbb{R}^3))$ for any $r \in (0, \rho)$.

By virtue of (48), the function f^2 is Lipschitz continuous on $I' = \overline{I}_{2r}(w_0)$, whence Lemma 4 implies that y^1_w is of class $C^{0,1/2}(I')$. A repetition of the preceding argument with $\alpha = 1/2$ yields $y^1 \in C^{1,1/2}(\overline{S}_{2r}(w_0))$, and consequently $Y \in C^{1,1/2}(\overline{S}_{2r}(w_0), \mathbb{R}^3)$ for any $r \in (0, \rho)$. □

2.8 Higher Regularity in Case of Support Surfaces with Empty Boundaries. Analytic Continuation Across a Free Boundary

In this section we want to consider stationary points of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ whose support surface S has no boundary. We shall prove that any such surface X is of class $C^{m,\beta}(B \cup I, \mathbb{R}^3)$, provided that S is an admissible support surface of class $C^{m,\beta}$ with $m \geq 3$ and $\beta \in (0,1)$. Moreover, X will be seen to be real analytic on $B \cup I$ if S is real analytic, whence X can be continued analytically across its free boundary I.

Our key tool is the following

Proposition 1. *Let X be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ and suppose that S is of class $C^m, m \geq 2$. Then X is of class $C^{m-1,\alpha}(B \cup I, \mathbb{R}^3)$ for any $\alpha \in (0,1)$. Moreover, if S is of class $C^{m,\beta}$ for some $m \geq 2$ and some $\beta \in (0,1)$, then X is an element of $C^{m,\beta}(B \cup I, \mathbb{R}^3)$.*

Proof. Recall that, according to Definition 1 in Section 1.4, a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ is an element of $\mathcal{C}(\Gamma, S) \cap C^1(B \cup I, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which is harmonic in B, satisfies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0,$$

and intersects S perpendicularly along its free trace Σ given by the curve $X: I \to \mathbb{R}^3$.

Pick some $w_0 \in I$, and set $x_0 := X(w_0)$. Without loss of generality we can assume that $x_0 = 0$, and that for some cylinder

(1) $$C(R) := \{(x^1, x^2, x^3)\colon |x^1|^2 + |x^2|^2 \leq R^2, |x^3| \leq R\}$$

with $0 < R \ll 1$, the surface $S \cap C(R)$ is given by

(2) $$x^3 = f(x^1, x^2), \quad |x^1|^2 + |x^2|^2 \le R^2,$$

where f is a scalar function of class C^m or $C^{m,\beta}$ if S is of class C^m or $C^{m,\beta}$ respectively. Then S has the nonparametric representation

$$t(x^1, x^2) = (x^1, x^2, f(x^1, x^2)), \quad (x^1, x^2) \in \overline{B}_R(0),$$

with the surface normal

(3) $$n = \left(-\frac{f_1}{W}, -\frac{f_2}{W}, \frac{1}{W}\right) = (n^1, n^2, n^3),$$

where

(4) $$f_1 := f_{x^1}, \quad f_2 := f_{x^2}, \quad W := \sqrt{1 + f_1^2 + f_2^2}.$$

Now we choose some $r > 0$ such that $\overline{S}_r(w_0)$ is mapped by X into the cylinder $C(R)$. Since X_v is perpendicular to S, the vectors $X_v(w)$ and $n(X(w))$ are collinear for any $w \in I_r(w_0) := I \cap B_r(w_0)$. Consequently we have

$$X_v = \langle X_v, n(X)\rangle n(X) \quad \text{on } I_r(w_0),$$

that is,

$$x_v^j = x_v^k n^k(X) n^j(X) \quad \text{on } I_r(w_0) \text{ for } j = 1, 2, 3.$$

If we set

$$\xi^K := f_K / W^2, \quad K = 1, 2,$$

it follows that

(5) $$x_v^K = -\xi^K(x^1, x^2)\{x_v^3 - f_L(x^1, x^2) x_v^L\} \quad \text{on } I_r(w_0), K = 1, 2.$$

(Indices $K, L, M, \dots$ run from 1 to 2; repeated indices $K, L, M, \dots$ are to be summed from 1 to 2.) Let us introduce the function $y^3(w)$ by

(6) $$y^3(w) := x^3(w) - f(x^1(w), x^2(w)), \quad w \in \overline{S}_r(w_0).$$

Then we have the boundary condition "$X(w) \in S, w \in I$" transformed into

(7) $$y^3(w) = 0 \quad \text{for any } w \in I_r(w_0),$$

and (5) can be written as

(8) $$x_v^K = -\xi^K(x^1, x^2) y_v^3 \quad \text{on } I_r(w_0) \text{ for } K = 1, 2.$$

Moreover, from (6) and $\Delta X = 0$, we derive the equation

$$\Delta y^3 = -f_{KL}(x^1, x^2) D_\alpha x^K D_\alpha x^L \quad \text{in } S_r(w_0).$$

Thus we have the two boundary value problems

$$(*) \qquad \Delta y^3 = -f_{KL}(x^1, x^2) D_\alpha x^K D_\alpha x^L \quad \text{in } S_r(w_0), y^3 = 0 \quad \text{on } I_r(w_0)$$

with $f_{KL} := f_{x^K x^L}$, and

$$(**) \qquad \Delta x^K = 0 \quad \text{in } S_r(w_0), \quad x_v^K = -\xi^K(x^1, x^2) y_v^3 \quad \text{on } I_r(w_0), K = 1, 2.$$

Now we are going to bootstrap our regularity information by jumping back and forth from $(*)$ to $(**)$, assisted by the relation (6). To this end, we note that $f \in C^m$ or $C^{m,\beta}$; $f_K, \xi^K \in C^{m-1}$ or $C^{m-1,\beta}$; $f_{KL} \in C^{m-2}$ or $C^{m-2,\beta}$ if $S \in C^m$ of $C^{m,\beta}$, respectively.

We begin with the information $X \in C^1(\overline{S}_r(w_0), \mathbb{R}^3)$ assuming that $S \in C^2$. Then we infer from $(*)$ that

$$\Delta y^3 \in L_\infty(S_r(w_0)), \quad y^3 = 0 \quad \text{on } I_r(w_0).$$

whence $y^3 \in C^{1,\alpha}(\overline{S}_\rho(w_0))$ for any $\alpha \in (0,1)$ and $\rho \in (0, r)$. In the following, we shall always rename a number ρ with $0 < \rho < r$ in r; thus we actually obtain a sequence of decreasing numbers r.

Now we can infer from (8) that $x_v^K \in C^{0,\alpha}(I_r(w_0))$, and it follows from $(**)$ that $x^K \in C^{1,\alpha}(\overline{S}_r(w_0)), K = 1, 2$. By virtue of (6), we have

$$x^3 = y^3 + f(x^1, x^2) \tag{9}$$

whence $X \in C^{1,\alpha}(\overline{S}_r(w_0), \mathbb{R}^3)$ for any $\alpha \in (0, 1)$.

Suppose now that $S \in C^{2,\beta}$ holds for some $\beta \in (0, 1)$. Then we infer from $(*)$ that

$$\Delta y^3 \in C^{0,\beta}(\overline{S}_r(w_0)), \quad y^3 = 0 \quad \text{on } I_r(w_0),$$

whence $y^3 \in C^{2,\beta}(\overline{S}_r(w_0))$. Now it follows from $(**)$ that $x_v^K \in C^{1,\beta}(I_r(w_0))$ whence $x^K \in C^{2,\beta}(\overline{S}_r(w_0))$, $K = 1, 2$. Then we obtain from (9) that $X \in C^{2,\beta}(\overline{S}_r(w_0), \mathbb{R}^3)$.

Next we assume $S \in C^3$, whence $\Delta y^3 \in C^{0,\alpha}(\overline{S}_r(w_0))$, and $(*)$ yields $y^3 \in C^{2,\alpha}(\overline{S}_r(w_0))$ for all $\alpha \in (0, 1)$. Now $(**)$ implies $x_v^K \in C^{1,\alpha}(I_r(w_0))$, and therefore $x^K \in C^{2,\alpha}(\overline{S}_r(w_0))$ for any $\alpha \in (0, 1)$ whence $X \in C^{2,\alpha}(\overline{S}_r(w_0))$, taking (9) into account.

In this way we can proceed to prove the proposition. □

Recall that any stationary point X of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is a stationary minimal surface in $\mathcal{C}(\Gamma, S)$, provided that X is of class $C^1(B \cup I, \mathbb{R}^3)$ (cf. Section 1.4, Theorem 1). Hence from Proposition 1 we obtain the following result, by taking also Theorem 4 of Section 2.7 into account:

Theorem 1. *Let S be an admissible support surface of class C^m or $C^{m,\beta}$, $m \geq 3$, $\beta \in (0, 1)$. Then any stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is of class $C^{m-1,\alpha}(B \cup I, \mathbb{R}^3)$ for any $\alpha \in (0, 1)$ or of class $C^{m,\beta}(B \cup I, \mathbb{R}^3)$ respectively.*

Remark 1. The result of Theorem 4 in Section 2.7 is a by-product of the general discussion of that section, the main goal of which was to deal with surfaces S having a nonempty boundary. If ∂S is void, we can use a different method that avoids both the derivation of L_2-estimates and the use of the L_p-theory. This approach is more in the spirit of Section 2.3 and uses results which are closely related to those of Sections 2.1 and 2.2. To this end we choose Cartesian coordinates $x = (x^1, x^2, x^3)$ in the neighbourhood of $0 = X(w_0) \in S$ in such a way that S is given by a nonparametric representation

$$t(x^1, x^2) = (x^1, x^2, f(x^1, x^2)).$$

Moreover, we introduce the *signed distance function*

$$d(x) := \pm \text{dist}(x, S)$$

and the *foot* $a(x)$ of the *perpendicular* line from x onto S which has the *direction* $n(x), |n(x)| = 1$. Then, for all x in a sufficiently small neighbourhood of the origin 0, we have the representation

$$x = a(x) + d(x)n(x). \tag{10}$$

If $x \in S$, then clearly $x = a(x) = t(x^1, x^2)$. Note that $a(x), d(x), n(x)$ are of class C^{m-1} if $S \in C^m$, i.e., their degree of differentiability will in general drop by one. (In fact, it can be shown that $d \in C^m$.)

Let now X be the stationary point that we want to consider, and let $w_0 \in I$, $0 < r \ll 1$. Then we extend $X(w)$ from $\overline{S}_r(w_0)$ to $\overline{B}_r(w_0)$ by defining the extended surface $Z(w)$ as

$$Z(w) := \begin{cases} X(w) & \text{for } w \in \overline{S}_r(w_0), \\ a(X(\overline{w})) - d(X(\overline{w}))n(X(\overline{w})) & \text{for } \overline{w} \in \overline{S}_r(w_0). \end{cases} \tag{11}$$

It turns out that Z is a weak solution of an equation

$$\Delta Z = F(w)|\nabla Z|^2 \quad \text{in } B_r(w_0) \tag{12}$$

with some function $F \in L_\infty(B_r(w_0), \mathbb{R}^3)$, i.e. we have

$$\int_{B_r(w_0)} (\langle \nabla Z, \nabla \varphi \rangle + |\nabla Z|^2 \langle F, \varphi \rangle) \, du \, dv = 0 \tag{13}$$

for all $\varphi \in \overset{\circ}{H}{}_2^1 (B_r(w_0), \mathbb{R}^3) \cap L_\infty(B_r(w_0), \mathbb{R}^3)$. This is proved by first establishing (12) in $S_r(w_0)$ and in $S_r^*(w_0) := B_r(w_0) \setminus \overline{S}_r(w_0)$, and then multiplying (12) by φ. We integrate the resulting equation over $S_r(w_0) \cap \{\text{Im}\, w > \varepsilon\}$ and $S_r^*(w_0) \cap \{\text{Im}\, w < -\varepsilon\}, \varepsilon > 0$, and perform an integration by parts. The boundary terms on $\partial B_r(w_0)$ vanish because of $\varphi = 0$, and the remaining boundary terms cancel in the limit if we add the two equations and let $\varepsilon \to 0$; the resulting equation will be (13). The cancelling effect is derived from a *weak transversality relation* which expresses the fact that X is a stationary point of Dirichlet's integral. Concerning details of the computation, we refer the reader to Jäger [1], pp. 808–812.

Then, by a regularity theorem due to Heinz and Tomi [1] (see also the simplified version of Tomi [1]), it follows that $Z \in C^{1,\alpha}(B_{r'}(w_0), \mathbb{R}^3)$ for some $\alpha \in (0,1)$ and some $r' \in (0,r)$, whence $X \in C^{1,\alpha}(\overline{S}_r(w_0), \mathbb{R}^3)$, which was to be proved.

Remark 2. Another way to avoid L_p-estimates, $p > 2$, is the approach of Step 1 in Section 2.7. Assuming that S is of class C^4, we can estimate the L_2-norms of the third derivatives of a stationary point X up to the free boundary I, and this will imply $X \in C^1(B \cup I, \mathbb{R}^3)$. In fact, one can estimate $|D^s X|_{L_2}$ for any $s \geq 2$ thus obtaining $X \in C^{s-2,\alpha}(B \cup I, \mathbb{R}^3)$. Since one has to assume $S \in C^{s+1}$ to keep this method going, we essentially lose 2 derivatives passing from S to X. These derivatives can only be regained by potential-theoretic methods such as used in the beginning of this section. For details, we refer to Hildebrandt [3].

Analogously to Theorem 1, we obtain

Theorem 1′. *Let S be an admissible support surface of class C^m or $C^{m,\beta}$, $m \geq 3, \beta \in (0,1)$, and let B be the unit disk. Assume also that $X: B \to \mathbb{R}^3$ is a minimal surface of class $C^1(B \cup \gamma, \mathbb{R}^3)$ which maps some open subarc γ of ∂B into S, and which intersects S orthogonally along the trace curve $X: \gamma \to \mathbb{R}^3$. Then X is of class $C^{m-1,\alpha}(B \cup \gamma, \mathbb{R}^3)$ for any $\alpha \in (0,1)$, or of class $C^{m,\beta}(B \cup \gamma, \mathbb{R}^3)$ respectively.*

Now we come to the second main result of this section.

Theorem 2. *Let S be a real analytic support surface. Then any stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is real analytic in $B \cup I$ and can be extended across I as a minimal surface.*

Note that in Theorem 2 the parameter domain B is the semidisk $\{\operatorname{Im} w > 0, |w| < 1\}$ and I is the boundary interval $\{\operatorname{Im} w = 0, |w| < 1\}$.

Analogously we have

Theorem 2′. *Let S be a real analytic support surface in $\mathbb{R}^3$, and let B be the unit disk. Assume also that X is a minimal surface of class $C^1(B \cup \gamma, \mathbb{R}^3)$ for some open subarc γ of ∂B which is mapped by X into S, and suppose that X intersects S orthogonally along the trace curve $X: \gamma \to \mathbb{R}^3$. Then X is real analytic in $B \cup \gamma$ and can be extended across γ as a minimal surface.*

Since both results are proved in the same way, it is sufficient to give the

Proof of Theorem 2. By Proposition 1 we already know that X is of class $C^\infty(B \cup I, \mathbb{R}^3)$. Let $X^*(w)$ be the adjoint minimal surface to $X(w)$ in B, and let

$$f(w) := X(w) + iX^*(w) = (f^1(w), f^2(w), f^3(w))$$

be the holomorphic curve in $\mathbb{C}^3$ with $X = \operatorname{Re} f$ and $X^* = \operatorname{Im} f$, satisfying

$$\langle f'(w), f'(w) \rangle = 0 \quad \text{on } B.$$

We have to show that, for any $u_0 \in I$, there is some $\delta > 0$ such that $f(w)$ can be extended across $I_\delta(u_0) = I \cap B_\delta(u_0)$ as a holomorphic mapping from $B_\delta(u_0)$ into $\mathbb{C}^3$. Without loss of generality we can assume that $u_0 = 0$. Set $B_\delta := B_\delta(0)$, $I_\delta = I_\delta(0)$ and $S_\delta := B \cap B_\delta$. We can also achieve that $f(0) = 0$ holds true. Moreover, by a suitable choice of Cartesian coordinates in $\mathbb{R}^3$, we can accomplish that S in a suitable neighbourhood $\mathcal{U}$ of 0 is described by

$$S \cap \mathcal{U} = \{x = (x^1, x^2, x^3) \colon x^3 = \psi(x^1, x^2), |x^1|, |x^2| < R\}$$

for some $R > 0$, where

$$\psi(0,0) = 0, \quad \psi_{x^1}(0,0) = 0, \quad \psi_{x^2}(0,0) = 0.$$

Then there is some $\delta_0 > 0$ such that

$$|x^1(u)| < R, \quad |x^2(u)| < R \quad \text{for all } u \text{ with } |u| \le \delta_0.$$

The vector fields $T_K(x)$ defined by

$$T_1 := (1, 0, \psi_{x^1}), \quad T_2 := (0, 1, \psi_{x^2})$$

are tangent to S. Moreover X_v is orthogonal to X_u, and X_u is tangent to S along I. As X_v is orthogonal to S along I, we have

$$\langle T_K(X), X_v \rangle = 0 \quad \text{on } \overline{I}_{\delta_0} \text{ for } K = 1, 2,$$

whence

$$\langle T_K(X), X_u^* \rangle = 0 \quad \text{on } \overline{I}_{\delta_0} \text{ for } K = 1, 2,$$

and consequently

$$\langle T_K(X), X_u \rangle = \langle T_K(X), f' \rangle \quad \text{on } \overline{I}_{\delta_0},\ K = 1, 2.$$

This can be written as

$$x_u^K + \psi_{x^K}(x^1, x^2) x_u^3 = \frac{d}{dw} f^K + \psi_{x^K}(x^1, x^2) \frac{d}{dw} f^3$$

on $\overline{I}_{\delta_0}$ for $K = 1, 2$, and the identity

$$x^3(u) = \psi(x^1(u), x^2(u)) \quad \text{for all } u \in \overline{I}_{\delta_0}$$

yields

$$-\psi_{x^K}(x^1, x^2) x_u^K + x_u^3 = 0 \quad \text{on } \overline{I}_{\delta_0}$$

(summation with respect to K from 1 to 2 !).

Thus we obtain

(14)
$$\begin{pmatrix} 1 & 0 & \psi_{x^1}(x^1,x^2) \\ 0 & 1 & \psi_{x^2}(x^1,x^2) \\ -\psi_{x^1}(x^1,x^2) & -\psi_{x^2}(x^1,x^2) & 1 \end{pmatrix} \begin{pmatrix} x_u^1 \\ x_u^2 \\ x_u^3 \end{pmatrix} = \begin{pmatrix} f_w^1 + \psi_{x^1}(x^1,x^1) f_w^3 \\ f_w^2 + \psi_{x^2}(x^1,x^2) f_w^3 \\ 0 \end{pmatrix}$$

on $\overline{I}_{\delta_0}$. In matrix notation we may write

(15)
$$A(X)X_u = l(X, f') \quad \text{on } \overline{I}_{\delta_0}$$

with a 3×3-matrix $A(X)$, the determinant of which satisfies

$$\det A(X) = 1 + \psi_{x^1}^2(x^1,x^2) + \psi_{x^2}^2(x^1,x^2) \neq 0 \quad \text{on } \overline{I}_{\delta_0}$$

for $0 < \delta_0 \ll 1$. Thus we obtain

(16)
$$X_u = A^{-1}(X) l(X, f') \quad \text{on } \overline{I}_{\delta_0}.$$

Let us introduce the function $F(w,z)$ for $w \in \mathbb{C}$ and

$$z = (z^1, z^2, z^3) \in \mathbb{C}^3 \quad \text{with } |w| \le \rho_0,\ \operatorname{Im} w \ge 0,\ \text{and } |z| \le \rho_1$$

(i.e., $x \in \overline{B}^3_{\rho_1}$) by setting

(17)
$$F(w,z) := A^{-1}(z) l(z, f'(w)).$$

The mapping $F\colon \overline{S}_{\rho_0} \times \overline{B}^3_{\rho_1} \to \mathbb{C}^3$ is of class C^1 (differentiability meant in the "real sense" with respect to w) and holomorphic on $S_{\rho_0} \times B^3_{\rho_1}$.

Then we can write (16) in the form

(18)
$$\frac{d}{du} X(u) = F(u, X(u)) \quad \text{for all } u \in \overline{I}_{\delta_0}$$

if $\delta_0 \in (0, \rho_0]$ is sufficiently small. Since $X(0) = 0$, we obtain

(19)
$$X(u) = \int_0^u F(t, X(t))\, dt \quad \text{for all } u \in \overline{I}_{\delta_0}.$$

By a standard reasoning this integral equation has not more than one solution in $C^0(\overline{I}_{\delta_0}, \mathbb{R}^3)$ since $F(w,z)$ satisfies a Lipschitz condition with respect to $z \in \overline{B}^3_{\rho_1}$, uniformly for all $w \in \overline{S}_{\rho_0}$. By the same reasoning, the complex integral equation

(20)
$$Z(w) = \int_0^w F(\omega, Z(\omega))\, d\omega$$

has exactly one solution $Z(w)$, $w \in \overline{S}_\delta$, in the Banach space $\mathcal{A}(\overline{S}_\delta)$ of functions $Z\colon \overline{S}_\delta \to \mathbb{C}^3$ which are continuous on $\overline{S}_\delta$ and holomorphic in S_δ, provided that $\delta \in (0, \delta_0]$ is chosen sufficiently small (cf. the proof of Theorem 3 in Section 2.3, and in particular the footnote; one uses the standard Picard iteration). By the uniqueness principle, we have

$$Z(u) = X(u) \quad \text{for all } u \in I_\delta,$$

whence

$$\operatorname{Im} Z = 0 \quad \text{on } I_\delta.$$

Hence we can apply Schwarz's reflection principle, thus obtaining that

$$Z(w) := \overline{Z(\overline{w})} \quad \text{for } w \in B_\delta \text{ with } \operatorname{Im} w < 0$$

yields an analytic extension of Z across I_δ onto the disk B_δ centered at $u_0 = 0$.

Moreover, $f = X + iX^*$ is holomorphic in S_δ, continuous on $\overline{S}_\delta$, and

$$\operatorname{Re}(f - Z) = 0 \quad \text{on } I_\delta.$$

Thus we can extend $i(f - Z)$ analytically across I_δ by

$$i\{f(w) - Z(w)\} := -i\{\overline{f}(\overline{w}) - \overline{Z}(\overline{w})\} \quad \text{for } w \in B_\delta \text{ with } \operatorname{Im} w < 0.$$

Hence

$$f(w) := 2\overline{Z}(\overline{w}) - \overline{f}(\overline{w}) \quad \text{for } w \in B_\delta \text{ with } \operatorname{Im} w < 0$$

extends f analytically across I_δ. Now $f(w)$ is seen to be a holomorphic function on B_δ, and $X = \operatorname{Re} f$ defines the harmonic extension of X to B_δ which, by the principle of analytic continuation, has to be a minimal surface on B_δ. □

2.9 A Different Approach to Boundary Regularity

In this section we want to give a different proof of the Hölder continuity of ∇X where X is a stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$. This new proof merely requires that S is of class C^3. The first step is the same as in Section 2.7 and need not be repeated: one estimates the L_2-norms of $\nabla^2 X$ up to the free boundary. The other two steps are replaced by a new argument: We insert a suitable modification of the test function $\phi = \Delta_{-t}\{\eta^2 \Delta_t Y\}$ into the variational inequality

$$\delta E(Y, \phi) \geq 0.$$

This will lead us to a Morrey condition for $\nabla^2 X$ which, in turn, implies that ∇X is of class $C^{0,\alpha}$ on $B \cup I$ for some $\alpha \in (0, \frac{1}{2}]$. The essential new feature of this approach is that we shall explicitly use the first equation of Section 2.7, (35) which states that $y_v^1 = 0$ a.e. on $I_{2\rho}^+(w_0)$, and $y^1 = 0$ on $I_{2\rho}(w_0) \setminus I_{2\rho}^+(w_0)$.

Throughout this section we shall assume that S is an admissible support surface of class C^3 in the sense of Section 2.6, Definition 1.

As in Steps 2 and 3 of Section 2.1, we shall use a special boundary coordinate system satisfying (32) of Section 2.7. Note that, therefore, the defining diffeomorphisms g and h of the boundary coordinates are merely of class C^2.

We also assume that we have the same situation as in Section 2.6, that is:

$w_0 \in I, x_0 := X(w_0), \{\mathcal{U}, g\}$ is an admissible boundary coordinate system centered at $x_0, h = g^{-1}, Y := g(X), Y(w_0) = 0; \rho > 0$ is chosen in such a way that $|Y(w)| < R$ for all $w \in \overline{S}_{2\rho}(w_0)$; in addition, $\{\mathcal{U}, g\}$ is chosen in such a way that (32) of Section 2.7 holds true; we have

$$\begin{aligned} &y^1(w) \geq \sigma \quad \text{and} \quad y^3(w) = 0 \quad \text{for all } w \in I_{2\rho}(w_0), \\ &y^1_v(w) = 0 \quad \text{a.e. on } I^+_{2\rho}(w_0) := I_{2\rho}(w_0) \setminus \{w \colon y^1(w) = \sigma\}; \end{aligned}$$

finally, by Step 1 of Section 2.7, $Y \in H^2_2 \cap H^1_4(S_{2r}(w_0), \mathbb{R}^3)$ for any $r \in (0, \rho)$, as well as $Y \in C^{0,\alpha}(\overline{S}_{2r}(w_0), \mathbb{R}^3)$ for all $\alpha \in (0, 1)$.

Lemma 1. *Let $\phi = (\varphi^1, \varphi^2, \varphi^3) \in H^1_2 \cap L_\infty(S_{2\rho}(w_0), \mathbb{R}^3)$ be a test function with $\varphi^3 = 0$ on $I_{2\rho}(w_0)$,* supp $\phi \Subset S_{2\rho}(w_0) \cup I_{2\rho}(w_0)$*, and suppose that*

$$X_\varepsilon := h(Y + \varepsilon\phi), \quad 0 \leq \varepsilon < \varepsilon_0(\phi),$$

is an admissible type II*-variation of X in $\mathcal{C}(\Gamma, S)$. Then we have*

$$\int_B D_\alpha y^j D_\alpha \varphi^j \, du \, dv \geq \int_B \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k \varphi^l \, du \, dv. \tag{1}$$

For

$$\tilde{y}^j_1 := D_1 y^j - b^j, \quad \tilde{y}^j_2 := D_2 y^j - d^j$$

with $d^1 = d^2 = 0$ and arbitrary constants b^1, b^2, b^3, d^3, we also have

$$\int_B \tilde{y}^j_\alpha D_\alpha \varphi^j \, du \, dv \geq \int_B \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k \varphi^l \, du \, dv. \tag{2}$$

Proof. Because of (32) in Section 2.7, we infer that also $X^*_\varepsilon := h(Y + \varepsilon\Psi)$ with $\Psi = (\psi^1, \psi^2, \psi^3), \psi^j := g^{jk}(Y)\varphi^k, \phi = (\varphi^1, \varphi^2, \varphi^3)$ is an admissible type II-variation of X in $\mathcal{C}(\Gamma, S)$, with $\psi^3 = 0$ on $I_{2\rho}(w_0)$ and

$$\operatorname{supp} \psi \Subset S_{2\rho}(w_0) \cup I_{2\rho}(w_0).$$

Hence

$$\delta E(Y, \Psi) \geq 0,$$

and by computations similar to those in the beginning of Section 2.6, we obtain (1).

Secondly, an integration by parts yields the identities

$$\begin{aligned}\int_B \tilde{y}^j_\alpha D_\alpha \varphi^j \, du \, dv &= \int_B D_\alpha[\tilde{y}^j_\alpha \varphi^j] \, du \, dv - \int_B (\Delta y^j)\varphi^j \, du \, dv \\ &= -\int_I \tilde{y}^j_2 \varphi^j \, du - \int_B (\Delta y^j)\varphi^j \, du \, dv \\ &= -\int_I [(D_v y^1)\varphi^1 + (D_v y^2)\varphi^2] \, du - \int_B (\Delta y^j)\varphi^j \, du \, dv \\ &= -\int_I \langle D_v Y, \phi\rangle \, du - \int_B \langle \Delta Y, \phi\rangle \, du \, dv \\ &= \int_B \langle D_\alpha Y, D_\alpha \phi\rangle \, du \, dv = \int_B D_\alpha y^j D_\alpha \varphi^j \, du \, dv.\end{aligned}$$

Hence (2) is a consequence of (1). □

Next we shall prove a generalized version of Poincaré's inequality.

Lemma 2. *For any $\gamma > 0$, there is a constant $M > 0$ with the following property: If $w_0 \in \mathbb{R}, r > 0, T_{2r} := S_{2r}(w_0) \setminus S_r(w_0), \psi \in H^1_2(T_{2r})$ and*

$$\mathcal{H}^1\{w \in I_{2r}(w_0) \setminus I_r(w_0) \colon \psi(w) = 0\} \geq \gamma r,$$

then

$$\int_{T_{2r}} \psi^2 \, du \, dv \leq Mr^2 \int_{T_{2r}} |\nabla \psi|^2 \, du \, dv.$$

Proof. Suppose that $r = 1, \gamma > 0$, and let $\mathcal{C}_\gamma$ be the class of functions $\psi \in H^1_2(T)$, $T := T_2$, with $\mathcal{H}^1\{w \in I_2(w_0) \setminus I_1(w_0) \colon \psi(w) = 0\} \geq \gamma$. We claim that there is some number $M > 0$ such that

(3) $$\int_T \psi^2 \, du \, dv \leq M \int_T |\nabla \psi|^2 \, du \, dv$$

is satisfied for all $\psi \in \mathcal{C}_\gamma$. By a scaling argument we then obtain the assertion of the lemma.

Suppose now that there is no $M > 0$ with (3). Then there is a sequence of functions $\psi_k \in \mathcal{C}_\gamma, k \in \mathbb{N}$, such that

$$\int_T \psi_k^2 \, du \, dv > k \int_T |\nabla \psi_k|^2 \, du \, dv.$$

Without loss of generality we may assume that

(4) $$\int_T \psi_k^2 \, du \, dv = 1,$$

whence

(5) $$\int_T |\nabla \psi_k|^2 \, du \, dv < 1/k, \quad k \in \mathbb{N}.$$

Then, by a well-known compactness argument for bounded sequences in Hilbert spaces, there is a subsequence $\{\psi'_n\}$ of $\{\psi_k\}$ which converges weakly in $H^1_2(T)$ to some $\psi \in H^1_2(T)$. Then $\{\psi'_n\}$ converges strongly to ψ both in $L_2(T)$ and in $L_2(\partial T)$, on account of Rellich's theorem and a result by Morrey (cf. [8], pp. 75–77). As Dirichlet's integral is weakly lower semicontinuous, we infer from (5) that

$$\int_T |\nabla \psi|^2 \, du \, dv = 0.$$

Hence there is a constant c such that

$$\psi(w) = c \quad \text{a.e. on } T,$$

and, because of (4), we have $c \neq 0$.

On the other hand, we have

$$\begin{aligned} \int_{\partial T} |\psi'_n - c|^2 \, d\mathcal{H}^1 &\geq \int_{\partial T \cap \{\psi'_n = 0\}} |\psi'_n - c|^2 \, d\mathcal{H}^1 \\ &= c^2 \mathcal{H}^1(\partial T \cap \{\psi'_n = 0\}) \geq c^2 \gamma \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

As

$$\lim_{n \to \infty} \int_{\partial T} |\psi'_n - c|^2 \, d\mathcal{H}^1 = 0,$$

it follows that $c^2 \gamma = 0$, which is impossible since $c \neq 0$ and $\gamma > 0$. □

Now we want to use the generalized Poincaré inequality to establish the basic estimates of the stationary surface X in $\mathcal{C}(\Gamma, S)$, or rather of its transform $Y = g(X)$.

Lemma 3. *Set $T_{2r} := S_{2r}(w_0) \setminus S_r(w_0)$, and*

(6) $$\zeta(r) := \frac{1}{r^2} \min \left\{ \int_{T_{2r}} |D_u y^1|^2 \, du \, dv, \int_{T_{2r}} |D_v y^1|^2 \, du \, dv \right\}.$$

Then, for every $\delta \in (0,1)$, there is a constant $c = c(\delta) > 0$ such that the inequality

(7) $$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \leq c \left\{ \zeta(r) + r^{1+\delta} + \int_{T_{2r}} |\nabla^2 Y|^2 \, du \, dv \right\}$$

holds true for all $r \in (0, \rho)$.

Proof. Choose a cut-off function η as in Section 2.7, and replace the test function ϕ in Section 2.7, (1) by

(8) $$\phi := \Delta_{-t}\{\eta^2(\Delta_t Y - A)\}, \quad \phi = (\varphi^1, \varphi^2, \varphi^3),$$

where

$$A = (A^1, A^2, A^3) := (0, a, 0)$$

is a constant vector with an arbitrary constant $a \in \mathbb{R}$. We claim that ϕ is admissible for inequalities (1) and (2) in Lemma 1 provided that $|t| \ll 1$. In fact,

$$Y(w) + \varepsilon\phi(w) = \lambda_1 Y_t(w) + \lambda_2 Y_{-t}(w) + (1 - \lambda_1 - \lambda_2)Y(w) + \mu A,$$

$$\lambda_1 := \varepsilon t^{-2}\eta^2(w), \quad \lambda_2 := \varepsilon t^{-2}\eta^2_{-t}(w), \quad \mu := \varepsilon t^{-1}[\eta_t^2(w) - \eta^2(w)].$$

Thus $Y(w) + \varepsilon\phi(w)$ is a convex combination of the three points $Y(w), Y_t(w)$, $Y_{-t}(w)$ which is translated by μA, that is, in direction of the y^2-axis, provided that $0 \le \varepsilon \le \frac{1}{2}t^2$. Then, by a repetition of the reasoning used in the beginning of Section 2.7, we infer that $X_\varepsilon := h(Y + \varepsilon\phi), 0 \le \varepsilon \ll 1$, is an admissible variation of X in $\mathcal{C}(\Gamma, S)$ which is of type II. Thus we can insert ϕ in (1), whence

$$\int_B D_\alpha y^j D_\alpha \Delta_{-t}\{\eta^2(\Delta_t y^j - A^j)\}\, du\, dv$$
$$\ge \int_B \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k \Delta_{-t}\{\eta^2(\Delta_t y^l - A^l)\}\, du\, dv.$$

If we multiply this inequality by -1 and perform an integration by parts (cf. Section 2.7, Lemma 1), it follows that

$$\int_B \Delta_t D_\alpha y^j \{\eta^2(\Delta_t D_\alpha y^j) + 2\eta D_\alpha \eta(\Delta_t y^j - A^j)\}\, du\, dv$$
$$\le -\int_B \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k \{(\Delta_{-t}\eta^2)(\Delta_t y^l - A^l) + \eta_t^2 \Delta_{-t}\Delta_t y^l\}\, du\, dv.$$

As t tends to zero, we arrive at

$$\int_B \eta^2 |\nabla D_u Y|^2\, du\, dv$$
$$\le \int_B 2\eta|\nabla\eta||\nabla D_u Y||D_u Y - A|\, du\, dv$$
$$+ c\int_B |\nabla Y|^2 \eta|\nabla\eta||D_u Y - A|\, du\, dv + c\int_B |\nabla Y|^2\eta^2|D_u^2 Y|\, du\, dv.$$

Here and in the following, c will denote a canonical constant. Then, by means of the inequality

$$2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2,$$

we obtain that

$$\int_B \eta^2|\nabla D_u Y|^2\, du\, dv \le \varepsilon\int_B \eta^2|\nabla D_u Y|^2\, du\, dv + \frac{c}{\varepsilon}\int_B \eta^2|\nabla Y|^4\, du\, dv$$
$$+ \frac{c}{\varepsilon}\int_B |\nabla\eta|^2|D_u Y - A|^2\, du\, dv.$$

By choosing $\varepsilon = 1/2$, the first term on the right can be absorbed by the left-hand side, and it follows that

$$\int_{S_{2r}(w_0)} \eta^2 |\nabla D_u Y|^2 \, du \, dv \\ \leq cr^{-2} \int_{T_{2r}} |D_u Y - A|^2 \, du \, dv + c \int_{S_{2r}(w_0)} \eta^2 |\nabla Y|^4 \, du \, dv.$$

From the Euler equation

$$\Delta y^l + \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k = 0$$

we infer the inequality

$$|D_v^2 Y|^2 \leq |D_u^2 Y|^2 + c|\nabla Y|^4,$$

and consequently

$$\int_{S_{2r}(w_0)} \eta^2 |\nabla^2 Y|^2 \, du \, dv \\ \leq cr^{-2} \int_{T_{2r}} |D_u Y - A|^2 \, du \, dv + c \int_{S_{2r}(w_0)} \eta^2 |\nabla Y|^4 \, du \, dv. \tag{9}$$

Since we have already shown that

$$Y \in H_2^2 \cap H_p^1(\overline{Z}_d, \mathbb{R}^3), \quad 0 < d < 1,$$

for any $p \in (1, \infty)$, it follows that, for any $\delta \in (0, 1)$, there is a constant $c(\delta) > 0$ such that

$$\int_{S_{2r}(w_0)} |\nabla Y|^4 \, du \, dv \leq c(\delta) r^{1+\delta} \tag{10}$$

holds for all $r \in (0, \rho)$. Moreover, we have

$$\int_{T_{2r}} |D_u Y - A|^2 \, du \, dv = \int_{T_{2r}} (|D_u y^1|^2 + |D_u y^2 - a|^2 + |D_u y^3|^2) \, du \, dv.$$

Poincaré's inequality yields

$$\int_{T_{2r}} |D_u y^3|^2 \, du \, dv \leq cr^2 \int_{T_{2r}} |\nabla D_u y^3|^2 \, du \, dv \tag{11}$$

for all $r \in (0, \rho)$ since y^3 vanishes on $I_{2\rho}(w_0)$. Moreover, for

$$a := \fint_{T_{2r}} D_u y^2 \, du \, dv,$$

we infer from Poincaré's inequality that

$$\int_{T_{2r}} |D_u y^2 - a|^2 \, du \, dv \le c r^2 \int_{T_{2r}} |\nabla D_u y^2|^2 \, du \, dv \tag{12}$$

is satisfied for all $r \in (0, \rho)$.

Combining inequalities (9)–(12), we find that

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \le c(\delta) \left\{ r^{-2} \int_{T_{2r}} |D_u y^1|^2 \, du \, dv + \int_{T_{2r}} |\nabla^2 Y|^2 \, du \, dv + r^{1+\delta} \right\}. \tag{13}$$

By a similar reasoning, it follows that

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \le c(\delta) \left\{ r^{-2} \int_{T_{2r}} |D_v y^1|^2 \, du \, dv + \int_{T_{2r}} |\nabla^2 Y|^2 \, du \, dv + r^{1+\delta} \right\} \tag{14}$$

holds true if we insert the test function

$$\phi := \eta^2 \Delta_{-t} \Delta_t Y \tag{15}$$

in inequality (2) of Lemma 1. We leave it to the reader to check that (15) is an admissible test function for (2), and to carry out the derivation of (14) in detail.

Then the desired inequality (7) is a consequence of (13) and (14). □

Now we are going to prove our main result.

Theorem 1. *Let S be an admissible support surface of class C^3, and suppose that X is a stationary point of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$. Then there exists some $\alpha \in (0, 1/2)$ such that $X \in C^{1,\alpha}(B \cup I, \mathbb{R}^3)$.*

Proof. We have

$$y_v^1 = 0 \quad \text{a.e. on } I_{2r}^+(w_0) := \{w \in I_{2r}(w_0) \colon y^1(w) > \sigma\},$$
$$y_u^1 = 0 \quad \text{a.e. on } I_{2r}^0(w_0) := \{w \in I_{2r}(w_0) \colon y^1(w) = \sigma\}.$$

Hence, either

$$\mathcal{H}^1(I_{2r}^+(w_0) \setminus I_r(w_0)) \ge r$$

or

$$\mathcal{H}^1(I_{2r}^0(w_0) \setminus I_r(w_0)) \ge r$$

holds true, and we can apply Lemma 2 to $\psi = y_v^1$ or $\psi = y_u^1$ respectively, obtaining

$$\int_{T_{2r}} \psi^2 \, du \, dv \le Mr^2 \int_{T_{2r}} |\nabla \psi|^2 \, du \, dv.$$

Thus the function $\zeta(r)$, defined by (6), will satisfy

$$\zeta(r) \le M \int_{T_{2r}} |\nabla^2 y^1|^2 \, du \, dv \quad \text{for all } r \in (0, \rho),$$

and we infer from formula (7) of Lemma 3 that

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \le c \left\{ r^{1+\delta} + \int_{S_{2r}(w_0) \setminus S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \right\}$$

holds true for some $\delta \in (0, 1)$ and for all $r \in (0, \rho)$. Adding the term

$$c \int_{S_r(w_0)} |\nabla Y|^2 \, du \, dv$$

to both sides of the last inequality and dividing the result by $1 + c$, it follows that

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \le \theta \left\{ \int_{S_{2r}(w_0)} |\nabla^2 Y|^2 \, du \, dv + r^{1+\delta} \right\}$$

holds true for some $\delta \in (0, 1)$ and for all $r \in (0, \rho)$, where

$$\theta := \frac{c}{1 + c};$$

that is, $0 < \theta < 1$. Hence, by Lemma 6 of Section 2.6, we infer the existence of positive numbers k and $\alpha \le 1$ such that

$$\int_{S_r(w_0)} |\nabla^2 Y|^2 \, du \, dv \le kr^{2\alpha} \quad \text{for } 0 < r < \rho, \tag{16}$$

whence by

$$|\nabla^2 X|^2 \le c\{|\nabla^2 Y|^2 + |\nabla Y|^4\}$$

and (10) we obtain

$$\int_{S_r(w_0)} |\nabla^2 X|^2 \, du \, dv \le k^* r^{2\alpha} \quad \text{for } 0 < r < \rho$$

and some constant k^* depending on ρ but not on r. By virtue of Morrey's Dirichlet growth theorem we infer that $X \in C^{1,\alpha}(\overline{Z}_d, \mathbb{R}^3)$, for any $d \in (0, 1)$. □

2.10 Asymptotic Expansion of Minimal Surfaces at Boundary Branch Points and Geometric Consequences

We have seen that a minimal surface $X\colon B \to \mathbb{R}^3$ can be extended analytically and as a minimal surface across those parts of ∂B which are mapped by X into an analytic arc or which correspond to a free trace on an analytic support surface. Therefore, at a branch point of such a part of ∂B, the minimal surface X possesses an asymptotic expansion as described in Section 3.2 of Vol. 1. In this section we want to derive an analogous expansion of X at boundary branch points, assuming merely that Γ or S are of some appropriate class C^m. Our main tool will be a technique developed by Hartman and Wintner that is described in Chapter 3 in some detail. Presently we shall only sketch how the Hartman–Wintner technique can be used to obtain the desired expansions at boundary branch points.

Since in the preceding sections we have discussed stationary points of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$, that is, stationary minimal surfaces with a partially free boundary on I, we shall begin by considering such a minimal surface X. Thus we can assume that we have the same situation as in Section 2.6:

S is assumed to be an admissible support surface of class C^3; $w_0 \in I, x_0 := X(w_0)$; $\{\mathcal{U}, g\}$ is an admissible boundary coordinate system centered at $x_0, h = g^{-1}, Y = (y^1, y^2, y^3) := g(X), Y(w_0) = 0; \rho > 0$ is chosen in such a way that $|Y(w)| < R$ for all $w \in \overline{S}_{2\rho}(w_0)$; in addition, $\{\mathcal{U}, g\}$ is chosen in such a way that (32) of Section 2.7 holds true. We have

$$y^2_v = 0 \quad \text{and} \quad y^3 = 0 \quad \text{in } I_{2\rho}(w_0)$$

and

$$\Delta y^l + \Gamma^l_{jk}(Y) D_\alpha y^j D_\alpha y^k = 0 \quad \text{in } B.$$

Moreover, on account of

$$g_{jk}(Y) y^j_w y^k_w = 0 \quad \text{in } B,$$

it follows that

$$(1) \qquad |\nabla y^1|^2 \le c\{|\nabla y^2|^2 + |\nabla y^3|^2\}$$

and

$$(2) \qquad |\Delta y^2| + |\Delta y^3| \le c\{|\nabla y^2|^2 + |\nabla y^3|^2\}$$

holds is $S_{2\rho}(w_0)$ for some constant $c > 0$.

In Section 2.7 we have also proved that y^2 and y^3 are both of class $C^{1,\alpha}(\overline{S}_{2r}(w_0))$ and of class $H^2_p(S_{2r}(w_0))$ for any $\alpha \in (0,1), p \in (1, \infty)$, and

$r \in (0, \rho)$. Then the mapping $Z(w) = (z^1(w), z^2(w), z^3(w))$ defined by $z^1(w) := 0$ and by

$$z^2(w) := y^2(w), \quad z^3(w) := y^3(w) \quad \text{if } \operatorname{Im} w \geq 0,$$
$$z^2(w) := y^2(\overline{w}), \quad z^3(w) := -y^3(\overline{w}) \quad \text{if } \operatorname{Im} w < 0,$$

$w = u + iv \in B_\rho(w_0), \overline{w} = u - iv$, is of class $C^{1,\alpha}(B_\rho(w_0), \mathbb{R}^3)$ and of class C^2 in $B_\rho(w_0) \setminus I_\rho(w_0)$. Furthermore, for some constant $c > 0$, we have

$$(3) \qquad |Z_{w\overline{w}}| \leq c|Z_w| \quad \text{in } B_\rho(w_0) \setminus I_\rho(w_0).$$

Let Ω be an arbitrary subdomain of $B_r(w_0)$ for some $r \in (0, \rho)$ which has a piecewise smooth boundary $\partial\Omega$, and let $\phi = (\varphi^1, \varphi^2, \varphi^3)$ be an arbitrary function of class $C^1(\overline{\Omega}, \mathbb{C}^3)$. Then, by an integration by parts, we obtain that

$$(4) \qquad \frac{1}{2i} \int_{\partial\Omega} \langle Z_w, \phi \rangle \, dw = \int_\Omega (\langle Z_w, \phi_{\overline{w}} \rangle + \langle Z_{w\overline{w}}, \phi \rangle) \, d^2w,$$

where $dw = du + i\,dv, d^2w = du\,dv$.

Combining (3) and (4), we arrive at the inequality

$$(5) \qquad \left| \int_{\partial\Omega} \langle Z_w, \phi \rangle \, dw \right| \leq 2 \int_\Omega |Z_w|(|\phi_{\overline{w}}| + c|\phi|) \, d^2w$$

which holds for all $\phi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ and for all $\Omega \subset B_r(w_0)$ with a piecewise smooth boundary $\partial\Omega$. But this relation is the starting point for the Hartman–Wintner technique; cf. Chapter 3, Section 3.1.

Suppose now that $w_0 \in I$ is a branch point of X, that is,

$$|X_u(w_0)| = 0 \quad \text{and} \quad |X_v(w_0)| = 0.$$

Then we have

$$|Y_u(w_0)| = 0 \quad \text{and} \quad |Y_v(w_0)| = 0$$

and consequently

$$Y_w(w_0) = 0.$$

We claim that there is no $r \in (0, \rho)$ such that $Y_w(w) = 0$ for all $w \in S_r(w_0)$. In fact, suppose that $Y_w(w) \equiv 0$ on $S_r(w_0)$. Then we obtain $X_w(w) \equiv 0$ on $S_r(w_0)$, whence $X_w(w) \equiv 0$ on B; but this is impossible for any stationary point of the Dirichlet integral in $\mathcal{C}(\Gamma, S)$.

From $Y_w \not\equiv 0$ on $S_r(w_0)$ for any $r \in (0, \rho)$ it follows that $Z_w(w) \not\equiv 0$ on $S_r(w_0)$, on account of (1). Then by virtue of Theorem 1 of Section 3.1, there is some vector $P = (p^1, p^2, p^3) \neq 0$ in $\mathbb{C}^3$ and some number $\nu \in \mathbb{N}$ such that

$$Z_w(w) = P(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0.$$

Because of $z^1_w(w) \equiv 0$, it follows that $p^1 = 0$:

$$P = (0, p^2, p^3).$$

Now we consider the function $e(w), w \in S_\rho(w_0) \setminus \{w_0\}$, defined by

$$e(w) := (w - w_0)^{-\nu} f(w),$$

where $f(w)$ is defined by (42) in Section 2.7. Since $Y(w_0) = 0$ and

$$(g_{jk}(0)) = \begin{bmatrix} \mathcal{E}_0 & 0 & 0 \\ 0 & \mathcal{G}_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{E}_0 \neq 0, \quad \mathcal{G}_0 \neq 0,$$

we infer from formula (41) in Section 2.7 that $\lim_{w \to w_0} e^2(w)$ exists, and that

$$\begin{aligned} \lim_{w \to w_0} e^2(w) \ = \ & -\frac{\mathcal{G}_0}{\mathcal{E}_0} \lim_{w \to w_0} (w - w_0)^{-2\nu} (y_w^2(w))^2 \\ & -\frac{1}{\mathcal{E}_0} \lim_{w \to w_0} (w - w_0)^{-2\nu} (y_w^3(w))^2. \end{aligned}$$

Then, by Lemma 3 of Section 2.7, we see that $\lim_{w \to w_0} e(w)$ does exist. Set $F := (f^1, f^2, f^3)$, where

$$f^1 := \lim_{w \to w_0} e(w) = \lim_{w \to w_0} (w - w_0)^{-\nu} y_w^1(w), \quad f^2 := p^2, \quad f^3 := p^3.$$

It follows that

$$Y_w(w) = F(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0,$$

where $F \in \mathbb{C}^3$ satisfies $F \neq 0$ and $\langle\!\langle F, F \rangle\!\rangle = 0$, i.e.

$$g_{kl}(0) f^k f^l = 0.$$

Because of $X_w = h_y(Y) Y_w$, we obtain the following result:

Theorem 1. *Let S be an admissible support surface of class C^3 and X be a stationary point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$. Assume also that $w_0 \in I$ is a boundary branch point of X. Then there exist an integer $\nu \geq 1$ and a vector $A \in \mathbb{C}^3$ with $A \neq 0$ and*

$$\langle A, A \rangle = 0 \tag{6}$$

such that

$$X_w(w) = A(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0. \tag{7}$$

We call ν the order of the branch point w_0.

From this expansion we can draw the same geometric conclusions as in Section 3.2 of Vol. 1. To this end we write

$$A = \tfrac{1}{2}(\alpha - i\beta) \quad \text{with } \alpha, \beta \in \mathbb{R}^3.$$

Then it follows that

$$|\alpha| = |\beta| \neq 0, \quad \langle \alpha, \beta \rangle = 0$$

and

$$\begin{aligned} X_u(w) &= \alpha \operatorname{Re}(w - w_0)^\nu + \beta \operatorname{Im}(w - w_0)^\nu + o(|w - w_0|^\nu), \\ X_v(w) &= -\alpha \operatorname{Im}(w - w_0)^\nu + \beta \operatorname{Re}(w - w_0)^\nu + o(|w - w_0|^\nu) \end{aligned} \tag{8}$$

as $w \to w_0$, whence

$$X_u(w) \wedge X_v(w) = (\alpha \wedge \beta)|w - w_0|^{2\nu} + o(|w - w_0|^{2\nu}) \quad \text{as } w \to w_0.$$

This implies that the surface normal $N(w)$, given by

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v),$$

tends to a limit vector N_0 as $w \to w_0$:

$$\lim_{w \to w_0} N(w) = N_0 = |\alpha \wedge \beta|^{-1}(\alpha \wedge \beta). \tag{9}$$

Consequently, the Gauss map $N(w)$ of a stationary minimal surface $X(w)$ is well-defined on all of $B \cup I$ as a continuous mapping into S^2. Therefore *the surface $X(w)$ has a well-defined tangent plane at every boundary branch point on I*, and thus at every point $w_0 \in B \cup I$.

Consider now the trace curve $X\colon I \to \mathbb{R}^3$ of the minimal surface X on the supporting surface S. We infer from (8) that

$$X_u(w) = \alpha(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0, w \in I$$

and, writing $w = u, w_0 = u_0$ for $w, w_0 \in I$, we obtain for the unit tangent vector

$$t(u) := |X_u(u)|^{-1} X_u(u)$$

the expansion

$$t(u) = \frac{\alpha}{|\alpha|} \frac{(u - u_0)^\nu}{|u - u_0|^\nu} + o(1) \quad \text{as } u \to u_0. \tag{10}$$

Therefore *the nonoriented tangent moves continuously through any boundary branch point $u_0 \in I$. The oriented tangent $t(u)$ is continuous if the order ν of u_0 is even, but, for branch points of odd order, the direction of $t(u)$ jumps by* 180 *degrees when u passes through u_0.*

Finally, by choosing a suitable Cartesian coordinate system in $\mathbb{R}^3$, we obtain the expansion

$$(11)\quad \begin{aligned} x(w) + iy(w) &= (x_0 + iy_0) + a(w - w_0)^{\nu+1} + o(|w - w_0|^{\nu+1}), \\ z(w) &= z_0 + o(|w - w_0|^{\nu+1}) \end{aligned}$$

as $w \to w_0$, where $X(w_0) = (x_0, y_0, z_0)$ and $a > 0$; see Section 3.2, (6), of Vol. 1.

The same reasoning can be used for the investigation of X at a boundary branch point $w_0 \in \text{int } C$. We obtain again an expansion of the kind (7) with some $\nu \geq 1$ and some $A \in \mathbb{C}^3, A \neq 0, \langle A, A \rangle = 0$. As $X : C \to \Gamma$ is a monotonic mapping, *the tangent vector*

$$t(\varphi) := |X_\varphi(e^{i\varphi})|^{-1} X_\varphi(e^{i\varphi})$$

of this mapping has to be continuous, and we infer from (7) *that* ν *is even*, provided that Γ is of class C^2.

The same result can be proved for minimal surfaces $X \in \mathcal{C}(\Gamma)$ which solve Plateau's problem for a closed Jordan curve Γ of class C^2; cf. Chapter 4 of Vol. 1 for the definition of $\mathcal{C}(\Gamma)$. Thus we obtain

Theorem 2. *Let* Γ *be a closed Jordan curve of class* C^2 *in* $\mathbb{R}^3$*, and suppose that* $X \in \mathcal{C}(\Gamma)$ *is a minimal surface spanning* Γ*. Then every boundary branch point* $w_0 \in \partial B$ *is of even order* $\nu = 2p, p \geq 1$*, and we have the asymptotic expansion*

$$(12)\quad X_w(w) = A(w - w_0)^{2p} + o(|w - w_0|^{2p}) \quad \text{as } w \to w_0,$$

where $A \in \mathbb{C}^3, A \neq 0$*, and* $\langle A, A \rangle = 0$.

W. Jäger [3] has pointed out that $\Gamma \in C^{1,\mu}$ suffices to prove (12).

2.11 The Gauss–Bonnet Formula for Branched Minimal Surfaces

In Section 1.4 of Vol. 1 we have derived the Gauss–Bonnet formula

$$(1)\quad \int_X K dA + \int_\Gamma \kappa_g \, ds = 2\pi$$

for regular surfaces $X \in C^2(\overline{\Omega}, \mathbb{R}^3)$ defined on a simply connected bounded domain $\Omega \subset \mathbb{C}$ which map $\partial\Omega$ onto a Jordan curve Γ. The result as well as the proof given in Section 1.4 of Vol. 1 remain correct if X does not map $\partial\Omega$ bijectively onto a Jordan curve in $\mathbb{R}^3$ provided that we replace formula (1) by

$$(2)\quad \int_X K \, dA + \int_{\partial X} \kappa_g \, ds = 2\pi$$

or, precisely speaking, by

$$(3) \qquad \int_\Omega K|X_u \wedge X_v|\,du\,dv + \int_{\partial\Omega} \kappa_g |dX| = 2\pi.$$

Now we shall drop the assumption of regularity and, instead, admit finitely many branch points in the interior and on the boundary of the parameter domain Ω. To make our assumptions precise, we introduce the class $\mathcal{PR}(\overline{\Omega})$ of *pseudoregular surfaces* $X\colon \overline{\Omega} \to \mathbb{R}$ as follows:

A surface X is said to be of class $\mathcal{PR}(\overline{\Omega})$ if it satisfies the conditions
(i) $X \in C^2(\overline{\Omega}, \mathbb{R}^3)$ *and*

$$(4) \qquad |X_u| = |X_v|, \quad \langle X_u, X_v\rangle = 0.$$

(ii) *There is a continuous function $\mathcal{H}(w)$ on $\overline{\Omega}$ such that*

$$(5) \qquad \Delta X = 2\mathcal{H} X_u \wedge X_v.$$

(iii) *There is a finite set Σ_0 of points in $\overline{\Omega}$ such that $X_w(w) \neq 0$ for all $w \in \overline{\Omega} \setminus \Sigma_0$. For any point $w_0 \in \Sigma_0$ there is an integer $\nu \geq 1$ and a vector $A \in \mathbb{C}^3$ satisfying $A \neq 0$ and $\langle A, A\rangle = 0$ such that*

$$(6) \qquad X_w(w) = A(w - w_0)^\nu + o(|w - w_0|^\nu) \quad \text{as } w \to w_0.$$

We call Σ_0 the singular set *of $X \in \mathcal{PR}(\overline{\Omega})$.*

Remark 1. The set $\Omega_0 := \{w \in \Omega\colon X_w(w) \neq 0\}$ of regular points of X in Ω is open and, by Section 2.6 of Vol. 1, equations (4) yield the existence of a function $\mathcal{H} \in C^0(\Omega_0)$ such that (5) holds true on Ω_0. Moreover, the function $\mathcal{H}$ is the mean curvature of $X|_{\Omega_0}$. Thus condition (ii) is a consequence of (i) if we assume that $\mathcal{H}(w)$ can be extended from Ω_0 to $\overline{\Omega}$ as a continuous function. This extension is possible if, for some reason, we know that X is a solution of

$$(7) \qquad \Delta X = 2H(X) X_u \wedge X_v$$

in Ω, where $H \in C^0(\mathbb{R}^3)$.

If, on the other hand, $X \in C^2(\Omega, \mathbb{R}^3)$ is a solution of (4) and (7) for some $H \in C^1(\mathbb{R}^3)$, it is sometimes possible to extend X to a function of class $C^2(\overline{\Omega}, \mathbb{R}^3)$. For instance, the extendability can follow from suitable boundary conditions (e.g. from Plateau-type conditions or from free boundary conditions) as we have seen in the previous sections.

Finally if $X(w)$ is a nonconstant surface such that (4) and (5) hold for some $\mathcal{H} \in C^{0,\alpha}(\overline{\Omega}), 0 < \alpha < 1$, then the set of branch points of X defined by $\Sigma_0 := \{w \in \overline{\Omega} : X_w(w) = 0\}$ is finite (and possibly empty), and, for any $w_0 \in \Sigma_0$, the mapping X has an asymptotic expansion (6) as described in (iii). For minimal surfaces we have stated this result in Section 2.10. The general theory will be developed in Chapter 3, using Hartman–Wintner's technique.

Now we can formulate the Gauss–Bonnet theorem for pseudoregular surfaces; we shall immediately state it for multiply connected domains.

Theorem 1. *Let Ω be an m-fold connected domain in $\mathbb{C}$ bounded by m closed regular curves $\gamma_1, \dots, \gamma_m$ of class C^∞, and let $X : \overline{\Omega} \to \mathbb{R}^3$ be a pseudoregular surface with the area element $dA = |X_u \wedge X_v|\, du\, dv$, the singular set Σ_0, the Gauss curvature K in $\Omega \setminus \Sigma_0$, and the geodesic curvature κ_g of $X|_{\partial\Omega\setminus\Sigma_0}$. Suppose also that the total curvature integral $\int_X |K|\, dA$ of X exists as a Cauchy principle value. Then we obtain the generalized Gauss–Bonnet formula*

$$(8) \qquad \int_X K\, dA = 2\pi(2-m) + 2\pi \sum_{w\in\sigma'} \nu(w) + \pi \sum_{w\in\sigma''} \nu(w) - \int_{\partial\Omega} \kappa_g |dX|,$$

where $\sigma' := \Sigma_0 \cap \Omega$ is the set of interior branch points, $\sigma'' := \Sigma_0 \cap \partial\Omega$ the set of boundary branch points, and ν the order of a branch point $w \in \Sigma_0$.

For the proof of (8) we shall employ the reasoning of Section 1.4 of Vol. 1. To carry out these arguments in our present context, we need two auxiliary results.

Lemma 1. *Let $a > 0, I = (0, a]$, and be f a function of class $C^1(I)$ such that $|f(r)| \le m$ holds for all $r \in I$ and some constant $m \ge 0$. Then there is a sequence of numbers $r_k \in I$ satisfying $r_k \to 0$ and $r_k f'(r_k) \to 0$ as $k \to \infty$.*

Proof. Otherwise we could find two numbers $c > 0$ and $\varepsilon \in (0, a]$ such that

$$r|f'(r)| \ge c \quad \text{for all } r \in (0, \varepsilon].$$

Then we would either have

$$(i) \qquad f'(r) \ge c/r \quad \text{for all } r \in (0, \varepsilon]$$

or

$$(ii) \qquad f'(r) \ge -c/r \quad \text{for all } r \in (0, \varepsilon].$$

In case (i) we obtain

$$c \log\frac{\varepsilon}{r} = c \int_r^\varepsilon \frac{dr}{r} \le \int_r^\varepsilon f'(r)\, dr = f(\varepsilon) - f(r)$$

whence

$$\log\frac{1}{r} \le \frac{2m}{c} - \log \varepsilon \quad \text{for all } r \in (0, \varepsilon]$$

which yields a contradiction since $\log\frac{1}{r} \to \infty$ as $r \to +0$. Similarly case (ii) leads to a contradiction. □

Lemma 2. *Let Σ_0 be the singular set of a map $X \in \mathcal{PR}(\Omega)$. Then, for any $w_0 \in \Sigma_0$, there is a sequence of positive radii $r_k, k \in \mathbb{N}$, tending to zero such that*

$$(9) \qquad \lim_{k\to\infty} \int_{C(w_0, r_\kappa)} \frac{\partial}{\partial r} \log \sqrt{\Lambda}\, d\sigma = \begin{cases} 2\pi\nu & \text{if } w_0 \in \Omega, \\ \pi\nu & \text{if } w_0 \in \partial\Omega, \end{cases}$$

where $r = |w - w_0|, w = w_0 + re^{i\varphi}, \nu$ is the order of the branch point w_0 defined by the expansion (6), and $\Lambda = |X_u|^2$.

Proof. Let us write $C(w_0, r) = \{w \in \overline{\Omega}\colon |w - w_0| = r\}$ as

$$C(w_0, r) = \{w = w_0 + re^{i\varphi}\colon \varphi_1(r) \le \varphi \le \varphi_2(r)\}$$

for $0 < r \le \varepsilon \ll 1$, and set

$$f(r) := \int_{\varphi_1(r)}^{\varphi_2(r)} \log|X_w(w)||w - w_0|^{-\nu}\, d\varphi.$$

By Lemma 1, there is a sequence $r_k \to +0$ such that $r_k f'(r_k) \to 0$. Since $|X_w| = \sqrt{\Lambda}/\sqrt{2}$, we obtain

$$\log|X_w(w)||w - w_0|^{-\nu} = \log\sqrt{\Lambda(w)} - \log\sqrt{2} - \nu\log r$$

for $w \in C(w_0, r)$, whence

$$\frac{\partial}{\partial r}\log|X_w(w)||w - w_0|^{-\nu} = \frac{\partial}{\partial r}\log\sqrt{\Lambda(w)} - \frac{\nu}{r}.$$

Thus it follows that

$$\begin{aligned} r_k f'(r_k) \;=\; & \int_{\varphi_1(r_k)}^{\varphi_2(r_k)} \left(\frac{\partial}{\partial r}\log\sqrt{\Lambda}\right) r_k\, d\varphi - \nu \int_{\varphi_1(r_k)}^{\varphi_2(r_k)} d\varphi \\ & + r_k\varphi_2'(r_k)\log(|A| + \delta_k) - r_k\varphi_1'(r_k)\log(|A| + \delta_k^*), \end{aligned}$$

where $\{\delta_k\}$ and $\{\delta_k^*\}$ are two sequences tending to zero.

If $w_0 \in \Omega$, we can assume that $\varphi_1(r) = 0$ and $\varphi_2(r) = 2\pi$, whence

$$r_k f'(r_k) = \int_{C(w_0, r_k)} \left(\frac{\partial}{\partial r}\log\sqrt{\Lambda}\right) d\sigma - 2\pi\nu.$$

Because of $r_k f'(r_k) \to 0$, we then obtain

$$\lim_{k\to\infty}\int_{C(w_0, r_k)} \frac{\partial}{\partial r}\log\sqrt{\Lambda}\, d\sigma = 2\pi\nu.$$

If $w_0 \in \partial\Omega$, then the smoothness of $\partial\Omega$ implies

$$\varphi_1(r_k) - \varphi_1(r_k) \to \pi \quad \text{and} \quad r_k\{|\varphi_1'(r_k)| + |\varphi_2'(r_k)|\} \to 0 \quad \text{as } k \to \infty,$$

whence

$$\lim_{k\to\infty}\int_{C(w_0, r_k)} \frac{\partial}{\partial r}\log\sqrt{\Lambda}\, d\sigma = \pi\nu. \qquad \square$$

Now we turn to the

Proof of Theorem 1. Let $\sigma' = \{w_1, \ldots, w_N\}$ and $\sigma'' = \{\tilde{w}_1, \ldots, \tilde{w}_M\}$ be the sets of interior branch points and of boundary branch points respectively. We consider $N + M$ sequences $\{r_\alpha^{(j)}\}$ and $\{\tilde{r}_\beta^{(j)}\}, 1 \le \alpha \le N, 1 \le \beta \le M$, of positive numbers tending to zero as $j \to \infty$. Set

$$\Omega_j\colon = \{w \in \Omega\colon |w - w_\alpha| > r_\alpha^{(j)}, |w - \tilde{w}_\beta| > \tilde{r}_\beta^{(j)}, 1 \le \alpha \le N, 1 \le \beta \le M\}.$$

By formula (32) of Section 1.3 of Vol. 1 we have

$$-\int_{\Omega_j} K\, dA = \int_{\Omega_j} \Delta \log \sqrt{\Lambda}\, du\, dv$$

taking $|X_u \wedge X_v| = \Lambda$ into account, and an integration by parts yields

$$-\int_{\Omega_j} K\, dA = \int_{\partial\Omega_j} \left(\frac{\partial}{\partial n} \log \sqrt{\Lambda}\right) d\mathcal{H}^1, \tag{10}$$

where n denotes the exterior normal to $\partial\Omega_j$. (Actually, we should write $\int_{X_j} K\, dA$ instead of $\int_{\Omega_j} K\, dA$, with $X_j := X|_{\Omega_j}$.) According to Lemma 2, the sequences $\{r_\alpha^{(j)}\}$ and $\{\tilde{r}_\beta^{(j)}\}$ can be chosen in such a way that

$$\int_{C(w_\alpha, r_\alpha^{(j)})} \frac{\partial}{\partial n} \log \sqrt{\Lambda}\, d\mathcal{H}^1 \to 2\pi\nu(w_\alpha) \tag{11}$$

as $j \to \infty$, and that

$$\int_{C(\tilde{w}_\beta, \tilde{r}_\beta^{(j)})} \frac{\partial}{\partial n} \log \sqrt{\Lambda}\, d\mathcal{H}^1 \to \pi\nu(\tilde{w}_\beta), \tag{12}$$

where $\nu(w_\alpha)$ and $\nu(\tilde{w}_\beta)$ denote the orders of branch points w_α and $\tilde{w}_\beta$, respectively, which are defined by the corresponding expansions (6).

Moreover, let γ_k be one of the m closed curves, the union of which is $\partial\Omega$, and let $(a(\sigma), b(\sigma)), 0 \le \sigma \le L$, be a parameter representation of γ_k in terms of its parameter of arc length σ which orients $\partial\Omega$ in the positive sense with respect to Ω. Then we have $\dot{a}^2 + \dot{b}^2 = 1, a(0) = a(L), b(0) = b(L)$, and $n = (\dot{b}, -\dot{a})$ is the exterior normal to γ_k with respect to Ω. The geodesic curvature κ_g of the (oriented) curve $X \circ \gamma_k$ can, according to Vol. 1, Section 1.3, (46) be computed from the formula

$$\kappa_g \sqrt{\Lambda} = (\dot{a}\ddot{b} - \ddot{a}\dot{b}) + \frac{\partial}{\partial n} \log \sqrt{\Lambda}. \tag{13}$$

From $\dot{a}^2 + \dot{b}^2 = 1$ we infer that

$$\int_0^L (\dot{a}\ddot{b} - \ddot{a}\dot{b})\, d\sigma = \pm 2\pi, \tag{14}$$

where we have the plus-sign if γ_k is positively oriented with respect to its interior domain while otherwise the minus-sign is to be taken. As $\partial\Omega$ consists of the closed Jordan curves $\gamma_1, \gamma_2, \ldots, \gamma_m$, we can assume that γ_1 forms the outer boundary curve of $\partial\Omega$ whereas $\gamma_2, \ldots, \gamma_m$ lie in the interior domain of γ_1. Consequently we have the plus-sign for γ_1 and the minus-sign for $\gamma_2, \ldots, \gamma_m$, and we infer from (13) and (14) that

$$(15) \qquad -\int_0^L \frac{\partial}{\partial n} \log\sqrt{\Lambda}\, d\sigma = 2\pi\varepsilon_k - \int_0^L \kappa_g \sqrt{\Lambda}\, d\sigma,$$

where $\varepsilon_k := 1$ for $k = 1$ and $\varepsilon_k := -1$ for $2 \le k \le m$. (If there are branch points on γ_k, the integrals in γ_k are to be understood as Cauchy principal values.) Adding (15) from $k = 1$ to $k = n$, we obtain

$$(16) \qquad -\int_{\partial\Omega} \left(\frac{\partial}{\partial n} \log\sqrt{\Lambda} \right) d\mathcal{H}^1 = 2\pi(2-m) - \int_{\partial\Omega} \kappa_g |dX|.$$

Thus, letting j tend to infinity, we infer from (10) that

$$(17) \qquad \int_X K\, dA = 2\pi(2-m) + 2\pi \sum_{\alpha=1}^{N} \nu(w_\alpha) + \pi \sum_{\beta=1}^{M} \nu(\tilde{w}_\beta) - \int_{\partial\Omega} \kappa_g |dX|$$

provided that the integral $\int_X K\, dA$ exists as principal value

$$(18) \qquad \int_X K\, dA = \lim_{j\to\infty} \int_{\Omega_j} K\, dA. \qquad \square$$

In various instances it is superfluous to assume that the principle value (18) exists. Let us consider some instructive cases.

Suppose that $X \in \mathcal{PR}(\Omega)$ is a minimal surface. Then we have $K \le 0$, and we infer that $\int_X K\, dA$ exists, but it can have the value $-\infty$. If, however, X maps $\partial\Omega$ topologically onto $\Gamma = \bigcup_{j=1}^m \Gamma_j$ where $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are mutually disjoint and regular Jordan curves of class C^2, then the geodesic curvature κ_g of $X|_{\partial\Omega}$ is bounded by $|\kappa_g| \le \kappa$ where κ denotes the curvature of κ. Hence $\int_{\partial\Omega} \kappa_g |dX|$ exists and is finite, and we infer from (10) for $j \to \infty$ that $\int_X |K|\, dA$ exists and is finite. Thus Theorem 1 implies the following result.

Theorem 2. *Let Ω be an m-fold connected, bounded domain in $\mathbb{C}$ whose boundary consists of m closed, regular, disjoint curves $\gamma_1, \ldots, \gamma_m$. Secondly, let*

$$X \in C^0(\overline{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$$

be a minimal surface on Ω which maps the γ_j topologically onto closed regular and disjoint Jordan curves $\Gamma_j, 1 \le j \le m$, of class $C^{2,\alpha}, 0 < \alpha < 1$, with the curvature κ. Then $\int_X |K|\, dA$ and $\int_{\partial\Omega} \kappa_g |dX| = \int_\Gamma \kappa_g\, ds$ are finite, and we have

(19) $$\int_X K\,dA + \int_\Gamma \kappa_g\,ds = 2\pi(2-m) + 2\pi \sum_{w\in\sigma'} \nu(w) + \pi \sum_{w\in\sigma''} \nu(w),$$

where $ds = |dX|$ is the line element of $\Gamma := \bigcup_{j=1}^m \Gamma_j, \nu(w)$ is the order of a branch point $w \in \Sigma, \sigma' := \Omega \cap \Sigma_0, \sigma'' := \partial\Omega \cap \Sigma_0, \Sigma_0$ is the set of branch points of X in $\overline{\Omega}$, and κ_g is the geodesic curvature of Γ viewed as curve on the surface X. In particular, equation (19) implies that

(20) $$2 - m + \sum_{w\in\sigma'} \nu(w) + \frac{1}{2} \sum_{w\in\sigma''} \nu(w) \le \frac{1}{2\pi} \int_\Gamma \kappa\,ds.$$

Here we have used the fact that the assumption $\Gamma \in C^{2,\alpha}$ implies that $X \in \mathcal{PR}(\Omega)$, as we have seen in the previous sections of this chapter. We recall that *the order of boundary branch points has to be even* since X maps $\partial\Omega$ topologically onto Γ.

Remark 2. Because analogous regularity results hold for solutions $X \in C^0(\overline{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$ of

$$\Delta X = 2H(X) X_u \wedge X_v,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0,$$

where $H \in C^{0,\alpha}(\mathbb{R}^3)$ (see Section 2.3), we infer from

$$K \le H^2 \le h^2, \quad h := \sup_{w\in\Omega} H(X)$$

that $\int_X |K|\,dA$ and $\int_\Gamma \kappa_g\,ds$ are finite, and that we have formula (19) as well as the estimate

(21) $$2 - m + \sum_{w\in\sigma'} \nu(w) + \frac{1}{2} \sum_{w\in\sigma''} \nu(w) \le \frac{1}{2\pi} \int_\Gamma \kappa\,ds + h^2 A(X),$$

where $A(X) = D(X)$ denotes the area of X which in certain situations can be estimated in terms of the length of Γ by, say, by isoperimetric inequalities.

Remark 3. Let $X \in C^2 \cap H_2^1(\Omega, \mathbb{R}^3)$ be a minimal surface which is stationary with respect to a boundary configuration $\langle S_1, S_2, \dots, S_m\rangle$ consisting of m regular, sufficiently smooth surfaces S_j whose principal curvatures are bounded in absolute value by a constant $k > 0$, and suppose that Ω is an m-fold connected bounded domain. Then X is of class $\mathcal{PR}(\Omega)$ and intersects $S := \bigcup_{j=1}^m S_j$ perpendicularly. Moreover, the geodesic curvature κ_g of the free trace $\Sigma = X|_{\partial\Omega}$ can be written as

(22) $$\kappa_g = \pm\kappa_n^*,$$

where κ_n^* is the normal curvature of Σ viewed as curve(s) on S. By virtue of $|\kappa_n^*| \le k$ we then infer that

$$|\kappa_g| \le k.$$

Therefore we obtain the Gauss–Bonnet formula

$$\int_X K\,dA + \int_{\partial\Omega} \kappa_g |dX| = 2\pi(2-m) + 2\pi \sum_{w\in\sigma'} \nu(w) + \pi \sum_{w\in\sigma''} \nu(w), \tag{23}$$

where $\sigma' = \Sigma_0 \cap \Omega, \sigma'' = \Sigma_0 \cap \partial\Omega$, and Σ_0 is the set of branch points w_0 of $X, \nu(w_0)$ is the order of $w_0 \in \Sigma_0$, and we have the estimate

$$2 - m + \sum_{w\in\sigma'} \nu(w) + \frac{1}{2}\sum_{w\in\sigma''} \nu(w) \le \frac{k}{2\pi} L(\Sigma). \tag{24}$$

The length $L(\Sigma) = \int_\Sigma |dX|$ of the free trace Σ can possibly be estimated by other geometric expressions (see Sections 2.12 and 4.6).

If we want to state similar formulas for minimal surfaces solving partially free boundary problems, we have to take the angles at the corners of ∂X into account (see Section 1.4 of Vol. 1, (12) and (12′)). The necessary asymptotic expansions can be found in Chapter 3.

Remark 4. For (disk-type) minimal surfaces X solving a *thread problem* (see Chapter 5), the thread Σ has a fixed length $L(\Sigma)$ and a constant geodesic curvature κ_g if we view Σ as curve on X. Hence it follows that

$$\int_\Sigma \kappa_g |dX| = \kappa_g L(\Sigma).$$

This observation can be used to draw interesting conclusions from the Gauss–Bonnet formula.

Remark 5. It is not difficult to carry over the Gauss–Bonnet formula (14) of Section 1.4 in Vol. 1 to minimal surfaces $\mathcal{X} : M \to \mathbb{R}^3$ with branch points which are defined on a compact Riemann surface M with nonempty boundary. Suppose that ∂M consists of m disjoint, regular, smooth Jordan arcs $\gamma_1, \dots, \gamma_m$ which are topologically mapped by $\mathcal{X}$ onto a system Γ of disjoint, regular, smooth Jordan arcs $\Gamma_1, \dots, \Gamma_m$, and let g be the genus of the orientable surfaces $\mathcal{X}$. Then we have

$$\int_{\mathcal{X}} K\,dA + \int_\Gamma \kappa_g\,ds + 4\pi(g-1) + 2\pi m = 2\pi \sum_{w\in\sigma'} \nu(w) + \pi \sum_{w\in\sigma''} \nu(w),$$

where σ' and σ'' denote the sets of interior and of boundary branch points, and $\nu(w)$ is the order of any $w \in \sigma' \cup \sigma''$.

2.12 Scholia

1. The first results concerning the boundary behaviour of minimal surfaces are the reflection principles of Schwarz; see Sections 3.4 and 4.8 of Vol. 1.

They insure that a minimal surface can be extended analytically across any straight part of its boundary, or across any part of its boundary where the surface meets some plane perpendicularly. Schwarz's reasoning is described in Section 3.4 of Vol. 1; cf. Schwarz [2], vol. I, p. 181. (As Schwarz mentions, he learned this reasoning from Weierstrass.) Our discussion in Section 4.8 of Vol. 1 follows the exposition in Courant [15], pp. 118–119 and pp. 218–219.

2. Another important result, found rather early, is Tsuji's theorem that a minimal surface $X \in H_2^1(B, \mathbb{R}^3)$ has boundary values $X|_{\partial B}$ of class $H_1^1(\partial B, \mathbb{R}^3)$ if their total variation is finite, i.e., if

$$\int_{\partial B} |dX| < \infty.$$

(Here we have used the parameter domain $B := \{w\colon |w| < 1\}$.) The importance of this result, which remained unnoticed for a long time, has been emphasized in the work of Nitsche, see [28]. Tsuji's paper [1] appeared in 1942; it is based on a classical result by F. and M. Riesz [1] from 1916 concerning the boundary values of holomorphic functions. We have presented Tsuji's result in Section 4.7 of Vol. 1.

3. The result stated as Theorem 3 in Section 2.3 is H. Lewy's celebrated regularity result from 1951; see Lewy [5]. It is the direct generalization of Schwarz's reflection principle guaranteeing that any minimal surface can be extended analytically across an analytic part of its boundary. In Courant's monograph [15], this problem was still quoted as an open question (see [15], p. 118). Lewy succeeded in proving his result without using Tsuji's theorem. Our proof essentially agrees with that of Lewy except that we use the fact that X is of class C^∞ on $B \cup \gamma$ if γ is a subarc of ∂B which is mapped by X into a real analytic arc Γ of $\mathbb{R}^3$. One can, however, avoid the use this fact (which follows from the results of Section 2.3); see Lewy [5], or Nitsche [28], pp. 297–302.

4. Hildebrandt [1] has in a first paper derived a priori estimates for minimal surfaces assuming them to be smooth up to the boundary. In conjunction with Lewy's result, one then obtains the following:

Let $X \in C^0(\overline{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$ be a minimal surface which is bounded by a closed Jordan arc Γ. Suppose that $\Gamma \in C^{m,\mu}, m \geq 4, \mu \in (0,1)$, and that there is a sequence of real analytic curves Γ_n with

$$(1) \qquad |\Gamma - \Gamma_n|_{C^{m,\mu}} \to 0 \quad \text{as } n \to \infty.$$

Assume also that there is a sequence of minimal surfaces X_n bounded by Γ_n such that

$$|X - X_n|_{C^0(\overline{B})} \to 0 \quad \text{as } n \to \infty.$$

Then X is smooth up to the boundary, i.e., $X \in C^{m,\mu}(\overline{B}, \mathbb{R}^3)$.

However, it might be possible that not every solution of Plateau's problem for Γ satisfies this approximation condition; it certainly holds true for isolated local minima of Dirichlet's integral; cf. Hildebrandt [1]. By approximating a given smooth curve Γ in the sense of (1) by real analytic curves Γ_n, and by solving the Plateau problem for each of the approximating curves Γ_n, the above result yields:

Every curve $\Gamma \in C^{m,\mu}, m \geq 4, \mu \in (0,1)$, *bounds at least one minimal surface* X *of class* $C^{m,\mu}(\overline{B}, \mathbb{R}^3)$.

As a given boundary Γ may be spanning many (and, possibly, infinitely many) minimal surfaces, this regularity result by Hildebrandt [1] is considerably weaker than Theorem 1 of Section 2.3 whose global version can be formulated as follows:

Let $\Gamma \in C^0(\overline{B}, \mathbb{R}^3)$ *be a minimal surface, i.e.,*

$$\Delta X = 0, \quad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B,$$

which is bounded by some Jordan curve Γ *of class* $C^{m,\mu}$ *with* $m \geq 1$ *and* $\mu \in (0,1)$. *Then* X *is of class* $C^{m,\mu}(\overline{B}, \mathbb{R}^3)$.

Assuming that $m \geq 4$, this result was first proved by Hildebrandt [3] in 1969. Some of the essential ideas of that paper are described in Step 1 of Section 2.7. Briefly thereafter, Heinz and Tomi [1] succeeded in establishing the result under the hypothesis $m \geq 3$, and both Nitsche [16] and Kinderlehrer [1] provided the final result for $m \geq 1$. Warschawski [6] verified that X has Dini-continuous first derivatives on $\overline{B}$, if the first derivatives of Γ with respect to arc length are Dini continuous; cf. also Lesley [1].

These results on the boundary behaviour of minimal surfaces hold for surfaces in $\mathbb{R}^n, n \geq 2$, and not only for $n = 3$; the proof requires no changes. For $n = 2$ these results include classical theorems on the boundary behaviour of conformal mappings due to Painlevé, Lichtenstein, Kellogg [2], and Warschawski [1–4]. (Concerning the older literature, we refer to Lichtenstein's article [1] in the Enzyklopädie der Mathematischen Wissenschaften; the most complete results can be found in the papers by Warschawski.)

As Nitsche has described his technique to prove boundary regularity in great detail in his monograph [28], Section 2.1, in particular pp. 283–284 and 303–312, we refer the reader to this source or to the original papers by Nitsche and Kinderlehrer quoted before. Instead we have presented a method by E. Heinz [15] which needs the slightly stronger hypothesis $m \geq 2$. By this method, Heinz could also treat H-surfaces, and Heinz and Hildebrandt [1] were able to handle minimal surfaces in Riemannian manifolds; cf. Section 2.3. The basic tools of Heinz's approach are the a priori estimates for vector-valued solutions X of differential inequalities

$$|\Delta X| \leq a|\nabla X|^2$$

which we have derived and collected in Section 2.2. They follow from classical results of potential theory which we have briefly but (more or less) completely proved in Section 2.1. The results of Section 2.2 and, in part, of Section 2.1 are taken from Heinz [2,5], and [15].

Closely related to this method is the approach of Heinz and Tomi [1] and the very useful regularity theorem of Tomi [1].

The first regularity theorem for surfaces of constant mean curvature was proved by Hildebrandt [4]; an essential improvement is due to Heinz [10]. The method of Heinz [15], described in the proof of Theorem 2 in Section 2.3, can be viewed as the optimal method. A very strong result was obtained by Jäger [3].

5. The possibility to obtain asymptotic expansions of minimal surfaces and, more generally, of H-surfaces by means of the Hartman–Wintner technique was first realized by Heinz (oral communication). A first application appeared in the paper by Heinz and Tomi [1].

6. In Theorem $2'$ of Section 2.8 we proved that any minimal surface X, meeting a real-analytic support surface S perpendicularly, can be extended analytically across S. The proof basically follows ideas from H. Lewy's paper [4], published in 1951. There it was proved that *any minimizing solution X of a free boundary problem can be continued analytically across the free boundary if S is assumed to be a compact real-analytic support surface.* In fact, Lewy first had to cut off a set of hairs from the minimizer by composing it with a suitable parameter transformation before he could apply his extension technique. (Later on it was proved by Jäger [1] that the removal of these hairs is not needed since they do not exist.)

7. Combining Lewy's theorem with new a priori estimates, Hildebrandt [2] proved that the Dirichlet integral possesses at least one minimizer in $\mathcal{C}(\Gamma, S)$ which is smooth up to its free boundary provided that S is smooth and satisfies a suitable condition at infinity which enables one to prove that solutions do not escape to infinity. (A very clean condition guaranteeing this property was later formulated by Hildebrandt and Nitsche [4].)

8. The first regularity theorem for minimal surfaces with a merely smooth, but not analytic support surface S was given by Jäger [1]. He proved for instance that *any minimizer X of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ is of class $C^{m,\mu}(B \cup I, \mathbb{R}^3)$, I being the free boundary of X, provided that $S \in C^{m,\mu}$ and $m \geq 3, \mu \in (0,1)$.* Part of Jäger's method we have described or at least sketched in Section 2.8. We have not presented his main contribution, the proof of $X \in C^0(B \cup I, \mathbb{R}^3)$, which requires S to be of class C^2. Instead, in Section 2.5, we have described a method to prove continuity of minimizers up to the free boundary that needs only a *chord-arc condition for S.* Because of the Courant–Cheung example, this result is the best possible one.

The approach of Section 2.5 follows more or less the discussion in Hildebrandt [9]. The sufficiency of the chord-arc condition for proving continuity of minimizers up to the free boundary was almost simultaneously discovered by Nitsche [22] and Goldhorn and Hildebrandt [1].

Later on, Nitsche [30] showed that Jäger's regularity theorem remains valid if we relax the assumption $m \geq 3$ to $m \geq 2$. Moreover, if we assume $S \in C^2$, then every minimizer in $\mathcal{C}(\Gamma, S)$ is of class $C^{1,\alpha}(B \cup I, \mathbb{R}^3)$ for all $\alpha \in (0, 1)$.

9. The regularity of stationary surfaces in $\mathcal{C}(\Gamma, S)$ up to their free boundaries was – almost simultaneously – proved by Grüter-Hildebrandt-Nitsche [1] and by Dziuk [3]. Both papers are based on the fundamental thesis of Grüter [1] (see also [2]) where interior regularity of weak H-surfaces is proved. The basic idea of Grüter's paper consists in deriving *monotonicity theorems* similar to those introduced by DeGiorgi and Almgren in geometric measure theory.

We have presented the method used in Grüter-Hildebrandt-Nitsche [1]; it has the advantage to be applicable to support surfaces with nonvoid boundary ∂S. Moreover we do not have to assume that

$$\lim_{w \to w_0} \operatorname{dist}(X(w), S) = 0 \quad \text{for any } w_0 \in I$$

as in Dziuk [5–7]. On the other hand, Dziuk's method is somewhat simpler than the other one since it reduces the boundary question to an interior regularity problem by applying Jäger's reflection method. This interior problem can be dealt with by means of the methods introduced in Grüter's thesis.

10. The results in Section 2.7 concerning the $C^{1,1/2}$-regularity of stationary minimal surfaces with a support surface S having a nonempty boundary ∂S are taken from Hildebrandt and Nitsche [1] and [2].

11. The proof of Proposition 1 in Section 2.8 is more or less that of Jäger [1], pp. 812–814.

12. The alternative method to attain the result of Step 2 in Section 2.7, given in Section 2.9, was worked out by Ye [1,4]. Ye's method is a quantitative version of the L_2-estimates of Step 2 in Section 2.7 which is based on an idea due to Kinderlehrer [6].

13. *Open questions*: (i) The regularity results for stationary minimal surfaces X with a free boundary are not yet in their final form. In particular one should prove that X is of class $C^{1,\mu}$ up to the free boundary if $S \in C^{1,\mu}_*$, and that $X \in C^{0,\alpha}$ for some $\alpha \in (0, 1)$ if S satisfies a chord-arc condition (this is only known for minimizers of the Dirichlet integral). Here we say that $S \in C^{1,\mu}_*$ if $S \in C^{1,\mu}$ and if S satisfies a *uniformity condition (B) at infinity* (see Section 2.6). Dziuk [7] and Jost [8] proved that X is of class $C^{1,\mu}$ up to the free boundary, $0 < \mu < 1$, if S is of class C^2 and satisfies a suitable uniformity condition.

(ii) It would be desirable to derive a priori estimates for stationary minimal surfaces, in particular for those of higher topological type. As in general there are no estimates depending only on the geometric data of the boundary configuration (cf. the examples in Section 2.6), one could try to derive estimates depending also on certain important data of the surfaces X in consideration such as the *area* (= *Dirichlet integral*) or the *length of the free trace.*

Such estimates could be useful for approximation theorems, for results involving the deformation of the boundary configuration, for building a Morse theory, and for deriving index theorems.

Note, however, that a priori estimates depending only on boundary data can be derived in certain favourable geometric situations, for instance if the support surface is only mildly curved. Results of this kind were found by Ye [2]. Let us quote a typical result:

Suppose that S is an orientable and admissible support surface of class $C^{3,\alpha}$, $\alpha \in (0,1)$, and let n_0 be a constant unit vector and σ be a positive number, such that the surface normal $n(p)$ of S satisfies

$$\langle n(p), n_0 \rangle \geq \sigma \quad \textit{for all } p \in S. \tag{2}$$

Then the length $l(\Sigma)$ of the free trace Σ of a stationary point X of Dirichlet's integral in $\mathcal{C}(\Gamma, S)$ without branch points on the free boundary I is estimated by the length of Γ via the formula

$$l(\Sigma) \leq l(\Gamma)/\sigma, \tag{3}$$

and the isoperimetric inequality yields the upper bound

$$D(X) \leq \frac{1}{4\pi}(1 + \sigma^{-2}) l^2(\Gamma) \tag{4}$$

for the Dirichlet integral of X.

Let us sketch the *proof of* (3), which is nothing but a simple variant of the reasoning used in Section 4.6.

By means of Green's formula we obtain

$$0 = \int_B \Delta X \, du \, dv = -\int_I X_v \, du + \int_C X_r \, d\varphi \tag{5}$$

with $w = u + iv = re^{i\varphi}$, where B stands for the usual semidisk. Because of (2) and of

$$X_v = |X_v| n(X) \quad \text{on } I$$

(where we possibly have to replace n by $-n$),

$$\begin{aligned} |X_r| &= |X_\varphi| \quad \text{on } C = \partial B \setminus I, \\ |X_u| &= |X_v| \quad \text{on } I, \end{aligned}$$

we then obtain

$$\begin{aligned} \sigma l(\Sigma) &= \sigma \int_I |X_u| \, du = \int_I \sigma |X_v| \, du \\ &\leq \int_I |X_v| \langle n(X), n_0 \rangle \, du = \int_I \langle X_u, n_0 \rangle \, du \\ &= \int_C \langle X_r, n_0 \rangle \, d\varphi \leq \int_C |X_r| \, d\varphi = \int_C |X_\varphi| \, d\varphi = l(\Gamma), \end{aligned}$$

i.e.,

$$\sigma l(\Sigma) \le l(\Gamma).$$

This proof of (3), (4) is not quite correct but it can easily be rectified by the reasoning in Section 4.6. We leave it to the reader to carry out the details. □

By way of an example Ye showed that the assumption $\sigma > 0$ in (2) is necessary if one wants to bound $l(\Sigma)$; see Ye [2], p. 101.

14. Now we briefly describe Courant's example of a configuration $\langle \Gamma, S \rangle$ with a continuous supporting surface S and a rectifiable Jordan arc Γ with end points on S which bounds infinitely many solutions of the corresponding free boundary problem with a discontinuous and even nonrectifiable trace curve; see Courant [15], p. 220. We firstly select a sequence of numbers $\varepsilon_n > 0, n \in \mathbb{N}$, with $\sum_{n=1}^{\infty} \varepsilon_n < 1/4$, and then we define set $A_n, B_n, C_n^1, C_n^2, D_n^1, D_n^2$ as follows:

$$\begin{aligned}
A_n &:= \left\{ (x,y,z) \colon z = 0, |x| < 1, \left| y - \frac{1}{n} \right| < \varepsilon_n^3 \right\}, \\
B_n &:= \left\{ (x,y,z) \colon z = -\varepsilon_n, |x| < 1, \left| y - \frac{1}{n} \right| \le \frac{\varepsilon_n}{2} \right\}, \\
C_n^1 &:= \left\{ (x,y,z) \colon |x| \le 1, z = \frac{2}{2\varepsilon_n^2 - 1} \left(y - \frac{1}{n} - \varepsilon_n^3 \right), -\varepsilon_n \le z \le 0 \right\}, \\
C_n^2 &:= \left\{ (x,y,z) \colon |x| \le 1, z = \frac{2}{1 - 2\varepsilon_n^2} \left(y - \frac{1}{n} + \varepsilon_n^3 \right), -\varepsilon_n \le z \le 0 \right\},
\end{aligned}$$

$$\begin{aligned}
D_n^{1,2} := &\text{ the compact region in } \{x = \pm 1\} \text{ which is bounded by} \\
&\{x = \pm 1\} \cap (\partial A_n \cup \partial B_n \cup \partial C_n^1 \cup \partial C_n^2),
\end{aligned}$$

see Fig. 1.

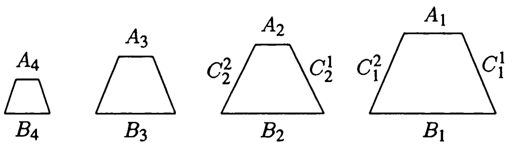

Fig. 1. Cross section of the sets A_n, B_n, C_n^1, C_n^2 at the levels $x = \pm 1$

Set

$$S_1 := \{z = 0\} \setminus \bigcup_{n=1}^{\infty} A_n,$$

and define S by

$$S := S_1 \cup \bigcup_{n=1}^{\infty} [B_n \cup C_n^1 \cup C_n^2 \cup D_n^1 \cup D_n^2]$$

(see Fig. 2).

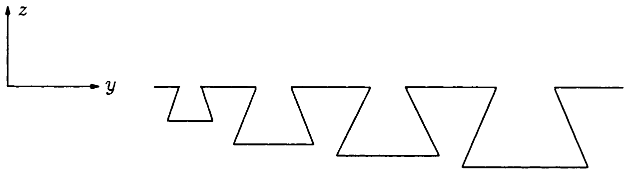

Fig. 2. Cross sections of the surface S at the levels $x = 0, \pm 1$

Then we observe the following: If Γ_1 denotes the straight segment $\{z = 0, x = 0, |y-1| \le \varepsilon_1^3\}$, then the associated free boundary problem $\mathcal{P}(\Gamma_1, S)$ has at least two solutions, namely the representations of the sets

$$A_1^+ := \{z = 0, 0 \le x \le 1, |y-1| \le \varepsilon_1^3\}$$

and

$$A_1^- := \{z = 0, -1 \le x \le 0, |y-1| \le \varepsilon_1^3\}.$$

In fact, there is still another stationary but not minimizing surface bounded by Γ_1 and S, namely the surface describing the compact region in $\{x = 0\}$ which is bounded by $\{x = 0\} \cap (\Gamma_1 \cup B_1 \cup C_1^1 \cup C_1^2)$. Similarly, if we set

$$\Gamma_n := \left\{ z = 0, x = 0, \left| y - \frac{1}{n} \right| \le \varepsilon_n^3 \right\},$$

then we obtain at leat two minimizing surfaces in $\mathcal{C}(\Gamma_n, S)$ which are determined by the sets

$$A_n^+ := \left\{ z = 0, 0 \le x \le 1, \left| y - \frac{1}{n} \right| \le \varepsilon_n^3 \right\}$$

and

$$A_n^- := \left\{ z = 0, -1 \le x \le 0, \left| y - \frac{1}{n} \right| \le \varepsilon_n^3 \right\}.$$

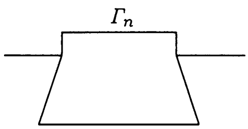

Fig. 3. The curve Γ_n lifted

Now let us lift the curves Γ_n to a height $z = \varepsilon_n^q, q \ge 4$, and connect the endpoints $P_n^1 = (0, \frac{1}{n} + \varepsilon_n^3, \varepsilon_n^q)$, $P_n^2 = (0, \frac{1}{n} - \varepsilon_n^3, \varepsilon_n^q)$ via vertical segments with S, see Fig. 3. Denoting again the lifted curves together with the vertical segments by Γ_n, it is reasonable to expect the existence of at least *two* solutions $X_n^1, X_n^2 \in \mathcal{C}(\Gamma_n, S)$ for the problem $\mathcal{P}(\Gamma_n, S)$, provided that q is large enough.

In particular, we can expect that the minimal surfaces X_n^1, X_n^2 converge to A_n^+ and A_n^- respectively, if q tends to infinity.

At a small height $z = \varepsilon < \varepsilon_{n+1}^q$, we connect Γ_n and Γ_{n+1} with a straight line segment parallel to $\{z = 0\}$ and omit the corresponding parts of the vertical segments, see Fig. 4.

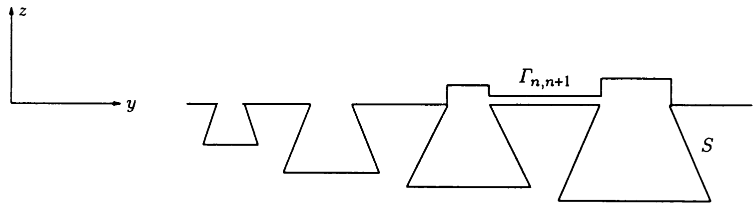

Fig. 4. Cross section of the boundary configuration $\langle \Gamma_{n,n+1}, S\rangle$ at the level $x = 0$

This way we obtain a Jordan arc $\Gamma_{n,n+1}$ with endpoints on S. Furthermore, let us assume the validity of the following bridge principle:

Given any two area-minimizing minimal surfaces $X_n \in \mathcal{C}(\Gamma_n, S)$ *and* $X_{n+1} \in \mathcal{C}(\Gamma_{n+1}, S)$*, there exists an area-minimizing minimal surface* $Y_\varepsilon \in \mathcal{C}(\Gamma_{n,n+1}, S)$ *which converges (in a geometric sense) to the union of* $X_n(B)$*,* $X_{n+1}(B)$ *as* ε *tends to zero.*

By means of this heuristic principle we obtain at least four stationary minimal surfaces in $\mathcal{C}(\Gamma_{n,n+1}, S)$ combining X_n^1 with X_{n+1}^1 or X_{n+1}^2, and X_n^2 with X_{n+1}^1 or X_{n+1}^2, respectively. Similarly we now define the Jordan arcs $\Gamma_{n,n+2}, \dots, \Gamma_{n,n+k}$ which bridge the ditches $A_n, A_{n+1}, \dots, A_{n+k}$. Then $\Gamma_{n,n+k}$ and S bound at least 2^{k+1} area-minimizing minimal surfaces which are stationary in $\mathcal{C}(\Gamma_{n,n+k}, S)$. Finally, let $\Gamma := \Gamma_{1,\infty}$ denote the rectifiable Jordan arc which bridges all the ditches and connects the points $(0, 0, 0)$ and $(0, 1 + \varepsilon_1, 0)$. Then it follows that there are infinitely (and even nondenumerably) many stationary minimal surfaces in $\mathcal{C}(\Gamma, S)$ each of which has a discontinuous and nonrectifiable trace curve.

Let us add that the previous reasoning is by no means rigorous; thus this example by Courant is merely of heuristic value.

15. Complementary to the existence result for the obstacle problem $\mathcal{P}(E, C)$, which we have described in the Scholia of Section 4 in Vol. 1 (see also Chapter 4 of the present volume), we want to mention some regularity properties of solutions for $\mathcal{P}(E, C)$, see also Chapter 4.

The problem $\mathcal{P} = \mathcal{P}(E, C)$ is a special case of a parametric obstacle problem which was first treated by Tomi [3,4], Hildebrandt [12,13], and Hildebrandt and Kaul [1]. Tomi's results are based on important earlier work by Lewy and Stampacchia [1,2], whereas Hildebrandt's approach uses the difference-quotient technique and some important observations due to Frehse [4].

For nonparametric obstacle problems we refer the reader to the treatise of Kinderlehrer and Stampacchia [1] and to the literature quoted there. Here we shall restrict our attention to the two-dimensional parametric case, basically following the papers by Hildebrandt and Hildebrandt-Kaul cited above. Consider the integral

$$E(X) = \int_B \{g_{ij}[X_u^i X_u^j + X_v^i X_v^j] + \langle Q(X), X_u \wedge X_v \rangle\}\, du\, dv,$$

and the class $C = C(K, \mathcal{C}^*) := \mathcal{C}^* \cap H_2^1(B, K)$, where $\mathcal{C}^*$ stands for $\mathcal{C}^*(\Gamma)$ or $\mathcal{C}^*(\Gamma, S)$ respectively and $K \subset \mathbb{R}^3$ denotes some closed set. Let us also introduce the variational problem

$$\mathcal{P}(E, C): \qquad E \to \min \text{in } C.$$

Using the abbreviation

$$e(x, p) = g_{ij}(x)\{p_1^i p_1^j + p_2^i p_2^j\} + \langle Q(x), p_1 \wedge p_2 \rangle$$

for $(x, p) \in K \times \mathbb{R}^6, p = (p_1, p_2)$, we assume that, for suitable constants $0 < m_0 \le m_1$, the coerciveness condition

$$(6) \qquad m_0|p|^2 \le e(x, p) \le m_1|p|^2$$

holds true for all $(x, p) \in K \times \mathbb{R}^6$.

Recall that the existence of a solution of $\mathcal{P}(E, C)$ can be proved under the mere assumption that K be a closed set. If we want to prove regularity, say Hölder continuity, we clearly have to add further assumptions on K. The concept of quasiregularity turns out to be of use.

Definition 1. *We call a set $K \subset \mathbb{R}^3$* quasiregular *if it is closed and if there are positive numbers δ_0, δ_1 and d such that for any point $x_0 \in K$, there exist a compact convex set K^* and a C^1-diffeomorphism g of an open neighbourhood of K^* which maps K^* onto $K \cap \overline{B}_d(x_0), B_d(x_0) = \{x \in \mathbb{R}^3 : |x - x_0| < d\}$, such that the matrix $\mathcal{H}(y) = (\frac{\partial g}{\partial y})^T \cdot (\frac{\partial g}{\partial y})$ satisfies*

$$(7) \qquad \delta_0|\xi|^2 \le \xi \mathcal{H}(y) \xi \le \delta_1|\xi|^2$$

for all $(y, \xi) \in K^* \times \mathbb{R}^3$. Here $\frac{\partial g}{\partial y}$ denotes the Jacobi matrix of g and $(\frac{\partial g}{\partial y})^T$ stands for its transpose.

Remarks. 1. Obviously, each closed convex set in $\mathbb{R}^3$ with nonvoid interior is quasiregular. Also, each compact three-dimensional submanifold of $\mathbb{R}^3$ with C^1-boundary is quasiregular.

2. The preceding Definition 1 and the following Theorem 1 extend to the case where K denotes some subset of $\mathbb{R}^N, N \ge 3$.

3. For our purposes it would be sufficient to assume that g is some bi-Lipschitz homeomorphism.

Theorem 1. *Suppose that* (6) *holds with functions* $g_{ij} \in C^0(K,\mathbb{R}), g_{ij} = g_{ji}$, *and* $Q \in C^0(K,\mathbb{R}^3)$. *In addition let* $K \subset \mathbb{R}^3$ *be a quasiregular set such that* $C(K,\mathcal{C}^*) = \mathcal{C}^* \cap H_2^1(B,K)$ *is nonempty. Then each solution* X *of* $\mathcal{P}(E,C)$ *satisfies a Morrey condition of the type*

$$D_{B_r(w_0)}(X) \le D_{B_R(w_0)}(X)\left(\frac{r}{R}\right)^{2\mu} \tag{8}$$

in $0 < r \le R$, *for each* $w_0 \in B_{1-R}(0)$ *and all* $R \in (0,1)$ *and some constant* $\mu > 0$. *Hence* X *is of class* $C^{0,\mu}(B,\mathbb{R}^3)$.

Furthermore, if $\mathcal{C}^* = \mathcal{C}^*(\Gamma)$, *then* X *is also of class* $C^0(\overline{B},\mathbb{R}^3)$, *and for* $\mathcal{C}^* = \mathcal{C}^*(\Gamma,S)$ *we infer that* $X \in C^0(\overline{B}\setminus\overline{I},\mathbb{R}^3)$.

The idea for proving Hölder continuity is to convexify the obstacle K locally by using the definition of quasiregularity, and then to fill in harmonic functions with the right boundary values. Elementary properties of harmonic functions will yield the estimate (8). The reasoning is similar to the argument used in the proof of Theorem 1, Section 2.5; for details we refer the reader to the original paper by Hildebrandt and Kaul [1].

We now give a brief discussion of higher regularity properties of X. First we need the following

Definition 2. *A set* $K \subset \mathbb{R}^3$ *is of class* C^s *if* K *is the closure of an open set in* $\mathbb{R}^3$, *and if, for each boundary point* $x_0 \in \partial K$, *there exists a neighbourhood* U *of* x_0 *and a* C^s*-diffeomorphism* ψ *of* $\mathbb{R}^3$ *onto itself which maps* $U \cap K$ *onto*

$$B_1^+(0) = \{x \in \mathbb{R}^3 \colon |x| < 1, x^3 > 0\},$$

$U \cap \partial K$ *onto*

$$B_1^0(0) = \{x \in \mathbb{R}^3 \colon |x| < 1, x^3 = 0\},$$

and x_0 *onto* 0.

We shall also assume that the integrand $e(x,p)$ has the following **property** (***E***):

There exist some open set $\mathcal{M} \subset \mathbb{R}^3$ *with* $K \subset \mathcal{M}$ *and functions* $Q \in C^2(\mathcal{M},\mathbb{R}^3)$ *and* $G = (g_{jk})_{j,k=1,2,3} \in C^2(\mathcal{M},\mathbb{R})$ *with* $g_{jk} = g_{kj}$ *such that*

$$e(x,p) = \sum_{\alpha=1}^{2} p_\alpha G(x) p_\alpha + \langle Q(x), p_1 \wedge p_2\rangle$$

and

$$m_0|p|^2 \le e(x,p) \le m_1|p|^2 \quad \textit{for all } (x,p) \in K \times \mathbb{R}^6, p = (p_1,p_2).$$

Theorem 2. *Suppose that* $e(x,p)$ *has property* (E), *and let* K *be quasiregular of class* C^3. *Then each solution* $X \in C(K,\mathcal{C}^*)$ *of* $\mathcal{P}(E,C)$ *is of class* $H_s^2(B',\mathbb{R}^3) \cap C^{1,\alpha}(B,\mathbb{R}^3)$ *for all* $B' \Subset B$ *and for all* $s \in [1,\infty)$ *and all* $\alpha \in (0,1)$.

Remarks. 1. Hölder continuity of the first derivatives is still valid for solutions of the elliptic variational problem

$$\int_\Omega f(u,v,X(u,v),\nabla X(u,v))\,du\,dv \to \min \quad \text{in } C = H_2^1(\Omega,K)\cap \mathcal{C}^*,$$

with a regular Lagrangian $f\colon \Omega\times\mathcal{M}\times\mathbb{R}^{2n}\to\mathbb{R}$, where $\mathcal{M}\subset\mathbb{R}^N$ denotes some open set containing K, and $\Omega\subset\mathbb{R}^2$ denotes the domain of definition. For details we refer to Hildebrandt [12,13].

2. Assuming the conditions of Theorem 2, Gornik [1] proved that each solution $X\in C = C(K,\mathcal{C}^*)$ of $\mathcal{P}(E,C)$ is in fact of class $C^{1,1}(B,\mathbb{R}^3)$. Simple examples show that this result will in general be the best possible one. Gornik's work is based on fundamental results due to Frehse [1], Gerhardt [1], and Brézis and Kinderlehrer [1] concerning $C^{1,1}$-regularity of solutions of scalar variational inequalities.

16. The first to estimate the total order of branch points of a minimal surface via the Gauss–Bonnet formula was Nitsche [6] who reversed an idea of Sasaki [1]. R. Schneider [1] later established the formula

$$1+\sum_{w\in\sigma'}\nu(w)\le\frac{1}{2\pi}\kappa(\Gamma)$$

for all disk-type minimal surfaces $X\colon B\to\mathbb{R}^3$ which are continuous in $\overline{B}$ and map ∂B monotonically (and hence topologically) onto an arbitrary closed Jordan curve Γ which has a generalized total curvature $\kappa(\Gamma)$.

The method of Section 2.11 and the generalization of the Nitsche–Sasaki formula is taken from a paper by Heinz and Hildebrandt [2].

17. Let $\Omega\subset\mathbb{R}^2$ be an open connected domain with smooth boundary and suppose $\psi\in C^2(\overline{\Omega})$ satisfies $\max_\Omega\psi>0$ and $\psi<0$ on $\partial\Omega$. Consider the convex set of comparison functions $K_\psi := \{v\in H_2^1(\Omega)\colon v\ge\psi \text{ in } \Omega,\ v=0 \text{ on } \partial\Omega\}$ and a solution $u\in K_\psi$ of the variational problem

$$D(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx \to \min \quad \text{in } K_\psi.$$

One readily verifies that a solution $u\in K_\psi$ satisfies the variational inequality

$$\int_\Omega D_i u D_i(v-u)\,dx \ge 0 \quad \text{for all } v\in K_\psi. \tag{9}$$

Lewy and Stampacchia [1] used the method of penalization together with suitable a priori estimates to show that u is of class $C^{1,\alpha}, \alpha<1$ (at least, if ψ is smooth and strictly concave). It is in fact true that u is of class $H^2_\infty(\Omega)$; cf. Frehse [1], Gerhardt [1], and Brézis and Kinderlehrer [1].

The set Ω may now be divided into two subsets, the coincidence set

$$\mathcal{I} = \mathcal{I}(u) = \{x\in\Omega\colon u(x)=\psi(x)\}$$

and its complement

$$\Omega \setminus \mathfrak{I} = \{x \in \Omega \colon u(x) > \psi(x)\}.$$

Of particular importance is a careful analysis of the boundary $\partial\mathfrak{I}$ of the set of coincidence $\mathfrak{I}$. Such investigations were initiated by H. Lewy and G. Stampacchia [1] and continued by Kinderlehrer [7] and Caffarelli and Rivière [1]. It was proved that the free boundary $\partial\mathfrak{I}$ is: (i) An analytic Jordan curve if ψ is strictly concave and analytic; (ii) a $C^{1,\beta}$-Jordan curve, $0 < \beta < \alpha$, if ψ is strictly concave and of class $C^{2,\alpha}$; (iii) a $C^{m-1,\alpha}$-Jordan curve if ψ is strictly concave and of class $C^{m,\alpha}$ with $m \geq 2$ and $0 < \alpha < 1$.

The investigation of $\partial\mathfrak{I}$ is more difficult if we span a nonparametric surface as a graph of a function u over some obstacle graph ψ such that it minimizes area. In other words, if Ω is a strictly convex domain in $\mathbb{R}^2$ with a smooth boundary, if ψ is given as above and K_ψ is the convex set of functions $v \in \mathring{H}^1_\infty(\Omega)$ satisfying $v \geq \psi$, we consider solutions of the variational problem

$$\int_\Omega \sqrt{1 + |\nabla u|^2}\, dx \to \min \quad \text{in } K_\psi.$$

The existence of a solution $u \in K_\psi$ was proved by Lewy and Stampacchia [2] and by Giaquinta and Pepe [1]. Moreover, these authors showed that the solution u is of class $H^2_q \cap C^{1,\alpha}(\Omega)$ for every $q \in [1, \infty)$ and any $\alpha \in (0, 1)$. Thus the set of coincidence $\mathfrak{I} = \{x \in \Omega \colon u(x) = \psi(x)\}$ is closed, and we have

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{in } \Omega \setminus \mathfrak{I} \quad \text{as well as}$$

$$(10) \qquad \int_\Omega (1 + |\nabla u|^2)^{-1/2} \langle \nabla u, \nabla(v - u) \rangle\, dx \geq 0 \quad \text{for all } v \in K_\psi.$$

Finally, using ideas of H. Lewy (see, for instance, the proof of Theorem 2 in Section 2.8), Kinderlehrer [6] proved that the curve of separation $\Gamma := \{(x^1, x^2, x^3) \colon x^3 = u(x) = \psi(x), x \in \partial\mathfrak{I}\}$ possesses a regular analytic parametrization provided that ψ is a strictly concave, analytic function.

Thin obstacle problems were treated by Lewy [6], Nitsche [19], and Giusti [2].

18. For solutions $X \in \mathcal{C}^*(\Gamma)$ of Plateau's problem satisfying a fixed three-point condition $X(w_j) = Q_j$, $j = 1, 2, 3$, $w_j \in \partial B$, $Q_j \in \Gamma$ and for $\Gamma \in C^{m,\mu}$, $m \geq 2$, $\mu \in (0, 1)$, there is a number $c(m, \mu)$, independent of X such that $\|X\|_{C^{m,\mu}(\overline{B}, \mathbb{R}^3)} \leq c(m, \mu)$. This a priori estimate is a quantitative version of Theorem 1 in Section 2.3, which also holds for $m = 1$ (cf. Jäger [3]).