Chapter 8

Introduction to the Douglas Problem

In this chapter we present an introduction to the *general problem of Plateau* that, justifiedly, is often called the *Douglas problem*. This is the question whether a configuration $\Gamma := \langle \Gamma_1, \dots, \Gamma_k \rangle$ of k nonintersecting closed Jordan curves Γ_j in $\mathbb{R}^3$ may bound multiply connected minimal surfaces of prescribed Euler characteristic and prescribed character of orientability. Here we treat only the simplest form of the Douglas problem, to find a minimal surface $X : \overline{\Omega} \to \mathbb{R}^3$ whose parameter domain Ω is a k-fold connected, bounded, open set in $\mathbb{R}^2$ whose boundary consists of k closed, nonintersecting Jordan curves. Since any such domain can be mapped conformally onto a domain B bounded by k circles, we may choose such k-circle domains as parameter domains for the desired minimal surfaces. However, different from the case $k = 1$ where all parameter domains are conformally equivalent, two admissible parameter domains will in general be of different conformal type if $k \geq 2$. Therefore we are no longer allowed to fix a k-circle domain B a priori as the parameter domain of any solution of the Douglas problem; instead, the determination of B is part of the problem since X has to fulfill the conformality relations.

After discussing some examples in Section 8.1, we state the main result. In Section 8.2 we show that from $\partial D(X, \eta) = 0$ for all C^1-vector fields $\eta : \overline{B} \to \mathbb{R}^2$ on the domain B of X one can derive the conformality relation $\langle X_w, X_w \rangle = 0$. The proof of this fact is a quite nontrivial generalization of the method used in Section 4.5.

Different from the Plateau problem ($k = 1$), the Douglas problem ($k \geq 2$) has in general no "connected" solution. For example, two parallel circles Γ_1 and Γ_2 contained in distinct planes do not bound a connected minimal surface if they are "too far apart". This phenomenon is discussed in Chapter 4 of Vol. 2. Douglas has exhibited a sufficient condition ensuring the existence of connected minimal surfaces bounded by $\Gamma_1, \dots, \Gamma_k$. However, this condition is somewhat difficult to deal with, while Courant's *condition of cohesion* is much easier to handle. This condition is described in Section 8.3, and it is

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shown that it leads to sequences of parameter domains which converge towards nondegenerate domains.

In Section 8.4 we solve the Douglas problem for k-fold connected minimal surfaces, assuming that the condition of cohesion is satisfied. Then, in Section 8.5, we prepare two useful tools which later on will be used to modify surfaces in a suitable way. These modifications were invented by Courant.

The main result is contained in Section 8.6 where we solve the Douglas problem, assuming the so-called Douglas condition. The solution is seen to be a simultaneous minimizer of the area A and the energy D in the class $\mathcal{C}(\Gamma)$ of admissible surfaces, which implies

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D.$$

The "necessary Douglas condition"

$$a(\Gamma) \le a^+(\Gamma)$$

and the "sufficient Douglas condition"

$$a(\Gamma) < a^+(\Gamma)$$

are studied in some detail in Sections 8.7 and 8.8; in particular, we present several examples. As a generalization of Riemann's mapping theorem to multiply connected planar domains we obtain *Koebe's mapping theorem.*

The Scholia (Section 8.9) contain some historical remarks and references to the literature.

8.1 The Douglas Problem. Examples and Main Result

In Chapter 4 we discussed the classical problem of Plateau as it was solved by Douglas and Radó, and we presented the solution found by Courant and, independently, by Tonelli. In the restricted sense formulated in Definition 1 of Section 4.2, Plateau's problem consists in finding a "disk-type" minimal surface spanning a prescribed closed Jordan curve Γ. This is to say, given Γ, we have to find a mapping $X : \overline{B} \to \mathbb{R}^3$ of the closure of the disk $B := \{w \in \mathbb{R}^2 \colon |w| < 1\}$ into $\mathbb{R}^3$ which is harmonic and conformal in B, continuous on $\overline{B}$, and maps ∂B topologically (i.e. homeomorphically) onto Γ.

As mentioned before, this is neither the most general nor the most natural way to formulate Plateau's problem, but merely the simplest and most convenient one, as we do not run into the difficulty that parameter domains of the same topological type may be of different conformal type. However, there is no need to restrict ourselves to minimal surfaces bounded by a single closed curve since boundary configurations consisting of several closed curves may bound multiply connected minimal surfaces. A classical example is the

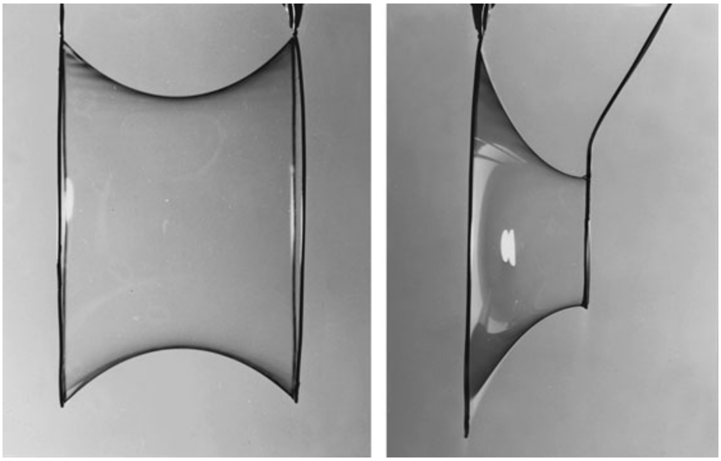

Fig. 1. A soap film experiment: Catenoids held by two coaxial circles

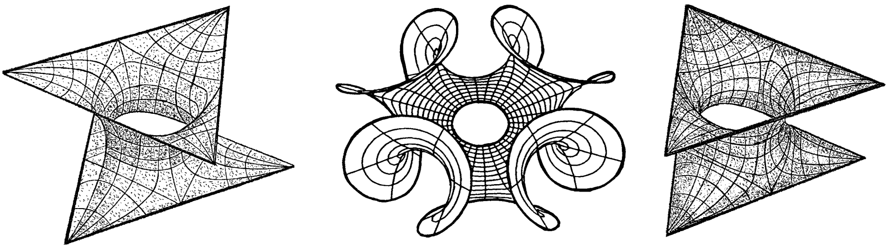

Fig. 2. Minimal surfaces bounded by two closed curves

catenoid, the minimal surface of revolution, which is bounded by two coaxial circles in parallel planes. Moreover, soap film experiments show that certain configurations may bound minimal surfaces of higher topological structure, even nonorientable ones such as surfaces of the type of the Möbius strip. Figures 1–5 depict several such contours as well as minimal surfaces spanning them. In certain cases it is not difficult to see that a topologically more complicated minimal surface may have a smaller area than any disk-type surface bounded by the same contour.

The first to state Plateau's problem in a general form was Jesse Douglas who attacked this question in a series of profound and pioneering papers. Hence many authors speak of the *Douglas problem* instead of what Douglas himself called the

General problem of Plateau. *Given a configuration $\Gamma = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$ in $\mathbb{R}^3$ consisting of k mutually disjoint Jordan curves Γ_j, find a minimal surface of prescribed topological type that spans Γ.*

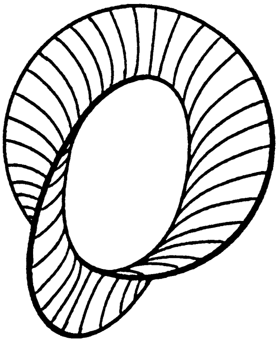

Fig. 3. An annulus-type minimal surface spanned by two interlocking curves

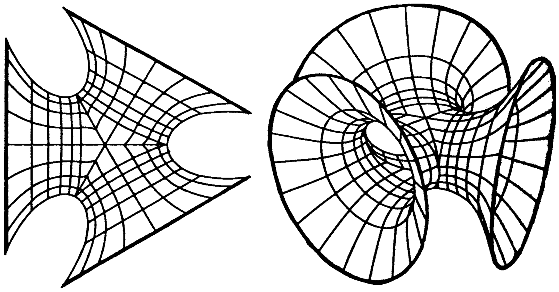

Fig. 4. Two views of a minimal surface of genus zero bounded by three closed curves. Courtesy of K. Polthier

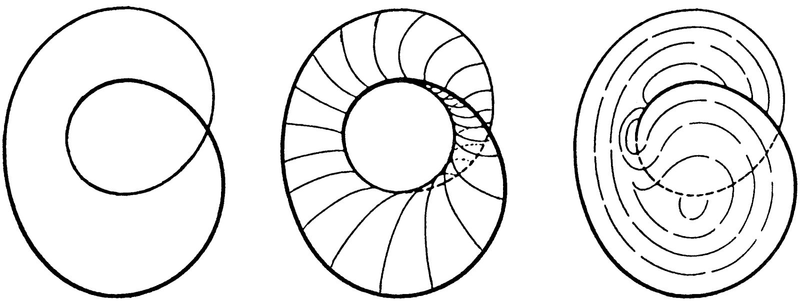

Fig. 5. A closed curve bounding a one-sided minimal surface. This curve also spans a disk-type minimal surface

What is a *surface*, and what is its *topological type*? One might think of a surface as a two-dimensional submanifold of $\mathbb{R}^3$, or better as an embedding $X : M \to \mathbb{R}^3$ of a two-dimensional manifold M with (or without) boundary into $\mathbb{R}^3$. This definition is too restrictive as we want to consider "surfaces" $S = X(M)$ with selfintersections; thus we might think of local embeddings such as immersions $X : M \to \mathbb{R}^3$. But even this class is too narrow as we want to study minimal surfaces with branch points. Moreover, in order to be able to use functional analytic arguments, we would like to operate with mappings X contained in a Sobolev space, which are in fact only equivalence classes of mappings $X : M \to \mathbb{R}^3$, and every representative of X is only determined up to a set of two-dimensional measure zero.

In such general cases we cannot define the topological type of the surface $S = X(M)$ in the usual way. Instead we use the following preliminary definition: *A surface in* $\mathbb{R}^3$ *is a mapping* $X : M \to \mathbb{R}^3$ *of a two-dimensional manifold* M *(with or without boundary), and the topological type of* X *is defined as the topological type of the "parameter manifold"* M. The image set $S := X(M)$ is called the *trace of* X *in* $\mathbb{R}^3$; occasionally one calls S instead of X a surface in $\mathbb{R}^3$, and S is said to be an *embedded* or *immersed surface* respectively if X is an embedding or an immersion. We note that X might be defined only up to a null set in M.

Suppose that M is a compact two-dimensional C^1-manifold whose boundary ∂M consists of k closed Jordan curves, and which is oriented ($\epsilon(M) := 1$) or nonoriented ($\epsilon(M) := -1$). Let $\chi(M) := \alpha_0 - \alpha_1 + \alpha_2$ be the Euler characteristic of M, with $\alpha_0, \alpha_1, \alpha_2$ the number of edges, wedges, and faces of any regular triangulation of M. Then the *topological type of* M, denoted by $[M]$, is defined as

$$[M] := \{\epsilon(M), r(M), \chi(M)\} \quad \text{with } r(M) := k,$$

and the *genus of* M, denoted by $g(M)$, is defined by

$$\chi(M) + r(M) =: \begin{cases} 2 - 2g(M) & \text{if } \epsilon(M) = 1, \\ 2 - g(M) & \text{if } \epsilon(M) = -1. \end{cases}$$

For instance, if M is a k-fold connected, compact region in $\mathbb{R}^2$, then $\epsilon(M) = 1$, $r(M) = k$, $\chi(M) = 2 - k$, and $[M] = \{1, k, 2 - k\}$, $g(M) = 0$. In the present chapter we want to consider surfaces $X : M \to \mathbb{R}^3$, whose parameter sets M have this topological type. The more general case $[M] = \{1, r(M), \chi(M)\}$ is treated in Chapter 4 of Vol. 3, while $[M] = \{-1, r(M), \chi(M)\}$ can be handled by passing to the double cover of M.

The *General Douglas Problem* then reads as follows:

Given a configuration $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *of* k *mutually disjoint Jordan curves* $\Gamma_1, \dots, \Gamma_k$*, find a minimal surface* $X : M \to \mathbb{R}^3$ *of prescribed topological type* $[M] = \{\epsilon(M), k, \chi(M)\}$ *that spans* Γ.

The idea to solve this task is to minimize Dirichlet's integral $D(X)$ among all surfaces $X : M \to \mathbb{R}^3$ bounded by Γ, and of given topological type $[M]$. For technical reasons it is inconvenient to allow all parameter sets M of fixed topological type for competition. Since D is invariant with respect to conformal mappings $\tau : \operatorname{int} M^* \to \operatorname{int} M$, it is sufficient to minimize D in a class of mappings $X : M \to \mathbb{R}^3$ with $M \in \mathcal{N}$ where $\mathcal{N}$ denotes a set of parameter manifolds M containing all conformal types with the fixed topological type $[M]$. In fact, it is not necessary to know a priori what all conformal representations for a given type $[M]$ are; it suffices to make a good guess and to verify that the method works. However, choosing a sequence $\{X_j\}$ of surfaces $X_j : M_j \to \mathbb{R}^3$, bounded by Γ, with $M_j \in \mathcal{N}$ and

$$D(X_j) \to \inf\{D(X)\colon X : M \to \mathbb{R}^3,\ \partial X = \Gamma,\ M \in \mathcal{N}\}$$

as $j \to \infty$ such that X_j converges in some sense to a mapping $X : M \to \mathbb{R}^3$, it is by no means clear that the limit set $M = \lim_{j\to\infty} M_j$ will belong to $\mathcal{N}$; in fact, M might very well jump out of the class $\mathcal{N}$. To prevent this, one has to take suitable precautions such as assuming the *condition of cohesion* or the *Douglas condition*.

Here we shall solve the simple kind of Douglas problem, namely: *Determine a minimal surface* $X : M \to \mathbb{R}^3$, *spanning* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, *defined on a schlicht parameter region* $M \subset \mathbb{R}^2$ *of type* $[M] = \{1, k, 2-k\}$. To obtain a solution, we minimize D in a suitable class $\mathcal{C}(\Gamma)$ of mappings $X : B \to \mathbb{R}^3$ with $B \in \mathcal{N}(k)$ where $\mathcal{N}(k)$ is the class of k-*circle domains* B in $\mathbb{R}^2$. Let us give a precise definition of this kind of domains.

As usual we identify the point $w = (u, v) \in \mathbb{R}^2$ with $w = u + iv \in \mathbb{C}$, and correspondingly $\mathbb{R}^2$ is identified with $\mathbb{C}$. For $q \in \mathbb{C}$ and $r > 0$ we define the disk $B_r(q)$ as

$$B_r(q) := \{w \in \mathbb{C}\colon |w - q| < r\};$$

it is a 1-*circle domain*. If $q = 0$ and $r = 1$, we call the unit disk $B_1(0)$ the *normed* 1-*circle domain*. For $k > 1$, a k-*circle domain* $B(q, r)$ with $q = (q_1, \dots, q_k) \in \mathbb{C}^k$ and $r = (r_1, \dots, r_k) \in \mathbb{R}^k$, $r_1 > 0, \dots, r_k > 0$, is a disk $B_{r_1}(q_1)$, from which $k-1$ closed disks $\overline{B}_{r_2}(q_2), \dots, \overline{B}_{r_k}(q_k)$ are removed which are contained in $B_{r_1}(q_1)$ and which do not intersect. That is,

$$B(q, r) = B_{r_1}(q_1) \setminus \{\overline{B}_{r_2}(q_2) \,\dot\cup \cdots \dot\cup\, \overline{B}_{r_k}(q_k)\},$$

and $|q_1 - q_j| + r_j > r_1$ for $1 < j \le k$ as well as

$$r_j + r_\ell < |q_j - q_\ell| \quad \text{for } j \ne \ell \text{ with } 2 \le j, \ell \le r.$$

If, in addition $q_1 = q_2 = 0$ and $r_1 = 1$, then $B(q, r)$ is called a *normed* k-*circle domain*. We set $C_j := \partial B_{r_j}(q_j)$.

Let $\mathcal{N}(k)$ be the *class of* k-*circle domains*, and $\mathcal{N}_1(k)$ be the class of normed k-circle domains.

For $X \in H_2^1(B, \mathbb{R}^3)$ with $B = \mathrm{dom}(X) \in \mathcal{N}(k)$ we define the *area functional* $A(X)$ and the *Dirichlet integral* $D(X)$ as

$$A(X) := \int_B |X_u \wedge X_v|\, du\, dv = \int_B \sqrt{|X_u|^2 |X_v|^2 - \langle X_u, X_v\rangle^2}\, du\, dv,$$

$$D(X) := \frac{1}{2} \int_B [|X_u|^2 + |X_v|^2]\, du\, dv.$$

Note that these integrals are extended over the *domain* B of X which may vary with X. If B' is a subdomain of $\mathrm{dom}(X) = B$ we write

$$A_{B'}(X) := \int_{B'} |X_u \wedge X_v| \, du \, dv, \quad D_{B'}(X) := \frac{1}{2} \int_{B'} |\nabla X|^2 \, du \, dv.$$

Recall that

$$A(X) \leq D(X) \quad \text{for any } X \in H_2^1(B, \mathbb{R}^3)$$

and

$$A(X) = D(X) \quad \text{if and only if} \quad \langle X_w, X_w \rangle = 0$$

where

$$X_w := \frac{1}{2}(X_u - iX_v), \quad X_{\overline{w}} := \frac{1}{2}(X_u + iX_v).$$

The real form of the *conformality relation* $\langle X_w, X_w \rangle = 0$ is

$$(1) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

For a boundary contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ of k mutually disjoint, closed Jordan curves $\Gamma_1, \dots, \Gamma_k$ we define the *Douglas class* $\mathcal{C}(\Gamma)$ of admissible mappings $X : B \to \mathbb{R}^3$ for the variational procedure that we are going to set up:

Definition 1. *A mapping* $X \in H_2^1(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ *with* $B = \mathrm{dom}(X) \in \mathcal{N}(k)$ *belongs to* $\mathcal{C}(\Gamma)$ *if the Sobolev trace* $X|_{\partial B}$ *maps* ∂B *in a weakly monotonic way onto* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$. *By this we mean the following: There is an enumeration* $C_1, \dots, C_k$ *of the boundary circles of* B *such that* $X|_{C_j}$ *maps* C_j *in a weakly monotonic way onto* Γ_j, $j = 1, \dots, k$.

If in the sequel we consider a mapping $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X)$ and $\partial B = C_1 \,\dot{\cup} \cdots \dot{\cup}\, C_k$, we tacitly assume the boundary circles C_j to be enumerated in such a way that

$$\Gamma_1 = X(C_1), \quad \dots, \quad \Gamma_k = X(C_k).$$

We note that $\mathcal{C}(\Gamma)$ is nonempty if $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ is rectifiable, which means that each of the curves $\Gamma_1, \dots, \Gamma_k$ is rectifiable.

Now we can formulate the principal result of this chapter.

Theorem 1. *Let* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *be a boundary contour consisting of* k *mutually disjoint, closed, rectifiable Jordan curves in* $\mathbb{R}^3$, *and suppose that* Γ *satisfies either Courant's condition of cohesion or the Douglas condition. Then the following holds true:*

(i) *There is a minimizer* $X \in \mathcal{C}(\Gamma)$ *of Dirichlet's integral in* $\mathcal{C}(\Gamma)$, *that is,*

$$(2) \qquad D(X) = \inf_{\mathcal{C}(\Gamma)} D.$$

Every such minimizer X *is a minimal surface, i.e.* X *is harmonic in* B *and satisfies the conformality relations* (1); *moreover,* X *is continuous on* $\overline{B}$ *and yields a topological mapping from* ∂B *onto* Γ.

(ii) *In addition, we have*

(3) $$\inf_{\mathcal{C}(\Gamma)} D = \inf_{\mathcal{C}(\Gamma)} A.$$

This implies that every minimizer of D in $\mathcal{C}(\Gamma)$ is also a minimizer of A in $\mathcal{C}(\Gamma)$. Conversely, every conformally parametrized minimizer of A in $\mathcal{C}(\Gamma)$ is also a minimizer of D in $\mathcal{C}(\Gamma)$.

(iii) *Set $\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$. Then we even have*

(4) $$\inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} A.$$

Courant's condition of cohesion and the Douglas condition will be stated in Sections 8.3 and 8.6 respectively.

Without proof we mention the following result that will be derived in Vol. 2, Section 2.3 (for $k \geq 2$):

Theorem 2. *Suppose that $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ is of class $C^{k,\alpha}$, $k \geq 1$, $\alpha \in (0,1)$. Then each minimal surface $X : B \to \mathbb{R}^3$ of class $\overline{\mathcal{C}}(\Gamma)$ is also of class $C^{k,\alpha}(\overline{B}, \mathbb{R}^3)$.*

8.2 Conformality of Minimizers of D in $\mathcal{C}(\Gamma)$

Following ideas of *R*. Courant and H. Lewy we shall prove:

Theorem 1. *If $X \in \mathcal{C}(\Gamma)$ is a minimizer of D in $\mathcal{C}(\Gamma)$ then it satisfies*

(1) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

We first recall the following result that was proved in Section 4.5:

Lemma 1. *If $X \in \mathcal{C}(\Gamma)$ minimizes D in $\mathcal{C}(\Gamma)$ then its inner variation $\partial D(X, \eta)$ vanishes for all vector fields $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ with $B = \mathrm{dom}(X)$.*

Therefore Theorem 1 follows from

Theorem 2. *If $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X)$ satisfies*

(2) $$\partial D(X, \eta) = 0 \quad \textit{for all } \eta \in C^1(\overline{B}, \mathbb{R}^2),$$

then the conformality relations (1) hold true.

Before we begin with the proof of this theorem, we will derive some auxiliary results. The first one was proved in Section 4.5:

Lemma 2. *If $X \in \mathcal{C}(\Gamma)$ with $B = \operatorname{dom}(X)$ then*

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v\rangle \tag{3}$$

are of class $L_1(B)$, and for any $\eta = (\eta^1, \eta^2) \in C^1(B, \mathbb{R}^2)$ we have

$$\partial D(X, \eta) = \frac{1}{2}\int_B [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, du\, dv. \tag{4}$$

Let X be of class $H^1_2(B, \mathbb{R}^3)$, and consider a conformal mapping $\nu : B^* \to B$ from $B^* \subset \mathbb{C}$ onto B. Then $X^* := X \circ \nu$ satisfies $D(X) = D(X^*)$, i.e.

$$\int_B |\nabla X|^2\, du\, dv = \int_{B^*} |\nabla X^*|^2\, du\, dv.$$

Since

$$\partial D(X, \eta) = \frac{d}{d\epsilon} D(X \circ \tau_\epsilon)\bigg|_{\epsilon=0}$$

where τ_ϵ denotes an "inner variation" of the form

$$\tau_\epsilon(w) = w - \epsilon\lambda(w) + o(\epsilon), \quad |\epsilon| \ll 1$$

we obtain

Lemma 3. *Let ν be a conformal mapping from $\overline{B}^*$ onto $\overline{B}$ and $X \in H^1_2(B, \mathbb{R}^2)$, $X^* = X \circ \tau$. Then*

$$\partial D(X, \eta) = 0 \quad \textit{for all } \eta \in C^1(\overline{B}, \mathbb{R}^2)$$

is equivalent to

$$\partial D(X, \zeta) = 0 \quad \textit{for all } \zeta \in C^1(\overline{B}^*, \mathbb{R}^2).$$

Lemma 4. *For any $B \in \mathcal{N}(k)$ there is a Möbius transformation f such that $f(B) \in \mathcal{N}_1(k)$.*

Proof. For $k = 1$, f is given by $f(w) := \frac{1}{r_1}(w - q_1)$ if $B = B_{r_1}(q_1)$. If $k \geq 2$ and $B = B_{r_1}(q_1) \setminus \{\overline{B}_{r_2}(q_2) \cup \cdots \cup \overline{B}_{r_k}(q_k)\}$ then $f := \varphi \circ \psi$ with

$$\varphi(w) := \frac{w_1 - q_1}{r_1}, \quad \psi(z) := \frac{z - p_2}{\overline{p}_2 z - 1} \quad \text{with } p_2 := \varphi(q_2)$$

solves the task. □

Lemma 5. *If $X \in \mathcal{C}(\Gamma)$ with $B = \operatorname{dom}(X)$ and*

$$\phi := 4\langle X_w, X_w\rangle = a - ib, \quad a = |X_u|^2 - |X_v|^2, \quad b = 2\langle X_u, X_v\rangle,$$

then (2) is equivalent to

$$(5) \qquad \int_B \eta_{\overline{w}}\phi \, du\, dv = 0 \quad \text{for all } \eta \in C^1(\overline{B}, \mathbb{C}).$$

Furthermore, if ν *is a Möbius transformation and* $B = \nu(B^*)$, $B, B^* \in \mathcal{N}(k)$ *as well as* $X^* = X \circ \nu$, $\phi^* := \langle X_w^*, X_w^* \rangle$, *then* $X^* \in \mathcal{C}(\Gamma)$ *with* $B^* = \mathrm{dom}(X^*)$ *satisfies*

$$(5') \qquad \int_{B^*} \zeta_{\overline{w}}\phi^* \, du\, dv = 0 \quad \text{for all } \zeta \in C^1(\overline{B}^*, \mathbb{C}).$$

(*Here and in the sequel,* C^1 *means continuously differentiable in the "real" sense, i.e.* $C^1(\overline{B}, \mathbb{C})$ *is identified with* $C^1(\overline{B}, \mathbb{R}^2)$, *etc.*)

Proof. The equivalence of (2) and (5) follows from (4) and the identity

$$\mathrm{Re}(\eta_{\overline{w}}\phi) = \tfrac{1}{2}[(\eta_u^1 - \eta_v^2)a + (\eta_u^2 + \eta_v^1)b].$$

Furthermore, equation (5) implies (5′) on account of Lemmas 2 and 3, using the first assertion of Lemma 5. □

Lemma 6. *If* $X \in \mathcal{C}(\Gamma)$ *with* $B = \mathrm{dom}(X)$ *satisfies*

$$(6) \qquad \int_B [a(\eta_u^1 - \eta_v^2) + b(\eta_u^2 + \eta_v^1)] \, du\, dv = 0 \quad \text{for all } \eta \in C_c^\infty(B, \mathbb{R}^2),$$

then $\phi := 4\langle X_w, X_w \rangle = a - ib$ *with* a, b *given by* (3) *is holomorphic in* B, *i.e.* $\phi_{\overline{w}}(w) = 0$ *for all* $w \in B$.

Proof. Let $\mu = (\mu_1, \mu_2) \in C_c^\infty(B, \mathbb{R}^2)$ and set $\eta := \mathcal{S}_\delta \mu = k_\delta * \mu$ where $\mathcal{S}_\delta$ is a mollifier with a symmetric kernel k_δ. Then $\eta \in C_c^\infty(B, \mathbb{R}^2)$ if $0 < \delta \ll 1$, and $a^\delta := \mathcal{S}_\delta a$, $b^\delta := \mathcal{S}_\delta b$ are of class $C^\infty(B)$, and we infer from (6) the relation

$$\int_B [a^\delta(\mu_u^1 - \mu_v^2) + b^\delta(\mu_u^2 + \mu_v^1)] \, du\, dv = 0.$$

An integration by parts yields

$$\int_B [-(a_u^\delta + b_v^\delta)\mu^1 + (a_v^\delta - b_u^\delta)\mu^2] \, du\, dv = 0$$

for all $\mu \in C_c^\infty(B', \mathbb{R}^2)$ with $B' \subset\subset B$ and $0 < \delta < \delta_0(B') \le \mathrm{dist}(B', \partial B)$. Hence $a^\delta, -b^\delta$ satisfy

$$a_u^\delta = (-b^\delta)_v, \quad a_v^\delta = -(-b^\delta)_u \quad \text{in } B'$$

and so $\phi^\delta := a^\delta - ib^\delta$ is holomorphic in $B' \subset\subset B$ for $0 < \delta < \delta_0(B')$. Since $\phi^\delta \to \phi$ in $L_1(B', \mathbb{C})$ as $\delta \to 0$ for $B' \subset\subset B$, we infer that ϕ is holomorphic in any $B' \subset\subset B$ and therefore also in B. □

Proof of Theorem 2. We have to show that the holomorphic function $\phi(w)$ vanishes identically in B. We shall proceed in five steps. First we prove:

(i) *Let α be a closed C^1-Jordan curve in B which partitions $B \setminus \alpha$ into two disjoint open sets B_1 and B_2, i.e. $B = B_1 \dot{\cup} \alpha \dot{\cup} B_2$. Suppose also that $\eta = (\eta^1, \eta^2) \in C^1(\overline{B}, \mathbb{R}^2)$, written in the complex form $\eta = \eta^1 + i\eta^2$, is holomorphic in B_1 and satisfies $\eta(w) = 0$ for any $w \in \partial B_2 \setminus \alpha$. Then we have*

$$\operatorname{Im} \int_\beta \eta(w)\phi(w)\, dw = 0 \tag{7}$$

for any closed C^1-curve β in B_1 that is homologous to α (where $\int_\beta \dots dw$ is the complex line integral along β).

In fact, $\eta_{\overline{w}} = 0$ on B_1 and (6) imply

$$\operatorname{Re} \int_{B_2} \eta_{\overline{w}} \phi \, du\, dv = 0,$$

whence

$$\int_{B_2} [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, du\, dv = 0.$$

Since $\eta = 0$ on $\partial B_2 \setminus \alpha$, an integration by parts yields

$$\begin{aligned} 0 = & \int_\alpha (a\eta^2 - b\eta^1) du + (a\eta^1 + b\eta^2)\, dv \\ & - \int_{B_2} [(a_u\eta^1 + b_u\eta^2) + (b_v\eta^1 - a_v\eta^2)]\, du\, dv. \end{aligned}$$

Furthermore,

$$2\operatorname{Re}(\eta\phi_{\overline{w}}) = (a_u\eta^1 + b_u\eta^2) + (b_v\eta^1 - a_v\eta^2),$$

and

$$\operatorname{Im}(\phi\eta\, dw) = (a\eta^2 - b\eta^1)\, du + (a\eta^1 + b\eta^2)\, dv.$$

Since $\phi_{\overline{w}} = 0$ in B it follows

$$\operatorname{Im} \int_\alpha \phi\eta\, dw = 0.$$

As $\phi\eta$ is holomorphic in B_1 we also have

$$\int_\alpha \phi\eta\, dw = \int_\beta \phi\eta\, dw$$

and so we obtain (7). Thus assertion (i) is proved.

For any M in $\mathbb{C}$ we define the "thickening" $B_\delta(M)$ by

$$B_\delta(M) := \{w \in \mathbb{C}\colon \operatorname{dist}(w, M) < \delta\},$$

and then the annuli $A_j(\delta)$ of width $\delta > 0$ about the circles $C_j = \partial B_{r_j}(q_j)$ which bound the domain $B \in \mathcal{N}(k)$ given by

$$B = B_{r_1}(q_1) \Big\backslash \bigcup_{j=2}^{k} \overline{B}_{r_j}(q_j)$$

with $\overline{B}_{r_j}(q_j) \subset B_{r_1}(q_1)$ and $\overline{B}_{r_j}(q_j) \cap \overline{B}_{r_\ell}(q_\ell) = \emptyset$ for $2 \le j, \ell \le k$, $j \ne \ell$:

$$A_j(\delta) := B \cap B_\delta(C_j), \quad j = 1, \ldots, k.$$

We have

$$A_j(\delta) \cap A_\ell(\delta) = \emptyset \quad \text{for } j \ne \ell,\ 1 \le j, \ell \le k,$$

provided that

$$\delta < \delta_0 := \tfrac{1}{2} \min\{\operatorname{dist}(C_j, C_\ell)\colon j \ne \ell,\ 1 \le j, \ell \le k\}.$$

Now we turn to the second step in the proof of Theorem 2, which consists in proving the following result:

(ii) *For any closed C^1-curve in $A_j(\delta)$, $0 < \delta < \delta_0$, which is homologous to C_j, we have*

$$\int_{\beta_j} \phi(w)\, dw = 0 \tag{8}$$

and

$$\int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \tag{9}$$

for $j = 1, \ldots, k$.

To prove this result, we fix some $j \in \{1, \ldots, k\}$ and consider three vector fields $\eta_1, \eta_2, \eta_3 \in C_c^\infty(B \cup C_j, \mathbb{C})$ with

$$\frac{\partial}{\partial \overline{w}} \eta_\ell(w) = 0 \quad \text{in } A_j(\delta), \quad \ell = 1, 2, 3,$$

satisfying

$$\eta_1(w) := \begin{cases} \zeta & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta), \end{cases}$$

where ζ is an arbitrary complex number,

$$\eta_2(w) := \begin{cases} w - q_j & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta), \end{cases}$$

$$\eta_3(w) := \begin{cases} -i(w - q_j) & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta). \end{cases}$$

Let C_j' be the circle $\partial A_j(\delta) \setminus C_j$ and apply step (i) to $\alpha := C_j'$ and $\eta := \eta_1$. Then, for any closed curve β_j in $A_j(\delta)$ homologous to α and therefore homologous to C_j, it follows that

$$\operatorname{Im}\left[\zeta \int_{\beta_j} \phi(w)\, dw\right] = 0 \quad \text{for all } \zeta \in \mathbb{C}.$$

This yields formula (8).

Applying the same reasoning to $\eta := \eta_2$ and $\eta := \eta_3$ respectively, we obtain

$$\operatorname{Im} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \quad \text{and} \quad \operatorname{Re} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0,$$

which proves formula (9).

Remark. One can as well choose

$$\eta_2(w) := (w - q_j)^n \quad \text{and} \quad \eta_3(w) := -i(w - q_j)^n \quad \text{on } \overline{A}_j(\delta)$$

with $n \in \mathbb{Z} \setminus \{0\}$ and

$$\eta_2(w) := 0 \quad \text{and} \quad \eta_3(w) := 0 \quad \text{on } \overline{B} \setminus \overline{A}_j(2\delta).$$

Then one obtains

$$\text{(10)} \qquad \int_{\beta_j} (w - q_j)^n \phi(w)\, dw = 0 \quad \text{for all } n \in \mathbb{Z}$$

and $\beta_j \subset A_j(\delta)$, $0 < \delta < \delta_0$. If $k = 2$ and

$$B = \{w \in \mathbb{C} \colon 0 < r < |w| < 1\} \in \mathcal{N}_1(2),$$

then $\phi(w)$ is holomorphic in B, and thus it can be expanded into a convergent Laurent series:

$$\phi(w) = \sum_{n=-\infty}^{\infty} a_n w^n \quad \text{for } w \in B.$$

Formula (10) then becomes

$$\int_{\beta} w^n \phi(w)\, dw = 0 \quad \text{for all } n \in \mathbb{Z}$$

and $\beta = \{w \in \mathbb{C} \colon |w| = \rho\}$ with $r < \rho < 1$, and we obtain $a_n = 0$ for all $n \in \mathbb{Z}$, i.e. $\phi(w) \equiv 0$. Since every $\tilde{B} \in \mathcal{N}(2)$ is equivalent to some $B \in \mathcal{N}_1(2)$, the assertion of Theorem 2 is proved in case that $k = 2$, and for $k = 1$ the proof follows in the same way. Thus the proof becomes really interesting for $k \geq 3$.

On account of Lemmas 4 and 5, it suffices to prove Theorem 2 under the *additional assumption*

$$B \in \mathcal{N}_1(k) \tag{11}$$

which from now on will be required. In other words, we assume that

$$r_1 = 1, \quad q_1 = q_2 = 0. \tag{11$'$}$$

Now we turn to the third step of the proof. We are going to show

(iii) *One has:* $(w - q_j)^2\phi(w)$ *is continuous on* $B \cup C_j$*, and*

$$\operatorname{Im}[(w - q_j)^2\phi(w)] = 0 \quad \text{for } w \in C_j, \quad 1 \leq j \leq k. \tag{12}$$

We will first verify (12) for the case $j = 1$ where $q_1 = 0$ and $r_1 = 1$; by a suitable Möbius transformation any of the cases $j = 2, \ldots, k$ will be reduced to $j = 1$.

Fix some $\delta \in (0, \delta_0)$, and let ψ be an arbitrary real valued function with $\psi \in C^1(\overline{B})$ and

$$\psi(w) = 0 \quad \text{for } w \in \overline{B} \text{ with } |w| \leq 1 - 2\delta.$$

Set

$$\eta(w) := -i[w\psi(w)] \quad \text{for } w \in \overline{B}.$$

By (6) we have

$$0 = \operatorname{Re} \int_B \eta_{\overline{w}}\phi \, du \, dv = \lim_{R \to 1-0} \operatorname{Re} \int_{B \cap B_R(0)} \eta_{\overline{w}}\phi \, du \, dv.$$

As in the proof of step (i) it follows that

$$0 = -\lim_{R \to 1-0} \operatorname{Im} \int_{\partial B_R(0)} iw\psi(w)\phi(w) \, dw.$$

With $w = \mathrm{Re}^{i\theta}$ and $dw = iw \, d\theta$ we obtain

$$0 = \lim_{R \to 1-0} \int_0^{2\pi} \psi(\mathrm{Re}^{i\theta}) h(\mathrm{Re}^{i\theta}) \, d\theta \tag{13}$$

if we denote by $h : B \to \mathbb{R}$ the harmonic function

$$h(w) := \operatorname{Im}[w^2\phi(w)], \quad w \in B.$$

Suppose now that ψ depends also on a further parameter $z \in \overline{B}_\rho(0)$ such that $\psi(w,z)$ is of class C^1 for (w,z) satisfying $1-\delta \le |w| \le 1$, $|z| \le \rho \le 1-\sigma$ for $\sigma \in (0, 2\delta)$. Then we obtain for $f := \mathrm{Re}[\eta_{\overline{w}}(\cdot, z)\phi]$ that

$$\begin{aligned}\left|\int_{B\cap B_R(0)} f\,du\,dv\right| &= \left|\int_B f\,du\,dv - \int_{B\setminus B_R(0)} f\,du\,dv\right| \\ &= \left|\int_{B\setminus B_R(0)} f\,du\,dv\right| \\ &\le M\cdot \int_{B\setminus B_R(0)} |\phi|\,du\,dv \quad \text{for } R > 1-\sigma\end{aligned}$$

where

$$M := \sup\{|\eta_{\overline{w}}(w,z)| \colon 1-\delta \le |w| \le 1,\ |z| \le \rho\} < \infty.$$

Thus we achieve the uniform convergence of $\int_{B\cap B_R(0)} f(w,z)\,du\,dv$ to zero as $R \to 1-0$ for $z \in \overline{B}_\rho(0)$, i.e.

$$\mathrm{Re}\int_{B\cap B_R(0)} \eta_{\overline{w}}(w,z)\phi(w)\,du\,dv \to 0 \quad \text{uniformly in } z \in \overline{B}_\rho(0) \text{ as } R \to 1-0,$$

since $|\phi| \in L_1(B)$. This implies that the convergence in (13) is uniform with respect to $z \in \overline{B}_\rho(0)$, i.e.

$$(14)\qquad \int_0^{2\pi} \psi(\mathrm{Re}^{i\theta}, z)h(\mathrm{Re}^{i\theta})\,d\theta \to 0 \quad \text{uniformly in } z \in \overline{B}_\rho(0) \text{ as } R \to 1-0.$$

For $0 \le r \le \rho < 1-\sigma < R < 1$ and $w = \mathrm{Re}^{i\theta}$, $z = re^{i\vartheta}$ we introduce the Poisson kernel $K(w,z)$ of the ball $B_R(0)$ with respect to $w \in \partial B_R(0)$ and $z \in \overline{B}_\rho(0)$,

$$K(w,z) := \frac{R^2 - r^2}{2\pi[R^2 - 2rR\cos(\theta - \vartheta) + r^2]}.$$

Furthermore let ξ be a radial cut-off function of class $C^\infty(\mathbb{R})$ with $\xi(r) = 1$ for $r \ge 1-\sigma/2$ and $\xi(r) = 0$ for $r \le 1-\sigma$, $0 < \sigma < 2\delta$, and set

$$\psi(w,z) := \xi(|w|)K(w,z)$$

for $z \in \overline{B}_\rho(0)$, $0 < \rho < 1-\sigma$, and $1-2\delta < 1-\sigma \le |w| \le 1$. Then $\psi(w,z)$ has the properties required above, and for $R = |w| \ge 1-\sigma/2$ one has $\xi(|w|) = 1$. Consequently it follows from (14) that

$$H_R(z) := \int_0^{2\pi} K(\mathrm{Re}^{i\theta}, z)h(\mathrm{Re}^{i\theta})d\theta, \quad z \in B_R(0),$$

satisfies

$$(15)\qquad \|H_R\|_{C^0(\overline{B}_\rho(0))} \to 0 \quad \text{as } R \to 1-0 \text{ for any } \rho < 1-\sigma,\ 0 < \sigma < 2\delta.$$

By Poisson's formula and Schwarz's theorem it follows that H_R is harmonic in the disk $B_R(0)$ and can be extended to a continuous function on $\overline{B}_R(0)$ satisfying

$$H_R(w) = h(w) \quad \text{for } w \in \partial B_R(0). \tag{16}$$

In the sequel, $A(r, r')$ denotes the annulus

$$A(r, r') := \{w \in \mathbb{C} \colon r < |w| < r'\} \quad \text{for } 0 < r < r' < \infty.$$

For $R_0 := 1 - 2\delta < R < 1$ we now consider the excess function $E_R : \overline{A}(R_0, R) \to \mathbb{R}$ defined by

$$E_R(w) := h(w) - H_R(w) \quad \text{for } w \in \overline{A}(R_0, R),$$

which is continuous on $\overline{A}(R_0, R)$, harmonic in $A(R_0, R)$, and vanishes on the circle $\partial B_R(0)$ according to (16). By reflection in this circle we can extend E_R to a continuous function on $\overline{A}(R_0, R')$ with $R' := R^2/R_0$ which is harmonic in $A(R_0, R')$ and satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| = \max_{\partial B_{R'}(0)} |E_R|. \tag{17}$$

Set

$$C = C(R_0) := 2 \max_{\partial B_{R_0}(0)} |h|, \quad R_0 = 1 - 2\delta,$$

and for arbitrarily chosen $\epsilon > 0$ we pick a number σ with

$$0 < \sigma < \min\left\{\frac{\delta}{2}, \frac{\epsilon\delta}{2C}\right\}. \tag{18}$$

Because of (15) there is a number $R_1 \in (1 - (\sigma/2), 1)$ such that

$$\max_{\partial B_{R_0}(0)} |H_R| < C/2 \quad \text{for all } R \in (R_1, 1),$$

and so $E_R = h - H_R$ satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| < C \quad \text{for all } R \in (R_1, 1).$$

In conjunction with (17) the maximum principle then implies

$$\max_{\overline{A}(R_0, R')} |E_R| < C \quad \text{for all } R \in (R_1, 1) \tag{19}$$

where $R_0 = 1 - 2\delta$ and $R' = R^2/R_0$.

For $R \in (R_1, 1)$ we have $1 - \sigma/2 < R < 1$ and therefore $R - (1 - \sigma) > \sigma/2 > 0$. For any $w \in A(1 - \sigma, R)$ it follows that

$$\operatorname{dist}(w, \partial A(R_0, R')) > (1-\sigma) - R_0 = 2\delta - \sigma > \delta.$$

Applying Cauchy's estimate to ∇E_R on $A(1-\sigma, R)$ we then infer from (19) that

$$\max_{\overline{A}(1-\sigma,R)} |\nabla E_R| \le \frac{C(R_0)}{\delta} \quad \text{for } R \in (R_1, 1).$$

Since $E_R(w) = 0$ for $|w| = R$, we can write

$$|E_R((1-\sigma)e^{i\theta})| \le \int_{1-\sigma}^{R} |\partial_r E_R(re^{i\theta})|\, dr \le \sigma \frac{C}{\delta} < \frac{\epsilon\delta}{2C} \cdot \frac{C}{\delta}$$

whence

$$|E_R(w)| < \frac{\epsilon}{2} \quad \text{for } |w| = 1-\sigma \text{ and } R_1 < R < 1,$$

where $R_1 \in (1-\sigma/2, 1)$ was chosen above and σ is a fixed number satisfying (18).

Applying once more (15) it follows that for the chosen σ there is a number $R_2 \in [R_1, 1)$ such that

$$\max_{\overline{B}_{1-\sigma}(0)} |H_R| < \frac{\epsilon}{2} \quad \text{for all } R \in (R_2, 1).$$

Because of

$$h(w) = E_R(w) + H_R(w) \quad \text{for } w \in \overline{A}(R_0, R)$$

and $R_0 = 1 - 2\delta < 1 - \sigma < 1 - \sigma/2 < R_1 \le R_2 < R < 1$ we arrive at

$$|h(w)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for } |w| = 1-\sigma.$$

This implies for the harmonic function $h(w) = \operatorname{Im}[w^2\phi(w)]$ that

$$\lim_{\sigma \to +0} \max_{\partial B_{1-\sigma}(0)} |h| = 0,$$

and so we can extend h continuously to $B \cup C_1$ with $C_1 = \partial B_1(0)$ by setting

$$h(w) = 0 \quad \text{for } w \in C_1,$$

which completes the proof of (12) for $j = 1$.

Note that for the proof of (12) in the case $j = 1$ we only have used $q_1 = 0$, $r_1 = 1$ and the fact that $C_1 = \partial B_1(0)$ contains the other boundary circles $C_2, \dots, C_k$ in its interior domain $B_1(0)$. Therefore we can reduce the cases $j = 2, \dots, k$ to $j = 1$ by applying the Möbius transformation $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, defined by

$$z = \mu(w) := \frac{r_j}{w - q_j}$$

where $C_j = \partial B_{r_j}(q_j) = \{w \in \mathbb{C}\colon |w - q_j| = r_j\}$. The mapping μ maps B into another k-circle domain B^* whose exterior circle is $C_1 = \partial B_1(0)$, and $C_1 = \mu(C_j)$. Let $\nu := \mu^{-1}$ be the inverse of μ, and set

$$X^* := X \circ \nu \quad \text{with } \operatorname{dom}(X^*) = B^*.$$

Then by Lemma 5 we have

$$\int_{B^*} \zeta_{\overline{w}} \phi^* \, du \, dv = 0 \quad \text{for all } \zeta \in C^1(\overline{B}^*, \mathbb{R}^2),$$

and the above reasoning yields

$$\operatorname{Im}[z^2 \phi^*(z)] = 0 \quad \text{for } z \in C_1$$

and $\phi^*(z) = 4\langle X_z^*, X_z^* \rangle = a^*(z) - ib^*(z)$. A straight-forward computation yields

$$(w - q_j)^2 \phi(w) = z^2 \phi^*(z) \quad \text{for } z \in C_1 \text{ and } w = \nu(z) \in C_j.$$

Thus we have shown that $(w - q_j)^2 \phi(w)$ is continuous on $B \cup C_j$ and

$$\operatorname{Im}[(w - q_j)^2 \phi(w)] = 0 \quad \text{for } w \in C_j,\ 2 \le j \le k,$$

and so the proof of assertion (iii) is complete. □

Let us review the assertion of (iii). We have shown that each of the holomorphic functions

$$F_j(w) := (w - q_j)^2 \phi(w), \quad w \in B,$$

$1 \le j \le k$, has a harmonic imaginary part $h_j := \operatorname{Im} F_j$ which can continuously be extended to $B \cup C_j$ by setting $h_j = 0$ on C_j. Then the reflection principle for harmonic functions yields that h_j can be extended as a harmonic function beyond C_j. Inspecting the Cauchy–Riemann equations, it follows that F_j can be extended holomorphically across C_j, and therefore ϕ can be extended holomorphically to some domain G with $\overline{B} \subset G \subset \mathbb{C}$. This implies that either $\phi(w) \equiv 0$ in $\overline{B}$, or else ϕ has finitely many zeros in $\overline{B}$. Employing a method due to Hans Lewy we will show that the second case is impossible, thus verifying the assertion of Theorem 2. To this end we turn to the next step of the proof:

(iv) *If $\phi(w) \not\equiv 0$ in $\overline{B}$ then ϕ has at least four zeros on each boundary circle C_j of B.*

To prove this, let r, θ be polar coordinates around q_j defined by $w = q_j + re^{i\theta}$, and introduce the 2π-periodic functions

$$f_j(\theta) := r_j^2 e^{i2\theta} \phi(q_j + r_j e^{i\theta}), \quad j = 1, \dots, k,$$

that are real analytic in θ and satisfy $f_j(\theta) \in \mathbb{R}$ for $\theta \in \mathbb{R}$ on account of (12). By step (ii) applied to $\beta_j := C_j$ it follows that

$$i \int_0^{2\pi} f_j(\theta)\, d\theta = 0 \quad \text{and} \quad i r_j^{-1} \int_0^{2\pi} e^{-i\theta} f_j(\theta)\, d\theta = 0,$$

whence

$$(20) \quad \int_0^{2\pi} f_j(\theta)\, d\theta = 0, \quad \int_0^{2\pi} f_j(\theta) \cos\theta \, d\theta = 0, \quad \int_0^{2\pi} f_j(\theta) \sin\theta \, d\theta = 0.$$

Then $f_j(\theta) \not\equiv \text{const}$, because the first equation would imply $f_j(\theta) \equiv 0$ and therefore $\phi(w) \equiv 0$ on $\partial B_{r_j}(q_j)$ which is impossible since $\phi(w)$ has only finitely many zeros in $\overline{B}$. Moreover $\int_0^{2\pi} f_j(\theta)\, d\theta = 0$ shows that $f_j(\theta)$ must change its sign in $[0, 2\pi)$ at least once, and so it has a positive maximum and a negative minimum. Correspondingly $f_j(\theta)$ possesses two zeros $\theta_0, \theta_1 \in [0, 2\pi)$, i.e. $|\theta_0 - \theta_1| < 2\pi$ since f_j is periodic. By choosing the polar angle θ suitably we can assume that $f_j(\theta)$ has the two zeros θ_0 and $-\theta_0$ with some $\theta_0 \in (0, \pi)$, while the three equations (20) remain valid. This yields

$$(21) \quad \int_{-\pi}^{\pi} f_j(\theta)[\cos\theta - \cos\theta_0]\, d\theta = 0,$$

and so the function $f_j(\theta)[\cos\theta - \cos\theta_0]$ changes its sign in $(-\pi, \pi)$. Since $g(\theta) := \cos\theta - \cos\theta_0$ with $g'(\theta) = -\sin\theta$ satisfies $g'(\theta) > 0$ for $-\pi < \theta < 0$, $g'(\theta) < 0$ for $0 < \theta < \pi$, it follows that

$$g(\theta) < 0 \quad \text{on } (-\pi, -\theta_0) \cup (\theta_0, \pi), \quad g(\theta) > 0 \quad \text{on } (-\theta_0, \theta_0).$$

If $f_j(\theta)$ would have no other zero than θ_0 and $-\theta_0$ then $f_j(\theta)g(\theta)$ did not change its sign in $(-\pi, \pi)$, but this contradicts (21). Thus there is a third zero θ_3 of $f_j(\theta)$ in $(-\pi, \pi)$. We claim that there is even a fourth zero θ_4 of f_j in $(-\pi, \pi)$. In fact suppose that $f_j(\theta) \neq 0$ for $\theta \in (-\pi, \pi)$ with $\theta \neq \pm\theta_0, \theta_3$. If $\theta_3 \in (-\theta_0, \theta_0)$ then again $f_j(\theta)g(\theta)$ would not change its sign, a contradiction to (21). The other two cases $\theta_3 < -\theta_0$ and $\theta_0 < \theta_3$ can be transformed to the case $-\theta_0 < \theta_3 < \theta_0$ by a shift of θ which keeps (21) fixed because of (20). This completes the proof of assertion (iv). □

Now we turn to the final step in the proof of Theorem 2:

(v) *We have $\phi(w) \equiv 0$ in B.*

Suppose that this were false. Then $\phi(w)$ had only finitely many zeros in $\overline{B}$ as we have observed before. Let $w_m \in B$ be the interior zeros of ϕ with the multiplicities μ_m, $m = 1, \ldots, M$, and $\zeta_\ell \in \partial B$ be the boundary zeros of ϕ with the multiplicities ν_ℓ, $\ell = 1, \ldots, L$. Set $N := \mu_1 + \cdots + \mu_M$, and choose $\rho > 0$ sufficiently small. Then, by Rouché's formula, the number $N \geq 0$ is given by

$$N = \frac{1}{2\pi i} \int_{\partial G_\rho} \frac{\phi'(w)}{\phi(w)} \, dw, \quad G_\rho := B \setminus \bigcup_{\ell=1}^{L} \overline{B}_\rho(\zeta_\ell).$$

The boundary ∂G_ρ consists of $\beta_j(\rho) := C_j \cap \partial G_\rho$, $j = 1, \dots, k$, and of the circular arcs $\gamma_\ell(\rho) := \partial B_\rho(\zeta_\ell) \cap B$, $\ell = 1, \dots, L$. Recall also that $F_j(w) = (w - q_j)^2 \phi(w)$ is holomorphic in $B \cup C_j$ and *real valued on* C_j. Then we have

$$d \log F_j(w) = d \log(w - q_j)^2 + d \log \phi(w) \quad \text{on } \beta_j,$$

whence

$$\frac{\phi'(w)}{\phi(w)} \, dw = \frac{F_j'(w)}{F_j(w)} \, dw - \frac{2}{w - q_j} \, dw \quad \text{for } w \in \beta_j.$$

This implies

$$\frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{\phi'(w)}{\phi(w)} \, dw = I_j(\rho) \setminus K_j(\rho)$$

with

$$I_j(\rho) := \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{F_j'(w)}{F_j(w)} \, dw$$

and

$$K_j(\rho) := 2 \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{dw}{w - q_j}.$$

We have

$$\lim_{\rho \to +0} K_j(\rho) = \begin{cases} 2 & \text{for } j = 1, \\ -2 & \text{for } j = 2, \dots, k, \end{cases}$$

and it will be proved below that

$$\lim_{\rho \to +0} I_j(\rho) = 0. \tag{22}$$

Thus

$$N = \lim_{\rho \to +0} \sum_{j=1}^{k} [I_j(\rho) - K_j(\rho)] + \lim_{\rho \to +0} \sum_{\ell=1}^{L} P_\ell(\rho)$$

with

$$P_\ell(\rho) := \frac{1}{2\pi i} \int_{\gamma_\ell(\rho)} \frac{\phi'(w)}{\phi(w)} \, dw.$$

Since ϕ is mirror symmetric with respect to the inversion at C_j it follows that (for $\gamma_\ell^*(\rho)$ as reflection of $\gamma_\ell(\rho)$ at C_j)

$$\begin{aligned} \lim_{\rho \to +0} P_\ell(\rho) &= \frac{1}{4\pi i} \lim_{\rho \to +0} \int_{\gamma_\ell(\rho) \cup \gamma_\ell^*(\rho)} \frac{\phi'(w)}{\phi(w)} \, dw \\ &= \frac{1}{4\pi i} \lim_{\rho \to +0} \int_{-\partial B_\rho(\zeta_\ell)} \frac{\phi'(w)}{\phi(w)} \, dw = -\frac{\nu_\ell}{2} \end{aligned}$$

since the positive orientation of G_ρ implies that the circles $\partial B_\rho(\zeta_\ell)$ are to be taken as negatively oriented. Since $L \geq 4k$ and $\nu_\ell \geq 1$ it follows that

$$N = -2 + 2(k-1) - \frac{1}{2}\sum_{\ell=1}^{L} \nu_\ell \leq -4 + 2k - \frac{1}{2} \cdot 4k = -4,$$

a contradiction to $N \geq 0$. Therefore we obtain $\phi(w) \equiv 0$ on $\overline{B}$.

It remains to prove (22). Since

$$2\pi i I_j(\rho) = \int_{\beta_j(\rho)} d\log|F_j(w)| = \int_{\beta'_j(\rho)} d\log|\psi(\theta)|$$

with $\psi(\theta) := F_j(q_j + r_j e^{i\theta})$ and

$$\beta'_j(\rho) = [0, \theta_1 - \epsilon(\rho)] \cup \bigcup_{s=1}^{p-1} [\theta_s + \epsilon(\rho), \theta_{s+1} - \epsilon(\rho)] \cup [\theta_p + \epsilon(\rho), 2\pi],$$

where $\epsilon = \epsilon(\rho) \to +0$ as $\rho \to +0$, and $\zeta_s := e^{i\theta_s}$ are the zeros of F_j on C_j, we obtain

$$\int_{\beta'_j(\rho)} d\log|\psi(\theta)| = \sum_{s=1}^{p+1} [\log|\psi(\theta)|]_{a_s(\rho)}^{b_s(\rho)}$$

with

$$\begin{aligned}
&a_1(\rho) = 0, \quad a_2(\rho) = \theta_1 + \epsilon(\rho), \quad \ldots, \quad a_p(\rho) = \theta_{p-1} + \epsilon(\rho),\\
&a_{p+1}(\rho) = \theta_p + \epsilon(\rho),\\
&b_1(\rho) = \theta_1 - \epsilon(\rho), \quad b_2(\rho) = \theta_2 - \epsilon(\rho), \quad \ldots, \quad b_p(\rho) = \theta_p - \epsilon(\rho),\\
&b_{p+1}(\rho) = 2\pi.
\end{aligned}$$

Thus we infer from $\psi(0) = \psi(2\pi)$

$$\begin{aligned}
\int_{\beta'_j(\rho)} d\log|\psi(\theta)| &= \sum_{s=1}^{p} \big[\log|\psi(b_s(\rho))| - \log|\psi(a_{s+1}(\rho))|\big]\\
&= \sum_{s=1}^{p} \log\left|\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))}\right| \to 0 \quad \text{for } \rho \to +0
\end{aligned}$$

since

$$\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))} \to 1 \quad \text{as } \rho \to +0.$$

Thus we conclude $I_j(\rho) \to 0$ as $\rho \to +0$, and we have verified (22).

This completes the proof of Theorem 2. □

8.3 Cohesive Sequences of Mappings

Let $\{B_m\}$ be a sequence of k-circle domains

$$B_m = B(q^{(m)}, r^{(m)}) \in \mathcal{N}(k)$$

with $q^{(m)} = (q_1^{(m)}, \ldots, q_k^{(m)}) \in \mathbb{C}^k$, $r^{(m)} = (r_1^{(m)}, \ldots, r_k^{(m)}) \in \mathbb{R}^k$, $r_j^{(m)} > 0$.

We say that $\{B_m\}$ converges to the domain $B := B_{r_1}(q_1) \setminus \bigcup_{j=2}^k \overline{B_{r_j}(q_j)}$, symbol:

$$B_m \to B \quad \text{as } m \to \infty, \quad \text{or} \quad \lim_{m\to\infty} B_m = B,$$

if $q^{(m)} \to q$ in $\mathbb{C}^k$ and $r^{(m)} \to r$ in $\mathbb{R}^k$.

By $\overline{\mathcal{N}}(k)$ and $\overline{\mathcal{N}}_1(k)$ we denote the set of domains B in $\mathbb{C}$ that are limits of converging sequences $\{B_m\}$ in $\mathcal{N}(k)$ and $\mathcal{N}_1(k)$ respectively.

Clearly the limit B of a sequence $\{B_m\} \subset \mathcal{N}(k)$ need not be a k-circle domain again, i.e. B might be "degenerate" in the sense that $B \in \overline{\mathcal{N}}(k) \setminus \mathcal{N}(k)$. Let us investigate how the boundary circles

$$C_j^{(m)} := \partial B_{r_j^{(m)}}(q_j^{(m)}) \quad \text{of } B_m = B_{r_1^{(m)}}(q_1^{(m)}) \Big\backslash \bigcup_{j=2}^k \overline{B_{r_j^{(m)}}(q_j^{(m)})}$$

behave if the B_m converge to a degenerate domain with the "boundary circles" $C_j = \partial B_{r_j}(q_j)$. Here r_j might be zero; then C_j is just the point q_j, i.e. $C_j^{(m)} \to q_j$. Another form of degeneration is that two limit circles C_j and C_ℓ, $j \neq \ell$, are true circles which "touch" each other (this includes the possibility $C_j = C_\ell$).

We distinguish three kinds of degeneration:

Type 1. Two limits C_j and C_ℓ, $j \neq \ell$, are true circles which touch each other, i.e. either $C_j = C_\ell$ or $C_j \cap C_\ell = \{w_0\}$ for some $w_0 \in \overline{B}$.

Type 2. One limit C_ℓ is a point p which lies on a true limit circle C_j.

Type 3. One limit C_ℓ is a point p which does not lie on any true limit circle.

For our purposes it suffices to consider degenerate limits B of domains $B_m \in \mathcal{N}_1(k)$. Here we have for all $m \in \mathbb{N}$ that

$$C_1^{(m)} = C := \partial B_1(0), \quad C_2^{(m)} = \partial B_{r_2^{(m)}}(0), \quad 0 < r_2^{(m)} < 1.$$

Case (a): $k = 2$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to 0$, i.e. $C_1 = C_2 = C$ (type 1) or $C_2 = \{0\}$ (type 3), whereas type 2 cannot occur for a degenerate limit B.

Case (b): $k \geq 3$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to r \in [0, 1)$.

(b1) If $r_2^{(m)} \to 1$ then $C_1 = C_2 = C$ and $C_j = \{q_j\}$ with $j = 3, \dots, k$. Thus B is both of type 1 and 2.

(b2) If $r_2^{(m)} \to r_2$ with $0 \le r_2 < 1$, then $C_1 = C$ and either $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$ or $C_2 = \{0\}$. Here we have at least one of the following possibilities:

(i) B is of type 1 with $C_j \cap C_\ell = \{w_0\}$ for some $w_0 \in \overline{B}$, and possibly also of type 2 or type 3 or both.
(ii) B is not of type 1, but of type 2, or of type 3, or both of type 2 and 3.

The following result is obvious:

Lemma 1. *From any sequence of domains $B_m \in \mathcal{N}_1(k)$ we can extract a subsequence $\{B_{m_j}\}$ with $B_{m_j} \to B \in \overline{\mathcal{N}}_1(k)$ as $j \to \infty$.*

We now want to state conditions ensuring that the limit B of domains $B_m \in \mathcal{N}_1(k)$ is nondegenerate, that is $B \in \mathcal{N}_1(k)$. A first result in this direction is

Proposition 1. *Let $\{X_m\}$ be a sequence of mappings $X_m \in \mathcal{C}(\Gamma)$ with $B_m = \mathrm{dom}(X_m) \in \mathcal{N}_1(k)$, $k \ge 2$, where $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ is a contour consisting of k rectifiable, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$ and suppose that $B_m \to B$ for $m \to \infty$ as well as*

$$D(X_m) \le M \quad \text{for all } m \in \mathbb{N}$$

and some constant $M > 0$. Then $B \in \overline{\mathcal{N}}_1(k)$ cannot be degenerate of type 1.

Proof. Let $\mu(\Gamma)$ be the minimal distance of the curves $\Gamma_1, \dots, \Gamma_k$ from each other, i.e.

$$\mu(\Gamma) := \min\{\mathrm{dist}(\Gamma_j, \Gamma_\ell) \colon 1 \le j, \ell \le k,\ j \ne \ell\} > 0. \tag{1}$$

If B were of type 1, there would be $j, \ell \in \{1, \dots, k\}$ with $j \ne \ell$ such that $C_j^{(m)} \to C_j$ and $C_\ell^{(m)} \to C_\ell$ as $m \to \infty$, where C_j and C_ℓ are true circles with $C_j \cap C_\ell \ne \emptyset$. Let $w_0 \in C_j \cap C_\ell$, and introduce polar coordinates ρ, θ about $w_0 \colon w = w_0 + \rho e^{i\theta}$. There is a representative

$$Z_m(\rho, \theta) := X_m(w_0 + \rho e^{i\theta})$$

of X_m which, for almost all $\rho \in (0,1)$, is absolutely continuous in $\theta \in [\theta_1, \theta_2]$ along each arc $\gamma(\rho) := \{w_0 + \rho e^{i\theta} \colon \theta_1 \le \theta \le \theta_2\}$ contained in $\overline{B}_m$; we call $\gamma(\rho)$ X_m-admissible. The Courant–Lebesgue lemma yields: *For each $m \in \mathbb{N}$ and each $\delta \in (0,1)$ there is an X_m-admissible* arc $\gamma_m(\rho) = \{w_0 + \rho e^{i\theta} \colon \theta_1^m \le \theta \le \theta_2^m\}$ *in $\overline{B}_m$ with $\delta < \rho < \sqrt{\delta}$ such that*

$$\mathrm{osc}(Z_m, \gamma_m(\rho)) \le 2 \left\{ 2\pi M \left(\log \frac{1}{\delta} \right)^{-1} \right\}^{1/2}. \tag{2}$$

Furthermore, there is an $R > 0$ such that $\partial B_r(w_0)$ intersects C_j and C_ℓ for $0 < r < 2R$. Let δ be an arbitrary number with $0 < \sqrt{\delta} < R$. Since $C_j^{(m)} \to C_j$ and $C_\ell^{(m)} \to C_\ell$ as $m \to \infty$, there is a number $N(\delta, R) \in \mathbb{N}$ such that the following holds:

For $m > N(\delta, R)$ and $\delta < \rho < R$ the circle $\partial B_\rho(w_0)$ intersects $C_j^{(m)}$ and $C_\ell^{(m)}$.

Then there is an X_m-admissible subarc $\gamma_m(\rho)$ of $\partial B_\rho(w_0) \cap B_m$ satisfying $\delta < \rho < \sqrt{\delta}$ which has its endpoints on two circles $C_{j'}^{(m)}$ and $C_{\ell'}^{(m)}$ (which might be different from $C_j^{(m)}$ and $C_\ell^{(m)}$), and, moreover, which satisfies (2).

It follows that

$$\mu(\Gamma) \le \operatorname{dist}(\Gamma_{j'}, \Gamma_{\ell'}) \le 2\sqrt{\frac{2\pi M}{\log \frac{1}{\delta}}} \quad \text{for } 0 < \delta \ll 1$$

whence we obtain $\mu(\Gamma) = 0$ letting $\delta \to +0$, a contradiction to (1). □

Corollary 1. *Under the assumptions of Proposition 1, the limit $B \in \overline{\mathcal{N}}_1(k)$ of the domains $B_m \in \mathcal{N}_1(k)$ can only be degenerate of type 3 if $k = 2$. Moreover, if $k \ge 3$ then B can only be degenerate of type 2, or of type 3, or both.*

These two types of degeneration may indeed occur if we do not impose a further condition, namely a *condition of cohesion*.

If we operate with sequences in the class $\overline{\mathcal{C}}(\Gamma)$, defined by

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$$

one can conveniently use Courant's condition of cohesion. In this way one can solve the minimum problem

$$\text{“}D \to \min \quad \text{in } \overline{\mathcal{C}}(\Gamma)\text{”}.$$

In the same way one could also solve the problem

$$\text{“}A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma)\text{”}$$

where $A^\epsilon = (1-\epsilon)A + \epsilon D$, $0 < \epsilon \le 1$, but this would require a strong regularity theorem, which we here want to avoid in order to make the minimization procedure as transparent as possible. The prize for this is that we have to work with another condition of cohesion which is a bit more cumbersome to formulate than Courant's condition. This second condition is a simplified version of a stipulation introduced by M. Kurzke [1]; cf. also Kurzke and von der Mosel [1].

Definition 1. *A sequence $\{X_m\}$ of mappings $X_m \in \overline{H}_2^1(B_m, \mathbb{R}^3)$ with $B_m \in \mathcal{N}(k)$, $k \ge 2$, is said to be C-***cohesive** *if there is an $\epsilon > 0$ such that, for each $m \in \mathbb{N}$, any closed continuous curve $c : S^1 \to \mathbb{R}^2$ with $\gamma := c(S^1) \subset \overline{B}_m$ and $\operatorname{diam} X_m|_\gamma < \epsilon$ is homotopic to zero in $\overline{B}_m$.*

For $X \in H_2^1(B, \mathbb{R}^3)$, the composition $X \circ c$ of X with a closed curve $c \in C^0(S^1, \overline{B})$ is not defined in the usual sense. In order to give it a well-defined meaning we restrict ourselves to special curves c. Suppose that γ is a closed Jordan curve in $\overline{B}$, i.e. the image $\gamma = c(S^1)$ of S^1 under a homeomorphism $c : S^1 \to \gamma \subset \overline{B}$. If the inner domain G of γ is strong Lipschitz (i.e. $G \in C^{0,1}$) then X has a well-defined trace $Z =$ "$X|_\gamma$" on $\gamma = \partial G$, which is of class $L_2(\gamma, \mathbb{R}^3)$. If Z has a continuous representative $\gamma \to \mathbb{R}^3$ we denote it again by Z and call it the *continuous representative of X on γ*. Then $Z \circ c : S^1 \to \mathbb{R}^3$ is a well-defined, closed, continuous curve in $\mathbb{R}^3$. (Note that G need not be a subdomain of B.)

In applications G will be either (i) a disk, or (ii) a two-gon bounded by two circular arcs γ_1 and γ_2. In case (i), X is represented by a mapping $X^*(r, \theta)$ with respect to polar coordinates r, θ about the origin of the disk G of radius $R \in (0, 1)$ such that $X^*(r, \theta)$ is absolutely continuous with respect to $\theta \in \mathbb{R}$ for all $r \in (0, 1) \setminus N_1$ where N_1 is a 1-dimensional null set and $R \notin N_1$, and similarly $X^*(r, \theta)$ is absolutely continuous with respect to $r \in (\epsilon, 1 - \epsilon)$, $0 < \epsilon \ll 1$, for almost all $\theta \in \mathbb{R}$. Then the continuous representative $Z =$ "$X|_\gamma$" of X on the circle $\gamma = \partial G$ is given by $Z = X^*(R, \cdot)$. In case (ii), γ_1 is a subarc of ∂B, $B = \mathrm{dom}(X)$, and γ_2 is a circular subarc in B with the same endpoints as γ_1. Here the continuous representative $Z =$ "$X|_\gamma$" is the continuous trace of X on γ_1 (recall that for $X \in \mathcal{C}(\Gamma)$ we have "$X|_{\partial B}$" $\in C^0(\partial B, \mathbb{R}^3)$), while on γ_2 the trace $Z =$ "$X|_\gamma$" is given as in (i) by

$$Z(w_0 + Re^{i\theta}) = X^*(R, \theta) \quad \text{for } \theta_1 \le \theta \le \theta_2,$$

where $X^*(r, \theta)$ is a representation of X in polar coordinates around a point w_0 such that $X^*(R, \theta)$ is absolutely continuous in $\theta \in [\theta_1, \theta_2]$.

Definition 2. *A sequence $\{X_m\}$ of mappings $X_m \in H_2^1(B_m, \mathbb{R}^3)$ with $B_m = \mathrm{dom}(X_m) \in \mathcal{N}(k)$ is called* **separating** *if the following holds: For any $\epsilon > 0$ there is an $m_0(\epsilon) \in \mathbb{N}$ such that for any $m > m_0(\epsilon)$ there exists a closed Jordan curve γ_m in $\overline{B}_m$ bounding a strong Lipschitz interior B_m^* such that:*

(i) *X_m possesses a well-defined continuous trace $Z_m :=$ "$X_m|_{\gamma_m}$" on $\gamma_m = \partial B_m^*$;*
(ii) $\mathrm{diam}\, Z_m(\gamma_m) < \epsilon$;
(iii) *A homeomorphic representation $c_m : S^1 \to \gamma_m$ of γ_m is not homotopic to zero in $\overline{B}_m$.*

Definition 3. *A sequence $\{X_m\}$ of mappings $X_m \in H_2^1(B_m, \mathbb{R}^3)$ with $B_m = \mathrm{dom}(X_m) \in \mathcal{N}(k)$ is said to be* **cohesive** *if none of its subsequences is separating.*

An immediate consequence of these two definitions is

Proposition 2. *A sequence $\{X_m\}$ of mappings $X_m \in H_2^1(B_m, \mathbb{R}^3)$ with $B_m = \mathrm{dom}(X_m) \in \mathcal{N}(k)$ is cohesive if and only if the following holds: For* *every subsequence* $\{X_{m_j}\}$ *of* $\{X_m\}$ *there is an* $\epsilon > 0$ *and a further subsequence* $\{X_{m_{j_\ell}}\}$ *such that for each closed Jordan curve* γ *in* $\overline{B}_{m_{j_\ell}}$ *with a strong Lipschitz interior* G *the continuous trace* $Z_\ell :=$ "$X_{m_{j_\ell}}|_\gamma$" *satisfies* $\operatorname{diam} Z_\ell < \epsilon$, *but a homeomorphic representation* $c : S^1 \to \gamma$ *of* γ *is homotopic to zero in* $B_{m_{j_\ell}}$.

Comparing Proposition 2 with Definition 1 we obtain

Proposition 3. *Any* C*-cohesive sequence* $\{X_m\}$ *of mappings* X_m *of class* $\overline{H}{}^1_2(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$ *is also cohesive.*

Because of this, we in the sequel investigate only cohesive sequences.

Proposition 4. *Let* $\{X_m\}$ *be a sequence of mappings* $X_m \in H^1_2(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$, *and* $\{\sigma_m\}$ *be a sequence of Möbius transformations from* $\overline{B}{}^*_m$ *onto* $\overline{B}_m$, $B^*_m \in \mathcal{N}(k)$. *Then we have*:

(i) *If* $\{X_m\}$ *is separating, then also* $\{X_m \circ \sigma_m\}$.
(ii) *If* $\{X_m\}$ *is cohesive, then also* $\{X_m \circ \sigma_m\}$.

Proof. We only have to observe that every σ_m is a diffeomorphism from $\overline{B}{}^*_m$ onto $\overline{B}_m$; hence $X_m \circ \sigma_m \in H^1_2(\overline{B}{}^*_m, \mathbb{R}^3)$; furthermore, if γ is a Jordan curve in $\overline{B}_m$ bounding a strong Lipschitz domain, then $\sigma_m^{-1}(\gamma)$ is a Jordan curve in $\overline{B}{}^*_m$ bounding a strong Lipschitz domain. (We also note: If γ consists of circular arcs, then the same holds for $\sigma_m^{-1}(\gamma)$.) □

Theorem 1. *Let* $\{X_m\}$ *be a cohesive sequence of mappings* $X_m \in \mathcal{C}(\Gamma)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}_1(k)$, $k \geq 2$, *whose contour* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *consists of* k *rectifiable, mutually disjoint Jordan curves* $\Gamma_1, \dots, \Gamma_k$. *Suppose also that there is a constant* $M > 0$ *such that*

$$D(X_m) \leq M \quad \textit{for all } m \in \mathbb{N},$$

and that $B_m \to B$. *Then* B *is of class* $\mathcal{N}_1(k)$.

Proof. Clearly, $B \in \overline{\mathcal{N}}_1(k)$. If B were degenerate, it could not be of type 1 on account of Proposition 1; so we have to show that B can neither be of type 2 nor of type 3. □

Suppose first that B were of type 3, that is: *One or several circles shrink to a point* $p \in \overline{B}$ *which stays away from other limit points or limit circles.* Since $C_1^{(m)} \equiv C := \partial B_1(0)$ for all $m \in \mathbb{N}$, we have $C_1 = C$, and therefore $p \notin C$, i.e. $p \in \overline{B} \setminus C$. Thus the index set $I := \{\ell \in \mathbb{N} \colon 2 \leq \ell \leq k\}$ consists of two disjoint, nonempty sets I_1 and I_2 such that

$$C_j^{(m)} \to \{p\} \quad \text{as } m \to \infty \quad \text{for } j \in I_1,$$

$$C_\ell^{(m)} \to C_\ell \text{ (= point or circle)} \quad \text{as } m \to \infty \text{ with } p \notin C_\ell \quad \text{for } \ell \in I_2.$$

Then we can find a number $\rho_0 \in (0, 1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ the following holds true:

$$\begin{aligned} &C_j^{(m)} \subset B_{\rho_0}(p) \quad \text{for } j \in I_1, \\ &C_\ell^{(m)} \cap \overline{B}_{\rho_0}(p) = \emptyset \quad \text{for } \ell \in I_2. \end{aligned} \tag{3}$$

Secondly, for any $\rho_1 \in (0, \rho_0)$ there is an $m_1 = m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \geq m_0$ such that

$$C_j^{(m)} \subset B_{\rho_1}(p) \quad \text{for } j \in I_1 \text{ and } m > m_1(\rho_1).$$

We clearly have

$$\{w \in \mathbb{C} \colon \rho_1 \leq |w - p| \leq \rho_0\} \subset B_m \quad \text{for } m > m_1(\rho_1).$$

Furthermore, by virtue of a well-known extension theorem, there are Sobolev functions $Y_m \in H_2^1(B_1(0), \mathbb{R}^3)$ on the unit disk $B_1(0)$ satisfying

$$Y_m|_{B_m} = X_m \quad \text{for all } m \in \mathbb{N}.$$

We introduce polar coordinates r, θ about p, and choose representations $\tilde{Z}_m(r, \theta)$ of X_m, restricted to $B_{\rho_0}(p) \setminus B_{\rho_1}(p)$, for $m > m_1(\rho_1)$ which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. $\theta \in \mathbb{R}$. By the Courant–Lebesgue lemma we have:

(4) *For any $\epsilon > 0$ there is a number $\delta^*(\epsilon, M, \rho_0) \in (0, 1)$, depending only on ϵ, M, ρ_0, which has the following properties:*
(i) $\delta^* < \sqrt{\delta^*} \leq \rho_0$;
(ii) *for any $\rho_1 \in (0, \delta^*)$, any δ with $\rho_1 < \delta < \delta^*$, and all $m > m_1(\rho_1)$, there is a subset $J_m(\delta)$ of $(\delta, \sqrt{\delta})$ with* $\text{meas}\, J_m(\delta) > 0$ *and* $\text{osc}\, \tilde{Z}_m(r, \cdot) < \epsilon$ *for all $r \in J_m(\delta)$;*
(iii) *$\tilde{Z}_m(r, \cdot)$ is the trace of X_m on $\partial B_r(p)$ for any $r \in (\rho_1, \rho_0) \setminus \mathcal{S}_m$ where $\mathcal{S}_m$ is a one-dimensional null set, and so we can assume that $J_m(\delta) \subset (\rho_1, \rho_0) \setminus \mathcal{S}_m$.*

Let us now fix some $\epsilon > 0$ and then some $\rho_1 > 0$ with $\rho_1 < \delta^*(\epsilon, M, \rho_0)$. Furthermore we choose some $\delta > 0$ satisfying

$$\rho_1 < \delta < \delta^*(\epsilon, M, \rho_0).$$

Then

$$\{w \in \mathbb{C}\colon \delta < |w-p| < \sqrt{\delta}\} \subset B_{\rho_0}(p) \setminus B_{\rho_1}(p) \subset B_m \quad \text{for all } m > m_1(\rho_1).$$

For any $m > m_1(\rho_1)$ we choose some $r_m \in J_m(\delta)$ and set $\gamma_m := \partial B_{r_m}(p)$. Then γ_m is a Jordan curve in B_m which bounds the strong Lipschitz domain $B_m^* := B_{r_m}(p)$. By construction, Y_m is defined on B_m^*, and $X_m(w) = Y_m(w)$ for $w \in B_{\rho_0}(p) \setminus B_{\rho_1}(p)$. Thus X_m possesses an absolutely continuous representation $Z_m := \tilde{Z}_m(r_m, \cdot) =$ "$X_m|_{\gamma_m}$" with diam $Z_m(\gamma_m) < \epsilon$. Furthermore we have $C_j^{(m)} \subset B_m^*$ for $j \in I_1$. Therefore no homeomorphic representation $c_m : S^1 \to \gamma_m$ of γ_m is homotopic to zero in $\overline{B}_m$.

Since $\epsilon > 0$ can be chosen arbitrarily, we see that $\{X_m\}$ contains a separating subsequence, a contradiction, since $\{X_m\}$ was assumed to be cohesive.

Now we turn to the last possibility: Suppose that $B := \lim_{m\to\infty} B_m$ is of type 2. Then we have $k \geq 3$, see Corollary 1. Here we again have $C_1^{(m)} \equiv C = \partial B_1(0)$ for all $m \in \mathbb{N}$, whence $C_1 = C$, and either $C_2 = \{0\}$ or $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$. Furthermore, type 2 means that *one sequence of circles, say* $\{C_j^{(m)}\}$*, converges to a true circle* C_j*,* $1 \leq j \leq k$*, while one or several other sequences* $\{C_\ell^{(m)}\}$ *shrink to a point* p *of* C_j. Here we can decompose $I' := \{\ell \in \mathbb{N}\colon 1 \leq \ell \leq k,\ \ell \neq j\}$ into $I_1' := \{\ell \in I'\colon C_\ell^{(m)} \to \{p\} \text{ as } m \to \infty\}$ and $I_2' := I' \setminus I_1'$; then the limits C_ℓ of $C_\ell^{(m)}$ for $m \to \infty$ and $\ell \in I_2'$ are either points or circles which stay away from p.

We can find a number $\rho_0 \in (0,1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ the following holds true:

$$(*) \qquad \begin{cases} \partial B_{\rho_0}(p) \text{ intersects } C_j^{(m)} \text{ in exactly two points;} \\ C_\ell^{(m)} \subset B_{\rho_0}(p) \cap B_m := S_{\rho_0}^m(p) \quad \text{for } \ell \in I_1'; \\ C_\ell^{(m)} \cap \overline{B}_{\rho_0}(p) = \emptyset \quad \text{for } \ell \in I_2'. \end{cases}$$

Checking the three cases $j = 1$, $j = 2$, and $3 \leq j \leq k$, one realizes that both I_1' and I_2' are nonempty.

For any $\rho_1 \in (0, \rho_0)$ there is an $m_1 = m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \geq m_0$ such that

$$C_\ell^{(m)} \subset B_{\rho_1}(p) \cap B_m =: S_{\rho_1}^m(p) \quad \text{for } \ell \in I_1' \text{ and } m > m_1(\rho_1).$$

As in the preceding discussion we choose extensions $Y_m \in H_2^1(B_1(0), \mathbb{R}^3)$ of X_m from B_m to $B_1(0)$. Then we introduce polar coordinates r, θ about p, and choose representations $\tilde{Z}_m(r, \theta)$ of X_m, restricted to $S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$ for $m > m_1(\rho_1)$ which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. θ such that $w = p + re^{i\theta} \in S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$.

Applying the Courant–Lebesgue lemma, we obtain analogously to (4):

$$
(4') \quad \begin{cases} \textit{For any } \epsilon > 0 \textit{ there is a number } \delta^*(\epsilon, M, \rho_0) \in (0,1)\textit{, depending only} \\ \textit{on } \epsilon, M, \rho_0\textit{, which has the following properties:} \\ \text{(i) } \delta^* < \sqrt{\delta^*} \le \rho_0; \\ \text{(ii) } \textit{for any } \rho_1 \in (0, \delta^*)\textit{, any } \delta \textit{ with } \rho_1 < \delta < \delta^*\textit{, and all } m > \\ m_1(\rho_1)\textit{, there is a subset } J_m(\delta) \textit{ of } (\delta, \sqrt{\delta}) \textit{ with } \operatorname{meas} J_m(\delta) > 0\textit{, and} \\ \operatorname{osc} \tilde{Z}_m(r, \cdot) < \epsilon/2 \textit{ for all } r \in J_m(\delta); \\ \text{(iii) } \tilde{Z}_m(r, \cdot) \textit{ is the trace of } X_m \textit{ on } \partial B_r(p) \cap \overline{B}_m \textit{ for any } r \in \\ (\rho_1, \rho_0) \setminus \mathcal{S}_m \textit{ where } 1\text{-meas}\, \mathcal{S}_m = 0\textit{, and so we can assume that} \\ J_m(\delta) \subset (\rho_1, \rho_0) \setminus \mathcal{S}_m. \end{cases}
$$

Let us now fix some $\epsilon > 0$ and then $\rho_1 > 0$ with $\rho_1 < \delta^*(\epsilon, M, \rho_0)$. Furthermore, choose some $\delta > 0$ satisfying

$$
\rho_1 < \delta < \delta^*(\epsilon, M, \rho_0).
$$

Then it follows that, for $\rho \in (\delta, \sqrt{\delta})$ and $m > m_1(\rho_1)$, the circle $\partial B_\rho(p)$ meets $C_j^{(m)}$ in exactly two points $w'_m(\rho)$ and $w''_m(\rho)$, and that $\gamma'_m(\rho) := \partial B_\rho(p) \cap \overline{B}_m$ is a connected circular arc in $\overline{B}_m$ with the endpoints $w'_m(\rho)$ and $w''_m(\rho)$. Their image points $Q'_m(\rho)$ and $Q''_m(\rho)$ under $\tilde{Z}_m(\rho, \cdot)$ lie on Γ_j and decompose this curve into two arcs; denote the "smaller one" by $\Gamma^*(m, \rho)$. Then there is a function $\eta : (0, \infty) \to (0, \infty)$ with $\eta(t) \to +0$ as $t \to +0$ such that

$$
\operatorname{diam} \Gamma^*(m, \rho) < \eta(\rho) \quad \text{for } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta).
$$

We can arrange for

$$
\operatorname{diam} \Gamma^*(m, \rho) < \epsilon/2 \quad \text{for } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta)
$$

by choosing the number $\delta^*(\epsilon, M, \rho_0) > 0$ even smaller if necessary (see the application of the Courant–Lebesgue lemma in Section 4.3).

Instead of $(*)$, we even have

$$
(**) \quad \begin{cases} C_\ell^{(m)} \subset B_\rho(p) \cap B_m =: S_\rho^m(p) \quad \text{for } \ell \in I'_1, \\ C_\ell^{(m)} \cap \overline{B}_\rho(p) = \emptyset \quad \text{for } \ell \in I'_2, \\ \text{provided that } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta). \end{cases}
$$

Choose some $r_m \in J_m(\delta) \subset (\delta, \sqrt{\delta})$ and set

$$
\begin{aligned} \Gamma'_m &:= \text{image of } \gamma'_m(r_m) \text{ under the mapping } \tilde{Z}_m; \\ \Gamma''_m &:= \Gamma^*(m, r_m) = \text{image of } \gamma''_m(r_m) \text{ under } X_m; \end{aligned}
$$

here $\gamma''_m(r_m)$ is the connected arc on $C_j^{(m)}$, bounded by $w'_m(r_m), w''_m(r_m)$, which is mapped by the Sobolev trace $X_m|_{C_j^{(m)}}$ in a continuous way onto Γ''_m. Then we have

$$
\operatorname{diam} \Gamma'_m + \operatorname{diam} \Gamma''_m < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } m > m_1(\rho_1).
$$

Consider the closed Jordan curve $\gamma_m := \gamma'_m(r_m) \cup \gamma''_m(r_m)$ in $\overline{B}_m$, which bounds a two-gon B^*_m in $B_1(0)$; B^*_m is a strong Lipschitz domain. Because of $(**)$, one realizes that no homeomorphic representation $c_m : S^1 \to \gamma_m$ of γ_m is homotopic to zero in $\overline{B}_m$. There is a continuous representation Z_m of X_m on γ_m given by

$$Z_m := X_m(r_m, \cdot) \quad \text{on } \gamma'_m$$

and

$$Z_m := \text{trace of } X_m \quad \text{on } \gamma''_m.$$

Then it follows

$$\operatorname{diam} Z_m(\gamma_m) \le \operatorname{diam} \Gamma'_m + \operatorname{diam} \Gamma''_m < \epsilon \quad \text{for } m > m_1(\rho_1).$$

Since $\epsilon > 0$ is arbitrary, we obtain that $\{X_m\}$ contains a separating subsequence, and so it cannot be cohesive, a contradiction to the assumption.

Thus we have shown that B cannot be degenerate, i.e. $B \in \mathcal{N}_1(k)$. □

Proposition 5. *Let $\{X_m\}$ be a sequence of mappings $X_m \in H^1_2(B_m, \mathbb{R}^3)$ with $B_m = \text{dom}\ (X_m) \in \mathcal{N}_1(k)$, and suppose that $B_m \to B \in \mathcal{N}_1(k)$ and $D(X_m) \to L$ as $m \to \infty$. Then there is a sequence of diffeomorphisms σ_m from $\overline{B}$ onto $\overline{B}_m$ such that the following holds true*:

(i) *$X^*_m := X_m \circ \sigma_m \in H^1_2(B, \mathbb{R}^3)$ for all $m \in \mathbb{N}$, and if $X_m \in \mathcal{C}(\Gamma)$ then $X^*_m \in \mathcal{C}(\Gamma)$;*
(ii) *$D(X^*_m) \to L$ as $m \to \infty$;*
(iii) *$\{X^*_m\}$ is cohesive if and only if $\{X_m\}$ is cohesive;*
(iv) *If $X_m \in \overline{\mathcal{C}}(\Gamma)$ then $X^*_m \in \overline{\mathcal{C}}(\Gamma)$, and $\{X^*_m\}$ is C-cohesive if and only if $\{X_m\}$ is C-cohesive.*

Proof. Since the limit domain is nondegenerate, it is not difficult to prove that there is a sequence $\{\sigma_m\}$ of diffeomorphisms from $\overline{B}$ onto $\overline{B}_m$ which converges to the identity $id_{\overline{B}}$ on $\overline{B}$ with respect to the $C^1(\overline{B}, \mathbb{R}^2)$-norm. (This would not be true if $B \in \overline{\mathcal{N}}_1(k) \setminus \mathcal{N}_1(k)$). Setting $X^*_m := X_m \circ \sigma_m$, the assertions follow at once. □

Theorem 2. *Let $\{X_m\}$ be a cohesive sequence of mappings $X_m \in \mathcal{C}(\Gamma)$ with* $\text{dom}(X_m) \equiv B \in \mathcal{N}_1(k)$ *for all $m \in \mathbb{N}$, $k \ge 2$, whose boundary contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consists of k rectifiable, closed, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$. Suppose also that there is a constant $M > 0$ such that*

$$D(X_m) \le M \quad \textit{for all } m \in \mathbb{N}.$$

Then the boundary traces $X_m|_{\partial B}$ are equicontinuous on ∂B, and there is a subsequence $\{X_{m_\ell}\}$ of $\{X_m\}$ such that the traces $X_{m_\ell}|_{\partial B}$ converge uniformly on ∂B.

Proof. We can essentially proceed as in the proof of Theorem 1 of Section 4.3 noting that $X_m|_{C_j}$ maps C_j continuously and in a weakly monotonic way onto Γ_j. One only has to ensure that small arcs on C_j are mapped onto small subarcs of Γ_j. In the case $k = 1$ this was achieved by imposing a three-point condition upon $\{X_m\}$; for $k \geq 2$ the same will be attained by the cohesivity condition. In fact, mapping small arcs on C_j onto large arcs on Γ_j corresponds to mapping large arcs on C_j onto small arcs of Γ_j, and by the Courant-Lebesgue Lemma one would obtain Jordan curves γ_m in B bounding strong Lipschitz domains B_m^* such that the continuous trace $Z_m :=$ "$X_m|_{\gamma_m}$" of X_m on γ_m satisfies "$\operatorname{diam} Z_m(\gamma_m) = \text{small}$", but γ_m cannot be contracted continuously in $\overline{B}$ to some point of $\overline{B}$ since $\overline{B} \cap \overline{B}_m^*$ possesses at least one hole. □

Corresponding to Theorem 3 of Section 4.3 we obtain the following generalizations of Theorems 1 and 2 above:

Theorem 3. *The assertions of Theorems 1 and 2 remain valid if we replace the assumption "$X_m \in \mathcal{C}(\Gamma)$" by "$X_m \in \mathcal{C}(\Gamma^m)$" where $\Gamma^m = \langle \Gamma_1^m, \dots, \Gamma_k^m \rangle$ are boundary contours converging in the sense of Fréchet ("$\Gamma^m \to \Gamma$ as $m \to \infty$") to some contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consisting of k rectifiable, closed, mutually disjoint Jordan curves.*

8.4 Solution of the Douglas Problem

Using the results obtained in Sections 8.2 and 8.3 we can now solve the Douglas problem under the assumption that $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \geq 2$, bounds a cohesive minimizing sequence in $\mathcal{C}(\Gamma)$ for the Dirichlet integral.

Theorem 1. *Let $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \geq 2$, be a boundary configuration consisting of rectifiable, closed, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$ in $\mathbb{R}^3$, and suppose that Γ fulfills the following* **condition of cohesion**: *There is a cohesive sequence $\{X_m\}$ of surfaces $X_m \in \mathcal{C}(\Gamma)$ with*

$$D(X_m) \to d(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then there exists a minimizer X of the energy D in $\mathcal{C}(\Gamma)$ which is of class $C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and satisfies

$$(1) \qquad \Delta X = 0 \quad \textit{in } B$$

as well as

$$(2) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \textit{in } B.$$

Proof. Consider a cohesive sequence of $X_m \in \mathcal{C}(\Gamma)$ with $D(X_m) \to d(\Gamma)$ and $B_m = \mathrm{dom}(X_m) \in \mathcal{N}(k)$. By Lemma 4 of Section 8.2 there are Möbius transformations f_m mapping $\overline{B}^*_m \in \mathcal{N}_1(k)$ onto $\overline{B}_m$. Set $X^*_m := X_m \circ f_m \in \mathcal{C}(\Gamma)$; then $B^*_m = \mathrm{dom}(X^*_m) \in \mathcal{N}_1(k), D(X^*_m) \to d(\Gamma)$, and $\{X^*_m\}$ is cohesive too on account of Proposition 4 in Section 8.3. Furthermore, there is a constant $M_0 > 0$ such that $D(X^*_m) \le M_0$ for all $m \in \mathbb{N}$. By Lemma 1 of Section 8.3 we can extract a subsequence $\{B^*_{m_j}\}$ of $\{B^*_m\}$ such that $B^*_{m_j} \to B \in \overline{\mathcal{N}}_1(k)$. Applying Theorem 1 of Section 8.3 we infer that B is nondegenerate, i.e. $B \in \mathcal{N}_1(k)$, and by Proposition 5 of the same section we find diffeomorphisms σ_{m_j} from $\overline{B}$ onto $\overline{B}^*_{m_j}$ such that $X^{**}_{m_j} := X^*_{m_j} \circ \sigma_{m_j}$, $j \in \mathbb{N}$, defines a cohesive sequence of mappings $X^{**}_{m_j} \in \mathcal{C}(\Gamma)$ with $D(X^{**}_{m_j}) \to d(\Gamma)$. In virtue of Theorem 2 of Section 8.3, the boundary traces $X^{**}_{m_j}|_{\partial B}$ are compact in $C^0(\partial B, \mathbb{R}^3)$, and so we can assume without loss of generality that the cohesive minimizing sequence we have started with, satisfies also

(i) $\mathrm{dom}(X_m) \equiv B$ for all $m \in \mathbb{N}$;
(ii) $X_m|_{\partial B} \to \phi$ in $C^0(\partial B, \mathbb{R}^3)$.

If we replace X_m by the solution $H_m \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^2)$ of the Dirichlet problem

$$\Delta H_m = 0 \quad \text{in } B, \quad H_m|_{\partial B} = X_m|_{\partial B}$$

the sequence $\{H_m\}$ possesses all properties of $\{X_m\}$. Renaming H_m as X_m, we therefore obtain also

(iii) $X_m \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and $\Delta X_m = 0$ in B.

Because of (ii) and (iii) there is a mapping $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which is harmonic in B and satisfies

$$X_m \to X \quad \text{in } C^0(\overline{B}, \mathbb{R}^3). \tag{3}$$

Because of $D(X_m) \le M_0$ and $X_m \in \mathcal{C}(\Gamma)$ as well as (i) we can also assume that X_m converges weakly in $H^1_2(B, \mathbb{R}^3)$ to X, whence $X \in \mathcal{C}(\Gamma)$, and therefore

$$d(\Gamma) \le D(X).$$

Furthermore, the weak lower semicontinuity of D in $H^1_2(B, \mathbb{R}^3)$ yields

$$D(X) \le \lim_{n \to \infty} D(X_n) = d(\Gamma),$$

and so we obtain

$$D(X) = d(\Gamma).$$

That is, X minimizes D in $\mathcal{C}(\Gamma)$ and satisfies (1). Finally, Theorem 1 of Section 8.2 leads to the conformality relations (2), and so the proof is complete. □

Remark 1. If we assume the existence of a C-cohesive sequence of surfaces $X_m \in \overline{\mathcal{C}}(\Gamma)$ with

$$D(X_m) \to \overline{d}(\Gamma) := \inf_{\overline{\mathcal{C}}(\Gamma)} D,$$

a similar reasoning as above leads to a minimal surface $X \in \overline{\mathcal{C}}(\Gamma)$ minimizing D in $\overline{\mathcal{C}}(\Gamma)$. This is the original approach of Courant [15].

Remark 2. As we have noted earlier, C-cohesiveness implies cohesiveness. Using Theorem 1 one can also show the converse. Thus the two conditions actually are equivalent, and so they lead to the same result. Hence it seems superfluous to work with cohesiveness instead of C-cohesiveness, as it is more troublesome to work with. Its usefulness will become apparent when we will minimize

$$A^\epsilon = (1-\epsilon)A + \epsilon D$$

for some $\epsilon \in [0,1]$, in order to prove

(4) $$\inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} D.$$

Then it seems impossible, or at least much more cumbersome, to operate in $\overline{\mathcal{C}}(\Gamma)$, and it appears to be more natural to work in $\mathcal{C}(\Gamma)$.

The same holds true if one wants to minimize a Cartan functional under Plateau boundary conditions.

In the sequel we want to solve the Douglas problem assuming the ("sufficient") condition of Douglas, thereby verifying also (4). For this purpose we need two technical results that will be provided in the next section.

8.5 Useful Modifications of Surfaces

First we will show that we can replace small parts of a surface by the constant surface $X_0(w) \equiv 0$ without gaining much energy. This argument works for general functionals

$$\mathcal{F}(X) := \int_B F(X, \nabla X)\, du\, dv$$

and surfaces $X \in H_2^1(B, \mathbb{R}^3)$, $B = \operatorname{dom}(X) \in \mathcal{N}(k)$, with a Lagrangian $F(x,p) \in C^0(\mathbb{R}^3 \times \mathbb{R}^6)$ satisfying

$$0 \le F(x,p) \le \tfrac{1}{2}\mu|p|^2$$

for some constant $\mu > 0$. For $\Omega \subset B$ we set

$$\mathcal{F}_\Omega(X) := \int_\Omega F(X, \nabla X)\, du\, dv.$$

Proposition 1. *Suppose that $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}(k)$. Then, for any $\delta > 0$ and any point $p \in B$, there exists a number $r_0 \in (0, \mathrm{dist}(p, \partial B))$, depending on X, δ, p, and μ, such that for any $r \in (0, r_0)$ there is a surface $Z^r \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(Z^r) = B$ and*

$$\mathcal{F}(Z^r) < \mathcal{F}(X) + \delta \quad \textit{as well as} \quad Z^r(w) \equiv 0 \quad \textit{on } B_r(p).$$

Proof. Choose any $\delta > 0$ and $p \in B$; then there is some $R \in (0,1)$ with $R < \mathrm{dist}(p, \partial B)$ such that

$$(1) \qquad \int_{B_\rho(p)} |\nabla X|^2 \, du \, dv < \delta_0 := \frac{\delta}{2\mu} \quad \text{for all } \rho \in (0, R).$$

Then we take some $\rho \in (0, R)$ such that the trace $X|_{\partial B_\rho(p)}$ is absolutely continuous on $\partial B_\rho(p)$. Set

$$M := \sup_{\partial B_\rho(p)} |X|,$$

and choose some $H \in H_2^1(B_\rho(p), \mathbb{R}^3)$ with

$$\Delta H = 0 \quad \text{in } B_\rho(p), \quad H = X \quad \text{on } \partial B_\rho(p).$$

Then $H - X \in \mathring{H}_2^1(B_\rho(p), \mathbb{R}^3)$, and the maximum principle implies

$$(2) \qquad \sup_{B_\rho(p)} |H| = M.$$

Furthermore, Dirichlet's principle yields

$$(3) \qquad \int_{B_\rho(p)} |\nabla H|^2 \, du \, dv \le \int_{B_\rho(p)} |\nabla X|^2 \, du \, dv < \delta_0.$$

For some constant $\epsilon \in (0, \rho)$ to be fixed later we set

$$(4) \qquad \varphi(s, \epsilon^2) := \begin{cases} 1 & \text{for } \epsilon < s, \\ 0 & \text{for } 0 \le s \le \epsilon^2 \end{cases}$$

and

$$(5) \qquad \varphi(s, \epsilon^2) := 1 + \frac{\log \epsilon - \log s}{\log \epsilon} \quad \text{for } \epsilon^2 \le s \le \epsilon.$$

By means of $\varphi(\cdot, \epsilon^2) \in \mathrm{Lip}([0, \infty))$ we define $Y(\cdot, \epsilon^2)$ as

$$Y(w, \epsilon^2) := \begin{cases} X(w) & \text{for } |w - p| \ge \rho, \\ \varphi(|w - p|, \epsilon^2) H(w) & \text{for } |w - p| < \rho. \end{cases}$$

Writing

$$\phi(w) := \varphi(|w-p|, \epsilon^2),$$

we obtain

$$\begin{aligned}\int_{B_\rho(p)} |\nabla\phi|^2\, du\, dv &= |\log\epsilon|^{-2} \int_0^{2\pi}\int_{\epsilon^2}^{\epsilon} r^{-2} r\, dr\, d\theta \\ &= -\frac{2\pi}{\log\epsilon} =: \delta_1(\epsilon) > 0\end{aligned}$$

and then

$$\begin{aligned}\int_{B_\rho(p)} |\nabla Y(\cdot,\epsilon^2)|^2\, du\, dv &= \int_{B_\rho(p)} \{|\phi_u H + \phi H_u|^2 + |\phi_v H + \phi H_v|^2\}\, du\, dv \\ &\le 2M^2 \int_{B_\rho(p)} |\nabla\phi|^2\, du\, dv + 2\int_{B_\rho(p)} |\nabla H|^2\, du\, dv \\ &\le 2M^2\delta_1(\epsilon) + 2\delta_0 < 4\delta_0 \quad \text{for } 0<\epsilon<\epsilon_0\end{aligned}$$

if we choose $\epsilon_0 \in (0,\rho)$ so small that $M^2\delta_1(\epsilon) < \delta_0$ for $0<\epsilon<\epsilon_0$. Set $r := \epsilon^2$ with $0<\epsilon<\epsilon_0$ and $Z^r := Y(\cdot,\epsilon^2)$; then

$$\begin{aligned}\mathcal{F}(Z^r) &= \mathcal{F}_{B\setminus B_\rho(p)}(X) + \mathcal{F}_{B_\rho(p)}(Z^r) \\ &\le \mathcal{F}(X) + \frac{\mu}{2}\int_{B_\rho(p)} |\nabla Z^r|^2\, du\, dv \\ &< \mathcal{F}(X) + 2\delta_0\mu = \mathcal{F}(X) + \delta \quad \text{for } r\in(0,\epsilon_0^2),\end{aligned}$$

and similarly

$$\int_B |\nabla Z^r|^2\, du\, dv \le \int_B |\nabla X|^2\, du\, dv + 4\delta_0.$$

Since $|Z^r| \le |X|$, it follows $Z^r \in H_2^1(B,\mathbb{R}^3)$. Furthermore, $B_{\sqrt{r}}(p) \subset\subset B$, and

$$Z^r(w) \equiv 0 \quad \text{on } B_r(p), \qquad Z^r(w) \equiv X(w) \quad \text{on } B\setminus B_{\sqrt{r}}(p).$$

This implies $Z^r \in \mathcal{C}(\Gamma)$ since $X \in \mathcal{C}(\Gamma)$. Setting $r_0 := \epsilon_0^2$, the assertion is proved. □

Proposition 2 (Pinching method). *Let $\tilde\Gamma$ be a boundary configuration consisting of k rectifiable, closed, mutually disjoint Jordan curves in $\mathbb{R}^3$. Then, for given $K>0$ and $\delta>0$, there is a constant $\eta_0 \in (0,1)$ depending only on $\tilde\Gamma, K, \delta$, such that for every point $Q\in\mathbb{R}^3$ and any $\eta\in(0,\eta_0)$ there is a Lipschitz mapping $\Phi = \Phi_{\eta,Q}$ from $\mathbb{R}^3$ onto $\mathbb{R}^3$ with the following properties: If X is an arbitrary mapping of class $\mathcal{C}(\tilde\Gamma)$ with* $\mathrm{dom}(X) = B$ *and $D(X)\le K$, then we have*

(i) *$\Gamma^* := \Phi(\tilde\Gamma)$ consists of k rectifiable, closed, mutually disjoint Jordan curves such that the Fréchet distance $\Delta(\tilde\Gamma,\Gamma^*)$ of $\tilde\Gamma$ and Γ^* satisfies $\Delta(\tilde\Gamma,\Gamma^*)<\delta$;*

(ii) $\Phi \circ X \in \mathcal{C}(\Gamma^*)$, *and* $\mathrm{dom}(\Phi \circ X) = B$;
(iii) $\Phi(x) \equiv x$ *for* $x \in \mathbb{R}^3$ *with* $|x - Q| \geq \eta$;
(iv) $\Phi(x) \equiv Q$ *for* $x \in \mathbb{R}^3$ *with* $|x - Q| \leq \eta^2$;
(v) *For* $A^\epsilon := (1-\epsilon)A + \epsilon D$, $0 \leq \epsilon \leq 1$, *we have*

$$A^\epsilon(\Phi \circ X) \leq A^\epsilon(X) + \delta.$$

Proof. Choose $\eta_0 \in (0, 1/3)$ so small that

$$3|\log \eta_0|^{-1} < \delta/K \tag{6}$$

and

$$\eta_0 < \tfrac{1}{2} \min\{\mathrm{dist}(\tilde{\Gamma}_j, \tilde{\Gamma}_\ell) \colon j \neq \ell, j, \ell = 1, \dots, k\}$$

where $\tilde{\Gamma} = \langle \tilde{\Gamma}_1, \dots, \tilde{\Gamma}_k \rangle$. For $\eta \in (0, \eta_0)$ we define the Lipschitz function $\varphi_\eta : [0, \infty) \to \mathbb{R}$ by

$$\varphi_\eta(s) := \begin{cases} 1 & \text{for } \eta < s, \\ 2 - \frac{\log s}{\log \eta} & \text{for } \eta^2 \leq s \leq \eta, \\ 0 & \text{for } 0 \leq s < \eta^2. \end{cases}$$

Then, fixing an arbitrary point $Q \in \mathbb{R}^3$, we define the mapping $\Phi_{\eta,Q} \equiv \Phi_\eta : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Phi_\eta(x) := Q + \varphi_\eta(|x - Q|)\{x - Q\} \quad \text{for } x \in \mathbb{R}^3.$$

Clearly, Φ_η is a Lipschitz map from $\mathbb{R}^3$ onto itself which "pinches" the ball $K_{\eta^2}(Q) := \{x \in \mathbb{R}^3 \colon |x - Q| \leq \eta^2\}$ to the point Q and maps $\mathbb{R}^3 \setminus K_{\eta^2}(Q)$ in a $1-1$ way onto $\mathbb{R}^3 \setminus \{Q\}$. This immediately implies the properties (i)–(iv) of $X^* := \Phi_\eta(X)$. It remains to show (v). We first note that

$$\begin{aligned} &X^*(w) = Q \quad \text{and} \quad \nabla X^*(w) = 0 \quad \text{a.e. on } B' := \{w \in B \colon |X(w) - Q| \leq \eta^2\}, \\ &X^*(w) = X(w) \quad \text{and} \quad \nabla X^*(w) = \nabla X(w) \\ &\quad \text{a.e. on } B'' := \{w \in B \colon |X(w) - Q| \geq \eta\}. \end{aligned}$$

Thus we have to compute ∇X^* on $R := \{w \in B \colon \eta^2 < |X(w) - Q| < \eta\}$. Set

$$e(w) := |X(w) - Q|^{-1}\{X(w) - Q\} \quad \text{for } w \in R;$$

then $|e(w)| = 1$ on R. Furthermore, we have on R:

$$\begin{aligned} &X^* = Q + \varphi_\eta(|X - Q|)\{X - Q\}, \quad \varphi_\eta(|X - Q|) = 2 - \frac{\log|X - Q|}{\log \eta}, \\ &\frac{\partial}{\partial u}\varphi_\eta(|X - Q|) = \frac{-e \cdot X_u}{(\log \eta)|X - Q|}, \quad \frac{\partial}{\partial v}\varphi_\eta(|X - Q|) = \frac{-e \cdot X_v}{(\log \eta)|X - Q|}. \end{aligned}$$

Then,

(7)
$$X_u^* = \varphi_\eta(|X-Q|)X_u - \frac{1}{\log\eta}(e\cdot X_u)e,$$
$$X_v^* = \varphi_\eta(|X-Q|)X_v - \frac{1}{\log\eta}(e\cdot X_v)e \quad \text{on } R,$$

whence by $0 \le \varphi_\eta(|X-Q|) \le 1$, $|e| = 1, -\log\eta = |\log\eta| > 1$ we obtain on R:

$$\begin{aligned} |X_u^*|^2 &\le |X_u|^2 - 2(\log\eta)^{-1}|X_u|^2 + |\log\eta|^2|X_u|^2 \\ &\le (1+3|\log\eta|^{-1})|X_u|^2, \\ |X_v^*|^2 &\le (1+3|\log\eta|^{-1})|X_v|^2. \end{aligned}$$

On account of (6), this leads to

(8)
$$D_R(X^*) \le D_R(X) + (\delta/K)D(X) \le D_R(X) + \delta.$$

Now we are going to estimate $A_R(X^*)$. From (7) we infer by setting $\psi := \varphi_\eta(|X-Q|)$ that

$$X_u^* \wedge X_v^* = \psi^2 X_u \wedge X_v + \psi|\log\eta|^{-1}\{(e\cdot X_v)(e\wedge X_u) + (e\cdot X_u)(e\wedge X_v)\}$$

whence

$$\begin{aligned} |X_u^* \wedge X_v^*| &\le |X_u \wedge X_v| + |\log\eta|^{-1}2|X_u||X_v| \\ &\le |X_u \wedge X_v| + |\log\eta|^{-1}|\nabla X|^2 \quad \text{on } R. \end{aligned}$$

This implies

(9)
$$\begin{aligned} A_R(X^*) &\le A_R(X) + |\log\eta|^{-1}2D_R(X) \\ &\le A_R(X) + (\delta/K)D(X) \le A_R(X) + \delta. \end{aligned}$$

From (8), (9), and $A_R^\epsilon = (1-\epsilon)A_R + \epsilon D_R$ we infer

$$A_R^\epsilon(X^*) \le A_R^\epsilon(X) + \delta \quad \text{for any } \epsilon \in [0,1].$$

Furthermore,
$$A_{B'}^\epsilon(X^*) = 0, \quad A_{B''}^\epsilon(X^*) = A_{B''}^\epsilon(X).$$

Since $B = B' \dot\cup R \dot\cup B''$, we arrive at

$$\begin{aligned} A^\epsilon(X^*) &= A_{B'}^\epsilon(X^*) + A_R^\epsilon(X^*) + A_{B''}^\epsilon(X^*) \\ &\le 0 + A_R^\epsilon(X) + \delta + A_{B''}^\epsilon(X) \le A^\epsilon(X) \end{aligned}$$

for any $\epsilon \in [0,1]$. This completes the proof. □

8.6 Douglas Condition and Douglas Problem

For $\epsilon \in [0,1]$ we consider the conformally invariant functionals

$$A^\epsilon(X) := (1-\epsilon)A(X) + \epsilon D(X)$$

which satisfy

$$A^0(X) = A(X), \quad A^1(X) = D(X)$$

and

(1) $$A(X) \le A^\epsilon(X) \le D(X) \quad \text{for any } \epsilon \in [0,1].$$

Furthermore, for $0 < \epsilon \le 1$ we have

(2) $$A(X) = A^\epsilon(X) = D(X) \quad \text{if and only if} \quad \langle X_w, X_w \rangle = 0,$$

and

$$\langle X_w, X_w \rangle = 0 \quad \Leftrightarrow \quad |X_u|^2 = |X_v|^2 \quad \text{and} \quad \langle X_u, X_v \rangle = 0.$$

As a first result we shall prove that the problem

$$A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma)$$

has a solution $X^\epsilon \in \mathcal{C}(\Gamma)$ for any $\epsilon \in (0, \epsilon_0]$ with $0 < \epsilon_0 \ll 1$ provided that $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \ge 2$, satisfies the Douglas condition. The proof follows essentially the same lines as in Section 8.4, but it is somewhat more involved.

In order to define the Douglas condition for $k > 1$ we have to consider the class of mappings $X : B \to \mathbb{R}^3$ whose domains B are disconnected. Precisely speaking we assume that B is a set $\{B^1, \dots, B^s\}$, $s > 1$, of k_ν-circle domains $B^\nu \in \mathcal{N}(k_\nu)$ with

$$k = k_1 + k_2 + \cdots + k_s,$$

and X is a collection $\{X^{(1)}, \dots, X^{(s)}\}$ of mappings

$$X^{(\nu)} \in H^{1,2}(B^\nu, \mathbb{R}^3) \cap C^0(\partial B^\nu, \mathbb{R}^3)$$

such that $X^{(\nu)}|_{\partial B^\nu}$ is a weakly monotonic mapping of ∂B^ν onto a configuration of k_ν disjoint closed, rectifiable Jordan curves $\Gamma_1, \dots, \Gamma_{k_\nu}$. The set $\mathcal{C}^+(\Gamma)$ of such maps X is called the **class of splitting mappings bounded by** Γ.

Now we define $A^\epsilon(X)$ for $X = \{X^{(1)}, \dots, X^{(s)}\} \in \mathcal{C}^+(\Gamma)$ by

$$A^\epsilon(X) := A^\epsilon(X^{(1)}) + \cdots + A^\epsilon(X^{(s)}),$$

and then

$$d(\Gamma, \epsilon) := \inf_{\mathcal{C}(\Gamma)} A^\epsilon, \quad d^+(\Gamma, \epsilon) := \inf_{\mathcal{C}^+(\Gamma)} A^\epsilon,$$

in particular

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad a^+(\Gamma) := \inf_{\mathcal{C}^+(\Gamma)} A,$$

that is, $a(\Gamma) = d(\Gamma, 0)$ and $a^+(\Gamma) = d^+(\Gamma, 0)$.

Definition 1. *The* **Douglas condition** *is the hypothesis*

$$a(\Gamma) < a^+(\Gamma).$$

In the following discussion we need a third function of ϵ *besides* $d(\Gamma, \epsilon)$ *and* $d^+(\Gamma, \epsilon)$*, namely*

$$d^*(\Gamma, \epsilon) := \inf\Big\{ \liminf_{m\to\infty} A^\epsilon(X_m) \colon \{X_m\} = \textit{separating sequence} \\ \textit{of } X_m \in \mathcal{C}(\Gamma)\Big\}.$$

Lemma 1. *The infima* $d(\Gamma, \epsilon), d^+(\Gamma, \epsilon), d^*(\Gamma, \epsilon)$ *are nondecreasing functions of* $\epsilon \in [0, 1]$*, and*

$$d(\Gamma, 0) = \lim_{\epsilon\to+0} d(\Gamma, \epsilon), \quad d^+(\Gamma, 0) = \lim_{\epsilon\to+0} d^+(\Gamma, \epsilon). \tag{3}$$

Proof. Since $A \le D$ we obtain for $0 < \epsilon \le \epsilon'$ that

$$\begin{aligned} A^\epsilon(X) &= A(X) + \epsilon[D(X) - A(X)] \\ &\le A(X) + \epsilon'[D(X) - A(X)] = A^{\epsilon'}(X), \end{aligned}$$

which shows that $d(\Gamma, \cdot), d^+(\Gamma, \cdot), d^*(\Gamma, \cdot)$ are nondecreasing, whence in particular

$$d(\Gamma, 0) \le \lim_{\epsilon\to+0} d(\Gamma, \epsilon).$$

Suppose that

$$\delta := \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - d(\Gamma, 0) > 0.$$

Then there is a mapping $X \in \mathcal{C}(\Gamma)$ such that

$$A(X) \le d(\Gamma, 0) + \frac{\delta}{2} = \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2}.$$

Choosing $\epsilon^* \in (0, 1)$ so small that

$$0 \le \epsilon^*[D(X) - A(X)] \le \frac{\delta}{4},$$

it follows

$$\begin{aligned} A^{\epsilon^*}(X) &= A(X) + \epsilon^*[D(X) - A(X)] \le A(X) + \frac{\delta}{4} \\ &\le \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2} + \frac{\delta}{4} \\ &\le d(\Gamma, \epsilon^*) - \frac{\delta}{4} \le A^{\epsilon^*}(X) - \frac{\delta}{4}, \end{aligned}$$

a contradiction. Thus we have $\delta = 0$ and therefore $d(\Gamma, \epsilon) \to d(\Gamma, 0)$ as $\epsilon \to 0$. Analogously, the second relation in (3) is proved. □

Lemma 2. *Let $\epsilon \in (0,1]$ and $M \geq 0$, and consider a sequence $\{\Gamma^m\}$ of boundary configurations converging to the configuration Γ in the sense of Fréchet ($\Gamma^m \to \Gamma$) as $m \to \infty$, where Γ^m and Γ consist of k closed, disjoint, rectifiable Jordan curves. Then for any cohesive sequence $\{X_m\}$ of mappings $X_m \in \mathcal{C}(\Gamma^m)$ with*

$$D(X_m) \leq M \quad \textit{for all } m \in \mathbb{N} \tag{4}$$

there exists a mapping $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}_1(k)$ such that

$$d(\Gamma, \epsilon) \leq A^\epsilon(X) \leq \liminf_{m\to\infty} A^\epsilon(X_m). \tag{5}$$

Proof. For $k = 1$ each sequence of mappings $X_m \in \mathcal{C}(\Gamma^m)$ is cohesive, and the assertion follows using the results of Chapter 4. Thus we now suppose $k \geq 2$. There is a subsequence $\{X_{m_j}\}$ such that $\{A^\epsilon(X_{m_j})\}$ converges and

$$\lim_{j\to\infty} A^\epsilon(X_{m_j}) = \liminf_{m\to\infty} A^\epsilon(X_m). \tag{6}$$

Because of (4) we can also achieve that

$$D(X_{m_j}) \to L \in [0, M_0] \quad \text{as } j \to \infty. \tag{6'}$$

Applying the results of Section 8.3 (and using Theorem 3 of that section instead of Theorems 1 and 2), we obtain by the reasoning used in the proof of Theorem 1 of Section 8.4 that we can also assume that $\{X_{m_j}\}$ is a cohesive sequence with $\mathrm{dom}(X_{m_j}) = B \in \mathcal{N}_1(k)$ for all $j \in \mathbb{N}$, while (6) and (6′) remain unaltered.

Using (4) and a suitable variant of Poincaré's theorem we can in addition assume that

$$X_{m_j} \rightharpoonup X \quad \text{in } H_2^1(B, \mathbb{R}^3)$$

and

$$X_{m_j}|_{\partial B} \to X|_{\partial B} \quad \text{in } L_2(B, \mathbb{R}^3)$$

as $j \to \infty$, and by Theorem 3 of Section 8.3 also

$$X_{m_j}|_{\partial B} \to X|_{\partial B} \quad \text{in } C^0(\partial B, \mathbb{R}^3).$$

Since $X_m \in \mathcal{C}(\Gamma^m)$ and $\Gamma^m \to \Gamma$ it follows $X \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(X) = B \in \mathcal{N}_1(k)$, and the lower semicontinuity of A^ϵ with respect to weak convergence of sequences in $H_2^1(B, \mathbb{R}^3)$ yields

$$A^\epsilon(X) \leq \liminf_{j\to\infty} A^\epsilon(X_{m_j}). \tag{7}$$

Then we infer (5) from (6), (7), and the fact that $X \in \mathcal{C}(\Gamma)$ implies $d(\Gamma, \epsilon) \leq A^\epsilon(X)$. □

Lemma 3. *For all* $\epsilon \in [0,1]$ *we have*

$$d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon).$$

Proof. For any separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ we have

$$d(\Gamma,\epsilon) \le A^\epsilon(X_m) \quad \text{for all } m \in \mathbb{N},$$

which implies

$$d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon).$$

Thus we have to prove

$$d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon). \tag{8}$$

For $k = 1$, nothing is to be proved since then $d^+(\Gamma,\epsilon) = \infty$ as $\mathcal{C}^+(\Gamma) = \emptyset$. Thus we assume $k \ge 2$. We have to show: For any partition $\{\Gamma^1, \dots, \Gamma^s\}$ of $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ with $s \ge 2$ one has

$$d^*(\Gamma,\epsilon) \le \sum_{j=1}^s d(\Gamma^j,\epsilon).$$

This is equivalent to the following assertion:

For every number $\eta > 0$ *there is a separating sequence* $\{X_m\}$ *of mappings* $X_m \in \mathcal{C}(\Gamma)$ *such that*

$$\liminf_{m\to\infty} A^\epsilon(X_m) \le \sum_{j=1}^s d(\Gamma^j,\epsilon) + \eta. \tag{9}$$

We begin with $s = 2$ and an arbitrary partition $\{\Gamma^1, \Gamma^2\}$ of Γ. For an arbitrary chosen $\delta > 0$ there are $X^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B_\nu = \mathrm{dom}(X^\nu) \in \mathcal{N}(k_\nu), \nu = 1,2$, $k_1 + k_2 = k$, such that

$$A^\epsilon(X^{(\nu)}) \le d(\Gamma^\nu,\epsilon) + \delta \quad \text{for } \nu = 1,2.$$

Applying Proposition 1 of Section 8.5 to $\mathcal{F} := A^\epsilon$ we construct new mappings $Z_\nu \in \mathcal{C}(\Gamma^\nu)$ with $\mathrm{dom}(Z_\nu) = B^\nu \in \mathcal{N}(k_\nu)$ and

$$Z_\nu|_{B_{2r}(p_\nu)} = 0 \quad \text{for some disks } B_{2r}(p_\nu) \subset\subset B^\nu$$

such that

$$A^\epsilon(Z_\nu) \le A^\epsilon(X^{(\nu)}) + \delta \quad \text{for } \nu = 1,2.$$

Shifting B^2 suitably we may assume that $p_1 = p_2$; set

$$p := p_1 = p_2.$$

Let ρ be the inversion with respect to the circle $\partial B_{2r}(p)$ and set

$$B_2^* := \rho(B^2 \setminus B_{2r}(p)).$$

Furthermore, let C^* be the "outer" boundary circle of B_2^*, and B^* be the disk bounded by C^*. Set

$$B_1^* := B^1 \setminus B^*$$

and

$$Z_1^* := Z_1|_{B_1^*}, \quad Z_2^* := Z_2 \circ \rho^{-1}|_{B_2^*}.$$

Then

$$X^* := \begin{cases} Z_1^* & \text{on } B_1^*, \\ Z_2^* & \text{on } B_2^* \end{cases}$$

defines a mapping $X^* \in \mathcal{C}(\Gamma)$ with

$$\operatorname{dom}(X^*) = B_1^* \cup B_2^* \in \mathcal{N}(k).$$

Since A^ϵ is conformally invariant, it follows that

$$\begin{aligned} A^\epsilon(X^*) &= A^\epsilon(Z_1^*) + A^\epsilon(Z_2^*) \\ &= A^\epsilon(Z_1|_{B_1^*}) + A^\epsilon(Z_2|_{B^2\setminus B_{2r}(p)}) \\ &= A^\epsilon(Z_1) + A^\epsilon(Z_2) \\ &= A^\epsilon(X^{(1)}) + \delta + A^\epsilon(X^{(2)}) + \delta \\ &\le d(\Gamma^1, \epsilon) + d(\Gamma^2, \epsilon) + 4\delta. \end{aligned}$$

Given $\eta > 0$ we choose $\delta := \eta/4$ and $X_m := X^*$ for all $m \in \mathbb{N}$. Then $\{X_m\}$ is a separating sequence satisfying (9) for a partition $\{\Gamma^1, \Gamma^2\}$ of Γ.

Similarly, if Γ is partitioned as $\{\Gamma^1, \dots, \Gamma^s\}$, we fix $\delta > 0$ and choose $X^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B^\nu = \operatorname{dom}(X^{(\nu)}) \in \mathcal{N}(k_\nu)$, $k_1 + \cdots + k_s = k$, such that

$$A^\epsilon(X^{(\nu)}) \le d(\Gamma^\nu, \epsilon) + \delta, \quad \nu = 1, \dots, s.$$

By the above procedure, carried out $(s-1)$ times, we find a mapping $X^* \in \mathcal{C}(\Gamma)$ with $\operatorname{dom}(X^*) \in \mathcal{N}(k)$ satisfying

$$A^\epsilon(X^*) \le \sum_{\nu=1}^{s} A^\epsilon(X^\nu) + 2^s\delta$$

whence

$$A^\epsilon(X^*) \le \sum_{\nu=1}^{s} d(\Gamma^\nu, \epsilon) + (s + 2^s)\delta.$$

Choosing $\delta := (s+2^s)^{-1}\eta$ and considering the separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with $X_m := X^*$ for all $m \in \mathbb{N}$, we obtain (9), and the proof of (8) is complete. □

Lemma 4. (a) *Let $\Gamma^m \to \Gamma$ as $m \to \infty$ in the Fréchet sense, and $\{X_m\}$ be a sequence of mappings $X_m \in \mathcal{C}(\Gamma^m)$ with*

$$\Gamma^m = \langle \Gamma_1^{(m)}, \dots, \Gamma_k^{(m)} \rangle, \quad \Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$$

consisting of k rectifiable, closed, mutually disjoint Jordan curves. Then

$$d(\Gamma, \epsilon) \le \liminf_{m\to\infty} A^\epsilon(X_m) \quad \textit{for any } \epsilon \in (0,1]. \tag{10}$$

(b) *For any $\epsilon \in (0,1]$ we have*

$$d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon). \tag{11}$$

Proof. We fix ϵ with $0 < \epsilon \le 1$.

(a) Inequality (10) is trivially satisfied if the right-hand side is $= \infty$. Thus we may assume that $\{A^\epsilon(X_m)\}$ converges as $m \to \infty$, i.e.

$$\liminf_{m\to\infty} A^\epsilon(X_m) = \lim_{m\to\infty} A^\epsilon(X_m) < \infty. \tag{12}$$

Since $D(X_m) \le \epsilon^{-1} A^\epsilon(X_m)$ we have

$$D(X_m) \le M_0 \quad \text{for all } m \in \mathbb{N} \tag{13}$$

and some constant $M_0 = M_0(\epsilon) < \infty$. Then (10) follows from Lemma 2 if $\{X_m\}$ is cohesive, $k \ge 2$, and for $k = 1$ one infers (10) for any sequence on account of Chapter 4.

Now we are going to prove (10) by induction over k where we can restrict ourselves to noncohesive sequences $\{X_m\}$.

Induction hypothesis. Suppose that (10) is satisfied for boundary configurations consisting of at most $k-1$ closed curves.

Consider now a noncohesive sequence $\{X_m\}$ of $X_m \in \mathcal{C}(\Gamma^m)$ with $\mathrm{dom}(X_m) \in \mathcal{N}(k)$ satisfying (12) and therefore also (13). As $\{X_m\}$ is noncohesive, it possesses a separating subsequence which we may again call $\{X_m\}$. By Lemma 4 of Section 8.2 and Proposition 4 of Section 8.3 we can also assume that $B_m \in \mathcal{N}_1(k)$. Then there exist points $Q_m \in \mathbb{R}^3$, numbers $\eta_m > 0$ with $\eta_m \to 0$, and closed rectifiable Jordan curves γ_m in $\overline{B}_m$ bounding a strong Lipschitz interior B_m^* in $\mathbb{R}^2$ such that X_m possesses a well-defined continuous trace $Z_m =$ "$X_m|_{\gamma_m}$" on $\gamma_m = \partial B_m^*$ with

$$\sup_{\gamma_m} |Z_m - Q_m| \le \eta_m^2,$$

and any topological representation $c_m : S^1 \to \gamma_m$ of γ_m is not homotopic to zero in $\overline{B}_m$.

Then we choose a sequence of numbers $\delta_j > 0$ with $\delta_j \to 0$ and apply Proposition 2 of Section 8.5 with $\delta := \delta_j$ and $K := M_0(\epsilon)$. Let $\eta_{0,j}$ be the corresponding numbers $\eta_0 \in (0,1)$. For a suitable sequence $\{m_j\}$ of $m_j \in \mathbb{N}$ with $m_1 < m_2 < m_3 < \cdots$ we have $\eta_{m_j} < \eta_{0,j}$ for all $j \in \mathbb{N}$. Renaming $X_{m_j}, Q_{m_j}, Z_{m_j}, \eta_{m_j}$ as X_j, Q_j, Z_j, η_j respectively, it follows

$$\eta_j < \eta_{0,j} \quad \text{for all } j \in \mathbb{N},$$

and there are mappings

$$\Phi_j := \Phi_{\eta_j, Q_j} = \Phi_{\eta_j} : \mathbb{R}^3 \to \mathbb{R}^3$$

with the following properties:

(i) $\Gamma^{j*} := \Phi_j(\Gamma^j)$ is a configuration of k closed, disjoint Jordan curves such that the Fréchet distance $\Delta(\Gamma^j, \Gamma^{j*})$ of Γ^j and Γ^{j*} satisfies

$$\Delta(\Gamma^j, \Gamma^{j*}) < \delta_j \quad \text{for all } j \in \mathbb{N};$$

(ii) $\Phi_j \circ X_j \in \mathcal{C}(\Gamma^{j*})$ and $\text{dom}(\Phi_j \circ X_j) = B_j$;
(iii) $\Phi_j(x) \equiv x$ for $x \in \mathbb{R}^3$ with $|x - Q_j| \geq \eta_j$;
(iv) $\Phi_j(x) \equiv Q_j$ for $x \in \mathbb{R}^3$ with $|x - Q_j| \leq \eta_j^2$;
(v) $A^\epsilon(\Phi_j \circ X_j) \leq A^\epsilon(X_j) + \delta_j$.

In particular we have

$$\Phi_j \circ Z_j = Q_j \quad \text{for all } j \in \mathbb{N}.$$

Then we define

$$B_j^1 := B_j \cap B_j^*, \quad B_j^2 := B_j \setminus \overline{B}_j^1$$

where B_j^* is the "inner domain" of γ_j. This means: Cutting along γ_j we decompose B_j into

$$B_j = B_j^1 \,\dot\cup\, \gamma_j \,\dot\cup\, B_j^2,$$

where B_j^1, B_j^2 are disjoint subdomains of B_j. Since γ_j cannot be contracted in $\overline{B}_j$ to a point, both B_j^1 and B_j^2 contain at least one of the boundary circles of B_j. Thus there is a circle β_j in B_j^1 whose center does not lie in $\overline{B}_j$. Let ρ_j be the inversion with respect to β_j, and set

$$\begin{aligned} E_j^1 &:= \overline{B}_j^{**} \cup \rho_j(B_j^1) \quad \text{with } B_j^{**} := \text{"inner domain" of } \rho_j(\gamma_j), \\ E_j^2 &:= \overline{B}_j^* \cup B_j^2. \end{aligned}$$

We note that $E_j^1 \in \mathcal{N}(k')$, $E_j^2 \in \mathcal{N}(k'')$ with $1 \leq k', k'' < k$ and $k = k' + k''$. Now we define new mappings $X_j^1 \in H_2^1(E_j^1, \mathbb{R}^3)$, $X_j^2 \in H_2^1(E_j^2, \mathbb{R}^3)$ by

$$X_j^1 := \begin{cases} \Phi_j \circ X_j \circ \rho_j^{-1} & \text{on } \rho_j(B_j^1), \\ Q_j & \text{on } \overline{B}_j^{**}, \end{cases}$$

$$X_j^2 := \begin{cases} \Phi_j \circ X_j & \text{on } B_j^2, \\ Q_j & \text{on } \overline{B}_j^*. \end{cases}$$

Roughly speaking, this process amounts to "pinching" X_j to a point in the neighborhood of the closed curve γ_j and to decomposing the resulting surface into two surfaces of "lower topological type" by cutting through γ_j.

Then there is a decomposition $\Gamma = \{\tilde{\Gamma}^1, \tilde{\Gamma}^2\}$ of Γ and correspondingly a decomposition $\Gamma^j = \{\tilde{\Gamma}^{j,1}, \tilde{\Gamma}^{j,2}\}$ of Γ^j such that

$$X_j^1 \in \mathcal{C}(\Phi_j(\tilde{\Gamma}^{j,1})), \quad X_j^2 \in \mathcal{C}(\Phi_j(\tilde{\Gamma}^{j,2}))$$

and

$$\Phi_j(\tilde{\Gamma}^{j,1}) \to \tilde{\Gamma}^1, \quad \Phi_j(\tilde{\Gamma}^{j,2}) \to \tilde{\Gamma}^2 \quad \text{in the sense of Fréchet.}$$

Furthermore, the construction yields

$$\begin{aligned} A^\epsilon(X_j^1) + A^\epsilon(X_j^2) &= A^\epsilon(\Phi_j \circ X_j|_{B_j^1}) + A^\epsilon(\Phi_j \circ X_j|_{B_j^2}) \\ &= A^\epsilon(\Phi_j \circ X_j), \end{aligned}$$

and the induction hypothesis implies

$$d(\Gamma^\ell, \epsilon) \le \liminf_{j\to\infty} A^\epsilon(X_j^\ell) \quad \text{for } \ell = 1, 2.$$

The partition $\Gamma = \{\tilde{\Gamma}^1, \tilde{\Gamma}^2\}$ leads to

$$d^+(\Gamma, \epsilon) \le d(\tilde{\Gamma}^1, \epsilon) + d(\tilde{\Gamma}^2, \epsilon).$$

Therefore

$$\begin{aligned} d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) &\le d(\tilde{\Gamma}^1, \epsilon) + d(\tilde{\Gamma}^2, \epsilon) \\ &\le \liminf_{j\to\infty} A^\epsilon(X_j^1) + \liminf_{j\to\infty} A^\epsilon(X_j^2) \\ &\le \liminf_{j\to\infty} [A^\epsilon(X_j^1) + A^\epsilon(X_j^2)] \\ &= \liminf_{j\to\infty} A^\epsilon(\Phi_j \circ X_j) \\ &\le \liminf_{j\to\infty} [A^\epsilon(X_j) + \delta_j]. \end{aligned}$$

Since $\delta_j \to 0$, we arrive at

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \le \liminf_{m\to\infty} A^\epsilon(X_m), \tag{14}$$

which completes the proof by induction, and so we have verified assertion (a).

(b) For $k = 1$ we have $d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon) = \infty$, and so (11) holds true.

If $k \ge 2$ then Lemma 3 yields $d^*(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) < \infty$. Thus it suffices to show $d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon)$. In fact, for given $\delta > 0$ there is a separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with

$$\liminf_{m\to\infty} A^\epsilon(X_m) \le d^*(\Gamma, \epsilon) + \delta.$$

By the same proof as in (a) we obtain (14) for this sequence. Thus,

$$d^+(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon) + \delta \quad \text{for any } \delta > 0,$$

whence

$$d^+(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon),$$

which finishes the proof of (b). □

Theorem 1. *If the Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied, $k \ge 2$, then there is an $\epsilon_0 \in (0,1]$ such that for each $\epsilon \in (0,\epsilon_0]$ there exists a mapping $X^\epsilon \in \mathcal{C}(\Gamma)$ with*

$$A^\epsilon(X^\epsilon) = d(\Gamma,\epsilon) \tag{15}$$

and

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad \langle X_u^\epsilon, X_v^\epsilon\rangle = 0. \tag{16}$$

Proof. Since

$$\lim_{\epsilon\to+0} d(\Gamma,\epsilon) = d(\Gamma,0) = a(\Gamma) < a^+(\Gamma) = d^+(\Gamma,0) = \lim_{\epsilon\to+0} d^+(\Gamma,\epsilon),$$

there is an ϵ_0 with $0 < \epsilon_0 \le 1$ such that

$$d(\Gamma,\epsilon) < d^+(\Gamma,\epsilon) \quad \text{for } 0 < \epsilon \le \epsilon_0. \tag{17}$$

Fix some $\epsilon \in (0,\epsilon_0]$ and choose a sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with

$$A^\epsilon(X_m) \to d(\Gamma,\epsilon) \quad \text{as } m \to \infty.$$

If $\{X_m\}$ were not cohesive, there would exist a separating subsequence $\{X_{m_j}\}$, whence

$$d^*(\Gamma,\epsilon) \le \lim_{j\to\infty} A^\epsilon(X_{m_j}) = d(\Gamma,\epsilon),$$

and by (11) we would have

$$d^+(\Gamma,\epsilon) = d^*(\Gamma,\epsilon) \le d(\Gamma,\epsilon),$$

a contradiction to (17). Thus $\{X_m\}$ has to be cohesive, and $D(X_m) \le M_0(\epsilon)$ for all $m \in \mathbb{N}$ since $A^\epsilon(X_m) \le$ const and $\epsilon D(X_m) \le A^\epsilon(X_m)$. Hence we can apply Lemma 2 to $\Gamma^m \equiv \Gamma$ for all $m \in \mathbb{N}$, and consequently there is a $X^\epsilon \in \mathcal{C}(\Gamma)$ such that

$$d(\Gamma,\epsilon) \le A^\epsilon(X^\epsilon) \le \liminf_{m\to\infty} A^\epsilon(X_m) = d(\Gamma,\epsilon),$$

which yields (15), i.e.

$$A^\epsilon(X^\epsilon) \le A^\epsilon(X) \quad \text{for all } X \in \mathcal{C}(\Gamma).$$

By Theorem 1 of Section 8.2 this implies (16). □

Now we can prove the main result (cf. Theorem 1 in Section 8.2):

Theorem 2. *Suppose that the Douglas condition $a(\Gamma) < a^+(\Gamma)$ holds. Then there is a mapping $X \in \overline{\mathcal{C}}(\Gamma)$ with*

$$A(X) = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D = D(X) \tag{18}$$

satisfying the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B$$

as well as $X \in C^2(B, \mathbb{R}^3)$ and

$$\Delta X = 0 \quad \text{in } B. \tag{19}$$

Furthermore, X maps ∂B homeomorphically onto Γ.

Proof. Let $\epsilon_0 > 0$ be as in Theorem 1 and consider the mapping $X^\epsilon \in \mathcal{C}(\Gamma)$, $0 < \epsilon \le \epsilon_0$, satisfying (15) and (16). Then $A(X^\epsilon) = D(X^\epsilon)$, and consequently

$$d(\Gamma, \epsilon) = A^\epsilon(X^\epsilon) = A(X^\epsilon) = D(X^\epsilon) \quad \text{for } 0 < \epsilon \le \epsilon_0.$$

For an arbitrary $Y \in \mathcal{C}(\Gamma)$ we have

$$A^\epsilon(X^\epsilon) \le A^\epsilon(Y) \le D(Y),$$

whence

$$d(\Gamma) \le D(X^\epsilon) = A^\epsilon(X^\epsilon) \le A^\epsilon(Y) \le D(Y) \quad \text{for all } Y \in \mathcal{C}(\Gamma).$$

This yields

$$d(\Gamma) \le D(X^\epsilon) \le d(\Gamma)$$

and therefore

$$d(\Gamma) = D(X^\epsilon) \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

Then it follows for all $Y \in \mathcal{C}(\Gamma)$ and any $\epsilon, \epsilon' \in (0, \epsilon_0]$:

$$a(\Gamma) \le A(X^\epsilon) = A^\epsilon(X^\epsilon) = A^{\epsilon'}(X^{\epsilon'}) \le A^{\epsilon'}(Y).$$

Since $A^{\epsilon'}(Y) \to A(Y)$ as $\epsilon' \to +0$, we arrive at

$$a(\Gamma) \le A(X^\epsilon) \le a(\Gamma),$$

which implies

$$a(\Gamma) = A(X^\epsilon).$$

Thus we have

$$A(X^\epsilon) = D(X^\epsilon) = a(\Gamma) = d(\Gamma) \quad \text{for } 0 < \epsilon \le \epsilon_0,$$

that is,

$$A(X^\epsilon) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = D(X^\epsilon). \tag{20}$$

Fix some $\epsilon \in (0, \epsilon_0]$ and set $X = X^\epsilon$. From

$$D(X) = \inf_{\mathcal{C}(\Gamma)} D$$

it follows that X is harmonic in B, and by virtue of $X \in H_2^1(B, \mathbb{R}^3)$ and $X|_{\partial B} \in C^0(\partial B, \mathbb{R}^3)$ we conclude that $X \in C^0(\overline{B}, \mathbb{R}^3)$, i.e. $X \in \overline{\mathcal{C}}(\Gamma)$. On account of (20) we obtain

$$A(X) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D = \inf_{\mathcal{C}(\Gamma)} D = D(X).$$

Finally one proves in the same way as for Theorem 3 in Section 4.5 that X maps ∂B homeomorphically onto Γ. This completes the proof of the theorem. □

8.7 Further Discussion of the Douglas Condition

We had formulated the Douglas condition as the assumption that

$$a(\Gamma) < a^+(\Gamma) \tag{1}$$

holds true. Jesse Douglas [28] noted that (1) is equivalent to the assumption

$$d(\Gamma) < d^+(\Gamma) \tag{2}$$

where $d(\Gamma)$ and $d^+(\Gamma)$ are defined by

$$d(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D, \quad d^+(\Gamma) := \inf_{\mathcal{C}^+(\Gamma)} D.$$

Using the notation of the previous section, this means

$$d(\Gamma) = d(\Gamma, 1), \quad d^+(\Gamma) = d^+(\Gamma, 1).$$

In fact, Douglas pointed out that the gist of his method to find a minimal surface X bounded by $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consisted in using exclusively Dirichlet's integral D instead of the area, replacing condition (1) by (2), cf. [28], p. 232, and all later authors proceeded in the same way. In order to prove that his solution is area minimizing, Douglas showed

$$a(\Gamma) = d(\Gamma), \tag{3}$$

and the proof of this identity he based on a theorem by P. Koebe, according to which every polyhedral surface possesses an a.e.-conformal representation of the same topological type. Our proof of (3) in Theorem 2 of Section 8.6 required no such tool, but was based on the assumption (1). Now we want to show that $a(\Gamma) = d(\Gamma)$ and $a^+(\Gamma) = d^+(\Gamma)$ holds for any contour Γ, without using any conformal mapping theorem. This in turn will yield the equivalence of the conditions (1) and (2) which are often called the *sufficient condition of Douglas*.

First, however, we note that for any contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ one has the two inequalities

$$a(\Gamma) \le a^+(\Gamma) \quad \text{and} \quad d(\Gamma) \le d^+(\Gamma), \tag{4}$$

which are sometimes denoted as *necessary condition of Douglas*. Clearly, (4) follows from the inequality

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \quad \text{for } \epsilon \in [0, 1], \tag{5}$$

which was established in Lemma 3 of Section 8.6.

Furthermore we recall (cf. Theorem 2 of Section 8.6):

$$\textit{Inequality (1) implies } a(\Gamma) = d(\Gamma). \tag{6}$$

Theorem 1. *We have*

$$a(\Gamma) = d(\Gamma) \quad \textit{for } k \ge 1 \tag{7}$$

and

$$a^+(\Gamma) = d^+(\Gamma) \quad \textit{for } k \ge 2 \tag{8}$$

Proof. (i) For $k = 1$, the identity (7) was proved in Chapter 4.

(ii) Let $k = 2$ and $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$.

(α) $a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2) \overset{\text{(i)}}{=} d(\Gamma_1) + d(\Gamma_2) = d^+(\Gamma)$.

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$ by (6).

(γ) If $a(\Gamma) = a^+(\Gamma)$ then $a(\Gamma) \overset{(\alpha)}{=} d^+(\Gamma) \overset{(4)}{\ge} d(\Gamma)$, and trivially we have

$$a(\Gamma) \le d(\Gamma) \quad \text{for any } k \ge 1 \tag{9}$$

because of $A \le D$. Thus $a(\Gamma) = d(\Gamma)$ also in case (γ), and by (4) it follows $a(\Gamma) = d(\Gamma)$ in any case if $k = 2$.

(iii) Let $k = 3$ and $\Gamma = \langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$.

$$(\alpha) \qquad \begin{aligned} a^+(\Gamma) &= \min\{a(\Gamma_\mu) + a(\Gamma_\nu) + a(\Gamma_\rho), a(\Gamma_\mu, \Gamma_\nu) + a(\Gamma_\rho) : \\ &\qquad (\mu, \nu, \rho) \sim (1, 2, 3)\} \\ &= \min\{d(\Gamma_\mu) + d(\Gamma_\nu) + d(\Gamma_\rho), d(\Gamma_\mu, \Gamma_\nu) + d(\Gamma_\rho) : \\ &\qquad (\mu, \nu, \rho) \sim (1, 2, 3)\} \\ &= d^+(\Gamma). \end{aligned}$$

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$ by (6).

(γ) If $a(\Gamma) = a^+(\Gamma)$, then by (α), (4) and (9) it follows

$$a(\Gamma) = d^+(\Gamma) \geq d(\Gamma) \geq a(\Gamma),$$

whence $a(\Gamma) = d(\Gamma)$ in any case on account of (4), if $k = 3$.

(iv) The general case is proved by induction: Suppose that (7) is verified for $k \leq N$. Then we obtain for $k = N + 1$:

(α) $a^+(\Gamma) = d^+(\Gamma)$. In fact,

$$\begin{aligned} a^+(\Gamma) &= \min\{a(\Gamma^1) + \cdots + a(\Gamma^s): \\ &\qquad \{\Gamma^1, \ldots, \Gamma^s\} = \text{partition of } \Gamma \text{ with } s \geq 2\}, \\ d^+(\Gamma) &= \min\{d(\Gamma^1) + \cdots + d(\Gamma^s): \\ &\qquad \{\Gamma^1, \ldots, \Gamma^s\} = \text{partition of } \Gamma \text{ with } s \geq 2\}, \end{aligned}$$

and Γ^ℓ consists of k_ℓ closed curves, $k_1 + \cdots + k_s = N + 1$, whence $k_\ell \leq N$ for $\ell = 1, \ldots, s$, and since (7) holds for $k \leq N$, we obtain $a(\Gamma^1) = d(\Gamma^1), \ldots, a(\Gamma^s) = d(\Gamma^s)$; therefore we have (8) for $k = N + 1$.

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$.

(γ) If $a(\Gamma) = a^+(\Gamma)$, then by (α), (4), and (9):

$$a(\Gamma) = d^+(\Gamma) \geq d(\Gamma) \geq a(\Gamma)$$

whence $a(\Gamma) = d(\Gamma)$ in any case on account of $a(\Gamma) \leq a^+(\Gamma)$. □

Similarly one proves

Theorem 2. *For any $\epsilon \in [0, 1]$ we have*

(10) $$a(\Gamma, \epsilon) = d(\Gamma, \epsilon) = a(\Gamma) = d(\Gamma) \quad \text{if } k \geq 1$$

and

(11) $$a^+(\Gamma, \epsilon) = d^+(\Gamma, \epsilon) = a^+(\Gamma) = d^+(\Gamma) \quad \text{if } k \geq 2;$$

therefore also

(12) $$a^+(\Gamma) - a(\Gamma) = d^+(\Gamma) - d(\Gamma) = d^+(\Gamma, \epsilon) - d(\Gamma, \epsilon) \quad \text{if } k \geq 2.$$

Corollary 1. *The conditions (1) and (2) are equivalent.*

Corollary 2. *In Theorem 1 of Section 8.6 we can choose $\epsilon_0 = 1$.*

8.8 Examples

We now exhibit some examples when the sufficient Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied.

$\boxed{1}$ Let $k = 2$, and consider two closed, rectifiable, disjoint Jordan curves Γ_1 and Γ_2 that lie in planes Π_1 and Π_2 which intersect in a straight line L. By S_1 and S_2 we denote the two bounded planar domains in Π_1 and Π_2 with the boundary contours Γ_1 and Γ_2 respectively. Then

$$a(\Gamma_1) = \text{area}(S_1), \quad a(\Gamma_2) = \text{area}(S_2).$$

Suppose that $S_1 \cap S_2$ is nonempty. Then S_1 and S_2 intersect in a closed interval I contained in L. The line L decomposes S_1 and S_2 into the pieces S_1^+, S_1^- and S_2^+, S_2^- respectively with $S_1^+ \cap S_1^- := I_1 \subset L$ and $S_2^+ \cap S_2^- =: I_2 \subset L$. Take an interior point $P \in L$, a bisectrix L' of one of the angles between Π_1 and Π_2 meeting L at P perpendicularly, and consider a sufficiently small circular cylinder Z with the axis L'. Then Z intersects $S_1^+, S_1^-, S_2^+, S_2^-$ in closed curves $\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-$ consisting of semi-ellipses $\epsilon_1^+, \epsilon_1^-, \epsilon_2^+, \epsilon_2^-$ and an interval $j \subset I$. Let $E_1^+, E_1^-, E_2^+, E_2^-$ be the "full" semi-ellipses bounded by $\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-$ respectively. Then $\gamma_1 := \gamma_1^+ \cup \gamma_2^+$ spans a nonparametric minimal surface M_1 with

$$\text{area}(M_1) < \text{area}(E_1^+ \cup E_2^+),$$

and $\gamma_2 := \gamma_1^- \cup \gamma_2^-$ spans a nonparametric minimal surface M_2 with

$$\text{area}(M_2) < \text{area}(E_1^- \cup E_2^-).$$

Then the set

$$\Sigma := (S_1 \cup S_2 \cup M_1 \cup M_2) \setminus (E_1^+ \cup E_2^+ \cup E_1^- \cup E_2^-)$$

has an area less than that of $S_1 \cup S_2$, i.e.

$$\text{area}(\Sigma) < \text{area}(S_1 \cup S_2) = \text{area}(S_1) + \text{area}(S_2).$$

We can construct a mapping $X \in \mathcal{C}(\Gamma)$, $\Gamma := \langle \Gamma_1, \Gamma_2 \rangle$, such that

$$\Sigma = X(\overline{B}), \quad B = \text{dom}(X) \in \mathcal{N}(2),$$

and thus we have

$$a(\Gamma) \le A(X) = \text{area}(\Sigma) < a(\Gamma_1) + a(\Gamma_2) = a^+(\Gamma).$$

Hence we have

Proposition 1. $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ *satisfies the Douglas condition if* Γ_1 *and* Γ_2 *fulfill the assumptions stated above. In particular, we have* $a(\Gamma) < a^+(\Gamma)$ *for* $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ *if* Γ_1 *and* Γ_2 *are closed, rectifiable, disjoint planar Jordan curves in* $\mathbb{R}^3$ *which are linked.*

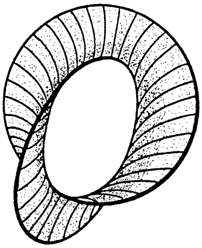

Fig. 1. An annulus-type minimal surface bounded by two interlocking closed curves

Actually, it is irrelevant that Γ_1 and Γ_2 are planar, and a similar reasoning as above yields

Theorem 1. *Suppose that the contour $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ consists of two closed, rectifiable, disjoint Jordan curves in $\mathbb{R}^3$ which are linked. Then $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$, and so there is a minimal surface $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}(2)$ and $A(X) = a(\Gamma)$, i.e. X is an area-minimizing minimal surface of annulus type bounded by two linked closed curves Γ_1 and Γ_2.*

J. Douglas (cf. [13], p. 351) obtained Theorem 1 as a corollary of the following

Theorem 2. *Let Γ_1 and Γ_2 be two nonintersecting, closed, rectifiable Jordan curves in $\mathbb{R}^3$, and suppose that there are minimal surfaces $X_1 \in \mathcal{C}(\Gamma_1), X_2 \in \mathcal{C}(\Gamma_2)$ with $A(X_1) = a(\Gamma_1)$, $A(X_2) = a(\Gamma_2)$ such that $X_1(w_1) = X_2(w_2)$ for some $w_1, w_2 \in B = B_1(0) = \mathrm{dom}(X_1) = \mathrm{dom}(X_2)$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ satisfies the Douglas condition $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$, and so there is an annulus-type minimal surface $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$.*

Remark 1. Instead of giving a geometric proof for $a(\Gamma) < a^+(\Gamma)$, Douglas derived the inequality $d(\Gamma) < d^+(\Gamma)$ in an analytic way working with the Dirichlet integral and arranging for $w_1 = w_2 = 0$. Using the harmonic mapping $H : \{r < |w| < 1\} \to \mathbb{R}^3$, $0 < r < 1$, with the boundary values $H(w) = X_1(w)$ for $|w| = 1, H(w) = X_2(w)$ for $|w| = r$. Then it can be shown that

$$D(H) < D(X_1) + D(X_2) = d(\Gamma_1) + d(\Gamma_2) \quad \text{for } 0 < r \ll 1,$$

which implies $d(\Gamma) < d^+(\Gamma)$ for $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$, and we know that this inequality is equivalent to $a(\Gamma) < a^+(\Gamma)$.

Essentially the same proof can be found in J.C.C. Nitsche [28], pp. 531–533.

Remark 2. Both Douglas and Nitsche assumed in addition that w_1 is not a branch point of X_1 and w_2 is not a branch point of X_2. These requirements are now superfluous because of the Osserman–Alt–Gulliver result.

Remark 3. Note that for planar Γ_1 and Γ_2 the result of Theorem 2 is essentially contained in Proposition 1. Furthermore, the proof of this proposition can be modified to yield Theorem 2.

$\boxed{2}$ Obviously the Douglas condition $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$ for $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is satisfied if Γ_1 and Γ_2 bound a doubly connected surface S with

$$\text{area}(S) < a(\Gamma_1) + a(\Gamma_2),$$

say, the lateral surfaces of a conical frustum, or a cylindrical surface. This simple observation was used in the construction of a one-parameter family of triply-connected minimal surfaces bounded by three coaxial circles $\Gamma_1, \Gamma_2, \Gamma_3$; see Section 4.15.

$\boxed{3}$ Finally we note that the Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied for $\Gamma = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$, $k \geq 2$, if the distinct, closed, rectifiable Jordan curves $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ form the boundary of a bounded, k-fold connected domain Ω in $\mathbb{R}^2$:

$$\partial\Omega = \Gamma_1 \,\dot{\cup}\, \Gamma_2 \,\dot{\cup} \cdots \dot{\cup}\, \Gamma_k.$$

In fact, each contour Γ_j bounds a simply connected, bounded domain Ω_j in $\mathbb{R}^2$, and we may assume that

$$\Omega = \Omega_1 \setminus \{\Omega_2 \cup \dots \cup \Omega_k\},$$

i.e. Γ_1 is the "exterior" boundary curve of Ω. Then

$$\text{(1)} \qquad \text{area}(\Omega) = \text{area}(\Omega_1) - \left\{ \sum_{j=2}^{k} \text{area}(\Omega_j) \right\}.$$

Let $\Gamma = \{\Gamma^1, \dots, \Gamma^s\}$ be an arbitrary partition of the boundary curves $\Gamma_1, \dots, \Gamma_k$, $s \geq 2$. We may assume that Γ_1 belongs to Γ^1, i.e. $\Gamma^1 = \langle \Gamma_1, \Gamma_{j_2}, \dots, \Gamma_{j_\ell} \rangle$ with $1 < j_2 < \dots < j_\ell$ and $1 \leq \ell < k$. Then

$$a^+(\Gamma) := \inf\{a(\Gamma^1) + \dots + a(\Gamma^s) \colon \{\Gamma^1, \dots, \Gamma^s\} = \text{partition of } \Gamma\}$$

whence

$$\text{(2)} \qquad a^+(\Gamma) \geq a(\Gamma^1) = \text{area}(\Omega_1) - \sum_{\nu=2}^{\ell} \text{area}(\Omega_{j_\nu})$$

($\sum_{\nu=2}^{\ell} = 0$ if $\ell = 1$). From (1) and (2) we infer

$$a^+(\Gamma) > \text{area}(\Omega) = a(\Gamma).$$

Thus we can apply Theorem 2 of Section 8.6. Combining this with the reasoning that was used in Section 4.11 to prove Riemann's mapping theorem (cf. Theorem 1 in Section 4.11), we obtain **Koebe's mapping theorem**:

Theorem 3. *Let Ω be a k-fold connected domain in $\mathbb{C}$ whose boundary consists of k closed, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$. Then there exists a homeomorphism f from $\overline{B}$ onto $\overline{\Omega}$, $B \in \mathcal{N}(k)$, which is holomorphic in B and satisfies $f'(w) \neq 0$ for all $w \in B$.*

P. Koebe also proved that f is uniquely determined up to a Möbius transformation, i.e. if f^* is another mapping like f from $\overline{B}^*$ onto $\overline{\Omega}$, $B^* \in \mathcal{N}(k)$, then there is a Möbius transformation τ from $\overline{B}^*$ onto $\overline{B}$ with $f^* = f \circ \tau$. An elegant proof of this fact can be found in Courant and Hurwitz [1], pp. 517–519. In another form, a uniqueness result is stated and proved in R. Courant [15], pp. 187–191: $f^* = f$ if $f, f^* \in \mathcal{N}_1(k)$ and $f(\zeta) = f^*(\zeta)$ for a fixed point $\zeta \in \partial B_1(0)$.

8.9 Scholia

1. The first to study general Plateau problems for minimal surfaces of higher topological type was Jesse Douglas; his work was truly pioneering, and his ideas and insights are as exciting and important nowadays as at the time when they were published, more than half a century ago. It seems that Douglas was the first to grasp the idea that a minimizing sequence could be degenerating in topological type, and he interpreted such a conceivable degeneration as a change in the conformal structure. He based his notion of degeneration on the representation of Riemann surfaces as branched coverings of the sphere. Then degeneration meant "disappearance of branch cuts". The intuitive meaning of degeneration is the shrinking of handles and the tendency to separate the Riemann surface into several components. Since degeneration is unavoidable in general, Douglas had the idea of minimizing not over surfaces of a *fixed* topological type but also over all possible reductions of the given type. In this set of Riemann surfaces of varying topological type, Douglas introduced a notion of convergence as convergence of branch points in the representation of the surfaces as branched coverings of the sphere. The compactness of this set of Riemann surfaces seemed to be a trivial matter to him since his whole argument reads: *"This is because the set can be referred to a finite number of parameters, e.g., the position of the branch points ..."*. This reasoning is, however, rather inaccurate since the position of branch points alone does not determine the structure of the surface. Douglas also argued on a rather intuitive level when it came to the lower semicontinuity of Dirichlet's integral with respect to the convergence of surfaces. Taking the compactness of the above set of Riemann surfaces and the lower semicontinuity of Dirichlet's integral for granted, it is then obvious that an absolute minimum of Dirichlet's integral in the class of surfaces considered by Douglas must be achieved, either in a surface of desired (highest) topological type or in one of reduced type. In this way Douglas was led to his celebrated solution of the **general Plateau problem**:

Given a boundary configuration $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ consisting of $k \geq 1$ closed, rectifiable, mutually disjoint Jordan curves $\Gamma_1, \ldots, \Gamma_k$ in $\mathbb{R}^3$, there is a connected minimal surface X of prescribed Euler characteristic and prescribed character of orientability, bounded by Γ, provided that the infimum $a(\Gamma)$ of area for all admissible surfaces is less than the infimum $a^+(\Gamma)$ of Dirichlet's integral or of the sum of Dirichlet integrals for surfaces of lower type bounded by Γ.

Here a possibly disconnected surface Y bounded by Γ is called of *lower type* if at least one of the following degenerations occurs:

(i) Y has a smaller Euler characteristic than prescribed;
(ii) Y is disconnected and consists of several connected pieces of total characteristic (= sum of the characteristics of the connected pieces) not greater than prescribed, and each piece is bounded by complementary subsets of $\{\Gamma_1, \ldots, \Gamma_k\}$ which together make up Γ.

J. Douglas published this most general result in his 1939 paper [28]. Already in 1931 he had treated the case $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ for annulus-type minimal surfaces (cf. [18]), and one-sided minimal surfaces in a given contour he had discussed 1932 in his paper [15]. Further work dealing with the general Plateau problem are his papers [27,29] and [31].

2. R. Courant [9,11], and M. Shiffman [3,5] put the pioneering work of Douglas on a solid basis by solving the variational problem "$D \to \min$" within a class of surfaces of fixed topological type. In this context we also mention H. Lewy's lecture notes [3] from 1939.

Courant gave a very clear exposition of his method in his treatise [15] from 1950 for minimal surfaces $X : B \to \mathbb{R}^3$ with $B \in N$, where the class N of parameter domains comprises either (a) schlicht k-circle domains, or (b) slit domains, or (c) Riemann domains over the w-plane bounded by k unit circles and having branch points of total multiplicity $2k - 2$ (cf. Courant [15], pp. 144–145, 149); other types are briefly discussed in [15], pp. 164–166.

3. Douglas has based his investigations on the use of symmetric Riemann surfaces without boundary which are obtained as doubles of Riemann surfaces of genus g with k boundary curves. This idea is also employed in the study of the general Plateau problem by F. Tomi and A. Tromba [5], which will be presented in Chapter 4 of Vol. 3.

An exposition of how to solve the Douglas problem for surfaces of higher topological type or for nonorientable surfaces is presented in Courant [15], pp. 160–164. In particular, the existence proof for surfaces of the topological type of the Möbius strip with one boundary contour is worked out in detail. For the general case, Courant refers to Shiffman [3].

Another presentation of the work of Douglas is given in the treatise [6] of J. Jost. In Nitsche's *Vorlesungen* [28], the Douglas problem for annulus-type minimal surfaces with two boundary curves is treated. C.B. Morrey [8], Chapter 9, described a solution of the Douglas problem for k-fold connected minimal surfaces.

The Douglas problem for H-surfaces was studied by H. Werner [1] for $H = \text{const}$, and for variable H by S. Luckhaus [1].

Beautiful soap film experiments with minimal surfaces are described in papers by Courant [10] and by Almgren and Taylor [1].

4. Douglas also treated the case of configurations $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ with *non-rectifiable* curves. In this regard we refer to Section 17 of his paper [28], pp. 279–287.

5. The idea to prove Koebe's mapping theorem via the solution of the general Plateau problem was also conceived by Douglas in [11] and [28]. Courant presented an elaboration of this approach in Chapter 5, pp. 167–198, of his treatise [15].

A generalization of Lichtenstein's mapping theorem to Riemannian metrics on multiply connected domains is due to J. Jost [6] and [17]; the original approach by Morrey [8] is incorrect. A new proof in the spirit of Section 4.11 was given in the paper [8] by Hildebrandt and von der Mosel. Jost [6] treated the Douglas problem for orientable minimal surfaces in a Riemannian manifold; see also Morrey [3] and [8]. The nonorientable case was worked out by F. Bernatzki [1].

6. The presentation of this chapter is based on the work of Courant [15] and on the papers of Kurzke [1], Kurzke and von der Mosel [1], and Hildebrandt and von der Mosel [6,8].