Chapter 1

Minimal Surfaces with Free Boundaries

This chapter is centered on the proof of existence theorems for minimal surfaces with completely free boundaries. We approach the problem by applying the direct methods of the calculus of variations, thus establishing the existence of minimizers with a boundary on a given supporting surface S. However, this method does not yield the existence of stationary minimal surfaces which are not area minimizing. As certain kinds of supporting surfaces are not able to hold nontrivial minimizers, our method is restricted by serious topological limitations. For example, it does not furnish existence of nontrivial stationary minimal surfaces within a closed convex surface. It seems that the techniques of geometrical measure theory are best suited to handle this problem. Unfortunately they are beyond the scope of our lecture notes, but we shall at least present a survey of the pertinent results in Section 1.8 as well as an existence result for the particular case of S being a tetrahedron. There the reader will also find references to the literature.

In the following we shall describe Courant's method for proving the existence of a nontrivial and minimizing minimal surface whose boundary lies on a given closed supporting surface. This problem is more difficult than the Plateau problem or the semifree problem treated in Chapter 4 of Vol. 1 because an arbitrary minimal sequence will shrink to a single point. In order to exclude this phenomenon, we have to impose suitable topological conditions on the boundary values of admissible surfaces. For instance, one could assume that the boundary values are continuous curves on S which are contained in a prescribed homotopy class. This approach would, however, lead to a rather difficult problem. One would first have to prove that a suitable minimizing sequence tends to a limit with continuous boundary values, and then one would have to show that these boundary values lie in a prescribed homotopy class. Therefore we abandon this idea.

Instead we show in Section 1.1 how a kind of homotopy class can be set up for surfaces X which are of class $H_2^1(B, \mathbb{R}^3)$ and have their boundary values on S. We shall also prove by way of example that the problem of prescribed

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homotopy class need not have a solution. In Section 1.2 we set up the classes of admissible functions for which we can solve the minimum problem and in which we are able to find nondegenerate solutions.

The free boundary problem will be solved in Section 1.3; the supporting set S may look as bizarre as the one in Fig. 1 or as simple as the catenoid. The gist of our reasoning consists in an indirect argument showing that the limit of a suitable minimizing sequence satisfies the prescribed topological condition, and therefore it will be a nondegenerate solution of the minimum problem.

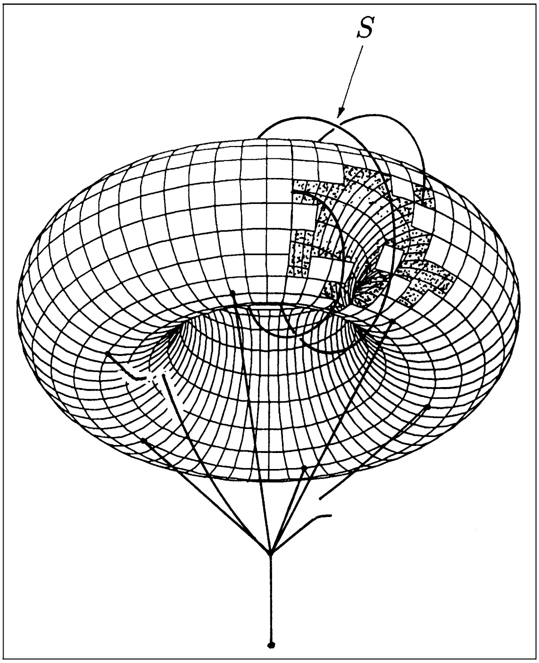

Fig. 1. A bizarre supporting set

The remaining part of the chapter will deal with additional properties of minimal surfaces with free boundaries.

In Section 1.4 we give a precise definition of a *stationary minimal surface* X whose free boundary lies on a given support surface S. Here we do not require X to be a minimizer. It will be investigated how the condition of being stationary is linked with the condition that X intersects S perpendicularly at its free trace Σ, provided Σ does not touch the boundary of S. This discussion is used in Section 1.5 to set up necessary conditions for the existence of stationary minimal surfaces with boundary on S. This will lead us to a class of non-existence results which explain, for example, why soap films in a funnel always run to its narrow end.

In Section 1.6 we prove the existence of three embedded stationary surfaces with their boundaries on a tetrahedron, following the discussion of B. Smyth. This is a case where the minimizing approach cannot be used.

Section 1.7 is concerned with stationary surfaces whose boundaries lie on a sphere. We shall prove Nitsche's result that flat disks are the only solutions to this problem that are of the type of the disk.

After a report on the existence of stationary minimal surfaces with boundaries on a convex surface (Section 1.8), in Section 1.9 we shall present some results concerning uniqueness and nonuniqueness of minimal surfaces with a free boundary on a given support surface. In particular, we construct a family of minimizing minimal surfaces with boundaries on a regular, real analytic surface of the topological type of a torus which are nonisometric to each other. Moreover, we discuss some finiteness results of Alt & Tomi for minimizers with boundaries on a real analytic supporting surface.

1.1 Surfaces of Class H_2^1 and Homotopy Classes of Their Boundary Curves. Nonsolvability of the Free Boundary Problem with Fixed Homotopy Type of the Boundary Traces

Let us fix some closed set S in $\mathbb{R}^3$. Then we want to define the class $\mathcal{C}(S)$ of surfaces $X \in H_2^1(B, \mathbb{R}^3)$ with boundary values $X|_{\partial B}$ on S. The parameter domain B will be chosen as the unit disk:

$$B = \{w = u + iv \colon |w| < 1\}.$$

In the following we shall usually pick an ACM-representative[1] for a given Sobolev mapping X. If we work with polar coordinates r, θ about the origin, i.e., $w = re^{i\theta}$, this means that we choose a representative $X(r, \theta)$ such that $X(r, \cdot)$ is absolutely continuous for almost all $r \in (0, 1)$, and that $X(\cdot, \theta)$ is absolutely continuous for almost all $\theta \in (0, 2\pi)$. Thus X is in particular a continuous function on almost all circles $C_r = \{w \in \mathbb{C} \colon |w| = r\}$.

Any function $X \in H_2^1(B, \mathbb{R}^3)$ possesses a *trace* (or *boundary values*) ξ on ∂B which is of class $L_2(C, \mathbb{R}^3), C := \partial B$, and we have both

(1) $$\lim_{r \to 1-0} X(r, \varphi) = \xi(\varphi) \quad \text{for almost all } \varphi \in [0, 2\pi]$$

and

(2) $$\lim_{r \to 1-0} \int_0^{2\pi} |X(r, \varphi) - \xi(\varphi)|^2 \, d\varphi = 0.$$

[1] ACM stands for absolutely continuous in the sense of Morrey; cf. Morrey [8], Lemma 3.1.1.

However, the trace $\Sigma = \{\xi(\varphi)\colon \varphi \in [0, 2\pi]\}$ of an arbitrary Sobolev function $X \in H_2^1(B, \mathbb{R}^3)$ will in general not be a continuous curve, whereas the curves

$$\Sigma_r := \{X(r, \varphi)\colon 0 \le \varphi \le 2\pi\}$$

are absolutely continuous for a.a. $r \in (0, 1)$. As we cannot formulate topological conditions for a possibly noncontinuous curve Σ, we shall use the continuous curves Σ_r as a substitute. In view of (1) and (2) we can expect that conditions on curves Σ_r close to Σ express conditions on Σ in an appropriate sense.

We begin, however, by defining the class $\mathcal{C}(S)$ of surfaces with boundary values on a supporting set S. *We assume once and for all that supporting sets S are closed, proper, and nonempty subsets of $\mathbb{R}^3$. However, if a boundary configuration contains other parts besides S, we allow S to be empty.*

Definition 1. *Let S be a supporting set in $\mathbb{R}^3$. Then we denote by $\mathcal{C}(S)$ the class of functions $X \in H_2^1(B, \mathbb{R}^3)$ whose L_2-trace $\xi := X|_C$ sends almost every $w \in C = \partial B$ into S.*

For any closed set S in $\mathbb{R}^3$, $S \ne \emptyset$, and for any number $\mu > 0$, we define the *tubular μ-neighbourhood $T_\mu = T_\mu(S)$* of S by

$$T_\mu(S) := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, S) < \mu\}. \tag{3}$$

Then we can formulate our first result on surfaces of class $\mathcal{C}(S)$ which will shed some light on their boundary behaviour.

Theorem 1. *Let S be a supporting set in $\mathbb{R}^3$, and suppose that X belongs to $\mathcal{C}(S)$. Then, for every $\mu > 0$ and every $\varepsilon > 0$, there is a subset $\mathcal{I} \subset (1 - \varepsilon, 1)$ of positive measure such that, for all $r \in \mathcal{I}$, the curve $\Sigma_r = \{X(r, \varphi)\colon 0 \le \varphi \le 2\pi\}$ is a closed continuous curve which is contained in the tubular neighbourhood $T_\mu(S)$ of S.*

Note that other curves Σ_r, $r \in (1 - \varepsilon, 1) \setminus \mathcal{I}$, may stay arbitrarily far from $T_\mu(S)$ as can be shown by simple examples; cf. Fig. 1.

We shall prove Theorem 1 in several steps.

Lemma 1. *For any closed set S in $\mathbb{R}^3$, the function $d_s := \operatorname{dist}(\cdot, S)$ is Lipschitz continuous on $\mathbb{R}^3$ with a Lipschitz constant less than or equal to one.*

Proof. For arbitrary points $P_1, P_2 \in \mathbb{R}^3$ there exist points $Q_1, Q_2 \in S$ such that

$$d_s(P_1) = |P_1 - Q_1| = \inf_{Q \in S} |P_1 - Q|,$$
$$d_s(P_2) = |P_2 - Q_2| = \inf_{Q \in S} |P_2 - Q|.$$

Therefore we obtain

$$d_S(P_2) \le |P_2 - Q_1|$$

and

$$d_S(P_2) - d_S(P_1) \le |P_2 - Q_1| - |P_1 - Q_1| \le |P_2 - P_1|,$$

and analogously

$$d_S(P_1) - d_S(P_2) \le |P_1 - P_2|.$$

Therefore we have

$$|d_S(P_1) - d_S(P_2)| \le |P_1 - P_2| \quad \text{for all } P_1, P_2 \in \mathbb{R}^3. \qquad \square$$

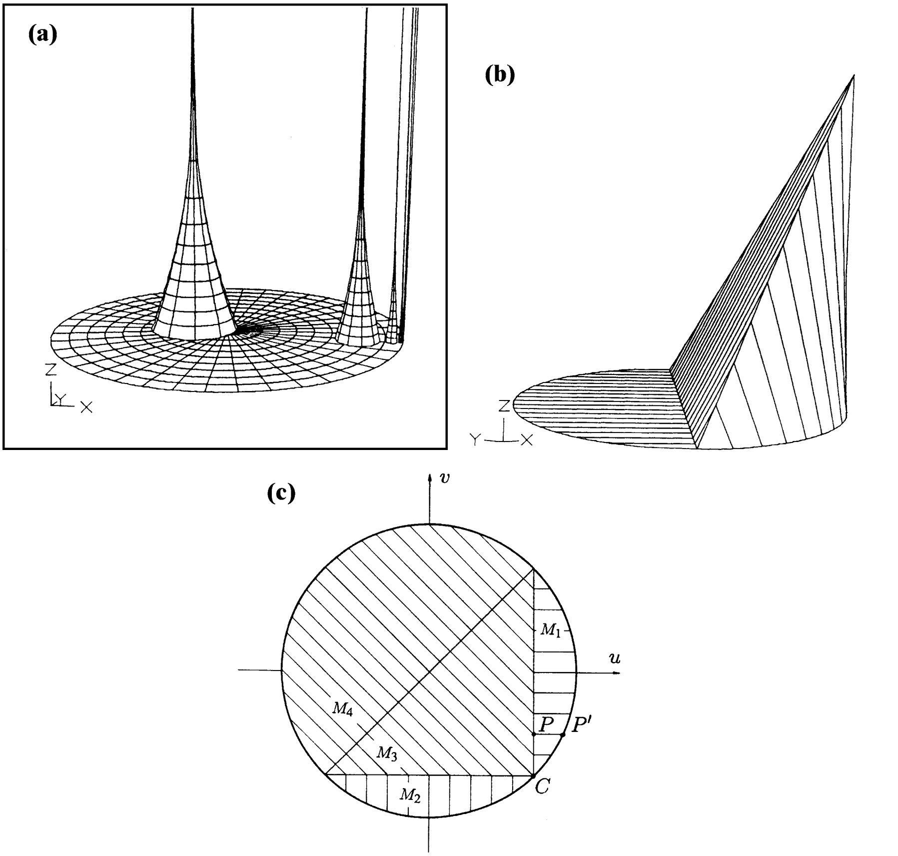

Fig. 1. (a) The graph of a bizarre function $f \in \mathring{H}_2^1(B)$ which has infinitely many peaks congruent to a part of the graph of $\log|\log|w||$. These peaks converge to the point $w = 1$ on ∂B. Given $\varepsilon > 0$ and $\delta > 0$, there is a set of values $r \in (1-\delta, 1)$ of positive measure such that the absolute values of f on C_r remain less that ε; see Lemma 4. This is a borderline case of the boundary behaviour of functions of class $\mathring{H}_p^1(B)$. For $p > 2$ they are continuous up to ∂B, and therefore their values on *all* circles sufficiently close to ∂B remain close to zero. For $p < 2$ there may be *no* such circle, as is shown by the function depicted in **(b)** and **(c)** which belongs to $\mathring{H}_p^1(B)$ for all $p \in (1, 2)$ and has a discontinuity at $C \in \partial B$

Lemma 2. *A function $X \in H_2^1(B, \mathbb{R}^3)$ belongs to $\mathcal{C}(S)$ if and only if the scalar function $d_S \circ X$ is an element of the space $\overset{\circ}{H}{}_2^1(B)$ of functions $f \in H_2^1(B)$ with generalized boundary values zero.*

Proof. Note that $X \in H_2^1(B, \mathbb{R}^3)$ implies $d_S \circ X \in H_2^1(B)$. Then the assertion follows from well-known properties of functions of class $\overset{\circ}{H}{}_2^1(B)$ (see Gilbarg and Trudinger [1]). □

Lemma 3. *Let X belong to $H_2^1(B, \mathbb{R}^N), N \geq 1$. Then, for any two numbers $\mu > 0$ and $\delta > 0$, there is an $\varepsilon > 0$ with the following property:*

If $\mathcal{I}' = [\theta_1, \theta_2]$ is an angular interval with $\theta_2 - \theta_1 = \delta$, then there exists a subset $\sigma \subset \mathcal{I}'$ of positive measure such that

$$|X(1, \theta) - X(r, \theta)| \leq \mu$$

holds for all $\theta \in \sigma$ and for all $r \in (1 - \varepsilon, 1)$. In fact, we can choose ε as

$$\varepsilon = \min\left\{\frac{1}{2}, \frac{1}{4}\frac{\mu^2\delta}{D(X)}\right\}. \tag{4}$$

Proof. From

$$r\int_{\theta_1}^{\theta_2}\int_r^1 |X_\rho(\rho, \theta)|^2\, d\rho\, d\theta \leq 2D(X)$$

we conclude that there is a subset $\sigma \subset [\theta_1, \theta_2]$ of positive measure such that

$$\int_r^1 |X_\rho(\rho, \theta)|^2\, d\rho \leq \frac{2}{r\delta} D(X)$$

holds for all $\theta \in \sigma$ and for $\delta = \theta_2 - \theta_1$. Moreover, we have

$$\begin{aligned} |X(1, \theta) - X(r, \theta)| &\leq \int_r^1 |X_\rho(\rho, \theta)|\, d\rho \\ &\leq \sqrt{1 - r}\left(\int_r^1 |X_\rho(\rho, \theta)|^2\, d\rho\right)^{1/2} \end{aligned}$$

for $\theta \in \sigma$ and $0 < r < 1$, whence

$$|X(1, \theta) - X(r, \theta)| \leq \{2r^{-1}(1 - r)\delta^{-1}D(X)\}^{1/2} \quad \text{for } \theta \in \sigma.$$

Choosing ε as in (4), the assertion follows at once. □

Lemma 4. *Let f belong to $\overset{\circ}{H}{}_2^1(B)$. Then, for any $\mu > 0$ and any $\varepsilon > 0$, the set $\mathcal{I} := \{r\colon 1 - \varepsilon < r < 1, |f|_{0,C_r} < \mu\}$ has positive measure.*

Proof. Suppose that the assertion were false. Then we would have $D(f) > 0$, and there were numbers $\varepsilon > 0$ and $\mu > 0$ such that

$$|f|_{0,C_r} \geq \mu \tag{5}$$

for almost all $r \in (1-\varepsilon, 1)$. Without loss of generality we can assume that

$$0 < \mu < \sqrt{D(f)}$$

holds true.

Because of (6) we infer that, for almost all $r \in (1-\varepsilon, 1)$, there is an angle $\theta(r)$ such that

$$|f(re^{i\theta(r)})| \geq \mu.$$

Furthermore we choose some $\delta \in (0,1)$ such that

$$\varepsilon' := \min\left\{\frac{1}{2}, \frac{\mu^2\delta}{16D(f)}\right\} \tag{6}$$

satisfies $0 < \varepsilon' < \varepsilon$. By Lemma 3, every angular interval $\mathcal{I}'$ of width δ contains an angle θ' such that $f(\cdot, \theta')$ is absolutely continuous and that

$$|f(re^{i\theta'})| < \tfrac{1}{2}\mu \quad \text{for all } r \in (1-\varepsilon', 1).$$

Conclusion: For almost all $r \in (1-\varepsilon', 1)$, there exist angles $\theta(r)$ and $\theta'(r)$ with $|\theta(r) - \theta'(r)| < \delta$ and

$$|f(re^{i\theta(r)})| \geq \mu, \quad |f(re^{i\theta'(r)})| \leq \frac{\mu}{2}.$$

Thus

$$\frac{\mu}{2} \leq \left| \int_{\theta(r)}^{\theta'(r)} |f_\theta(re^{i\theta})| \, d\theta \right|$$

and consequently

$$\frac{\mu^2}{4\delta} \leq \int_0^{2\pi} f_\theta^2(re^{i\theta}) \, d\theta.$$

Thus

$$\begin{aligned}\int_{\{1-\varepsilon'<|w|<1\}} |\nabla f|^2 \, du \, dv &\geq \int_{1-\varepsilon'}^1 \int_0^{2\pi} \frac{1}{r^2} f_\theta^2(re^{i\theta}) r \, d\theta \, dr \\ &\geq \int_{1-\varepsilon'}^1 \left(\int_0^{2\pi} f_\theta^2(re^{i\theta}) \, d\theta \right) dr \geq \frac{\varepsilon'\mu^2}{4\delta}.\end{aligned}$$

Because of (6), we have

$$\int_{\{1-\varepsilon'<|w|<1\}} |\nabla f|^2 \, du \, dv \geq \frac{\mu^4}{64D(f)}$$

for $0 < \delta \ll 1$, and $\varepsilon' \to 0$ as $\delta \to +0$. This is impossible for an H_2^1-function. □

Proof of Theorem 1. The assertion of Theorem 1 is now an immediate consequence of the Lemmata 1–4. □

Remark 1. The assertion of Lemma 4 holds for trivial reasons if $f \in \mathring{H}_p^1(B)$ and $p > 2$, because Sobolev's embedding theorem yields that $f \in C^0(\overline{B})$ and $f = 0$ on ∂B. The assertion turns out to be false if $p < 2$, as one can find examples of functions $f \in \mathring{H}_p^1(B), p < 2$, such that near ∂B the function $|f(w)|$ is bounded away from zero by an arbitrary constant (cf. Fig. 1).

Now we want to give a reasonable definition for a homotopy class of a boundary mapping $\xi(\theta) = X(1, \theta)$ of a surface X of class $\mathcal{C}(S)$ which is not necessarily continuous on B. To this end we consider the curves $\Sigma_r = \{X(r, \theta)\colon 0 \le \theta \le 2\pi\}$ for r close to one which are absolutely continuous and lie in a tubular neighbourhood T_μ of S. By Theorem 1, there exist sufficiently many of them: In fact, for any number $\varepsilon \in (0, 1)$ there is a set $\mathcal{I} \subset (1 - \varepsilon, 1)$ of positive measure such that, for every $r \in \mathcal{I}$, the mapping $X(r, \cdot)$ is absolutely continuous and $\Sigma_r \subset T_\mu$.

Now we can state the following result:

Theorem 2. *Let T_μ be the μ-neighbourhood of some closed set S in $\mathbb{R}^3$, and suppose that $X \in \mathcal{C}(S)$. Then for $\delta := \frac{1}{4}\pi\mu^2 > 0$, the following holds true:*

If $r_1, r_2 \in (0, 1)$ are two radii such that

(i) *the Dirichlet integral of X over the annulus*

$$\Omega(r_1, r_2) := \{w \in \mathbb{C}\colon r_1 < |w| < r_2\}$$

is at most δ;

(ii) *the curves $X|_{C_1}$ and $X|_{C_2}$ with $C_k := C_{r_k} = \{w\colon |w| = r_k\}$ are absolutely continuous, and their traces $\Sigma_k := X(C_k)$ are contained in $T_{\mu/2}$;*

(iii) *there is an angle θ such that the curve $X(r, \theta), r_1 \le r \le r_2$, connecting Σ_1 and Σ_2 is absolutely continuous and that its trace lies in $T_{\mu/2}$; then the curves $X|_{C_1}$ and $X|_{C_2}$ are homotopic in T_μ.*

Recall that two closed continuous curves $\gamma_1\colon C \to T_{\mu/2}$ and $\gamma_2\colon C \to T_{\mu/2}$ are *homotopic in* T_μ if there is a continuous map $H\colon C \times [0, 1] \to T_\mu$ such that $H(\cdot, 0) = \gamma_1$ and $H(\cdot, 1) = \gamma_2$. The mapping H is called a *homotopy*.

Furthermore, a closed curve $\gamma\colon C \to T_{\mu/2}$ is *contractible in* T_μ if it is homotopic in T_μ to a constant map or, equivalently, if it extends to a continuous map $\overline{B} \to T_\mu$.

Remark 2. Close to $C = \partial B$, the angle θ appearing in condition (iii) can be found by virtue of Lemma 3.

The *proof of Theorem 2* can be reduced to proving the following

Lemma 5. *Let T_μ be the μ-neighbourhood of some closed set S in $\mathbb{R}^3$, and set $\delta := \frac{1}{4}\pi\mu^2$. Suppose, moreover, that X is a mapping of class $H_2^1(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ whose boundary curve $X|_{\partial B}$ is contained in $T_{\mu/2}$ and which satisfies $D(X) < \delta(\mu)$. Then the curve $X|_{\partial B}$ is contractible in T_μ.*

In fact, let r_1 and r_2 be two radii as in Theorem 2, and let $\theta \in [0, 2\pi)$ be an angle as in (iii) of the theorem. Then we consider a conformal map τ of B onto the slit annulus

$$\{w = re^{i\varphi}: r_1 < r < r_2, \varphi \in [0, 2\pi), \varphi \neq \theta\}$$

and apply Lemma 5 to the surface $Z := X \circ \tau$, thus obtaining that $Z|_{\partial B}$ is contractible in T_μ. A straightforward reasoning now implies that the curves $X|_{C_1}$ and $X|_{C_2}$ are homotopic in T_μ.

Proof of Lemma 5. We begin by choosing a mapping $Y: \overline{B} \to \mathbb{R}^3$ which is harmonic in B, continuous on $\overline{B}$, of class $H_2^1(B, \mathbb{R}^3)$ and satisfies $Y - X \in \mathring{H}_2^1(B, \mathbb{R}^3)$ and $Y = X$ on ∂B. We know that $D(Y) \leq D(X)$. Since $X(\partial B) = Y(\partial B)$ is contained in $T_{\mu/2}$, there exists a strip $U = \{w: 1 - \varepsilon < |w| \leq 1\}$ about the boundary $C = \partial B$ such that $Y(U) \subset T_{\mu/2}$. Then we can find a regular, real analytic curve $Y|_{C_r}, r \in (1 - \varepsilon, 1)$, which is homotopic to $X|_C = Y|_C$ in $T_{\mu/2}$. Thereafter we can find a sequence $\{\Gamma_k\}$ of smooth closed Jordan curves Γ_k given by smooth topological mappings $\Phi_k: C \to \Gamma_k$ such that $|\Phi_k - \Phi|_{2,C} \to 0$ as $k \to \infty$ holds for the mapping $\Phi: C \to \mathbb{R}^3$ defined by $\Phi(e^{i\theta}) := Y(re^{i\theta})$.

Now let $Z(w) := Y(rw)$ and $Z_k(w)$ be the harmonic extensions to B of the boundary values Φ and Φ_k respectively, and let X_k be a solution of the variational problem $\mathcal{P}(\Gamma_k)$. Then the maximum principle implies $|Z_k - Z|_{0,\overline{B}} \to 0$ as $k \to \infty$ and, applying the estimate of Lemma 7 in Section 2.1 together with the Arzelà–Ascoli theorem, we also obtain $|Z_k - Z|_{1,\overline{B}} \to 0$ as $k \to \infty$. This implies

$$\lim_{k \to \infty} D(Z_k) = D(Z).$$

Consequently we have

$$A(X_k) = D(X_k) \leq D(Z_k) \to D(Z) = D_{B_r}(Y) \leq D(Y) \leq D(X).$$

By assumption, we have also

$$D(X) < \delta(\mu) = \tfrac{1}{4}\pi\mu^2,$$

whence

$$A(X_k) = D(X_k) < \pi(\mu/2)^2 \tag{7}$$

is satisfied for k sufficiently large.

If for one of these k the minimal surface X_k were not contained in T_μ, then there would exist some $w \in B$ such that $X_k(w) \notin T_\mu$. We choose a conformal selfmapping of B satisfying $\tau(0) = w$ and note that all the boundary values of $X_k \circ \tau$ lie outside the ball of radius $\mu/2$ centered at $X_k(w) = X_k(\tau(0))$. Then we infer from Vol. 1, Section 3.2, Proposition 2 that

$$A(X_k) \geq \pi(\mu/2)^2$$

which contradicts (7). Thus we have shown that

(8) $$X_k(\overline{B}) \subset T_\mu \quad \text{for all } k \gg 1.$$

Moreover, every minimal surface X_k furnishes a topological mapping of C onto Γ_k (see Vol. 1, Section 4.5, Theorem 3). Thus $X_k|_C$ furnishes a parameter representation of Γ_k equivalent to Φ_k, and we infer from (9) that Φ_k is contractible in T_μ for $k \gg 1$. Since $X|_C$ is homotopic in T_μ to all of the Γ_k with $k \gg 1$, we infer that $X|_C$ is contractible in T_μ. □

Recall now that $\mathcal{C}(S)$ has been defined as the class of all surfaces $X \in H_2^1(B, \mathbb{R}^3)$ having their boundary values $X|_C$ on a closed subset S in $\mathbb{R}^3$ (see Definition 1).

We now denote by $\tilde{\Pi}_1(S)$ the *set of all homotopy classes of closed paths* in S. (For details, we refer for instance to Schubert [1], or to Greenberg [1].)

Assumption (A). *Suppose that there is a number $\mu > 0$ such that the inclusion map $S \to T_\mu$ of the closed set S into its μ-neighbourhood T_μ induces a bijection from $\tilde{\Pi}_1(S)$ to $\tilde{\Pi}_1(T_\mu)$.*

For example, this assumption is fulfilled for sufficiently small $\mu > 0$ if S is a smooth compact submanifold of $\mathbb{R}^3$.

Let $\mu > 0$ be a number as in Assumption (A), and recall that the curves $X|_{C_r}$ are absolutely continuous for almost all $r \in (0, 1)$.

If $X \in \mathcal{C}(S)$, then there is a number $\varepsilon > 0$ such that any two curves $X|_{C_r}$ and $X|_{C_{r'}}$ contained in $T_{\mu/2}$ and with $r, r' \in (1-\varepsilon, 1)$ define the same homotopy class in $\tilde{\Pi}_1(T_\mu)$; this homotopy class will be viewed as *homotopy class of the boundary values* $X|_C$. It is denoted by $[X|_C]$ and will be called *the boundary class of a surface* $X \in \mathcal{C}(S)$. Because we have a bijection

$$\tilde{\Pi}_1(T_\mu) \leftrightarrow \tilde{\Pi}_1(S),$$

we can view the class $[X|_C]$ as an element of $\tilde{\Pi}_1(S)$. If the mapping $X\colon C \to \mathbb{R}^3$ is continuous, then $[X|_C]$ coincides with the usual homotopy class of $X|_C$.

Note that the definition of the homotopy class $[X|_{\partial B}]$ does not depend on the particular *ACM*-representative of X that we have chosen since any two of them coincide on almost all circles C_r.

Moreover, the definition $[X|_C]$ is even independent of μ in the following sense: Suppose that the inclusion maps $S \to T_\mu$ and $S \to T_{\mu'}$ induce two bijections $\tilde{\Pi}_1(S) \leftrightarrow \tilde{\Pi}_1(T_\mu)$ and $\tilde{\Pi}_1(S) \leftrightarrow \tilde{\Pi}_1(T_{\mu'})$. Then both constructions with respect to μ and μ' lead to the same class $[X|_C]$ in $\tilde{\Pi}_1(S)$.

Indeed, according to the definition we first have to choose an $\varepsilon > 0$ such that any two of the curves $X|_{C_r}, r \in (1-\varepsilon, 1)$, lying completely in $T_{\mu/2}$ (or in $T_{\mu'/2}$) are homotopic in T_μ (or in $T_{\mu'}$). This ε may be the same for μ and μ' because decreasing ε does not change the class $[X|_{\partial B}]$. If, say, $\mu' \le \mu$, then we find in $(1-\varepsilon, 1)$ a subset $\mathcal{J}'$ of positive measure or radii r such that the curves $X|_{c_r}, r \in \mathcal{J}'$, are completely contained in $T_{\mu'/2}$ and that any two of them are homotopic in $T_{\mu'}$. Therefore all these curves $X|_{C_r}$ define a homotopy class α' in $\tilde{\Pi}_1(T_{\mu'})$ which corresponds to the boundary class $[X|_{\partial B}]' \in \tilde{\Pi}_1(S)$ which is constructed by means of $T_{\mu'}$.

On the other hand, all curves $X|_{C_r}, r \in \mathfrak{J}'$, are contained in $T_{\mu/2} \supset T_{\mu'/2}$, and any two of them are homotopic in $T_\mu \supset T_{\mu'}$. Therefore all these curves $X|_{C_r}, r \in \mathfrak{J}'$, define a homotopy class $\alpha \in \tilde{\Pi}_1(T_\mu)$ which by the definition of ε corresponds to the boundary class $[X|_{\partial B}] \in \tilde{\Pi}_1(S)$ defined by means of T_μ. Since the inclusion $T_{\mu'} \to T_\mu$ induces a bijection $\tilde{\Pi}_1(T_{\mu'}) \to \tilde{\Pi}_1(T_\mu)$ which maps α' to α, the boundary classes $[X|_{\partial B}]$ and $[X|_{\partial B}]'$ and identical. □

Collecting our results and inspecting Chapter 4 of Vol. 1, we obtain the following

Theorem 3 (Natural boundary classes). *Let S be a subset of $\mathbb{R}^3$ such that for some $\mu > 0$ the inclusion $S \to T_\mu$ induces a bijection $\tilde{\Pi}_1(S) \to \tilde{\Pi}_1(T_\mu)$ between the corresponding sets $\tilde{\Pi}_1$ of homotopy classes of closed paths in S and T_μ respectively.*

(i) *Then for every surface $X \in \mathcal{C}(S)$ a boundary homotopy class $[X|_{\partial B}] \in \tilde{\Pi}_1(S)$ is defined in a natural way.*

(ii) *If σ is a closed curve in S which is not contractible in S and if $[\sigma] \in \tilde{\Pi}_1(S)$ denotes its homotopy class, then every minimizer of the Dirichlet integral $D(X) = \frac{1}{2}\int_B |\nabla X|^2\, du\, dv$ in the class*

$$\mathcal{C}(\sigma, S) := \{X \in \mathcal{C}(S) \colon [X|_{\partial B}] = [\sigma]\} \tag{9}$$

is a minimal surface.

Let us denote the minimum problem

$$D(X) \to \min \quad \text{in } \mathcal{C}(\sigma, S) \tag{10}$$

by $\mathcal{P}(\sigma, S)$.

In general one encounters serious difficulties if one tries to solve the problem $\mathcal{P}(\sigma, S)$. For instance, the classes $\mathcal{C}(\sigma, S)$ are not necessarily closed with respect to weak convergence in H_2^1; yet this fact was crucial for the existence proof carried out in Section 4.6 of Vol. 1.

All basic difficulties of this problem can already be seen in the comparatively simple case that we shall consider next. The reader who is not interested in the details of the following discussion may very well skip it since it is not anymore needed in the later sections.

Let us choose a *torus* T in $\mathbb{R}^3$ as the prescribed supporting surface, and consider the corresponding variational problem

$$\mathcal{P}(\sigma, T): \qquad D(X) \to \min \ \text{ in } \ \mathcal{C}(\sigma, T).$$

To be precise, let T be the torus in $\mathbb{R}^3$ which is obtained by revolving the circle

$$\{(x, y, z) : y = 0, (x - R)^2 + z^2 = r^2\}, \quad 0 < r < R,$$

about the z-axis (see Fig. 2). Denote by $\sigma_1, \sigma_2 \colon [0, 2\pi] \to T$ the two circles

$$\sigma_1(t) = (R - r\cos t, 0, -r\sin t)$$

and

$$\sigma_2(t) = ((R - r)\cos t, (R - r)\sin t, 0).$$

Finally let $P = \sigma_1(0) = \sigma_2(0) = (R - r, 0, 0)$ be the base point of T.

Note that in this case the assumption made in the construction of the boundary classes $[X|_{\partial B}]$ of a surface $X \in \mathcal{C}(T)$, namely that the inclusion map $T \to T_\mu$ induces a bijection $\tilde{\Pi}_1(T) \leftrightarrow \tilde{\Pi}_1(T_\mu)$, is satisfied for all sufficiently small μ since for these μ the above inclusion $T \to T_\mu$ is a homotopy equivalence.

In general the set $\tilde{\Pi}_1(M)$ of all equivalence classes of (freely) homotopic closed curves in a topological space M is different from its fundamental group $\Pi_1(M, *)$; but if Π_1 is Abelian and if M is connected, then the canonical map $\Pi_1(M, *) \to \tilde{\Pi}_1(M), [\sigma] \to [\sigma]$, is indeed a bijection (cf. Schubert [1]).

The fundamental group of the torus T is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and it is freely generated by $[\sigma_1]$ and $[\sigma_2]$. Therefore, in the case of the torus, the class $\mathcal{C}(T)$ of all H_2^1-surfaces with boundary values on T is the disjoint union of the classes $\mathcal{C}^{k,l}, k, l \in \mathbb{Z}$, of surfaces $X \in \mathcal{C}(T)$ whose boundary class $[X|_{\partial B}]$ can be represented by the closed path $\sigma_1^k \cdot \sigma_2^l$. (First k-times along σ_1, then l times along σ_2, negative powers denote reversal of orientation.)

Now we can state our *nonexistence result.*

Theorem 4. *Let T be the torus defined before.*

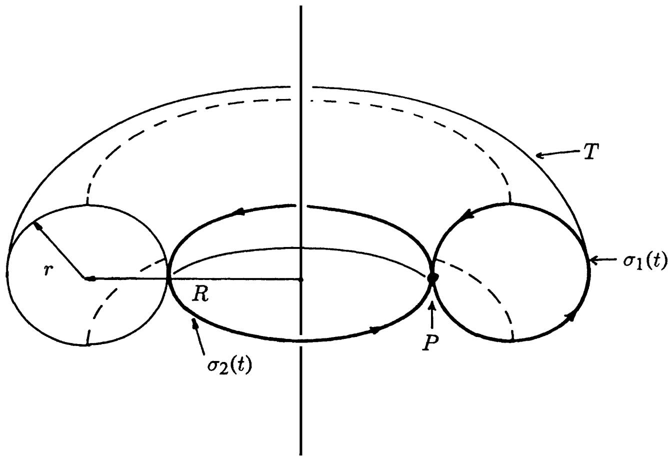

Fig. 2. The points and curves on a torus T used in the study of minimizing sequences for the Dirichlet integral of surfaces with free boundaries on T whose boundary curves have a prescribed homotopy class

(i) *For all* $k, l \in \mathbb{Z}$ *the numbers* $d_{k,l} := \inf\{D(X)\colon X \in \mathcal{C}^{k,l}\}$ *are given by*

$$d_{k,l} = \pi\{|k|r^2 + |l|(R-r)^2\}.$$

(ii) *The variational problem*

$$D(X) \to \min \quad \text{in } \mathcal{C}^{\,k,l}$$

has a solution if and only if $k = 0$ *or* $l = 0$.

For the proof of Theorem 4 we shall need the following

Lemma 6 (A formula for the oriented area). *Assume that the boundary values of a mapping* $X = (X^1, X^2) \in H_2^1(B, \mathbb{R}^2)$ *are contained in* $\mathbb{R}^2 \setminus B_\rho(w_0)$. *Then the boundary class* $[X|_{\partial B}] \in \tilde{\Pi}_1(\mathbb{R}^2 \setminus B_\rho(w_0))$ *is well defined, and it is characterized by the winding number* $U([X|_{\partial B}], w_0)$. *If* $\Omega := \{w \in B\colon X(w) \in B_\rho(w_0)\}$, *then we have for the oriented area*

$$A^0_\Omega(X) := \int_\Omega X_u \wedge X_v \, du\, dv$$

of the mapping X *the formula*

$$A^0_\Omega(X) := \int_\Omega \{X^1_v X^2_v - X^1_v X^2_u\}\, du\, dv = \pi\rho^2 U([X|_{\partial B}], w_0).$$

Proof of Lemma 6. Approximating H_2^1-mappings $Z \in H_2^1(B, \mathbb{R}^2)$ by smooth mappings, we obtain the following two formulas that are well known for smooth maps:

(i) For almost all $R \in (0,1)$, the oriented surface area of Z is given by

$$A^0_{B_R}(Z) = \frac{1}{2}\int_0^{2\pi} \{Z^1 Z^2_\theta - Z^2 Z^1_\theta\}\, d\theta.$$

(ii) If Z is absolutely continuous on ∂B_R, then

$$U(Z|_{\partial B_R}, 0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{Z^1 Z^2_\theta - Z^2 Z^1_\theta}{|Z|^2}\, d\theta$$

unless $Z = 0$ somewhere on ∂B_R. Of course, $0 < R < 1$ and $Z = Z(Re^{i\theta})$, etc.

Let us now prove the lemma. We may assume without loss of generality that $w_0 = 0$. Moreover, for $0 < \varepsilon < \rho$, let $\pi_\varepsilon\colon \mathbb{R}^2 \to \overline{B}_{\rho-\varepsilon}(0)$ denote the radial projection

$$Z \mapsto \begin{cases} Z & \text{if } |Z| < \rho - \varepsilon, \\ \frac{Z}{|Z|}(\rho - \varepsilon) & \text{otherwise,} \end{cases}$$

and set $Y^\varepsilon := \pi_\varepsilon \circ X$, which is again of class $H_2^1(B, \mathbb{R}^2)$ since π_ε is Lipschitz continuous. The boundary values of Y^ε are contained in $\mathbb{R}^2 \setminus B_{\rho-\varepsilon}(0)$, and we have

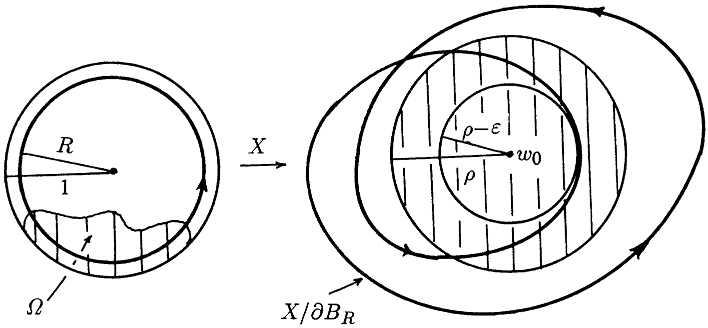

Fig. 3. The area functional $A(X)$ of a map $X\colon B \to \mathbb{R}^2$, whose boundary curve winds around a disk in $\mathbb{R}^2$, can be calculated from the radius of the disk and the winding number of the boundary curve, cf. Lemma 6

$$U([X|_{\partial B}], 0) = U([Y^\varepsilon|_{\partial B}], 0).$$

Now we choose R so close to 1 that $X|_{\partial B_R} \subset \mathbb{R}^2 \setminus B_{\rho-\varepsilon}(0)$ represents the boundary class $[X|_{\partial B}]$ and that the integration-by-parts-formula (i) holds true. Then we conclude that

$$\begin{aligned} U([X|_{\partial B}], 0) &= U(X|_{\partial B_R}, 0) = U(Y^\varepsilon|_{\partial B_R}, 0) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{Y^\varepsilon \wedge Y_\theta^\varepsilon}{|Y^\varepsilon|^2}\bigg|_{r=R} d\theta = \frac{1}{\pi(\rho-\varepsilon)^2}\frac{1}{2}\int_0^{2\pi} Y^\varepsilon \wedge Y_\theta^\varepsilon \, d\theta \\ &= \frac{1}{\pi(\rho-\varepsilon)^2} A^0_{B_R}(Y^\varepsilon). \end{aligned}$$

Now, on the one hand, $Y_r^\varepsilon \wedge Y_\theta^\varepsilon = 0$ almost everywhere on $\Omega_\varepsilon = \{|X| \geq \rho - \varepsilon\}$ since both Y_r^ε and Y_θ^ε are tangential to $\partial B_{\rho-\varepsilon}(0)$. On the other hand, we have $Y_r^\varepsilon \wedge Y_\theta^\varepsilon = X_r \wedge X_\theta$ almost everywhere on $\Omega'_\varepsilon = \{|X| < \rho - \varepsilon\}$. Therefore

$$A^0_{B_R}(Y^\varepsilon) = A^0_{\Omega'_\varepsilon \cap B_R}(X).$$

Thus at last, if ε decreases to zero, the radii R chosen above tend to one whence

$$A^0_{B_R}(Y^\varepsilon) \to A^0_\Omega(X),$$

and the lemma is proved. □

Now we turn to the

Proof of Theorem 4. Let $\pi_1\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection onto the x, y-plane given by

$$(x, y, z) \to (x, y),$$

and denote by $\pi_2\colon \mathbb{R}^3 \to \mathbb{R}^2$ the projection mapping

$$(x, y, z) \to (\rho, z) \quad \text{with } \rho = \sqrt{x^2 + y^2}$$

which maps (x, y, z) onto the point $(\rho, z) \in \mathbb{R}^2$ defined by the two cylinder coordinates ρ and z. Note that π_2 is Lipschitz continuous and, for $\rho \neq 0$, even real analytic.

Then, for any $X \in H_2^1(\Omega, \mathbb{R}^3)$, we have the following inequalities:[2]

(I) $D(\pi_1 \circ X) \leq D(X)$, and the equality sign holds if and only if $\nabla z(w) = 0$ a.e. on Ω, where $z(w)$ is the third component of $X(w)$.

(II) $D(\pi_2 \circ X) \leq D(X)$, and the equality sign holds if and only if $\nabla \varphi(w) = 0$ a.e. in Ω, where $\varphi(w) := \arctan \frac{y(w)}{x(w)}$ is the angle belonging to the cylinder coordinates ρ, φ, z. For the assertion of (II) to hold we have to assume that $X(\Omega) \Subset \mathbb{R}^3 \setminus \mathcal{H}$ where $\mathcal{H}$ is some halfplane in $\mathbb{R}^3$ having the z-axis as its boundary.

Now, given $X = (x, y, z) \in \mathcal{C}^{k,l}$, let us consider the sets

$$\Omega_1 := \{w \in B \colon x^2(w) + y^2(w) < (R - r)^2\}$$

and

$$\Omega_2 := \{w \in B \colon |\pi_2(X(w)) - (R, 0)| < r\}$$

which are the pre-images of the cylinder $\{0 \leq \rho < R - r\}$ and of the open solid torus T, respectively. The sets Ω_1 and Ω_2 are disjoint.

From (I), (II) and Lemma 6 we infer

$$\begin{aligned} D(X) &\geq D_{\Omega_1}(X) + D_{\Omega_2}(X) \geq D_{\Omega_1}(\pi_1 \circ X) + D_{\Omega_2}(\pi_2 \circ X) \\ &\geq |A^0_{\Omega_1}(\pi_1 \circ X)| + |A^0_{\Omega_2}(\pi_2 \circ X)| \geq \pi |l| (R - r)^2 + \pi |k| r^2, \end{aligned}$$

that is,

$$\text{(III)} \qquad \pi(|l|(R - r)^2 + |k| r^2) \leq D(X).$$

In order to complete the proof of the first part of the theorem, we construct a minimizing sequence as follows. For $0 < \rho \ll 1$, we introduce the set

$$\Omega_\rho := B_1(0) \cup B_\rho(1) \cup B_1(2).$$

As Ω_ρ is conformally equivalent to the unit disk B (see Fig. 4), we can choose Ω_ρ as parameter domain. For $k, l \geq 0$, we define

$$X_{\rho(w)} := \begin{cases} ((R - r)\operatorname{Re} w^l, (R - r)\operatorname{Im} w^l, 0) & \text{if } w \in \overline{B}_1(0) \setminus B_\rho(1), \\ (R - r\operatorname{Re}(2 - w)^k, 0, -r\operatorname{Im}(2 - w)^k) & \text{if } w \in \overline{B}_2(0) \setminus B_\rho(1). \end{cases}$$

If $k < 0$, we replace in the definition of X_ρ the variable $w \in \overline{B}_1(0) \setminus B_\rho(1)$ by $\overline{w}$, and for $l < 0$ we substitute $w \in \overline{B}_2(0) \setminus B_\rho(1)$ by $\overline{w}$.

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[2] The proof of the second fact is not totally trivial. It can be derived by choosing an ACM-representation of $\pi_2 \circ X$ in conjunction with Fubini's theorem.

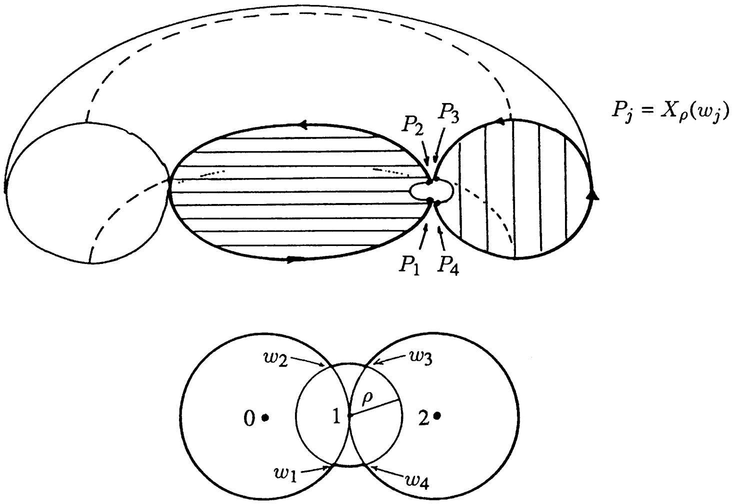


Fig. 4. Construction of a minimizing sequence for the Dirichlet integral for surfaces with free boundaries on T and a boundary class homotopic to σ_1 followed by σ_2

Now let $w_1, \dots, w_4$ be the four vertices in Ω_ρ. Then we connect every two of the points $P_j := X(w_j)$ by geodesic lines on the torus T such that the curve $X_\rho|_{\partial\Omega_\rho}$ is homotopic to $\sigma_1^k \cdot \sigma_2^l$. These geodesics are parametrized in proportion to the arc length by means of the boundary pieces of $\partial\Omega_\rho$ between w_1 and w_4, w_2 and w_3.

Having thus defined $X|_{\partial B_\rho(1)}$, one completes the construction by filling in a harmonic surface in $B_\rho(1)$ with the boundary values X on $\partial B_\rho(1)$. Since $D_B(w^n) = \pi n$ and $D_{B_\rho(1)}(X_\rho)$ tends to zero with ρ, we have found a minimizing sequence.

In order to show part (ii) of the theorem, we consider a minimizer X in $\mathcal{C}^{k,l}$. Then X is harmonic in B and equality holds in (III). Our initial remarks (I) and (II) imply that $X(B)$ lies in a plane which either contains the z-axis (in which case $l = 0$) or is orthogonal to the z-axis (implying $k = 0$). Finally, minimizers in $\mathcal{C}^{k,0}$ and $\mathcal{C}^{0,l}$ can be constructed again using powers of w. □

1.2 Classes of Admissible Functions. Linking Condition

If we enlarge the class of admissible functions in a suitable way, the minimum problem becomes solvable. The difficulty consists in finding a proper class $\tilde{\mathcal{C}}$ of surfaces between $\mathcal{C}(\sigma, S)$ and $\mathcal{C}(S)$ such that the Dirichlet integral has a nondegenerate minimizer in $\tilde{\mathcal{C}}$. In this section we want to set up several of such classes $\tilde{\mathcal{C}}$ which serve this purpose.

To this end we shall assume throughout that S is a closed, proper, nonempty subset of $\mathbb{R}^3$ satisfying *Assumption* (A) of Section 1.1: There is a $\mu > 0$ such that the inclusion $S \to T_\mu$ induces a bijection $\tilde{\Pi}_1(S) \leftrightarrow \tilde{\Pi}_1(T_\mu)$. Then we can define

$$\mathcal{C}^+(S) := \bigcup_{[\sigma]\neq[\text{const}]} \mathcal{C}(\sigma, S), \tag{1}$$

where the union is to be taken over all closed curves σ in S which are not homotopic in S to a constant map. In other words, $\mathcal{C}^+(S)$ consists of all those surfaces $X \in \mathcal{C}(S)$ whose boundary class $[X|_{\partial B}]$ is not represented by a constant map.

Clearly, the position of the competing surfaces $X \in \mathcal{C}^+(S)$ is not particularly restricted. Therefore the minimizer in $\mathcal{C}^+(S)$ will always fill the smallest hole in S.

In order to specify the position of the boundary values of the competing surfaces more precisely, we choose some polygon Π (that is, a piecewise linear image of ∂B) which does not meet the tubular neighbourhood T_μ of S.

Then we introduce the variational class $\mathcal{C}(\Pi, S)$ of all surfaces $X \in \mathcal{C}(S)$ whose boundary class $[X|_{\partial B}]$ is linked with the polygon Π, that is, whose *linking number* $\mathcal{L}([X|_{\partial B}], \Pi)$ is nonzero:

$$\mathcal{C}(\Pi, S) := \{X \in \mathcal{C}(S) \colon \mathcal{L}([X|_{\partial B}], \Pi) \neq 0\}.$$

The classes $\mathcal{C}^+(S)$ and $\mathcal{C}(\Pi, S)$ will be the two sets on which we want to minimize the Dirichlet integral in order to obtain nondegenerate minimal surfaces with a free boundary on S. The minimizing procedures will be carried out in the next section.

For the convenience of the reader we shall in the following sketch the main features of the linking number. For proofs and further details we refer to the treatise of Alexandroff and Hopf [1].

Definition and Properties of the Linking Number

(I) First we define the intersection number of two oriented simplices $e^p = (a_0, \dots, a_p)$ and $f^q = (b_0, \dots, b_q)$ for two particular cases.

(α) If the corresponding geometric simplices furnished by the convex hulls of $\{a_0, \dots, a_p\}$ and $\{b_0, \dots, b_q\}$ are disjoint, then we define the intersection number $\emptyset(e^p, f^q)$ to be zero.

(β) If $p + q = 3$, and if the intersection of the corresponding geometrical simplices is neither empty nor does it contain any vertex of e^p, f^q, we define the intersection number $\emptyset(e^p, f^q)$ to be one if the ordered base $(a_1 - a_0, \dots, a_p - a_0, b_1 - a_0, \dots, b_q - a_0)$ has the same orientation as the standard simplex $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ of $\mathbb{R}^3$, and we set $\emptyset(e^p, f^q) = -1$ if the orientations are different.

(II) Secondly we define the linking number of two disjoint closed polygons Π_1 and Π_2.

Assume that Π_1 and Π_2 have r (resp. s) corners $P_{r+1} = P_1, \dots, P_r$ and $Q_{s+1} = Q_1, \dots, Q_s$, and choose a point $P \in \mathbb{R}^3$ such that any pair of simplices $e_j := (P, P_j, P_{j+1})$ and $f_k = (Q_k, Q_{k+1}), j = 1, \dots, r; k = 1, \dots, s$, satisfies one of the above conditions $(\alpha), (\beta)$ in (I). Then we define

$$\mathfrak{L}(\Pi_1, \Pi_2) := \sum_{j=1}^{r} \sum_{k=1}^{s} \emptyset(e_j, f_k)$$

as the *linking number of the two polygons* Π_1 *and* Π_2*.*

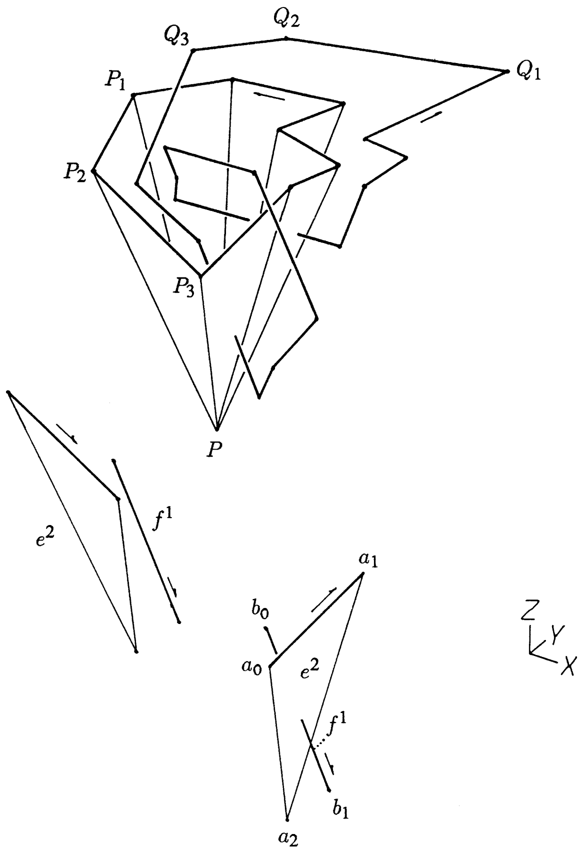


Fig. 1. The definition of the linking number of two closed polygons $P_1, P_2, \dots$ and $Q_1, Q_2, \dots$ is reduced to the intersection numbers of the faces (2-dimensional simplices) of a cone erected over the first polygon with the line segments (1-dimensional simplices) of the second. The intersection number is 0 if the simplices are disjoint, and +1 or −1 otherwise depending on their orientations. The resulting linking number for the polygons shown here is −2

(III) Finally, if c_1 and c_2 are two closed curves $\partial B \to \mathbb{R}^3$ with disjoint traces $c_i(\partial B)$, say, $\operatorname{dist}(c_1(\partial B), c_2(\partial B)) = \delta > 0$, then we choose two closed polygons Π_1 and Π_2 such that

$$|c_1 - \Pi_1|_{0,\partial B}, \quad |c_2 - \Pi_2|_{0,\partial B} < \frac{\delta}{2},$$

and define the *linking number of* c_1 *and* c_2 as

$$\mathfrak{L}(c_1, c_2) := \mathfrak{L}(\Pi_1, \Pi_2).$$

(IV) Some of its *properties* are:

(i) The definition of the linking number of two disjoint closed curves is independent of all choices made above (see Alexandroff and Hopf [1], p. 423).

(ii) *Deformation invariance.* If $h_1(t,\theta)$ and $h_2(t,\theta)\colon [0,1] \times \partial B \to \mathbb{R}^3$ are two homotopies of closed curves such that for every $t \in [0,1]$ the supports of the deformed curves are disjoint, then

$$\mathfrak{L}(h_1(0,\cdot), h_2(0,\cdot)) = \mathfrak{L}(h_1(1,\cdot), h_2(1,\cdot))$$

(see Alexandroff and Hopf [1], p. 424).

(iii) *Additivity of linking numbers.* If c_1, c_2 and c are three closed curves such that c_1 and c_2 have the same end points and that

$$c_i(\partial B) \cap c(\partial B) = \emptyset \quad \text{for } i = 1, 2,$$

then we have for the composite curve $c_1 \cdot c_2$

$$\mathfrak{L}(c_1 \cdot c_2, c) = \mathfrak{L}(c_1, c) + \mathfrak{L}(c_2, c).$$

This follows immediately from the construction (see Alexandroff and Hopf [1], p. 418).

(V) In view of the homotopy invariance of the linking numbers, the linking number of a boundary class $[X|_{\partial B}]$ with a polygon Π at a distance greater than μ from S is well defined:

$$\mathfrak{L}([X|_{\partial B}], \Pi) := \mathfrak{L}(X|_{C_R}, \Pi),$$

where $X|_{C_R}, R \in (0,1)$, is any curve in $T_{\mu/2}$ which represents the boundary class $[X|_{\partial B}]$.

1.3 Existence of Minimizers for the Free Boundary Problem

Let us now treat some free boundary problems for minimal surfaces with a prescribed supporting surface. We shall minimize the Dirichlet integral

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv$$

both in $\mathcal{C}^+(S)$ and $\mathcal{C}(\Pi, S)$, the classes introduced in the previous sections. We shall describe some geometric conditions on S such that the two variational problems

$$\mathcal{P}(\Pi, S): \qquad D(X) \to \min \quad \text{in } \mathcal{C}(\Pi, S)$$

and

$$\mathcal{P}^+(S): \qquad D(X) \to \min \quad \text{in } \mathcal{C}^+(S)$$

are solvable.

By definition of the classes $\mathcal{C}^+(S)$ and $\mathcal{C}(\Pi, S)$, every solution of $\mathcal{P}(\Pi, S)$ and $\mathcal{P}^+(S)$ is nondegenerate.

Theorem 1. *Let S be a supporting set in $\mathbb{R}^3$ satisfying Assumption (A) of Section 1.1, i.e. there is some $\mu > 0$ such that the inclusion map $S \to T_\mu$ of S into its μ-neighbourhood T_μ induces a bijection from $\tilde{\Pi}_1(S)$ to $\tilde{\Pi}_1(T_\mu)$. Then we have:*

(i) *If there is a closed polygon Π in $\mathbb{R}^3$ which does not meet T_μ and for which $\mathcal{C}(\Pi, S)$ is nonempty, then there exists a solution of $\mathcal{P}(\Pi, S)$.*

(ii) *If S is compact and $\mathcal{C}^+(S)$ is nonempty, then there is a solution of $\mathcal{P}^+(S)$.*

(iii) *Any solution X of $\mathcal{P}(\Pi, S)$ or of $\mathcal{P}^+(S)$ is a minimal surface. That is, X is a nonconstant mapping of class $C^2(B, \mathbb{R}^3)$ and satisfies the equations*

$$\Delta X = 0, \tag{1}$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{2}$$

in B.

It is of great importance to investigate the boundary behaviour of solutions of $\mathcal{P}(\Pi, S)$ and $\mathcal{P}^+(S)$. If X is a solution of one of these problems that is smooth up to its boundary (say, $X \in C^1(\overline{B}, \mathbb{R}^3)$), and if S is a smooth surface with an empty boundary ∂S, then we shall prove in the next section that X meets S perpendicularly along its free trace $\Sigma = X(\partial B)$ on the supporting surface S. However, if ∂S is nonempty, then it may very well happen that Σ touches ∂S (this phenomenon is studied in Chapter 2, and in the Chapters 1, 2 of Vol. 3); then one cannot anymore expect that X meets S perpendicularly everywhere along Σ. In fact, a right angle between X and S is generally formed only at those parts of Σ which do not coincide with ∂S.

Moreover, we have to answer the question as to whether a solution of $\mathcal{P}(\Pi, S)$ or of $\mathcal{P}^+(S)$ is smooth on the closure $\overline{B}$ of its parameter domain B, so that we can apply the succeeding results of Section 1.4. A detailed discussion of this and related problems is given in Chapter 2. There and in

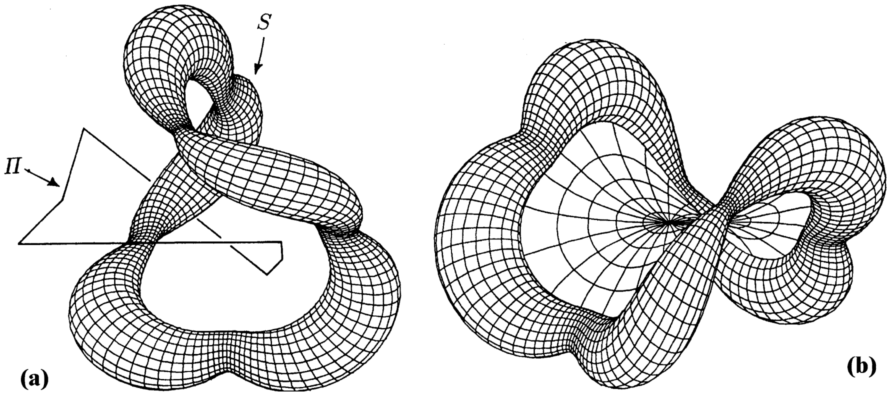


Fig. 1. **(a)** A closed smooth surface S linked with a polygon Π for which the class of surfaces $\mathcal{C}(\Pi, S)$ is non-empty. **(b)** A solution of the corresponding free boundary value problem $\mathcal{P}(\Pi, S)$

Chapter 3, we also investigate how a solution X and its trace curve Σ behave in the neighbourhood of a boundary branch point.

Let us now turn to the proof of Theorem 1. We need the notion of the *greatest distance* $g(A, B)$ of a closed set A of $\mathbb{R}^3$ to another closed set B of $\mathbb{R}^3$ which is defined by

$$g(A, B) := \sup\{\text{dist}(x, B) \colon x \in A\}. \tag{3}$$

Clearly, we have $0 \le g(A, B) \le \infty$.

Lemma 1. *Let S_k and S be closed sets in $\mathbb{R}^3$ such that $\lim_{k\to\infty} g(S_k, S) = 0$, and suppose that $\{X_k\}$ is a sequence of surfaces $X_k \in \mathcal{C}(S_k)$ which tends weakly in $H_2^1(B, \mathbb{R}^3)$ to some surface X. Then X is of class $\mathcal{C}(S)$.*

Proof. By passing to a suitable subsequence of $\{X_k\}$ and renumbering, we can assume that the $L_2(\partial B, \mathbb{R}^3)$-boundary values converge pointwise almost everywhere on ∂B to $X|_{\partial B}$ (cf. Morrey [8], Theorem 3.4.5). Then we obtain

$$\text{dist}(X(1, \theta), S) \le |X(1, \theta) - X_k(1, \theta)| + g(S_k, S) \to 0$$

as $k \to \infty$, for almost all $\theta \in [0, 2\pi]$. □

Proof of Theorem 1. (i) Suppose that X is a surface of class $\mathcal{C}(S')$, where S' is a closed set with $g(S', S) < \mu/4$. Because of Assumption (A), we can define a boundary class $[X|_{\partial B}]$ which can be viewed as element of $\tilde{\Pi}_1(S)$.

Definition 1. *A sequence of surfaces $X_k \in H_2^1(B, \mathbb{R}^3)$ is said to be a* generalized admissible sequence *for the problem $\mathcal{P}(\Pi, S)$ if there is a sequence of closed sets $S_k \subset \mathbb{R}^3$ such that $\lim_{k\to\infty} g(S_k, S) = 0$ and $X_k \in \mathcal{C}(\Pi, S_k)$, $k \in \mathbb{N}$, holds true.*

We set

$$e := \inf\{D(X)\colon X \in \mathcal{C}(\Pi, S)\} \tag{4}$$

and

$$e^* := \inf\Big\{\liminf_{k\to\infty} D(X_k)\colon \{X_k\} \text{ is a generalized admissible sequence for } \mathcal{P}(\Pi,S)\Big\}. \tag{5}$$

Evidently we have

$$e^* \le e. \tag{6}$$

Now we pick a sequence $\{\mathcal{S}^l\}$ of generalized admissible sequences $\mathcal{S}^l = \{Z_k^l\}_{k\in\mathbb{N}}$ for $\mathcal{P}(\Pi, S)$ such that

$$\lim_{l\to\infty} \liminf_{k\to\infty} D(Z_k^l) = e^*.$$

From the sequences $\mathcal{S}^l$ we can extract a sequence $\mathcal{S} = \{Z_k\}$ of surfaces Z_k which is a generalized admissible sequence for $\mathcal{P}(\Pi, S)$ and satisfies

$$\lim_{k\to\infty} D(Z_k) = e^*.$$

Definition 2. *Such a sequence $\mathcal{S}$ of surfaces $Z_k \in \mathcal{C}(\Pi, S_k)$ is said to be a* generalized minimizing sequence *for the minimum problem $\mathcal{P}(\Pi, S)$.*

Next we choose radii $\rho_k \in (0,1)$ with $\rho_k \to 1$ having the following properties on the circles $C_k := C_{\rho_k}$:

(α) The curve $Z_k|_{C_k}$ is absolutely continuous, $Z_k(C_k)$ lies in $T_{\mu/2}$ and is linked with the polygon Π, i.e. $\mathcal{L}(Z_k|_{C_k}, \Pi) \ne 0$.

(β) The sequence of surfaces $Y_k(w) := Z_k(\rho_k w), w \in \overline{B}$, with boundary values on $S_k := Z_k(C_k)$ is a generalized minimizing sequence for $\mathcal{P}(\Pi, S)$. Thus we have in particular

$$\lim_{k\to\infty} D(Y_k) = e^*.$$

In addition, all $Y_k|_{\partial B}$ are continuous curves whose greatest distance from S converges to zero as k tends to infinity.

Now we pass from the sequence $\{Y_k\}$ to the sequence of harmonic mappings $X_k\colon B \to \mathbb{R}^3$ which are continuous on $\overline{B}$ and have the boundary values $Y_k|_{\partial B}$ on ∂B. We know that $X_k - Y_k \in \mathring{H}_2^1(B, \mathbb{R}^3)$ and

$$D(X_k) \le D(Y_k).$$

Therefore, also $\{X_k\}$ is a generalized minimizing sequence, and we have in particular

$$\lim_{k\to\infty} D(X_k) = e^*, \tag{7}$$

whence there is a constant M such that

$$D(X_k) \le M \quad \text{for all } k \in \mathbb{N}. \tag{8}$$

By virtue of the mean value theorem for harmonic functions, there is a constant c such that

$$|\nabla X_k(w)| \le c\sqrt{M}\rho^{-1} \quad \text{for all } k \in \mathbb{N} \text{ and for } |w| \le 1-\rho, \tag{9}$$

where $\rho \in (0,1)$.

Without loss of generality we can also assume that $X_k(0)$ lies on the closed polygon Π for all $k \in \mathbb{N}$, since we can replace X_k by $X_k \circ \tau_k$, where τ_k is a conformal selfmapping of B that maps $w = 0$ onto some point $w_k^* \in B$ with $X_k(w_k^*) \in \Pi$, and such a point can always be found since the polygon Π and the curve $X_k|_{\partial B}$ are linked.

In conjunction with (9) we infer that the harmonic mappings $X_k, k \in \mathbb{N}$, are uniformly bounded on every subset $\Omega \Subset B$. Applying a standard compactness result for harmonic mappings, there is a subsequence of $\{X_k\}$ that converges uniformly on every set $\Omega \Subset B$. By renumbering this subsequence we can achieve that the sequence X_k tends to a harmonic mapping $X\colon B \to \mathbb{R}^3$ on every compact subset of B. In conjunction with (8), we obtain that the $H_2^1(B)$-norms of the surfaces X_k are uniformly bounded, and thus we may also assume that the X_k tend weakly in $H_2^1(B, \mathbb{R}^3)$ and strongly in $L_2(\partial B, \mathbb{R}^3)$ to X which then is of class $H_2^1(B, \mathbb{R}^3)$.

From Lemma 1 we infer that $X \in \mathcal{C}(S)$, and the relations $X_k(0) \in \Pi$ imply in the limit that $X(0) \in \Pi$. Thus the harmonic mapping X is certainly not a constant, and therefore $X(w) \neq \text{const}$ on any open subset Ω of B. Hence

$$D_\Omega(X) > 0 \quad \text{for every nonempty open set } \Omega \Subset B. \tag{10}$$

The lower semicontinuity of the Dirichlet integral with respect to weak convergence in $H_2^1(B, \mathbb{R}^3)$ yields

$$D(X) \le \liminf_{k\to\infty} D(X_k),$$

and together with (6) and (7) we arrive at

$$D(X) \le e^* \le e.$$

As X is of class $\mathcal{C}(S)$, we shall expect X to be a solution of $\mathcal{P}(\Pi, S)$. However, it remains to be shown that X lies in $\mathcal{P}(\Pi, S)$. To this end we have to prove that the linking number of the polygon Π with the boundary class $X|_{\partial B}$ does not vanish. This will be proved by contradiction.

Hence we suppose that $\mathcal{L}([X|_{\partial B}], \Pi) = 0$. Then a sequence of radii $r_k \in (1/2, 1)$ with $r_k \to 1$ can be found such that

$$\xi_k(\theta) := X(r_k, \theta), \quad 0 \le \theta \le 2\pi,$$

represents the boundary class $[X|_{\partial B}]$ of X, and that both the conditions

$$\mathcal{L}(\xi_k, \Pi) = 0, \quad k \in \mathbb{N},$$

and

$$\lim_{k\to\infty} g(\xi_k, S) = 0$$

hold true.

Recall that $\{X_k\}$ converges to X uniformly on every $\Omega \Subset B$. Then, by passing to another subsequence of X_k and renumbering it, we may assume that

$$\max_{0\le\theta\le 2\pi} |X_k(r_k, \theta) - \xi_k(\theta)| \to 0 \quad \text{as } k \to \infty.$$

Set

$$\xi_k^*(\theta) := X_k(r_k, \theta), \quad 0 \le \theta \le 2\pi.$$

Then we infer that

$$\mathcal{L}(\xi_k^*, \Pi) = 0 \quad \text{for } k \in \mathbb{N} \tag{11}$$

and

$$\lim_{k\to\infty} g(\xi_k^*, S) = 0. \tag{12}$$

Moreover, it follows as in the proof of Lemma 3 in Section 1.1 that there is an angle $\theta_k \in [0, 2\pi]$ such that

$$|X_k(r, \theta_k) - X_k(1, \theta_k)| \le \left(\frac{2M}{\pi}\right)^{1/2} \sqrt{1 - r} \tag{13}$$

is satisfied for $1/2 \le r \le 1$ and for all $k \in \mathbb{N}$.

Finally we choose conformal mappings τ_k from B onto the slit annuli

$$\{w = re^{i\theta} \in B \colon r_k < r < 1, \theta \ne \theta_k\}.$$

We use these mappings to define a new sequence of surfaces $\hat{X}_k := X_k \circ \tau_k$. On account of the additivity of linking numbers, of (11), (12), and of $\mathcal{L}([X_k|_{\partial B}], \Pi) \ne 0$, it follows that

$$\mathcal{L}([\hat{X}_k|_{\partial B}], \Pi) \ne 0 \quad \text{for } k \in \mathbb{N}.$$

Moreover, we infer from (12) and (13) that the surfaces $\hat{X}_k$ are of class $C^0(\overline{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$ and have boundary values on closed sets $\Sigma_k := \hat{X}_k(\partial B)$ with $g(\Sigma_k, S) \to 0$ as $k \to \infty$. Consequently, $\{\hat{X}_k\}$ is a generalized admissible sequence for the problem $\mathcal{P}(\Pi, S)$, and we obtain from (5) that

$$e^* \le \liminf_{k\to\infty} D(\hat{X}_k). \tag{14}$$

On the other hand, the conformal invariance of the Dirichlet integral yields

$$D(\hat{X}_k) = D(X_k) - D_{B_{r_k}}(X_k) \leq D(X_k) - D_{B_{1/2}}(X_k),$$

where $B_r := \{w \in \mathbb{C}\colon |w| < r\}$. By a classical result on harmonic mappings, we infer from

$$\lim_{k\to 0} |X - X_k|_{0,\Omega} = 0 \quad \text{for any } \Omega \Subset B$$

that also

$$\lim_{k\to 0} |\nabla X - \nabla X_k|_{0,\Omega} = 0 \quad \text{for any } \Omega \Subset B$$

holds true, whence

$$\lim_{k\to\infty} D_{B_{1/2}}(X_k) = D_{B_{1/2}}(X).$$

In conjunction with (7), we conclude that

$$\liminf_{k\to\infty} D(\hat{X}_k) \leq e^* - D_{B_{1/2}}(X),$$

and now (14) yields

$$e^* \leq e^* - D_{B_{1/2}}(X).$$

This is a contradiction to (10). Consequently we obtain $\mathcal{L}([X|_{\partial B}], \Pi) \neq 0$, whence $X \in \mathcal{C}(\Pi, S)$ and therefore

$$e \leq D(X).$$

In view of (6) and (7) it follows that

$$D(X) = e = e^* \tag{15}$$

which shows that X is a solution of the minimum problem $\mathcal{P}(\Pi, S)$. This completes the proof of part (i) of the theorem.

(ii) The proof of part (ii) essentially follows the same lines of reasoning if we replace the conditions "$\mathcal{L}(\ldots, \Pi) \neq 0$" by "the boundary class of ... is not contractible", and if the relations "$X_k(0) \in \Pi$" are substituted by the assumption "S is compact". Then we have

$$|X_k|_{0,\partial B} \leq M' \quad \text{for all } k \in \mathbb{N},$$

and the maximum principle for harmonic functions yields

$$|X_k|_{0,\overline{B}} \leq M' \quad \text{for all } k \in \mathbb{N}.$$

Now we may carry on as before.

(iii) The third assertion of the theorem follows in the same way as for solutions of the Plateau problem; cf. Chapter 4 of Vol. 1. This completes the proof of Theorem 1. □

Theorem 2. *Every minimizer of the Dirichlet integral $D(X)$ in the class $\mathcal{C}(\Pi, S)$ (or in $\mathcal{C}^+(S)$) is a surface of least area in $\mathcal{C}(\Pi, S)$ (or in $\mathcal{C}^+(S)$).*

The *proof* of this result can be carried out in the same way as that of Theorem 4 in Section 4.5 of Vol. 1. An alternative method to establish

$$\inf_{\mathcal{C}(\Pi,S)} A = \inf_{\mathcal{C}(\Pi,S)} D$$

and

$$\inf_{\mathcal{C}^+(S)} A = \inf_{\mathcal{C}^+(S)} D$$

without using Morrey's lemma on ϵ-conformal mappings consists in applying the technique of Section 4.10 in Vol. 1, namely, to minimize $A^\epsilon := (1-\epsilon)A + \epsilon D$. □

1.4 Stationary Minimal Surfaces with Free or Partially Free Boundaries and the Transversality Condition

In the preceding chapters we have considered minimal surfaces which minimize Dirichlet's integral in suitable classes of admissible surfaces. However, the definition of minimal surfaces does not require them to be minimizers, and thus we are led to study also minimal surfaces that are only stationary within a given free boundary configuration. This, roughly speaking, means that the first order change of Dirichlet's integral is zero if we change the stationary surface in such a way that the boundary values remain on the prescribed supporting surface S. It will turn out that stationary minimal surfaces essentially are minimal surfaces which intersect S perpendicularly at their trace curves on S, provided that S is smooth and the boundary of S is empty. However, we also want to consider the case when ∂S is nonempty and consists of smooth regular curves.

By definition we want two distinguish two types of stationary minimal surfaces. The *first type* is defined by the differential equations

(1) $$\Delta X = 0$$

(2) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

which have to hold in B, and by a *natural boundary condition* which is to be satisfied on the free part of ∂B.

The *second type* will be described as critical points of the Dirichlet integral with respect to *inner* and *outer variations.*

Then we shall prove that both types of stationary minimal surfaces are the same provided that both S and ∂S are sufficiently smooth.

Let us begin by defining minimal surfaces in a partially free boundary configuration $\langle \Gamma, S \rangle$. In the following we use the notation of Section 4.6 of Vol. 1; in particular we define the class $\mathcal{C}(\Gamma, S)$ of admissible surfaces for the partially free problem as in that Section.

At present we assume that S and ∂S are of class C^1 and that Γ is a rectifiable arc.

For any point P on S, we denote by $T_p(S)$ the tangent plane of S at P. If $P \in \partial S$, then $T_p(S)$ is divided by the tangent $T_p(\partial S)$ of ∂S at P into two halfplanes. If $N_{\partial S}(P) \in T_P(S)$ denotes the outward unit normal of ∂S at $P \in \partial S$, then we call all tangent vectors $V \in T_P(S)$ with $\langle V, N_{\partial S}(P) \rangle \le 0$ *interior tangent vectors of S at $P \in \partial S$.*

Definition 1. *A* stationary minimal surface *in $\mathcal{C}(\Gamma, S)$ is an element of $\mathcal{C}(\Gamma, S)$ satisfying*

(i) $X \in C^1(B \cup I, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$, *where $I = \partial B \cap \{\text{Im}\, w < 0\}$ denotes the free boundary (of the parameter domain $B := B_1(0)$) of X.*

(ii) *In B we have the equations* (1) *and* (2).

(iii) Along $I_1 := \{w \in I : X(w) \in \text{ int } S\}$, the exterior normal derivative $\frac{\partial X}{\partial \nu}$ is perpendicular to S. (Using polar coordinates r, θ about the origin $w = 0$, we have $\frac{\partial X}{\partial \nu} = \frac{\partial X}{\partial r}$.)

(iv) *For any w belonging to $I_2 := \{w \in I : X(w) \in \partial S\}$ and every interior tangent vector $V \in T_{X(w)}S$, we have $\langle \frac{\partial X}{\partial \nu}(w), V \rangle \ge 0$.*

Definition 2. *An element $X \in \mathcal{C}(\Gamma, S)$ is called a* critical *(or* stationary*) point of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$ if*

$$\lim_{\varepsilon \to +0} \frac{1}{\varepsilon} \{D(X_\varepsilon) - D(X)\} \ge 0 \tag{3}$$

holds for all admissible variations $X_\varepsilon, |\varepsilon| < \varepsilon_0$, *of X. A family $\{X_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ of surfaces $X_\varepsilon \in \mathcal{C}(\Gamma, S)$ is said to be an* admissible variation *of X, if it is of one of the following two types.*

Type I (inner variations). $X_\varepsilon = X \circ \sigma_\varepsilon$ *where $\{\sigma_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$, $\varepsilon_0 > 0$, is a differentiable family of diffeomorphisms $\sigma_\varepsilon : \overline{B}^*_\varepsilon \to \overline{B}$ which are defined as inverse mappings of the diffeomorphisms $\tau_\varepsilon : \overline{B} \to \overline{B}^*_\varepsilon$ defined by*

$$\tau_\varepsilon(w) = w - \varepsilon\lambda(w), \quad \lambda \in C^1(\mathbb{R}^2, \mathbb{R}^2),$$

cf. Section 4.5 *of Vol.* 1.

Type II (outer variations). $X_\varepsilon = X + \varepsilon\phi(\cdot, \varepsilon)$ *for all $\varepsilon \in (0, \varepsilon_0)$ and some $\varepsilon_0 > 0$, where the following holds:*

(α) *the Dirichlet integrals of the mappings $\phi(\cdot, \varepsilon)$ are uniformly bounded, i.e.,*

$$D(\phi(\cdot, \varepsilon)) \le \text{const} \quad \text{for all } \varepsilon \in (0, \varepsilon_0);$$

(β) *the functions* $\phi(\cdot,\varepsilon)$ *converge pointwise a.e. in* B *to some function* $\phi_0 \in H_2^1 \cap L_\infty(B,\mathbb{R}^3)$ *as* $\varepsilon \to +0$.

By Proposition 2 of Vol. 1, Section 4.5, we obtain from the inner variations that a critical point X of Dirichlet's integral in the class $\mathcal{C}(\Gamma, S)$ satisfies

$$\partial D(X,\lambda) = 0 \quad \text{for all } \lambda = (\mu,\nu) \in C^1(\overline{B},\mathbb{R}^2). \tag{4}$$

Here $\partial D(X,\lambda)$ denotes the first inner variation of the Dirichlet integral, given by

$$2\partial D(X,\lambda) = \int_B \{a(\mu_u - \nu_v) + b(\mu_v + \nu_u)\}\, du\, dv, \tag{5}$$

where a and b denote the functions

$$a = |X_u|^2 - |X_v|^2, \quad b = 2\langle X_u, X_v\rangle. \tag{6}$$

Moreover, applying outer variations, it follows that

$$\lim_{\varepsilon\to+0} \frac{1}{\varepsilon}\{D(X_\varepsilon) - D(X)\} = \int_B \langle \nabla X, \nabla\phi_0\rangle\, du\, dv. \tag{7}$$

In fact, if $\{\varepsilon_n\}$ is a sequence of positive numbers tending to zero, then $\phi_n(w) := \phi(w,\varepsilon_n) \to \phi_0(w)$ a.e. on B.

By Egorov's theorem, for any $\delta > 0$ there is a compact subset B_δ of B with meas $(B \setminus B_\delta) < \delta$ such that $\lim_{n\to\infty} |\phi_0 - \phi_n|_{0,B_\delta} = 0$. By virtue of ($\alpha$) and of Poincaré's inequality (see Morrey [8], Theorem 3.6.4) we then infer that the $H_2^1(B)$-norms of the ϕ_n are uniformly bounded. From this we deduce that the sequence ϕ_n converges weakly in $H_2^1(B,\mathbb{R}^3)$ to ϕ_0, and this implies relation (7).

The following result states that the two kinds of stationary minimal surfaces are identical if S and ∂S are sufficiently smooth.

Theorem 1. *Assume that* S *and* ∂S *are of class* C^1 *and that* Γ *is rectifiable. Then every stationary minimal surface in* $\mathcal{C}(\Gamma, S)$ *is a stationary point of Dirichlet's integral in* $\mathcal{C}(\Gamma, S)$. *If* S *and* ∂S *are of class* $C^{3,\beta}, \beta \in (0,1)$, *then also the converse holds true, that is, every stationary point of Dirichlet's integral in* $\mathcal{C}(\Gamma, S)$ *furnishes a stationary minimal surface in* $\mathcal{C}(\Gamma, S)$.

For the proof we need the following auxiliary result:

Lemma 1. *Let* $X_\varepsilon = X + \varepsilon\phi(\cdot,\varepsilon), 0 \le \varepsilon \le \varepsilon_0$, *be an outer variation (i.e. an admissible variation of type II) of a surface* $X \in \mathcal{C}(\Gamma, S)$, *and let* $\phi_0 = \phi(\cdot, 0)$. *Then we have:*

(i) *For almost all* $w \in I$, *the vector* $\phi_0(w)$ *is a tangent vector of* S *at* $X(w)$. *If* $X(w)$ *lies on* ∂S, *then* $\phi_0(w)$ *is an interior tangent vector.*

(ii) *If, in addition to our general assumption, the arc* Γ *is of class* C^1 *or if* X *is a stationary minimal surface, then* $\phi_0(w)$ *is tangent to* Γ *at* $X(w)$ *for almost all* $w \in C = \partial B \setminus I$.

Proof. Choose a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to +0$ such that $\phi(w, \varepsilon_n) \to \phi_0(w)$ for a.e. $w \in \partial B$. Then assertion (i) follows from

$$\phi(w, \varepsilon_n) = \frac{1}{\varepsilon_n}\{X_{\varepsilon_n}(w) - X(w)\}$$

and from $X(w), X_{\varepsilon_n}(w) \in S$ for a.a. $w \in \partial B$.

(ii) is verified in the same way. (If Γ is only rectifiable we note that it has a tangent everywhere except at countably many points. Moreover, if X is a minimal surface then it follows from Theorem 1 in Vol. 1, Section 4.7 that, for almost all $w \in C = \partial B \setminus I$, the curve Γ has a tangent at the point $X(w)$.) □

Now we turn to the

Proof of Theorem 1. (i) Let X be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$. In order to show that X is a stationary point of the Dirichlet integral, we have to verify (3) for all admissible variations $\{X_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ of X. Since the case of inner variations (type I) has already been settled in Section 4.5 of Vol. 1, it suffices to consider variations of type II. In view of (7) we have to show

$$\int_B \langle \nabla X, \nabla \phi_0 \rangle \, du \, dv \geq 0. \tag{8}$$

The Courant–Lebesgue lemma (see Section 4.4 of Vol. 1) shows that, given any $\delta \in (0,1)$, there are two radii $r_1(\delta)$ and $r_2(\delta)$ with $\delta \leq r_1, r_2 \leq \sqrt{\delta}$ such that

$$\int_{\gamma_k} \left| \frac{\partial X}{\partial \nu} \right| ds = \int_{\gamma_k} \left| \frac{\partial X}{\partial t} \right| ds \leq \frac{M}{\{\log \frac{1}{\delta}\}^{1/2}} \tag{9}$$

holds true for $\gamma_1 := \overline{B} \cap \partial B_{r_1}(1)$ and $\gamma_2 := \overline{B} \cap \partial B_{r_2}(-1)$, where $M = \text{const}\sqrt{D(X)}$. (Here ν and t denote unit normal and unit tangent to γ_1 and γ_2, respectively.)

On account of Theorem 2 in Section 4.7 of Vol. 1 we have

$$\int_{\Omega_\delta} \langle \nabla X, \nabla \phi_0 \rangle \, du \, dv = \int_{\partial \Omega_\delta} \left\langle \frac{\partial X}{\partial \nu}, \phi_0 \right\rangle ds, \tag{10}$$

where $\Omega_\delta := B \setminus \{\overline{B}_{r_1}(1) \cup \overline{B}_{r_2}(-1)\}$. Letting δ tend to zero, we infer from (9) and (10) that

$$\int_B \langle \nabla X, \nabla \phi_0 \rangle \, du \, dv = \int_{\partial B} \left\langle \frac{\partial X}{\partial r}, \phi_0 \right\rangle ds. \tag{11}$$

By (iii) and (iv) of Definition 1 it follows that

$$\left\langle \frac{\partial X}{\partial r}, \phi_0 \right\rangle \geq 0 \quad \text{on } I = \partial B \cap \{\operatorname{Im} w < 0\}$$

holds true, whereas in view of Lemma 1 we obtain

$$\left\langle \frac{\partial X}{\partial r}, \phi_0 \right\rangle = 0 \quad \text{a.e. on } C = \partial B \cap \{\operatorname{Im} w \geq 0\}.$$

Consequently we have

$$\int_{\partial B} \left\langle \frac{\partial X}{\partial r}, \phi_0 \right\rangle ds \geq 0,$$

whence the identity (11) implies (8).

(ii) Let us now consider a stationary point X of Dirichlet's integral. By the results of Chapter 4 such a mapping X is a minimal surface, that is, equations (1) and (2) are satisfied in B (cf. equations (5)–(7)). The regularity results of 2.4 imply that $X \in C^1(B \cup I, \mathbb{R}^3)$. Thus it remains to prove conditions (iii) and (iv) of Definition 1. This will be carried out by applying the fundamental lemma of the calculus of variations to the equation

$$\int_{\partial B} \left\langle \frac{\partial X}{\partial r}, \phi_0 \right\rangle ds \geq 0 \tag{12}$$

which follows from (3), (7) and (11). As we shall see it will be enough to consider outer variations

$$X_\varepsilon = X + \varepsilon \phi(\cdot, \varepsilon), \quad 0 \leq \varepsilon < \varepsilon_0,$$

with

$$\text{support } \phi(\cdot, \varepsilon) \subset B \cup I.$$

Then also supp $\phi_0 \subset B \cup I$, and (12) reduces to

$$\int_I \langle X_r, \phi_0 \rangle \, ds \geq 0. \tag{13}$$

Consider now an arbitrary function $V \in C_c^1(I, \mathbb{R}^3)$ with $V(w) \in T_{X(w)}S$ for all $w \in I$ and

$$\langle V(w), N_{\partial S}(X(w)) \rangle < 0 \quad \text{for all } w \in I_2. \tag{14}$$

Here and in the sequel, the subsets I_1 and I_2 of I be defined in the same way as in Definition 1.

Then we solve the initial value problem

$$\frac{D}{d\varepsilon} \frac{d}{d\varepsilon} Z(w, \varepsilon) = 0,$$

$$Z(w, 0) = X(w), \quad \frac{dZ}{d\varepsilon}(w, 0) = V(w)$$

for fixed $w \in I$ and $0 \leq \varepsilon < \varepsilon_0$ with $0 < \varepsilon_0 \ll 1$, where $\frac{D}{d\varepsilon}$ denotes the covariant derivative on S. In other words, we define $Z(w, \varepsilon)$ as the geodesic flow on S starting at $X(w)$ in direction of $V(w), w \in I$. This flow exists for $0 \le \varepsilon < \varepsilon_0$ and $w \in I$, where ε_0 denotes a sufficiently small positive number, and we have

$$Z(w,\varepsilon) = X(w) + \varepsilon\Psi(w,\varepsilon) = X(w) + \varepsilon V(w) + o(\varepsilon)$$

for any $w \in I$, as well as

$$Z(w,\varepsilon) = X(w) \quad \text{for } w \in I \setminus \operatorname{supp} V \text{ and } 0 \le \varepsilon < \varepsilon_0.$$

Therefore we obtain

$$\Psi(\cdot,\varepsilon) \in C_c^1(I,\mathbb{R}^3), \quad \Psi(w,0) = V(w) \quad \text{for } w \in I$$

and

$$Z(w,\varepsilon) \in S \quad \text{for } (w,\varepsilon) \in I \times [0,\varepsilon_0).$$

Then we extend $\Psi(\cdot,\varepsilon)$ to functions $\phi(\cdot,\varepsilon)$ for class $C_c^1(B \cup I, \mathbb{R}^3)$ which depend smoothly on $(w,\varepsilon) \in B \cup I \times [0,\varepsilon_0)$, and set

$$X_\varepsilon(w) := X(w) + \varepsilon\phi(w,\varepsilon) \quad \text{for } w \in \overline{B} \text{ and } 0 \le \varepsilon < \varepsilon_0.$$

By construction we have $X_\varepsilon \in \mathcal{C}(\Gamma, S)$, and the function $\phi_0 := \phi(\cdot,0)$ satisfies $\phi_0(w) = V(w)$ for all $w \in I$. Consequently, the relation (13) holds true. This implies

$$\int_I \langle X_r, V\rangle \, ds \ge 0 \tag{15}$$

for every $V \in C_c^1(I,\mathbb{R}^3)$ with $V(w) \in T_{X(w)}S$ for all $w \in I$ which, in addition, satisfies $\langle V(w), N_{\partial S}(X(w))\rangle \le 0$ for $w \in I_2$. (In contrast to (14), we may admit the equality sign in the last inequality as can be proved by a straightforward approximation argument.)

Let us write

$$X_r = X_r' + X_r'', \quad X_r' \in T_X S, \quad X_r'' \perp T_X S;$$

then (15) is equivalent to

$$\int_I \langle X_r', V\rangle \, ds \ge 0. \tag{16}$$

Suppose now that $w_0 \in I_1$. Then there exists some $\rho > 0$ such that $I_\rho(w_0) := I \cap B_\rho(w_0)$ is contained in I_1, and we infer that

$$\int_{I_\rho(w_0)} \langle X_r', V\rangle \, ds \ge 0$$

is satisfied for every $V \in C_c^1(I_\rho(w_0),\mathbb{R}^3)$ with $V(w) \in T_{X(w)}S, w \in I_\rho(w_0)$, and since the same inequality holds if we replace V by $-V$, we even have

$$\int_{I_\rho(w_0)} \langle X_r', V\rangle \, ds = 0. \tag{17}$$

The fundamental lemma of the calculus of variations yields $X_r' = 0$ on $I_\rho(w_0)$, whence $X_r(w_0) = X_r''(w_0)$. Consequently the normal derivative $X_r(w_0)$ is perpendicular to $T_{X(w_0)}S$ for every $w_0 \in I_1$, and we have verified property (iii) of Definition 1.

Similarly we infer from (15) that condition (iv) of Definition 1 is fulfilled. We leave it as an exercise for the reader to carry out the details. □

Remarks and Generalizations

(i) Analogous to the Definitions 1 and 2 one can define stationary minimal surfaces in $\mathcal{C}(S)$ (or $\mathcal{C}^+(S)$ or $\mathcal{C}(\Pi, S)$) as well as *stationary points of the Dirichlet integral in* $\mathcal{C}(S)$ (or in $\mathcal{C}^+(S)$, or $\mathcal{C}(\Pi, S)$, respectively). We only have to replace I by $\partial B, C$ by the empty set, and $\mathcal{C}(\Gamma, S)$ by $\mathcal{C}(S)$ (or by $\mathcal{C}^+(S)$, or $\mathcal{C}(\Pi, S)$); all statements about Γ are now to be omitted. Then, analogous to Theorem 1, we obtain

Theorem 2. *Assume that S and ∂S are of class C^1. Then every stationary minimal surface in $\mathcal{C}(S)$ (or in $\mathcal{C}^+(S)$ or $\mathcal{C}(\Pi, S)$) is a stationary point of Dirichlet's integral in $\mathcal{C}(S)$ (or in $\mathcal{C}^+(S)$ or $\mathcal{C}(\Pi, S)$). If S and ∂S are of class $C^{3+\beta}, \beta \in (0, 1)$, also the converse holds true.*

A stationary minimal surface in $\mathcal{C}(S)$ will also be called *stationary minimal surface with respect to S, or: with a free boundary on S.*

If $\partial S = \emptyset$ and $S \in C^{2,\beta}$, $0 < \beta < 1$, then a stationary point of Dirichlet's integral is even of class $C^{2,\beta}$ up to its free boundary, according to results by Dziuk and Jost. In this case, the second statements of the Theorems 1 and 2 also hold under the assumption $S \in C^{2,\beta}$.

(ii) If we want to define *stationary minimal surfaces $X\colon \Omega \to \mathbb{R}^3$ with a free boundary on S and critical points of the Dirichlet integral with a free boundary on S* which are defined on multiply connected parameter domains Ω or even on Riemann surfaces, the matter is slightly more complicated. We are not anymore allowed to fix Ω, but only the *conformal type of Ω* can be prescribed. Then the boundary behaviour of the minimal surface does not only depend on S but also on the boundary $\partial\Omega$ of the parameter domain. However, we never have a real problem. For instance, a theorem of Koebe [1] states that every k-fold connected domain in $\mathbb{C}$ is conformally equivalent to a bounded domain in $\mathbb{C}$ whose boundary consists of k disjoint circles. Therefore we can essentially proceed as before.

(iii) In Section 1.6 we shall also consider stationary minimal surfaces $X\colon \Omega \to \mathbb{R}^3$ having their boundaries on a *simplex* or, more generally, on a *polyhedron*. In this case we shall call a *minimal surface X stationary* if for some finite subset M of ∂B the surface X is of class $H_2^1(B, \mathbb{R}^3) \cap C^1(\overline{B} \setminus M, \mathbb{R}^3)$ and if X meets the interiors of the faces of the polyhedron orthogonally along $\partial B \setminus M$.

(iv) In the special case that Γ is a rectifiable Jordan curve and that $\mathcal{C}(\Gamma)$ is defined as in Chapter 4 of Vol. 1, we obtain the following result: The solutions of Plateau's problem within the class $\mathcal{C}(\Gamma)$ which are not necessarily minimizers of the area are precisely the stationary points of the Dirichlet integral in $\mathcal{C}(\Gamma)$, i.e., those elements X of $\mathcal{C}(\Gamma)$ which satisfy

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{D(X_\varepsilon) - D(X)\} = 0$$

for all admissible variations $\{X_\varepsilon\}_{|\varepsilon| < \varepsilon_0}$ of X which are of class $\mathcal{C}(\Gamma)$. These admissible variations are defined as in Definition 2 if we replace S and I by the empty set, and C by ∂B.

Let us close this section with a simple

Example. If S is the boundary of an open, convex and bounded subset $\mathcal{K}$ of $\mathbb{R}^3$ of class C^1, and if E is a plane which intersects S orthogonally (e.g., a plane of symmetry of $\mathcal{K}$), then a conformal map X from B onto $E \cap \mathcal{K}$ defines a stationary minimal surface having $S \cap E$ as its trace. Therefore *the plane disks bounded by the great circles of a sphere S are stationary minimal surfaces in S. As we shall see in Section 1.7, they are the only stationary disk-type surfaces in the sphere.*

Moreover, the ellipses in an *ellipsoid* $\mathcal{E}$, having two of the axes of $\mathcal{E}$ as their principal axes, are *three stationary minimal surfaces in* $\mathcal{E}$. It is unknown whether they are the only stationary surfaces in $\mathcal{E}$ which are of the type of the disk.

1.5 Necessary Conditions for Stationary Minimal Surfaces

Let us agree that throughout this section we consider minimal surfaces $X : \Omega \to \mathbb{R}^3$, the parameter domain Ω of which will be bounded by finitely many disjoint circles $C_1, C_2, \dots, C_k$. Moreover, the surfaces X will be stationary minimal surfaces with a free boundary on a polyhedron S or on a closed, orientable, regular C^1-surface S.

Now, if V is an arbitrary constant vector in $\mathbb{R}^3$, then an integration by parts shows that

$$\int_{\partial\Omega} \left\langle \frac{\partial X}{\partial \nu}, V \right\rangle ds = \int_\Omega \langle \Delta X, V \rangle \, ds = 0.$$

(If S is a polyhedron this can be justified as in Section 1.4.) Since X is stationary, we know that (almost) everywhere on $\partial\Omega$

$$\begin{aligned}\frac{\partial X}{\partial \nu}(w) &= \pm N_S(X(w)) \cdot \left|\frac{\partial X}{\partial \nu}(w)\right| \\ &= \pm N_S(X(w)) \cdot \left|\frac{\partial X}{\partial t}(w)\right|,\end{aligned}$$

where N_S is the surface normal of S, and $\frac{\partial}{\partial \nu}(\frac{\partial}{\partial t})$ denotes the normal (tangential) derivative along $\partial\Omega$. Furthermore, s denotes the parameter of the arc length on $\partial\Omega$.

Therefore we have obtained the following

Proposition 1. *If $X\colon \Omega \to \mathbb{R}^3$ is a stationary minimal surface with respect to a surface S, X and S satisfying the general assumption, then*

$$\int_{\partial\Omega} N_S(X(w)) \cdot \left|\frac{\partial X}{\partial t}(w)\right| ds = 0 \tag{1}$$

unless $\langle \frac{\partial X}{\partial \nu}, N_S(X)\rangle$ changes its sign on $\partial\Omega$.

Remarks. (i) If we denote by $\Sigma := X|_{\partial\Omega}$ the trace of a stationary minimal surface X, then formula (1) could also be written as

$$\int_\Sigma \mu(P) N_S(P)\, d\mathcal{H}^1 = 0,$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff-measure in $\mathbb{R}^3$ and $\mu(P)$ the number of points $w \in \partial\Omega$ such that $X(w) = P$.

(ii) The geometric interpretation of formula (1) is that the integral of the normal N_S over the trace Σ of the stationary minimal surface X vanishes.

(iii) Here are some conditions implying that

$$\left\langle \frac{\partial X}{\partial \nu}, N_S(X)\right\rangle \quad \textit{does not change its sign on } \partial\Omega. \tag{2}$$

A first condition is

(I) *S is smooth and X has no branch points on $\partial\Omega$.*

Recall that $w \in \overline{\Omega}$ is a branch point of X if $|\nabla X(w)|^2 = 0$. Since by conformality we have the identity $|\frac{\partial X}{\partial \nu}|^2 = \frac{1}{2}|\nabla X|^2$ on $\partial\Omega$, property (2) follows from (I).

(II) More generally, *for surfaces S of class C^4 (or $C^{3,\beta}, \beta \in (0,1)$), the absence of branch points of odd order on $\partial\Omega$ is also sufficient for* (2).
This follows from the expansion formula in Section 2.10 which describes the asymptotic behavior of the minimal surface near a boundary branch point.

(III) *The surface $X(\overline{\Omega})$ stays on one side of S.*
Clearly, (2) follows from this property of X which, in turn, is true if

(IV) *S is the boundary of a convex body $\mathcal{K} \in \mathbb{R}^3$*, as we can infer from the maximum principle, or if

(V) *S is the graph of a C^2-function defined on the 2-sphere S^2 such that the mean curvature of S with respect to the inward unit normal N_S is nowhere negative.*
This follows from a maximum principle to be stated in the Chapter 4.

(iv) A system $\Gamma = \{\Gamma_1, \dots, \Gamma_k\}$ of rectifiable curves Γ_j on a surface S is sometimes called a *system of balanced curves on S* if

$$\int_\Gamma N_S \, d\boldsymbol{s} = 0,$$

that is, if

$$\sum_{j=1}^{k} \int_{\Gamma_j} N_S \, d\boldsymbol{s} = 0$$

holds true (here $\boldsymbol{s}$ denotes the parameter of the arc length of Γ).

According to Proposition 1, formula (1), we have

$$\int_\Sigma N_S \, d\boldsymbol{s} = 0$$

for the free trace Σ of a minimal surface X with a free boundary on S and satisfying (2), since $d\boldsymbol{s} = |X_t| \, ds$ if s denotes the arc length on $\partial\Omega$. Consequently, we can read Proposition 1 as follows:

The free trace Σ of a stationary minimal surface in $\mathcal{C}(S)$ is a system of balanced curves.

We shall now draw several conclusions from Proposition 1.

Corollary 1. *Let $X\colon \Omega \to \mathbb{R}^3$ be a stationary minimal surface with respect to a support surface S which is the boundary of an open set $\mathcal{K}$ in $\mathbb{R}^3$, and assume that $X(\overline{\Omega})$ is contained in $\overline{\mathcal{K}}$. Then the free trace $X(\partial\Omega)$ of X on S cannot be contained in a subset of S which is mapped by the Gauss map $N_S\colon S \to S^2$ of S into an open hemisphere of S^2.*

Corollary 2. *In particular there are no stationary minimal surfaces which have their boundaries on a paraboloid or on one sheet of a hyperboloid of two sheets. Likewise there is no stationary minimal surface with respect to a simplex whose trace intersects only three of the four faces.*

Corollary 3. *If X and S are as in Corollary 1 and if $X(\partial\Omega)$ is contained in a subset $\mathcal{U}$ of S whose Gauss image $N_S(\mathcal{U})$ is contained in a closed hemisphere H of S^2, then $N_S(X(\partial\Omega))$ is the great circle ∂H.*

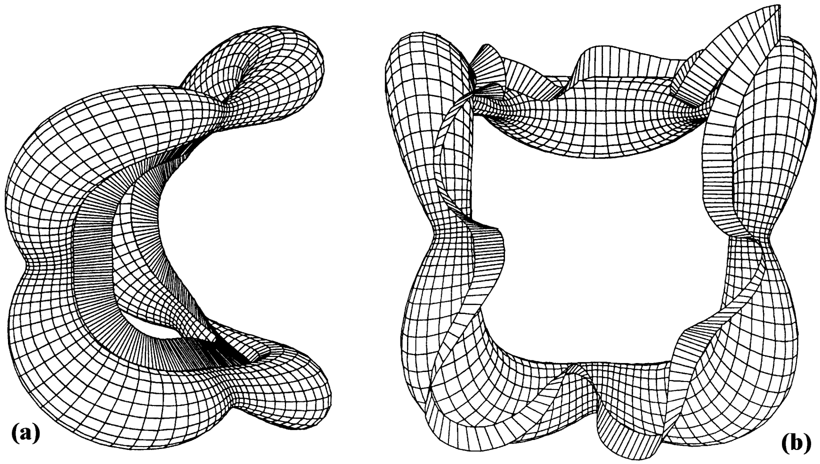


Fig. 1. **(a)** The integral of the unit normal bundle along a balanced curve on a smooth surface vanishes. **(b)** The integral of the unit normal bundle along this unbalanced curve on the same surface has a non-vanishing component pointing to the reader

Corollary 4. *If X and S are as in Corollary 1, and if Ω is simply connected, S is of class C^1, and if the image of the trace $X(\partial\Omega)$ under the Gauss map $N_S\colon S \to S^2$ is a great circle, then Σ is a plane curve, and X is a plane minimal surface.*

Proof. We can now assume that Ω is the unit disk B. Let X^* be the adjoint minimal surface of X. Then we have

$$X_u = X_v^* \quad \text{and} \quad X_v = -X_u^* \quad \text{in } B, \tag{3}$$

or, in polar coordinates,

$$X_r^* = -\frac{1}{r} X_\theta \quad \text{and} \quad X_\theta^* = r X_r,$$

hence for $0 \le \theta \le 2\pi$

$$\begin{aligned} X^*(1,\theta) &= X^*(1,0) + \int_0^\theta X_\theta^*(1,\varphi)\,d\varphi \\ &= X^*(1,0) + \int_0^\theta X_r(1,\varphi)\,d\varphi. \end{aligned} \tag{4}$$

By assumption, the normals of S along Σ are contained in a plane. Consequently the vectors $X_r(w) = \pm|X_r(w)| N_S(X(w)), w \in \partial B$, lie in a plane, and (4) implies that $X^*(1,\theta)$ is contained in a parallel plane. The maximum principle now yields that $X^*(\overline{\Omega})$ lies in this plane, and the assertion follows from (3). $\square$

Corollary 5. *Stationary minimal surfaces of the type of the disk (i.e., $\Omega = B$) with their free boundary on a cylinder are plane disks orthogonal to the cylinder axis.*

This is an immediate consequence of the preceding corollary. Note that the infinite strip is excluded, as our definition of “stationary” implies “finite area”.

1.6 Existence of Stationary Minimal Surfaces in a Simplex

The examples of stationary minimal surfaces with a free boundary on a supporting surface S have been rather trivial since all of them were planar surfaces. The first nontrivial example of a minimal surface with a free boundary on a tetrahedron was found by H.A. Schwarz in 1872 (cf. Math. Abhandlungen [2], vol. 1, pp. 149–150); we have copied Schwarz's drawing in Fig. 1. Schwarz obtained this surface as an adjoint of the minimal surface bounded by four consecutive edges of a regular tetrahedron. In the following we describe a result of B. Smyth [1] which may be viewed as a generalization of the Schwarz surface to arbitrary simplices in $\mathbb{R}^3$.

Theorem 1. *Let S be the boundary of a simplex in $\mathbb{R}^3$. Then there are exactly three stationary minimal surfaces of disk-type having connected intersections with each of the four faces of S. They neither have branch points in B nor on the arcs of ∂B which are mapped into the faces of S.*

Remark. Exactly three means, of course, exactly three except for reparametrizations.

Proof of Theorem 1. First of all, in order to prove existence, choose a fixed order $H_1, \dots, H_4$ of the faces H_i of S and let $N_1, \dots, N_4$ be their outward unit normals. Next choose four real numbers l_i such that $\sum_{i=1}^4 l_i N_i = 0$ (note that all l_i are different from zero). Now let Γ be the quadrilateral which is determined by the four vectors $l_1 N_1, \dots, l_4 N_4$ just in this order, i.e., $\Gamma(t) := 4 l_1 N_1 \cdot t$ for $0 \le t \le \frac{1}{4}$, $\Gamma(t) = l_1 N_1 + 4(t - \frac{1}{4}) \cdot l_2 N_2$ for $\frac{1}{4} \le t \le \frac{1}{2}$, etc. Since Γ can be projected onto a convex curve in a plane, there is exactly one solution Y of the Plateau problem $\mathcal{P}(\Gamma)$ (see Section 4.9 of the Vol. 1). By the reflection principle (cf. Vol. 1, Section 4.8), Y is of class $H_2^1(B, \mathbb{R}^3) \cap C^\omega(\overline{B} \setminus M, \mathbb{R}^3)$, where M contains the four points of ∂B corresponding to the corners of Γ. Then the minimal surface $\hat{X} := -Y^*$ is stationary with respect to a simplex $\hat{S}$ similar to S. After a suitable choice of $a > 0$ and $A_0 \in \mathbb{R}^3$, the surface $X := A_0 + a\hat{X}$ is a stationary minimal surface with respect to the given simplex S, which crosses the faces $H_1, \dots, H_4$ in this order.

Now note that, since a stationary minimal surface has to cross all four faces of the simplex (Corollary 2 of Section 1.5), one can select any of them

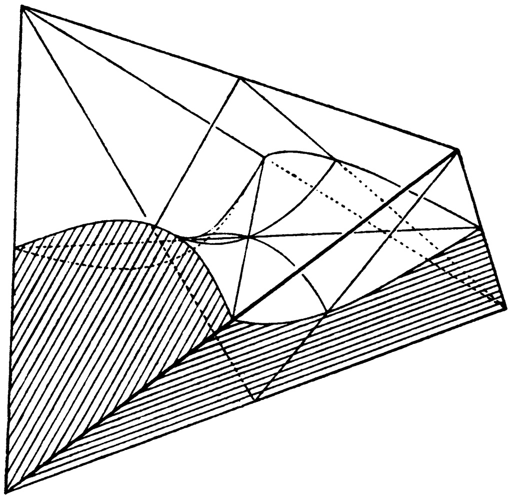


Fig. 1. Schwarz's stationary minimal surface in a tetrahedron

as the first to be crossed. But then the three possible choices of the face to be crossed next but one lead to three geometrically different stationary minimal surfaces. This proves the existence of at least three stationary minimal surfaces.

Before we show the uniqueness part of the theorem let us show that *the length of the trace* $X|_{\partial B}$ *of any stationary minimal surface in the simplex* S *having connected intersections with the faces is finite.*

Denote by $C_1, \dots, C_4$ the four open subarcs of ∂B which are mapped by X into the interiors of the faces H_1, H_2, H_3, H_4 of the simplex. Next, note that the adjoint minimal surface X^* of X also belongs to $H_2^1(B, \mathbb{R}^3) \cap C^\omega(\overline{B} \setminus M, \mathbb{R}^3)$, where $M = \partial B \setminus \bigcup_1^4 C_i$. By virtue of the maximum principle and the boundary condition, we obtain

$$\frac{\partial X^*}{\partial \theta} = \frac{\partial X}{\partial r} = \left| \frac{\partial X^*}{\partial r} \right| \cdot N_i \quad \text{on } C_i,$$

whence we see that X^* maps the four arcs C_i monotonically onto four mutually nonparallel straight lines $\hat{L}_i$ parallel to N_i. The Courant–Lebesgue lemma now implies that $\hat{L}_i$ intersects $\hat{L}_{i+1(\text{mod } 4)}$, and that X^* is continuous on $\overline{B}$ and bounded by the quadrilateral Γ given by the line segments L_i on $\hat{L}_i$ between the intersections of $\hat{L}_i$ with $\hat{L}_{i-1(\text{mod } 4)}$ and $\hat{L}_{i+1(\text{mod } 4)}$.

In particular we have for $i = 1, \dots, 4$ that

$$l_i = \int_{C_i} |X_\theta| \, ds = \int_{C_i} |X_\theta^*| \, ds < \infty$$

(one can now also show that X is continuous in $\overline{B}$). Since the boundary curve $X|_{\partial B}$ is balanced (see Section 1.5), we have

$$\sum_{i=1}^{4} l_i N_i = 0.$$

This equation shows that the lengths of the intercepts of X with the faces are determined up to a constant, since there is only one linear relation between four vectors in $\mathbb{R}^3$, no three of which are dependent.

Consequently, if Y is another stationary minimal surface in the simplex S which intersects the faces in the same order as X, then the bounding quadrilaterals of X^* and Y^* are homothetic, hence also $X^*(B)$ and $Y^*(B)$, as follows from the uniqueness theorem in Section 4.9 of Vol. 1. Therefore $X(B)$ and $Y(B)$ are homothetic too, and they even coincide since they are bounded by the same simplex.

Hence we have shown that a particular choice of the order in which the faces are crossed determines the minimal surface uniquely. Hence only three essentially different stationary minimal surfaces remain. This proves the assertion. □

A stationary minimal surface X in the simplex S has no interior branch points since X^* has none (Theorem 1 in Vol. 1, Section 4.9). The simplex is convex; therefore X stays on one side of each of the faces H_i. Hence we may first continue X by reflection across H_i as a minimal surface and then exclude branch points on C_i by means of the expansion formulas stated in Section 3.2 of Vol. 1.

Remark. By means of the *theorem of Krust* presented at the end of Section 3.3 of Vol. 1, it follows that the three stationary solutions of Smyth are graphs, since their adjoints are graphs; thus, in particular, they are embedded minimal surfaces.

1.7 Stationary Minimal Surfaces of Disk-Type in a Sphere

In this section we shall prove that plane disks are the only stationary minimal surfaces of disk type that have their boundaries on a sphere.

Theorem. *Let $X \in C^1(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ satisfy*

(1) $$\Delta X = 0 \quad \text{in } B,$$

(2) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B.$$

Moreover, assume that $X(\partial B)$ is contained in a sphere S and that the normal derivative $\frac{\partial X}{\partial \nu}$ is orthogonal to S along ∂B. Then $X(\overline{B})$ is a plane disk.

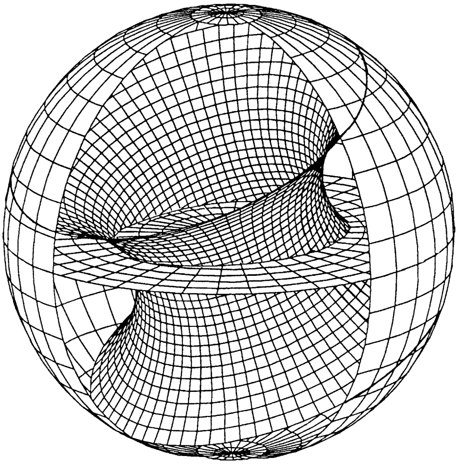


Fig. 1. The catenoid yields a doubly connected non-planar minimal surface intersecting a sphere perpendicularly. On the other hand, all simply connected stationary minimal surfaces in a sphere are planar surfaces

Remark. Note that the theorem is false if we admit minimal surfaces of a different topological type. For example, any sphere S bounds a catenoid intersecting S orthogonally along its trace.

Proof of the Theorem. We shall prove in Chapter 2 that

(3) X *is real analytic in* $\overline{B}$.

Then the arguments used in Chapter 3 of Vol. 1 show the following:

(4) *X has only finitely many isolated branch points in* $\overline{B}$. The surface normal $N(w)$ of $X(w)$and hence also the coefficients of the second fundamental form of X can be extended continuously to all of $\overline{B}$.

Let $M \subset \overline{B}$ denote the set of branch points, i.e. of points w with $|X_u(w)| = |X_v(w)| = 0$. Then, by H. Hopf's observation (cf. Vol. 1, Section 1.3), the function $f(w) = \frac{1}{2}(\mathcal{L} - \mathcal{N}) - i\mathcal{M}$ is holomorphic in $B \setminus M$ and continuous on $\bar{B}$. Consequently all interior singularities of f are removable and f is holomorphic in all of B. Let us now assume without loss of generality that $S = S^2$, and consider the boundary condition, which is equivalent to

$$\begin{aligned} X_\rho(e^{i\theta}) &= \lambda(e^{i\theta}) N_S(X(e^{i\theta})) \\ &= \lambda(e^{i\theta}) X(e^{i\theta}), \end{aligned}$$

where $N_S(X)$ denotes the outward unit normal of S at X and $\lambda(\theta) := \sqrt{\mathcal{E}(e^{i\theta})}$. Next we differentiate this equation in $\partial B \setminus M$ with respect to θ. We obtain ($' = \frac{\partial}{\partial\theta}$): $X_{\rho\theta} = \lambda' X + \lambda X_\theta = \frac{\lambda'}{\lambda} X_\rho + \lambda X_\theta$, and a comparison with formula (36) of Section 1.3 in Vol. 1 shows that the boundary values of the imaginary part β of $g := w^2 f$ vanish. Hence $\beta \equiv 0$ in $\overline{B}$, and therefore $\alpha :=$ Re g is identically constant in B, whence $\alpha \equiv \alpha(0) = 0$. Thus $\mathcal{L} = \mathcal{N}$ and $\mathcal{M} = 0$ in B. Now Weingarten's equations (cf. Vol. 1, Section 1.2, (38) ff. and (42)) imply that

$$\nabla N = -H \nabla X = 0 \quad \text{in } B \setminus M.$$

Therefore $N \equiv$ const in B and $X(B)$ is contained in a plane orthogonal to N. □

Remark. Suppose that $X \in C^1(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ is a disk-type surface of constant mean curvature H. Then our previous reasoning shows that $f(w) = \frac{1}{2}(\mathcal{L} - \mathcal{N}) - i\mathcal{M}$ is again holomorphic, and the same arguments as before yield that $\mathcal{L} = \mathcal{N}$ and $\mathcal{M} = 0$, as one has the same asymptotic expansion about branch points as for minimal surfaces (see Heinz [15]). Then it is fairly easy to prove that X *is a parametrization of a spherical cap*. This result as well as Theorem 1 are due to Nitsche [35]. Furthermore, one can construct an example where this spherical cap actually covers a whole sphere of radius $1/|H|$.

1.8 Report on the Existence of Stationary Minimal Surfaces in Convex Bodies

Let $S \subset \mathbb{R}^3$ be an embedded submanifold of $\mathbb{R}^3$ without boundary and of genus $g \geq 1$, that is, S has at least one hole to be spanned. Then there exists a closed polygon Π such that the class $\mathcal{C}(\Pi, S)$ is nonempty, and we can prove the existence of a stationary minimal surface X which has its boundary on S and such that $X|_{\partial B}$ is not contractible in $\mathbb{R}^3 \setminus \Pi$, see Theorem 1 in Section 1.3. Such a surface X is constructed as a solution to the variational problem $\mathcal{P}(\Pi, S)$: $D_B(X) \to \min$ in $\mathcal{C}(\Pi, S)$. In Section 1.6, on the other hand, we have considered the case of a simplex S, and we have proved the existence of three (distinct) stationary minimal surfaces in $\mathcal{C}(S)$. Clearly these surfaces cannot be solutions of the minimum problem $\mathcal{P}(S)$: $D_B(\cdot) \to$ *minimum in* $\mathcal{C}(S)$ as the constant surfaces have a smaller Dirichlet integral, and the classes $\mathcal{C}^+(S)$ and $\mathcal{C}(\Pi, S)$ are void. Thus we cannot use a minimizing procedure to obtain nondegenerate minimal surfaces in S.

Consider now the ellipsoid E given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, with $a > b > c$. As we have already noted, there exist at least three geometrically distinct, stationary minimal surfaces inside E which are of the type of the disk, namely, the parts of the coordinate planes $\{x = 0\}, \{y = 0\}$, and $\{z = 0\}$ lying in the interior of E.

Thus, if S is the boundary of a convex body $\mathcal{K} \subset \mathbb{R}^3$, it is tempting to conjecture that there exist at least three geometrically different stationary minimal surfaces with boundary on S. As mentioned before, we cannot obtain these surfaces by a minimum procedure. Hence more refined *minimax procedures* (or *saddle-point methods*) have to be used if we want to find such surfaces which are not minimizers. As a first result in this direction Struwe [3] proved the following

Theorem 1. *For any embedded surface S of class C^4 which is diffeomorphic to the unit sphere S^2 in $\mathbb{R}^3$, there exists a stationary minimal surface $X \in \mathcal{C}(S)$ of the type of the disk which has its free boundary on S.*

Struwe's proof applies a minimax principle from Palais [1] to a modified class of variational problems $\mathcal{P}_\alpha, \alpha > 1$, which satisfy the Palais–Smale condition and hence admit a saddle-type solution X_α. A nonconstant stationary minimal surface is obtained by passing to the limit $\alpha \to 1$ via a suitable subsequence of the surfaces X_α. This approach can be viewed as an adaptation of a method due to K. Uhlenbeck (see, for instance, Sacks and Uhlenbeck [1]).

Struwe's theorem does not answer the question as to whether one can find an *embedded* stationary minimal surface with its free boundary on the surface S of some convex body $\mathcal{K}$, or if there is at least an *immersed* stationary minimal surface in $\mathcal{C}(S)$. In case that S is the boundary of a strictly convex subset $\mathcal{K} \subset \mathbb{R}^3$ of class C^4, Grüter and Jost [1] have found the following stronger result.

Theorem 2. *There exists an embedded, stationary disk-type minimal surface having its free boundary on S (and values in $\mathcal{K}$).*

The proof of this theorem uses methods from geometric measure theory which have not been treated in these notes. Let us only mention some main ingredients of the arguments used by Grüter and Jost. First the minimax methods from Pitts [1] are employed to obtain a so-called *almost minimizing varifold* in the sense of Pitts [1] and Simon and Smith [1], which meets S transversally along its trace. The regularity of this varifold at its free boundary relies on an extension of Allard's regularity results to free boundary value problems due to Grüter and Jost [2]. Finally, Simon and Smith proved the existence of a minimally embedded two-sphere in any manifold diffeomorphic to the three-sphere. The methods of these authors are used in an essential way to show that the above varifold is both embedded and simply connected, that is, the minimizing varifold is of the type of the disk or of a collection of disks.

Theorem 2 also extends to Riemannian manifolds if one adapts methods by Pitts [1] and by Meeks, Simon, and Yau [2].

The following theorem due to Jost [15] (cf. also [9] for an earlier, more restricted result) shows that a closed convex surface S bounds in fact three different stationary minimal surfaces.

Theorem 3. *Let S be the boundary of a strictly convex body $\mathcal{K} \subset \mathbb{R}^3$ of class C^5. Then there exist three geometrically different, stationary, embedded* *minimal surfaces in $\mathcal{K}$ which are of disk type and have their free boundaries on S.*

In fact, Jost [15] proved that the assertion still holds true if S is merely *H-convex*.

A generalization of Theorem 2 to convex polyhedral surfaces S was established by Jost [15]. His result to be stated next contains a part of the Theorem of B. Smyth as a special case.

Theorem 4. *Let S be a compact convex polyhedron in $\mathbb{R}^3$. Then there exists an embedded minimal surface X of the type of the disk meeting S perpendicularly along its boundary such that no segment of any edge of S is contained in the boundary of X.*

1.9 Nonuniqueness of Solutions to a Free Boundary Problem. Families of Solutions

Examples of minimal surfaces with free or partially free boundaries on a prescribed supporting surface S were already investigated during the last century. The first geometric problem leading to minimal surfaces with free boundaries was posed by the French mathematician Gergonne [1] in 1816, but a correct solution was only found by H.A. Schwarz in 1872 (see [2], pp. 126–148, and Tafel 4 at the end of vol. I).

Gergonne's problem consists in finding a minimal surface spanning a frame $\langle \Gamma_1, \Gamma_2, S_1, S_2 \rangle$ that consists of two parallel faces S_1 and S_2 of some cube and of two straight arcs Γ_1 and Γ_2 lying on opposite faces of the cube.[3] As depicted in Fig. 1, we assume that the two diagonals Γ_1 and Γ_2 are perpendicular to each other. In contrast to his predecessors, Schwarz arrived at correct stationary surfaces spanning the configuration $\langle \Gamma_1, \Gamma_2, S_1, S_2 \rangle$ since he had discovered the proper free boundary condition: each stationary surface has to meet the two supporting surfaces at a right angle. In addition to an area minimizing solution which is depicted in Fig. 1, Schwarz discovered infinitely many other *non-congruent* stationary minimal surfaces in the frame $\langle \Gamma_1, \Gamma_2, S_1, S_2, S_3, S_4 \rangle$ consisting of the four vertical faces S_i and the two horizontal arcs Γ_1, Γ_2. In other words, *a partially free boundary problem may have infinitely many distinct (i.e. noncongruent) solutions.*

Let us set up the definition of free or partially free boundary problems in some more generality than in Section 1.4. We consider boundary configurations $\langle \Gamma, S \rangle$ in $\mathbb{R}^3$ consisting of a system Γ of Jordan curves $\Gamma_1, \dots, \Gamma_m$ and of a system S of surfaces $S_1, \dots, S_n$. Each of the curves Γ_i is either a closed curve or else a Jordan arc with end points on S. We shall call S the *free part*

[3] In fact, the original form of this problem is somewhat different; it was stated as a *partition problem* for the cube.

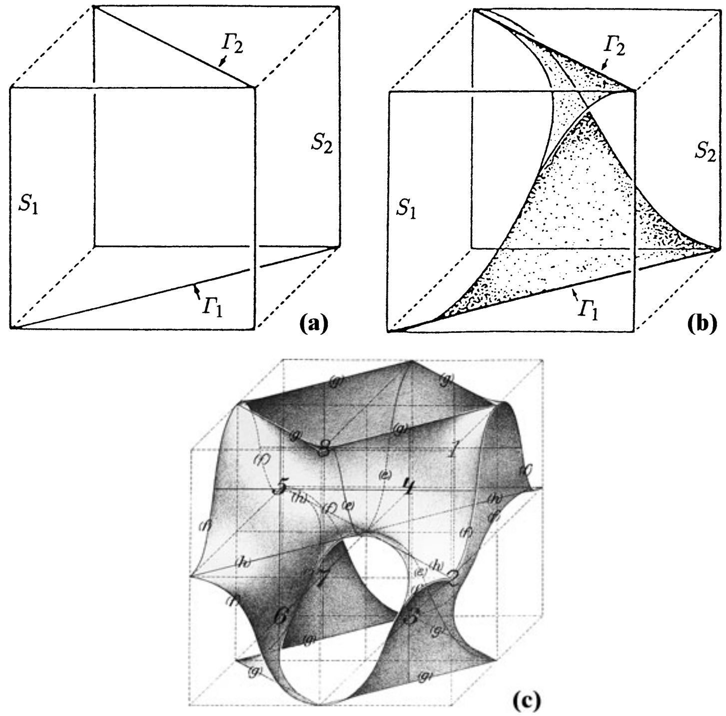


Fig. 1. **(a)** The Schwarzian chain $\langle \Gamma_1, \Gamma_2, S_1, S_2 \rangle$ forming the boundary frame for Gergonne's problem. **(b)** Gergonne's surface, the area minimizing solution of Gergonne's problem discovered by Schwarz. **(c)** Gergonne's surface generates the fifth periodic minimal surface known to Schwarz (Lithograph by H.A. Schwarz)

of the configuration $\langle \Gamma, S \rangle$. The fixed part Γ of the boundary frame may be empty.

A minimal surface $\mathcal{M}$ is said to be *stationary within the configuration* $\langle \Gamma, S \rangle$ if the boundary of $\mathcal{M}$ lies on $\Gamma \cup S$ and, moreover, if $\mathcal{M}$ meets S orthogonally at the part $\Sigma = \partial\mathcal{M} \cap S$ of its boundary. As usual, we shall call Σ the *free trace* of $\mathcal{M}$ on S.

Remark. If this definition is to make sense we have to assume that each of the support surfaces S_j is a regular surface of class C^1. Furthermore we shall suppose the each Γ_k is a piecewise smooth regular arc. Similarly, $\mathcal{M}$ is supposed to be smooth except for finitely many points. Note that in this section we assume that $\mathcal{M}$ meets S *everywhere* at a right angle (except for at most finitely many points). In other words, we essentially exclude the case that $\partial\mathcal{M}$ attaches in segments (i.e. intervals) to ∂S since in this case the two surfaces $\mathcal{M}$ and S need not include an angle of ninety degrees.

The *free-boundary problem* of a configuration $\langle \Gamma, S \rangle$ is the problem to determine a stationary minimal surface within $\langle \Gamma, S \rangle$. As before, such a problem

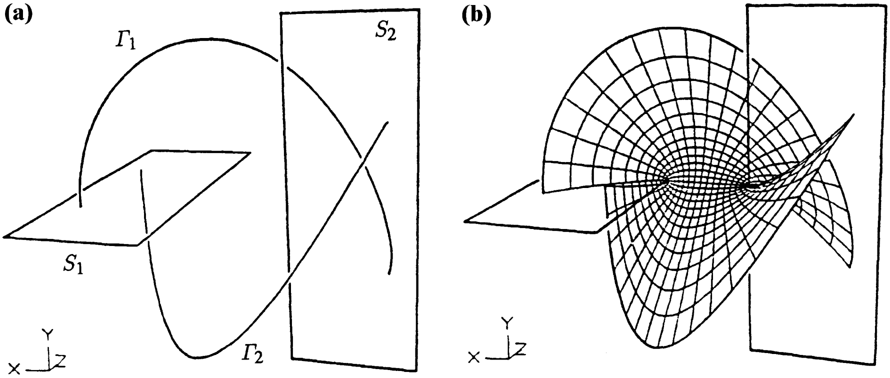


Fig. 2. **(a)** A partially free boundary problem for a frame $\langle \Gamma_1, \Gamma_1, S_1, S_2 \rangle$. **(b)** A part of Henneberg's surface forms a disk-type solution of the problem. Note that S_1 and S_2 are surfaces with boundary. As in the present case, this can lead to singularities of the free boundary of a solution (see Chapters 1 and 2 of Vol. 3)

is said to be *partially free* if Γ is nonvoid; otherwise we call it *completely free* or simply *free*.

As usual we describe minimal surfaces $\mathcal{M}$ by mappings X from a planar parameter domain Ω or from a Riemann surface $\mathcal{R}$ into $\mathbb{R}^3$; $\partial\Omega$ and $\partial\mathcal{R}$ are assumed to be piecewise smooth, and X will be smooth in $\overline{\Omega}$ or $\overline{\mathcal{R}}$ except for at most finitely many points on $\partial\Omega$ or $\partial\mathcal{R}$.

It is trivial to find supporting surfaces S which bound continua of stationary minimal surfaces. The sphere, the cylinder, or the torus furnish simple examples. In these cases, however, all minimal surfaces belonging to the same continuum are congruent to each other.

Therefore it is of interest to see that there are free or even partially free boundary problems which possess denumerably many noncongruent solutions, or even continua of noncongruent solutions.

As we have mentioned above, Gergonne's problem furnishes an example of such a free boundary problem. In fact, using the helicoids, Schwarz was able to exhibit an even simpler and completely elementary example of such a boundary configuration. Consider a boundary frame $\langle \Gamma_1, \Gamma_2, S \rangle$ consisting of a cylinder surface S and two straight arcs Γ_1 and Γ_2 which are perpendicular to each other as well as to the cylinder axis and pass through the axis at different heights. This configuration bounds denumerably many left and right winding helicoids which meet the cylinder S at a right angle (Fig. 3). Only two of these helicoids are area minimizing, the others are only stationary.

A slight modification of the previous example yields a boundary frame $\langle \Gamma, S \rangle$ consisting of a cylinder S as surface of support and of a polygon Γ made of a piece A of a cylinder axis and of two straight segments A_1 and A_2 which connect A with S; we assume that A_1 and A_2 are perpendicular to each other. There are again infinitely many stationary surfaces for $\langle \Gamma, S\rangle$, all of which are helicoidal surfaces (cf. Fig. 4).

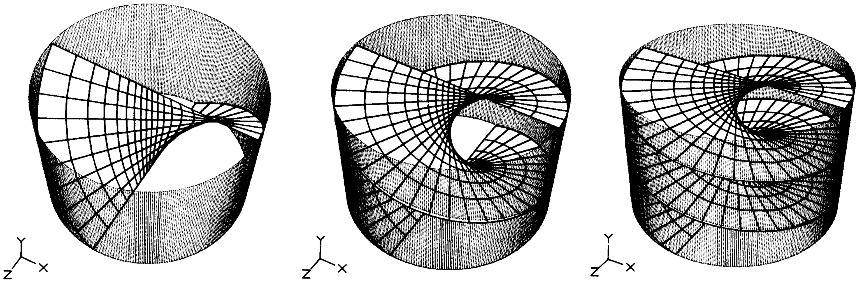


Fig. 3. Three of infinitely many noncongruent minimal surfaces that are stationary within a configuration $\langle \Gamma_1, \Gamma_2, S\rangle$

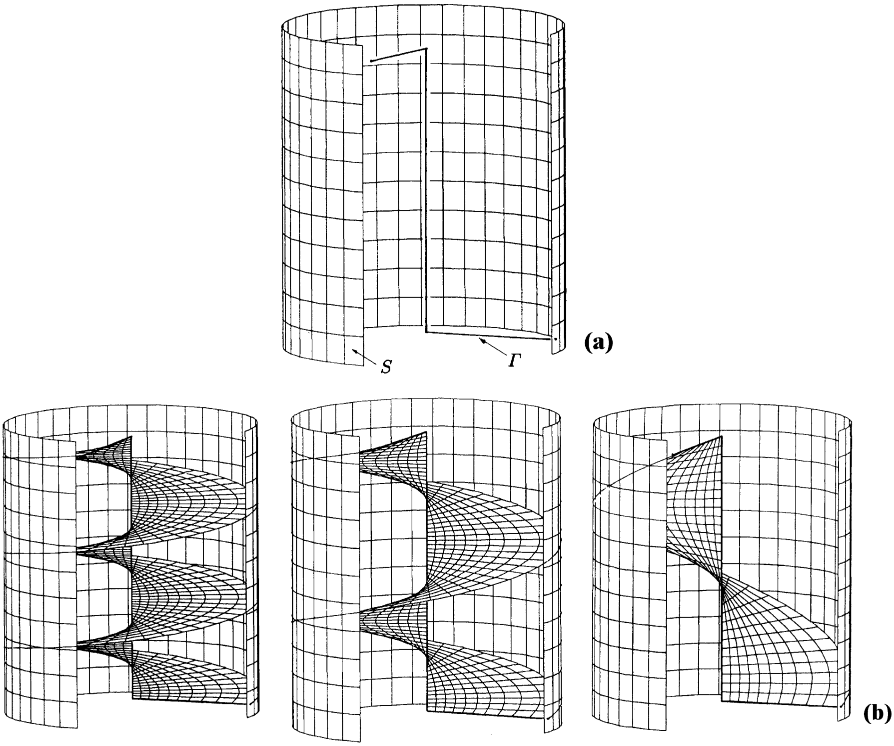


Fig. 4. A boundary configuration $\langle \Gamma, S\rangle$ **(a)** bounding infinitely many stationary minimal surfaces of the type of the disk; these are pieces of helicoids **(b)**

Next we consider a configuration $\langle \Gamma, S\rangle$ consisting of a circle Γ and of a supporting surface S which bounds a continuum of noncongruent and even area minimizing minimal surfaces. It turns out that such an example can be derived from the classical calculus of variation. In the following we freely use some of these results, see Bolza [1] (Beispiel I); Bliss [1], pp. 85–127; Carathèodory [3], pp. 340–341, 360–367; Giaquinta and Hildebrandt [1].

Let $x(t), y(t), t_1 \leq t \leq t_2$, be the parameter representation of a curve contained in the upper half plane $\{y > 0\}$. The surface area of its surface of revolution about the x-axis is given by the integral $2\pi \int_{t_1}^{t_2} y\sqrt{dx^2 + dy^2}$. Thus the minimal surfaces of revolution are described by the extremals of the functional $\int y\sqrt{dx^2 + dy^2}$, $y > 0$, which are the parallels to the positive y-axis,

$$x = x_0, \quad y > 0,$$

and the catenaries

$$y = a\cosh\left(\frac{x - x_0}{a}\right), \quad -\infty < x < \infty, \tag{1}$$

which form a 2-parameter family of nonparametric curves, $a > 0, -\infty < x_0 < \infty$. The point (x_0, a) is the vertex of the catenary (1).

Let us consider all catenaries passing through some fixed point $P = (0, b), b > 0$, on the y-axis. They must satisfy $b = a\cosh(\frac{x_0}{a})$ or $b = a\cosh\lambda$, if we introduce the new parameter $\lambda = -\frac{x_0}{a}$. Then there is a 1-1 correspondence between all real values of the parameter λ and all catenaries passing through $(0, b)$ which is given by

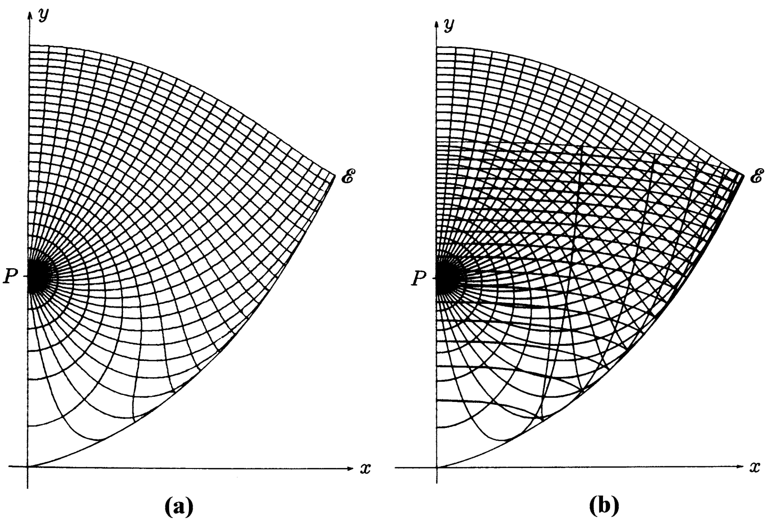


Fig. 5. (a) Catenaries emanating from P to the right, and their wave fronts. **(b)** A complete figure: The stable catenaries emanating from P and terminating at their envelope $\mathcal{E}$, together with their wave fronts

(2)
$$y = g(x,\lambda) := a(\lambda)\cosh\left(\lambda + \frac{x}{a(\lambda)}\right), \quad x \in \mathbb{R},$$
$$a(\lambda) := \frac{b}{\cosh\lambda}, \quad \lambda \in \mathbb{R}.$$

We can also write

$$g(x,\lambda) = b\cosh\frac{x}{a(\lambda)} + a(\lambda)\sinh\lambda\sinh\frac{x}{a(\lambda)},$$

and $\sinh\lambda = \pm\sqrt{b^2 - a^2(\lambda)}/a(\lambda)$.

We now consider the branches $y = g(x,\lambda), x \geq 0$, lying in the first quadrant of the x,y-plane. There exists exactly one conjugate point $Q(\lambda) = (\xi(\lambda),\eta(\lambda))$ with respect to P on each catenary (2). The points $Q(\lambda), \lambda \in \mathbb{R}$, form a real-analytic curve $\mathcal{E}$ that resembles a branch of a parabola extending from the origin to infinity. The curve $\mathcal{E}$ is given by the condition

$$\frac{\partial}{\partial\lambda}g(x,\lambda) = 0$$

and describes the envelope of the catenary arcs

$$C_\lambda = \{(x, g(x,\lambda))\colon 0 \leq x \leq \xi(\lambda)\}, \quad \lambda \in \mathbb{R}.$$

The domain $\Omega = \{(x,y)\colon 0 < x < \xi(\lambda), y > \eta(\lambda) \text{ for some } \lambda\}$ is simply covered by the open arcs $\mathring{C}_\lambda = C_\lambda \setminus \{P, Q(\lambda)\}$.

Consider the *wavefronts* $W_c, c > 0$, emanating from P. The curves W_c are the real analytic level lines $\{\mathcal{S}(x,y) = c\}$ of the wave function $\mathcal{S}(x,y)$ that satisfies the Hamilton–Jacobi equation

$$\mathcal{S}_x^2 + \mathcal{S}_y^2 = y^2$$

and is given by

$$\mathcal{S}(x, g(x,\lambda)) = \mathcal{I}(x,\lambda), \quad 0 \leq x \leq \xi(\lambda),$$

where the right-hand side is defined by

$$\mathcal{I}(x,\lambda) = \int_0^x g(u,\lambda)\sqrt{1 + g'(u,\lambda)^2}\,du,$$

and $g'(u,\lambda) = \frac{\partial}{\partial u}g(u,\lambda)$.

The two families of curves $C_\lambda, \lambda \in \mathbb{R}$, and $W_c, c > 0$, form the *complete figure* (in sense of Carathéodory) associated with the variational problem

$$\int y\sqrt{dx^2 + dy^2} \to \text{Extr}, \quad y(0) = b,$$

in $x \geq 0, y > 0$, see Fig. 5.

By Adolf Kneser's transversality theorem, the curves W_c intersect the catenaries C_λ orthogonally. Two curves W_{c_1} and W_{c_2}, $c_1 < c_2$, cut a piece $C_\lambda(c_1, c_2)$ out of each curve C_λ such that

$$\int_{C_\lambda(c_1,c_2)} y\sqrt{dx^2+dy^2} = c_2 - c_1,$$

and $c_2 - c_1$ is the infimum of the integral $\int y\sqrt{dx^2+dy^2}$ along all paths joining W_{c_1} and W_{c_2} within Ω. In particular, if $C_{\lambda,c} = \{(x, g(x,\lambda)) \colon 0 \leq x \leq x_0(\lambda, c)\}$ denotes the subarc of the catenary that connects P with W_c, then $\mathcal{I}(x_0(\lambda,c),\lambda)$ is the infimum of the integral $\int y\sqrt{dx^2+dy^2}$ taken along all curves joining P and W_c within Ω. If we now rotate the whole configuration shown in Fig. 5 about the x-axis, the wavefront W_c generates a surface of revolution S_c, and each catenary $C_{\lambda,c}$ produces a minimal catenoid $K_{\lambda,c}$ with the area $2\pi c$. The catenoid $K_{\lambda,c}$ is bounded by two parallel coaxial circles Γ and $\Sigma_{\lambda,c}$ centered on the x-axis. Γ is generated by the rotation of P, and $\Sigma_{\lambda,c}$ by the rotation of the intersection point of C_λ with W_c. Each catenoid $K_{\lambda,c}$ intersects S_c orthogonally and, therefore, is a stationary minimal surface within the configuration $\langle \Gamma, S_c \rangle$. All catenoids $K_{\lambda,c}$, c fixed, have the same area and minimize area among all surfaces of revolution bounded by $\langle \Gamma, S \rangle$ which lie in the open set $\mathcal{H}$ generated by rotating $\Omega \cup \Omega^* \cup \{x = 0, y > 0\}$ about the x-axis. Here Ω^* is the mirror image of Ω at the y-axis in the x, y-plane (cf. Fig. 6).

In fact, it turns out that the catenoids $K_{\lambda,c}$ even minimize area among all orientable surfaces $\mathcal{F}$ bounded by $\langle \Gamma, S_c \rangle$ that are contained in $\mathcal{H}$. A well-known projection argument shows that it suffices to prove $\text{Area}(K_{\lambda,c}) \leq \text{Area}(\mathcal{F})$ for all oriented surfaces $\mathcal{F}$ with boundary on $\Gamma \cup S_c$ that are contained in $\mathcal{H}^+ = \mathcal{H} \cap \{x \geq 0\}$.

Let now $\mathcal{F}$ be such a surface with $\gamma = \partial\mathcal{F} \cap S_c$. Then there is a region T in the surface S_c with integer multiplicities, the boundary of which equals $\gamma - \Sigma_{\lambda,c}$. Therefore $K_{\lambda,c} - \mathcal{F} + T$ is a cycle, and it follows that there is a three-dimensional region $\mathcal{R}$ with integer multiplicities such that the boundary of $\mathcal{R}$ is $K_{\lambda,c} - \mathcal{F} + T$. Gauss's theorem yields

$$\int_{\mathcal{R}} \operatorname{div} X \, \text{dvol} = \int_{\partial\mathcal{R}} \langle X, N_{\partial\mathcal{R}} \rangle \, dA, \tag{3}$$

where $N_{\partial R}$ is the oriented unit normal to $\partial\mathcal{R}$. Let $X = X(x, y, z)$ be a field of unit vectors normal to the foliation formed by the catenoids $K_{\lambda,c}$. Then we infer from Vol. 1, Section 2.7, in particular from formula (3) that

$$\operatorname{div} X = -2H,$$

H being the mean curvature of the leaves of the foliation. Since $H \equiv 0$, the vector field X is divergence free. Since $\langle X, N_T \rangle = 0$ and X can be chosen in such a way that $\langle X, N_{K_{\lambda,c}} \rangle = 1$, we obtain

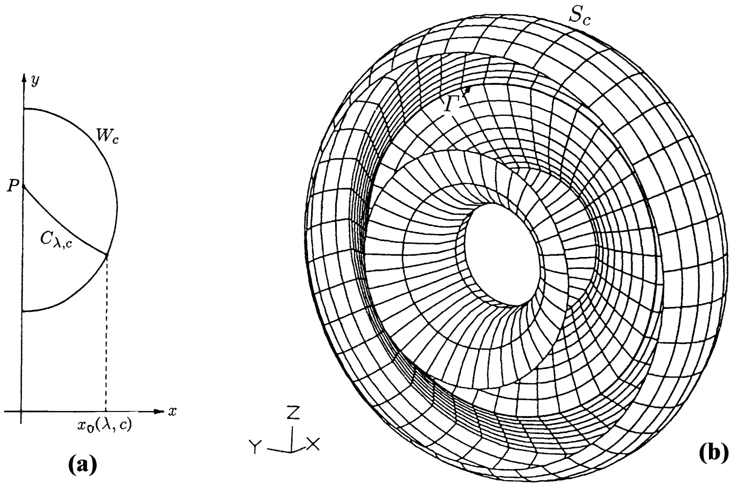


Fig. 6. **(a)** Rotation of the wave front W_c about the x-axis yields half the surface S_c; the whole surface S_c is then obtained by reflection at the plane $x = 0$. The curve Γ is a circle obtained by rotating P about the x-axis. **(b)** This drawing depicts the configuration $\langle \Gamma, S_c \rangle$ and two of the minimal leaves within $\langle \Gamma, S_c \rangle$. A part of S_c is removed to permit a glimpse into the interior

$$\text{Area}(K_{\lambda,c}) = \int_{K_{\lambda,c}} \langle X, N_{K_{\lambda,c}} \rangle \, dA = \int_{\mathcal{F}} \langle X, N_{\mathcal{F}} \rangle \, dA.$$

Because of $\langle X, N_{\mathcal{F}} \rangle \le 1$, the term on the right-hand side is estimated from above by Area($\mathcal{F}$). Thus we have proved:

Theorem 1. *There exists a configuration $\langle \Gamma, S_c \rangle$ consisting of a circle Γ and a real analytic surface of revolution S_c that bounds a family $\{K_{\lambda,c}\}$ of stationary and even area-minimizing minimal surfaces of annulus-type that are really distinct in the sense that, for any two different values λ_1, λ_2, the surfaces $K_{\lambda_1,c}$ and $K_{\lambda_2,c}$ are not congruent.*

A simple modification of the previous example leads to *boundary configurations S as shown in Fig. 7 that bound continua $\mathcal{C}$ of noncongruent stationary surfaces of annulus type which have a completely free boundary on S. The surfaces of $\mathcal{C}$ are even area minimizing within the class $\mathcal{C}^*$ of annulus type surfaces whose free boundaries are homologous to those of the surfaces of $\mathcal{C}$.*

For this purpose, we take two wavefront curves W_{c_1} and $W_{c_2}, c_1, c_2 > 0$, contained in $x > 0, y > 0$. If c_1 and c_2 are chosen sufficiently small, both curves terminate at the positive y-axis and meet this axis orthogonally. Reflecting both arcs at the y-axis, we obtain two closed real analytic curves Γ_{c_1} and Γ_{c_2}, and their rotation about the x-axis leads to two closed torus-type surfaces S_1 and S_2 that are orthogonally met by a family of catenoids, generated by the catenary arcs $C_\lambda(c_1, c_2)$. These catenoids are stationary annulus-type

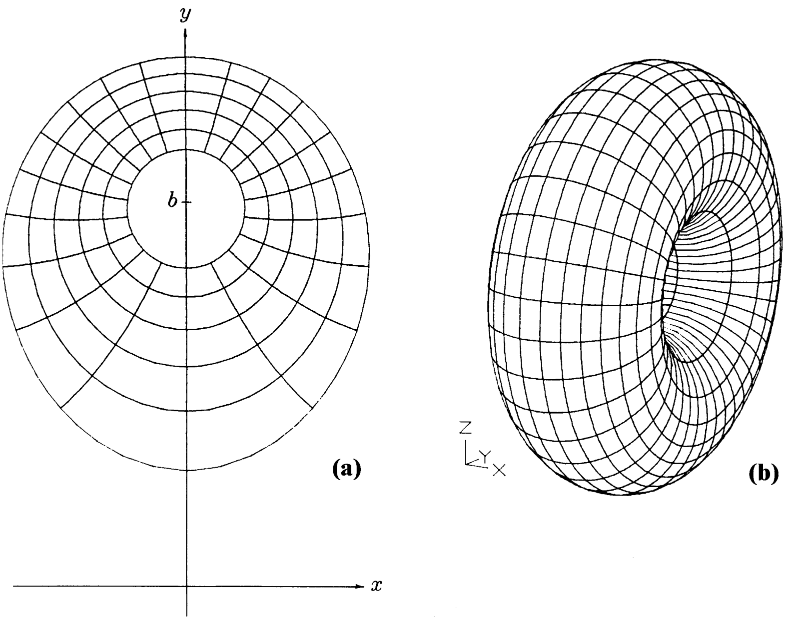


Fig. 7. A modification of the example depicted in Fig. 6. Any of the closed curves W_c in **(a)** generates a real analytic and rotationally symmetric surface S_c as depicted in **(b)**. Any configuration $\langle S_1, S_2 \rangle$ with $S_i := S_{c_i}$ bounds a one parameter family $\mathcal{C}$ of annulus-type minimal leaves which are parts of catenoids. **(c)**, **(d)** Parts of the configuration $\langle S_1, S_2 \rangle$. **(e)**, **(f)** Three surfaces of the family $\mathcal{C}$ outside and within $\langle S_1, S_2 \rangle$

minimal surfaces within the configuration $\langle S_1, S_2 \rangle$, and a reasoning similar to the previous one shows that they even minimize area within $\mathcal{C}^*$ (cf. Fig. 7).

A somewhat different example, which is not rotationally symmetric, leads to a free-boundary problem for minimal surfaces of the type of the disk, with their boundary lying on a given real analytic torus-like surface. Let $K_\lambda, \lambda \in \mathbb{R}$, be the catenoids obtained by rotating the arc C_λ about the x-axis, and let K_λ^* be the surface obtained from K_λ by reflection at the y, z-plane. Moreover, let $K_{-\infty}$ be the disk interior to the circle Γ in the y, z-plane, and let K_∞ be the plane domain exterior to Γ. We may think of $K_{\pm\infty}$ as degenerate catenoids obtained for $\lambda \to \pm\infty$. Then the surfaces $K_\lambda, K_\lambda^*, -\infty \leq \lambda \leq \infty$, describe a minimal foliation, singular at Γ, of the rotationally symmetric domain $\mathcal{H}$.

We now introduce cylindrical coordinates (x, r, θ), where $y = r\cos\theta, z = r\sin\theta$. For each $r \in (0, b)$, there exists exactly one value $c(r) > 0$ such that the closed, real analytic curve $\Gamma_{c(r)}$ in the plane $\theta = 0$, obtained from the wavefront $W_{c(r)}$ as described before, passes through $(0, r, 0)$.

Denote by $L_{r,\theta}$ the closed curve that is obtained by rotating $\Gamma_{c(r)}$ about the angle θ around the x-axis. The curves $L_{r,\theta}, 0 < r < b, 0 \leq \theta < 2\pi$, meet the plane $x = 0$ orthogonally at the points $(0, r, \theta)$ and sweep out an open subdomain $\mathcal{H}_0$ of $\mathcal{H}$.

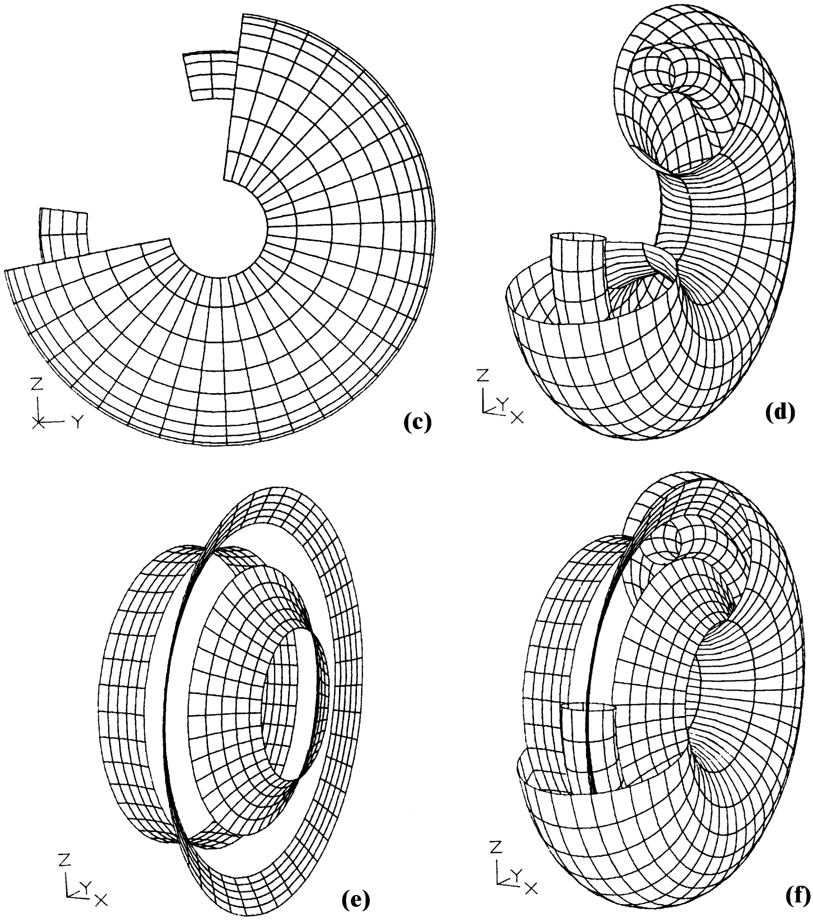


Fig. 7. (Continued. Captions see preceding page)

Let γ_0 be a real analytic Jordan curve in the plane $x = 0$, say, a circle, which is contained in the open disk $K_{-\infty}$ (the interior of Γ) and does not wind about the origin. As the point $(0, r, \theta)$ traverses the curve γ_0, the curves $L_{r,\theta}$ sweep out a toruslike surface S which bounds a tube G. This tube is foliated by a family $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*, -\infty \le \lambda \le \infty$, of minimal surfaces that are cut by S out of the catenoids K_λ, K_λ^*. The surfaces $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*$ are of the type of the disk and meet S perpendicularly; hence they are stationary within S (cf. Figs. 8 and 9). Moreover, the unit normal vectors to $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*$ form a divergence free vector field on the set $\mathcal{H} \setminus \Gamma$ containing G which is tangent to S. Then, by an argument parallel to the previous reasoning, all surfaces $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*$ have equal area, and each oriented surface $\mathcal{F}$ contained in $\mathcal{H} \setminus \Gamma$ and with a boundary γ homologous in S to γ_0 has area larger than the leaves $\mathcal{M}_\lambda, \mathcal{M}_\lambda^*$ unless it coincides with one of these surfaces. Thus we have shown:

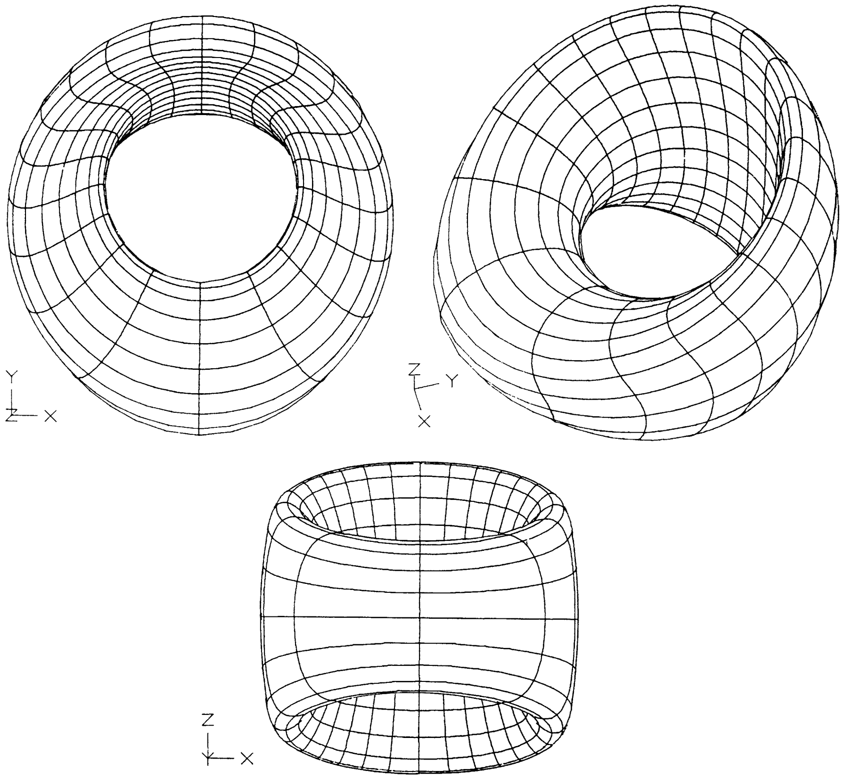


Fig. 8. Three views of a real analytic topological torus bounding a 1-parameter family of minimal disks whose traces on S are depicted in the figures by curvy lines

Theorem 2. *There exists a real analytic, embedded surface S of the type of the torus, and a homology class $[\gamma_0]$ in $H_1(S;\mathbb{Z})$, so that S bounds a family of stationary minimal surfaces of the type of the disk which have smallest area among all oriented surfaces in $\mathcal{H} \setminus \Gamma$ having their boundaries lying on S and homologous in S to γ_0.*

In view of the two examples described in Theorems 1 and 2, the following two *theorems* will be rather surprising.

Theorem 3 (F. Tomi). *If a compact analytic H-convex body M in $\mathbb{R}^3$ has the property that there is closed Jordan curve in M which cannot be contracted in M and, secondly, that the free boundary problem for ∂M admits infinitely many minimizing solutions of disk-type contained in M, then M must be homeomorphic to a solid torus, and the set of all such solutions is an analytic S^1-family of minimal embeddings of the disk.*

For the proof of Theorem 3, we refer the reader to Tomi's paper [10]. There one also finds the following interesting observation:

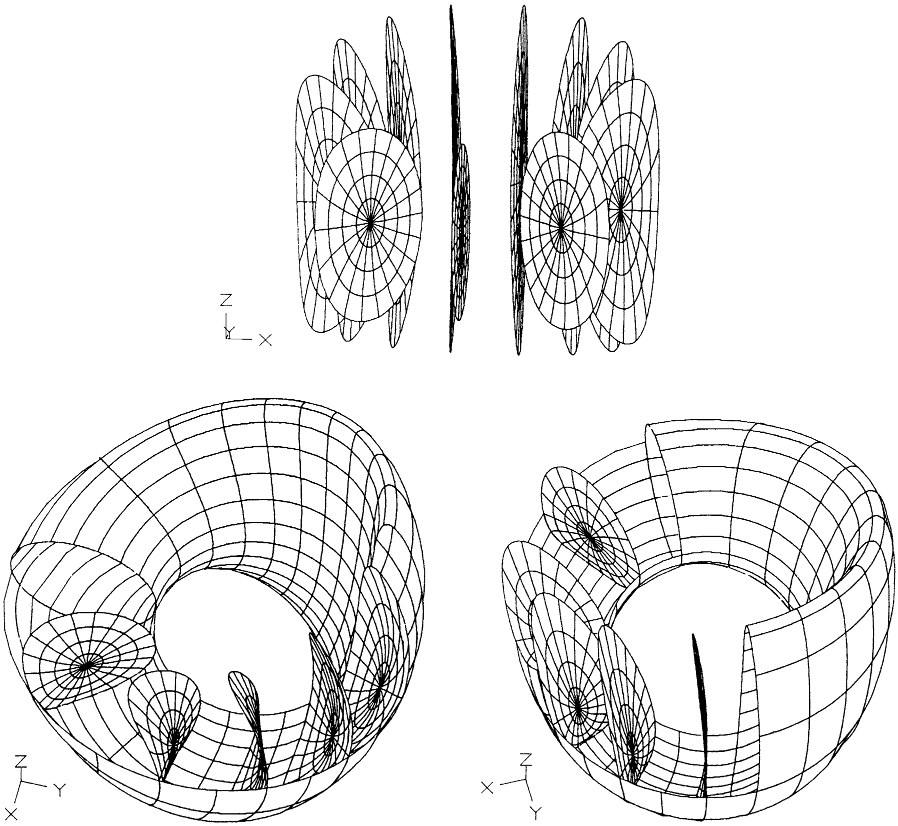


Fig. 9. Samples of minimal leaves $\mathcal{M}_\lambda, \mathcal{M}^*_\lambda$, two of which are flat, and how they fit into the surface S shown in Fig. 8. A part of S has been removed to permit a glimpse into the interior of S

If a torus M is foliated by a smooth S^1-family of plane disk-type minimal surfaces being orthogonal to ∂M, then all surfaces in the family are congruent.

In contrast, we obtain from Theorem 2 the following result:

There exist real analytic (topological) tori admitting families of non-flat disk-type minimal surfaces which intersect the tori at a right angle, and secondly, the surfaces in such a family need not be congruent (nor isometric).

Related to Theorem 3 there is a *finiteness theorem* due to Alt and Tomi [1] that will be stated as Theorem 4. We shall outline a proof of this result. As their techniques are closely related to the methods used for proving Theorem 3, the reader will obtain a good idea of how such results are proved.

Let S be a compact, embedded, real analytic surface in $\mathbb{R}^3$, and let $\varPi$ be a homotopically nontrivial closed polygon in the unbounded component of $\mathbb{R}^3 \setminus S$.

As in Section 1.2 we define the class $\mathcal{C}(\varPi, S)$ by

$$\mathcal{C}(\varPi, S) := \{X \in \mathcal{C}(S) \colon \mathcal{L}([X|_{\partial B}], \varPi) \neq 0\};$$

that is, $\mathcal{C}(\varPi, S)$ is defined as the set of all $X \in \mathcal{C}(S)$ whose boundary values are not contractible in $\mathbb{R}^3 \setminus \varPi$. Then we obtain the following finiteness result contrasting with Theorem 2:

Theorem 4. *There are only finitely many geometrically different minimal surfaces which are minimizers of Dirichlet's integral in* $\mathcal{C}(\varPi, S)$.

As a by-product of the proof of Theorem 4 we obtain the following result which is of independent interest.

Theorem 5. *Let* $X \in \mathcal{C}(\varPi, S)$ *be a strong relative minimum of* $\mathcal{P}(\varPi, S)$, *i.e. we have* $D(X) \le D(Y)$ *for all surfaces* $Y \in \mathcal{C}(\varPi, S)$ *with* $Y(\bar{B}) \subset \mathcal{U}, \mathcal{U}$ *being an open neighbourhood of* $X(\bar{B})$. *Then* X *is immersed up to the boundary, that is,* $|X_u(w)| = |X_v(w)| \neq 0$ *for all* $w \in \bar{B}$.

Anticipating the regularity results of Chapter 2 we may assume that each minimizer X – and even each stationary point – can be continued analytically across the boundary ∂B. Moreover, if $w_0 \in \bar{B}$ is a branch point (i.e., $X_u(w_0) = X_v(w_0) = 0$), then we obtain as in Section 3.2 of Vol. 1 and Section 2.10 of this volume in suitable (new) coordinates x^1, x^2, x^3 the representation

$$\begin{aligned} x^1(w) + ix^2(w) &= A(w - w_0)^m + O(|w - w_0|^{m+1}), \\ x^3(w) &= O(|w - w_0|^{m+1}) \end{aligned} \tag{4}$$

with $A \in \mathbb{C}^3 \setminus \{0\}$. Next we infer from Lemma 5 of Section 5.3 the existence of a C^1-diffeomorphism $F \colon \mathcal{U} \to V$ defined on a neighbourhood $\mathcal{U}$ of w_0 such that for some function $\varphi \in C^2(V)$ we have

$$\begin{aligned} x^1(w) + ix^2(w) &= [F(w)]^m \\ x^3(w) &= \varphi(F(w)). \end{aligned} \tag{5}$$

Moreover it follows from the proof of Lemma 5 of Section 5.3 that $F(w_0) = 0, F \in C^\omega(\mathcal{U} \setminus \{w_0\})$ and $\varphi \in C^\omega(V \setminus \{0\})$, V being a suitable neighbourhood of $0 \in \mathbb{R}^2 \mathrel{\widehat{=}} \mathbb{C}$. Of course we may assume that V is a disk $B_r(0)$ of a sufficiently small radius $r > 0$. The representation (5) permits us to introduce the new variable $\tilde{w} = F(w) \in V$, and we have

$$\nabla^k \varphi(\tilde{w}) = O(|\tilde{w}|^{m+1-k}) \quad \text{as } \tilde{w} \to 0$$

for $k = 0, 1, 2$.

We have to distinguish true and false branch points of a given minimal surface $X(w)$; the surface has different geometric properties in a neighbourhood of different kinds of branch points.

We call a branch point w_0 of X a *false branch point* if in some neighbourhood of w_0 the surface $X(w)$ can be reparametrized as an immersed surface. This is true if and only if φ is a function of $(\tilde{w})^m$. Otherwise w_0 is called a *true branch point*. It is shown in Chapter 5 how to exclude true branch points on the boundary by using only the minimum property of X. Since the argument is similar for true interior branch points, we refrain from repeating the procedure and refer to Section 5.3 as well as to the original papers by Osserman [12], Alt [1] and Gulliver [2]. Another possibility could be to apply Tromba's technique, which is presented in Chapter 6. We are going to outline the discussion for false branch points. Note that by analytic continuation we may assume X to be defined on some open neighbourhood $B_R, R > 1$, of the closed unit disk $\bar{B}$. Denoting the new function again by X, we may in addition assume that all branch points of X lie in $\bar{B}$ and that (5) continues to hold. Moreover, we can define a continuous unit normal $N(w)$ for $X(w)$ on all of $\bar{B}$.

Definition 1. *Two points $z, w \in B_R$ are called* equivalent*, $z \sim w$, if there are fundamental systems of open neighbourhoods $\mathcal{U}_n(z), V_n(w), n \in \mathbb{N}$ such that $X(\mathcal{U}_n) = X(V_n)$ for all n. We also define the* equivalent boundary $\tilde{\partial} B$ *by $\tilde{\partial} B = \{z \in \overline{B} : z \sim w \text{ for some } w \in \partial B\}$.*

Proposition 1. *Suppose $z_k \to z, w_k \to w$ and $z_k \sim w_k$. Then $z \sim w$. In particular, the equivalent boundary $\tilde{\partial} B$ is closed.*

Proposition 1 is a consequence of

Lemma 1. *Let z and w be two points in $B_R, R > 1$, such that $X(z) = X(w)$ and $N(z) = \pm N(w)$. Furthermore denote by $\mathcal{U}$ and V coordinate neighbourhoods of z and w such that a representation (5) holds, and suppose that there is an open subset $\mathcal{U}'$ of $\mathcal{U}$ with the property that $X(\mathcal{U}') \subset X(V)$. Then it follows that $z \sim w$.*

Proof. From (5) we infer the existence of small positive number r and s such that,

$$
\begin{aligned}
X(\mathcal{U}) &= \{(x^1, x^2, x^3) \colon x^1 + ix^2 = (\tilde{w})^m, x^3 = \varphi(\tilde{w}), |\tilde{w}| < r\},\\
X(V) &= \{(x^1, x^2, x^3) \colon x^1 + ix^2 = (\tilde{\omega})^n, x^3 = \psi(\tilde{\omega}), |\tilde{\omega}| < s\}.
\end{aligned}
$$

Since $X(\mathcal{U}) \subset X(V)$, it follows that for some open set of numbers z contained in $\{w \colon 0 < |w| < \min(\sqrt[n]{r}, \sqrt[m]{s})\}$ the relation $\varphi(z^n) = \psi(\eta_n z^m)$ holds true where η_n denotes some n-th root of unity. By the analyticity of φ and ψ in $0 < |z| < r$ and $0 < |z| < s$ respectively we conclude that $\varphi(z^n) = \psi(\eta_n z^m)$ holds for all z with $|z| < \min(\sqrt[n]{r}, \sqrt[m]{s})$. Next define

$$\mathcal{U}_\varepsilon := \{z \in \mathcal{U} \colon |\tilde{w}(z)| < \sqrt[m]{\varepsilon}\}, \quad V_\varepsilon := \{z \in V \colon |\tilde{\omega}(z)| < \sqrt[n]{\varepsilon}\}.$$

Then it follows that $X(\mathcal{U}_\varepsilon) = X(V_\varepsilon)$ for suitably small $\varepsilon > 0$. □

For the formulation of the next result the following definition will be clarifying.

Definition 2. *An analytic arc in* $\mathbb{R}^n, n \geq 2$, *emanating from a point* p *of* $\mathbb{R}^n$, *is the image of a closed interval* $[0, \delta]$ *under a nonconstant, real analytic map* α *which is defined on an open interval containing* $[0, \delta]$ *and satisfies* $\alpha(0) = p$.

Lemma 2. *Let* $\varphi \colon \mathcal{U} \to \mathbb{C}$ *be an analytic function defined on some neighbourhood* $\mathcal{U}$ *of the origin* 0 *in* $\mathbb{C}$, *and suppose that*

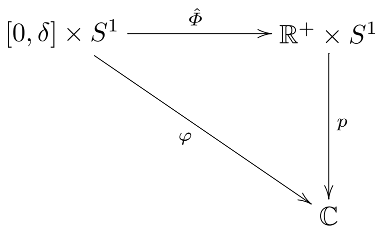
$$\varphi(z) = az^m + O(|z|^{m+1}) \quad \text{as } z \to 0, \tag{6}$$

for some $m \in \mathbb{N}, a \in \mathbb{C}, a \neq 0$. *Furthermore, let* $\alpha \colon [0, \tau] \to \mathbb{C}$ *be a regular analytic arc emanating from* $0 \in \mathbb{C}$. *Then there exists some* $\tau_0 \in (0, \tau]$ *such that* $\varphi^{-1}(\alpha[0, \tau_0])$ *consists of* m *analytic arcs emanating from* 0.

Proof. Choose some neighbourhood $B_\delta = B_\delta(0) \subset \mathbb{C}$ with $\varphi \neq 0$ for all $z \in B_\delta \setminus \{0\}$, and introduce polar coordinates $(r, \xi) \in [0, \delta] \times S^1$. Without loss of generality we assume that $a = 1$. Then (6) implies that

$$\varphi(r, \xi) = r^m \xi^m \{1 + \varphi_1(r, \xi)\} \tag{7}$$

with some analytic function φ_1 satisfying $\varphi_1(0, \xi) = 0$. Let $\hat{\Phi} \colon [0, \delta] \times S^1 \to \mathbb{R}^+ \times S^1$ be a mapping so that the following diagram commutes:



where $p(r, \xi) := r \cdot \xi$. Hence $\hat{\Phi}(r, \xi) = (|\varphi(r, \xi)|, \frac{\varphi(r,\xi)}{|\varphi(r,\xi)|})$. Similarly, let $\hat{\alpha} : [0, \tau] \to \mathbb{R}^+ \times S^1$ be chosen in such a way that $p \circ \hat{\alpha}(t) = \alpha(t)$, i.e., $\hat{\alpha}(t) = (\rho(t), \gamma(t))$ with real analytic functions $\rho(t) \geq 0$ and $\gamma(t) \in S^1$. In fact, replacing $\alpha(t)$ by the mapping $\tilde{\alpha}(t) := \alpha(t^{2m})$ which parametrizes the same arc, we may even assume that $\hat{\alpha}(t) = (\rho_1^m(t), \gamma(t))$ with analytic functions $\rho_1(t)$ and $\gamma(t)$. Note that $\gamma(0) = |\dot{\alpha}(0)|^{-1}\dot{\alpha}(0)$ is the direction of α at zero. We infer from (7) that

$$\hat{\Phi}(0, \xi) = (0, \xi^m) \tag{8}$$

and that $\hat{\Phi}$ can be continued analytically onto $[-\delta, \delta] \times S^1$ for some suitable $\delta > 0$. We can now define an analytic map $\tilde{\Phi}$ by

$$\tilde{\Phi}(r, \xi) := (\sqrt[m]{|\varphi(r, \xi)|}, |\varphi(r, \xi)|^{-1}\varphi(r, \xi)).$$

We are interested in the pre-image of $\hat{\alpha}$ under $\hat{\Phi}$, or equivalently, in the pre-image of $(\rho_1(t), \gamma(t))$ under $\tilde{\Phi}$. By virtue of (8) we infer that $\tilde{\Phi}^{-1}(0, \gamma(0))$ consists of the m points $(0, \gamma_1), \ldots, (0, \gamma_m)$ where $\gamma_1, \ldots, \gamma_m$ denote the m-th roots of $\gamma(0)$.

From the properness of $\tilde{\Phi}$ we infer that, for any given ε_0-neighbourhood $U_{\varepsilon_0}(0, \gamma_j)$ of $(0, \gamma_j)$ in $[0, \delta] \times S^1$, there exists a number $\varepsilon > 0$ such that

$$\tilde{\Phi}^{-1}(U_\varepsilon(0, \gamma(0))) \subset \bigcup_{j=1}^{m} U_{\varepsilon_0}(0, \gamma_j).$$

We choose ε_0 in such a way that $\tilde{\Phi}$ is an analytic diffeomorphism on each rectangle $\{(r, \xi)\colon |r| < \varepsilon_0, |\xi - \gamma_j| < \varepsilon_0\}$. Finally we select $\tau_0 > 0$ so small that $\rho_1(t) < \varepsilon$ and $|\gamma(t) - \gamma(0)| < \varepsilon$ holds for all $t \in [0, \tau_0]$. Then $\tilde{\Phi}^{-1}(\rho_1(t), \gamma(t))|_{[0,\tau_0]}$ consists of m analytic arcs emanating from $(0, \gamma_1), \ldots,$ $(0, \gamma_m)$. Therefore the set $\varphi^{-1}(\alpha[0, \tau_0])$ consists of m disjoint arcs starting at 0 with the directions $\gamma_1, \ldots, \gamma_m$. □

Lemma 3. *The equivalent boundary $\tilde{\partial} B$ is the union of finitely many analytic arcs.*

Proof. By Proposition 1, the set $\tilde{\partial} B$ is compact, and hence we may argue locally. First we claim that, for arbitrary $z_0 \in \bar{B}$, the pre-image of $P_0 := X(z_0)$ consists of only finitely many points. In fact, assuming the contrary, we would obtain a sequence $\{Z_j\}_{j \in \mathbb{N}} \in X^{-1}(P_0)$ with $z_j \to w$ whence, by continuity of X, we would have $w \in X^{-1}(P_0)$. However this would contradict (5) since any neighbourhood of w would contain points z_j with $X(z_j) = X(w) = X(z_0)$. Thus there are only finitely many points $z_1, \ldots, z_n \in \partial B$ which are equivalent to a given $z_0 \in \tilde{\partial} B$. For given (small) neighbourhoods $\mathcal{U}_j = \mathcal{U}_j(z_j)$ we can find a neighbourhood $\mathcal{U}$ of z_0 with

$$\{w \in \partial B\colon w \sim z \in \mathcal{U}\} \subset \bigcup_{j=1}^{m} \mathcal{U}_j(z_j) \cap \partial B. \tag{9}$$

Otherwise there would exist a sequence of points $\xi_k \in B_R, R > 1$, with $\xi_k \to z_0$, and another sequence of points $w_k \in \partial B$ with $w_k \sim \xi_k$ but $w_k \notin \bigcup_{j=1}^n \mathcal{U}_j(z_j)$. Passing to a sequence, we could assume that $w_k \to w \in \partial B \setminus \bigcup_{j=1}^n \mathcal{U}_j(z_j)$. Because of Proposition 1 we would have $w \sim z_0$ or $w = z_j$ for some $j \in \{1, \ldots, n\}$, an obvious contradiction. Since $z_0, z_1, \ldots, z_n$ are equivalent, we may assume that $X(z_0) = X(z_1) = \cdots = X(z_n) = 0$ and that the common tangent plane is the (x^1, x^2)-plane. Denote by φ the mapping

$P \circ X$, where $P\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the orthogonal projection onto the x^1, x^2-plane. Then (9) implies

$$\tilde{\partial} B \cap \mathcal{U} \subset \bar{B} \cap \mathcal{U} \cap \varphi^{-1}\left(\bigcup_{j=1}^{n} \varphi(\mathcal{U}_j \cap \partial B)\right).$$

The set $\mathcal{U}_j \cap \partial B$ consists of two circular arcs γ_j^+, γ_j^- emanating from z_j in opposite directions. Also, $\bigcup_{j=1}^{n} \varphi(\mathcal{U}_j \cap \partial B)$ is a finite union of analytic arcs starting from the origin. Now we apply Lemma 2, choosing possibly smaller neighbourhoods $\mathcal{U}_j$ and $\mathcal{U}$, and conclude that $\varphi^{-1}(\bigcup_{j=1}^{n} \varphi(\mathcal{U}_j \cap \partial B)) \cap \mathcal{U}$ is a collection of analytic arcs $\alpha_1, \ldots, \alpha_N$, all starting at z_0. The lemma is proved if we can show that every arc α_k containing one point $z \in \tilde{\partial} B$ different from z_0, already belongs to $\tilde{\partial} B$. To this end let $\varphi(\alpha_k) \subset \varphi(\gamma_j^+)$ for some j and suppose that $z \in \alpha_k \setminus \{z_0\}$ is equivalent to $w \in \gamma_j^+ \setminus \{z_j\}$. We infer from (6) that we can write $X(\sigma \cap \mathcal{U}_j)$ as a graph over the plane domain $\varphi(\sigma \cap \mathcal{U}_j)$, where σ denotes the open sector

$$\{z_j + re^{i\theta}\colon r > 0, |\theta - \theta_j| < \varepsilon\}, \quad \theta_j = \arg z_j \pm \pi/2.$$

Since $\varphi(\alpha_k) \subset \varphi(\gamma_j^+)$ and X is continuous, we can find another open sector

$$\sigma_0 = \{z_0 + re^{i\theta}\colon r > 0, |\theta - \theta_0| < \delta\},$$

$e^{i\theta_0}$ being the direction of α_k at z_0, such that we have $\varphi(\sigma_0 \cap \mathcal{U}) \subset \varphi(\sigma \cap \mathcal{U}_j)$ and $\alpha_k \setminus \{z_0\} \subset \sigma_0$ for sufficiently small $\mathcal{U}$.

Also $X|_{\sigma_0 \cap \mathcal{U}_S}$ is a graph over $\varphi(\sigma_0 \cap \mathcal{U})$. Since $z \in \alpha_k \setminus \{z_0\}$ and $w \in \gamma_j^+ \setminus \{z_j\}$ are equivalent, we infer from the analyticity of minimal graphs that $X(\sigma_0 \cap \mathcal{U}) \subset X(\sigma \cap \mathcal{U}_j)$. In particular, we have $\alpha_k \subset \tilde{\partial} B$. □

Lemma 4. *Denote by $\tilde{\partial}_1 B$ the connected component of $\tilde{\partial} B$ which contains ∂B. Then $B \setminus \tilde{\partial}_1 B$ is connected.*

Proof. Lemma 3 implies that $B \setminus \tilde{\partial}_1 B$ consists of finitely many connected components $B_1, \ldots, B_n$, having piecewise analytic boundaries, whence

$$X|_{\partial B} = \sum_{k=1}^{n} X|_{\partial B_k}$$

and

$$X(\partial B_k) \subset X(\tilde{\partial} B) \subset X(\partial B) \subset S.$$

Choose some j so that $X|_{\partial B_j}$ is linked with Π, and then select some conformal map $\tau\colon B \to B_j$ of B onto B_j. If n were greater than 1, we would have

$$D(X \circ \tau) = D(X|_{B_j}) < D(X),$$

which contradicts the minimality of X. □

Consider now a (relative) minimizer X to the variational problem $\mathcal{P}(\Pi, S)$. We claim that, for a suitable reparametrization $\tilde{X} = X \circ \tau$ of X, we obtain another minimizer $\tilde{X}$ with $\tilde{\partial}_1 B = \partial B$. In fact, Lemmata 3 and 4 imply that $\tilde{\partial}_1 B$ consists of ∂B together with a finite number of trees growing out of certain points on ∂B. Let $\tau\colon B \to B \setminus \tilde{\partial}_1 B$ be a conformal map. Then the loop $X \circ \tau|_{\partial B}$ is homotopic to $X|_{\partial B}$ on S, whence $\tilde{X} = X \circ \tau \in \mathcal{C}(\Pi, S)$. We also have $D(\widetilde{X}) = D(X)$ and $\tilde{\partial}_1 B = \partial B$.

Note that the conformal reparametrization $\tau\colon B \to B \setminus \tilde{\partial}_1 B$ produces boundary branch points for the surface $\tilde{X}\colon B \to \mathbb{R}^3$ at those points $w \in \partial B$ which correspond to an endpoint $z \in \tilde{\partial}_1 B \cap B$ since, at these points, the boundary mapping runs back and forth in its own trace. Thus we have proved the following

Proposition 2. *Suppose that each strong relative minimizer $X \in \mathcal{C}(\Pi, S)$ which in addition satisfies $\tilde{\partial}_1 B = \partial B$, is immersed up to the boundary. Then the relation $\tilde{\partial}_1 B = \partial B$ holds for any strong relative minimizer $X \in \mathcal{C}(\Pi, S)$ of the variational problem $\mathcal{P}(\Pi, S)$.*

Let us now consider a minimizer X which satisfies $\tilde{\partial}_1 B = \partial B$.

Lemma 5. *Suppose that for a strong relative minimizer $X \in \mathcal{C}(\Pi, S)$ the relation $\tilde{\partial}_1 B = \partial B$ holds true. Then it follows that $\tilde{\partial} B = \partial B$.*

Proof. We argue by contradiction. Assume that the set

$$\partial_0 B := \{z \in \partial B\colon z \sim z_0 \in B\}$$

were not empty. From the definition of $\sim$ we then infer that $\partial_0 B$ is open in ∂B. The set $\partial_0 B$ in ∂B is also closed because of Proposition 1 and the assumption $\tilde{\partial}_1 B = \partial B$. In fact, let $z_n \in \partial_0 B$ be a sequence with $z_n \to z \in \partial B$ and $z_n \sim z_{0_n} \in B$. Without loss of generality, let $z_{0_n} \to z_0$. Because of Proposition 1 we obtain that $z_0 \in \tilde{\partial} B$, and since $\tilde{\partial}_1 B = \partial B$ it follows that $z_0 \in B$. Clearly, we have $z_0 \sim z$, whence $z \in \partial_0 B$. We conclude that $\partial B = \partial_0 B$ which means that X maps some neighbourhood of ∂B into $X(B)$. Thus $X(B)$ would be a compact minimal surface in $\mathbb{R}^3$, which is impossible because of the maximum principle. □

Proposition 3. *Let $X \in \mathcal{C}(\Pi, S)$ be a strong relative minimizer of $\mathcal{P}(\Pi, S)$ such that $\tilde{\partial}_1 B = \partial B$ holds true. Then X is immersed up to the boundary.*

Sketch of the proof. As we have already mentioned before, we only show the absence of false branch points. We argue by contradiction and assume first that $z_0 \in B$ is a false (interior) branch point of order m. Let $\gamma_1(t), t \in [0, 1]$, be an analytic Jordan arc which avoids branch points and points equivalent to branch points and has the following properties:

$$\gamma_1(0) = z_0, \quad \gamma_1(1) \in \partial B, \quad \gamma_1([0, 1)) \subset B.$$

We claim that there exist Jordan arcs $\gamma_k(t), t \in [0,1], k = 2, \ldots, m$, with

$$\begin{aligned} &\gamma_k([0,1)) \subset B, \quad \gamma_k(0) = z_0, \quad \gamma_k(1) \in \partial B, \\ &\gamma_k(t) \sim \gamma_l(t) \qquad\qquad\qquad \text{for } 1 \le k \le l \le m, \\ &\gamma_k((0,1)) \cap \gamma_l((0,1)) = \emptyset \qquad \text{for } k \ne l. \end{aligned}$$

In fact, the local existence of γ_k follows from the representation (5), while global existence is secured by analytic continuation. The inclusion $\gamma_k(1) \in \partial B$ follows from Lemma 1. An additional argument is required to show that γ_1 can be chosen in such a way that all γ_k are free of intersections; for details, see Alt [1] and Alt and Tomi [1]. Now let $z_1 = \gamma_1(1), \ldots, z_m = \gamma_m(1)$ denote consecutive points on ∂B in positive orientation. For convenience, we put $z_{m+1} = z_1$. If σ_k denotes the arc of ∂B bounded by z_k and z_{k+1}, we see that $X(\sigma_k)$ is a closed loop on S and that $X|_{\partial B} = \sum_{k=1}^m X|_{\sigma_k}$. We choose k such that $X|_{\sigma_k}$ is not contractible in $\mathbb{R}^3 \setminus \Pi$ and denote by B_k the subdomain of B bounded by γ_k, σ_k and γ_{k+1}. There exists a conformal map $\tau\colon \bar{B} \setminus [0,1] \to B_k \cup \overset{\circ}{\sigma}_k$ with the property that

$$\lim_{\substack{z \to t \\ \operatorname{im} z > 0}} \tau(z) = \gamma_k(t) \quad \text{for all } t \in [0,1],$$

and that

$$\lim_{\substack{z \to t \\ \operatorname{im} z < 0}} \tau(z) = \gamma_{k+1}(t) \quad \text{for all } t \in [0,1].$$

Since $\gamma_k(t) \sim \gamma_{k+1}(t)$, we infer that $X \circ \tau$ is continuous in $\bar{B}$ and that $X \circ \tau$ is contained in $\mathcal{C}(\Pi, S)$. If m were larger than 1, we had $D(X \circ \tau) < D(X)$, a contradiction to the minimum property of X.

Next we consider a false branch point z_0 on the boundary ∂B which is of order $m \ge 2$. Here it is convenient to map the closed disk $\bar{B}$ conformally onto the half plane $(\operatorname{im} z \ge 0\} \cup \{\infty\}$ and z_0 onto 0. Denote the open half plane by B, and let X be the corresponding minimal surface. Then we may also assume that $X(0) = 0$, and that the tangent plane at $X(0)$ is the x^1, x^2-plane, applying a suitable motion in $\mathbb{R}^3$. Suppose also that the direction of the curve $X(\mathbb{R}^+) \subset S$ at 0 is given by $(1,0,0)$. We want to show the existence of a curve $\alpha\colon [0,1] \to \bar{B}$ with $\alpha(0) = z_0 = 0, \alpha((0,1)) \subset B$, and $X(\alpha[0,1]) \subset S$. From the representation formula (5) we infer the existence of numbers $r, R > 0$ and $\theta \in (\frac{\pi}{m}, \frac{2\pi}{m})$ such that the image of the sector $S_{r,\theta} := \{\rho e^{i\varphi}\colon 0 < \rho < r, 0 < \varphi < \theta\}$ under the mapping $\phi = P \circ X$ covers the half disk

$$H_R = \left\{\rho e^{i\varphi}\colon \frac{\pi}{2} < \varphi < \frac{3\pi}{2}, 0 < \rho < R\right\},$$

and $X(S_{r,\theta})$ is a graph over $\phi(S_{r,\theta})$. Then $X(S_{r,\theta})$ intersects S along an analytic arc $\hat{\alpha}\colon [0,1] \to \mathbb{R}^3$ with $\hat{\alpha}(t) \to 0$ as $t \to 0$ and $\frac{\hat{\alpha}'(t)}{|\hat{\alpha}'(t)|} \to (-1,0,0)$ as $t \to 0$. Thus the arc $\alpha := X^{-1}_{|S_{r,\theta}}(\hat{\alpha})$ has all the desired properties.

Next let $B' = B \setminus \alpha([0,1])$, and consider some conformal map $\tau\colon B \to B'$ of B onto B'. If we put $X' := X \circ \tau$, then $D(X') = D(X)$ and $X' \in \mathcal{C}(\Pi, S)$, whence X' is a solution to $\mathcal{P}(\Pi, S)$. It follows that X' intersects S orthogonally along ∂B, which means that X and S intersect perpendicularly along α. The curve α can be continued analytically until it hits ∂B. Moreover, by analyticity, the surface X remains orthogonal to S along α. We also note that α cannot have any double points in B, since X is a local embedding in B and hence intersects S in an embedded arc. Thus we have shown that $B \setminus \alpha$ consists of two simply connected domains B_1 and B_2 such that

$$X|_{\partial B} = X|_{\partial B_1} + X|_{\partial B_2}$$

holds true.

Suppose that $X|_{\partial B_1}$ is not contractible in $\mathbb{R}^3 \setminus \Pi$, and let $\tau\colon B \to B_1$ be a conformal equivalence. Then we obtain $X \circ \tau \in \mathcal{C}(\Pi, S)$ and $D(X \circ \tau) < D(X)$, which contradicts the minimality of X, and Proposition 3 is proved. □

Proof of Theorem 5. By virtue of Proposition 3 we only have to show that $\tilde{\partial}_1 B = \partial B$ holds for any strong relative minimizer $X \in \mathcal{C}(\Pi, S)$. But this immediately follows from Proposition 3 in conjunction with Proposition 2. □

Sketch of the proof of Theorem 4. By Theorem 5 we can assume that each minimizer $X \in \mathcal{C}(\Pi, S)$ is an immersion of $\bar{B}$ into $\mathbb{R}^3$.

If we apply a suitable conformal selfmapping of the disk B, we can also achieve the normalization $X(0) \in \Pi$. This condition ensures compactness of minimizers in C^k. In fact, we have

Proposition 4. *Let $\mathcal{C}^* \subset \mathcal{C}(\Pi, S)$ be the set of all minimizing minimal surfaces with $X(0) \in \Pi$. Then $\mathcal{C}^*$ is uniformly bounded in $C^{k,\alpha}(\bar{B})$, for any $k \geq 2, \alpha \in (0,1)$.*

Proof. In order to apply the results of Chapter 2, in particular Theorem 1 of Section 2.5, we wish to verify the following condition which is to hold uniformly in $\mathcal{C}^*$:

For each $\delta > 0$ there exists some $\varepsilon > 0$ such that

$$D_{B \setminus B_{1-\varepsilon}(0)}(X) < \delta \quad \text{for all } X \in \mathcal{C}^*. \tag{10}$$

Suppose on the contrary that there exist $\delta > 0$ and sequences $X_n \in \mathcal{C}^*, \varepsilon_n \to 0$ with $D_{B \setminus B_{1-\varepsilon_n}(0)}(X_n) \geq \delta$. Then all X_n are harmonic and bounded and hence a subsequence, again denoted by X_n, converges to some harmonic X uniformly in $C^k(\Omega)$, for all $\Omega \Subset B, k \in \mathbb{N}$. Because of

$$D(X_n) = d := \inf_{Y \in \mathcal{C}(\Pi, S)} D(Y),$$

we infer that

$$D(X) \leq d - \delta \quad \text{and} \quad X(0) \in \Pi. \tag{11}$$

Recalling the argument in the proof of Theorem 1 of Section 1.3, we conclude that $X \in \mathcal{C}(\Pi, S)$, whence $D(X) \geq d$, contradicting (11). Hence the relation (10) holds true.

On the other hand, (10) enables us to employ the regularity results of Chapter 2. First we see from the proof of Theorem 1 of Section 2.5 that the elements X in $\mathcal{C}^*$ satisfy a uniform global Hölder condition. Once having established a uniform Hölder condition, one can easily derive the higher order estimates by applying Theorem 1′ in Section 2.8. □

Suppose now that there are infinitely many geometrically different minimizing surfaces in $\mathcal{C}(\Pi, S)$. By Proposition 4, we can select a sequence $\{X_n\}$ that converges in $C^k(\bar{B})$ to some $X^* \in \mathcal{C}(\Pi, S)$ which must again be minimizing. By virtue of the immersed character of X^*, it can be shown as in Tomi [10] that there even exists a one-parameter family $F(t), |t| < \varepsilon$, of area minimizing surfaces in $\mathcal{C}(\Pi, S)$ with $F(0) = X^*$, and $F'(0)$ is a nonvanishing normal field along X^*. Furthermore, each solution of $\mathcal{P}(\Pi, S)$ sufficiently close to X^* belongs to the family F (after a suitable reparametrization).

Now let Σ^* denote the connected component of X^* in the set of minimizing surfaces. Then the set

$$U^* := U \cap \left\{ \bigcup_{X \in \Sigma^*} X(B) \right\}$$

must be open and nonempty in the unbounded component U of $\mathbb{R}^3 \setminus S$. On the other hand, the set U^* must be bounded and closed in U according to Proposition 4. Thus we infer $U = U^*$ which clearly is impossible. □

1.10 Scholia

1. The first existence theorem for minimal surfaces with free boundaries was given by Courant [6] and [9] in the years 1938–40. At that time these results were considerable mathematical achievements comparable to the solution of Plateau's problem by Douglas and Radó. We also mention a paper by Courant and Davids [1] as well as a generalization of these results to *generalized Schwarzian chains* $\langle \Gamma_1, \ldots, \Gamma_k, S_1, \ldots, S_m \rangle$ given by Ritter [1]. A comprehensive treatment can be found in Courant [15] and in Nitsche [28].

2. Our exposition in the Sections 1.1–1.3 follows Küster [1]. The reader who is familiar with Courant's treatise [15] will have noticed that we have replaced Courant's condition

$$\lim_{w \to w_0} \operatorname{dist}(X(w), S) = 0 \quad \text{for all } w_0 \in \partial B \tag{1}$$

by the simpler condition $X \in \mathcal{C}(S)$. It is somewhat easier to define the linking condition

$$\mathfrak{L}(X|_{\partial B}, \Pi) \neq 0$$

for surfaces satisfying (1). However, one then has to verify a compactness theorem that will ensure the condition (1) to hold in the limit, whereas our Lemma in Section 1.3 is close to trivial.

Moreover, our approach has the additional advantage that it can easily be carried over to *obstacle* problems with only modest smoothness assumptions on the obstructions, and it can also be used to handle more general functionals than the Dirichlet integral.

The proof that the curves $X|_{C_r}$ and $X|_{C_{r'}}$ are homotopic if r and r' are sufficiently close together (Section 1.1) has been adapted from an analogous theorem due to Schoen and Yau [2].

3. Let us mention some related existence results. Davids [1] proved the existence of multiply connected minimal surfaces with free boundaries. Hildebrandt [6] and Küster [1] treated surfaces of prescribed mean curvature, Lipkin [1] studied 2-dimensional parametric integrals, and F.P. Harth [1] proved existence of minimal surfaces with free boundaries in Riemannian manifolds. Meeks and Yau [1] dealt with Riemannian manifolds as ambient spaces.

P. Tolksdorf [2] stated that any non-trivial homotopy class in $\tilde{\Pi}_1(S)$ can be decomposed into finitely many nontrivial homotopy classes for which the problem of prescribed homotopy class has a solution, assuming that S is a smooth compact surface in $\mathbb{R}^3$. However, R. Ye [6] has pointed but that Tolksdorf's reasoning is faulty.

R. Ye [5,6] proved the existence of a minimal surface with prescribed boundary homotopy class α provided that α satisfies some Douglas-type condition. His method generalizes to Riemannian manifolds as well.

We also refer to remarks by E. Kuwert [5], p. 6, concerning the papers of Tolksdorf and Ye.

4. The existence proof of three different stationary minimal surfaces in a simplex presented in Section 1.6 is due to Smyth [1]. Smyth also stated that each of the three stationary surfaces possesses a non-parametric representation with respect to suitably chosen coordinates.

5. The uniqueness result for stationary minimal surfaces of disk-type in a sphere proved in Section 1.7 is due to Nitsche [35].

6. The examples in Section 1.9 of foliations given by 1-parameter families of minimizing minimal surfaces with their boundaries on a real analytic supporting surface S of the topological type of the torus are due to Gulliver and Hildebrandt [1], and we have followed their exposition quite closely.

7. Concerning detailed proofs of the *finiteness results* of Tomi [10] and Alt and Tomi [1] described in Section 1.9 we refer the reader to the original papers.

8. The most exciting recent development in the theory of minimal surfaces with free boundaries are the beautiful existence results for stationary minimal surfaces in convex bodies some of which we have listed in Section 1.8. We emphasize the importance of the contributions by Sacks and Uhlenbeck [1,2],

Struwe [3], Grüter and Jost [1], Pitts [1], Simon and Smith [1], and Jost [9, 13,15].

9. We also mention a paper by Karcher, Pinkall, and Sterling [1] on new examples of compact embedded minimal surfaces in the 3-sphere which generalizes the important earlier work by Lawson [4]. The Karcher–Pinkall–Sterling approach is closely related to the ideas of Smyth presented in Section 1.6, as their main construction consists in solving free boundary problems in S^3 (instead of $\mathbb{R}^3$).

10. Finally we shall briefly describe the work of E. Kuwert [5–7] on minimizers of Dirichlet's integral among disk-type surfaces $X \in H_2^1(B, \mathbb{R}^n)$ *whose boundary curves* $X|_{\partial B}$ *represent a given homotopy class* α of free loops on a closed configuration S in $\mathbb{R}^n$, $n \geq 3$.

Kuwert's work is an important and far reaching generalization of the theory presented before in this chapter. It deals with the problem of minimizing Dirichlet's integral among all disk-type surfaces $X : B \to \mathbb{R}^n, n \geq 2$, whose boundary values lie on a given configuration S and satisfy certain homotopy constraints. Here one observes *degeneration*, just as in the Douglas problem, and this causes *concentrations of the parametrization*, which in Kuwert's setting can occur only at the boundary of the disk B and leads to a separation of disks. It is proved that *any minimizing sequence has a subsequence which decomposes in the limit into a finite or countably infinite collection of disk-type surfaces, each of which is a minimizer with respect to its own homotopy class.* Here S can be any compact set in $\mathbb{R}^n$, or an unbounded closed set satisfying a suitable condition that prevents the escape of components to infinity.

Kuwert takes the view of Jesse Douglas and considers minimal surfaces as critical points of Dirichlet's integral within the class of *harmonic surfaces* $X \in H_2^1(B, R^n)$ satisfying the prescribed boundary conditions. Since such surfaces are uniquely determined by their "boundary values" $x = X|_{\partial B}$ (i.e. by their "Sobolev trace" on ∂B), the minimum problem is reduced to the minimization of Douglas's functional $A_0(x)$ among all admissible boundary curves x, since for any harmonic extension X of x one has $A_0(x) = D(X)$. However, we have seen before that, for free boundary value problems, it is not feasible to work with continuous boundary values $x(\theta) = X(e^{i\theta})$, since there is no a priori certainty that the minimization procedure leads to a continuous minimizer. To overcome this difficulty, Kuwert applies Courant's artifice of using sequences $\boldsymbol{x} = \{x_k\}$ with $x_k(\theta) := X(r_k e^{i\theta})$, $r_k \to 1-0$, which approximate x in $H_2^1(\partial B, \mathbb{R}^n)$ and satisfy $x_k \in H_2^1(\partial B, \mathbb{R}^n) \cap C^0(\partial B, \mathbb{R}^n)$; for the x_k it is possible to impose homotopy conditions. Finally, one altogether forgets the origin of $\boldsymbol{x}$ and operates with suitable sequences $\boldsymbol{x} = \{x_k\}$ of continuous curves x_k. Keeping this idea in mind, we turn to the technicalities needed to formulate Kuwert's results.

Let S be a nonempty closed set in $\mathbb{R}^n$, and denote by $U_\delta(S)$ the δ-neighbourhood of S in $\mathbb{R}^n$, $\delta > 0$:

$$U_\delta(S) := \{p \in \mathbb{R}^n : d(p,S) < \delta\}, d(p,S) := \operatorname{dist}(p,S).$$

Throughout we assume that $U_\delta(S)$ *is connected for* any $\delta > 0$.

Let $\pi_1(S)$ be the set of homotopy classes $[x]$ of free loops $x \in C^0(\mathbb{R}/2\pi, S)$. In order to define the sequence space $\Pi_1(S)$, we have to introduce the equivalence relations "$x \overset{\delta}{\sim} y$" between two curves $x, y \in C^0(\mathbb{R}/2\pi, \mathbb{R}^n)$, which means: *Both curves lie in* $U_\delta(S)$ *and are freely homotopic to each other in* $U_\delta(S)$. Then we set

$$\Pi_1(S) := \{\boldsymbol{x} = \{x_k\} : x_k \in C^0(\mathbb{R}/2\pi, \mathbb{R}^n), \text{ and for any } \delta > 0$$
$$\text{there is a } k_0 \in \mathbb{N} \text{ with } x_k \overset{\delta}{\sim} x_l \text{ for all } k, l > k_0\}.$$

For any $\boldsymbol{x} \in \Pi_1(S)$ we denote the smallest possible $k_0 \in \mathbb{N}$ by $k(\boldsymbol{x}, \delta)$.

On $\Pi_1(S)$ we introduce the equivalence relation "$\boldsymbol{x} \sim \boldsymbol{y}$" by:

For any $\delta > 0$ *there is a* $k_0 \in \mathbb{N}$ *such that* $x_\sim y_l$ *for all* $k, l \geq k_0$.

The quotient

$$\hat{\pi}_1(S) := \Pi_1(S)/\sim$$

will be the substitute for $\pi_1(S)$, if we operate with sequences $\boldsymbol{x} = \{x_k\}$ of loops x_k close to S instead of loops x on S. It turns out that $\hat{\pi}_1(S)$ is the inverse limit of the set $\pi_1(U_\delta(S))$ of homotopy classes of free loops in $U_\delta(S)$, i.e.

$$\hat{\pi}_1(S) = \lim_{\delta \to 0} \pi_1(U_\delta(S)).$$

We obtain the maps

$$i : \pi_1(S) \to \hat{\pi}_1(S) \quad \text{with } [x] \mapsto [\{x_k \equiv x\}]$$

and

$$i_\delta : \hat{\pi}_1(S) \to \pi_1(U_\delta(S)) \quad \text{with } [\boldsymbol{x}] \mapsto [x_k], k = k(\boldsymbol{x}, \delta).$$

For $\alpha, \beta \in \hat{\pi}_1(S)$ we define $\delta(\alpha, \beta) \in [0, \infty]$ by

$$\delta(\alpha, \beta) := \inf\{\delta > 0 : i_\delta(\alpha) = i_\delta(\beta)\};$$

this is a complete generalized metric on $\hat{\pi}_1(S)$, except that $\delta(\alpha, \beta) = \infty$ if S is unbounded, and $\hat{\pi}_1(S)$ is arcwise totally disconnected. Moreover i is injective if S is a retract of $U_{\delta_0}(S)$ for some $\delta_0 > 0$, and if S is a uniform deformation retract of $U_{\delta_0}(S)$, then i, i_{δ_0} are bijective and $\delta(\alpha, \beta) \geq \delta_0$ for $\alpha, \beta \in \hat{\pi}_1(S)$ with $\alpha \neq \beta$.

Finally we define $|\alpha|$ for $\alpha \in \hat{\pi}_1(S)$ by

$$|\alpha| := \inf\{\delta > 0 : \ i_\delta(\alpha) \text{ contains a constant map}\},$$

and we call $\alpha \in \hat{\pi}_1(S)$ *trivial* if $|\alpha| = 0$, otherwise *nontrivial*.

Now we consider the space H of Fourier series

$$x \sim a_0 + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta), \quad a_m, b_m \in \mathbb{R}^n,$$

satisfying

$$E(x) := \frac{\pi}{2} \sum_{m=1}^{\infty} m(|a_m|^2 + |b_m|^2) < \infty.$$

The harmonic extension of X with $X(e^{i\theta}) = x(\theta)$ satisfies

$$D(X) = E(x) = A_0(x).$$

The space H with the norm $\|x\|_H$, defined

$$\|x\|_H^2 := |a_0|^2 + E(x),$$

can be identified with the Hilbert space $H^{1/2,2}(\mathbb{R}/2\pi, \mathbb{R}^n)$. Set

$$\begin{aligned} \mathcal{H}(S) := \{\boldsymbol{x} = \{x_k\} : x_k \in H \cap C^0(\partial B, \mathbb{R}^n), d(x_k, S) \to 0, \\ \{x_k\} \text{ is a Cauchy sequence in } H\}, \end{aligned}$$

where $d(x_k, S) := \sup\{\alpha(x_k(\theta), S) : 0 \le \theta \le 2\pi\}$.

For $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}(S)$ we write

$$\boldsymbol{x} \sim \boldsymbol{y} \quad \text{if and only if} \quad \|x_k - y_k\|_H \to 0.$$

We define the quotient space

$$H(S) := \mathcal{H}(S)/\sim$$

and note that $H(S)$ can be identified isometrically with $(W(S), \|\cdot\|_H)$, where

$$W(S) := \{x \in H : x(\theta) \in S \text{ for a.e. } \theta \in [0, 2\pi]\},$$

i.e.

$$W(S) = H(S),$$

and for $\boldsymbol{x} = \{x_k\} \in \mathcal{H}(S)$ we have $\|x_k - x\|_H \to 0$ for some $x \in W(S)$; then the equivalence class of $\boldsymbol{x}$ is identified with x, and $\|x\|_H = \lim_{k\to\infty} \|x_k\|_H =: \|\boldsymbol{x}\|_{\mathcal{H}(S)}$.

One obtains the following topological substitute for a Sobolev embedding of H into $C^0(\partial B, \mathbb{R}^n)$ which is essentially due to Courant (cf. Section 1.1); a proof can be found in B. White [7] and Kuwert [5,7].

Theorem A. *The set $\mathcal{H}(S)$ is a subset of $\Pi_1(S)$, and the inclusion $\mathcal{H}(S) \subset \Pi_1(S)$ induces a well-defined assignment from any $x \in W(S)$ to a homotopy class in $\hat{\pi}_1(S)$ which will be denoted by $[x]$. The mapping $x \mapsto [x]$ from $W(S) = H(S)$ into $\hat{\pi}_1(S)$ is continuous.*

The following can be seen: *For any $x \in W(S)$, the harmonic extension X satisfies*

$$\lim_{|w|\to 1} d(X(w), S) = 0,$$

and for any sequence $\{r_k\}$ with $r_k \to 1-0$ the sequence $x_k(\theta) := X(r_k e^{i\theta})$, $k \in \mathbb{N}$, can be used for the definition of $[x] \in \hat{\pi}_1(S)$.

Now we formulate the *minimization problem for a given class $\alpha \in \hat{\pi}_1(S)$*. We set

$$E_*(\alpha) := \inf\Big\{\liminf_{k\to\infty} E(x_k) : \boldsymbol{x} = \{x_k\} \text{ with } [\boldsymbol{x}] \in \alpha\Big\}.$$

The function $E_* : \hat{\pi}_1(S) \to [0, \infty]$ is lower semicontinuous and satisfies $E_*(\alpha) \geq \pi|\alpha|^2$ as well as: $E_*(\alpha) = 0 \Leftrightarrow \alpha$ is trivial. Furthermore:

For any $\alpha \in \hat{\pi}_1(S)$ there is always a sequence $\boldsymbol{x} = \{x_k\}$ with $[\boldsymbol{x}] \in \alpha$ and $x_k \in C^\infty(\mathbb{R}/2\pi, \mathbb{R}^n)$ such that

$$\lim_{k\to\infty} E(x_k) = E_*(\alpha).$$

Definition 1. (i) *A minimizing sequence for $\alpha \in \hat{\pi}_1(S)$ is a sequence $\boldsymbol{x} = \{x_k\} \in \Pi_1(S)$ with $[x] \in \alpha$ satisfying $E(x_k) \to E_*(\alpha)$.*

(ii) *Any $x \in W(S)$ with $[x] = \alpha$ and $E(x) = E_*(\alpha)$ is called a minimizer of E.*

We set

$$\mathcal{F}(S) := \{\boldsymbol{x} = \{x_k\} : x_k \in C^0(\mathbb{R}/2\pi, \mathbb{R}^n), d(x_k, S) \to 0\};$$

in particular we have $\Pi_1(S) \subset \mathcal{F}(S)$.

A sequence $\boldsymbol{x} = \{x_k\}$ is said to be *trivial*, if for any $\delta > 0$ there is a $k_1(\delta) \in \mathbb{N}$ such that x_k is contractible in $U_\delta(S)$ for all $k \geq k_1(\delta)$; otherwise $\boldsymbol{x}$ is called *nontrivial.* Then we introduce $\epsilon_*(S)$ and $\epsilon_0(S) \in \mathbb{R}$ with $0 \leq \epsilon_*(S) < \epsilon_0(S)$ by

$$\begin{aligned} \epsilon_*(S) &:= \inf\Big\{\epsilon > 0 : \text{ There is a nontrivial sequence } \boldsymbol{x} = \{x_k\} \in \mathcal{F}(S) \\ &\qquad\qquad \text{with } \limsup_{k\to\infty} E(x_k) \leq \epsilon\Big\}, \\ \epsilon_0(S) &:= \inf\{E_*(\alpha) : \alpha \in \hat{\pi}_1(S) \text{ is nontrivial}\}. \end{aligned}$$

Note that $\alpha \in \hat{\pi}_1(S)$ is nontrivial if $\alpha = [\boldsymbol{x}]$ with $\boldsymbol{x} \in \Pi_1(S)$ and $\boldsymbol{x}$ is nontrivial. Observe also that $\epsilon_*(S)$ is only defined if there is a nontrivial sequence in $\mathcal{F}(S)$, and $\epsilon_0(S)$ is only defined if there is a nontrivial $\alpha \in \hat{\pi}_1(S)$.

For a sequence $\boldsymbol{M} = \{M_k\}$ of sets $M_k \subset \mathbb{R}^n$ we define the closed set of accumulation points of $\boldsymbol{M}$ by

$$\begin{aligned} \mathcal{A}(\boldsymbol{M}) := \{p \in \mathbb{R}^n : &\text{ There is a subsequence } \{k_l\} \text{ and a sequence} \\ &\text{ of point } p_l \in M_{k_l} \text{ with } p_l \to p\}. \end{aligned}$$

Furthermore, define the subset $\mathcal{F}'$ of $\mathcal{F}$ by

$$\mathcal{F}' := \{\boldsymbol{x} = \{x_k\} \in \mathcal{F} : x_k \in H, E(x_k) \to e \text{ for some } e \in [0,\infty)\}$$

and set

$$\mathcal{A}(\boldsymbol{x}) := \mathcal{A}(\{\operatorname{im} x_k\}), \quad \mathcal{A}(\boldsymbol{X}) := \mathcal{A}(\{\operatorname{im} X_k\})$$

for $\boldsymbol{x} = \{x_k\} \in \mathcal{F}'$ and $\boldsymbol{X} := \{X_k\}$, $X_k =$ harmonic extension of x_k,

$$\operatorname{im} x_k = \text{ image of } x_k, \quad \operatorname{im} X_k = \text{image of } X_k.$$

One has $\mathcal{A}(\boldsymbol{x}) \subset S \cap \mathcal{A}(X)$ and

$$\mathcal{A}(\boldsymbol{X}) \subset \operatorname{clos}(U_{\delta(e)}(\mathcal{A}(\boldsymbol{x}) \cap S)) \quad \text{for } \boldsymbol{x} = \{x_k\} \in \mathcal{F}'$$

where $\delta(e) := \sqrt{e/\pi}$, and $e = \lim_{k\to\infty} E(x_k)$. Furthermore, $\mathcal{A}(\boldsymbol{X}) = \emptyset$ if and only if $\mathcal{A}(\boldsymbol{x}) = \emptyset$.

One of Kuwert's main tools is a decomposition result for sequences $\boldsymbol{x} = \{x_k\} \in \mathcal{F}'$ which is formulated as Lemma 3 in Section 2 of his paper [5], but is too involved to be stated here.

Now the following compactness question is raised: *Given a sequence $\boldsymbol{z} = \{z_k\} \in \mathcal{F}'$, is there always a subsequence $\boldsymbol{x} = \{x_l\}, x_l = z_{k_l}$ with $\boldsymbol{x} \in \Pi_1(S)$?*

It turns out that the answer is negative in general. To clarify the situation, some topological notions are needed. Recall first that $U_\delta(S)$ is assumed to be connected for any $\delta > 0$, which is the case if S is connected. Therefore there is a unique trivial element o in $\hat{\pi}_1(S)$ which is represented by any sequence $\{x_k\}$ of constant loops $x_k(\theta) \equiv p_k$ with $p_k \to S$, and $|\alpha| = \delta(\alpha, 0)$.

Now we consider m-tupel $\underline{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^m)$ of homotopy classes $\alpha^j \in \hat{\pi}_1(S)$ which are either finite, $m \in \mathbb{N}$, or infinite, $m = \infty$. We also require that, for any $S > 0$, the set $I(\underline{\alpha}, \delta) = \{j : |\alpha^j| > \delta\}$ is finite with the number $m(\underline{\alpha}, \delta)$ of elements (which may be zero). Next we introduce the *sequence space* $\Pi_1(\underline{\alpha})$ as follows:

$$\begin{aligned}\Pi_1(\underline{\alpha}) := \{\boldsymbol{x} = \{x_k\} \in \mathcal{F} : \;&\text{For any } \delta > 0 \text{ there is a smallest possible} \\ &k(\boldsymbol{x}, \delta) \in \mathbb{N} \text{ such that the homotopy class } [x_k] \in \pi_1(U_\delta(S)) \\ &\text{belongs to the composition set of the classes } i_\delta(\alpha^j), j \in I(\underline{\alpha}, \delta), \\ &\text{for all } k \geq k(\boldsymbol{x}, \delta)\}.\end{aligned}$$

We say that $\alpha \in \hat{\pi}_1(S)$ belongs to the *composition set* $\mathcal{C}(\underline{\alpha})$ of $\underline{\alpha} = (\alpha^1, \ldots, \alpha^m)$ if and only if $i_\delta(\alpha)$ belongs to the composition set of the finite $\boldsymbol{m}(\alpha, \delta)$-tupel of the $i_\delta(\alpha^j)$ with $j \in I(\underline{\alpha}, \delta)$, for all $\delta > 0$. Equivalently we say: $\underline{\alpha}$ is a decomposition of $\alpha, \underline{\alpha} \in \mathcal{D}(\alpha)$.

This means: Given $\boldsymbol{x} = \{x_k\} \in \Pi_1(\underline{\alpha})$, $\delta > 0$, and representatives $\boldsymbol{x}^j = \{x_k^j\}$ of α^j, then the boundary data $x_k, x^j_{k(\boldsymbol{x},\delta)}$ with $j \in I(\underline{\alpha}, \delta)$ can be extended to an $(m(\underline{\alpha}, \delta) + 1)$-fold connected domain by a map into $U_\delta(S)$ for $k \geq k(\boldsymbol{x}, \delta)$. If α is finite then $m(\alpha, \delta) \equiv \text{const}$ for $0 < \delta \ll 1$.

For unbounded S it can happen that disks escape to infinity. This will be excluded by imposing an energy condition. For this purpose we define

$$\epsilon_\infty(S) := \inf\Big\{\epsilon > 0 : \text{There is a nontrivial sequence } \boldsymbol{x} = \{x_k\} \in \mathcal{F}(S) \text{ with } \mathcal{A}(X) \subset S \text{ and } \limsup_{k\to\infty} E(x_k) \le \epsilon\Big\}.$$

Clearly, $\epsilon_*(S) \le \epsilon_\infty(S)$, and $\epsilon_\infty(S) = \epsilon_0(S)$ if S is compact.

We have the following answer to the "compactness question" raised above:

Theorem B. *Let $z = \{z_k\} \in \mathcal{F}'$ be a sequence with $E(x_k) \to e < \epsilon_\infty$. Then there exist an m-tupel $\underline{\alpha} = (\alpha^1, \dots, \alpha^m)$ of $\alpha^j \in \hat{\pi}_1(S)$, $|\alpha^j| > 0$, with $|\alpha^j| \to 0$ as $j \to \infty$ if $m = \infty$, and a subsequence $\boldsymbol{x} = \{x_l\}, x_l = z_{k_l}$, such that $\boldsymbol{x} \in \Pi_1(\underline{\alpha})$, and it addition*

$$\sum_j E_*(\alpha^j) \le e \quad \textit{and} \quad m \le e/\epsilon_0(S).$$

Moreover, if $e < \min\{2\epsilon_0(S), \epsilon_\infty(S)\}$ then $\boldsymbol{x} \in \Pi_1(S)$, i.e. $\boldsymbol{x}$ defines a homotopy class. Finally, $\epsilon_*(S) = \epsilon_0(S)$ provided that $\epsilon_*(S) < \epsilon_0(S)$.

Theorem C. *Let $\alpha \in \hat{\pi}_1(S)$ be a nontrivial homotopy class with $E_*(\alpha) < \infty$, and $\boldsymbol{x} = \{x_k\}$ be a minimizing sequence for $\alpha \in \hat{\pi}_1(S)$ which converges weakly in H to $x \in W(S)$. Then x is a minimizer with respect to its own homotopy class.*

The hypothesis on α can be verified if α can be represented by a sequence of equibounded length. While $\{x_k\}$ will not converge strongly in general, it is often possible to extract a nonconstant weak limit.

The next theorem is the main result of Kuwert [5–7]. It states that *any minimizing sequence contains a subsequence which decomposes in the limit both in homotopy and in energy into a union of minimizing disks.*

Theorem D. *Let $\alpha \in \hat{\pi}_1(S)$ be a given nontrivial homotopy class with $E_*(\alpha) < \epsilon_\infty(S)$, and let $\boldsymbol{z} = \{z_k\}$ with $[z] \in \alpha$ be a given minimizing sequence. Then there are a subsequence $\{z_{k_l}\}$, a number $m \in \mathbb{N} \cup \{\infty\}$, a sequence $\{h_l^1\}$ of conformal automorphisms of B, topological disks $D_l^j, l \in \mathbb{N}$, $1 \le j \le m$, and Riemann mapping functions $g_l^j : B^j \to D_l^j$ such that the loops $x_l := z_{k_l} \circ h_l^1$ satisfy:*

(i) *$D_l^j \Subset B, \overline{D}_l^j \cap \overline{D}_j^k = \emptyset$ for $j \ne k$; ∂D_l^j is regular and real analytic; $D_j^1 = \{w \in B : |w| < r_l^1\}$ with $r_l^1 \to 1 - 0$; $g_l^1(w) = r_l^1 w$.*

(ii) *For any j, the sequence $\{X_l \circ g_l^j\}_{l\in\mathbb{N}}$ converges strongly in $H_2^1(B^j, \mathbb{R}^n)$ to a nontrivial minimizer $X^j : B^j \to \mathbb{R}^n$ with the boundary values $x^j \in W(S)$,*

(B_j identified with B), and $\alpha^j := \{x^j\}$ is an element of $\hat{\pi}_1(S) \setminus \{0\}$. Each mapping X^j is a (possibly branched) minimal surface.

(iii) $E_*(\alpha) = \lim_{k\to\infty} E(z_k)$ *can be written as*

$$E_*(\alpha) = \sum_{j=1}^{m} E_*(\alpha^j) = \sum_{j=1}^{\infty} E(x^j).$$

(iv) *The m-tupel $\underline{\alpha} := (\alpha^1, \alpha^2, \dots, \alpha^m)$ is a decomposition of the given class α.*

(v) *For $M_l := B \setminus \bigcup_{j=1}^m D_l^j$ we have $d(X_l|_{M_l}, S) \to 0$, and the Dirichlet integrals $D_{M_l}(X_l)$ of X_l over M_l tend to zero as $l \to \infty$.*

(vi) *If $\epsilon_0(S) > 0$ then $m \le E_*(\alpha)/\epsilon_0(S) < \infty$.*

We mention that this result can be used to generalize H.W. Alt's solution of the so-called *thread problem*, treated in Chapter 5 of this volume; cf. Kuwert [5], pp. 51–52. Kuwert's approach allows to consider threads whose endpoints are fixed at support surfaces (instead of arcs).

Now we want to collect several applications of Theorem D in case that $\alpha \in \hat{\pi}_1(S)$ satisfies a *sufficient Douglas condition*. This means: *For any proper decomposition $\underline{\alpha} = (\alpha^1, \alpha^2, \dots, \alpha^m) \in \mathcal{D}(\alpha)$ we have the strict inequality*

$$E_*(\alpha) < \sum_{j=1}^{m} E_*(\alpha^j).$$

Note however that, so far, this condition can only be verified for specific homotopy classes.

Theorem E. *Let $\alpha \in \hat{\pi}_1(S)$ satisfy the sufficient Douglas condition, and assume that $E_*(\alpha) < \epsilon_\infty(S)$. Then we have:*

(i) *For any minimizing sequence $\boldsymbol{z} = \{z_k\}$ representing α there are a subsequence $\{z_{k_l}\}$ as well as conformal automorphisms h_l of B such that $\boldsymbol{x} = \{x_l\}$ with $x_l := z_{k_l} \circ h_l$ converges strongly in H to some minimizer $x \in W(S)$ for α, and $E_*(\alpha)$ is attained.*

(ii) *If a minimizing sequence converges weakly to a nonconstant map $x \in W(S)$, then it converges strongly to x.*

(iii) *The nonempty set $\mathcal{M}(\alpha)$ of minimizers for α is compact in $W(S)$ modulo the conformal automorphism group* $\mathrm{Aut}(B)$ *of B.*

Kuwert also shows ([5], pp. 57–60) how the results presented in the following Chapter 2 can be used to show regularity of minimal surfaces X defined by minimizers x for α and to prove compactness for $\mathcal{M}(\alpha)$ with respect to $C^{0,\beta}$ or $C^{k,\beta}$.