Problems

Here we formulate some major open problems for minimal surfaces, mostly in the context of Plateau's or Douglas's problem. Many of them are unsolved since a long time, see e.g. J.C.C. Nitsche [28,37].

Let Γ be a rectifiable curve in $\mathbb{R}^3$, and $\mathcal{C}(\Gamma)$ be the class of admissible curves for Plateau's problem defined in Section 4.2. Furthermore, for any $X \in \mathcal{C}(\Gamma)$ let $A(X)$ be the area functional and $D(X)$ the Dirichlet integral of X. One has

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D.$$

Hence an absolute minimizer of D is one of A, and conversely, a conformally parametrized absolute minimizer of A is an absolute minimizer of D.

1. Is it true that a relative minimizer of D in $\mathcal{C}(\Gamma)$ is also a relative minimizer of A in $\mathcal{C}(\Gamma)$ with respect to some suitable norm on $\mathcal{C}(\Gamma)$, say, $\|\cdot\|_{C^0}$, $\|\cdot\|_{C^0} + \sqrt{D(\cdot)}$, $\|\cdot\|_{C^1}$, $\|\cdot\|_{C^{1,\alpha}}, \dots$? The converse holds for conformally parametrized relative minimizers of A. The problem might be easier to solve if one assumes $\Gamma \in C^{m,\alpha}$. (See also Appendix 1.)
   A similar question can be raised for the Douglas problem, for H-surfaces, or for minimal surfaces in a Riemann surface.

Let $\mathcal{C}^*(\Gamma)$ be the class of surfaces in $\mathcal{C}(\Gamma)$ which satisfy a preassigned three-point condition.

2. How many minimal surfaces of class $\mathcal{C}^*(\Gamma)$ are bounded by a given "well-behaved" closed Jordan curve Γ? Here well-behaved might be interpreted as *regular and real analytic*, or as *regular and of class* C^k, or as *piecewise linear* (i.e. Γ be a *polygon*), or in another suitable way.
   An analogous question can be asked for the Douglas problem, or for cmc-surfaces.
3. Is the number of immersed, stable minimal surfaces in $\mathcal{C}^*(\Gamma)$ finite?
4. Can one give upper or lower bounds on the number $N(\Gamma)$ of minimal surfaces in $\mathcal{C}^*(\Gamma)$ in terms of bounds on the geometric data of X? Can

U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 339,
DOI 10.1007/978-3-642-11698-8_9, © Springer-Verlag Berlin Heidelberg 2010

one find such bounds for the number $N_s(\Gamma)$ of stable minimal surfaces in $\mathcal{C}^*(\Gamma)$? Similarly for the number $N_i(\Gamma)$ of immersed minimal surfaces in $\mathcal{C}^*(\Gamma)$.

A closed, connected component $\mathcal{K}_c$ of a level set $\mathcal{M}_c(\Gamma) := \{X$ is minimal surface of class $\mathcal{C}^*(\Gamma)$ with $D(X) = A(X) = c\}$, $c \in \mathbb{R}$, is called a *block of minimal surfaces.*

5. Do blocks of minimal surfaces in $\mathbb{R}^3$ always consist of single elements? If not, what is their topological and analytic structure?
6. Can one solve Plateau's problem for minimal surfaces in a Riemann manifold M, if Γ is an arbitrary Jordan curve in M? The same question can be posed for the Douglas problem, or for H-surfaces (cf. Section 4.12, Remark 1)?
7. Is it possible to solve Plateau's problem within the class of immersed minimal surfaces, either by a continuity method or by a variational method?
8. Can minimizers of D in $\mathcal{C}(\Gamma)$ possess boundary branch points?
9. Can one show the existence of immersed, but unstable minimal surfaces in $\mathcal{C}^*(\Gamma)$ via a mountain pass theorem?
10. Is it possible to derive general uniqueness theorems which include and combine those of Radó and Nitsche as well as that for small perturbations of certain planar curves?
11. Let $\mathcal{M}(\Gamma)$ be the class of minimal surfaces in $\mathcal{C}(\Gamma)$. What is the interrelation between the classes $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ if Γ is a regular C^k-curve and Γ' is a polygon or another C^k-curve close to Γ? In particular, how does $N(\Gamma)$ change under perturbations of Γ? What happens to $N(\Gamma)$, $N_i(\Gamma)$ and $N_s(\Gamma)$ if the total curvature of Γ changes beyond 6π?
12. Is there an index formula for polygons similar to the formula established by Böhme/Tromba and Tomi/Tromba for smooth contours?
13. Given a closed polygon Γ, can one give a classification of minimal surfaces in $\mathcal{C}^*(\Gamma)$ by Courant's and Shiffman's functions, using ideas of Heinz and Sauvigny? Is Courant's function of class C^2? Can one prove an index-sum formula in the nondegenerate situation?
14. Is it possible to estimate the modulus of continuity of the normal of a minimal surface up to and including the boundary, at least for special classes of minimal surfaces?
15. Can one derive estimates of the Gaussian curvature for stable minimal immersions, possibly up to and including the boundary?