Chapter 7

Graphs with Prescribed Mean Curvature

This chapter is devoted to nonparametric surfaces of prescribed mean curvature H, that is, to ***H*-surfaces** which can be represented as graphs over planar domains. Nonparametric minimal surfaces, i.e. graphs with $H = 0$, were already considered in Section 2.2, and the celebrated two-dimensional Bernstein theorem was described in Section 2.4. Generalizations of this result are presented in Volume 3 of this treatise.

One can find a wealth of theorems on nonparametric minimal surfaces and H-surfaces in the monographs of J.C.C. Nitsche [28], D. Gilbarg and N. Trudinger [1], U. Massari and M. Miranda [1], E. Giusti [4], as well as in the notes [8] of L. Simon, in his survey paper [9], and in his encyclopaedia article [17], IV. Clearly the abundance of this material deserves a thorough and comprehensive presentation which exceeds the scope of the present book. For this reason we merely describe some *existence and uniqueness results* for the *nonparametric Plateau problem* (i.e. the *Dirichlet problem*) for minimal surfaces and, more generally, for H-surfaces, which can be derived from the solution of the *parametric Plateau problem* for minimal surfaces, studied in Chapter 4, and for H-surfaces that will be treated in Vol. 2.

We shall base our investigations on the results of Chapter 5 concerning stable minimal- and H-surfaces, and so we will use the same notations as in Chapter 5. The discussion ends in Section 7.3 with a presentation of some basic estimates for nonparametric H-surfaces, namely Heinz's *maximal radius theorem*, Serrin's *maximal height theorem*, and *Finn's area estimate*. Furthermore a gradient estimate for nonparametric H-surfaces is derived. The section closes with an energy estimate for the difference of two solutions of the H-surface equation, which can be used to prove *unique* solvability of the H-surface equation even in cases when the classical maximum principle fails. An application of this estimate is a theorem about the removability of isolated singularities of nonparametric H-surfaces which generalizes *Bers's celebrated result* that isolated singularities of solutions for the minimal surface equation can be removed.

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The basic feature of this chapter is the Gaussian approach viewing graphs as regular parametric surfaces whose normals $N = (N^1, N^2, N^3)$ point into the upper hemisphere

$$S^2_+ := \{x \in \mathbb{R}^3 \colon \langle x, e\rangle > 0\}$$

where e denotes some unit vector in $\mathbb{R}^3$. Applying a rotation we can assume that $e = e_3 = (0,0,1)$, and then $N(B) \subset S^2_+$ means $N^3 > 0$.

7.1 H-Surfaces with a One-to-One Projection onto a Plane, and the Nonparametric Dirichlet Problem

In Section 4.9 Radó's result on minimal surfaces with a 1–1 projection onto a plane was presented, using H. Kneser's lemma. Now we take up these considerations following F. Sauvigny [1,2], and the textbook [16], where in Chapter XII, §9, the Dirichlet problem for the nonparametric H-surface equation is solved by a continuity method.

For the following we assume that $H(x, y, z)$ is a real-valued function on $\mathbb{R}^3$ of class $C^{1,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$, satisfying

$$\sup_{\mathbb{R}^3} |H| \le h_0 \quad \text{and} \quad H_z(x,y,z) \ge 0 \quad \text{on } \mathbb{R}^3 \tag{1}$$

for some $h_0 \in (0, \infty)$. Set

$$r_0 = \frac{1}{2h_0}. \tag{2}$$

Let $\mathbb{R}^2$ be the x, y-plane with the points $p = (x, y)$. The Euclidean distance of two points $p = (x, y)$ and $p' = (x', y')$ is denoted by

$$|p - p'| := \sqrt{(x - x')^2 + (y - y')^2}.$$

The disk with radius $r > 0$ and center $p_0 = (x_0, y_0)$ is

$$B_r(p_0) := \{p \in \mathbb{R}^2 \colon |p - p_0| < r\}.$$

Specifically we introduce the disk

$$\Omega_0 := B_{r_0}(0) = \{p \in \mathbb{R}^2 \colon |p| < r_0\} \tag{3}$$

of radius r_0 about the origin, and the closed circular cylinder

$$\mathcal{Z} := \overline{\Omega}_0 \times \mathbb{R} = \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in \overline{\Omega}_0,\ z \in \mathbb{R}\}. \tag{4}$$

Definition 1. (i) *A bounded open set Ω of $\mathbb{R}^2$ is called a* **Jordan domain** *if it is bounded by a closed Jordan curve.*

(ii) *A Jordan domain Ω in $\mathbb{R}^2$ with $0 \in \Omega \subset \Omega_0$ is said to be* $2h_0$**-convex** *if for every point $p' \in \partial\Omega$ there is a closed disk $S_0 := \overline{B}_{r_0}(p_0)$ such that*

$$\overline{\Omega} \subset S_0 \quad \text{and} \quad p' \in \partial\Omega \cap \partial S_0. \tag{5}$$

We call S_0 a **support disk** *of Ω at the point $p' \in \partial\Omega$.*

Remark 1. A Jordan domain Ω with $0 \in \Omega \subset \Omega_0$ with $\partial\Omega \in C^{2,\alpha}, 0 < \alpha < 1$, is $2h_0$-convex if and only if the curvature κ of the positive-oriented boundary $\partial\Omega$ satisfies $\kappa(p) \geq 1/r_0 = 2h_0$ at each point $p \in \partial\Omega$.

Let Γ be a rectifiable closed Jordan curve in $\mathbb{R}^3$, and recall that $\mathcal{C}(\Gamma)$ denotes the class of surfaces $X : B \to \mathbb{R}^3$ bounded by Γ. We fix a *three-point condition*

$$X(\zeta_k) = Q_k \quad \text{for } k = 1, 2, 3, \tag{$*$}$$

with $\zeta_k = \exp(\frac{2\pi k}{3} i)$ and three given distinct points $Q_k \in \Gamma$, thereby expressing the orientation of Γ. As usual we denote by $\mathcal{C}^*(\Gamma)$ the class of surfaces $X \in \mathcal{C}(\Gamma)$ satisfying $(*)$.

Now we consider regular curves $\Gamma \in C^{3,\alpha}$ which lie as *graphs above the boundary* $\partial\Omega$ of a $2h_0$-convex Jordan domain. This means the following: *There is a function $\gamma \in C^{3,\alpha}(\partial\Omega)$ above the boundary $\partial\Omega \in C^{3,\alpha}$ such that*

$$\Gamma = \{(p, \gamma(p)) \in \mathbb{R}^3 \colon p \in \partial\Omega\}. \tag{6}$$

Then we write:

$$\Gamma = \operatorname{graph} \gamma. \tag{7}$$

Furthermore we assume that $Q_k = (q_k, \gamma(q_k))$, $q_k \in \partial\Omega$, holds where q_1, q_2, q_3 induce a positive orientation of $\partial\Omega$.

Theorem 1. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with $0 \in \Omega \subset \Omega_0$ which is $2h_0$-convex, $\partial\Omega \in C^{3,\alpha}$, and suppose that $\Gamma \in C^{3,\alpha}$ is given as a graph γ for some $\gamma \in C^{3,\alpha}(\partial\Omega)$, whereas $H \in C^{1,\alpha}$ satisfies (1). Then there exists exactly one stable H-surface $X \in \mathcal{C}^*(\Gamma)$. This surface is an immersion and even an embedding of $\overline{\Omega}$ into $\mathbb{R}^3$, and it can be represented nonparametrically as* graph ζ, *where $\zeta \in C^{3,\alpha}(\overline{\Omega})$ is a solution of the boundary value problem*

$$\begin{aligned} &\mathcal{M}\zeta = 2H(\cdot, \zeta)(1 + |\nabla\zeta|^2)^{3/2} \quad \text{in } \Omega, \\ &\zeta = \gamma \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

and $\mathcal{M}\zeta$ denotes the minimal surface operator

$$\mathcal{M}\zeta := (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1 + \zeta_x^2)\zeta_{yy}. \tag{9}$$

Proof. (i) Consider the vector field

$$Q(x,y,z) := \frac{1}{2}\bigg(\int_0^x H(t,y,z)\,dt, \int_0^y H(x,t,z)\,dt, 0 \bigg),$$

satisfying $\operatorname{div} Q = H$ on $\mathcal{Z}$, and the associated functional

$$E(X) := \int_B \bigg(\frac{1}{2}|\nabla X|^2 + 2[Q(X), X_u, X_v] \bigg)\,du\,dv$$

defined by formula (4) of Section 5.3.

By minimizing E among all $X \in \mathcal{C}^*(\Gamma)$ with $X(\overline{B}) \subset \mathcal{Z}$ one obtains an H-surface X contained in $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $X(B) \subset \operatorname{int} \mathcal{Z}$ (cf. Gulliver and Spruck [1], Hildebrandt [10]; these results are described in Chapter 4 of Vol. 2). On account of 5.3, Theorem 1, *the H-surface X is stable* since it also minimizes

$$F(X) := \int_B (|X_u \wedge X_v| + 2[Q(X), X_u, X_v])\,du\,dv$$

in the class $\{X \in \mathcal{C}^*(\Gamma) : X(B) \subset \mathcal{Z}\}$ and satisfies $X(B) \subset \operatorname{int} \mathcal{Z}$.

(ii) Now we consider an arbitrary stable H-surface X of class $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $X(\overline{B}) \subset \mathcal{Z}$. We write

$$X(w) = (X^1(w), X^2(w), X^3(w)) = (f(w), X^3(w))$$

where $f : \overline{B} \to \mathbb{R}^2$ denotes the associated planar mapping

$$f(w) := (X^1(w), X^2(w)), \quad w \in \overline{B}. \tag{10}$$

One realizes that $f|_{\partial B}$ maps ∂B homeomorphically onto $\partial \Omega$. We claim that

$$f(B) \subset \Omega. \tag{11}$$

□

Otherwise we could find a point $\tilde{w} \in B$ with $f(\tilde{w}) \notin \Omega$. Then there is a support disk S_0 of Ω at some $p' \in \partial\Omega \cap \partial S_0$ such that $f(\tilde{w}) \notin \operatorname{int} S_0$, and $\Omega \subset \operatorname{int} S_0$. Let $S_0 = \overline{B}_{r_0}(p_0)$ and consider the family $\Phi(w,\lambda)$, $w \in \overline{B}$, $\lambda \in [0,1]$, of functions

$$\Phi(w,\lambda) := |f(w) - \lambda p_0|^2, \quad w \in \overline{B},$$

which satisfy

$$\Phi(w,\lambda) \le r_0^2 \quad \text{for } w \in \partial B \text{ and } 0 \le \lambda \le 1.$$

On account of $X(B) \subset \operatorname{int} Z$ we have

$$\Phi(w,0) < r_0^2 \quad \text{for } w \in B$$

whereas $\Phi(\tilde{w}, 1) > r_0^2$. Then there is a $\lambda^* \in (0,1)$ and a point $w^* \in B$ with

$$\Phi(w^*, \lambda^*) = r_0^2 \quad \text{and} \quad \Phi(w, \lambda^*) \le r_0^2 \quad \text{on } \overline{B}. \tag{12}$$

The conformality relation $X_w \cdot X_w = 0$ implies $|\nabla X^3|^2 \le |\nabla f|^2$, and so

$$\begin{aligned} \Delta\Phi(\cdot, \lambda^*) &= 2|\nabla f|^2 + 2\langle f - \lambda^* p_0, \Delta f\rangle \\ &\ge 2|\nabla f|^2 - 2|f - \lambda^* p_0||\Delta f| \ge 2|\nabla f|^2 - 2r_0|\Delta X| \\ &\ge 2|\nabla f|^2 - 2r_0 \cdot 2h_0|X_u \wedge X_v| \ge 2|\nabla f|^2 - 2|X_u||X_v| \\ &= 2|\nabla f|^2 - |\nabla X|^2 \ge 2|\nabla f|^2 - |\nabla f|^2 - |\nabla X^3|^2 \ge 0, \end{aligned}$$

that is,

$$\Delta\Phi(\cdot, \lambda^*) \ge 0 \quad \text{in } B. \tag{13}$$

By virtue of the maximum principle we infer from (12) and (13) that $\Phi(w, \lambda^*) \equiv r_0^2$ for $w \in \overline{B}$, which evidently is not true. Thus (11) is valid.

(iii) For each point $w' \in \partial B$ with the image $p' := f(w') \in \partial\Omega$ we consider the support disk $S_0 = \overline{B}_{r_0}(p_0)$ and define the auxiliary function $\Phi : \overline{B} \to \mathbb{R}^2$ defined by

$$\Phi(w) := |f(w) - p_0|^2$$

which satisfies

$$\Phi(w) \le r_0^2 \quad \text{in } \overline{B} \quad \text{and} \quad \Phi(w') = r_0^2.$$

By the same reasoning as before we have

$$\Delta\Phi \ge 0 \quad \text{in } B.$$

Then the boundary point lemma of E. Hopf yields

$$\frac{\partial\Phi}{\partial\nu}(w') = 2\left\langle f(w') - p_0, \frac{\partial f}{\partial\nu}(w')\right\rangle > 0 \tag{14}$$

for the derivative in direction of the exterior normal ν to ∂B at $w' \in \partial B$. This immediately implies

$$\frac{\partial X}{\partial\nu}(w') \ne 0 \quad \text{for all } w' \in \partial B, \tag{15}$$

and consequently the H-surface X has no boundary branch points.

Furthermore, Φ assumes its maximum at $w' \in \partial B$. Therefore

$$\frac{\partial\Phi}{\partial\tau}(w') = 0 \tag{16}$$

holds true where $\frac{\partial}{\partial\tau}$ denotes the tangential derivative to ∂B at w'.

Equation (16) implies

$$\left\langle f(w') - p_0, \frac{\partial f}{\partial \tau}(w') \right\rangle = 0, \quad p_0 = (x_0^1, x_0^2),$$

whence

$$X_\tau^1(w') = -\lambda[X^2(w') - x_0^2], \quad X_\tau^2(w') = \lambda[X^1(w') - x_0^1] \tag{17}$$

for some $\lambda \in \mathbb{R}$.

Because of

$$|X_\tau(w')|^2 = |X_\nu(w')|^2 > 0$$

and

$$|X_\tau^3(w')|^2 \leq c|f_\tau(w')|^2$$

for some constant c we arrive at

$$|f_\tau(w')|^2 > 0. \tag{18}$$

Since f is positive-oriented it follows that (17) holds with some $\lambda > 0$, and we infer from (14) that the Jacobian

$$J_f = \det(f_u, f_v) = \det(f_\nu, f_\tau)$$

satisfies

$$J_f(w') = (X_\nu^1 X_\tau^2 - X_\tau^1 X_\nu^2)(w') = \frac{\lambda}{2}\Phi_\nu(w') > 0.$$

Thus we have found

$$J_f(w') > 0 \quad \text{for all } w' \in \partial B \tag{19}$$

which is equivalent to

$$N^3(w') = \langle N(w'), e_3 \rangle > 0 \quad \text{for all } w' \in \partial B. \tag{20}$$

Invoking the fundamental Theorem 2 of Section 5.3 on stable H-surfaces, we arrive at

$$N^3(w) = \langle N(w), e_3 \rangle > 0 \quad \text{for all } w \in \overline{B}. \tag{21}$$

(iv) Now we want to show that X has no branch points in B, using formula (21) and applying an index-sum argument to the mapping $f : \overline{B} \to \mathbb{R}^2$ (see Sauvigny [16], Chapter III).

We use the asymptotic expansion of an H-surface X at an interior branch point $w_0 \in B$ which is obtained by the Hartmann–Wintner technique (cf. Vol. 2, Chapter 3) and has the same form as for minimal surfaces: There is a vector $A \in \mathbb{C}^3$ with $A \neq 0$ and $A \cdot A = 0$, and an integer $n \geq 1$ such that

$$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n) \quad \text{as } w \to w_0. \tag{22}$$

If w_0 is a regular point of X, i.e. if $X_w(w_0) \neq 0$, then the same formula holds with $n = 0$. As explained in Section 5.1, the normal

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v)$$

satisfies

$$\lim_{w \to w_0} N(w) = |a \wedge b|^{-1}(a \wedge b) = |a|^{-2}(a \wedge b),$$

where $A = a - ib$; $a, b \in \mathbb{R}^3 \setminus \{0\}$, $|a| = |b|$, $\langle a, b\rangle = 0$. Since $H \in C^{1,\alpha}$, it follows $X \in C^{3,\alpha}(B, \mathbb{R}^3)$, and by 5.1, Theorem 1, we have: N is of the class $C^{3,\alpha}(B, \mathbb{R}^3)$ and satisfies equation (12) of 5.1. Set

$$a := (a^1, a^2, a^3), \quad b = (b^1, b^2, b^3).$$

Then (21) yields

$$a^1 b^2 - a^2 b^1 > 0. \tag{23}$$

We integrate the first two equations of (22),

$$\left.\begin{aligned} X_w^1(w) &= A^1(w - w_0)^n + o(|w - w_0|^n) \\ X_w^2(w) &= A^2(w - w_0)^n + o(|w - w_0|^n) \end{aligned}\right. \quad \text{as } w \to w_0,$$

$A^1 = a^1 - ib^1$, $A^2 = a^2 - ib^2$. This leads to

$$\begin{aligned} X^1(w) &= X^1(w_0) + \frac{1}{n+1}[A^1(w - w_0)^{n+1} + \overline{A}^1(\overline{w} - \overline{w}_0)^{n+1}] \\ &\quad + o(|w - w_0|^{n+1}), \\ X^2(w) &= X^2(w_0) + \frac{1}{n+1}[A^2(w - w_0)^{n+1} + \overline{A}^2(\overline{w} - \overline{w}_0)^{n+1}] \\ &\quad + o(|w - w_0|^{n+1}) \end{aligned}$$

as $w \to w_0$. Using polar coordinates r, φ with $w = w_0 + re^{i\varphi}$, it follows

$$\begin{aligned} X^1(w_0 + re^{i\varphi}) &= X^1(w_0) + \frac{2}{n+1}[a^1 \cos(n+1)\varphi + b^1 \sin(n+1)\varphi] r^{n+1} \\ &\quad + o(r^{n+1}), \\ X^2(w_0 + re^{i\varphi}) &= X^2(w_0) + \frac{2}{n+1}[a^2 \cos(n+1)\varphi + b^2 \sin(n+1)\varphi] r^{n+1} \\ &\quad + o(r^{n+1}) \end{aligned}$$

as $r \to 0$.

When $l : \mathbb{C} \to \mathbb{C}$ denotes the mapping given by the matrix

$$\frac{2}{n+1}\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix}, \tag{24}$$

we obtain for $f(w) = X^1(w) + iX^2(w)$ the expansion

$$f(w) = f(w_0) + l((w - w_0)^{n+1}) + o(|w - w_0|^{n+1}) \quad \text{as } w \to w_0. \tag{25}$$

From (23)–(25) we infer that

$$f(w) \neq f(w_0) \quad \text{for } 0 < |w - w_0| < \epsilon \ll 1.$$

Furthermore, the "topological index" $i(f, w_0)$ of f at w_0 is given by

$$i(f, w_0) = n + 1. \tag{26}$$

The mapping $f : B \to \mathbb{C}$ is open and satisfies (11): $f(B) \subset \Omega$. Since $f|_{\partial B}$ yields a homeomorphism of ∂B onto $\partial \Omega$ and $f \in C^0(\overline{B}, \mathbb{R}^2)$, $\mathbb{R}^2 \hat{=} \mathbb{C}$, it follows that $f(B) = \Omega$. Then an arbitrarily chosen point $z^* \in \Omega$ has at least one and at most finitely many pre-images $w_1, \ldots, w_k$ in B, i.e.

$$f(w_\nu) = z^* \quad \text{for } \nu = 1, \ldots, k.$$

As $f|_{\partial B}$ is positive-oriented, the index-sum formula yields

$$\sum_{\nu=1}^{k} i(f, w_\nu) = 1$$

which together with (26) implies $k = 1$ and $i(f, w^*) = 1$ for $w^* := w_1$. Therefore $f|_B$ is a one-to-one mapping of B onto Ω, and (23)–(25) imply that the Jacobian $J_f(w^*)$ of f at $w^* \in B$ satisfies $J_f(w^*) > 0$. Thus $f|_B$ is a diffeomorphism from B onto Ω with $J_f(w) > 0$ for all $w \in B$, i.e. $f|_B$ is orientation preserving.

(v) Now we introduce $\zeta \in C^{3,\alpha}(\overline{\Omega})$ by

$$\zeta := X^3 \circ f^{-1}, \tag{27}$$

which solves the Dirichlet problem (8). Using (1): $H_z \geq 0$, the maximum principle implies that the solution of (8) is uniquely determined; see e.g. F. Sauvigny [16], Chapter VI, pp. 365–370, or Gilbarg–Trudinger [1]. Therefore, any two stable H-surfaces within the class $\{X \in \mathcal{C}^*(\Gamma)\colon X(B) \subset \mathcal{Z}\}$ coincide.

Remark 2. Mutatis mutandis, Theorem 1 remains valid if the bounding contour Γ is allowed to creep vertically along the z-axis finitely many times. The planar map f then possesses finitely many intervals of constancy on ∂B which correspond to the creeping intervals of $X|_{\partial B}$. However, the parametric H-surface X has no branch points on $\overline{B}$ and is uniquely determined within the class of stable H-surfaces $\in \mathcal{C}^*(\Gamma)$. The Dirichlet boundary values of $\zeta := X^3 \circ f^{-1}$ on $\partial \Omega$ jump finitely often. Even in this case one can verify the unique solvability of the Dirichlet problem (8) by an "energy method" due to J.C.C. Nitsche.

Remark 3. S. Hildebrandt and F. Sauvigny [4–7] have studied the phenomenon that minimal surfaces with a free boundary on a surface $\mathcal{S}$ having edges that may creep along such edges. This work is described in Vol. 3. Generalizations of these results to H-surfaces can be found in papers by F. Müller [5–11].

Via a simultaneous approximation of the "projection domain Ω" and the boundary values one can derive the following result from Theorem 1:

Theorem 2 (Nonparametric Dirichlet problem). *Let $\gamma \in C^0(\partial\Omega)$ be prescribed boundary values on a $2h_0$-convex Jordan domain Ω with $0 \in \Omega \subset \Omega_0$. Then the Dirichlet problem (8) possesses exactly one solution $\zeta \in C^0(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$.*

Proof. The uniqueness of a solution of (8) is proved in the same way as before, using the maximum principle. Another way to establish unique solvability of (8) is to apply Corollary 1 of Section 7.3.

Hence we only have to show the existence of a solution. This will be achieved with the aid of a suitable approximation procedure, approximating Ω by smoothly bounded Ω_n and $\gamma : \partial\Omega \to \mathbb{R}$ by smooth functions $\gamma_n : \partial\Omega_n \to \mathbb{R}$, and applying Theorem 1 to the "approximating problems"

$$\begin{aligned} \mathcal{M}\zeta_n &= 2H(\cdot, \zeta_n)(1 + |\nabla\zeta_n|^2)^{3/2} \quad \text{in } \Omega_n, \\ \zeta_n &= \gamma_n \quad \text{on } \partial\Omega_n. \end{aligned} \tag{28}$$

Let us sketch this approach.

(i) First we construct a sequence $\{\Omega_n\}$ of $2h_0$-convex domains Ω_n with $\partial\Omega_n \in C^{3,\alpha}$ and $0 \in \Omega_n \subset \Omega$ such that

$$\operatorname{dist}(\partial\Omega_n, \partial\Omega) \to 0 \quad \text{as } n \to \infty \tag{29}$$

and

$$\operatorname{length}(\partial\Omega_n) \nearrow \operatorname{length}(\partial\Omega) \quad \text{as } n \to \infty$$

(see F. Sauvigny [1,2] for details). We can write

$$\partial\Omega = \omega(I), \quad \partial\Omega_n = \omega_n(I), \quad I := [0, 2\pi]$$

where ω and ω_n are 2π-periodic mappings $\mathbb{R} \to \mathbb{R}^2$ which provide monotonic, positive-oriented representations of $\partial\Omega$ and $\partial\Omega_n$ respectively such that $\omega \in \operatorname{Lip}(\mathbb{R}, \mathbb{R}^2)$, $\omega_n \in C^{3,\alpha}(\mathbb{R}, \mathbb{R}^2)$. Using polar coordinates about the origin, we can write ω_n and ω in the form

$$\omega_n(\theta) = (r_n(\theta)\cos\theta, r_n(\theta)\sin\theta), \quad \omega(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \tag{30}$$

where $r_n(\theta)$ and $r(\theta)$ are 2π-periodic. Because of (29) we can assume that

$$\omega_n(\theta) \rightrightarrows \omega(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty; \quad \text{equivalently:} \quad r_n(\theta) \rightrightarrows r(\theta). \tag{31}$$

$\square$

Since the $2h_0$-convex curve ω fulfills a chord-arc condition, we can choose the ω_n in such a way that the ω_n satisfy a uniform chord-arc condition, i.e. there is an $\epsilon > 0$ and an $M_0 > 0$ such that

$$(32) \qquad \int_{\theta_1}^{\theta_2} |\dot{\omega}_n(\theta)| d\theta \le M_0 |\omega_n(\theta_2) - \omega_n(\theta_1)| \quad \text{for all } n \in \mathbb{N}$$
$$\text{and all } \theta_1, \theta_2 \in \mathbb{R} \text{ with } \theta_1 \le \theta_2 \text{ and } |\omega_n(\theta_1) - \omega_n(\theta_2)| \le \epsilon.$$

Now we interpret the boundary values $\gamma : \partial\Omega \to \mathbb{R}$ as a continuous, 2π-periodic function $\gamma(\theta)$ of the polar angle θ, and we approximate γ uniformly on $\mathbb{R}$ by 2π-periodic functions $\gamma_n(\theta)$, $\theta \in \mathbb{R}$, which are of class $C^{3,\alpha}(\mathbb{R})$:

$$(33) \qquad \gamma_n(\theta) \rightrightarrows \gamma(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty.$$

Set

$$(34) \qquad \psi_n(\theta) := (\omega_n(\theta), \gamma_n(\theta)), \quad \psi(\theta) := (\omega(\theta), \gamma(\theta)), \quad \theta \in \mathbb{R}.$$

Then we obtain the Jordan contours

$$(35) \qquad \Gamma_n := \psi_n(I) \in C^{3,\alpha}, \quad \Gamma := \psi(I), \quad I = [0, 2\pi],$$

whose representations ψ_n and ψ satisfy

$$(36) \qquad \psi_n(\theta) \rightrightarrows \psi(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty.$$

This yields the following *auxiliary statement*: For each $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that

$$(37) \quad |\psi_n(\theta_1) - \psi_n(\theta_2)| \le \epsilon \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R} \text{ with } |\theta_1 - \theta_2| \le \delta(\epsilon),\ n \in \mathbb{N}.$$

(ii) On account of Theorem 1 we obtain: For each $n \in \mathbb{N}$ there is an $X_n \in \mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, satisfying

$$(38) \qquad \Delta X_n = 2H(X_n) X_{n,u} \wedge X_{n,v} \quad \text{and} \quad X_{n,w} \cdot X_{n,w} = 0,$$

which admits an equivalent representation

$$Z_n(x, y) = (x, y, \zeta_n(x, y)), \quad (x, y) \in \Omega_n.$$

Here $\zeta_n \in C^{3,\alpha}(\overline{\Omega}_n)$ is a solution of the equation

$$(39) \qquad \mathcal{M}\zeta_n = 2H(\cdot, \zeta_n)(1 + |\nabla \zeta_n|^2)^{3/2} \quad \text{in } \Omega_n,$$

which is obtained by

$$(40) \qquad \zeta_n = X_n^3 \circ f_n^{-1},$$

where $f_n : \overline{B} \to \mathbb{R}^2$ is a diffeomorphism from $\overline{B}$ onto $\overline{\Omega}_n$ with $f_n \in C^{3,\alpha}(\overline{B}, \mathbb{R}^2)$. By (33) we have

$$m_0 := \sup\{|\gamma_n(\theta)| : \theta \in \mathbb{R}, n \in \mathbb{N}\} < \infty.$$

Then it follows from Theorem 4 in Section 7.3 that

$$\sup_{\Omega_n} |\zeta_n| \le m_0 + h_0^{-1} \quad \text{for all } n \in \mathbb{N}, \tag{41}$$

and

$$\begin{aligned} D(X_n) &= A(X_n) = A(Z_n) \\ &\le 3 \operatorname{meas} \Omega + m_0[2h_0 \operatorname{meas} \Omega + \operatorname{length}(\partial\Omega)] =: c_0. \end{aligned} \tag{42}$$

(iii) Now we want to prove a result that will be used to prove equicontinuity of the sequence $\{X_n\}$. To this end we consider an arbitrary mapping $X = (X^1, X^2, X^3) \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ satisfying $X(B) \subset \Omega_0 \times \mathbb{R} = \operatorname{int} \mathcal{Z}$, $\mathcal{Z} = \overline{\Omega}_0 \times \mathbb{R}$, $\Omega_0 = B_{r_0}(0)$, $r_0 = (2h_0)^{-1}$, and

$$\Delta X = 2H(X) X_u \wedge X_v \quad \text{and} \quad X_w \cdot X_w = 0 \quad \text{in } B$$

with $\sup_{\mathbb{R}^3} |H| \le h_0$. Let $f := (X^1, X^2)$ be the associated planar mapping; it satisfies

$$f(B) \subset \Omega_0$$

and

$$|\nabla X|^2 \le 2|\nabla f|^2 \quad \text{in } B.$$

Lemma 1. *Let $0 < \epsilon < r_0$, $p^* \in \Omega_0$, $\Omega := \Omega_0 \cap B_\epsilon(p^*)$, G a subdomain of B, and suppose that $f(\partial G) \subset \overline{B}_\epsilon(p^*) = \{p \in \mathbb{R}^2 : |p - p^*| \le \epsilon\}$. Then we have*

$$f(G) \subset \Omega.$$

Proof. We essentially apply the same reasoning as in part (ii) of the proof of Theorem 1. Suppose that the assertion is not valid. Then there is a point $\tilde{w} \in G$ with $f(\tilde{w}) \notin \Omega$. Since Ω is $2h_0$-convex, there exists a support disk $S_0 = \overline{B}_{r_0}(p_0)$ at some point $p' \in \partial\Omega \cap \partial S_0$ such that $f(\tilde{w}) \notin B_{r_0}(p_0)$ and $\Omega \subset B_{r_0}(p_0)$. Set

$$\Phi(w, \lambda) := |f(w) - \lambda p_0|^2 \quad \text{for } w \in \overline{G} \text{ and } 0 \le \lambda \le 1.$$

For $w \in \partial G$ it follows that $|f(w) - p_0| \le r_0$ and $|f(w)| \le \epsilon$ whence

$$|f(w) - \lambda p_0| \le \lambda |f(w) - p_0| + (1 - \lambda)|f(w)| \le \lambda r_0 + (1 - \lambda)\epsilon < r_0,$$

and therefore

$$\Phi(w, \lambda) < r_0^2 \quad \text{for all } w \in \partial G \text{ and } \lambda \in [0, 1].$$

Furthermore, $f(G) \subset f(B) \subset \Omega_0 = B_{r_0}(0)$ implies

$$\Phi(w,0) < r_0^2 \quad \text{for all } w \in G,$$

and $f(\tilde{w}) \notin B_{r_0}(p_0)$ yields

$$\Phi(\tilde{w},1) > r_0^2.$$

Then there exists some $\lambda^* \in (0,1)$ and some $w^* \in G$ with

$$\Phi(w^*,\lambda^*) = r_0^2 \quad \text{and} \quad \Phi(w,\lambda^*) \le r_0^2 \quad \text{for all } w \in \overline{G}.$$

By virtue of

$$\begin{aligned} \Delta\Phi(\cdot,\lambda^*) &= 2|\nabla f|^2 + 2\langle f - \lambda^* p_0, \Delta f\rangle \\ &\ge 2\{|\nabla f|^2 - |f-\lambda^* p_0||\Delta f|\} \ge 2\{|\nabla f|^2 - r_0|\Delta X|\} \\ &\ge 2\{|\nabla f|^2 - 2h_0 r_0 |X_u \wedge X_v|\} \\ &\ge 2\{|\nabla f|^2 - |X_u||X_v|\} \ge 2\{|\nabla f|^2 - \tfrac{1}{2}|\nabla X|^2\} \ge 0, \end{aligned}$$

the function $\Phi(\cdot,\lambda^*)$ is subharmonic in G and assumes its maximum at some point $w^* \in G$. This yields $\Phi(w,\lambda^*) \equiv r_0^2$ for all $w \in G$, a contradiction to $\Phi(w,\lambda^*) < r_0^2$ for all $w \in \partial G$. □

The next result is evident:

Lemma 2. *Let G be a subdomain of B such that $\operatorname{osc}_{\partial G} X \le \epsilon$. Then there is a point $P^* = (p^*, z^*) \in \mathcal{Z}$ such that*

$$X(\partial G) \subset K_\epsilon(P^*) := \{P \in \mathbb{R}^3 \colon |P - P^*| \le \epsilon\},$$

and, in particular, $f(\partial G) \subset \overline{B}_\epsilon(p^*)$.

For $P^* = (p^*, z^*) \in \mathbb{R}^2 \times \mathbb{R}$ we introduce the *spherical box* $N_{\epsilon,\mu}(P^*)$ with $0 < \epsilon < h_0^{-1}$ and $\mu > 0$ by

$$N_{\epsilon,\mu}(P^*) := \{P = (p,z) \in \mathbb{R}^2 \times \mathbb{R} \colon |p - p^*| \le \epsilon,\ |z - z^*| \le \mu + \eta(p - p^*, \epsilon)\}$$

with

$$\eta(p - p^*, \epsilon) := \sqrt{h_0^{-2} - |p - p^*|^2} - \sqrt{h_0^{-2} - \epsilon^2} \quad \text{for } h_0 > 0$$

and $\eta := 0$ for $h_0 = 0$.

If $h_0 > 0$, the boundary of $N_{\epsilon,\mu}(P^*)$ consists of the cylinder

$$\{(p,z) \in \mathbb{R}^3 \colon |p - p^*| = \epsilon, |z - z^*| \le \mu\}$$

and the two spherical caps

$$\begin{aligned} F^+_{\epsilon,\mu}(P^*) &:= \{(p,z) \in \mathbb{R}^3 \colon |p - p^*| \le \epsilon, z = z^* + \mu + \eta(p - p^*, \epsilon)\}, \\ F^-_{\epsilon,\mu}(P^*) &:= \{(p,z) \in \mathbb{R}^3 \colon |p - p^*| \le \epsilon, z = z^* - \mu - \eta(p - p^*, \epsilon)\}. \end{aligned}$$

Lemma 3. *We have*

$$K_\epsilon(P^*) \subset N_{\epsilon,\epsilon}(P^*) \subset K_{2\epsilon}(P^*)$$

for $h_0 = 0$ *as well as for* $h_0 > 0$ *provided that* $\epsilon < r_0 = \frac{1}{2}h_0^{-1}$.

Proof. The first inclusion is evident, and the second is evident for $h_0 = 0$, hence we have to verify it for $h_0 > 0$. We may assume that $P^* = 0$.

Suppose now $P = (p, z) \in N_{\epsilon,\epsilon}(0)$, i.e. $|p|^2 \le \epsilon^2$ and $|z| \le \epsilon + \eta(p, \epsilon)$. Then

$$\begin{aligned} |z| &\le \epsilon + \sqrt{h_0^{-2} - |p|^2} - \sqrt{h_0^{-2} - \epsilon^2} \le \epsilon + \sqrt{h_0^{-2}} - \sqrt{h_0^{-2} - \epsilon^2} \\ &\le \epsilon + \frac{h_0^{-2} - (h_0^{-2} - \epsilon^2)}{\sqrt{h_0^{-2}}} = (1 + \epsilon h_0)\epsilon < \frac{3}{2}\epsilon. \end{aligned}$$

Therefore,

$$|p|^2 + z^2 \le \epsilon^2 + \tfrac{9}{4}\epsilon^2 < 4\epsilon^2,$$

and so $P \in K_{2\epsilon}(0)$. □

Lemma 4. *Let* $0 < \epsilon < r_0 = (2h_0)^{-1}$, *and suppose that* $\mathrm{osc}_{\partial G} X \le \epsilon$ *holds true for some subdomain* G *of* B. *Then we have*:

(i) *There is a point* $P^* = (p^*, z^*) \in \mathcal{Z}$ *such that*

$$X(\partial G) \subset K_\epsilon(P^*) \quad \textit{and} \quad f(\partial G) \subset \overline{B}_\epsilon(p^*).$$

(ii) *We have* $f(G) \subset B_\epsilon(p^*)$.

(iii) *Finally we obtain*

$$X(G) \subset N_{\epsilon,\epsilon}(P^*) \subset K_{2\epsilon}(P^*).$$

Proof. Assertion (i) follows from Lemma 2, and (ii) is a consequence of Lemma 1. Because of Lemma 3 it suffices to prove $X(G) \subset N_{\epsilon,\epsilon}(P^*)$. If $h_0 = 0$, this is implied by the maximum principle for harmonic mappings. Thus we may assume $h_0 > 0$. By (ii) we have $|f(w) - p^*| < \epsilon$ for $w \in G$; therefore we only have to show

$$|X^3(w) - z^*| \le \epsilon + \eta(f(w) - p^*, \epsilon) \quad \text{for all } w \in \overline{G}.$$

If this were not true, we could find a number $\mu > \epsilon$ and some point $w' \in \overline{G}$ such that

$$|X^3(w') - z^*| = \mu + \eta(f(w') - p^*, \epsilon)$$

and

$$|X^3(w) - z^*| \le \mu + \eta(f(w) - p^*, \epsilon) \quad \text{for all } w \in \overline{G}.$$

Furthermore, we infer from $X(\partial G) \subset K_\epsilon(P^*) \subset N_{\epsilon,\epsilon}(P^*)$ that

$$|X^3(w) - z^*| \le \epsilon + \eta(f(w) - p^*, \epsilon) \quad \text{for } w \in \partial G. \tag{43}$$

Thus we obtain $w' \in G$. Consequently, $X(\overline{G})$ either lies entirely below $F^+_{\epsilon,\mu}(P^*)$ or above $F^-_{\epsilon,\mu}(P^*)$ and touches the corresponding cap at some point $X(w')$ with $w' \in G$. It suffices to consider the first case. Then we have

$$X^3(w) - z^* \le \mu + \sqrt{h_0^{-2} - |f(w) - p^*|^2} - \sqrt{h_0^{-2} - \epsilon^2} \quad \text{for } w \in \overline{G}$$

and equality for $w = w'$. Setting

$$\Phi(w) := |f(w) - p^*|^2 + \left| X^3(w) - z^* - \mu + \sqrt{h_0^{-2} - \epsilon^2} \right|^2,$$

this means

$$\Phi(w) \le h_0^{-2} \quad \text{for all } w \in \overline{G} \quad \text{and} \quad \Phi(w') = h_0^{-2}, \quad w' \in G.$$

We have

$$|\Delta X| \le 2h_0 |X_u \wedge X_v| \le h_0 |\nabla X|^2$$

and

$$\Delta \Phi = 2|\nabla X|^2 + 2\langle Y, \Delta X \rangle$$

with

$$Y := \left(f - p^*, X^3 - z^* - \mu + \sqrt{h_0^{-2} - \epsilon^2} \right).$$

This yields

$$|Y(w)| = \sqrt{\Phi(w)} \le h_0^{-1} \quad \text{for } w \in \overline{G},$$

whence

$$\begin{aligned} \Delta\Phi &\ge 2|\nabla X|^2 - 2|Y||\Delta X| \\ &\ge 2|\nabla X|^2 - 2h_0^{-1} h_0 |\nabla X|^2 = 0 \quad \text{in } G. \end{aligned}$$

Thus Φ is subharmonic in G and satisfies

$$\Phi(w') = h_0^{-2} = \max_{\overline{G}} \Phi \quad \text{for some } w' \in G,$$

whence $\Phi(w) \equiv h_0^{-2}$ holds true for all $w \in \overline{G}$. This, however, is a contradiction to the property (43) which implies $\Phi(w) < h_0^{-2}$ for $w \in \partial G$ on account of $\epsilon < \mu$. □

(iv) Now we use (37), (42), and the Courant–Lebesgue lemma to make the oscillation $\operatorname{osc}_{\partial G} X_n$ of the X_n uniformly small for appropriate subdomains G of B whose boundaries are either circles or two-gons. By Lemma 4(iii), it follows that the X_n, $n \in \mathbb{N}$, are *equicontinuous* on $\overline{B}$. Furthermore, the f_n are *uniformly bounded* on B since $f_n(\overline{B}) \subset \overline{\Omega}_0$, and (40), (41) imply

$$\sup_{\overline{B}} |X_n^3| \le m_0 + h_0^{-1} \quad \text{for all } n \in \mathbb{N}.$$

(This can also be proved by a reasoning similar to (iii).) Thus the X_n are also *uniformly bounded* on $\overline{B}$. By Arzelà–Ascoli's theorem we may then assume that the X_n converge uniformly on $\overline{B}$ to some $X \in C^0(\overline{B}, \mathbb{R}^3)$, and on account of (42) we may also assume that

$$X_n \rightharpoonup X \quad \text{in } H_2^1(B, \mathbb{R}^3).$$

This implies $X \in \overline{\mathcal{C}}(\Gamma)$.

(v) From (38) we infer

$$|\Delta X_n| \le h_0 |\nabla X_n|^2 \quad \text{in } B \text{ for all } n \in \mathbb{N}.$$

In conjunction with $X_n(w) \rightrightarrows X(w)$ on $\overline{B}$, an a priori estimate due to E. Heinz yields:

For any $B' \subset\subset B$ there is a number $c(B') > 0$ such that

$$\sup_{B'} |\nabla X_n| \le c(B') \quad \textit{for all } n \in \mathbb{N} \tag{44}$$

holds true; cf. Vol. II, Section 2.2, Proposition 1.

Then we infer from (38) and (44) by a standard reasoning that

$$\|X_n\|_{C^{3,\alpha}(B', \mathbb{R}^3)} \le c^*(B', \alpha) \quad \text{for all } n \in \mathbb{N}$$

and all $B' \subset\subset B$, and we obtain $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{3,\alpha}(B, \mathbb{R}^3)$ as well as $X_n \to X$ in $C^{3,\beta}(B', \mathbb{R}^3)$, $0 < \beta < \alpha$, for all $B' \subset\subset B$ and

$$\Delta X = 2H(X) X_u \wedge X_v \quad \text{in } B.$$

Moreover, (38) yields also

$$X_w \cdot X_w = 0 \quad \text{in } B.$$

Thus X is an H-surface of class $\mathcal{C}(\Gamma)$.

Let N and N_n be the normals of X and X_n respectively. From $N_n^3(w) > 0$ on B we infer

$$N^3(w) \ge 0 \quad \text{in } B, \tag{45}$$

and Theorem 1 in Section 5.1 yields

$$\Delta N + 2pN = -2\Lambda \operatorname{grad} H(X).$$

Since $H_z \ge 0$ it follows

$$\Delta N^3 + 2pN^3 \le 0. \tag{46}$$

Invoking a reasoning due to E. Heinz [5], Lemma 6, we infer from (45) and (46) that

$$N^3(w) > 0 \quad \text{in } B. \tag{47}$$

Another possibility to verify (47) is to invoke Moser's inequality (cf. Sauvigny [16], vol. 2, p. 369).

Now we proceed as in the proof of Theorem 1 and conclude that X has no branch points in B and that $f := (X^1, X^2)$ furnishes a homeomorphic mapping from $\overline{B}$ onto $\overline{\Omega}$ which is diffeomorphic from B onto Ω, and $f \in C^{3,\alpha}(B, \mathbb{R}^2)$. Then $\zeta := X^3 \circ f^{-1}$ is of class $C^0(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ and solves the Dirichlet problem (8).

Remark 4. Since $Z_n^3 := X_n^3 \circ f_n^{-1}$ and $f_n \rightrightarrows f$ in B, one can derive the equicontinuity of the X_n^3 from formula (9) in Section 7.3.

7.2 Unique Solvability of Plateau's Problem for Contours with a Nonconvex Projection onto a Plane

In this section we consider closed Jordan curves Γ in $\mathbb{R}^3$ which possess a one-to-one projection onto a closed Jordan curve $\underline{\Gamma}$ lying in a plane Π, which we identify with $\mathbb{R}^2$. The points in $\mathbb{R}^2$ are described by $p = (x, y)$, and $P = (x, y, z)$ denote the points in $\mathbb{R}^3$.

Radó's theorem states: *If $\underline{\Gamma}$ is convex then there exists exactly one minimal surface of class $\mathcal{C}^*(\Gamma)$, and this surface is nonparametric.* The existence follows from Theorem 2 in Section 7.1, and the uniqueness was proved in Section 4.9. Inspecting this proof, we realize that only planes were used as comparison surfaces for a given minimal surface $X \in \mathcal{C}(\Gamma)$ in order to derive a nonparametric representation

$$Z(x, y) = (x, y, \zeta(x, y)), \quad (x, y) \in \overline{\Omega},$$

of X. Now we shall substitute the plane by *Scherk's first surface* from Section 3.5.6, restricted to its fundamental domain (see also Sauvigny [16], pp. 272–273). This comparison surface leads to a new uniqueness theorem for Plateau's problem in the case that $H = 0$, established by F. Sauvigny [12]. To formulate this result we first repeat the definition of Scherk's surface in a form that we will use, and then we define the *Scherkian tongs* which will replace the ordinary half-space in our considerations.

Definition 1. *For each parameter value $a > 0$ we consider the open square $Q(a) := \{(x, y) \in \mathbb{R}^2 \colon |x|, |y| < \pi/(2a)\}$, where* Scherk's surface $\mathcal{S}(a)$ *is defined as the minimal graph*

$$\mathcal{S}(a) := \{(x, y, \sigma(x, y)) \colon (x, y) \in Q(a)\} \tag{1}$$
$$\textit{with } \sigma(x, y) := \frac{1}{a}[\log \cos(ax) - \log \cos(ay)].$$

Then $\sigma_x(x,y) = -\tan(ax)$, $\sigma_y(x,y) = \tan(ay)$; the surface element

$$\omega := \sqrt{1 + \sigma_x^2 + \sigma_y^2} \tag{2}$$

and the upwards pointing unit normal

$$\Sigma := (-\sigma_x/\omega, -\sigma_y/\omega, 1/\omega) \tag{3}$$

are given by

$$\begin{aligned} \omega(x,y) &= \{1 + \tan^2(ax) + \tan^2(ay)\}^{1/2} \\ \Sigma(x,y) &= \omega^{-1}(x,y)(\tan(ax), -\tan(ay), 1) \end{aligned} \quad \text{for } (x,y) \in Q(a). \tag{4}$$

The intersection of $\mathcal{S}(a)$ and the x,z-plane is a principal-curvature line

$$\left(x, 0, \frac{1}{a}\log\cos(ax)\right), \quad |x| < \frac{\pi}{2a}, \tag{5}$$

with the oriented curvature

$$\kappa(x) = -a\cos(ax), \quad |x| < \frac{\pi}{2a}. \tag{6}$$

In the limit $a \to +0$ we obtain $\sigma(x,y) = 0$ and $Q(0) = \mathbb{R}^2$, i.e. the Scherkian surface tends to the x,y-plane $\{z = 0\}$.

Definition 2. *For all parameter values $a \geq 0$ we define the* Scherkian halfspace (*or* Scherkian tongs) $\mathcal{S}_+(a)$ *as the set*

$$\mathcal{S}_+(a) := \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in Q(a),\ x > \sigma(y,z)\}$$

whose boundary is the Scherk surface

$$\begin{aligned} \partial\mathcal{S}_+(a) &= \{(\sigma(y,z), y, z) \colon (y,z) \in Q(a)\} \\ \text{with } \sigma(y,z) &= \frac{1}{a}[\log\cos(ay) - \log\cos(az)], \end{aligned}$$

which lies over the y,z-plane.

Rotating $\mathcal{S}_+(a)$ about the z-axis such that the plane vector $e_1 = (1,0,0)$ is transformed into the vector

$$\nu = (\nu_1, \nu_2, 0) \in S^1 \times \{0\},$$

and translating the origin $0 \in \mathbb{R}^3$ into the point $P_0 = (x_0, y_0, z_0)$ of $\mathbb{R}^3$, we obtain the general Scherkian halfspace (*or* tongs)

$$\mathcal{S}_+(a, P_0, \nu) \subset \mathbb{R}^3.$$

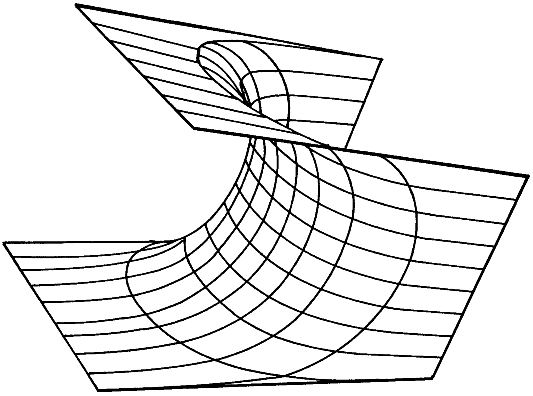

Fig. 1. Scherkian tongs

Note that the open set $\mathcal{S}_+(a, P_0, \nu)$ "emanates" from its boundary point $P_0 \in \partial\mathcal{S}_+(a, P_0, \nu)$ "in the direction ν" and possesses a square of side-length π/a as projection domain perpendicular to ν.

Now we can formulate the main result of this section, Sauvigny's uniqueness theorem.

Theorem 1. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with $\underline{\Gamma} := \partial\Omega \in C^{3,\alpha}$. Furthermore, consider boundary values $\gamma \in C^{3,\alpha}(\underline{\Gamma})$ and define the Jordan contour Γ in $\mathbb{R}^3$ by*

$$(7) \qquad \Gamma := \{(p, \gamma(p)) \in \mathbb{R}^3 : p \in \underline{\Gamma}\},$$

which has a 1–1 projection onto $\underline{\Gamma} = \partial\Omega$. Let $\nu : \partial\Omega \to S^1 \times \{0\}$ be the interior unit normal to $\partial\Omega$, and suppose that for each point $p_0 \in \partial\Omega$ there is a parameter value $a_0 = a(p_0)$ such that for $P_0 := (p_0, \gamma(p_0)) \in \Gamma$ and $\nu_0 := \nu(p_0)$ we have

$$(8) \qquad \Gamma \setminus \{P_0\} \subset \mathcal{S}_+(a_0, P_0, \nu_0).$$

As usual we fix a three-point condition $(*)$ on Γ and denote by $\mathcal{C}^*(\Gamma)$ the class of admissible surfaces $X : B \to \mathbb{R}^3$ satisfying $(*)$.

Then there exists exactly one minimal surface $X \in \mathcal{C}^*(\Gamma)$. This surface is a $C^{3,\alpha}$-immersion of $\overline{B}$ into $\mathbb{R}^3$ and possesses a nonparametric representation $(x, y, \zeta(x, y)), (x, y) \in \overline{\Omega}$, as graph of a solution $\zeta \in C^{3,\alpha}(\overline{\Omega})$ of the Dirichlet problem

$$(9) \qquad \mathcal{M}\zeta = 0 \quad \text{in } \Omega, \quad \zeta(p) = \gamma(p) \quad \text{on } \partial\Omega,$$

for the minimal surface equation.

The basic tool to be used in the proof of Theorem 1 is a *comparison principle* that will allow us to compare an arbitrary parametric minimal surface with one of Scherk's minimal graphs $\partial\mathcal{S}_+(a_0, P_0, \nu_0)$ as well as with other minimal graphs.

Theorem 2. *Let $X = (X^1, X^2, X^3) : B \to \mathbb{R}^3$ be a minimal surface with the associate planar mapping $f := (X^1, X^2) : B \to \mathbb{R}^2$, satisfying $f(B) \subset \Omega$, and the surface normal $N : B \to S^2$. Secondly, consider a solution $\eta \in C^2(\Omega)$ of the minimal surface equation $\mathcal{M}\eta = 0$ in some domain Ω of $\mathbb{R}^2$ with the normal $\Xi : \Omega \to S^2_+$ (= open upper hemisphere of S^2) and its pull-back*

$$T = (T^1, T^2, T^3) := \Xi \circ f : B \to S^2_+. \tag{10}$$

Then the auxiliary function

$$\Phi := X^3 - \eta(X^1, X^2) = X^3 - \eta \circ f \tag{11}$$

satisfies the elliptic differential equation

$$\frac{\partial}{\partial u}(T^3 \Phi_u) + \frac{\partial}{\partial v}(T^3 \Phi_v) - [e_3, T_v, N]\Phi_u - [T_u, e_3, N]\Phi_v = 0 \quad \text{in } B, \tag{12}$$

where $e_3 := (0, 0, 1)$, $T^3 = \langle T, e_3\rangle$, and $[a, b, c]$ denotes the triple product $\langle a, b \wedge c\rangle$.

Proof. (i) We have

$$\Delta X = 0 \quad \text{in } B \tag{13}$$

as well as

$$N \wedge X_u = X_v, \quad N \wedge X_v = -X_u \quad \text{in } B. \tag{14}$$

Now we consider the reparametrization

$$Y := (f, \eta \circ f) = (X^1, X^2, \eta(X^1, X^2)) \tag{15}$$

of the minimal graph $(x, y, \eta(x, y)), (x, y) \in \Omega$. Since the mean curvature of Y is identically zero on B, we obtain the parameter-invariant equation

$$Y_u \wedge T_v + T_u \wedge Y_v = 0 \quad \text{in } B, \tag{16}$$

whatever the sign and the zero set of the Jacobian $J_f = X^1_u X^2_v - X^2_u X^1_v$ might be.

(ii) Set $W := \sqrt{1 + \eta_x^2 \circ f + \eta_y^2 \circ f}$. Because of

$$\Xi = \{1 + \eta_x^2 + \eta_y^2\}^{-1/2}(-\eta_x, -\eta_y, 1)$$

we have

$$T = W^{-1} \cdot (-\eta_x \circ f, -\eta_y \circ f, 1), \quad \text{i.e.} \quad T^3 = 1/W.$$

This leads to

$$\begin{aligned}\Phi_u &= -(\eta_x \circ f)X_u^1 - (\eta_y \circ f)X_u^2 + X_u^3 = W\langle T, X_u\rangle,\\ \Phi_v &= -(\eta_x \circ f)X_v^1 - (\eta_y \circ f)X_v^2 + X_v^3 = W\langle T, X_v\rangle.\end{aligned} \tag{17}$$

Differentiating $Y = (f, X^3 + [\eta \circ f - X^3])$ we obtain

$$\begin{aligned}Y_u &= X_u + [(\eta_x \circ f)X_u^1 + (\eta_y \circ f)X_u^2 - X_u^3]e_3,\\ Y_v &= X_v + [(\eta_x \circ f)X_v^1 + (\eta_y \circ f)X_v^2 - X_v^3]e_3.\end{aligned}$$

On account of (17), we then arrive at

$$\begin{aligned}Y_u &= X_u - \Phi_u e_3,\\ Y_v &= X_v - \Phi_v e_3.\end{aligned} \tag{18}$$

From (17) we infer that the expression

$$L\Phi := \frac{\partial}{\partial u}(W^{-1}\Phi_u) + \frac{\partial}{\partial v}(W^{-1}\Phi_v) \tag{19}$$

satisfies

$$\begin{aligned}L\Phi &= \frac{\partial}{\partial u}\langle T, X_u\rangle + \frac{\partial}{\partial v}\langle T, X_v\rangle\\ &= \langle T, \Delta X\rangle + \langle T_u, X_u\rangle + \langle T_v, X_v\rangle.\end{aligned}$$

In virtue of (13) and (14) we get

$$L\Phi = [X_u, T_v, N] + [T_u, X_v, N],$$

and (18) then yields

$$\begin{aligned}L\Phi &= [Y_u + \Phi_u e_3, T_v, N] + [T_u, Y_v + \Phi_v e_3, N]\\ &= \langle Y_u \wedge T_v + T_u \wedge Y_v, N\rangle + [e_3, T_v, N]\Phi_u + [T_u, e_3, N]\Phi_v.\end{aligned}$$

By (16) it follows that

$$L\Phi = [e_3, T_v, N]\Phi_u + [T_u, e_3, N]\Phi_v. \tag{20}$$

From (19), (20), and $T^3 = 1/W$ we finally obtain (12). □

If we apply Theorem 2 to $\eta := \sigma$, defined by (1), we find:

Corollary 1. *If $X = (f, X^3) : B \to \mathbb{R}^3$, is a minimal surface with the normal N satisfying $f(B) \subset Q(a)$, then $\Phi := X^3 - \sigma \circ f$ satisfies*

$$\frac{\partial}{\partial u}(\Sigma^3\Phi_u) + \frac{\partial}{\partial v}(\Sigma^3\Phi_v) - [e_3, \Sigma_v, N]\Phi_u - [\Sigma_u, e_3, N]\Phi_v = 0 \quad \text{in } B. \tag{21}$$

Now we turn to the

Proof of Theorem 1. We proceed in four steps. First we show that any minimal surface $X \in \mathcal{C}^*(\Gamma)$ "lies above $\overline{\Omega}$", that means, $f(B) \subset \Omega$. Secondly we prove that X meets the bounding Scherkian graphs transversally. In the third step we show that a minimizer of D in $\mathcal{C}^*(\Gamma)$ possesses a nonparametric representation above Ω. Finally we use the comparison principle of Theorem 2 to identify any minimal surface $X \in \mathcal{C}^*(\Gamma)$ with this minimal graph.

(i) *Step 1* (*Inclusion Principle*). We claim that

$$f(B) \subset \Omega. \tag{22}$$

To verify this assertion we pick an arbitrary point $p_0 = (x_0, y_0) \in \partial\Omega$, set $P_0 = (p_0, \gamma(p_0))$, $a_0 = a(p_0)$, $\nu_0 = \nu(p_0)$, and note that

$$\Gamma \setminus \{P_0\} \subset \mathcal{S}_+(a_0, P_0, \nu_0)$$

an account of assumption (8). We want to show that

$$X(B) \subset \mathcal{S}_+(a_0, P_0, v_0). \tag{23}$$

By a translation in z-direction and a rotation about the z-axis we arrange for $P_0 = 0$ and $\nu_0 = e_1 = (1, 0, 0)$, and so (8) in combination with the boundary condition $X(\partial B) = \Gamma$ takes on the form

$$X(\partial B \setminus \{w_0\}) \subset \mathcal{S}_+(a_0) \quad \text{with } p_0 = f(w_0),\ w_0 \in \partial B. \tag{24}$$

Consider the auxiliary function $\Psi \in C^{3,\alpha}(\overline{B})$ which is defined by

$$\Psi(w) := X^1(w) - \sigma(X^2(w), X^3(w)) \quad \text{for } w \in \overline{B}. \tag{25}$$

This function is built in the same way as the function Φ in Corollary 1, only that the z-direction is interchanged with the x-direction. Therefore it satisfies an elliptic differential equation in B since the Scherk surface $\mathcal{S}_+(a_0)$ lies as a graph over a square $\{(y, z) \colon |y|, |z| < \pi/(2a_0)\}$ in the y, z-plane. This equation is of the same kind as (21), and by (24) we have

$$\Psi(w) > 0 \quad \text{for all } w \in \partial B \setminus \{w_0\}, \quad \text{and} \quad \Psi(w_0) = 0. \tag{26}$$

Then the maximum (or, rather, the minimum) principle yields

$$\Psi(w) > 0 \quad \text{for all } w \in B. \tag{27}$$

Thus the assumption (24) implies

$$X(B) \subset \mathcal{S}_+(a_0).$$

If we return to the original assumption (8), we obtain (23) for all $p_0 \in \partial\Omega$, and so we arrive at (22).

(ii) *Step 2* (*Transversality at the Boundary*). In the situation (26) and (27), the boundary point lemma of E. Hopf implies

$$\frac{\partial}{\partial n_0}\Psi(w_0) < 0 \quad \text{for } w_0 \in \partial B \quad \text{and} \quad n_0 = w_0, \tag{28}$$

and we also have $X(w_0) = P_0 = 0$. Without loss of generality we may assume that $w_0 = (0,1)$. Then (28) states that the function $\Psi(u,v)$ satisfies

$$\Psi_v(0,1) < 0.$$

Furthermore, we have

$$\begin{aligned}\Psi_v(0,1) &= X_v^1(0,1) - \sigma_y(0,0)X_v^2(0,1) - \sigma_z(0,0)X_v^3(0,1)\\ &= X_v^1(0,1),\end{aligned}$$

and therefore

$$X_v^1(0,1) < 0,$$

whence

$$|X_u(0,1)| = |X_v(0,1)| > 0,$$

and a reasoning analogous to that in the proof of Theorem 1 in Section 7.1 yields

$$(X_u^1 X_v^2 - X_v^1 X_u^2)(0,1) > 0.$$

Performing a rotation of B we finally obtain

$$|X_u(w_0)| = |X_v(w_0)| > 0 \quad \text{for all } w_0 \in \partial B \tag{29}$$

and

$$J_f(w_0) = (X_u^1 X_v^2 - X_v^1 X_u^2)(w_0) > 0 \quad \text{for all } w_0 \in \partial B. \tag{30}$$

This implies for the normal $N = (N^1, N^2, N^3)$ of X the inequality

$$N^3(w_0) = \langle N(w_0), e_3\rangle > 0 \quad \text{for all } w_0 \in \partial B. \tag{31}$$

Consequently, X meets the bounding Scherkian graphs transversally.

(iii) *Step 3.* Now we take a minimizer $\tilde{X}$ of D, and therefore also of A, in $\mathcal{C}^*(\Gamma)$. Then its normal $\tilde{N}$ satisfies

$$\langle \tilde{N}(w), e_3\rangle = \tilde{N}^3(w) > 0 \quad \text{on } \overline{B}$$

on account of Section 5.3, Theorem 2. Via the arguments in parts (iv) and (v) of the proof the Theorem 1 in Section 7.1, we see that the plane mapping $\tilde{f} = (\tilde{X}^1, \tilde{X}^2)$ yields a positive-oriented diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and $\tilde{\zeta} := \tilde{X}^3 \circ \tilde{f}^{-1}$ solves the boundary value problem (9).

(iv) *Step 4.* At last, we consider an *arbitrary* minimal surface $X \in \mathcal{C}^*(\Gamma)$, which might be nonstable, and compare it with the minimal graph

$$\{(x, y, \tilde{\zeta}(x, y)) \colon (x, y) \in \overline{\Omega}\}$$

that was obtained in (iii). It satisfies as well the inclusion property (22) and the transversality relations (29)–(31). To identify X with $\tilde{X}$ we consider the auxiliary function

$$\Phi := X^3 - \tilde{\zeta}(X^1, X^2) \in C^2(\overline{B})$$

from Theorem 2, which fulfills the boundary condition

$$\Phi(w) = 0 \quad \text{for all } w \in \partial B.$$

Since Φ satisfies the elliptic equation (12), we conclude that

$$\Phi(w) \equiv 0 \quad \text{on } \overline{B} \quad \Leftrightarrow \quad X^3 = \tilde{\zeta}(X^1, X^2). \tag{32}$$

This implies in particular for $f = (X^1, X^2)$ that

$$X_u^3 = \tilde{\zeta}_x(f)X_u^1 + \tilde{\zeta}_y(f)X_u^2, \quad X_v^3 = \tilde{\zeta}_x(f)X_v^1 + \tilde{\zeta}_y(f)X_v^2,$$

from which we infer in virtue of (31) that

$$N = \{(1 + \tilde{\zeta}_x^2 + \tilde{\zeta}_y^2)^{-1/2}(-\tilde{\zeta}_x, -\tilde{\zeta}_y, 1)\} \circ f$$

and therefore $N^3(w) > 0$ on $\overline{B}$. Now we conclude as in Step 4 that $f = (X^1, X^2)$ yields a positive-oriented diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and $\zeta := X^3 \circ f^{-1}$ solves (9). On the other hand, the identity (32) is equivalent to $X^3 = \tilde{\zeta} \circ f$ whence $\tilde{\zeta} = X^3 \circ f^{-1} = \zeta$. Consequently X and $\tilde{X}$ can only differ by a conformal mapping φ from $\overline{B}$ onto itself, i.e. $X = \tilde{X} \circ \varphi$, and this implies $X = \tilde{X}$ since both surfaces fulfill the same three-point condition $(*)$. This completes the proof of the theorem. □

Remark 1. In the paper [12] by Sauvigny, boundary values $\gamma : \partial\Omega \to \mathbb{R}$ are explicitly investigated for nonconvex domains with $\partial\Omega \in C^{3,\alpha}$ such that (9) is solvable. These boundary values satisfy a Lipschitz condition with a Lipschitz constant less than one.

We note that, according to a result by Osserman and Finn (see Finn [9]), (9) cannot be solved for all boundary values $\gamma \in C^0(\partial\Omega)$ if Ω is nonconvex; a detailed discussion of the pertinent results can be found in the treatise by J.C.C. Nitsche [28], §§406–411, and also §§648–653. For special classes of boundary values, a solution of the nonparametric problem (9) for nonconvex Ω was also provided by C.P. Lau [1], F. Schulz and G. Williams [1], and G. Williams [1].

Remark 2. H. Wenk [1] improved the results of this section substituting Scherk's surface by the catenoid as comparison surface. This approach is more intricate; however, multiply connected minimal surfaces are then accessible.

7.3 Miscellaneous Estimates for Nonparametric H-Surfaces

In the sequel we assume that Ω is a bounded Jordan domain in $\mathbb{R}^2$, and that $H : \mathbb{R}^3 \to \mathbb{R}$ denotes a mean curvature function of class $C^{1,\alpha}(\mathbb{R}^3)$.

We consider solutions $\zeta \in C^{3,\alpha}$ of the *nonparametric mean curvature equation* where the mean curvature is the prescribed curvature function $H(x, y, z)$, i.e. we consider nonparametric surfaces

$$\mathcal{S} := \operatorname{graph} \zeta = \{(x, y, \zeta(x, y)) \in \mathbb{R}^3 \colon (x, y)) \in \overline{\Omega}\},$$

the height function $z = \zeta(x, y)$ of which satisfies

$$\mathcal{M}\zeta(x, y) = 2H(x, y, \zeta(x, y))[1 + |\nabla\zeta(x, y)|^2]^{3/2} \quad \text{in } \Omega, \tag{1}$$

where $\mathcal{M}$ denotes the *minimal surface operator*

$$\mathcal{M}\zeta = (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1 + \zeta_x^2)\zeta_{yy}. \tag{2}$$

(Sometimes, weaker assumptions on H and X suffice.) We begin with the *Maximal Radius Theorem* due to E. Heinz [26] whose proof is almost elementary.

Theorem 1. *If there is a solution $\zeta \in C^2(\Omega)$ of (1) for a disk $\Omega = B_R(p_0)$ of radius $R > 0$, satisfying*

$$\inf_{\Omega} |H(x, y, \zeta(x, y))| \geq \beta > 0 \tag{3}$$

then it follows $R \leq 1/\beta$.

Proof. Condition (3) implies that either $H(\cdot, \zeta) > 0$ or $H(\cdot, \zeta) < 0$. The second case can be reduced to the first one by the reflection $(x, y, z) \mapsto (x, y, -z)$, and so we can assume that

$$H(x, y, \zeta(x, y)) \geq \beta > 0 \quad \text{for } (x, y) \in \Omega.$$

Let us write (1) in the form

$$\frac{\partial}{\partial x}\left(\frac{\zeta_x}{\mathcal{W}}\right) + \frac{\partial}{\partial y}\left(\frac{\zeta_y}{\mathcal{W}}\right) = 2H(\cdot, \zeta) \quad \text{in } \Omega, \quad \mathcal{W} := \sqrt{1 + \zeta_x^2 + \zeta_y^2}. \tag{4}$$

Integrating both sides over the disk $B_r := B_r(p_0)$, $0 < r < R$, we obtain

$$\begin{aligned} 2\pi r^2\beta &\leq \int_{B_r} 2H(x, y, \zeta(x, y))\, dx\, dy \\ &= \int_{\partial B_r} \left(\frac{\zeta_x}{\mathcal{W}}\, dy - \frac{\zeta_y}{\mathcal{W}}\, dx\right) \\ &\leq \int_{\partial B_r} \mathcal{W}^{-1}|\nabla\zeta|\sqrt{dx^2 + dy^2} \leq \int_{\partial B_r} ds = 2\pi r \end{aligned}$$

whence $r \leq 1/\beta$ for all $r \in (0, R)$. Letting $r \to R - 0$ we arrive at $R \leq 1/\beta$. $\square$

One can estimate the supremum of $|\zeta|$ for solutions ζ of (1) by their boundary values on sufficiently small disks; cf. F. Sauvigny [16], Vol. 2, Chap. XII, §9, Proposition 1. This is achieved by comparing the solution with a spherical cap, a technique proposed by S. Bernstein. With the aid of *Bonnet's parallel surface* from Section 5.2 we now estimate the height of solutions of (1), even on arbitrary domains, by their boundary values assuming that $H = \text{const}$. This device was used earlier by H. Liebmann to show that ovaloids of constant mean curvature are necessarily spheres. J. Serrin rediscovered Bonnet's surface in his investigation of the so-called large solutions to Plateau's problem with constant $H > 0$. We now derive Serrin's *Maximal Height Theorem* (cf. J. Serrin [5]).

Theorem 2. *Let $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of (1) for $H = \text{const} > 0$ which satisfies*

$$(5) \qquad |\zeta(x,y)| \le m \quad \textit{for all } (x,y) \in \partial\Omega$$

with a constant $m > 0$. Then ζ is estimated by

$$(6) \qquad -m - \frac{1}{H} \le \zeta(x,y) \le m \quad \textit{for all } (x,y) \in \Omega.$$

Proof. (i) we introduce conformal parameters into $Z : \overline{\Omega} \to \mathbb{R}^3$, given by $Z(x,y) := (x, y, \zeta(x,y))$, $(x,y) \in \overline{\Omega}$, using a positive-oriented uniformization map $f : \overline{B} \to \overline{\Omega}$ which is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$ and furnishes a conformal mapping from B onto Ω; see Section 4.11, or Sauvigny [16], Chapter VII, §§7–8. Set $X = (X^1, X^2, X^3) := Z \circ f$, and let N be the unit normal of X and Λ its surface element. Then $N^3 \ge 0$, and so the equation $\Delta X = 2H X_u \wedge X_v$ implies

$$\Delta X^3 = 2H(X_u^1 X_v^2 - X_v^1 X_u^2) \ge 0 \quad \text{in } B.$$

Thus X^3 is subharmonic, and therefore $X^3 = \zeta \circ f \le m$ on ∂B satisfies $X^3 \le m$ on B whence $\zeta = X^3 \circ f^{-1} \le m$ on Ω.

(ii) By Theorem 2 of Section 5.2, the parallel surface $Y := X + \frac{1}{H}N$ is again an H-surface satisfying

$$\Delta Y = 2H Y_u \wedge Y_v = -2H\Lambda(H^2 - K)N \quad \text{in } B.$$

The auxiliary function $\Phi := Y^3 = \langle Y, e_3 \rangle$ satisfies

$$\Delta\Phi = -2H\Lambda(H^2 - K)N^3 \le 0 \quad \text{in } B,$$

and so it is superharmonic in B. Since

$$\Phi(w) \ge -m + \frac{1}{H}N^3(w) \ge -m \quad \text{for } w \in \partial B,$$

we obtain $\Phi(w) \ge -m$ for $w \in B$, which means that

$$X^3(w) \geq -m - \frac{1}{H} N^3(w) \geq -m - \frac{1}{H} \quad \text{for all } w \in B$$

holds true. □

Next we derive an *Area Estimate* for nonparametric H-surfaces that repeatedly appears in the work of R. Finn.

Theorem 3. *The area $A(Z) := \int_\Omega \sqrt{1+|\nabla\zeta|^2}\, dx\, dy$ of an H-surface $Z(x,y) = (x,y,\zeta(x,y))$, corresponding to a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of (1) with $\sup_{\overline{\Omega}} |H(x,y,\zeta(x,y))| \leq h_0$, is bounded by*

$$A(Z) \leq \sup_{\partial\Omega} |\zeta| \cdot \operatorname{length}(\partial\Omega) + [1 + 2h_0 \sup_\Omega |\zeta|] \cdot \operatorname{meas} \Omega. \tag{7}$$

Proof. We shall verify (7) for domains Ω with a smooth boundary. Then the general result follows by approximation. Let us multiply (4) by ζ and integrate over Ω. Then

$$\begin{aligned} 2\int_\Omega \zeta H(\cdot,\zeta)\, dx\, dy &= \int_\Omega \zeta[(\mathcal{W}^{-1}\zeta_x)_x + (\mathcal{W}^{-1}\zeta_y)_y]\, dx\, dy \\ &= \int_\Omega [(\mathcal{W}^{-1}\zeta\zeta_x)_x + (\mathcal{W}^{-1}\zeta\zeta_y)_y]\, dx\, dy - \int_\Omega \mathcal{W}^{-1}|\nabla\zeta|^2\, dx\, dy \\ &= \int_{\partial\Omega} \mathcal{W}^{-1}\zeta(\zeta_x\, dy - \zeta_y\, dx) - \int_\Omega \mathcal{W}\, dx\, dy + \int_\Omega \mathcal{W}^{-1}\, dx\, dy. \end{aligned}$$

This leads to

$$\int_\Omega \mathcal{W}\, dx\, dy \leq \int_{\partial\Omega} |\zeta|\, ds + \operatorname{meas}\Omega + 2h_0 \int_\Omega |\zeta|\, dx\, dy \tag{8}$$

whence

$$\int_\Omega \mathcal{W}\, dx\, dy \leq \sup_{\partial\Omega} |\zeta| \cdot \operatorname{length}(\partial\Omega) + \left[1 + 2h_0 \sup_\Omega |\zeta|\right] \operatorname{meas}\Omega.$$ □

Let $\mu := \sup_{\partial\Omega} |\zeta|$ and

$$\eta^\pm := \pm\mu \pm \sqrt{h_0^{-2} - (x^2+y^2)} \pm \sqrt{h_0^{-2} - r_0^2} \quad \text{with } 0 < r_0 \leq h_0^{-1}.$$

The functions η^+ and η^- are spherical caps over $\Omega_0 := B_{r_0}(0)$. If $0 \in \Omega \subset B_{r_0}(0)$ then

$$\left.\begin{aligned} \mathcal{M}\eta^+ &= -2h_0(1+|\nabla\eta^+|^2)^{3/2} \leq 2H(\cdot,\eta^+)(1+|\nabla\eta^+|^2)^{3/2} \\ \mathcal{M}\eta^- &= 2h_0(1+|\nabla\eta^-|^2)^{3/2} \geq 2H(\cdot,\eta^-)(1+|\nabla\eta^-|^2)^{3/2} \end{aligned}\right\} \text{ in } \Omega.$$

Assuming $H_z \geq 0$ we can deduce differential inequalities in Ω for $\phi^+ := \zeta - \eta^+$ and $\phi^- := \zeta - \eta^-$ (see e.g. Sauvigny [16], Vol. 1, Chapter VI, §2). Then the maximum principle yields

$$\eta^-(w) \le \zeta(w) \le \eta^+(w) \quad \text{for } w \in \Omega.$$

Therefore

$$|\zeta(w)| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} + \sqrt{h_0^{-2} - r_0^2} \quad \text{for all } w \in \Omega. \tag{9}$$

In the maximal situation $r_0 = h_0^{-1}$ we attain

$$|\zeta(w)| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} \quad \text{in } \Omega.$$

In conjunction with Theorem 3 we obtain:

Theorem 4. *Suppose that* $0 \in \Omega \subset \Omega_0 = B_{r_0}(0)$, $r_0 = 1/h_0$, $H_z(x, y, z) \ge 0$, $|H| \le h_0$, *and let* $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *be a solution of* (1). *Then we have*

$$\sup_{\Omega} |\zeta| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} \tag{10}$$

and

$$\int_\Omega \sqrt{1 + |\nabla\zeta|^2}\, dx\, dy \le 3 \operatorname{meas} \Omega + [2h_0 \operatorname{meas} \Omega + \operatorname{length}(\partial\Omega)] \sup_{\partial\Omega} |\zeta|. \tag{11}$$

Theorem 5 (Gradient estimates for H-graphs). *Suppose that* $H \in C^{1,\alpha}(\mathbb{R}^3)$ *satisfies*

$$H_z(x, y, z) \ge 0 \quad \text{in } \mathbb{R}^3, \quad \sup_{\mathbb{R}^3} |H| \le h_0, \quad \sup_{\mathbb{R}^3} |\nabla H| \le h_1, \tag{12}$$

with positive constants h_0 *and* h_1. *Furthermore let* ζ *be a solution of* (1) *with* $\sup_\Omega |\zeta| \le M$ *for constant* $M > 0$. *Then there is a constant* $M_1 = M_1(h_0 R, h_1 R^2, M R^{-1}) > 0$, *depending only on the quantities* $h_0 R$, $h_1 R^2$, $M R^{-1}$, *such that*

$$|\nabla\zeta(p_0)| \le M_1 \tag{13}$$

holds true for any $p_0 \in \Omega$ *with* $B_R(p_0) \subset\subset \Omega$.

Proof. (i) Let $B_R := B_R(p_0) \subset\subset \Omega$; then by (7)

$$\int_{B_R} \sqrt{1 + |\nabla\zeta|}\, dx\, dy \le 2\pi R M + \pi R^2 + 2h_0 M \pi R^2. \tag{14}$$

Introduce conformal parameters for $Y(x, y) := (x, y, \zeta(x, y))$, $p = (x, y) \in B_R(p_0)$, via a uniformizing mapping $f \in C^{3,\alpha}(\overline{B}, B_R)$ with $f(0) = p_0 = (x_0, y_0)$, and set $X := Y \circ f \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$. Then the resulting H-surface X satisfies

$$D(X) = A(X) = A(Y) = \int_{B_R} \sqrt{1 + |\nabla\zeta|^2}\, dx\, dy,$$

and by (14) we obtain

$$(15) \qquad 2D(X) \le 4\pi RM + 2\pi R^2 + 4\pi h_0 MR^2.$$

(ii) Now we consider the *normalized plane mapping*

$$(16) \qquad F(w) := R^{-1}(X^1(w) - x_0, X^2(w) - y_0), \quad w \in \overline{B},$$

corresponding to $X(w) = (X^1(w), X^2(w), X^3(w))$. Clearly, $F \in C^{3,\alpha}(\overline{B}, \overline{B})$, $F(0) = 0$, and F is a diffeomorphism of $\overline{B}$ onto itself, which by (16) satisfies

$$(17) \quad \begin{aligned} 2D(F) &\le 2R^{-2}D(X) \\ &\le 4\pi MR^{-1} + 2\pi + 4\pi(MR^{-1})(h_0R) =: \tau(h_0R, MR^{-1}). \end{aligned}$$

Furthermore,

$$|\Delta F| \le R^{-1}|\Delta X| \le h_0R^{-1}|\nabla X|^2 = h_0R^{-1}(|\nabla X^1|^2 + |\nabla X^2|^2 + |\nabla X^3|^2),$$

and $X_w \cdot X_w = 0$ implies

$$|\nabla X^3|^2 \le |\nabla X^1|^2 + |\nabla X^2|^2 = R^2|\nabla F|^2.$$

This leads to

$$(18) \qquad |\Delta F| \le 2h_0R|\nabla F|^2 \quad \text{in } B.$$

Now we can apply a *distortion estimate* due to E. Heinz in the form derived in F. Sauvigny [16], Chap. XII, §5, formulae (29) and (28), using (17) and (18). This yields a number $\delta = \delta(h_0R, MR^{-1}) \in (0, 1)$ such that

$$(19) \qquad |F(w)| \ge 1/2 \quad \text{for all } w \in \partial B_{1-\delta}(0),$$

and numbers $\vartheta(h_0R, MR^{-1})$ and $\lambda(h_0R, MR^{-1})$ such that

$$(20) \qquad 0 < \vartheta \le |\nabla F(w)| \le \lambda \quad \text{for all } w \in B_{1-\delta}(0).$$

(iii) Next we consider the auxiliary function $\Phi \in C^{2,\alpha}(\overline{B})$ defined by

$$(21) \qquad \Phi := N^3 = \langle N, e_3 \rangle = \Lambda^{-1}(X_u^1X_v^2 - X_u^2X_v^1).$$

Since

$$\Lambda = 2^{-1}|\nabla X|^2 \le R^2|\nabla F|^2 \quad \text{and} \quad X_u^1X_v^2 - X_u^2X_v^1 = R^2J_F,$$

we obtain

$$(22) \qquad \Phi \ge |\nabla F|^{-2}J_F > 0 \quad \text{in } B,$$

where J_F is the Jacobian of F. On account of (19) it follows that $F(B_{1-\delta}(0)) \supset B_{1/2}(0)$; therefore

$$\int_{B_{1-\delta}(0)} J_F \, du \, dv = \operatorname{meas} F(B_{1-\delta}(0)) \geq \frac{1}{4}\pi.$$

Furthermore, (20) yields $|\nabla F(w)|^{-2} \geq \lambda^{-2}$ on $B_{1-\delta}(0)$, and so (22) implies

$$\int_{B_{1-\delta}(0)} \Phi \, du \, dv \geq \frac{\pi}{4}\lambda^{-2}(h_0 R, MR^{-1}). \tag{23}$$

By Theorem 1 of Section 5.1 we have

$$\Delta\Phi = -2p\Phi - 2\Lambda H_z(X)$$

with

$$p = 2\Lambda H^2(X) - \Lambda K - \Lambda\langle \operatorname{grad} H(X), N\rangle.$$

Then,

$$-2p = -2\Lambda[2H^2(X) - K] + 2\Lambda\langle \operatorname{grad} H(X), N\rangle \leq 0 + 2\Lambda h_1$$

and

$$-2\Lambda H_z(X) \leq 0.$$

Consequently we have

$$\begin{aligned} \Delta\Phi &\leq 2(R^{-2}\Lambda)(h_1 R^2)\Phi \leq 2|\nabla F|^2(h_1 R^2)\Phi \\ &\overset{(20)}{\leq} 2\lambda^2(h_0 R, MR^{-1})(h_1 R^2)\Phi \quad \text{on } B_{1-\delta}(0). \end{aligned}$$

Setting

$$\sigma(h_0 R, MR^{-1}, h_1 R^2) := 2\lambda^2(h_0 R, MR^{-1})(h_1 R^2),$$

we arrive at

$$\Delta\Phi \leq \sigma\Phi \quad \text{in } B_{1-\delta}(0).$$

(iv) Now we apply a quantitative version of Moser's inequality that in two dimensions had already been proved by E. Heinz [5], Lemma 6′ on p. 216; see also F. Sauvigny [16], Chap. X, §5, Theorem 1. This yields

$$\Phi(0) \geq \exp\left(-\frac{1}{4}(1-\delta)^2\sigma\right)[\pi(1-\delta)^2]^{-1}\int_{B_{1-\delta}(0)} \Phi \, du \, dv.$$

If we use (23) and define $M_1(h_0 R, h_1 R^2, MR^{-1})$ by

$$M_1^{-1} := \exp\left(-\frac{1}{4}(1-\delta)^2\sigma\right)[\pi(1-\delta)^2]^{-1}\frac{\pi}{4}\lambda^{-2}(h_0 R, MR^{-1}),$$

it follows that

$$\Phi(0) \geq 1/M_1.$$

On account of

$$\Phi(0) = N^3(0) = (1 + |\nabla\zeta(p_0)|^2)^{-1/2}$$

we obtain

$$|\nabla\zeta(p_0)| \leq \sqrt{1 + |\nabla\zeta(p_0)|^2} = 1/\Phi(0) \leq M_1,$$

which gives the desired gradient estimate (13). □

Remark 1. This proof of the gradient estimate is due to F. Sauvigny [11]. We note that only the estimate of $|\nabla F|$ from above by λ in (20) was used to derive a bound for $|\nabla\zeta(p_0)|$. If one wants to obtain curvature estimates then the lower bound by ϑ in (20) is needed as well. In Sauvigny [7,8], curvature estimates are derived for solutions ζ of (1), without assuming the monotonicity condition $H_z \geq 0$.

Remark 2. If $H(x, y, z) \equiv \text{const}$, then the graph of a solution ζ of (1) represents a stable *cmc*-surface, and Section 5.5 yields an estimate for the principal curvatures in this class.

Now we prove an estimate for the difference of two solutions of (1), using a similar idea as in the proof of Theorem 2. For $H = 0$, the estimate was derived by J.C.C. Nitsche (see [28], §585). It can be applied to prove uniqueness of solutions to the Dirichlet problem for (1) with discontinuous boundary values.

Theorem 6. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with a rectifiable boundary $\partial\Omega$, and suppose that $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ are two solutions of (1). Then, for any compact subset Q of Ω and with*

$$\mu(Q) := \max\Big\{\max_Q \sqrt{1 + |\nabla\zeta_1|^2}, \max_Q \sqrt{1 + |\nabla\zeta_2|^2}\Big\} \tag{24}$$

we have

$$\int_Q |\nabla\zeta_1 - \nabla\zeta_2|^2\, dx\, dy \leq 2\mu^3(Q) \int_{\partial\Omega} |\zeta_1 - \zeta_2|\, ds, \tag{25}$$

provided that $H_z(x, y, z) \geq 0$ on $\mathbb{R}^3$.

Proof. (i) Let $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be two solutions of (1), and set

$$p_j := \frac{\partial\zeta_j}{\partial x}, \quad q_j := \frac{\partial\zeta_j}{\partial y}, \quad W_j = \sqrt{1 + p_j^2 + q_j^2}, \quad j = 1, 2.$$

By (4) we have

$$\frac{\partial}{\partial x}\left(\frac{p_j}{W_j}\right) + \frac{\partial}{\partial y}\left(\frac{q_j}{W_j}\right) = 2H(\cdot, \zeta_j) \quad \text{in } \Omega, \quad j = 1, 2.$$

This leads to

$$\frac{\partial}{\partial x}\left(\frac{p_2}{W_2} - \frac{p_1}{W_1}\right) + \frac{\partial}{\partial y}\left(\frac{q_2}{W_2} - \frac{q_1}{W_1}\right) = 2H(\cdot, \zeta_2) - 2H(\cdot, \zeta_1).$$

If we multiply this equation by $\zeta_2 - \zeta_1$, integrate over $\Omega' \subset\subset \Omega$, and perform an integration by parts, we obtain

$$
\begin{aligned}
&-\int_{\Omega'}\left[(p_2-p_1)\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)+(q_2-q_1)\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)\right]dx\,dy\\
&\quad+\int_{\partial\Omega'}(\zeta_2-\zeta_1)\left[\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)dy-\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)dx\right]\\
&=\int_{\Omega'}2(\zeta_2-\zeta_1)[H(\cdot,\zeta_2)-H(\cdot,\zeta_1)]dx\,dy,
\end{aligned}
$$

provided that $\partial\Omega'$ is piecewise smooth.

(ii) The boundary integral is estimated by

$$\left|\int_{\partial\Omega'}(\zeta_2-\zeta_1)[\ldots]\right|\le 2\int_{\partial\Omega'}|\zeta_2-\zeta_1|\,ds,$$

and we observe

$$H(x,y,z_2)-H(x,y,z_1)=H_z(x,y,\tilde z)(z_2-z_1)$$

with an intermediate value $\tilde z$. Since $H_z\ge 0$, we obtain

$$(z_2-z_1)\cdot[H(x,y,z_2)-H(x,y,z_1)]=H_z(x,y,\tilde z)(z_2-z_1)^2\ge 0,$$

and therefore

$$\int_{\Omega'}2(\zeta_2-\zeta_1)[H(\cdot,\zeta_2)-H(\cdot,\zeta_1)]\,dx\,dy\ge 0.$$

Thus we arrive at

$$\int_{\Omega'}\left[(p_2-p_1)\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)+(q_2-q_1)\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)\right]dx\,dy\le 2\int_{\partial\Omega'}|\zeta_2-\zeta_1|\,ds.$$

(iii) For $0\le t\le 1$ we set

$$p(t):=p_1+t(p_2-p_1),\quad q(t):=q_1+t(q_2-q_1),\quad W(t):=\{1+p(t)^2+q(t)^2\}^{1/2},$$

$$f(t):=(p_2-p_1)\left[\frac{p(t)}{W(t)}-\frac{p_1}{W_1}\right]+(q_2-q_1)\left[\frac{q(t)}{W(t)}-\frac{q_1}{W_1}\right].$$

Note that $f(0)=0$. By the mean value theorem there is a value $t=t(x,y)\in(0,1)$ with $f(1)=f'(t)$, and a brief calculation yields

$$W''(t)=W^{-3}(t)\{|p'(t)|^2+|q'(t)|^2+[p(t)q'(t)-q(t)p'(t)]^2\}=f'(t),$$

whence

$$f'(t)\ge W^{-3}(t)[(p_2-p_1)^2+(q_2-q_1)^2]$$

and

$$W''(t)\ge 0.$$

Therefore

$$W(t) \le \max\{W_1, W_2\} \quad \text{for } 0 \le t \le 1,$$

and consequently

$$f(1) = f'(t) \ge (\max\{W_1, W_2\})^{-3}[(p_2 - p_1)^2 + (q_2 - q_1)^2].$$

(iv) Now we choose an arbitrary compact set Q in Ω and then an open set Ω' with $\partial\Omega' \in C^1$ and $Q \subset \Omega' \subset\subset \Omega$; set

$$\mu(Q) := \max\{W_1(x,y), W_2(x,y) \colon (x,y) \in Q\}.$$

Then $D_Q(\zeta_2 - \zeta_1) := \frac{1}{2}\int_Q |\nabla\zeta_2 - \nabla\zeta_1|^2\,dx\,dy$ is estimated by

$$D_Q(\zeta_2 - \zeta_1) \le \frac{1}{2}\mu^3(Q)\int_Q f(1)\,dx\,dy \le \mu^3(Q)\int_{\partial\Omega'} |\zeta_2 - \zeta_1|\,ds.$$

Approximating Ω from the interior by domains $\Omega' \subset\subset \Omega$ such that $\Omega' \nearrow \Omega$ and length $(\partial\Omega') \to$ length$(\partial\Omega)$, we find

$$D_Q(\zeta_2 - \zeta_1) \le \mu^3(Q)\int_{\partial\Omega} |\zeta_2 - \zeta_1|\,ds. \qquad \square$$

Corollary 1. *If $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ are two solutions of (1) in a Jordan domain Ω with a rectifiable boundary which satisfy $\zeta_1 = \zeta_2$ on $\partial\Omega$, then we have $\zeta_1 = \zeta_2$.*

Proof. The estimate (25) implies $\nabla\zeta_1|_Q = \nabla\zeta_2|_Q$ for any compact Q in Ω, whence $\nabla\zeta_1(p) = \nabla\zeta_2(p)$ for all $p \in \Omega$, and therefore $\zeta_1 - \zeta_2 =$ const. Since $\zeta_1(p) = \zeta_2(p)$ for $p \in \partial\Omega$, we obtain $\zeta_1 = \zeta_2$. $\square$

Remark 3. J.C.C. Nitsche (see [28], §586) has used the technique of the proof for Theorem 6 to establish a

General Maximum Principle. *Let $\zeta_1, \zeta_2 \in C^2(\Omega \setminus A), \Omega \subset \mathbb{R}^2$, be two solutions of $\mathcal{M}\zeta = 0$ in $\Omega \setminus A$ where A is a compact set in $\mathbb{R}^2$ with $\mathcal{H}^1(A) = 0$, $\mathcal{H}^1 =$ one-dimensional Hausdorff measure. Furthermore, suppose that*

$$\lim_{p \to p_0} [\zeta_1(p) - \zeta_2(p)] \le M \quad \textit{for all } p_0 \in \partial\Omega \setminus A.$$

Then we obtain $\zeta_1 - \zeta_2 \le M$ in $\Omega \setminus A$. Furthermore, if $\zeta_1(p') - \zeta_2(p') = M$ for a single point $p' \in \Omega \setminus A$, it follows $\zeta_1(p) - \zeta_2(p) \equiv M$.

Independently and at the same time, an n-dimensional version of the maximum principle was proved by De Giorgi and Stampacchia [1] in 1965. These authors as well as Nitsche also established the following result.

General Removability Theorem. *Let Ω be a domain in $\mathbb{R}^2$, A a compact subset of Ω with $\mathcal{H}^1(A) = 0$, and $\zeta \in C^2(\Omega \setminus A)$ be a solution of $\mathcal{M}\zeta = 0$ in $\Omega \setminus A$. Then there is exactly one extension $\zeta^* \in C^2(\Omega)$ of ζ such that $\mathcal{M}\zeta^* = 0$ in Ω.*

For a proof, see J.C.C. Nitsche [28], §§591–593. This result is a powerful generalization of a celebrated theorem by L. Bers [2], published in 1951: *An isolated singularity of a solution of the minimal surface equation* $\mathcal{M}\zeta = 0$ *is removable.*

We shall now generalize this to nonparametric H-surfaces. We will remove sets of exemption points which are specified in

Definition 1. *A subset A of a domain Ω in $\mathbb{R}^2$ is called* admissible singular subset of Ω, *if it is compact and has the following covering property: For each $\epsilon > 0$ there exist $N = N(\epsilon)$ open disks $B_k := \{p \in \mathbb{R}^2 \colon |p - p_k| < r_k\}$ with $0 < r_k < \epsilon$, $B_k \subset\subset \Omega$,*

$$A \subset \bigcup_{k=1}^{N} B_k, \quad \overline{B}_k \cap \overline{B}_\ell = \emptyset \quad \textit{for } k \neq \ell \tag{26}$$

and

$$\sum_{k=1}^{N} \text{length}(\partial B_k) \leq 2\pi\epsilon. \tag{27}$$

We call $\{B_k\}_{1 \leq k \leq N}$ an ϵ-covering of A.

Remark 4. Obviously, an admissible singular A in Ω is a two-dimensional null set in $\mathbb{R}^2$ which even satisfies $\mathcal{H}^1(A) = 0$; but in addition we require $\overline{B}_k \cap \overline{B}_\ell = \emptyset$. We note that, the *regular part* $\Omega' := \Omega \setminus A$ is connected, and thus Ω' is a domain. For example, any finite subset A of Ω is admissible. Also, any compact, denumerable subset A of Ω with at most finitely many accumulation points is admissible.

We can generalize Theorem 6 in the following way:

Theorem 7. *Let A be an admissible singular subset of a Jordan domain Ω with a rectifiable boundary, $H_z(x, y, z) \geq 0$ on $\mathbb{R}^3$, and suppose that $\zeta_1, \zeta_2 \in C^0(\overline{\Omega} \setminus A) \cap C^2(\Omega \setminus A)$ are solutions of*

$$\mathcal{M}\zeta_j = 2H(\cdot, \zeta_j)W_j^3 \quad \textit{in } \Omega \setminus A, \quad W_j := \sqrt{1 + |\nabla\zeta_j|^2}. \tag{28}$$

Then we have the weighted energy estimate

$$\int_{\Omega \setminus A} \mu |\nabla\zeta_1 - \nabla\zeta_2|^2 \, dx \, dy \leq 2 \int_{\partial\Omega} |\zeta_1 - \zeta_2| \, ds \tag{29}$$

with the positive, continuous weight function $\mu : \Omega \setminus A \to \mathbb{R}$ defined by

$$\mu(x, y) := [\max\{W_1(x, y), W_2(x, y)]^{-3} \quad \textit{for } (x, y) \in \Omega \setminus A. \tag{30}$$

Corollary 2. *Let the assumptions of Theorem 7 be satisfied and suppose also that $\zeta_1 = \zeta_2$ on $\partial\Omega$. Then we have*

$$\zeta_1 = \zeta_2 \quad \text{on } \overline{\Omega} \setminus A. \tag{31}$$

Proof. The weighted estimate (29) together with (30) imply that $\nabla\zeta_1 = \nabla\zeta_2$ in $\Omega \setminus A$. Since $\Omega \setminus A$ is connected we infer $\zeta_1 - \zeta_2 = \text{const}$ on $\overline{\Omega} \setminus A$, and the boundary condition $\zeta_1|_{\partial\Omega} = \zeta_2|_{\partial\Omega}$ finally yields (31). □

As an immediate application of Corollary 2 and of Theorem 1 in Section 7.1 we obtain the following **Theorem on Removable Singularities for *H*-Graphs**.

Theorem 8. *Let A be an admissible singular subset of the domain Ω in $\mathbb{R}^2$, and $\zeta \in C^2(\Omega \setminus A)$ be a solution of*

$$\mathcal{M}\zeta = 2H(\cdot, \zeta)\{1 + |\nabla\zeta|^2\}^{3/2} \quad \text{in } \Omega \setminus A, \tag{32}$$

where H satisfies $\sup_{\mathbb{R}^3} |H| \le h_0$ and $H_z(x,y,z) \ge 0$ on $\mathbb{R}^3$.

Then ζ can be extended to a function of class $C^2(\Omega)$ which satisfies (1).

Proof. Choose $0 < \epsilon < h_0$, and let $\{B_k\}_{1 \le k \le N}$ be an ϵ-covering of A. With the aid of Theorem 2 in Section 7.1 we obtain solutions $\zeta_k \in C^0(\overline{B}_k) \cap C^2(B_k)$ of

$$\begin{aligned} \mathcal{M}\zeta_k &= 2H(\cdot, \zeta_k)\{1 + |\nabla\zeta_k|^2\}^{3/2} \quad \text{in } B_k, \\ \zeta_k &= \zeta \quad \text{on } \partial B_k. \end{aligned}$$

Corollary 2 can be applied to the pair $\{\zeta|_{\overline{B}_k}, \zeta_k\}$, and we obtain $\zeta = \zeta_k$ on $\overline{B}_k \setminus A$, $k = 1, \dots, N(\epsilon)$. Thus it follows $\zeta \in C^2(\Omega)$, and (1) is now an immediate consequence of (32). □

It remains to establish Theorem 7.

Proof of Theorem 7. (i) We first assume that $\partial\Omega \in C^1$. Then we write (28) in the form

$$\operatorname{div}(W_j^{-1}\nabla\zeta_j) = 2H(\cdot, \zeta_j) \quad \text{in } \Omega' := \Omega \setminus A, \quad j = 1, 2.$$

Subtracting the two equations from each other we obtain

$$\begin{aligned} \frac{1}{2}\operatorname{div}[(W_1^{-1}\nabla\zeta_1) - (W_2^{-1}\nabla\zeta_2)] &= H(\cdot, \zeta_1) - H(\cdot, \zeta_2) \\ &= (\zeta_1 - \zeta_2)\int_0^1 H_z(\cdot, \zeta_2 + t(\zeta_1 - \zeta_2))\,dt \quad \text{on } \Omega'. \end{aligned} \tag{33}$$

For any $\zeta \in C^2(\overline{\Omega} \setminus A)$ we define the truncated function $[\zeta]_M$, $0 < M < \infty$, by

$$[\zeta]_M(x,y) := \begin{cases} M & \text{for } \zeta(x,y) \geq M, \\ \zeta(x,y) & \text{for } |\zeta(x,y)| < M, \\ -M & \text{for } \zeta(x,y) \leq -M, \end{cases} \qquad (x,y) \in \overline{\Omega} \setminus A.$$

Clearly, $[\zeta_1 - \zeta_2]_M \in H^1_{2,\mathrm{loc}}(\Omega') \cap L_\infty(\Omega')$. Moreover we infer from (33) that

$$\begin{aligned} (34) \qquad 0 &\leq 2[\zeta_1 - \zeta_2]_M[\zeta_1 - \zeta_2] \int_0^1 H_z(\cdot, \zeta_2 + t(\zeta_1 - \zeta_2))\, dt \\ &= [\zeta_1 - \zeta_2]_M \operatorname{div}[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2] \\ &= \operatorname{div}\{[\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2] \\ &\quad - \langle \nabla[\zeta_1 - \zeta_2]_M, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle \quad \text{on } \Omega'. \end{aligned}$$

We note that the open sets

$$\Omega_M := \{(x,y) \in \Omega' : |\zeta_1(x,y) - \zeta_2(x,y)| < M\}, \quad M > 0,$$

exhaust Ω' monotonically, i.e. $\Omega_M \nearrow \Omega'$ as $M \to \infty$, in the sense that $\Omega_M \subset \Omega_{\tilde{M}}$ for $M < \tilde{M}$ and $\Omega' = \bigcup_{M=1}^\infty \Omega_M$.

(ii) Let $\epsilon > 0$, and choose an ϵ-covering $\{B_k\}_{1 \leq k \leq N}$ of A. Define the subdomain Ω_ϵ of Ω' by

$$\Omega_\epsilon := \Omega \setminus \{\overline{B}_1 \cup \cdots \cup \overline{B}_N\}.$$

The Ω_ϵ exhaust the regular domain $\Omega' = \Omega \setminus A$, i.e. $\Omega_\epsilon \to \Omega'$ for $\epsilon \to +0$, but the exhaustion need not be monotonic.

For any $\epsilon > 0$, the vector field $\eta := [\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2]$ belongs to the class $H_2^1(\Omega_\epsilon, \mathbb{R}^2) \cap C^0(\overline{\Omega}_\epsilon, \mathbb{R}^2)$. Thus we may apply an integration by parts to (34) integrated over Ω_ϵ, thereby obtaining

$$\begin{aligned} (35) \qquad &\int_{\Omega_\epsilon \cap \Omega_M} \langle \nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle\, dx\, dy \\ &\quad \leq \int_{\partial\Omega_\epsilon} \langle [\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2], \nu\rangle\, ds, \end{aligned}$$

where ν denotes the exterior unit normal to the domain Ω_ϵ, which is of class C^1. Since $\eta \in L^\infty(\Omega')$ and

$$\sum_{k=1}^{N(\epsilon)} \operatorname{length}(\partial B_k) \leq 2\pi\epsilon,$$

we infer from (35) for $\epsilon \to 0$ that

$$\begin{aligned} (36) \qquad &\int_{\Omega_M} \langle \nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle\, dx\, dy \\ &\quad \leq 2\int_{\partial\Omega} |\zeta_1 - \zeta_2|\, ds \quad \text{for all } M > 0. \end{aligned}$$

Now we could use the reasoning from part (iii) in the proof of Theorem 6 to derive the following estimate from (36):

$$(37) \qquad \int_{\Omega_M} \mu |\nabla\zeta_1 - \nabla\zeta_2| \, dx\, dy \le 2 \int_{\partial\Omega} |\zeta_1 - \zeta_2| \, ds \quad \text{for all } M > 0.$$

Instead it might be welcome to the reader if we present the following detailed computation, because it gives some geometric insight. Consider the function $F(p) = \sqrt{1 + |p|^2}$ on $\mathbb{R}^2$. Setting $p = (\alpha, \beta)$, we have $F(p) = \sqrt{1 + \alpha^2 + \beta^2}$, and the Hessian $F_{pp}(p)$ of F is given by

$$F_{pp}(p) = F^{-3}(p) C(p) \quad \text{with } C(p) := \begin{pmatrix} 1+\beta^2 & -\alpha\beta \\ -\alpha\beta & 1+\alpha^2 \end{pmatrix}.$$

With $\gamma = (\xi, \eta) \in \mathbb{R}^2$ we obtain for the quadratic form associated with $C(p)$ that

$$\begin{aligned} \langle \gamma, C(p)\gamma \rangle &= \xi^2 + \beta^2\xi^2 - 2\alpha\beta\xi\eta + \eta^2 + \alpha^2\eta^2 \\ &= \xi^2 + \eta^2 + (\alpha\eta - \beta\xi)^2 \ge \xi^2 + \eta^2 = |\gamma|^2. \end{aligned}$$

Therefore,

$$(38) \qquad \langle \gamma, F_{pp}(p)\gamma \rangle \ge F^{-3}(p)|\gamma|^2.$$

For $p_1, p_2 \in \mathbb{R}^2$ we obtain

$$F_p(p_1) - F_p(p_2) = \int_0^1 F_{pp}(p_2 + t(p_1 - p_2))(p_1 - p_2)\, dt,$$

whence by (38),

$$\begin{aligned} &\langle p_1 - p_2, F_p(p_1) - F_p(p_2) \rangle \\ &\quad = \int_0^1 \langle p_1 - p_2, F_{pp}(p_2 + t(p_1 - p_2))(p_1 - p_2) \rangle \, dt \\ &\quad \ge \left(\int_0^1 F^{-3}(p_2 + t(p_1 - p_2))\, dt \right) |p_1 - p_2|^2. \end{aligned}$$

By (38), the function $F(p)$ is convex; hence

$$F(p_2 + t(p_1 - p_2)) \le \max\{F(p_1), F(p_2)\} \quad \text{for } 0 \le t \le 1.$$

Then it follows

$$(39) \qquad \langle p_1 - p_2, F_p(p_1) - F_p(p_2) \rangle \ge [\max\{F(p_1), F(p_2)\}]^{-3} |p_1 - p_2|^2.$$

With $p_1 := \nabla\zeta_1(x, y)$ and $p_2 := \nabla\zeta_2(x, y)$ we obtain

$$F_p(p_1) = W_1^{-1}(x, y)\nabla\zeta_1(x, y), \quad F_p(p_2) = W_2^{-1}(x, y)\nabla\zeta_2(x, y),$$

and then

$$\langle \nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle \geq \mu|\nabla\zeta_1 - \nabla\zeta_2|^2.$$

In conjunction with (36), this implies (37).

Letting M tend to infinity and recalling $\Omega_M \nearrow \Omega'$, we infer with the aid of B. Levi's theorem on monotone convergence the desired inequality (29) with μ given by (30), provided that $\partial\Omega \in C^1$.

(ii) If $\partial\Omega$ is merely a rectifiable Jordan curve, we exhaust Ω by domains Ω_j with $A \subset \Omega_j \subset\subset \Omega$, $\partial\Omega_j \in C^1$, and length $(\partial\Omega_j) \to$ length $(\partial\Omega)$ as $j \to \infty$. Then the desired estimate is obtained from the estimate for Ω_j in the limit $j \to \infty$. $\square$

7.4 Scholia

1. In this chapter we presented an approach to the Dirichlet problem for the minimal surface equation $\mathcal{M}\zeta = 0$ and, more generally, for the nonparametric H-surface equation

$$\mathcal{M}\zeta = 2H(\cdot,\zeta)[1 + |\nabla\zeta|^2]^{3/2} \quad \text{in } \Omega \tag{1}$$

in two dimensions. The special feature of our method is to start with a solution X of the Plateau problem for the parametric equation

$$\Delta X = 2H(X)X_u \wedge X_v \quad \text{in } B \tag{2}$$

and then to show that X possesses an equivalent nonparametric representation $Y(x,y) = (x, y, \zeta(x,y))$ with ζ solving (1), provided that Γ is a graph above the boundary of a $2h_0$-convex domain Ω in $\mathbb{R}^2$ and that $|H| \leq h_0$ as well as $H_z \geq 0$. The transition from the parametric problem to the nonparametric one is based on the *projection theorem* by F. Sauvigny [1,2]. For the minimal surface equation this idea was invented by T. Radó [21] in the proof of his uniqueness theorem for Plateau's problem, see Section 4.9. For the general case, the uniqueness is restricted to stable H-surfaces. Sauvigny's ideas were generalized to the study of free boundary value problems for minimal surfaces (cf. S. Hildebrandt and F. Sauvigny [1–7]; see Vol. 3) and also for H-surfaces (F. Müller [5–11]).

Besides the treatises of Nitsche [28], Gilbarg and Trudinger [1], and Sauvigny [16] we also refer to the monograph on capillarity problems by R. Finn [11] as well as to later work by this author.

2. An independent proof of the removability theorem of Bers was given by R. Finn [1]. Finn's result extends to isolated singularities of solutions to equations of the minimal surface type. L. Bers [5] gave another proof of Finn's theorem using the uniformization theorem, and eventually Finn [6] strengthened Bers's method, thereby obtaining a removability for a more general type of nonlinear elliptic equations. Nitsche's removability theorem appeared first in his paper [12]. De Giorgi and Stampacchia [1] proved: *If* $\zeta \in C^2(\Omega \setminus K)$ *is a solution of the* n*-dimensional minimal surface equation in* $\Omega \setminus K$ *where* Ω *is an open set in* $\mathbb{R}^n$ *and* K *a compact subset of* Ω *with* $\mathcal{H}^{n-1}(K) = 0$, *then* u *extends to a* C^2*-solution on the whole of* Ω. L. Simon [3] showed that it is in fact only necessary for K to be a *locally compact subset* of Ω, and therefore K can extend to the boundary of Ω. Furthermore, Simon's method carries over to equations of the form

$$\sum_{j=1}^{n} D_j F_{p_j}(x, -D\zeta, 1) = H(x),$$

where $F(x,p)$ is a positive definite, elliptic Lagrangian satisfying $\lambda F(x,p) = F(x, \lambda p)$ for $\lambda > 0$. We also refer to work of M. Miranda [1], G. Anzellotti [1], and Hildebrandt and Sauvigny [8].