Chapter 6

Unstable Minimal Surfaces

In this chapter we want to show that the existence of two minimal surfaces in a closed rectifiable contour Γ, which are local minimizers of Dirichlet's integral D, guarantees the existence of a third minimal surface bounded by Γ, which is unstable, i.e. of non-minimum character. Results of this kind were first proved by M. Shiffman [2] and simultaneously by M. Morse and C. Tompkins [1,2]. Here we present an approach to the result stated above that is due to R. Courant [13] (a detailed presentation is given in his treatise [15], Chapter VI, Sections 7 and 8). Courant's method proceeds by reduction of the problem to a finite-dimensional one for a function $\Theta : T \to \mathbb{R}^n$ provided the boundary Γ is a closed polygon.[1] In Section 6.1 we describe *Courant's reduction method* in a modified version due to E. Heinz [13]. Then, in 6.2, we prove several results concerning the existence of unstable critical points for a function $f \in C^1(\Omega)$ defined on a bounded, open, connected set of $\mathbb{R}^n$. The prototype is the following theorem: *If f possesses two strict local minimizers $x_1, x_2 \in \Omega$ and satisfies $f(x) \to \infty$ as x tends to $\partial\Omega$, then there exists a third critical point x_3 which is of non-minimum type.* The proof of such a result uses a maximum-minimum principle that is nowadays called the *mountain pass lemma*.

In 6.3 this result is used to show that a polygonal contour bounds an unstable minimal surface if it bounds two surfaces which are *separated by a wall*, for instance if it spans two strict local minimizers with respect to the "strong norm"

$$\|X\|_{1,B} := \|X\|_{C^0(B,\mathbb{R}^3)} + \sqrt{D(X)}.$$

Shiffman [4] extended Courant's approach from polygons to general rectifiable contours using convergence results for the area functional A and for Dirichlet's integral D. These results are presented in Section 6.4. One kind of convergence employs the *Douglas functional* A_0 which is seen to coincide with D on harmonic mappings. The other kind of convergence uses computations and

[1] In Section 4.15, No. 5, the Courant function was denoted by d just as in Courant's original work.

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estimates which also lead to an *isoperimetric inequality for harmonic surfaces* H of class $\overline{\mathcal{C}}(\Gamma)$,

$$A(H) \le \tfrac{1}{4} L^2(\Gamma),$$

where $L(\Gamma)$ is the length of the boundary contour Γ, or more generally,

$$A(H) \le \frac{1}{4}\left(\int_{\partial B} |dX|\right)^2$$

for $H \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ if $X|_{\partial B}$ is not monotonic. Shiffman's passage to the limit from polygons to general rectifiable contours satisfying a chord-arc condition is worked out in Section 6.6, also using ideas due to Heinz [14] and a topological reasoning that we have taken from R. Jakob [1] and [2]; this part is presented in Section 6.5. It should be pointed out that in 6.6 we have to work with the weaker norm $\|X\|_{C^0(\overline{B},\mathbb{R}^3)}$ since the convergence results of 6.4 do not suffice to carry over the results for polygons to the case of general boundaries in full strength.

6.1 Courant's Function Θ

Let Γ be a simple closed polygon (i.e. a piecewise linear and closed Jordan curve) in $\mathbb{R}^3$ with $N+3$ (≥ 4) consecutive vertices

$$Q_0, A_1, \dots, A_l, Q_1, A_{l+1}, \dots, A_m, Q_2, A_{m+1}, \dots, A_N, Q_0.$$

Set $\psi_k := \frac{2k\pi}{3}$ for $k = 0, 1, 2, 3$ and consider the set T of points $t = (t^1, \dots, t^N) \in \mathbb{R}^N$ satisfying

$$\psi_0 < t^1 < \cdots < t^l < \psi_1 < t^{l+1} < \cdots < t^m < \psi_2 < t^{m+1} < \cdots < t^N < \psi_3.$$

Clearly T is a bounded, open, and convex subset of $\mathbb{R}^N$. We define $\overline{\mathcal{C}}(\Gamma)$ as in Chapter 4, and the subclass $\overline{\mathcal{C}}^*(\Gamma)$ is to consist of those $X \in \overline{\mathcal{C}}(\Gamma)$ which satisfy the three-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, with $w_k := e^{i\psi_k}$.

Definition 1. *With every $t \in T$ we associate the set*

(1) $$U(t) = \{X \in \overline{\mathcal{C}}^*(\Gamma) \colon X(e^{it^k}) = A_k,\ k = 1, \dots, N\}.$$

Then we define the **Courant function** $\Theta : T \to \mathbb{R}$ *by*

(2) $$\Theta(t) := \inf\{D(X) \colon X \in U(t)\}.$$

Proposition 1. *For every $t \in T$ there is exactly one $X \in U(t)$ such that*

$$D(X) = \Theta(t).$$

This X is harmonic in B, continuous on $\overline{B}$, and the quadratic differential $dX \cdot dX$ is holomorphic, that is, the function $f := a - ib$ with $a := |X_u|^2 - |X_v|^2$, $b := 2\langle X_u, X_v\rangle$ is holomorphic in B.

Proof. (i) The existence of a solution of *Courant's minimum problem*

$$D \to \min \quad \text{in } U(t) \tag{3}$$

is proved in the same way as the existence of a solution to Plateau's problem "$D \to \min$ in $\overline{\mathcal{C}}^*(\Gamma)$", and also the regularity properties follow in the same manner. Since $D(X) = \Theta(t)$ implies

$$\partial D(X, \lambda) = 0 \quad \text{for any } \lambda \in C_c^\infty(B, \mathbb{R}^2)$$

the function $f = a - ib$ is holomorphic in B (see Chapter 4).

(ii) Suppose now that $X_1, X_2 \in U(t)$ are two solutions of (3), i.e.

$$D(X_1) = D(X_2) = \Theta(t).$$

Set $Y_1 := \frac{1}{2}(X_1 - X_2)$, $Y_2 := \frac{1}{2}(X_1 + X_2)$. Since $U(t)$ is evidently convex we have $Y_2 \in U(t)$, and so

$$\Theta(t) \le D(Y_2).$$

The parallelogram law yields

$$D(X_1) + D(X_2) - 2D(Y_2) = 2D(Y_1)$$

whence $D(X_1 - X_2) = 0$. This implies $\nabla(X_1 - X_2) = 0$, and so $X_1 - X_2 = \text{const}$. As $X_j(w_k) = Q_k$, $k = 0, 1, 2$, for both $j = 1$ and $j = 2$, we arrive at $X_1 = X_2$. □

Definition 2. *We introduce the* **Courant mapping** $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$ *as the mapping* $t \mapsto Z(t)$ *for* $t \in T$ *where* $Z(t)$ *is the uniquely determined element in* $U(t)$ *such that*

$$\Theta(t) = D(Z(t)). \tag{4}$$

For $w = u + iv \widehat{=} (u, v) \in \overline{B}$ *we write*

$$Z(t, u, v) = Z(t, w) := Z(t)(w).$$

There is a close connection between the Courant function Θ, the Courant map Z, and the minimal surfaces bounded by Γ. In fact we shall see that the minimal surfaces of class $\overline{\mathcal{C}}^*(\Gamma)$ are in one-to-one correspondence to the critical points t of Θ, and they are given by the values $Z(t)$ of Z at these t. Precisely speaking we shall prove:

Theorem 1. (i) *The Courant function* Θ *is of class* $C^1(T)$, *and* $\Theta(t)$ *tends to infinity if* t *approaches the boundary* ∂T.

(ii) *If* X *is a minimal surface of class* $\overline{\mathcal{C}}^*(\Gamma)$ *then* $X = Z(t)$ *for exactly one* $t \in T$.

(iii) *For* $t \in T$ *the harmonic surface* $Z(t)$ *is a minimal surface if and only if* t *is a critical point of* Θ. *Thus the set of minimal surfaces in* $\overline{\mathcal{C}}^*(\Gamma)$ *is in* 1–1 *correspondence to the set of critical points of* Θ.

To verify this result we proceed in several steps. We begin by proving

Lemma 1. *Suppose that* $X \in U(t)$ *is a minimal surface. Then* X *is real analytic on* $B' := \overline{B} \setminus \{e^{is_1}, \dots, e^{is_{N+4}}\}$ *where* $s_1 < s_2 < \dots < s_{N+4}$ *stand for the* $N+4$ *parameters*

$$\psi_0 < t^1 < \dots < t^l < \psi_1 < t^{l+1} < \dots < t^m < \psi_2 < t^{m+1} < \dots < t^N < \psi_3$$

corresponding to the vertices $Q_0, A_1, \dots, A_l, Q_1, \dots, A_N, Q_0$ *of* Γ. *Moreover, transforming* X *to polar coordinates* r, φ *around the origin by* $Y(r, \varphi) := X(re^{i\varphi})$ *we find for any* $j \in \{1, 2, \dots, N+3\}$ *an orthonormal triple of constant vectors* $p_1, p_2, p_3 \in \mathbb{R}^3$ *such that*

$$(5)\quad Y(r,\varphi) = Y(1, s_j) + \alpha_1(r,\varphi)p_1 + \alpha_2(r,\varphi)p_2 + \alpha_3(r,\varphi)p_3 \quad \textit{for } re^{i\varphi} \in B'$$

and

$$(6)\qquad \alpha_1(1,\varphi) = 0, \quad a_2(1,\varphi) = 0, \quad a_{3,r}(1,\varphi) = 0 \quad \textit{for } s_j < \varphi < s_{j+1}.$$

Proof. If X is a minimal surface of class $U(t)$, the assertions follow from the reflection principle, see Section 4.8, Theorem 1. □

Proposition 2. *Any minimal surface* X *of class* $U(t)$ *coincides with the minimizer* $Y := Z(t)$ *of* D *in* $U(t)$.

Proof. Consider the domain

$$\Omega := B \Big\backslash \bigcup_{j=1}^{N+3} B_{\varepsilon_j}(\tilde{w}_j), \quad \tilde{w}_j := e^{is_j}, \ \varepsilon_j > 0,$$

with $s_1, \dots, s_{N+3}$ as in Lemma 1. For $0 < \varepsilon_j \ll 1$ the domain Ω is simply connected, and $\partial\Omega$ consists of subarcs γ_j of ∂B and of circular subarcs C_j of $\partial B_{\varepsilon_j}(\tilde{w}_j)$. For $\phi := Y - X$ we have

$$D_\Omega(Y) = D_\Omega(X) + D_\Omega(\phi) + 2D_\Omega(X, \phi),$$

and

$$D_\Omega(X, \phi) = \frac{1}{2} \int_{\partial\Omega} X_\nu \cdot \phi \, d\mathcal{H}^1, \quad \nu := \text{exterior normal to } \partial\Omega,$$

since $X \in C^2(\overline{\Omega}, \mathbb{R}^3)$, $X, \phi \in C^0(\overline{\Omega}, \mathbb{R}^3)$, and $\Delta X = 0$ in B. By Lemma 1 it follows that $X_\nu \cdot \phi = 0$ on $\gamma_1, \gamma_2, \dots, \gamma_{N+3}$ whence

$$D_\Omega(X, \phi) = \sum_{j=1}^{N+3} \int_{C_j} X_\nu \phi \, d\mathcal{H}^1$$

and therefore

$$|D_\Omega(X,\phi)| \le \text{const} \sum_{j=1}^{N+3} \int_{C_j} |X_\nu| \, d\mathcal{H}^1.$$

Choosing $\varepsilon_1, \dots, \varepsilon_{N+3}$ appropriately we can make the right hand side as small as we like (using the conformality relations and the Courant–Lebesgue Lemma, see Section 4.4), and so we arrive at

$$D(Y) = D(X) + D(Y - X) \ge D(X).$$

Since Y is assumed to be the uniquely determined minimizer of D in $U(t)$ we obtain $X = Y$. □

Proof of part (ii) of Theorem 1. Since X is of class $\overline{\mathcal{C}}^*(\Gamma)$ it satisfies the 3-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, and $X|_{\partial B}$ is (weakly) monotonic. Thus there is an n-tuple $t \in T$ such that $X \in U(t)$. By Proposition 2 it follows that $X = Z(t)$. Suppose that there is another $t' \in T$ with $t' \ne t$ such that $X = Z(t')$. Then there exist values s and s' with $0 \le s < s' < 2\pi$ such that $\gamma := \{e^{i\varphi} \colon s < \varphi < s'\}$ lies in B' and $X(1,\varphi) \equiv \text{const}$ on γ whence $\nabla X \equiv 0$ on γ because of the conformality relations. As the branch points of a nonconstant minimal surface are isolated we obtain $X(w) \equiv \text{const}$ on $\overline{B}$ which contradicts the 3-point condition. Therefore $t = t'$. □

This shows that the minimal surfaces within $\overline{\mathcal{C}}^*(\Gamma)$ are in one-to-one correspondence with a nonempty subset T_0 of T. We want to prove that T_0 is the set of critical points of Θ. This, in particular, requires to show that Θ is of class $C^1(T)$.

An important technical tool is a formula for the inner variation of Dirichlet's integral, for which we shall state a certain generalization. First we introduce an important class of diffeomorphisms $\sigma_\varepsilon = \sigma(\cdot, \varepsilon)$ of $\overline{B}$ onto itself that was already used in Chapter 4; see 4.5, Supplementary Remarks.

Lemma 2. *There exist two constants $\delta_0 > 0$ and $\kappa_0 > 0$ with the following properties*:

(i) *For every $\varepsilon \in (-2, 2)$ and any real-valued function $\mu \in C^1(\overline{B})$ with $|\mu|_{C^1(\overline{B})} < \delta_0$, the mapping $\tau_\varepsilon = \tau(\cdot, \varepsilon)$ of $\overline{B}$ into $\mathbb{R}^2$ defined by*

$$\tau_\varepsilon(w) = \tau(w, \varepsilon) := w e^{i\varepsilon\mu(w)}, \quad w \in \overline{B}, \tag{7}$$

is a C^1-diffeomorphism of $\overline{B}$ onto itself which maps any circle $C_r := \{w \in \mathbb{C} \colon |w| = r\}$, $0 < r \le 1$, onto itself, in particular $\sigma_\varepsilon(\partial B) = \partial B$. Denote by $\sigma_\varepsilon = \sigma(\cdot, \varepsilon) := \tau_\varepsilon^{-1}$ the inverse mapping to τ_ε. If we view $w \mapsto \sigma_\varepsilon(w)$ and $\omega \mapsto \tau_\varepsilon(w)$ as one-parameter families of diffeomorphisms from $\overline{B}$ onto itself, we have

$$\begin{aligned} &\tau_\varepsilon(w) = w - \varepsilon\lambda(w) + \rho(w, \varepsilon), \\ &\lambda(w) = -iw\mu(w), \quad \rho(w, \varepsilon) = \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} w i^n \mu^n(w). \end{aligned} \tag{8}$$

Writing $\lambda(w) = \lambda^1(w) + i\lambda^2(w)$ *in the real form* $\lambda(u,v) = (\lambda^1(u,v), \lambda^2(u,v))$ *we obtain*

$$\lambda^1(u,v) = v\mu(u,v), \quad \lambda^2(u,v) = -u\mu(u,v). \tag{9}$$

Clearly $\lambda(u,v)$ *is tangential to* ∂B *at* $w = (u,v)$.

(ii) *For* $X \in \mathcal{C}(\Gamma) \cap C^1(B, \mathbb{R}^3)$, $|\varepsilon| < 2$ *and* $|\mu|_{C^1(\overline{B})} < \delta_0$ *we can represent* $D(X \circ \sigma_\varepsilon)$ *in the following way*:

$$D(X \circ \sigma_\varepsilon) = D(X) + \varepsilon \partial D(X, \lambda) + \varepsilon^2 R(X, \mu) \tag{10}$$

with

$$\begin{aligned} \partial D(X,\lambda) = & \frac{1}{2}\int_B [a(\lambda_u^1 - \lambda_v^2) + b(\lambda_v^1 + \lambda_u^2)]\, du\, dv, \\ & a := |X_u|^2 - |X_v|^2,\ b := 2\langle X_u, X_v\rangle, \end{aligned} \tag{11}$$

and

$$|R(X,\mu)| \le \kappa_0 D(X) |\mu|^2_{C^1(\overline{B})}. \tag{12}$$

Furthermore we can write $\partial D(X,\lambda)$ *in the form*

$$\partial D(X,\lambda) = 4\int_B \operatorname{Im}[w\mu_{\overline{w}} X_w \cdot X_w]\, du\, dv =: V(X,\mu). \tag{13}$$

Proof. Part (i) is fairly obvious and can be left to the reader. Assertion (ii) follows by the computations of Section 4.5. Note that the functions φ, μ, ν in 4.5, (26)–(28) have in (10)–(13) been replaced by μ, λ^1, λ^2 respectively. Thus we have

$$\tau_\varepsilon(w) = we^{i\varepsilon\mu(w)} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} w i^n \mu^n(w) = w - \varepsilon\lambda(w) + \cdots \quad \text{as } \varepsilon \to 0,$$
$$\sigma_\varepsilon(\omega) = \omega + \varepsilon\lambda(\omega) + \cdots \quad \text{as } \varepsilon \to 0, \quad \lambda(w) = \lambda^1(w) + i\lambda^2(w) = -iw\mu(w),$$

i.e.

$$\lambda^1(u,v) = v\mu(u,v), \quad \lambda^2(u,v) = -u\mu(u,v),$$

and so

$$\begin{aligned} \lambda_u^1 &= v\mu_u, & \lambda_v^1 &= \mu + v\mu_v, \\ \lambda_u^2 &= -\mu - u\mu_u, & \lambda_v^2 &= -u\mu_v, \end{aligned}$$

whence

$$a(\lambda_u^1 - \lambda_v^2) + b(\lambda_v^1 + \lambda_u^2) = a(v\mu_u + u\mu_v) + b(v\mu_v - u\mu_u)$$

and

$$2w\mu_{\overline{w}} = (u+iv)(\mu_u + i\mu_v) = (u\mu_u - v\mu_v) + i(v\mu_u + u\mu_v).$$

In conjunction with $4X_w \cdot X_w = a - ib$ the last equation yields

$$\begin{aligned} 8\,\mathrm{Im}[w\mu_{\overline{w}} X_w \cdot X_w] &= a(u\mu_v + v\mu_u) + b(v\mu_v - u\mu_u) \\ &= a(\lambda^1_u - \lambda^2_v) + b(\lambda^1_v + \lambda^2_u). \end{aligned}$$

Therefore equation (13) is equivalent to (11). □

Proposition 3. *We have $\Theta \in C^0(T)$.*

Proof. For $t_p = (t^1_p, \ldots, t^N_p) \in T$ with $t_p \to t \in T$ as $p \to \infty$, $t = (t^1, \ldots, t^N)$ we have to show that $\Theta(t_p) \to \Theta(t)$. To this end we choose functions $\nu_n \in C^1(\overline{B})$, $n = 1, \ldots, N$, satisfying

$$(14) \qquad \nu_n(w_k) = 0, \quad k = 0, 1, 2, \quad \text{and} \quad \nu_n(\zeta_j) = \delta_{nj}, \quad \zeta_j := e^{it^j},$$

δ_{nj} = Kronecker symbol, and set

$$(15) \qquad \mu_p(w) := \sum_{n=1}^{N} (t^n_p - t^n)\nu_n(w) \quad \text{for } w \in \overline{B}.$$

Then $\mu_p \in C^1(\overline{B})$ and $|\mu_p|_{C^1(\overline{B})} \to 0$ as $p \to \infty$, in particular

$$|\mu_p|_{C^1(\overline{B})} < \delta_0 \quad \text{for } p \gg 1.$$

Hence the mappings $\sigma_p := \tau_p^{-1}$ with τ_p defined by

$$(16) \qquad \tau_p(w) := we^{i\mu_p(w)}, \quad w \in \overline{B},$$

satisfy the assumptions of Lemma 2 for $p \gg 1$, and $\sigma_p(w_k) = w_k$ and $\sigma_p(\zeta^p_j) = \zeta_j$, $\zeta^p_j := e^{it^j_p}$, whence $X_p := Z(t) \circ \sigma_p \in U(t_p)$. Therefore

$$\Theta(t_p) \le D(X_p), \quad \Theta(t) = D(Z(t)),$$

and by Lemma 2 we obtain

$$(17) \qquad \Theta(t_p) \le \Theta(t) + \kappa D(Z(t))|\mu_p|^2_{C^1(\overline{B})}$$

for $p \gg 1$ and some constant $\kappa > 0$. Since $\Theta(t_p) = D(Z(t_p))$ it follows that

$$(18) \qquad D(Z(t_p)) \le \Theta(t)[1 + \kappa\delta_0^2] \quad \text{for } p \gg 1.$$

On the other hand, replacing μ_p by μ'_p with

$$\mu'_p(w) := \sum_{n=1}^{N} (t^n - t^n_p)\nu^p_n(w)$$

with $\nu_n^p(w_k) = 0$, $0 \le k \le 2$, $\nu_n^p(\zeta_j^p) = \delta_{nj}$ for $1 \le j, n \le N$ and $\nu_n^p \in C^1(\overline{B})$ as well as $|\nu_n^p|_{C^1(\overline{B})} \le c$ for some constant c and all $p \in \mathbb{N}$, $1 \le n \le N$, we consider $\sigma_p' := (\tau_p')^{-1}$ with

$$\tau_p'(w) := we^{i\mu_p'(w)} \quad \text{for } w \in \overline{B}.$$

Then $Y_p := Z(t_p) \circ \sigma_p' \in U(t)$ whence

$$\Theta(t) \le D(Y_p), \quad \Theta(t_p) = D(Z(t_p)),$$

and Lemma 2 yields

$$\Theta(t) \le \Theta(t_p) + \kappa D(Z(t_p))|\mu_p'|^2_{C^1(\overline{B})} \quad \text{for } p \gg 1 \tag{19}$$

with the same κ as in (17). On account of (17)–(19) we arrive at

$$|\Theta(t) - \Theta(t_p)| \le \kappa\Theta(t)\Big\{|\mu_p|^2_{C^1(\overline{B})} + [1 + \kappa\delta_0^2]|\mu_p'|^2_{C^1(\overline{B})}\Big\}$$

for $p \gg 1$. Since $|\mu_p|_{C^1(\overline{B})} \to 0$ and $|\mu_p'|_{C^1(\overline{B})} \to 0$ as $p \to \infty$ we obtain

$$\Theta(t_p) \to \Theta(t) \quad \text{as } t_p \to t. \tag{20}$$

□

Because of (18) and $|Z(t_p)|_{C^0(\overline{B},\mathbb{R}^3)} \le \text{const}$ for $p \in \mathbb{N}$, it follows that

$$|Z(t_p)|_{H_2^1(B,\mathbb{R}^3)} \le \text{const} \quad \text{for all } p \in \mathbb{N}.$$

Then, for any subsequence of $\{Z(t_p)\}$ we may extract another subsequence $\{Z(t_{p_k})\}$ such that $Z(t_{p_k}) \rightharpoonup Y$ in $H_2^1(B,\mathbb{R}^3)$ for some $Y \in H_2^1(B,\mathbb{R}^3)$. By the Courant–Lebesgue Lemma and the maximum principle we may also assume that

$$|Y - Z(t_{p_k})|_{C^0(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } k \to \infty.$$

This implies $Y \in U(t) \cap C^0(\overline{B},\mathbb{R}^3)$; therefore $\Theta(t) \le D(Y)$. On the other hand we infer from $Z(t_{p_k}) \rightharpoonup Y$ in $H_2^1(B,\mathbb{R}^3)$ that

$$D(Y) \le \varliminf_{k\to\infty} D(Z(t_{p_k})) = \lim_{k\to\infty} \Theta(t_{p_k}) = \Theta(t).$$

Thus we have $D(Y) = \Theta(t)$. In virtue of Proposition 1 and Definition 2 it follows that $Y = Z(t)$, whence $Z(t_{p_k}) \to Z(t)$ in $H_2^1(B,\mathbb{R}^3)$. By a standard reasoning we obtain

$$\big|Z(t) - Z(t_p)\big|_{H_2^1(B,\mathbb{R}^3)} + \big|Z(t) - Z(t_p)\big|_{C^0(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } t_p \to t, \tag{21}$$

and well-known estimates for harmonic mappings yield

$$\big|Z(t) - Z(t_p)\big|_{C^s(\Omega,\mathbb{R}^3)} \to 0 \quad \text{as } t_p \to t \tag{22}$$

for any $\Omega \subset\subset B$ and any $s \in \mathbb{N}$. Thus we have found:

Proposition 4. *The Courant mapping* $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$ *is continuous in the sense of* (21) *and* (22).

Lemma 3. *Suppose that* $\mu \in C^1(\overline{B})$ *satisfies* $|\mu|_{C^1(\overline{B})} < \delta_0$ *and* $\mu(w_k) = 0$, $k = 0, 1, 2$, *where* δ_0 *is the constant from Lemma* 2. *Then for every* $t = (t^1, \ldots, t^N) \in T$ *and* $u = (u^1, \ldots, u^N)$ *with* $u^j := \mu(\zeta_j)$ *and* $\zeta_j = e^{it^j}$ *we have*

$$\Theta(t + u) \le \Theta(t) + V(Z(t), \mu) + R(Z(t), \mu) \tag{23}$$

with

$$\big|R(Z(t), \mu)\big| \le \kappa_0 D(Z(t))|\mu|^2_{C^1(\overline{B})}. \tag{24}$$

Proof. Consider the diffeomorphism $\sigma = \tau^{-1}$ defined by $\tau(w) := we^{i\mu(w)}$, and let $Z(t)$ be the minimizer of D in $U(t)$. Then $Z' := Z(t) \circ \sigma \in U(t + u)$ and (10)–(13) implies

$$D(Z') = D(Z(t)) + V(Z(t), \mu) + R(Z(t), \mu)$$

where $R(Z(t), \mu)$ is estimated by (24), see Lemma 2. Furthermore $\Theta(t) = D(Z(t))$ and $\Theta(t + u) \le D(Z')$, and so we obtain (23). □

Proposition 5. *For any* $t \in T$ *and any* $u \in \mathbb{R}^N$ *the directional derivative*

$$\frac{\partial}{\partial u}\Theta(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\Theta(t + \varepsilon u) - \Theta(t)] \tag{25}$$

exists and can be computed as follows: Choose some real-valued function $\mu \in C^1(\overline{B})$ *satisfying* $\mu(w_k) = 0$, $k = 0, 1, 2$, *and* $\mu(\zeta_j) = u^j$, $1 \le j \le N$, *for* $\zeta_j := \exp(it^j)$. *Then*

$$\frac{\partial}{\partial u}\Theta(t) = V(Z(t), \mu). \tag{26}$$

Setting $\lambda(w) := -iw\mu(w)$ *we can equivalently write*

$$\frac{\partial}{\partial u}\Theta(t) = \partial D(Z(t), \lambda). \tag{27}$$

Proof. Given $t \in T$ and $u \in \mathbb{R}^N$ we choose a function $\mu \in C^1(\overline{B})$ satisfying

(I) $\mu(w_k) = 0$, $k = 0, 1, 2$, and $\mu(\zeta_j) = u^j$ for $\zeta_j := e^{it^j}$, $1 \le j \le N$.

(II_r) There is some number $r > 0$ such that

$$\mu(w) \equiv \mu(\zeta_j) \quad \text{for } w \in \overline{B} \cap B_r(w_j),\ 1 \le j \le N.$$

Then we choose some $\varepsilon_0 > 0$ such that

$$\varepsilon_0 |\mu|_{C^1(\overline{B})} < \delta_0.$$

If we apply Lemma 3 to t and εu (instead of t and u) we obtain

$$\Theta(t+\varepsilon u) \le \Theta(t) + \varepsilon V(Z(t),\mu) + R(Z(t),\varepsilon\mu)$$

provided that $|\varepsilon| < \varepsilon_0$. Here we only have used (I). To establish the next inequality we employ (II_r). For this purpose we apply Lemma 3 to $t' := t+\varepsilon u$ and $-\varepsilon u$ instead of t and u respectively. Then

$$\Theta(t) \le \Theta(t+\varepsilon u) - \varepsilon V\big(Z(t+\varepsilon u),\mu) + R(Z(t+\varepsilon u), -\varepsilon\mu)$$

for $|\varepsilon| < \varepsilon_1(r)$ and some $\varepsilon_1(r) \in (0,\varepsilon_0)$. Thus, for $0 < |\varepsilon| < \varepsilon_1(r)$,

$$\begin{aligned} &\big|\Theta(t+\varepsilon u) - \Theta(t) - \varepsilon V\big(Z(t),\mu)\big| \\ &\quad \le |\varepsilon|\big|V(Z(t+\varepsilon u),\mu) - V(Z(t),\mu)\big| + \big|R(Z(t),\varepsilon\mu\big| + \big|R(Z(t+\varepsilon u),-\varepsilon\mu)\big|. \end{aligned}$$

By (13) and (21) we see that

$$\big|V(Z(t+\varepsilon u),\mu) - V(Z(t),\mu)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Furthermore we have $|Z(t+\varepsilon u)|_{H_2^1(B,\mathbb{R}^3)} \le c = \text{const}$ for $|\varepsilon| \ll 1$, and so

$$|\varepsilon|^{-1}\big\{\big|R(Z(t),\varepsilon\mu)\big| + \big|R(Z(t+\varepsilon u),-\varepsilon\mu)\big|\big\} \le 2c|\mu|^2_{C^1(\overline{B})}\cdot|\varepsilon| \quad \text{for } |\varepsilon| \ll 1.$$

We then conclude that

$$\left|\frac{1}{\varepsilon}[\Theta(t+\varepsilon u) - \Theta(t)] - V(Z(t),\mu)\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we have proved (26) for functions $\mu \in C^1(\overline{B})$ satisfying (I) and (II_r). In order to show that (II_r) is superfluous we approximate a given $\mu \in C^1(\overline{B})$ satisfying (I) by functions $\mu_p \in C^1(\overline{B})$ satisfying (I) and (II_{r_p}) with $r_p \to +0$, as well as

$$|\mu - \mu_p|_{C^1(\Omega)} \to 0 \quad \text{for } p \to \infty \text{ and any } \Omega \subset\subset B.$$

By (13) we obtain

$$V(Z(t),\mu_p) \to V(Z(t),\mu) \quad \text{as } p \to \infty.$$

On the other hand we have already proved that $\frac{\partial}{\partial u}\Theta(t)$ exists and that

$$\frac{\partial}{\partial u}\Theta(t) = V(Z(t),\mu_p) \quad \text{for all } p \in \mathbb{N}.$$

Letting p tend to infinity we obtain (26) for any $\mu \in C^1(\overline{B})$ satisfying (I). □

Proposition 6. *The Courant function Θ is of class $C^1(T)$. Moreover, for any $t = (t^1,\dots,t^N) \in T$, $\zeta_j = \exp(it^j)$, and any $\mu \in C^1(\overline{B})$ satisfying $\mu(w_k) = 0$, $k = 0,1,2$, we have*

(28) $$\sum_{j=1}^{N} \Theta_{t^j}(t)\mu(\zeta_j) = V(Z(t), \mu)$$

and, equivalently, for $\lambda(w) := -wi\mu(w)$:

(29) $$\sum_{j=1}^{N} \Theta_{t^j}(t)\mu(\zeta_j) = \partial D(Z(t), \lambda).$$

Proof. Let $u = (1, 0, \ldots, 0)$ and choose $\mu, \mu' \in C^1(\overline{B})$ with

$$\mu(w_k) = 0, \quad \mu'(w_k) = 0, \quad k = 0, 1, 2,$$

and

$$\mu(\zeta_1) = \mu'(\zeta_1') = 1, \quad \mu(\zeta_j) = \mu'(\zeta_j') = 0 \quad \text{for } 2 \le j \le N,$$

where $\zeta_j = \exp(it^j)$, $\zeta_j' = \exp(it'^j)$. According to (26) we have

$$\frac{\partial \Theta}{\partial t^1}(t) = V(Z(t), \mu), \quad \frac{\partial \Theta}{\partial t^1}(t') = V(Z(t'), \mu'),$$

and Proposition 4 yields $|Z(t') - Z(t)|_{H_2^1(B,\mathbb{R}^3)} \to 0$ as $t' \to t$.

Furthermore we can construct μ' as a one-parameter family of functions $\mu'(t', \cdot) \in C^1(\overline{B})$, $|t' - t| \ll 1$, with

$$\lim_{t' \to t} \left| \mu'(t', \cdot) - \mu \right|_{C^1(\overline{B})} = 0.$$

Then we obtain

$$\lim_{t' \to t} \left| \Theta_{t^1}(t') - \Theta_{t^1}(t) \right| = 0,$$

i.e. $\Theta_{t^1} \in C^0(T)$, and similarly it follows that $\Theta_{t^2}, \ldots, \Theta_{t^N} \in C^0(T)$. Hence we have proved that $\Theta \in C^1(T)$, and this implies

$$\frac{\partial \Theta}{\partial u}(t) = \sum_{j=1}^{N} \Theta_{t^j}(t) u_j.$$

On account of Proposition 5, we now obtain (28) and (29). □

Proposition 7. *We have* $\Theta(t) \to \infty$ *as* $\operatorname{dist}(t, \partial T) \to 0$.

Proof. Otherwise there is a sequence $\{t_p\}$ of points $t_p \in T$ with $t_p \to t \in \partial T$ and $\Theta(t_p) = D(Z(t_p)) \le \text{const}$, and by the Courant–Lebesgue Lemma we may assume that the mappings $Z(t_p)|_{\partial B}$ are uniformly convergent on ∂B. This clearly contradicts the fact that $t_p \to t \in \partial T$, which means that at least one of the sequences of intervals

$$[\psi_0, t_p^1], \; [t_p^1, t_p^2], \; \ldots, \; [t_p^N, \psi_2], \quad p \in \mathbb{N},$$

shrinks to one point, whereas each of these intervals is mapped by $Z(t_p)|_{\partial B}$ onto one of the sides of Γ. □

Proof of Theorem 1. Part (i) of the assertion follows from Propositions 6 and 7, and Part (ii) is already proved. Thus it remains to prove Part (iii):

(I) If $Z(t)$ is a minimal surface we have $Z(t)_w \cdot Z(t)_w = 0$, and consequently $V(Z(t), \mu) = 0$ for any $\mu \in C^1(\overline{B})$, which implies $\nabla\Theta(t) = 0$ by virtue of Propositions 5 and 6 respectively.

(II) If $\nabla\Theta(t) = 0$ we infer from Proposition 6 that

$$V(Z(t), \mu) = 0 \quad \text{for any } \mu \in C^1(\overline{B}) \text{ with } \mu(w_k) = 0, \quad k = 0, 1, 2. \tag{30}$$

By Proposition 8 to be proved consequently we obtain $Z(t)_w \cdot Z(t)_w = 0$, and therefore $Z(t)$ is a minimal surface since $Z(t)$ is harmonic.

(III) By assertion (ii) of Theorem 1 we know that for every minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ there is exactly one $t \in T$ such that $X = Z(t)$. Hence the set of minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$ is in one-to-one correspondence to the set of critical points of Θ. □

Proposition 8. *Suppose that (30) is satisfied. Then*

$$V(Z(t), \mu) = 0 \quad \textit{for any } \mu \in C^1(\overline{B}), \tag{31}$$

and so $Z(t)$ satisfies the conformality relation

$$Z(t)_w \cdot Z(t)_w = 0, \tag{32}$$

i.e. $Z(t)$ is a minimal surface.

Proof. For the sake of brevity we set $X := Z(t)$. We have

$$V(X, \mu) = \lim_{r \to 1-0} V_r(X, \mu)$$

with

$$V_r(X, \mu) := 4 \operatorname{Im} \int_{B_r} w X_w \cdot X_w \mu_{\overline{w}} \, du \, dv,$$

$B_r := \{w \in B \colon |w| < r\}$, $0 < r < 1$. Set

$$f(w) := w X_w(w) \cdot X_w(w) \mu(w), \quad g(w) := w X_w(w) \cdot X_w(w) \mu_{\overline{w}}(w).$$

Since $X_w \cdot X_w$ is holomorphic we have

$$f_{\overline{w}} = g,$$

and Gauß's theorem yields

$$\int_{B_r} g(w) \, du \, dv = \frac{1}{2i} \int_{\partial B_r} f(w) \, dw = \frac{1}{2} \int_0^{2\pi} \tilde{f}(re^{i\varphi}) \mu(re^{i\varphi}) \, d\varphi$$

with

$$\tilde{f}(w) := w^2 X_w(w) \cdot X_w(w).$$

Therefore,

$$V_r(X,\mu) = \int_0^{2\pi} h(re^{i\varphi})\mu(re^{i\varphi})\,d\varphi$$

with

$$h(w) := 2\,\mathrm{Im}\,\tilde{f}(w) = \sum_{k=2}^{\infty}(a_k w^k + \overline{a}_k \overline{w}^k).$$

Let

$$\mu_0(w) := \mathrm{Re}(a + bw + c\overline{w})$$

for arbitrarily chosen $a, b, c \in \mathbb{C}$. Then

$$\int_0^{2\pi} h(re^{i\varphi})\mu_0(re^{i\varphi})\,d\varphi = 0 \quad \text{for } 0 < r < 1$$

and therefore

$$V_r(X,\mu) = V_r(X,\mu - \mu_0).$$

With $r \to 1-0$ we arrive at

$$V(X,\mu) = V(X,\mu - \mu_0).$$

We can choose $a, b, c \in \mathbb{C}$ in such a way that $\mu(w_k) = \mu_0(w_k)$ for $k = 0, 1, 2$ whence $V(X, \mu - \mu_0) = 0$ on account of (30), and so $V(X,\mu) = 0$; i.e. we have verified (31).

Now we can argue as in Section 4.5, Supplementary Remark 1, to obtain $X_w \cdot X_w = 0$.

Another way to verify this equation is to apply the relation

$$0 = \lim_{r \to 1-0} \int_0^{2\pi} h(re^{i\varphi})\mu(re^{i\varphi})\,d\varphi, \quad h(w) = \sum_{k=2}^{\infty}(a_k w^k + \overline{a}_k \overline{w}^k),$$

to $\mu(w) := \frac{1}{2}(w^k + \overline{w}^k)$ as well as to $\mu(w) := \frac{1}{2i}(w^k - \overline{w}^k)$. This leads to $a_k + \overline{a}_k = 0$ and $a_k - \overline{a}_k = 0$, i.e. $a_k = 0$ for $k \geq 2$, and so $h(w) \equiv 0$ on B. Since

$$h(w) := 2\,\mathrm{Im}[w^2 X_w(w) \cdot X_w(w)]$$

it follows that

$$w^2 X_w(w) \cdot X_w(w) \equiv \text{const} =: c.$$

Therefore $X_w(w) \cdot X_w(w) = cw^{-2}$ on $B \setminus \{0\}$, which implies $c = 0$ since the left-hand side is holomorphic in B. □

6.2 Courant's Mountain Pass Lemma

In this section we want to prove several versions of the mountain pass lemma that can essentially be found in Courant's treatise [15], VI.7.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following two properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$;
(ii) *there are two distinct strict local minimizers* $x_1, x_2 \in \Omega$ *of* f.

Then f possesses a third critical point $x_3 \in \Omega$ that is "unstable" in the sense that x_3 is not a local minimizer of f. Furthermore x_3 has the following "saddle point property":

$$f(x_3) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x) =: c \tag{1}$$

where $\mathcal{P}$ denotes the set of all compact connected subsets $\boldsymbol{p}$ of Ω with $x_1, x_2 \in \boldsymbol{p}$ (i.e. the set of all "paths" in Ω connecting x_1 and x_2).

Proof. Because of (i) there is an $\varepsilon > 0$ such that

$$f(x) > c + 1 \quad \text{for all } x \in \Omega \text{ with } \operatorname{dist}(x, \partial\Omega) < \varepsilon. \tag{2}$$

We choose a sequence $\{\boldsymbol{p}_m\}$ of paths $\boldsymbol{p}_m \in \mathcal{P}$ with

$$c_m := \max_{\boldsymbol{p}_m} f \le c + 1 \quad \text{for all } m \in \mathbb{N} \quad \text{and} \quad \lim_{m \to \infty} c_m = c,$$

and then we set

$$\boldsymbol{p}_m^* := \text{closure}\,(\boldsymbol{p}_m \cup \boldsymbol{p}_{m+1} \cup \boldsymbol{p}_{m+2} \cup \cdots), \quad \boldsymbol{p}^* := \boldsymbol{p}_1^* \cap \boldsymbol{p}_2^* \cap \boldsymbol{p}_3^* \cap \cdots.$$

By (2), the compact sets $\boldsymbol{p}_m^*$ are contained in Ω, and $\boldsymbol{p}_1^* \supset \boldsymbol{p}_2^* \supset \boldsymbol{p}_3^* \supset \cdots$. Therefore $\boldsymbol{p}^*$ is a compact subset of Ω. Since x_1 and x_2 are contained in all $\boldsymbol{p}_m$ it follows that all sets $\boldsymbol{p}_m^*$ are connected, and so $\boldsymbol{p}^*$ is connected (see e.g. Alexandroff and Hopf [1], p. 118). Hence $\boldsymbol{p}^* \in \mathcal{P}$ and so

$$\max_{\boldsymbol{p}^*} f \ge \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f = c. \tag{3}$$

On the other hand, $\boldsymbol{p}^* = \limsup_{m \to \infty} \boldsymbol{p}_m :=$ set of all points $x \in \mathbb{R}^n$ with $x = \lim_{j \to \infty} z_j$ of points $z_j \in \boldsymbol{p}_{m_j}$ with $m_j \to \infty$. Thus any point $y \in \boldsymbol{p}^*$ is the limit of a sequence of points $z_j \in \boldsymbol{p}_{m_j}$ with $m_j \to \infty$. Therefore $f(z_j) \to f(y)$ and

$$f(z_j) \le \max_{\boldsymbol{p}_{m_j}} f = c_{m_j} \to c,$$

whence $f(y) \le c$, and consequently $\max_{\boldsymbol{p}^*} f \le c$. By virtue of (3) we obtain

$$\max_{\boldsymbol{p}^*} f = c \quad \text{with } c := \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x). \tag{4}$$

On account of (ii) we have also

$$\max\{f(x_1), f(x_2)\} < c. \tag{5}$$

Now we want to show that there is a critical point x_3 of f with $x_3 \in \boldsymbol{p}^*$ and $f(x_3) = c$. To prove this we consider the level set L_c in $\boldsymbol{p}^*$, defined by

$$L_c := \{x \in \boldsymbol{p}^* \colon f(x) = c\},$$

which is compact and nonvoid. We claim that $\nabla f(x_3) = 0$ for some $x_3 \in L_c$. Otherwise, $|\nabla f(x)| \geq 2\varepsilon > 0$ for all $x \in L_c$. Since $f \in C^1(\Omega)$ there would exist a number $\delta > 0$ such that

$$|\nabla f(x)| > \varepsilon \text{ in } \mathcal{U} := \{x \in \Omega \colon \operatorname{dist}(x, L_c) < \delta\} \subset\subset \Omega.$$

By virtue of (5) we can also choose $\delta > 0$ so small that $x_1, x_2 \notin \mathcal{U}$. Let $\mathcal{V}$ be an open subset of the open set $\mathcal{U}$ such that

$$L_c \subset\subset \mathcal{V} \subset\subset \mathcal{U} \subset\subset \Omega.$$

By Tietze's theorem there is a function $\eta \in C_c^0(\Omega)$ with $0 \leq \eta \leq 1$, $\eta(x) \equiv 1$ on $\mathcal{V}$, and $\operatorname{supp} \eta \subset \mathcal{U}$. Then we define $\varphi \in C^0(\Omega \times \mathbb{R}, \mathbb{R}^n)$ by

$$\varphi(x,t) := x - t\eta(x)\nabla f(x).$$

Clearly, $\varphi(x, \cdot) \in C^1(\mathbb{R}, \mathbb{R}^n)$ for any $x \in \Omega$. Since $\mathcal{U} \subset\subset \Omega$ and $\eta(x) = 0$ for $x \in \Omega \setminus \mathcal{U}$, there is a number $t_0 > 0$ such that $\varphi(x,t) \in \Omega$ for any $x \in \Omega$ and $|t| \leq t_0$. Thus $f \circ \varphi$ is defined on $\Omega \times [-t_0, t_0]$, and

$$\begin{aligned} \frac{d}{dt} f(\varphi(x,t)) &= -\eta(x)\langle \nabla f(\varphi(x,t)), \nabla f(x) \rangle \\ &= -\eta(x)|\nabla f(x)|^2 - \eta(x)\langle a(x,t) - \nabla f(x), \nabla f(x)\rangle \end{aligned}$$

with

$$a(x,t) := \nabla f(\varphi(x,t)).$$

By making $t_0 > 0$ sufficiently small we can achieve that

$$|a(x,t) - \nabla f(x)| \leq \frac{\varepsilon}{2} \leq \frac{1}{2}|\nabla f(x)|$$

for $x \in \overline{\mathcal{U}}$ and $|t| \leq t_0$ whence

$$-\eta(x)\langle a(x,t) - \nabla f(x), \nabla f(x)\rangle \leq \frac{\eta(x)}{2}|\nabla f(x)|^2$$

for $x \in \overline{\mathcal{U}}$ and $|t| \leq t_0$. Since $\eta(x) = 0$ for $x \in \Omega \setminus \mathcal{U}$ this inequality is also satisfied for $x \in \Omega \setminus \mathcal{U}$ and $|t| \leq t_0$, and so

$$\frac{d}{dt} f(\varphi(x,t)) \le -\frac{\eta(x)}{2}|\nabla f(x)|^2 \quad \text{for } x \in \Omega \text{ and } |t| \le t_0.$$

Let $\boldsymbol{p}_0$ be the compact, connected set

$$\boldsymbol{p}_0 := \varphi(\boldsymbol{p}^*, t_0) = \{\varphi(x,t_0) \colon x \in \boldsymbol{p}^*\}.$$

Since $x_1, x_2 \in \boldsymbol{p}^*$ and $x_1, x_2 \notin \mathcal{U}$ we obtain $x_1, x_2 \in \boldsymbol{p}_0$ on account of $\varphi(x,t) = x$ for $x \in \Omega \setminus \mathcal{U}$, and so we see that $\boldsymbol{p}_0 \in \mathcal{P}$. Consequently we have

$$\max_{\boldsymbol{p}_0} f \ge c. \tag{6}$$

Furthermore, for any $x \in \boldsymbol{p}^*$ and $z := \varphi(x,t_0)$ we can write

$$f(z) - f(x) = f(\varphi(x,t_0)) - f(\varphi(x,0)) = \int_0^{t_0} \frac{d}{dt} f(\varphi(x,t))\, dt$$

and therefore

$$f(z) \le f(x) - \frac{t_0}{2}\eta(x)|\nabla f(x)|^2.$$

For $x \in L_c$ we then obtain

$$f(z) \le f(x) - \frac{t_0}{2}\,\varepsilon^2 < f(x) = c,$$

and for $x \in \boldsymbol{p}_0 \setminus L_c$ we have $f(x) < c$ and therefore $f(z) \le f(x) < c$. This implies $f(z) < c$ for all $z \in \boldsymbol{p}_0$, whence $\max_{\boldsymbol{p}_0} f < c$ which is a contradiction to (6), and so we infer that $\min_{L_c} |\nabla f| = 0$. Hence L_c contains a critical point of f. Let K_c be the set of critical points of f contained in L_c, i.e.

$$K_c = \{x \in \boldsymbol{p}^* \colon f(x) = c \text{ and } \nabla f(x) = 0\}.$$

Clearly K_c is a closed subset of the compact set $\boldsymbol{p}^*$. Since x_1 and x_2 are contained in $\boldsymbol{p}^* \setminus L_c$ there is a boundary point x_3 of K_c (viewed as a subspace of the connected topological space $\boldsymbol{p}^*$). Then, in any neighborhood $\mathcal{N}$ of x_3 there is a point $y \in \boldsymbol{p}^* \setminus K_c$. By $f(x) \le c$ for all $x \in \boldsymbol{p}^*$ we either have $f(y) < c$ or $f(y) = c$. In the second case we have $\nabla f(y) \ne 0$; consequently there is a point $z \in \mathcal{N}$ with $f(z) < f(y) = c$. Thus any neighborhood $\mathcal{N}$ of x_3 contains a point x with $f(x) < f(x_3)$, i.e. x_3 is not a local minimizer of f. □

The first part of the preceding proof yields the following result.

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ satisfies $f(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ for $x \in \Omega$. Then, for any $x_1, x_2 \in \Omega$, there exists a compact, connected set $\boldsymbol{p}^* \subset \Omega$ with $x_1, x_2 \in \boldsymbol{p}^*$ such that*

$$\max_{\boldsymbol{p}^*} f = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f$$

where $\mathcal{P}$ denotes the set of all compact connected sets $\boldsymbol{p}$ in Ω with $x_1, x_2 \in \boldsymbol{p}$. We call $\boldsymbol{p}^*$ a **minimal path of** *$\mathcal{P}$.*

Remark 1. We note that the unstable critical point x_3 of f determined in the proof of Theorem 1 lies on a minimal path $\boldsymbol{p}^*$ of $\boldsymbol{p}$.

The result of Theorem 1 can be extended in the following way:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$.
(ii*) *There are two distinct points $x_1, x_2 \in \Omega$ such that*

$$\max_{\boldsymbol{p}} f > \max\{f(x_1), f(x_2)\} \quad \text{for all } \boldsymbol{p} \in \mathcal{P}$$

where $\mathcal{P}$ denotes the set of all compact, connected subsets $\boldsymbol{p}$ of Ω containing x_1 and x_2.

Then f possesses a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}$ and an unstable critical point x_3 such that $x_3 \in \boldsymbol{p}^*$, $f(x_3) = \max_{\boldsymbol{p}^*} f$, and

$$f(x_3) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x).$$

Proof. By Proposition 1 one shows that there is a "minimal path" $\boldsymbol{p}^*$ in $\mathcal{P}$ satisfying (4). Then (ii*) implies (5), and we can proceed as before. □

Remark 2. If (ii*) holds we say that x_1 *and* x_2 *are separated by a wall.* This is, for instance, the case if there exist numbers c and $r > 0$ such that $|x_1 - x_2| > r$, $f(x) \geq c$ for $x \in \Omega$ with $|x - x_1| = r$, and $f(x_1), f(x_2) < c$.

Now we want to discuss the situation that f possesses two local minimizers which are not necessarily separated by a wall.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following two properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$;
(ii) *there are two distinct local minimizers $x_1, x_2 \in \Omega$ of f.*

Then either

(1°) *there is a compact connected set $\boldsymbol{p}^*$ in Ω containing x_1 and x_2 such that*

$$f(x_1) = f(x_2) =: c \quad \text{and} \quad f(x) \equiv c, \quad \nabla f(x) \equiv 0 \quad \text{for } x \in \boldsymbol{p}^*,$$

or else

(2°) *f possesses a third critical point $x_3 \in \Omega$ which is unstable.*

Proof. As before let $\mathcal{P}$ be the set of "paths" $\boldsymbol{p}$ containing x_1 and x_2 and set

$$c := \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f.$$

We may assume that $f(x_1) \le f(x_2)$. By Proposition 1 we show that there is a minimal path $\boldsymbol{p}^* \in \mathcal{P}$ such that

$$\max_{\boldsymbol{p}^*} f = c.$$

If $f(x_2) < c$ we can proceed as before and obtain (2°). Therefore it suffices to consider the case $f(x_2) = c$. Since x_2 is a local minimizer of f, there is a $\delta > 0$ such that $f(x) \ge c$ on the ball $\mathcal{U}_\delta(x_2) := \{x \in \mathbb{R}^n : |x - x_2| < \delta\}$. Since $f(x) \le c$ for $x \in \boldsymbol{p}^*$ we have $f(x) \equiv c$ for $x \in \boldsymbol{p}^* \cap \mathcal{U}_\delta(x_2)$, which implies $\nabla f(x) \equiv 0$ for $x \in \boldsymbol{p}^* \cap \mathcal{U}_\delta(x_2)$. Set

$$L_c := \{x \in \boldsymbol{p}^* : f(x) = c\}, \quad K_c := \{x \in L_c : \nabla f(x) = 0\}.$$

If $K_c = \boldsymbol{p}^*$ we obtain assertion (1°), and we finally have to consider the case that $\boldsymbol{p}^* \setminus K_c$ is nonempty. Then there is a boundary point x_3 of K_c (viewed as a subspace of the connected topological space $\boldsymbol{p}^*$), and it follows as in the proof of Theorem 1 that x_3 is an unstable critical point of f. □

6.3 Unstable Minimal Surfaces in a Polygon

Now we return to the situation considered in 6.1 where Γ is a simple closed polygon in $\mathbb{R}^3$ with $N+3$ vertices. As before $\overline{\mathcal{C}}^*(\Gamma)$ denotes the subset of surfaces X in $\overline{H}_2^1(B, \mathbb{R}^3) := H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ that map ∂B monotonically onto Γ (in the sense of 4.2, Definition 3) and fulfill a fixed 3-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, as described in 6.1. We equip $\overline{H}_2^1(B, \mathbb{R}^3)$ with the norm

(1) $$\|X\|_{1,B} := \|X\|_{C^0(\overline{B}, \mathbb{R}^3)} + \sqrt{D(X)}$$

and the corresponding distance function

(2) $$d_1(X, Y) := \|X - Y\|_{1,B}, \quad X, Y \in \overline{H}_2^1(B, \mathbb{R}^3).$$

Clearly $(\overline{H}_2^1(B, \mathbb{R}^3), \|\cdot\|_{1,B})$ is a Banach space, and $\overline{\mathcal{C}}^*(\Gamma)$ is a closed subset of this space. Therefore $(\overline{\mathcal{C}}^*(\Gamma), d_1)$ is a complete metric space.

In this section all topological concepts concerning subsets of $\overline{\mathcal{C}}^*(\Gamma)$ will refer to the metric d_1. (In 6.6 we shall change to a weaker metric, to d_0).

A *path* P in $\overline{\mathcal{C}}^*(\Gamma)$ is defined as a compact, connected subset of $\overline{\mathcal{C}}^*(\Gamma)$. We say that P *joins* (or *connects*) X_1 and X_2 if X_1, X_2 are contained in P, and $\mathbb{P}(X_1, X_2)$ denotes the *set of paths* in $\overline{\mathcal{C}}^*(\Gamma)$ joining X_1 and X_2.

Furthermore let $\mathcal{H}^*(\Gamma)$ be the subset of $X \in \overline{\mathcal{C}}^*(\Gamma)$ that are harmonic in B, and $\mathcal{W}^*(\Gamma)$ be the image of the bounded, open, convex subset T of $\mathbb{R}^N$ introduced in 6.1 under the Courant mapping $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$. Then we have

(3) $$\mathcal{W}^*(\Gamma) := Z(T) \subset \mathcal{H}^*(\Gamma) \subset \overline{\mathcal{C}}^*(\Gamma).$$

For $t_1, t_2 \in T$ we denote by $\mathcal{P}(t_1, t_2)$ the set of paths $\boldsymbol{p}$ in T joining t_1, t_2, i.e. the set of compact, connected subsets $\boldsymbol{p}$ of T with $t_1, t_2 \in \boldsymbol{p}$. Moreover, $\mathbb{P}'(X_1, X_2)$ be the set of all paths $P \in \mathbb{P}(X_1, X_2)$ with $P \subset \mathcal{H}^*(\Gamma)$, and $\mathbb{P}''(X_1, X_2)$ be the set of paths $P \in \mathbb{P}(X_1, X_2)$ with $P \in \mathcal{W}^*(\Gamma)$.

The set T is connected, and Z is continuous according to 6.1, Proposition 4. Hence $\mathcal{W}^*(\Gamma)$ is connected, and the image $Z(\boldsymbol{p})$ of any path $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ is a path in $\mathcal{W}^*(\Gamma)$, i.e.

(4) $$Z(\boldsymbol{p}) \in \mathbb{P}''(X_1, X_2) \quad \text{for } \boldsymbol{p} \in \mathcal{P}(t_1, t_2) \text{ and } X_1 := Z(t_1),\ X_2 := Z(t_2).$$

For any $t \in T$ the set is convex. Hence, for any $X \in U(t)$ the mapping $R(t, X) : [0, 1] \to U(t)$, given by

(5) $$R(t, X)(\lambda) := \lambda Z(t) + (1 - \lambda)X, \quad 0 \le \lambda \le 1,$$

defines a continuous arc in $U(t)$ which connects X with $Z(t)$, and so the segment

(6) $$\Sigma(t, X) := \{R(t, X)(\lambda) \colon 0 \le \lambda \le 1\}$$

is a path in $\overline{\mathcal{C}}^*(\Gamma)$ joining X and $Z(t)$, i.e.

(7) $$\Sigma(t, X) \in \mathbb{P}(X, Z(t)) \quad \text{for } X \in U(t).$$

Lemma 1. *For any $X \in U(t)$ we have*

$$\max_{\Sigma(t,X)} D = D(X).$$

Proof. For $0 \le \lambda \le 1$ we set $Y(\lambda) := R(t, X)(\lambda)$, i.e.

$$Y(\lambda) = Z(t) + (1 - \lambda)\phi \quad \text{with } \phi := X - Z(t).$$

Then

$$D(Y(\lambda)) = D(Z(t)) + 2(1 - \lambda)D(Z(t), \phi) + (1 - \lambda)^2 D(\phi),$$

and consequently

(8) $$\frac{d}{d\lambda} D(Y(\lambda)) = -2D(Z(t), \phi) - 2(1 - \lambda)D(\phi) \quad \text{for } 0 \le \lambda \le 1.$$

On the other hand we have

$$D(Y(\lambda)) \ge D(Z(t)) = D(Y(1)) \quad \text{for } 0 \le \lambda \le 1$$

since $Y(\lambda) \in U(t)$ and $Z(t)$ is the minimizer of D in $U(t)$. It follows that

$$\frac{D(Y(1)) - D(Y(\lambda))}{1-\lambda} \le 0 \quad \text{for } 0 \le \lambda < 1,$$

whence

$$\frac{d}{d\lambda} D(Y(\lambda))\Big|_{\lambda=1} \le 0.$$

From (8) we infer for $\lambda = 1$ that

$$D(Z(t), \phi) \ge 0,$$

and so (8) yields

$$\frac{d}{d\lambda} D(Y(\lambda)) \le 0 \quad \text{for } 0 \le \lambda \le 1.$$

Thus the function $\lambda \mapsto D(Y(\lambda))$ is decreasing for $0 \le \lambda \le 1$, whence $D(X) = D(Y(0)) \ge D(Y(\lambda))$ for $0 \le \lambda \le 1$. □

Lemma 2. *For any $X_1 \in U(t_1)$ and $X_2 \in U(t_2)$ there exists a path $P^* \in \mathbb{P}(X_1, X_2)$ such that*

$$\max_{X \in P^*} D(X) \le \max\Big\{D(X_1), D(X_2), \max_{t \in \boldsymbol{p}} \Theta(t)\Big\} \tag{9}$$

holds for any $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$. Moreover, if $X_1, X_2 \in \mathcal{H}^*(\Gamma)$ then $P^* \in \mathbb{P}'(X_1, X_2)$.

Proof. By Lemma 1 we have

$$D(X) \le D(X_j) \quad \text{for } X \in \Sigma(t_j, X_j) \text{ with } j = 1, 2,$$

and

$$\max_{X \in Z(\boldsymbol{p})} D(X) = \max_{t \in \boldsymbol{p}} \Theta(t) \quad \text{for } \boldsymbol{p} \in \mathcal{P}(t_1, t_2).$$

On account of 6.2, Proposition 1, there is a path $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$ such that

$$\max_{t \in \boldsymbol{p}^*} \Theta(t) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1,t_2)} \max_{t \in \boldsymbol{p}} \Theta(t).$$

Then $P^* := \Sigma(t_1, X_1) \cup Z(\boldsymbol{p}^*) \cup \Sigma(t_2, X_2)$ is a path in $\overline{\mathcal{C}}^*(\Gamma)$ joining X_1 and X_2 which, in addition, satisfies (9). Moreover, $P^* \subset \mathcal{H}^*(\Gamma)$ if $X_1, X_2 \in \mathcal{H}^*(\Gamma)$. □

Remark 1. Consequently $\mathbb{P}(X_1, X_2)$ is nonempty for any $X_1, X_2 \in \mathcal{C}^*(\Gamma)$ since there are points $t_1, t_2 \in T$ such that $X_1 \in U(t_1)$ and $X_2 \in U(t_2)$. Correspondingly, $\mathbb{P}'(X_1, X_2)$ is nonvoid for any $X_1, X_2 \in \mathcal{H}^*(\Gamma)$.

Now we want to establish the existence of unstable minimal surfaces spanning the polygon Γ using the results from 6.2. To this end we recall that Courant's function $\Theta := D \circ Z$ is of class $C^1(T)$ and satisfies $\Theta(t) \to \infty$ as $\text{dist}(t, \partial T) \to 0$ for $t \in T$. Therefore, taking $\Omega := T$, $n := N$, and $f := \Theta$, we see that f satisfies assumption (i) of Theorem 1–3 in 6.2, which we will now apply to the present situation.

Definition 1. *A minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ is said to be unstable if for any $\rho > 0$ there is a mapping $Y \in \overline{\mathcal{C}}^*(\Gamma)$ such that $d_1(Y, X) < \rho$ and $D(Y) < D(X)$.*

Remark 2. Precisely speaking, a minimal surface X as in the preceding definition should be called D-*unstable*, whereas it could be called A-*unstable* if for any $\rho > 0$ there is a $Y \in \overline{\mathcal{C}}^*(\Gamma)$ such that $d_1(Y, X) < \rho$ and $A(Y) < A(X)$. We have: *Any D-unstable minimal surface X in $\overline{\mathcal{C}}^*(\Gamma)$ is also A-unstable.* In fact, the inequality $D(Y) < D(X)$ implies $A(Y) < A(X)$ because of

$$A(Y) \le D(Y) < D(X) = A(X).$$

Theorem 1. *Let X_1 and X_2 be two distinct minimal surfaces which are strict local minimizers of Dirichlet's integral on $(\overline{\mathcal{C}}^*(\Gamma), d_1)$. Then there exists an unstable minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$.*

Proof. By assumption there is an $\varepsilon_0 > 0$ such that

$$D(X_j) < D(X) \quad \text{for all } X \in \overline{\mathcal{C}}^*(\Gamma) \text{ with } 0 < d_1(X, X_j) < \varepsilon_0, \quad j = 1, 2.$$

Furthermore there are two points $t_1, t_2 \in T$ with $t_1 \neq t_2$ and $X_1 = Z(t_1)$, $X_2 = Z(t_2)$. Since Z is continuous there is a $\delta_0 > 0$ such that

$$d_1(Z(t), Z(t_j)) < \varepsilon_0 \quad \text{if } t \in T \text{ satisfies } |t - t_j| < \delta_0, \quad j = 1, 2.$$

Because of Theorem 1(ii) of 6.1 we have $Z(t) \neq Z(t_j)$ for $t \neq t_j$. It follows that

$$D(Z(t_j)) < D(Z(t)) \quad \text{for } t \in T \text{ with } 0 < |t - t_j| < \delta_0, \quad j = 1, 2,$$

which is equivalent to

$$\Theta(t_j) < \Theta(t) \quad \text{for } t \in T \text{ satisfying } 0 < |t - t_j| < \delta_0, \quad j = 1, 2.$$

Then, by 6.2, Theorem 1, there is an unstable critical point $t_3 \in T$ of Θ, i.e. $\nabla\Theta(t_3) = 0$, and for any $\delta > 0$ there is a point $t_\delta \in T$ with $|t_\delta - t_3| < \delta$ and $\Theta(t_\delta) < \Theta(t_3)$. Moreover, given $\rho > 0$, we have $d_1(Z(t), Z(t_3)) < \rho$ if $|t - t_3| < \delta \ll 1$. Setting $X_3 := Z(t_3)$ and $Y := Z(t_\delta)$ it follows that

$$D(Y) < D(X_3) \quad \text{and} \quad d_1(Y, X_3) < \rho.$$

By taking Theorem 1 of 6.1 into account we see that X_3 is an unstable minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$. □

On account of 6.2, formula (1), the unstable minimal surface X_3 of Theorem 1 has the following *saddle point property*:

Corollary 1. *If the two strict local minima X_1, X_2 of D are given by $X_1 = Z(t_1)$, $X_2 = Z(t_2)$, then*

$$D(X_3) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1,t_2)} \max_{t \in \boldsymbol{p}} D(Z(t)). \tag{10}$$

The preceding theorem can be generalized as follows:

Theorem 2. *Suppose that $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are "separated by a wall", i.e. it is assumed that $X_1 \neq X_2$ and*

$$\max_{X \in P} D(X) > \max\{D(X_1), D(X_2)\} \quad \textit{for all } P \in \mathbb{P}(X_1, X_2). \tag{11}$$

Then there exists an unstable minimal surface X_3 in $\overline{\mathcal{C}}^*(\Gamma)$.

Proof. There are two points $t_1, t_2 \in T$ with $X_1 \in U(t_1)$, $X_2 \in U(t_2)$. Here, t_1 and t_2 are not necessarily uniquely determined by X_1 and X_2 respectively. However, $t_1 \neq t_2$ since $t_1 = t_2$ would imply that $P := \Sigma(t_1, X_1) \cup \Sigma(t_2, X_2)$ is a path contained in $\mathbb{P}(X_1, X_2)$ such that $\max_{X \in P} D(X) = \max\{D(X_1), D(X_2)\}$ if we take Lemma 1 into account; but this were a contradiction to (11).

We claim that

$$\max_{\boldsymbol{p}} \Theta > \max\{\Theta(t_1), \Theta(t_2)\} \quad \text{for all } \boldsymbol{p} \in \mathcal{P}(t_1, t_2). \tag{12}$$

Otherwise we would have for all $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ that

$$\max_{\boldsymbol{p}} \Theta = \max\{\Theta(t_1), \Theta(t_2)\} \leq \max\{D(X_1), D(X_2)\},$$

since $\Theta(t_1) \leq D(X_1)$ and $\Theta(t_2) \leq D(X_2)$. Then it follows from Lemma 2 that there is a path $P^* \in \mathbb{P}(X_1, X_2)$ such that

$$\max_{X \in P^*} D(X) = \max\{D(X_1), D(X_2)\},$$

a contradiction to (11). Thus we have verified (12), and by 6.2, Theorem 2, there is an unstable critical point $t_3 \in T$ of the Courant function. Setting $X_3 := Z(t_3)$, we see as in the proof of Theorem 1 that X_3 is an unstable minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$. □

As before we obtain the saddle point property (9) for X_3:

Corollary 2. *If $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are separated by a wall, there is an unstable minimal surface $X = Z(\bar{t}) \in \overline{\mathcal{C}}^*(\Gamma)$ such that*

$$D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1,t_2)} \max_{t \in \boldsymbol{p}} D(Z(t))$$

for some critical point $\bar{t}$ of Θ.

Next we want to consider the case where $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are two local minimizers of Dirichlet's integral which are not necessarily separated by a wall.

Theorem 3. *Suppose that X_1 and X_2 are two distinct minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$ both of which are local minimizers of D on $\overline{\mathcal{C}}^*(\Gamma)$.*

Then either

1° *There is a path $P^* \in \mathbb{P}''(X_1, X_2)$ such that*

$$D(X) \equiv \text{const} =: c \quad \textit{for all } X \in P^*,$$

or else

2° *D possesses a third critical point X_3 in $\overline{\mathcal{C}}^*(\Gamma)$ which is an unstable minimal surface.*

Proof. There are uniquely determined points $t_1, t_2 \in T$ with $t_1 \neq t_2$ such that $X_1 = Z(t_1)$ and $X_2 = Z(t_2)$. The assumption of the theorem implies that t_1 and t_2 are distinct local minimizers of Θ. By virtue of 6.2, Theorem 3, there is a path $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$ such that either

$$\Theta(t) \equiv \text{const} =: c \quad \text{and} \quad \nabla\Theta(t) \equiv 0 \quad \text{for } t \in \boldsymbol{p}^*,$$

or else Θ possesses a third critical point $t_3 \in T$ which is unstable. In the first case we have 1° for $P^* = Z(\boldsymbol{p}^*) \in \mathbb{P}''(X_1, X_2)$, and in the second we obtain 2° for $X_3 := Z(t_3)$. □

In 6.6 we shall use the following variant of the preceding results.

Theorem 4. *For $t_1, t_2 \in T$ with $t_1 \neq t_2$ there is a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}(t_1, t_2)$ satisfying*

$$\max_{X \in Z(\boldsymbol{p}^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{X \in Z(\boldsymbol{p})} D(X). \tag{13}$$

If, in addition,

$$\max_{X \in Z(\boldsymbol{p}^*)} D(X) > \max\{D\big(Z(t_1)\big), D(Z(t_2))\} \tag{14}$$

then there is an unstable minimal surface X_3 in $\overline{\mathcal{C}}^*(\Gamma)$ such that $X_3 = Z(t_3)$ for $t_3 \in \boldsymbol{p}^*$, i.e. $X_3 \in Z(\boldsymbol{p}^*)$, and

$$D(X_3) = \max_{X \in Z(\boldsymbol{p}^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{X \in Z(\boldsymbol{p})} D(X).$$

Proof. The assertions follow immediately from 6.2, Proposition 1 and Theorem 2, if we take Theorem 1 of 6.1 into account. □

The next result will not be needed for the proof of the final theorems in Section 6.6; yet they are of independent interest.

Proposition 1. *If* $X_1 = Z(t_1)$ *and* $X_2 = Z(t_2)$ *for* $t_1, t_2 \in T$ *then*

$$d(X_1, X_2) := \inf_{P \in \mathbb{P}''(X_1, X_2)} \max_{X \in P} D(X) \tag{15}$$

and

$$\sigma(t_1, t_2) := \inf_{P \in Z(\mathcal{P}(t_1, t_2))} \max_{X \in P} D(X) \tag{16}$$

satisfy

$$d(X_1, X_2) = \sigma(t_1, t_2). \tag{17}$$

Proof. From $Z(\mathcal{P}(t_1, t_2)) \subset \mathbb{P}''(X_1, X_2)$ it follows that

$$d(X_1, X_2) \le \sigma(t_1, t_2).$$

Thus it remains to show

$$\sigma(t_1, t_2) \le d(X_1, X_2). \tag{18}$$

This is not obvious since the pre-image $Z^{-1}(P)$ of $P \in \mathbb{P}''(X_1, X_2)$ might not contain a path $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$. Instead we prove a weaker result, stated in the next proposition, which suffices to verify (18). □

Proposition 2. *For any* $P \in \mathbb{P}''(Z(t_1), Z(t_2))$ *there exists a* $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ *such that*

$$\max_{X \in Z(\boldsymbol{p})} D(X) \le \max_{X \in P} D(X). \tag{19}$$

Proof. We first note that the pre-image $m := Z^{-1}(P)$ of a given $P \in \mathbb{P}''(Z(t_1), Z(t_2))$ is closed. In fact, if $t_j \in m$ for all $j \in \mathbb{N}$ and $t_j \to t_0$ then $t_0 \in \Omega$ since $t_0 \in \partial\Omega$ would imply $D(Z(t_j)) = \Theta(t_j) \to \infty$ whereas $Z(t_j) \in P$ yields $D(Z(t_j)) \le \text{const} < \infty$. Since Z is continuous we have $Z(t_j) \to Z(t_0)$ whence $Z(t_0) \in P$. Hence m is closed and therefore compact. If m is connected we set $\boldsymbol{p} := m$ and obtain (19). Thus we now assume that m is disconnected and write m as disjoint union $m = \bigcup_{\alpha \in J} m_\alpha$ of its compact connected components m_α.

Consider two such components m_α and m_β, $\alpha \ne \beta$, for which $Z(m_\alpha) \cap Z(m_\beta)$ is nonvoid. Then there are points $t \in m_\alpha$ and $\bar{t} \in m_\beta$ such that $Z(t) = Z(\bar{t})$. Let $j \in \{1, \dots, N\}$ be the first index such that $t^j \ne \bar{t}^j$, say, $t^j < \bar{t}^j$. Then it follows that

$$Z(t)(e^{i\varphi}) \equiv A_j \quad \text{for } \varphi \in [t^j, \bar{t}^j].$$

Consider the path $\gamma_1 := \{(t^1, \dots, t^{j-1}, s, t^{j+1}, \dots, t^N) \colon t^j \le s \le \bar{t}^j\}$ and set $Y(t_1) = Z(t)$ for $t_1 \in \gamma_1$. Then $Y(t_1) \in U(t_1)$ for $t_1 \in \gamma_1$. If $t^{j+1} =$

$\bar{t}^{j+1}, \ldots, t^N = \bar{t}^N$, the path γ_1 connects t and $\bar{t}$ in T, and $Y(t_1) \equiv Z(t) = Z(\bar{t})$ for all $t_1 \in \gamma_1$. Otherwise we proceed in the same way for the next index k with $t^k \neq \bar{t}^k$ and obtain a path γ_2 that connects $(t^1, \ldots, t^{j-1}, \bar{t}^j, t^{j+1}, \ldots, t^k, \ldots, t^N)$ with $(t^1, \ldots, t^{j-1}, \bar{t}^j, t^{j+1}, \ldots, \bar{t}^k, \ldots, t^N)$, and $Y(t_2) \equiv Z(t) = Z(\bar{t})$ for $t_2 \in \gamma_2$. After at most N steps we have constructed a path $\gamma_{\alpha\beta} \in \mathcal{P}(t, \bar{t})$ in T with $D(Z(\tau)) \leq D(Z(t)) = D(Z(\bar{t}))$ for all $\tau \in \gamma_{\alpha\beta}$. Then $m_{\alpha\beta} := m_\alpha \cup m_\beta \cup \gamma_{\alpha\beta} \in \mathcal{P}(t, \bar{t})$, and

$$\max_{X \in Z(m_{\alpha\beta})} D(X) \leq \max_{X \in P} D(X). \tag{20}$$

On the other hand, if $Z(m_\alpha) \cap Z(m_\beta) = \emptyset$ we set $m_{\alpha\beta} := m_\alpha \cup m_\beta$; in this case (20) is clearly satisfied. Set

$$m' := \bigcup_{(\alpha,\beta) \in J \times J} m_{\alpha\beta}.$$

Then

$$\sup_{X \in Z(m')} D(X) \leq \max_{X \in P} D(X) < \infty. \tag{21}$$

Since $\Theta(t) \to \infty$ as $\operatorname{dist}(t, \partial T) \to 0$ for $t \in T$, we conclude that $\boldsymbol{p} := \overline{m'}$ is a compact subset of T. Moreover we infer from the connectedness of P and the above construction that m' is connected, whence $\boldsymbol{p}$ is connected, since the closure of a connected set is connected. Hence $\boldsymbol{p}$ is an element of $\mathcal{P}(t_1, t_2)$, and (21) implies (19) because of the continuity of Z. □

Corollary 3. *If $X_1 = Z(t_1)$, $X_2 = Z(t_2)$ for $t_1, t_2 \in T$, $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$, and $\max_{t \in \boldsymbol{p}^*} D(Z(t)) = \sigma(t_1, t_2)$ then $P^* := Z(\boldsymbol{p}^*)$ satisfies $\max_{X \in P^*} D(X) = d(X_1, X_2)$. This means: The image $P^* = Z(\boldsymbol{p}^*)$ of a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}(t_1, t_2)$ is a minimal path of $P(X_1, X_2)$.*

Proof. The assertion is an immediate consequence of Proposition 1. □

In particular we obtain:

Corollary 4. *The saddle point property*

$$D(X_3) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{t \in \boldsymbol{p}} D(Z(t))$$

in the Corollaries 1 and 2 is equivalent to

$$D(X_3) = \inf_{P \in \mathbb{P}''(X_1, X_2)} \max_{X \in P} D(X). \tag{22}$$

6.4 The Douglas Functional. Convergence Theorems for Harmonic Mappings

Let $C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ be the class of continuous mappings $\xi : \mathbb{R} \to \mathbb{R}^3$ that are 2π-periodic, i.e. which satisfy $\xi(\theta + 2\pi) = \xi(\theta)$ for any $\theta \in \mathbb{R}$. Then the **Douglas functional** A_0 is a function $A_0 : C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3) \to \mathbb{R}$ defined by

$$(1) \qquad A_0(\xi) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\xi(\theta) - \xi(\varphi)|^2}{4 \sin^2 \frac{1}{2}(\theta - \varphi)} \, d\theta \, d\varphi \le \infty.$$

Because of

$$|e^{i\theta} - e^{i\varphi}|^2 = 4 \sin^2 \frac{1}{2}(\theta - \varphi)$$

we can write $A_0(\xi)$ as

$$(2) \qquad A_0(\xi) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\xi(\theta) - \xi(\varphi)|^2}{|e^{i\theta} - e^{i\varphi}|^2} \, d\theta \, d\varphi.$$

We recall the following well-known result:

Lemma 1. *Let $\xi \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$. Then the uniquely determined mapping $H \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with*

$$\Delta H = 0 \quad \text{in } B, \quad H(e^{i\theta}) = \xi(\theta) \quad \text{for } \theta \in \mathbb{R},$$

is given by

$$(3) \quad H(\rho e^{i\theta}) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta), \quad 0 \le \rho \le 1, \, \theta \in \mathbb{R},$$
$$a_n := \frac{1}{\pi} \int_0^{2\pi} \xi(\theta) \cos n\theta \, d\theta, \quad b_n := \frac{1}{\pi} \int_0^{2\pi} \xi(\theta) \sin n\theta \, d\theta.$$

We call H the "harmonic extension of ξ".

Theorem 1. *Let $H \in C^0(\overline{B}, \mathbb{R}^3)$ be the harmonic extension of $\xi \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$. Then*

$$(4) \qquad D(H) = A_0(\xi).$$

Proof. H is given by (3). Then

$$|a_n|^2 = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \xi(\theta) \xi(\varphi) \cos n\theta \cos n\varphi \, d\theta \, d\varphi,$$

$$|b_n|^2 = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \xi(\theta) \xi(\varphi) \sin n\theta \sin n\varphi \, d\theta \, d\varphi,$$

whence

$$|a_n|^2 + |b_n|^2 = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \xi(\theta)\xi(\varphi) \cos n(\theta - \varphi)\, d\theta\, d\varphi \quad \text{for } n \geq 1.$$

Because of

$$\int_0^{2\pi} \cos n(\theta - \varphi)\, d\varphi = \int_0^{2\pi} \cos n(\theta - \varphi)\, d\theta = 0$$

we obtain

$$|a_n|^2 + |b_n|^2 = -\frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |\xi(\theta) - \xi(\varphi)|^2 \cos n(\theta - \varphi)\, d\theta\, d\varphi, \quad n \geq 1.$$

Furthermore,

$$D_{B_r}(H) = \frac{1}{2} \int_{B_r} |\nabla H|^2\, du\, dv \quad \text{with } B_r := \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < r^2\}$$

is computed as

$$D_{B_r}(H) = \frac{\pi}{2} \sum_{n=1}^{\infty} n r^{2n} (|a_n|^2 + |b_n|^2) \quad \text{for } 0 < r < 1.$$

Setting

$$\text{(5)} \qquad Q(r,\alpha) := \begin{cases} -\sum_{n=1}^{\infty} n r^{2n} \cos n\alpha & \text{for } 0 \leq r < 1, \\ \frac{1}{4 \sin^2 \frac{1}{2}\alpha} & \text{for } r = 1, \end{cases}$$

we obtain

$$\text{(6)} \quad D_{B_r}(H) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2\, d\theta\, d\varphi \quad \text{for } 0 < r < 1.$$

Furthermore,

$$-2Q(r,\alpha) = \sum_{n=1}^{\infty} n r^{2n} e^{in\alpha} + \sum_{n=1}^{\infty} n r^{2n} e^{-in\alpha}.$$

Setting $z := r^2 e^{i\alpha}$, we find that

$$\begin{aligned} -2Q(r,\alpha) &= z \sum_{n=1}^{\infty} n z^{n-1} + \overline{z} \sum_{n=1}^{\infty} n \overline{z}^{n-1} = \frac{z}{(1-z)^2} + \frac{\overline{z}}{(1-\overline{z})^2} \\ &= \frac{z(1-\overline{z})^2 + \overline{z}(1-z)^2}{(1-z)^2(1-\overline{z})^2} = \frac{(z+\overline{z}) - 4|z|^2 + (z+\overline{z})|z|^2}{[1-(z+\overline{z}) + z\overline{z}]^2}, \end{aligned}$$

and so

$$Q(r,\alpha) = r^2 \frac{a-b}{(a+b)^2}, \quad a := (1+r^2)^2 \sin^2 \frac{\alpha}{2}, \quad b := (1-r^2)^2 \cos^2 \frac{\alpha}{2}.$$

Hence

$$\frac{Q(r,\alpha)}{Q(1,\alpha)} = \frac{4r^2 \sin^2 \frac{\alpha}{2}}{a+b} \, \frac{a-b}{a+b} \quad \text{for } \alpha \not\equiv 0 \bmod 2\pi$$

which implies

$$\begin{aligned} &Q(r,\alpha) \le Q(1,\alpha) \quad \text{for } 0 \le r < 1, \\ &\lim_{r\to 1-0} Q(r,\alpha) = Q(1,\alpha) \quad \text{for } \alpha \not\equiv 0 \bmod 2\pi. \end{aligned} \tag{7}$$

If $A_0(\xi) < \infty$ then

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \to A_0(\xi) \quad \text{as } r \to 1 - 0$$

on account of Lebesgue's convergence theorem. Since

$$D_{B_r}(H) \to D(H) \quad \text{as } r \to 1 - 0$$

we infer from (6) that $D(H) = A_0(\xi)$.

Conversely, if $D(H) < \infty$, $0 < \varepsilon < \pi$, and

$$R(\varepsilon) := \{(\theta, \varphi) \in [0, 2\pi] \times [0, 2\pi] \colon |e^{i\theta} - e^{i\varphi}| > \varepsilon\}$$

we have

$$\begin{aligned} &\int_{R(\varepsilon)} Q(1, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad \le \lim_{r\to 1-0} \int_{R(\varepsilon)} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad \le \lim_{r\to 1-0} \int_0^{2\pi} \int_0^{2\pi} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad = \lim_{r\to 1-0} 4\pi D_{B_r}(H) = 4\pi D(H) < \infty. \end{aligned}$$

With $\varepsilon \to +0$ we obtain

$$A_0(\xi) = \lim_{\varepsilon\to +0} \frac{1}{4\pi} \int_{R(\varepsilon)} Q(1, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi < \infty,$$

and then the reasoning above yields $A_0(\xi) = D(H)$. Thus we have proved (4), since our arguments imply that $D(H) = \infty$ if and only if $A_0(\xi) = \infty$. □

Corollary 1. *Let* $\{\xi_j\}$ *be a sequence in* $C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ *with the following properties*:

(i) $\xi_j(\theta) \rightrightarrows 0$ *on* $\mathbb{R}$ *as* $j \to \infty$.
(ii) *There is a mapping* $\eta \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ *such that* $A_0(\eta) < \infty$ *and*

$$(8) \qquad |\xi_j(\theta) - \xi_j(\varphi)| \le |\eta(\theta) - \eta(\varphi)| \quad \textit{for all } j \in \mathbb{N} \textit{ and } \theta, \varphi \in \mathbb{R}.$$

Then we have the relation

$$\lim_{j\to\infty} A_0(\xi_j) = 0.$$

Proof. As $Q(1, \theta - \varphi)|\eta(\theta) - \eta(\varphi)|^2$ is an L^1-majorant of the functions $Q(1, \theta - \varphi)|\xi_j(\theta) - \xi_j(\varphi)|^2$ on $[0, 2\pi] \times [0, 2\pi]$, the assertion is an immediate consequence of Lebesgue's convergence theorem. □

Let $\overline{\mathcal{H}}(B, \mathbb{R}^3)$ be the class of mappings $H \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with $\Delta H = 0$ in B. For any $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$ we define the value $D_0(H)$ by

$$(9) \qquad D_0(H) := A_0(\xi) \quad \text{where } \xi(\theta) := H(e^{i\theta}),\ \theta \in \mathbb{R}.$$

The function $D_0 : \overline{\mathcal{H}}(B, \mathbb{R}^3) \to \mathbb{R}$ is also denoted as **Douglas functional**. Because of (2) we can as well write

$$(10) \qquad D_0(H) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|H(e^{i\theta}) - H(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2}\, d\theta\, d\varphi.$$

An immediate consequence of Theorem 1 is

Corollary 2. *For any* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$ *we have*

$$(11) \qquad D(H) = D_0(H).$$

For $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ with $\xi(\theta) := H(e^{i\theta})$ for $\theta \in \mathbb{R}$ we define the norm

$$(12) \qquad \|H\|_{1,B} := \|\xi\|_{C^0([0,2\pi],\mathbb{R}^3)} + \sqrt{A_0(\xi)};$$

in virtue of (9), (11), and the maximum principle it agrees with the norm

$$(13) \qquad \|H\|_{1,B} := \|H\|_{C^0(\overline{B},\mathbb{R}^3)} + \sqrt{D(H)}$$

introduced in 6.3, (1); hence $\overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ equipped with the norm $\|\cdot\|_{1,B}$ is complete, i.e. a Banach space.

From Corollaries 1 and 2 we infer the following important result:

Theorem 2 (E. Heinz [14]). *Let* $\{H_j\}$ *be a sequence in* $\overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ *with the boundary values* $\{\xi_j\}$, $\xi_j(\theta) = H_j(e^{i\theta})$ *for* $\theta \in \mathbb{R}$, *and assume the following*:

(i) $\xi_j(\theta) \rightrightarrows \xi(\theta)$ *on* $\mathbb{R}$ *for* $j \to \infty$ *with* $A_0(\xi) < \infty$.
(ii) *There is a number* $\kappa > 0$ *such that*

$$|\xi_j(\theta) - \xi_j(\varphi)| \le \kappa |\xi(\theta) - \xi(\varphi)| \quad \textit{for all } j \in \mathbb{N} \textit{ and } \theta, \varphi \in \mathbb{R}.$$

Then we have

$$\|H_j - H\|_{1,B} = \|H_j - H\|_{C^0(\overline{B},\mathbb{R}^3)} + \sqrt{D(H_j - H)} \to 0 \quad \textit{as } j \to \infty,$$

where H *is the harmonic extension of* ξ, *and in particular* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$ *and* $H_j \to H$ *in* $H_2^1(B, \mathbb{R}^3)$.

Now we want to prove a second kind of convergence theorem for harmonic mappings. We begin with deriving an *isoperimetric inequality for harmonic surfaces* due to M. Morse and C. Tompkins [3].

Theorem 3. *For any* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) := C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \cap \{\Delta H = 0 \textit{ in } B\}$ *we have*

$$A(H) \le \frac{1}{4}\left(\int_{\partial B} |dH|\right)^2. \tag{14}$$

Proof. We may assume that $\int_{\partial B} |dH|$ is finite, because otherwise (14) is certainly true. The area $A(H)$ of H is defined as

$$A(H) = \int_B |H_u \wedge H_v| \, du \, dv.$$

We transform $H(u, v)$ to polar coordinates r, θ around the origin by setting

$$X(r, \theta) := H(r \cos\theta, r \sin\theta), \quad 0 \le r \le 1, \ 0 \le \theta \le 2\pi,$$

and obtain

$$A(H) = \int_0^1 \int_0^{2\pi} |X_r \wedge X_\theta| \, d\theta \, dr.$$

Poisson's integral formula yields

$$X(r, \theta) = \int_0^{2\pi} K(r, \varphi - \theta) \xi(\varphi) \, d\varphi, \quad \xi(\varphi) := X(1, \varphi),$$

where $K(r, \alpha)$ denotes the Poisson kernel

$$K(r, \alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}.$$

As in the proof of 4.7, Proposition 1, we obtain

$$X_\theta(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{\omega(r, \theta, \varphi)} \, d\xi(\varphi)$$

where

$$\omega(r,\theta,\varphi) := 1 - 2r\cos(\theta-\varphi) + r^2.$$

By using the computation of the proof of 4.7, Proposition 2, we find in addition that

$$X_r(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(\varphi-\theta)}{\omega(r,\theta,\varphi)}\, d\xi(\varphi).$$

Therefore,

$$X_r(r,\theta)\wedge X_\theta(r,\theta) = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)\sin(\varphi-\theta)}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, d\xi(\varphi)\wedge\, d\xi(\psi).$$

Interchanging φ and ψ on the right-hand side, the left-hand side remains the same. Adding the two expressions, dividing by 2, and noting the relation $d\xi(\varphi)\wedge d\xi(\psi) = -d\xi(\psi)\wedge d\xi(\varphi)$, we arrive at

$$\begin{aligned} &X_r(r,\theta)\wedge X_\theta(r,\theta) \\ &\quad = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)[\sin(\varphi-\theta)-\sin(\psi-\theta)]}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, d\xi(\varphi)\wedge d\xi(\psi). \end{aligned}$$

Furthermore, the identity

$$\sin\varphi - \sin\psi = 2\cos\frac{\varphi+\psi}{2}\sin\frac{\varphi-\psi}{2}$$

implies

$$\sin(\varphi-\theta)-\sin(\psi-\theta) = 2\cos\left[\tfrac{1}{2}(\varphi+\psi)-\theta\right]\sin\tfrac{1}{2}(\varphi-\psi)$$

whence

$$|\sin(\varphi-\theta)-\sin(\psi-\theta)| \le 2|\sin\tfrac{1}{2}(\varphi-\psi)|,$$

and therefore

$$|X_r(r,\theta)\wedge X_\theta(r,\theta)| \le \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)|\sin\frac{1}{2}(\varphi-\psi)|}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, |d\xi(\varphi)||d\xi(\psi)|.$$

For $0<\varepsilon<\rho<1$ we set

$$a(\varepsilon,\rho) := \int_\varepsilon^\rho\int_0^{2\pi} |X_r(r,\theta)\wedge X_\theta(r,\theta)|\, d\theta\, dr.$$

Then

$$a(\varepsilon,\rho) \le \int_0^{2\pi}\int_0^{2\pi} \mathfrak{I}(\varphi,\psi)|\sin\tfrac{1}{2}(\varphi-\psi)||d\xi(\varphi)||d\xi(\psi)|$$

with

$$\begin{aligned}\mathcal{I}(\varphi,\psi) &:= \frac{1}{\pi}\int_\varepsilon^\rho \mathcal{I}^*(r,\varphi,\psi)\,dr,\\ \mathcal{I}^*(r,\varphi,\psi) &:= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\,d\theta.\end{aligned}$$

Fix $\psi \in [0,2\pi]$ and $r \in (\varepsilon,\rho)$, and consider a harmonic function f in the unit disk B with $f \in C^0(\overline{B})$ which has the boundary values

$$f(e^{i\theta}) := \frac{1}{\omega(r,\theta,\psi)} = \frac{1}{1-2r\cos(\psi-\theta)+r^2}.$$

For $0 \le R \le 1$ we write

$$h(R,\varphi) := f(Re^{i\varphi}).$$

Then Poisson's integral formula yields

$$h(R,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-R^2}{\omega(R,\theta,\varphi)\omega(r,\theta,\psi)}\,d\theta,$$

whence

$$\mathcal{I}^*(r,\varphi,\psi) = h(r,\varphi).$$

In order to determine the function h, we recall that for fixed r with $0<r<1$ the Poisson kernel

$$g(R,\theta) := \frac{R^2-r^2}{R^2-2rR\cos(\psi-\theta)+r^2}$$

is a harmonic function (written in polar coordinates) of R,θ in $\{R>1\}$, i.e. in the exterior of $B=\{w\in\mathbb{C}\colon |w|<1\}$, and $g(1,\theta)=(1-r^2)h(1,\theta)$. Hence h is obtained from $(1-r^2)^{-1}g$ by reflection at the unit circle $\partial B = \{R=1\}$, that is, by replacing R by $\frac{1}{R}$:

$$h(R,\varphi) = \frac{1}{1-r^2}\;\frac{R^{-2}-r^2}{R^{-2}-2rR^{-1}\cos(\psi-\varphi)+r^2}.$$

Thus we infer

$$\mathcal{I}^*(r,\varphi,\psi) = \frac{1+r^2}{1-2r^2\cos(\psi-\varphi)+r^4}$$

whence

$$\begin{aligned}\mathcal{I}(\varphi,\psi) &= \frac{1}{\pi}\int_\varepsilon^\rho \frac{1+r^2}{1-2r^2\cos(\psi-\varphi)+r^4}\,dr\\ &= \frac{1}{2\pi}\int_\varepsilon^\rho\left\{\frac{1}{1-2r\cos\frac{1}{2}(\psi-\varphi)+r^2}+\frac{1}{1+2r\cos\frac{1}{2}(\psi-\varphi)+r^2}\right\}dr\\ &= \frac{1}{2\pi}\frac{1}{|\sin\frac{1}{2}(\psi-\varphi)|}\;[S(r,\varphi,\psi)]_\varepsilon^\rho\end{aligned}$$

with

$$S(r,\varphi,\psi) := \operatorname{arctg}\left(\frac{r-\cos\frac{1}{2}(\psi-\varphi)}{|\sin\frac{1}{2}(\psi-\varphi)|}\right) + \operatorname{arctg}\left(\frac{r+\cos\frac{1}{2}(\psi-\varphi)}{|\sin\frac{1}{2}(\psi-\varphi)|}\right).$$

Using the formula

$$\operatorname{arctg} a + \operatorname{arctg} b = \operatorname{arctg}\frac{a+b}{1-ab}$$

we obtain

$$\mathcal{I}(\varphi,\psi) = \left[\frac{1}{2\pi|\sin\frac{1}{2}(\psi-\varphi)|}\operatorname{arctg}\left(\frac{2r|\sin\frac{1}{2}(\psi-\varphi)|}{1-r^2}\right)\right]_{r=\varepsilon}^{r=\rho}.$$

Therefore,

$$\text{(15)}\qquad a(\varepsilon,\rho) \le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left[\operatorname{arctg}\left(\frac{2r|\sin\frac{1}{2}(\psi-\varphi)|}{1-r^2}\right)\right]_{r=\varepsilon}^{r=\rho}|d\xi(\varphi)||d\xi(\psi)|.$$

Since $[\dots] \to \frac{\pi}{2}$ as $\varepsilon \to +0$ and $\rho \to 1-0$, and $a(\varepsilon,\rho) \to A$, we finally see that

$$\text{(16)}\qquad A \le \frac{1}{4}L^2 \quad \text{with } L := \int_0^{2\pi}|d\xi(\theta)|. \qquad \square$$

Now we prove a convergence theorem for the area of harmonic mappings discovered by M. Morse and C. Tompkins [3].

Theorem 4. *Let $\{H_j\}$ be a sequence of harmonic mappings in $\overline{\mathcal{H}}(B,\mathbb{R}^3)$ with the following two properties*:

(i) $\|H_j - H\|_{C^0(\overline{B},\mathbb{R}^3)} \to 0$ *as* $j\to\infty$ *for some* $H \in \overline{\mathcal{H}}(B,\mathbb{R}^3)$;
(ii) $\int_{\partial B}|dH_j| \to \int_{\partial B}|dH|$ *as* $j \to \infty$.

Then the area of H_j tends to the area of H, i.e.

$$\text{(17)}\qquad \lim_{j\to\infty} A(H_j) = A(H).$$

Proof. Analogously to the preceding proof we introduce X_j, X and ξ_j, ξ by

$$X_j(r,\theta) := H_j(re^{i\theta}), \quad X(r,\theta) := H(re^{i\theta}),$$
$$\xi_j(\theta) := X_j(1,\theta), \quad \xi(\theta) := X(1,\theta).$$

For $\alpha,\beta \in \mathbb{R}$ with $0 < \beta - \alpha < 2\pi$ we set

$$L_j(\alpha,\beta) := \int_\alpha^\beta |d\xi_j(\theta)|, \quad L(\alpha,\beta) := \int_\alpha^\beta |d\xi(\theta)|.$$

1° *Claim*: $L_j(\alpha,\beta) \to L(\alpha,\beta)$ *as* $j\to\infty$ *uniformly in* α,β.

Otherwise there would be an $\varepsilon > 0$ and a subsequence of indices $j_p \to \infty$ as $p \to \infty$ and sequences $\alpha_{j_p} \to \alpha$, $\beta_{j_p} \to \beta$ such that

$$|L_j(\alpha_j, \beta_j) - L(\alpha_j, \beta_j)| \geq 2\varepsilon \quad \text{for all } j = j_p,\ p \in \mathbb{N}.$$

Since $L(\alpha_{j_p}, \beta_{j_p}) \to L(\alpha, \beta)$ for $p \to \infty$ we may assume that

$$|L(\alpha_j, \beta_j) - L(\alpha, \beta)| < \varepsilon \quad \text{for all } j = j_p,$$

and so we obtain

$$|L_j(\alpha_j, \beta_j) - L(\alpha, \beta)| > \varepsilon \quad \text{for all } j = j_p.$$

Since the arc length is lower semicontinuous with respect to uniform convergence we obtain

$$\liminf_{p \to \infty} L_{j_p}(\alpha_{j_p}, \beta_{j_p}) \geq L(\alpha, \beta) + \varepsilon.$$

By passing to a suitable subsequence of $\{j_p\}$, which will again be denoted by $\{j_p\}$, we may even assume that

$$\lim_{p \to \infty} L_{j_p}(\alpha_{j_p}, \beta_{j_p}) \geq L(\alpha, \beta) + \varepsilon. \tag{18}$$

On the other hand, if γ_j and γ are the complementary arcs to $\{e^{i\theta} \colon \alpha_j \leq \theta \leq \beta_j\}$ and $\{e^{i\theta} \colon \alpha \leq \theta \leq \beta\}$ respectively in ∂B, and

$$L_j := \int_{\gamma_j} |dH_j|, \quad L := \int_\gamma |dH|,$$

we get

$$\liminf_{p \to \infty} L_{j_p} \geq L. \tag{19}$$

Adding (18) and (19) we would obtain

$$\liminf_{p \to \infty} \int_{\partial B} |dH_{j_p}| \geq \int_{\partial B} |dH| + \varepsilon,$$

which contradicts assumption (ii). Thus the claim 1° is proved.

2° Set

$$l(\sigma) := \sup\left\{ \int_\gamma |dH|, \int_\gamma |dH_j| \colon \gamma \subset \partial B,\ \text{length } \gamma = \sigma, j \in \mathbb{N} \right\}. \tag{20}$$

Because of 1° we obtain

$$l(\sigma) \to 0 \quad \text{as } \sigma \to +0. \tag{21}$$

Furthermore there is a number $\lambda > 0$ such that

$$\int_{\partial B} |dH_j| \le \lambda \quad \text{for all } j \in \mathbb{N}.$$

Set

$$a_j(R,\rho) := \int_R^\rho \int_0^{2\pi} |X_{j,r}(r,\theta) \wedge X_{j,\theta}(r,\theta)|\, d\theta\, dr$$

for $0 < R < \rho < 1$. By (15) we have

$$a_j(R,\rho) \le \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} [\chi(r,\psi,\varphi)]_{r=R}^{r=\rho} |d\xi_j(\varphi)||d\xi_j(\psi)|$$

with

$$\chi(r,\psi,\varphi) := \operatorname{arctg}\left(\frac{2r|\sin \frac{1}{2}(\psi - \varphi)|}{1 - r^2} \right).$$

We decompose the domain of integration $\Omega := \{(\varphi,\psi)\colon 0 < \varphi, \psi < 2\pi\}$ into the disjoint sets Ω_1 and Ω_2 defined by

$$\Omega_1 := \{(\varphi,\psi) \in \Omega\colon \|\psi - \varphi\| \le \sigma\}, \quad \Omega_2 := \{(\varphi,\psi) \in \Omega\colon \|\psi - \varphi\| > \sigma\},$$

where $\|\psi - \varphi\|$ denotes the length of the shorter arc on ∂B with the endpoints $e^{i\varphi}$ and $e^{i\psi}$. Then

$$a_j(R,\rho) \le I_j^1(R,\rho) + I_j^2(R,\rho)$$

with

$$I_j^k(R,\rho) := \frac{1}{2\pi} \int_{\Omega_k} [\chi(r,\psi,\varphi)]_{r=R}^{r=\rho} |d\xi_j(\varphi)||d\xi_j(\psi)|, \quad k = 1, 2.$$

On Ω_1 we estimate $[\chi(r,\psi,\varphi)]_R^\rho$ from above by $\frac{\pi}{2}$ and obtain

$$I_j^1(R,\rho) \le \frac{1}{2\pi} \cdot \frac{\pi}{2} \int_{\Omega_1} |d\xi_j(\varphi)||d\xi_j(\psi)| \le \frac{1}{4}\, \lambda l(\sigma).$$

On Ω_2 we find

$$[\chi(r,\psi,\varphi)]_R^\rho \le \frac{\pi}{2} - \operatorname{arctg}\left\{ \frac{2R|\sin(\sigma/2)|}{1 - R^2} \right\}.$$

For $0 < \sigma < 1$ we certainly have $\sin(\sigma/2) \ge \sigma/4$, and so

$$[\chi(r,\psi,\varphi)]_R^\rho \le \frac{\pi}{2} - \operatorname{arctg} \frac{R\sigma}{2(1 - R^2)}$$

whence

$$I_j^2(R,\rho) \le \frac{\lambda^2}{2\pi} \left\{ \frac{\pi}{2} - \operatorname{arctg} \frac{R\sigma}{2(1 - R^2)} \right\}.$$

Then we obtain for $a_j(R,1) := \lim_{\rho\to 1-0} a_j(R,\rho)$ the estimate

$$a_j(R,1) \le \frac{\lambda}{4} l(\sigma) + \frac{\lambda^2}{2\pi}\left\{\frac{\pi}{2} - \operatorname{arctg}\frac{R\sigma}{2(1-R^2)}\right\} \quad \text{for } 0<\sigma<1.$$

Now we choose an arbitrary $\varepsilon > 0$; then there is some $\sigma \in (0,1)$ such that $l(\sigma) < \varepsilon/(2\lambda)$. Moreover we can find an $R \in (0,1)$ depending on ϵ and σ such that

$$\frac{\lambda^2}{2\pi}\left\{\frac{\pi}{2} - \operatorname{arctg}\frac{R\sigma}{2(1-R^2)}\right\} < \frac{\varepsilon}{8}.$$

Then we obtain

$$A_{B\setminus B_R}(H_j) = a_j(R,1) < \frac{\varepsilon}{4} \quad \text{for } B_R = \{w\colon |w|<R\},\ B := B_1,$$

and the same reasoning yields

$$A_{B\setminus B_R}(H) < \frac{\varepsilon}{4}.$$

On B_R we have $\nabla H_j \rightrightarrows \nabla H$; therefore there is a number $j_0 \in \mathbb{N}$ such that

$$|A_{B_R}(H) - A_{B_R}(H_j)| < \frac{\varepsilon}{2} \quad \text{for } j > j_0(\varepsilon).$$

It follows that

$$\begin{aligned} |A(H) - A(H_j)| &\le |A_{B_R}(H) - A_{B_R}(H_j)| + A_{B\setminus B_R}(H) + A_{B\setminus B_R}(H_j) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \quad \text{for } j > j_0(\varepsilon), \end{aligned}$$

and so: $A(H_j) \to A(H)$ as $j \to \infty$. □

For minimal surfaces we obtain a stronger convergence result:

Theorem 5. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are continuous on $\overline{B}$ and satisfy*

(i) $\|X_j - X\|_{C^0(\overline{B},\mathbb{R}^3)} \to 0$ *for some* $X \in C^0(\overline{B},\mathbb{R}^3)$;
(ii) $\int_{\partial B} |dX_j| \to \int_{\partial B} |dX|$ *as* $j \to \infty$.

Then X is a minimal surface in B, and

(22) $$\lim_{j\to\infty} D(X_j) = D(X).$$

Moreover, $X_j \to X$ in $H_2^1(B,\mathbb{R}^3)$.

Proof. By assumption we have in B

(23) $$\Delta X_j = 0 \quad \text{and} \quad |D_u X_j|^2 = |D_v X_j|^2, \quad \langle D_u X_j, D_v X_j\rangle = 0 \quad \text{in } B.$$

Furthermore relation (i) implies $\nabla^s X_j \rightrightarrows \nabla^s X$ on every $B' \subset\subset B$ and for any $s \geq 1$. Therefore (23) implies

(24) $$\Delta X = 0 \quad \text{and} \quad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B,$$

i.e. X is a minimal surface in B. From (23) and (24) we infer

$$D(X_j) = A(X_j) \quad \text{and} \quad D(X) = A(X),$$

and (17) of Theorem 4 yields $A(X_j) \to A(X)$. This implies (22). Finally, a standard reasoning shows that $X_j \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. Then, in conjunction with (22), we obtain $X_j \to X$ in $H_2^1(B, \mathbb{R}^3)$. □

An immediate consequence of this theorem are the next two results:

Corollary 3. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are of class $\overline{\mathcal{C}}^*(\Gamma)$ and satisfy $X_j \rightrightarrows X$ on $\overline{B}$. Then X is a minimal surface in B of class $\overline{\mathcal{C}}^*(\Gamma)$, and*

$$\|X - X_j\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Corollary 4. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are of class $\overline{\mathcal{C}}(\Gamma_j)$ and satisfy $X_j \rightrightarrows X$ on $\overline{B}$. We also assume that $\Gamma, \Gamma_1, \Gamma_2, \ldots$ are closed rectifiable Jordan curves in $\mathbb{R}^3$ such that $\Gamma_j \to \Gamma$ (in the sense of Fréchet), and that the lengths $L(\Gamma_j)$ of Γ_j tend to the length $L(\Gamma)$ of Γ. Then X is a minimal surface in B of class $\overline{\mathcal{C}}(\Gamma)$, and $D(X_j) \to D(X) < \infty$ as well as*

$$\|X - X_j\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Remark 1. If we in Theorem 3 of 4.3 assume in addition that $L(\Gamma_n) \to L(\Gamma)$ then the extracted subsequence $\{X_{n_p}\}$ of the quoted theorem also satisfies $\|X - X_{n_p}\|_{H_2^1(B,\mathbb{R}^3)} \to 0$ as $p \to \infty$.

6.5 When Is the Limes Superior of a Sequence of Paths Again a Path?

In the next section we need a generalization of the reasoning used in the proof of Theorem 1 of 6.2 to prove the existence of a minimizing path $\boldsymbol{p}^*$ joining two minimizers. Since we shall operate in the metric space $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, we shall formulate this generalization in the context of a general metric space (E, d) with a distance function d.

Let $\{M_j\}$ be a sequence of subsets M_j of E. Following the example of Hausdorff we define the **Limes Inferior of $\{M_j\}$** by

$$\liminf_{j\to\infty} M_j := \big\{x \in E\colon \text{there is a sequence of points } x_j \in M_j,\ j \in \mathbb{N},\ \text{with } d(x, x_j) \to 0\big\},$$

and the **Limes Superior of $\{M_j\}$** by

$$\limsup_{j\to\infty} M_j := \big\{x \in E\colon \text{there is an increasing sequence of indices } j_l \to \infty \text{ and a sequence of points } x_l \in M_{j_l} \text{ with } d(x, x_l) \to 0\big\}.$$

Proposition 1. *We have*

$$\limsup_{j\to\infty} M_j = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{j\geq k} M_j}. \tag{1}$$

Proof. Set

$$M := \limsup_{j\to\infty} M_j \quad \text{and} \quad M^* := \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{j\geq k} M_j}.$$

(i) Let $x \in M$; then $d(x, x_l) \to 0$ for some sequence of points $x_l \in M_{j_l}$ with increasing $j_l \to \infty$. Given $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $j_l \geq k$ for all $l \geq N$ whence $x_l \in M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots$ for $l \geq N$, and therefore $x \in \text{closure}(M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots)$ for any $k \in \mathbb{N}$. Hence $x \in M^*$, and consequently $M \subset M^*$.

(ii) Conversely let $x \in M^*$. Then $x \in \text{closure}(M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots)$ for all $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ we can find a point $x_k \in M_k \cup M_{k+1} \cup \cdots$ with $d(x, x_k) < 2^{-k}$. By induction we can now extract a subsequence $\{x_{j_l}\}$ of points $x_{j_l} \in M_{j_l}$ where $\{j_l\}$ is an increasing sequence of indices $j_l \to \infty$. Clearly, $d(x, x_{j_l}) \to 0$, and so $x \in M$, whence $M^* \subset M$. □

The following results are well known:

Proposition 2. *If $\{M_j\}_{j\in\mathbb{N}}$ is a family of connected subsets of E with the property $M_j \cap M_k \neq \emptyset$ for all $j, k \in \mathbb{N}$ then $\bigcup_{j\in\mathbb{N}} M_j$ is connected.*

Proposition 3. *If M is a connected subset of E then also $\overline{M}$.*

Proposition 4. *If $\{M_j\}$ is a sequence of compact, connected subsets M_j of E with $M_1 \supset M_2 \supset M_3 \supset \cdots$ then $\bigcap_{j\in\mathbb{N}} M_j$ is connected.*

Proof. See e.g. Alexandroff and Hopf [1], p. 118. □

A straight-forward consequence of Propositions 1–4 is:

Theorem 1. *If $\{M_j\}$ is a sequence of compact, connected subsets of (E, d) such that $M_j \cap M_k$ is nonempty for all $j, k \in \mathbb{N}$, and that $\bigcup_{j\geq k} M_j$ is relatively compact for any $k \in \mathbb{N}$, then $\limsup_{j\to\infty} M_j$ is connected and compact.*

We note that it was this result that we have used in 6.2 to establish the existence of a minimal path $\boldsymbol{p}^*$. Now we prove the following generalization of Theorem 1 that will be employed in 6.6. We use the following notation: A **path in $\boldsymbol{E}$** *is a nonempty compact, connected subset of E.*

Theorem 2. *Let* $\{M_n\}$ *be a sequence of paths in* (E, d) *such that* $\bigcup_{j\in\mathbb{N}} M_j$ *is relatively compact and* $\liminf_{j\to\infty} M_j$ *is nonempty. Then also* $\limsup_{j\to\infty} M_j$ *is a path in* (E, d).

Proof. Set $M := \limsup_{j\to\infty} M_j$. By (1), M is a closed subset of the compact subset closure$(M_1 \cup M_2 \cup M_3 \cup \cdots\}$, and so M is compact and nonempty, as $\liminf_{j\to\infty} M_j \subset M$.

Suppose now that M were not connected. Then there are two open sets Ω' and Ω'' in E such that the sets $M' := M \cap \Omega'$ and $M'' := M \cap \Omega''$ are nonvoid as well as disjoint and satisfy $M = M' \cup M''$. Clearly M' and M'' are compact subsets of E whence $\delta := \text{dist}(M', M'') > 0$. Set $\varepsilon := \delta/4$ and define the sets

$$M'_\varepsilon := \{x \in E\colon \text{ dist}(x, M') < \varepsilon\}, \quad M''_\varepsilon := \{x \in E\colon \text{ dist}(x, M'') < \varepsilon\}.$$

Moreover let x be an arbitrary point of $\liminf M_j$. Then there is a sequence $\{x_j\}$ of points $x_j \in M_j$ with $\text{dist}(x, x_j) \to 0$. We may assume that x is contained in M', because the case $x \in M''$ can be handled analogously. Then there is a number $N(\varepsilon) \in \mathbb{N}$ such that

$$M_j \cap M'_\varepsilon \neq \emptyset \quad \text{for all } j > N(\varepsilon).$$

Furthermore, since M'' is nonvoid there is a subsequence $\{M_{j_l}\}$ such that $M_{j_l} \cap M''_\varepsilon \neq \emptyset$ for all $l \in \mathbb{N}$; in addition we can assume that $j_l > N(\varepsilon)$ for all $l \in \mathbb{N}$. In this way we obtain

$$M_{j_l} \cap M'_\varepsilon \neq \emptyset \quad \text{and} \quad M_{j_l} \cap M''_\varepsilon \neq \emptyset \quad \text{for all } l \in \mathbb{N}.$$

Thus, for any $l \in \mathbb{N}$, we can choose points $x'_l \in M_{j_l} \cap M'_\varepsilon$ and $x''_l \in M_{j_l} \cap M''_\varepsilon$. Fix l; since M_{j_l} is connected there is a finite set $\{z_1, \dots, z_m\}$ of points in M_{j_l} with $z_1 = x'_l$, $z_m = x''_l$, and $d(z_k, z_{k+1}) < \varepsilon$ for $k = 1, \dots, m-1$ (see e.g. Querenburg [1], p. 46). Since $\text{dist}(M'_\varepsilon, M''_\varepsilon) > \delta - 2\varepsilon = 2\varepsilon$, at least one of the points $z_1, \dots, z_m$ is not contained in $M'_\varepsilon \cup M''_\varepsilon$, we call it y_l. Then $\{y_l\}$ is a sequence of points with $y_l \in M_{j_l}$ and

$$\text{dist}(y_l, M) \geq \varepsilon \quad \text{for } l \in \mathbb{N}. \tag{2}$$

As $\bigcup_{j\in\mathbb{N}} M_j$ is relatively compact, there is a subsequence $\{y_{l_k}\}$ of $\{y_l\}$ that converges to some point y, i.e. $d(y_{l_k}, y) \to 0$ as $k \to \infty$. By definition of M we have $y \in M$, contrary to (2). Thus the compact set is also connected, and so M is a path. □

6.6 Unstable Minimal Surfaces in Rectifiable Boundaries

Now we want to carry over the results of Section 6.3 to rectifiable closed Jordan curves Γ in $\mathbb{R}^3$ that satisfy a (global) *chord-arc condition* of the following kind:

There is a positive constant μ such that for any two points x_1, x_2 of Γ we have

$$(1) \qquad \ell(x_1, x_2) \le \mu |x_1 - x_2|$$

where $\ell(x_1, x_2)$ is the length of the shorter one of the two subarcs of Γ bounded by x_1 and x_2.

This assumption will be denoted as **Condition (μ)**.

By $\overline{\mathcal{C}}^*(\Gamma)$ we denote the class of surfaces $X \in \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$ which satisfy some fixed preassigned 3-point condition

$$(2) \qquad X(w_k) = Q_k, \quad k = 0, 1, 2, \quad w_k := \exp(i\psi_k), \quad \psi_k = \frac{2\pi k}{3},$$

where Q_0, Q_1, Q_2 are three fixed points on Γ.

Furthermore let $\mathcal{H}^*(\Gamma)$ be the subset of mappings $X \in \overline{\mathcal{C}}^*(\Gamma)$ which are harmonic in B. As in 6.3 we could equip both $\overline{\mathcal{C}}^*(\Gamma)$ and $\mathcal{H}^*(\Gamma)$ with the distance function

$$(3) \qquad d_1(X, Y) := \|X - Y\|_{1,B}, \quad X, Y \in \overline{H}{}_2^1(B, \mathbb{R}^3),$$

which is derived from the norm

$$(4) \qquad \|X\|_{1,B} := \|X\|_{C^0(\overline{B}, \mathbb{R}^3)} + \sqrt{D(X)}$$

of the Banach space $\overline{H}{}_2^1(B, \mathbb{R}^3) := H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$. Unfortunately we have to work with the metric space $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_0)$,

$$(5) \qquad d_0(X, Y) := \|X - Y\|_{C^0(\overline{B}, \mathbb{R}^3)},$$

as we are unable to apply Theorem 2 of 6.5 in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_1)$. Instead we can obtain a version of this result in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_0)$; this will be stated as Lemma 2. We note that this deficiency is the reason why we cannot carry over the results obtained for polygons in full strength to general boundary contours Γ. We begin our discussion of the general case with a suitable approximation device.

Lemma 1. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Then there exists a sequence $\{\Gamma_j\}$ of simple, closed polygons Γ_j in $\mathbb{R}^3$ and a sequence of homeomorphisms $\phi_j : \Gamma \to \Gamma_j$ from Γ onto Γ_j such that the following holds*:

(i) $Q_1, Q_2, Q_3 \in \Gamma_j$ *for all* $j \in \mathbb{N}$.
(ii) Γ_j *has* N_j+3 ($\ge$4) *consecutive vertices which lie on* Γ, *given by* $Q_0, A_1(j), \dots, A_{l_j}(j), Q_1, A_{l_j+1}(j), \dots, A_{m_j}(j), Q_2, A_{m_j+1}(j), \dots, A_{N_j}(j), Q_0$.
(iii) $\Delta(\Gamma_j) \to 0$ *where* $\Delta(\Gamma_j)$ *denotes the length of the largest edge of* Γ_j.

(iv) *The length* $L(\Gamma_j)$ *of the polygons* Γ_j *tends to the length* $L(\Gamma)$ *of* Γ.

(v) *We have*

$$\max_{x \in \Gamma} |x - \phi_j(x)| \to 0 \quad \textit{as } j \to \infty$$

and $\phi_j(x) = x$ *if* x *is a vertex of* Γ_j.

(vi) *The subarc on* Γ *bounded by two consecutive vertices of* Γ_j *is the shorter one of the two subarcs of* Γ *bounded by these vertices if it contains no other vertex of* Γ_j.

(vii) *For any* $x', x'' \in \Gamma$ *and any* $j \in \mathbb{N}$ *we have*

$$|\phi_j(x') - \phi_j(x'')| \le l(x', x'')$$

where $l(x', x'')$ *is the length of the shorter arc on* Γ *with endpoints* x' *and* x''.

We call $\{\Gamma_j\}$ *an* **approximating sequence of inscribed polygons for** $\boldsymbol{\Gamma}$ **and** $\boldsymbol{Q_0, Q_1, Q_2}$.

The proof of this lemma is tedious, but elementary, and will therefore be omitted.

Lemma 2. *Let* $\{\Gamma_j\}$ *be an approximating sequence of inscribed polygons for* Γ, *and* $Q_0, Q_1, Q_2 \in \Gamma$, *and* $\{P_j\}$ *be a sequence of paths (i.e. compact and connected sets)* P_j *in* $(\mathcal{H}^*(\Gamma_j), d_0)$ *such that*

(6) $$\sup\{D(X) \colon X \in P_j\} \le \kappa \quad \textit{for all } j \in \mathbb{N} \textit{ and some } \kappa > 0.$$

Moreover, suppose that there is a sequence $\{Y_j\}$ *of points* $Y_j \in \mathcal{H}^*(\Gamma_j)$ *with*

(7) $$d_0(Y_j, Y) \to 0 \quad \textit{as } j \to \infty \textit{ for some } Y \in \overline{H}{}_2^1(B, \mathbb{R}^3).$$

Then $P := \limsup_{j\to\infty} P_j$ *is a path in* $(\mathcal{H}^*(\Gamma), d_0)$, *and*

(8) $$\sup\{D(X) \colon X \in P\} \le \kappa.$$

Proof. By Theorem 3 of 4.3 we obtain that $\bigcup_{j \in \mathbb{N}} P_j$ is relatively compact in $(\overline{H}{}_2^1(B, \mathbb{R}^N), d_0)$, and (7) implies that $\liminf_{j\to\infty} P_j$ is nonempty. Then we infer from 6.5, Proposition 1 and Theorem 2, and 4.3, Theorem 3, that $P := \limsup_{j\to\infty} P_j$ is a path in $(\mathcal{H}^*(\Gamma), d_0)$, and (8) follows from (6) since D is sequentially weakly lower semicontinuous on $H_2^1(B, \mathbb{R}^3)$. □

Remark 1. A path in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_1)$ is also a path in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_0)$. This ensues from the following two statements:

1° A d_1-compact set in $\overline{H}{}_2^1(B, \mathbb{R}^3)$ is also d_0-compact.

2° A d_1-connected set in $\overline{H}{}_2^1(B, \mathbb{R}^3)$ is as well d_0-connected.

Proof. (a) We first recall that a subset of a metric space is compact if and only if it is sequentially compact.

Let $M \subset \overline{H}_2^1(B, \mathbb{R}^3)$ be d_1-compact, and $\{x_j\}$ be a sequence in M. Then there exists a subsequence $\{x_{j_k}\}$ with $d_1(x_{j_k}, x) \to 0$ for some $x \in M$. It follows that $d_0(x_{j_k}, x) \to 0$; consequently M is d_0-compact.

(b) Suppose now that $M \subset \overline{H}_2^1(B, \mathbb{R}^3)$ is d_1-connected, but not d_0-connected. Then there exist two d_0-closed sets M' and M'' which are nonvoid and satisfy $M = M' \cup M''$ and $M' \cap M'' = \emptyset$. We claim that both M' and M'' are d_1-closed. For instance, if $\{x_j\}$ is a sequence in M' with $d_1(x_j, x) \to 0$ then $d_0(x_j, x) \to 0$, and therefore $x \in M'$ since M' is d_0-closed. Analogously we see that M'' is d_1-closed. Consequently M is d_1-disconnected, contrary to our assumption. □

Remark 2. In virtue of Remark 1 we can carry over the results of 6.3, Theorems 1–3, from $(\overline{\mathcal{C}}^*(\Gamma), d_1)$ to $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, Γ being a closed simple polygon. We merely have to replace expressions of the kind $\max_P D$ for paths P by $\sup_P D$ since D is no longer continuous on a d_0-path. Of course, $\sup_P D$ could be infinite, and it might be infinite for any path P containing two given points $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$. It will be seen later that the latter does not occur; cf. Lemma 6.

Let us now consider an arbitrary boundary contour Γ satisfying Condition (μ), and three points $Q_0, Q_1, Q_2 \in \Gamma$. We choose an approximating sequence $\{\Gamma_j\}$ of inscribed polygons Γ_j for Γ and Q_0, Q_1, Q_2. As in 6.1 we define for each Γ_j the set T_j of points $t = (t^1, \dots, t^{N(j)}) \in \mathbb{R}^{N(j)}$ satisfying

$$\psi_0 < t^1 < \cdots < t^{l_j} < \psi_1 < t^{l_j+1} < \cdots < t^{m_j} < \psi_2 < t^{m_j+1} < \cdots < t^{N_j} < \psi_3,$$

$\psi_k := \frac{2k\pi}{3}$, $k = 0, 1, 2, 3$, and $\overline{\mathcal{C}}^*(\Gamma_j)$ and $\mathcal{H}^*(\Gamma_j)$ are the subsets of mappings X of class $\overline{\mathcal{C}}(\Gamma_j)$ or $\mathcal{H}(\Gamma_j)$ respectively satisfying (2).

With every $t \in T_j$ we associate the set

$$(9) \qquad U_j(t) := \{X \in \overline{\mathcal{C}}^*(\Gamma_j) \colon X(\exp(it^k)) = A_k(j),\ k = 1, \dots, N_j\},$$

and the corresponding Courant function $\Theta_j : T_j \to \mathbb{R}$ is defined by

$$(10) \qquad \Theta_j(t) := \inf\{D(X) \colon X \in U_j(t)\} \quad \text{for } t \in T_j.$$

Furthermore the associated Courant mapping $Z_j : T_j \to \overline{\mathcal{C}}^*(\Gamma_j)$ is the mapping $t \mapsto Z_j(t)$ where $Z_j(t)$ is the uniquely determined minimizer of D in $U_j(t)$, i.e.

$$(11) \qquad \Theta_j(t) = D(Z_j(t)) \quad \text{for } t \in T_j.$$

The beautiful properties of Z_j and Θ_j are discussed in 6.1 and 6.3. We set

$$(12) \qquad \mathcal{W}_j^*(\Gamma_j) := Z_j(T_j), \quad j \in \mathbb{N}.$$

Furthermore Θ_j is of class $C^1(T_j)$, and the minimal surfaces of class $\overline{\mathcal{C}}^*(\Gamma_j)$ are in 1–1 correspondence with the critical points of Θ_j in T_j.

Lemma 3. *Given $H \in \mathcal{H}^*(\Gamma)$ there are points $t_j = (t_j^1, \dots, t_j^{N_j}) \in T_j$ such that*

$$H(\exp(it_j^k)) = A_k(j) \quad \text{for } 1 \le k \le N_j \text{ and for all } j \in \mathbb{N}. \tag{13}$$

Then

$$d_1(Z_j(t_j), H) \to 0 \quad \text{as } j \to \infty. \tag{14}$$

Proof. The first statement is obvious since H satisfies the Plateau boundary condition.

Let now H_j be the harmonic extension of $\phi_j(H|_{\partial B})$ to B, and set $Y_j := H_j - H$. Then, by (vii) of Lemma 1 and (1), we obtain for any $\alpha, \beta \in \mathbb{R}$ that

$$\begin{aligned} |Y_j(e^{i\alpha}) - Y_j(e^{i\beta})| &\le |H_j(e^{i\alpha}) - H_j(e^{i\beta})| + |H(e^{i\alpha}) - H(e^{i\beta})| \\ &\le l(H(e^{i\alpha}), H(e^{i\beta})) + |H(e^{i\alpha}) - H(e^{i\beta})| \\ &\le (\mu + 1)|H(e^{i\alpha}) - H(e^{i\beta})|. \end{aligned}$$

Furthermore, $\phi_j(H|_{\partial B}) \to H|_{\partial B}$ in $C^0(\partial B, \mathbb{R}^3)$, i.e. $Y_j \to 0$ in $C^0(\partial B, \mathbb{R}^3)$. Then 6.4, Theorem 2 implies

$$d_1(H_j, H) \to 0 \quad \text{as } j \to \infty. \tag{15}$$

From (13) it follows that $H_j \in U_j(t_j)$, $j \in \mathbb{N}$, and then

$$D(Z_j(t_j)) \le D(H_j) \le \text{const} \quad \text{for all } j \in \mathbb{N} \tag{16}$$

because of (10), (11), and (15). Since $Z_j(t_j)$ and H_j lie in $U_j(t_j)$, we infer from Lemma 1(iii) that

$$\|Z_j(t_j) - H_j\|_{C^0(\partial B, \mathbb{R}^3)} \le \Delta(\Gamma_j) \to 0 \quad \text{as } j \to \infty.$$

In conjunction with (15) we arrive at

$$d_0(Z_j(t_j), H) \to 0 \quad \text{as } j \to \infty. \tag{17}$$

Because of (16) we can extract from any subsequence of $\{Z_j(t_j)\}$ another subsequence $\{Z_{j_k}(t_{j_k})\}$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ and therefore strongly in $L_2(B, \mathbb{R}^3)$ to some element X, and (17) implies $X = H$. Hence

$$Z_{j_k}(t_{j_k}) \rightharpoonup H \quad \text{in } H_2^1(B, \mathbb{R}^3) \text{ as } k \to \infty, \tag{18}$$

and consequently

$$D(H) \le \liminf_{k \to \infty} D(Z_{j_k}(t_{j_k})).$$

On the other hand,

$$\limsup_{k \to \infty} D(Z_{j_k}(t_{j_k})) \le D(H)$$

in virtue of (15) and (16), and so we obtain

$$D(Z_{j_k}(t_{j_k})) \to D(H).$$

In conjunction with (18) we arrive at $Z_{j_k}(t_{j_k}) \to H$ in $H_2^1(B, \mathbb{R}^3)$, and then a standard reasoning implies

$$\|Z_j(t_j) - H\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Together with (17) we finally have (14). □

Lemma 4. *Equip* $\overline{\mathcal{C}}^*(\Gamma)$ *with the metric* d_0; *then there is a continuous mapping* $(X, \lambda) \mapsto R(X, \lambda)$ *from* $\overline{\mathcal{C}}^*(\Gamma) \times [0, 1]$ *into* $\overline{\mathcal{C}}^*(\Gamma)$ *such that* $R(X, 0) = X$, $R(X, 1) = H \in \mathcal{H}^*(\Gamma)$ *with* $X|_{\partial B} = H|_{\partial B}$, *and* $d(\lambda) := D(R(X, \lambda))$ *decreases from* $d(0) = D(X)$ *to* $d(1) = D(H)$.

Proof. Choose $H \in \mathcal{H}^*(\Gamma)$ with $H|_{\partial B} = X|_{\partial B}$ for some $X \in \overline{\mathcal{C}}^*(\Gamma)$, and set

$$R(X, \lambda) := \lambda H + (1 - \lambda)X = H + (1 - \lambda)(X - H) \quad \text{for } 0 \le \lambda \le 1.$$

By Dirichlet's principle we have $D(H, \phi) = 0$ for all $\phi \in H_2^1(B, \mathbb{R}^3)$ with $\phi|_{\partial B} = 0$ whence

$$D(R(X, \lambda)) = D(H) + (1 - \lambda)^2 D(X - H) \quad \text{for } 0 \le \lambda \le 1. \qquad \square$$

Let $\mathbb{P}(X_1, X_2)$ be the set of all paths P in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$ with $X_1, X_2 \in P$.

Lemma 5. *Let* $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$, *and* $H_1, H_2 \in \mathcal{H}^*(\Gamma)$ *be the harmonic mappings with* $H_k|_{\partial B} = X_k|_{\partial B}$, $k = 1, 2$. *Then we have*:

(i) $\mathbb{P}(X_1, X_2)$ *is nonvoid if and only if* $\mathbb{P}(H_1, H_2)$ *is nonvoid.*
(ii) *Assume that* $\mathbb{P}(H_1, H_2)$ *is nonvoid. Then*

$$\sup_P D > \max\{D(X_1), D(X_2)\} \quad \text{for all } P \in \mathbb{P}(X_1, X_2) \tag{19}$$

implies

$$\sup_P D > \max\{D(H_1), D(H_2)\} \quad \text{for all } P \in \mathbb{P}(H_1, H_2). \tag{20}$$

Proof. If $P \in \mathbb{P}(X_1, X_2)$ then $R(P, 1) \in \mathbb{P}(H_1, H_2)$. Conversely, if $P \in \mathbb{P}(H_1, H_2)$ and $P_1 := \{R(X_1, \lambda) \colon 0 \le \lambda \le 1\}$, $P_2 := \{R(X_2, \lambda) \colon 0 \le \lambda \le 1\}$ then $\tilde{P} := P_1 \cup P \cup P_2 \in \mathbb{P}(X_1, X_2)$, and so (i) is proved.

Suppose that there is some $P \in \mathbb{P}(H_1, H_2)$ with $\sup_P D \le \max\{D(H_1), D(H_2)\}$. Then $\sup_{\tilde{P}} D \le \max\{D(X_1), D(X_2)\}$ on account of Lemma 4. Hence (19) implies (20). □

Lemma 6. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Then any two points H_1 and H_2 of $\mathcal{H}^*(\Gamma)$ with $H_1 \neq H_2$ can be joined by a path P^* (i.e. a compact connected subset) in $(\mathcal{H}^*(\Gamma), d_0)$ such that*

$$\sup_{P^*} D \leq \max\left\{ D(H_1), D(H_2), \frac{1}{4\pi} L^2(\Gamma) \right\}. \tag{21}$$

Proof. Let $\{\Gamma_j\}$ be an approximating sequence of inscribed polygons Γ_j for Γ and Q_0, Q_1, Q_2 with the associated points $t_{j,1}$ and $t_{j,2}$ in $T_j \subset \mathbb{R}^{N_j}$ for H_1 and H_2 such that $\phi_j(H_1|_{\partial B})$ and $\phi_j(H_2|_{\partial B})$ are the boundary values of harmonic mappings $H_{j,1}$ and $H_{j,2}$ in $U_j(t_{j,1})$ and $U_j(t_{j,2})$ respectively; see Lemma 3. Set

$$Y_{j,1} := Z_j(t_{j,1}), \quad Y_{j,2} := Z_j(t_{j,2}), \quad j \in \mathbb{N}.$$

In virtue of Lemma 3, (14) we have

$$d_1(Y_{j,1}, H_1) \to 0 \quad \text{and} \quad d_1(Y_{j,2}, H_2) \to 0 \quad \text{as } j \to \infty, \tag{22}$$

in particular

$$D(Y_{j,1}) \to D(H_1) \quad \text{and} \quad D(Y_{j,2}) \to D(H_2). \tag{23}$$

Consider the set $\mathcal{P}_j := \mathcal{P}_j(t_{j,1}, t_{j,2})$ of all paths $\boldsymbol{p}$ in T_j joining $t_{j,1}$ and $t_{j,2}$. By (22) and $H_1 \neq H_2$ we may assume that $t_{j,1} \neq t_{j,2}$ for all $j \in \mathbb{N}$. In virtue of 6.3, Theorem 4, there is a minimal path $\boldsymbol{p}_j^*$ in $\mathcal{P}_j$ such that

$$c_j := \max_{X \in Z_j(\boldsymbol{p}_j^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}_j} \max_{X \in Z_j(\boldsymbol{p})} D(X), \quad j \in \mathbb{N}.$$

We claim that for all $j \in \mathbb{N}$

$$c_j \leq \max\left\{ D(Y_{j,1}), D(Y_{j,2}), \frac{1}{4\pi} L^2(\Gamma_j) \right\}. \tag{24}$$

In fact, suppose that

$$c_j > \max\{D(Y_{j,1}), D(Y_{j,2})\}.$$

Then it follows from Theorem 4 of 6.3 that there is an unstable minimal surface $Y_j \in P_j^* := Z(\boldsymbol{p}_j^*)$ with $c_j = D(Y_j)$. The isoperimetric inequality yields

$$c_j \leq \frac{1}{4\pi} L^2(\Gamma_j),$$

and so we have (24). On the other hand, if $c_j \leq \max\{D(Y_{j,1}), D(Y_{j,2})\}$, (24) is clearly fulfilled, and so we have verified (24) for all $j \in \mathbb{N}$.

In conjunction with (23) and $L(\Gamma_j) \to L(\Gamma)$ we conclude that the sequence $\{c_j\}$ is bounded, and so (passing to a subsequence and renaming it) we may assume that $c_j \to \kappa$ for some $\kappa \geq 0$. Then H_1 and H_2 lie in

$\liminf_{j\to\infty} P_j^*$, and Lemma 2 implies that $P^* := \limsup_{j\to\infty} P_j^*$ is a path in $(\mathcal{H}^*(\Gamma), d_0)$ joining H_1 and H_2. The weak lower semicontinuity D with respect to weak convergence in $H_2^1(B, \mathbb{R}^3)$ in conjunction with the definition of P^* yields $\sup_{P^*} D \le \kappa$, and so we arrive at

$$\text{(25)} \qquad \sup_{P^*} D \le \kappa = \lim_{j\to\infty} c_j \le \max\left\{ D(H_1), D(H_2), \frac{1}{4\pi} L^2(\Gamma) \right\}. \qquad \square$$

Lemma 7. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Suppose also that H_1, H_2 are two different points of $(\mathcal{H}^*(\Gamma), d_0)$ such that*

$$\text{(26)} \qquad \sup_{P'} D > \max\{D(H_1), D(H_2)\} \quad \textit{for all } P' \in \mathbb{P}'(H_1, H_2)$$

where $\mathbb{P}'(H_1, H_2)$ denotes the set of all paths in $(\mathcal{H}^*(\Gamma), d_0)$ joining H_1 and H_2. Then there exists some path $P^* \in (\mathcal{H}^*(\Gamma), d_0)$ and some minimal surface $H_3 \in P^*$ with

$$\text{(27)} \qquad D(H_3) = c := \sup_{P^*} D$$

which is d_0-unstable, i.e. in every d_0-neighborhood of H_3 there exists an $X \in \overline{\mathcal{C}}^*(\Gamma)$ such that $D(X) < D(H_3)$.

Proof. Let $P^* \in \mathbb{P}'(H_1, H_2)$ be the path constructed in the proof of Lemma 6. Then

$$c := \sup_{P^*} D > \max\{D(H_1), D(H_2)\}$$

in virtue of (26). By (25) we have

$$c \le \kappa = \lim_{j\to\infty} c_j, \quad c_j := \max_{P_j^*} D.$$

In conjunction with (23) we then obtain

$$\max_{P_j^*} D > \max\{D(Y_{j,1}), D(Y_{j,2})\} \quad \text{for } j \gg 1.$$

Using the proof of Theorem 4 in 6.3 we conclude that for $j \gg 1$ there is a minimal surface $X_j \in P_j^*$ satisfying $D(X_j) = c_j$, and a standard reasoning (cf. 4.3, Theorem 3) shows that there is a subsequence $\{X_{j_k}\}$ and a minimal surface $X_0 \in P^*$ such that $d_0(X_{j_k}, X_0) \to 0$ and $X_{j_k} \rightharpoonup X_0$ in $H_2^1(B, \mathbb{R}^3)$. Furthermore, $L(\Gamma_j) \to L(\Gamma)$. Then we infer $D(X_{j_k}) \to D(X_0)$ on account of 6.4, Theorem 5. Therefore

$$\kappa = \lim_{k\to\infty} c_{j_k} = D(X_0) \le c.$$

Thus the minimal surface $X_0 \in P^*$ satisfies $D(X_0) = c$. In order to obtain an unstable minimal surface $H_3 \in P^*$ with $D(H_3) = c$ we consider the set K_c

of all minimal surfaces $H \in P^*$ with $D(H) = c$ which is a closed subset of (P^*, d_0) on account of 4.3, Theorem 3, and 6.4, Theorem 5. Furthermore K_c is a nonvoid and proper subset of P^* since $X_0 \in K_c$ and $H_1, H_2 \notin K_c$. Hence, on account of the connectedness of P^*, there exists a boundary point H_3 of K_c with $H_3 \in K_c$, therefore

$$\mathcal{N}_\varepsilon := (P^* \setminus K_c) \cap \{X : d_0(X, H_3) < \varepsilon\} \neq \emptyset \quad \text{for every } \varepsilon > 0.$$

If for any $\varepsilon > 0$ there is an $X \in \mathcal{N}_\varepsilon$ with $D(X) < c$, we have shown that H_3 is unstable. It remains to consider the case when we have $D(X) = c$ for all $X \in \mathcal{N}_\varepsilon$ and any $\varepsilon \in (0, \varepsilon_0)$ for some positive ε_0. Pick some $X \in \mathcal{N}_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_0)$. Then $D(X) = c$, and X is harmonic as $\mathcal{N}_\varepsilon \subset P^* \subset \mathbb{P}'$. Since $X \notin K_c$ we conclude that X is not conformal; therefore we have

$$\partial D(X, \lambda) \neq 0 \tag{28}$$

for some vector field $\lambda \in C^2(\overline{B}, \mathbb{R}^2)$ that is tangential at ∂B. Then we can find a C^1-family $\sigma(\cdot, t) : \overline{B} \to \overline{B}$ of diffeomorphisms of $\overline{B}$ onto itself such that $\sigma(w, 0) = w$ for all $w \in \overline{B}$ and

$$\frac{d}{dt} D(X \circ \sigma(\cdot, t))\bigg|_{t=0} < 0,$$

whence $D(X \circ \sigma(\cdot, t)) < D(X) = c$ for $0 < t \ll 1$. Thus $D(Y) < D(X)$ for $Y := X \circ \sigma(\cdot, t) \in \overline{\mathcal{C}}(\Gamma)$ as well as $d_0(Y, X) \ll 1$ for $0 < t \ll 1$, and therefore $d_0(Y, H_3) < \varepsilon$ for $0 < t \ll 1$. By the reasoning of 6.1, Proposition 8, one can achieve that (28) holds for some admissible λ with $\lambda(w_k) = 0$ for $w_k = \exp(i\psi_k)$, $k = 0, 1, 2$. Approximating λ in a suitable way one can construct $\sigma(\cdot, t)$ in such a way that also $\sigma(w_k, t) = w_k$ for $|t| \ll 1$ is fulfilled, and therefore Y lies even in $\overline{\mathcal{C}}^*(\Gamma)$. □

Now we can state the main result of this section.

Theorem 1. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let X_1, X_2 be two points of $(\overline{\mathcal{C}}^*(\Gamma), d_0)$ such that $X_1|_{\partial B} \neq X_2|_{\partial B}$ and*

$$\sup_P D > \max\{D(X_1), D(X_2)\} \quad \textit{for all } P \in \mathbb{P}(X_1, X_2). \tag{29}$$

Then there exists a D-unstable (and therefore also A-unstable) minimal surface $X_3 \in (\overline{\mathcal{C}}^*(\Gamma), d_0)$, i.e. for any $\varepsilon > 0$ there is an $X \in \overline{\mathcal{C}}^*(\Gamma)$ with $d_0(X, X_3) < \varepsilon$ and $D(X) < D(X_3)$.

Proof. Let $H_1, H_2 \in \mathcal{H}^*(\Gamma)$ be the harmonic surfaces with $H_k|_{\partial B} = X_k|_{\partial B}$, $k = 1, 2$. By Lemma 6 the set $\mathbb{P}'(H_1, H_2)$ is nonvoid, and so also $\mathbb{P}(X_1, X_2)$ is nonvoid according to Lemma 5(i). Thus assumption (29) makes sense, and Lemma 5(ii) yields

$$\sup_P D > \max\{D(H_1), D(H_2)\} \quad \text{for all } P \in \mathbb{P}(H_1, H_2).$$

Since $\mathbb{P}'(H_1, H_2) \subset \mathbb{P}(H_1, H_2)$ we also have

$$\sup_{P'} D > \max\{D(H_1), D(H_2)\} \quad \text{for all } P' \in \mathbb{P}'(H_1, H_2).$$

Moreover, $X_1|_{\partial B} \neq X_2|_{\partial B}$ implies $H_1 \neq H_2$. Now the assertion follows from Lemma 7 and from 6.3, Remark 2. □

As a corollary of the preceding result we obtain

Theorem 2. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ be two different minimal surfaces, each furnishing a strict local minimum for D in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, i.e.*

(30) $\quad D(X_k) < D(X) \quad$ *for any $X \in \overline{\mathcal{C}}^*(\Gamma)$ with $d_0(X, X_k) < \varepsilon$,* $\quad k = 1, 2,$

for any positive $\varepsilon \ll 1$. Then there is a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which is both D-unstable and A-unstable in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$.

Another corollary of Theorem 1 is the following result:

Theorem 3. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ be two minimal surfaces separated by a wall, i.e. which satisfy* (29)*. Then there exists a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which is both D-unstable and A-unstable in $\overline{\mathcal{C}}^*(\Gamma)$.*

6.7 Scholia

6.7.1 Historical Remarks and References to the Literature

The results of the present chapter are a special part of Morse theory that formerly ran under the headline "Theorem of the Wall". Nowadays one speaks of the "Mountain Pass Theorem", referring to the path-breaking work by A. Ambrosetti and P. Rabinowitz [1]. A presentation of applications of this theorem to various variational problems can be found in the texts of M. Struwe [13] and E. Zeidler [1]. Originally Morse theory worked very well for one-dimensional variational integrals, say, geodesics whereas already minimal surfaces lead to enormous difficulties. In a remarkable competition, the first results were found by M. Shiffman [2–5] and by M. Morse & C. Tompkins [1–5] almost simultaneously. Of particular interest is that work by Morse & Tompkins which uses the "theorem of the wall", while their general Morse-theoretic statements are more or less useless as they are based on topological assumptions which cannot be verified in a concrete situation. A general Morse theory for minimal surfaces in $\mathbb{R}^4$ was developed by M. Struwe [4,8] and J. Jost & M. Struwe [1]. Furthermore A. Tromba [10–12] obtained a Morse-theoretic result for minimal surfaces in three-dimensional space which is presented in the last chapter of Vol. 3.

Somewhat later than Shiffman and Morse & Tompkins, Courant found a new approach to the "theorem of the wall" that works for minimal surfaces bounded by a polygonal contour; cf. R. Courant [13] and [15], and Shiffman [4] showed how Courant's "polygonal theory" can also be used to establish the "theorem of the wall" for rectifiable boundary contours. This work was carried over by E. Heinz [12–14] to surfaces of constant mean curvature H with $|H| < \frac{1}{(2R)}$ which are contained in a ball of radius R. In his remarkable paper [2], Ströhmer was able to establish the "theorem of the wall" for surfaces of prescribed mean curvature $H(x)$ under the most general assumption $|H(x)| \le \frac{1}{R}$. Previously G. Ströhmer [1] had generalized the Courant–Shiffman theory to minimal surfaces in a Riemannian manifold of nonpositive sectional curvature. Further contributions by Ströhmer concern the semi-free problem [3], and in [4] the Plateau problem for more general integrals. M. Shiffman [8] developed a "mountain pass theorem" for general parametric integrals of the type

$$\mathcal{F}(X) = \int_B F(X_u \wedge X_v)\, du\, dv.$$

Unfortunately, Shiffman's reasoning is not stringent, as has been pointed out by R. Jakob (cf. [2], p. 403). Nevertheless, Shiffman's paper contains quite ingenious ideas which, combined with the technique developed by Courant and Heinz, enabled R. Jakob to establish a modified version of Shiffman's theory (see Jakob [1,2,4,5]).

6.7.2 The Theorem of the Wall for Minimal Surfaces in Textbooks

The first textbook presentation can be found in Courant [15], Chapter VI, Sections 7 and 8. The results of 6.6 can also be derived by using Courant's *pinching lemma* 6.10 (cf. pp. 236–237, 241–243) instead of Lemma 1 in 6.6.

J.C.C. Nitsche gave a detailed and very precise presentation of Shiffman's approach to unstable minimal surfaces in §§419–433 of his treatise [28], with applications to several examples in §§ 434–436.

An interesting and completely new approach to the "mountain pass theorem" for minimal surfaces, based on Douglas's functional, was given by M. Struwe [11], with a correction in Imbusch and Struwe [1]. In this work, an infinite-dimensional version of the mountain-pass lemma is used to prove the existence of unstable minimal surfaces directly for boundary contours Γ of class C^2, without the detour of approximating Γ by polygonal contours Γ_j. This enabled Struwe to work with a metric d_1 instead of d_0, just as we did in 6.3, which leads to somewhat stronger existence results than those in 6.6 for $\Gamma \in C^2$, i.e. to results as presented in 6.3 for polygonal Γ. It is a challenging problem to carry over Struwe's approach to related problems.

J. Jost [17], Corollary 4.4.11, proved the following result: *Let Γ be a closed Jordan curve in a compact Riemannian manifold that contains no minimal spheres (e.g. if the sectional curvature of M is nonpositive). Suppose that Γ bounds two homotopic minimal surfaces $X_1, X_2 : B \to M$ both of which are strict relative minima of Dirichlet's integral D (with respect to the C^0- or H_2^1-topology). Then there exists a third minimal surface $X_3 : B \to M$ bounded by Γ and satisfying*

$$D(X_3) > \max\{D(X_1), D(X_2)\}.$$

He also noted that for proving such a result the compactness of M is not really needed; it is sufficient to assume that $X_1(B)$ and $X_2(B)$ lie in a bounded, strictly convex subset of N, without further restrictions on M. Therefore one in particular obtains a corresponding result in $\mathbb{R}^n$. Moreover, Ströhmer's results in his papers [1] and [4] can be obtained in this way.

Another "instability result" of J. Jost [17] is his Theorem 4.6.1 which holds for *Jordan curves Γ of class C^2 in $\mathbb{R}^n$: Let $X_1, X_2 : B \to \mathbb{R}^n$ be minimal surfaces of class $\mathcal{C}^*(\Gamma)$ such that $X_1(B) \neq X_2(B)$.* (i) *If both X_1 and X_2 are strict local minimizers, then there is a third minimal surface in $\mathcal{C}^*(\Gamma)$ which is unstable.* (ii) *If both X_1 and X_2 are global minimizers, then one either has a third and unstable minimal surface X_3, or there is a continuous family $X(\cdot, t)$ with $X(\cdot, 0) = X_1$, $X(\cdot, 1) = X_2$, and $D(X(\cdot, t)) \equiv$ const.*

Generalizations of this result are indicated in Jost [17], p. 160, Remark (1).

6.7.3 Sources for This Chapter

In writing this chapter we have extensively used Courant's work in [15], Heinz's papers [13] and [14], as well as a first draft by R. Jakob. In addition, Jakob's papers [1,2] and several lectures that he gave to us were of help; we are very grateful for his support in drawing up the material and for his criticism of our first draft.

6.7.4 Multiply Connected Unstable Minimal Surfaces

In [8] Struwe used his approach from [11] to prove the existence of unstable minimal surfaces of annulus type. J. Hohrein [1] discussed the existence of unstable minimal surfaces of higher genus in Riemannian manifolds of non-positive curvature, employing ideas of Struwe [4]. Unstable minimal surfaces X of annulus type in a Riemannian manifold M were studied by H. Kim [1] assuming that the boundary of X lies in a ball $B_r(p)$ of normal range, which in particular means that the radius r of the ball satisfies $r < \pi/(2\sqrt{\kappa})$ where κ is an upper bound for the sectional curvature of M.

6.7.5 Quasi-Minimal Surfaces

It is not known whether the Courant function Θ associated with a polygon Γ is of class C^2, and so it is impossible to develop a Morse theory for Θ. To overcome this difficulty, Marx and Shiffman have set up a modified variational problem which leads to a modification Θ^* of Θ with a much better behavior; cf. Courant [15], pp. 235–236, and I. Marx [1]. The original work of Shiffman was never published, and the proofs given in Marx's paper are incomplete (see e.g. E. Heinz [20], p. 84, and [25], pp. 200–201). A satisfactory theory of the *variational problem of Marx–Shiffman* was developed only much later by E. Heinz [19–24], with further contributions by F. Sauvigny [1–3,6], and R. Jakob [6–10]. In the sequel we shall present a brief summary of this work.

Let Γ be a simple closed polygon with $N+3$ (≥ 4) consecutive vertices $Q_1, Q_2, \dots, Q_{N+3}$, and set $Q_{N+4} := Q_1$, $Q_0 := Q_{N+3}$. Consider the set of points $t = (t^1, t^2, \dots, t^N)$ with $0 < t^1 < t^2 < \cdots < t^N < \pi$, and set

$$t^{N+\nu} := \frac{1}{2}\pi(1+\nu) \quad \text{for } \nu = 1,2,3, \quad t^0 := 0.$$

We assume that the angles at the corners Q_j are neither 0 nor π, i.e. for $\xi_k := Q_k - Q_{k-1}$, any two vectors ξ_k, ξ_{k+1} are linearly independent. By Γ_k we denote the straight lines

$$\Gamma_k := \{s\xi_k \colon s \in \mathbb{R}\}.$$

For $t \in T$ we define $U^*(t)$ as the set of surfaces

$$X \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$$

which map the circular arcs $\gamma_k := \{e^{i\varphi} \colon t^k < \varphi < t^{k+1}\}$ into the straight lines $\Gamma_k' := Q_k + \Gamma_k$. Then $X(w_k) = Q_k$ for $w_k := e^{it^k}$ and $1 \leq k \leq N+3$. We want to minimize D in the class $U^*(t)$. This will be achieved by minimizing D in the class V, defined by

$$V := \{X \in H_2^1(B, \mathbb{R}^3) \colon X|_{\partial B}(\gamma_j) \subset \Gamma_j', \ j = 1, 2, \dots, N+3\},$$

and then proving that the minimizer in V actually belongs to $U^*(t)$. Since we have no control over the boundary values of elements of V we need a Poincaré inequality for the elements of V; in fact, such an inequality for the elements of $V \cap C^1(B, \mathbb{R}^3)$ will suffice. This will be achieved by formula (2) of the following

Lemma 1. *Let* $\tau_0 \in [0, 2\pi]$, $w_0 := e^{i\tau_0}$, $0 < \varepsilon_0 < \pi$, $0 < \varepsilon_1 < 1$, $\gamma^- := \{e^{i\varphi} \colon \tau_0 - \varepsilon_0 < \varphi < \tau_0\}$, $\gamma^+ := \{e^{i\varphi} \colon \tau_0 < \varphi < \tau_0 + \varepsilon_0\}$; $e^-, e^+ \in \mathbb{R}^3$ *with* $|e^-| = |e^+| = 1$ *and* $\langle e^-, e^+ \rangle \leq 1 - \varepsilon_1$, $\Gamma^+ := \{se^+ \colon s \in \mathbb{R}\}$, $\Gamma^- := \{se^- \colon s \in \mathbb{R}\}$; *finally suppose that* $Z \in H_2^1(B, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$ *and* $Z|_{\gamma^+}(w) \in \Gamma^+$ $\mathcal{H}^1$*-a.e. on* γ^+, $Z|_{\gamma^-}(w) \in \Gamma^-$ $\mathcal{H}^1$*-a.e. on* γ^-. *Then there are numbers* $c_1 = c_1(\varepsilon_0, \varepsilon_1)$, $c_2 = c_2(\varepsilon_0, \varepsilon_1)$, *and* $\delta_0 = \delta_0(\varepsilon_0) \in (0,1)$ *with the following properties*:

(i) *For any* $\delta \in (0, \delta_0]$ *there is a* $\delta^* \in (\delta, \sqrt{\delta})$ *with*

$$(1) \qquad |Z(w)| \leq c_1 \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \sqrt{D(Z)} \quad \textit{for } w \in B \textit{ with } |w - w_0| = \delta^*.$$

(ii) *We have*

$$\int_B |Z|^2 \, du \, dv \leq c_2 D(Z). \tag{2}$$

Proof. (i) By 4.4, Proposition 2, there is a $\delta^* \in (\delta, \sqrt{\delta})$ such that

$$|Z(w) - Z(w')| \leq 2\left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \sqrt{D(Z)} \tag{3}$$
$$\text{for } w, w' \in \overline{B} \quad \text{with } |w - w_0| = \delta^* \text{ and } |w' - w_0| = \delta^*.$$

Let w_1, w_2 be the two end points of the circular arc $\{w \in \overline{B} \colon |w - w_0| = \delta^*\}$. For $0 < \delta < \varepsilon_0$ we can assume that $w_1 \in \gamma^-$, $w_2 \in \gamma^+$. Then (3) and $\langle e^-, e^+\rangle \leq 1 - \varepsilon_1$ imply

$$|Z(w_1)| \leq c_0(\varepsilon_0, \varepsilon_1)\left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \cdot \sqrt{D(Z)}. \tag{4}$$

Now (1) follows from (3), (4), and $|Z(w)| \leq |Z(w_1)| + |Z(w) - Z(w_1)|$.

(ii) Fix some δ_0 with $0 < \delta_0 < \varepsilon$ and choose $\delta := \delta_0$ in (i). Since

$$\int_{1-\delta_0}^1 \left(\int_0^{2\pi} |Z_\varphi(re^{i\varphi})|^2 \, d\varphi\right) \frac{dr}{r} \leq 2D(Z),$$

there are numbers $\delta_1 \in (0, \delta_0)$ and $r_1 := 1 - \delta_1$ with $r_1 > 1 - \delta_0$ and

$$\int_0^{2\pi} |Z_\varphi(r_1 e^{i\varphi})|^2 \, d\varphi \leq \left(\int_{1-\delta_0}^1 \frac{dr}{r}\right)^{-1} \cdot 2D(Z) < \frac{2}{\delta_0} D(Z)$$

whence

$$|Z(r_1 e^{i\varphi}) - Z(r_1 e^{i\psi})| \leq [4\pi \delta_0^{-1} D(Z)]^{\frac{1}{2}} \quad \text{for } 0 \leq \varphi, \psi \leq 2\pi.$$

Since the arcs $\{w \in B \colon |w| = r_1\}$ and $\{w \in B \colon |w - w_0| = \delta^*\}$ intersect we obtain in conjunction with (1) that

$$|Z(w)| \leq \left[c_1 \cdot \left(\log \frac{1}{\delta_0}\right)^{-\frac{1}{2}} + \left(\frac{4\pi}{\delta_0}\right)^{\frac{1}{2}}\right] \sqrt{D(Z)} \tag{5}$$
$$\text{for } \{w \in B \colon |w| = r_1\}.$$

Choose some $\varepsilon > 0$ and consider the function

$$f(r) := \varepsilon + \int_0^{2\pi} |Z(re^{i\varphi})|^2 \, d\varphi, \quad r \in (0, 1).$$

From

$$\frac{d}{dr}\sqrt{f(r)} = \frac{f'(r)}{2\sqrt{f(r)}} = \frac{\int_0^{2\pi} 2\langle Z(re^{i\varphi}), Z_r(re^{i\varphi})\rangle \, d\varphi}{2\sqrt{f(r)}}$$

we infer by Schwarz's inequality that

$$\left|\frac{d}{dr}\sqrt{f(r)}\right| \le \left(\int_0^{2\pi} |Z_r(re^{i\varphi})|^2 \, d\varphi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\sqrt{f(r)} - \sqrt{f(r_1)} &\le \left|\int_{r_1}^r \frac{d}{dr}\sqrt{f(r)}\, dr\right| \le \left|\int_{r_1}^r \left|\frac{d}{dr}\sqrt{f(r)}\right| dr\right| \\ &\le \left|\int_{r_1}^r \frac{1}{\sqrt{r}} \cdot \left(\int_0^{2\pi} \left|Z_r(r\,e^{i\varphi})\right|^2 r d\varphi\right)^{\frac{1}{2}} dr\right|,\end{aligned}$$

and by Schwarz's inequality,

$$\sqrt{f(r)} \le \sqrt{f(r_1)} + \left(\sqrt{\log\frac{1}{r}} + \sqrt{\log\frac{1}{r_1}}\right)\sqrt{2D(Z)}.$$

Squaring and letting ε tend to zero we obtain the estimate

$$\int_0^{2\pi} |Z(re^{i\varphi})|^2 \, d\varphi \le 2\int_0^{2\pi} |Z(r_1 e^{i\varphi})|^2 \, d\varphi + 8\left(\log\frac{1}{r} + \log\frac{1}{r_1}\right) D(Z).$$

In virtue of $\frac{1}{r_1} < \frac{1}{1-\delta_0}$ and (5) we arrive at

$$\int_0^{2\pi} |Z(re^{i\varphi})|^2 \, d\varphi \le c(\varepsilon_0, \varepsilon_1) \cdot \left(1 + \log\frac{1}{r}\right) D(Z) \quad \text{for } 0 < r < 1.$$

Multiplying by r and integrating with respect to r from 0 to 1 we obtain (2). □

Now we fix some arbitrary $t \in T$. Depending on Γ there are numbers $q > 0$ and $\mu = \mu(t) \in (0, 1)$ such that

$$(6) \qquad \begin{gathered}|Q_k| \le q, \quad |t^j - t^k| \ge \mu, \quad |\langle \xi_k, \xi_{k+1}\rangle| \le 1 - \mu \\ \text{for } 1 \le j, k \le N+3,\ j \ne k.\end{gathered}$$

Proposition 1. *There exists a uniquely determined mapping $X \in U^*(t)$ with $D(X) = \inf_{U^*(t)} D$ which is harmonic in B.*

Proof. Set $d := \inf_V D$ and $d^* := \inf_{U^*(t)} D$. By $U^*(t) \subset V$ and (6) it follows that

$$(7) \qquad 0 \le d \le d^*(q, \mu) < \infty.$$

Choose a sequence of mappings $X_n \in V$ with $D(X_n) \to d$. By Dirichlet's principle we can assume that the X_n are harmonic in B, in particular $X_n \in C^1(B, \mathbb{R}^3)$. Since $Z := X_n - X_l$ satisfies the assumptions of Lemma 1 we have

$$\int_B |X_n - X_l|^2 \, du \, dv \le c_2 D(X_n - X_l) \quad \text{for any } n, l \in \mathbb{N}.$$

Furthermore, $\frac{1}{2}(X_k + X_l) \in V$ because V is a convex set, whence

$$D(X_n + X_l) = 4D\big(\tfrac{1}{2}(X_n + X_l)\big) \ge 4d,$$

and therefore

$$\begin{aligned} D(X_n - X_l) &= 2D(X_n) + 2D(X_l) - D(X_n + X_l) \\ &\le 2D(X_n) + 2D(X_l) - 4d \to 0 \quad \text{as } n, l \to \infty. \end{aligned}$$

Thus $\{X_n\}$ is a Cauchy sequence in $H_2^1(B, \mathbb{R}^3)$, and so there is an $X \in H_2^1(B, \mathbb{R}^3)$ with $X_n \to X$ in $H_2^1(B, \mathbb{R}^3)$ as $n \to \infty$. Then we also have $X_n \rightrightarrows X$ in B' for any $B' \subset\subset B$; hence X is harmonic in B. Since V is a closed subset of $H_2^1(B, \mathbb{R}^3)$ we see that $X \in V$, and $D(X_n) \to d$ yields $D(X) = d$. Consequently X is a minimizer of D in V.

Suppose that $Y \in V$ were another minimizer. Then

$$D(X - Y) = 2D(X) + 2D(Y) - D(X + Y) \le 2d + 2d - 4d \le 0$$

and consequently $D(X - Y) = 0$. By Lemma 1(ii), follows

$$\int_B |X - Y|^2 \, du \, dv = 0,$$

and therefore $X = Y$. Thus D possesses exactly one minimizer X in V. If we can show that $X \in C^0(\overline{B}, \mathbb{R}^3)$ it follows that X lies in $U^*(t)$, and consequently

$$\inf_V D = \inf_{U^*(t)} D$$

because of $U^*(t) \subset V$, and so it would be shown that X is the unique minimizer of D in $U^*(t)$.

Standard elliptic regularity theory yields that X is real analytic on the set $\overline{B} \setminus \{w_1, \dots, w_{N+3}\}$, $w_k := e^{it^k}$, and

$$\text{(8)} \qquad \begin{aligned} &\langle X_r(w), Q_{k+1} - Q_k \rangle = 0 \\ &R_k X(w) = X(w) \end{aligned} \qquad \text{for } w \in \gamma_k, \;\; k = 1, \dots, N+3.$$

Here $R_k : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the reflection of $\mathbb{R}^3$ in the straight line $\Gamma_k' = Q_k + \Gamma_k$.

It remains to prove the continuity of X at the points $w = w_k$. Set

$$X^+(w) := X(w) \quad \text{for } |w| < 1, \quad X^-(w) := X(\overline{w}^{-1}) \quad \text{for } |w| > 1.$$

Here $\overline{w}^{-1} = w/|w|^2$ is the mirror point of w with respect to the unit circle ∂B. Let P be the exterior of the convex hull of $w_1, \ldots, w_{N+3}$, and Ω_k be the subset of $B \cap P$ bounded by $\overline{\gamma}_k$ and the linear segment σ_k with the endpoints w_k and w_{k+1}. By (8) and Schwarz's reflection principle, X^- can be extended to a harmonic mapping in P, which will again be denoted by X^-, and one has

$$X^-(w) = R_k X^+(w) \quad \text{for } w \in \Omega_k. \tag{9}$$

Then for $0 < \rho^2 \le \delta_1(\mu) \ll 1$ the function $|X^- - Q_k|^2$ is subharmonic in $P \cap B_\rho(w_k)$, whereas $|X^+ - Q_k|^2$ is subharmonic in $B \cap B_\rho(w_k)$, and by (9) it follows that $|X^-(w) - Q_k|^2 = |X^+(w) - Q_k|^2$ holds both for $w \in \Omega_k$ and for $w \in \Omega_{k-1}$. Hence the function

$$g(w) := \begin{cases} |X^+(w) - Q_k|^2 & \text{for } w \in B_\rho(w_k) \cap B,\ w \ne w_k, \\ |X^-(w) - Q_k|^2 & \text{for } w \in B_\rho(w_k) \cap (\mathbb{C} \setminus B),\ w \ne w_k, \end{cases} \tag{10}$$

is subharmonic in the punctured disk $B'_\rho(w_k) := \{w \in \mathbb{R}^2 \colon 0 < |w - w_k| < \rho\}$. By (7) we have $D(X - Q_k) = D(x) \le d^*$, and so the Courant–Lebesgue Lemma yields together with $|\langle \xi_k, \xi_{k+1} \rangle| \le 1 - \mu$ for each k:

For any $\delta \in (0, \delta_1]$ and any k with $1 \le k \le N+3$ there is a number $\rho = \rho(\delta, k) \in (\delta, \sqrt{\delta})$ such that

$$|X(w) - Q_k| \le c_3(\mu)\sqrt{d^*} \cdot \left(\log \frac{1}{\delta}\right)^{-1/2} =: M_\delta \quad \text{for } w \in B \text{ with } |w - w_k| = \rho. \tag{11}$$

Hence for any $k = 1, \ldots, N+3$ there is a sequence $\{\rho_j\}$ of numbers $\rho_j > 0$ with $\rho_j \to 0$ as $j \to \infty$ such that

$$m_j := \max\{|X(w) - Q_k| \colon w \in \overline{B},\ |w - w_k| = \rho_j\} \to 0 \quad \text{as } j \to \infty. \tag{12}$$

Applying the maximum principle to the subharmonic function g on the annulus $A(\rho_j, \rho) := \{w \in \mathbb{C} \colon \rho_j \le |w - w_k| \le \rho\}$ for $j \gg 1$ we infer from (11) and (12) that

$$\max\{g(w) \colon w \in A(\rho_j, \rho)\} \le \max\{m_j^2, M_\delta^2\}.$$

Letting j tend to infinity it follows that

$$\max\{g(w) \colon w \in B'_\rho(w_k)\} \le M_\delta^2.$$

In conjunction with (10) we obtain that X is also continuous at the points $w_1, \ldots, w_{N+3}$, and therefore $X \in C^0(\overline{B}, \mathbb{R}^3)$ and $X \in U^*(t)$. $\square$

Definition 1. *The* **Marx–Shiffman mapping** $Z^* : T \to H_2^1(B, \mathbb{R}^3)$ *is defined by* $Z^*(t) := X$ *for* $t \in T$ *where* X *denotes the uniquely determined minimizer of* D *in* $U^*(t)$, *and the* **Marx–Shiffman function** $\Theta^* : T \to \mathbb{R}$ *is given by*

$$\Theta^*(t) := D(Z^*(t)) = \inf_{U^*(t)} D. \tag{13}$$

Any mapping $Z^*(t) : \overline{B} \to \mathbb{R}^3$ *is called a* **quasi-minimal surface** (*cf. I. Marx* [1]). *If we want to emphasize the dependence of* Z^* *and* Θ^* *on* t *and on the vertices* $Q_1, \dots, Q_{N+3}$ *we write* $Z^*(t, Q_1, \dots, Q_{N+3})$ *and as well as* $\Theta^*(t, Q_1, \dots, Q_{N+3})$.

Remark 1. The three point condition $X(w_k) = Q_k$, $k = N+1, N+2, N+3$ with $w_k = e^{it^k}$ and $t^{N+\nu} = \frac{1}{2}\pi(1+\nu)$ for $\nu = 1, 2, 3$ is only needed if we want to compare Θ^* with the Courant function Θ. Otherwise we can replace t by $t^* = (t^1, t^2, \dots, t^{N+3})$ and T by $T^* := \{t^* \in \mathbb{R}^{N+3} : t^1 < t^2 < \dots < t^{N+3} < t^1 + 2\pi\}$. Then the statements on Θ^* and Z^* as functions of $t \in T$ also hold (with obvious alterations) if we consider Θ^*, Z^* as functions of $t^* \in T^*$.

Corollary 1. *For* $t \in T$ *there is a number* $c_4 = c_4(q, \mu(t)) > 0$ *such that*

$$\max_{\overline{B}} |Z^*(t)| \le c_4.$$

Furthermore, for any $t \in T$ *and* $\varepsilon > 0$ *there is a number* $\delta_2 = \delta_2(q, \mu(t), \varepsilon) > 0$ *such that*

$$|Z^*(t)(w) - Z^*(t)(w')| < \varepsilon \quad \textit{for any } w, w' \in \overline{B} \textit{ with } |w - w'| < \delta_2.$$

Remark 2. This corollary together with Proposition 1 implies that the mappings $Z^*(t, Q_1, \dots, Q_{N+3})$ and $Z^*(t^*, Q_1, \dots, Q_{N+3})$ depend continuously on the data $t \in T$ and $t^* \in T^*$ respectively and on $Q_1, \dots, Q_{N+3}$.

Corollary 2. *For any* $t \in T$, *the mapping* $Z^*(t) \in U^*(t)$ *is of class* $C^0(\overline{B}, \mathbb{R}^3) \cap C^\omega(\overline{B} \setminus \{w_1, \dots, w_{N+3}\}, \mathbb{R}^3)$, *harmonic in* $\overline{B}' := \overline{B} \setminus \{w_1, \dots, w_{N+3}\}$, *and satisfies the boundary conditions* (8) *on* $\gamma_1 \cup \dots \cup \gamma_{N+3}$.

Proposition 2. *Let* Θ *and* Θ^* *be the Courant function and the Marx–Shiffman function associated with a given simple and closed polygon* Γ. *Then we have*:

(i) $\Theta^*(t) \le \Theta(t)$ *for all* $t \in T$;
(ii) $\Theta^*(t) = \Theta(t)$ *if* $t \in T$ *is a critical point of* Θ;
(iii) *There are polygons* Γ *such that* $\Theta^* \ne \Theta$, *i.e.*

$$\Theta^*(t) < \Theta(t) \quad \textit{for some } t \in T.$$

Proof. (i) Since $U(t) \subset U^*(t)$ for any $t \in T$, it follows that

$$\Theta^*(t) = \inf_{U^*(t)} D \le \inf_{U(t)} D = \Theta(t).$$

(ii) Let $t \in T$ be a critical point of Θ, and note that

$$\Theta(t) = D(Z(t)), \quad \Theta^*(t) = D(Z^*(t)).$$

By a similar reasoning as in the proof of Proposition 2 in 6.1 we infer that $Z(t) = Z^*(t)$.

(iii) The third assertion follows from the **Lewerenz examples**: *For any $N \in \mathbb{N}$ there is a closed simple polygon Γ with $N+3$ vertices such that the corresponding functions Θ and Θ^* do not coincide.*

It suffices to construct an example for $N = 1$; the other cases will be obtained by a slight modification of this example. So we are looking for a polygon with four vertices Q_1, Q_2, Q_3, Q_4 such that the corresponding functions Θ, Θ^* satisfy $\Theta^*(t) < \Theta(t)$ for some $t \in T$. (Note that T here reduces to an interval.) For the sake of convenience we parametrize all surfaces on the semidisk $B^+ := \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1 \text{ and } v > 0\}$ instead of the disk $B := \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1\}$. Let Γ_ε be the polygon determined by the four successive corners

$$Q_1^\varepsilon := (0,0,-\varepsilon), \quad Q_2^\varepsilon := (0,0,\varepsilon),$$
$$Q_3^\varepsilon = Q_3 := (1,1,-1), \quad Q_4^\varepsilon = Q_4 := (-1,1,1),$$

where $\varepsilon > 0$ is a parameter that will be fixed later on. Set

$$w_1 := (-1,0), \quad w_2 := (1,0), \quad w_3 := (\alpha,\alpha), \quad w_4 := (-\alpha,\alpha), \quad \alpha := \frac{1}{\sqrt{2}}.$$

By $Z_\varepsilon = (Z_\varepsilon^1, Z_\varepsilon^2, Z_\varepsilon^3)$ we denote the uniquely determined minimizer of D_{B^+},

$$D_{B^+}(X) := \frac{1}{2} \int_{B^+} |\nabla X|^2 \, du \, dv,$$

among all $X \in H_2^1(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$ which map ∂B^+ monotonically onto Γ_ε such that $X(w_1) = Q_1^\varepsilon$, $X(w_2) = Q_2^\varepsilon$, $X(w_3) = Q_3$, $X(w_4) = Q_4$, and $\mathcal{C}_\varepsilon$ be the class of all such X. Consider the surface Z'_ε defined by

$$Z'_\varepsilon(u,v) := \big(-Z_\varepsilon^1(-u,v), Z_\varepsilon^2(-u,v), -Z_\varepsilon^3(-u,v)\big).$$

One easily checks that $Z'_\varepsilon \in \mathcal{C}_\varepsilon$ and $D_{B^+}(Z_\varepsilon) = D_{B^+}(Z'_\varepsilon)$, whence we obtain $Z_\varepsilon = Z'_\varepsilon$. Thus, for any $(u,v) \in \overline{B^+}$, $Z_\varepsilon^1(-u,v) = -Z_\varepsilon^1(u,v)$, $Z_\varepsilon^2(-u,v) = Z_\varepsilon^2(u,v)$, $Z_\varepsilon^3(-u,v) = -Z_\varepsilon^3(u,v)$. In particular it follows that

$$Z_\varepsilon^1(0,v) = 0, \quad Z_\varepsilon^3(0,v) = 0 \quad \text{for } 0 \le v \le 1,$$

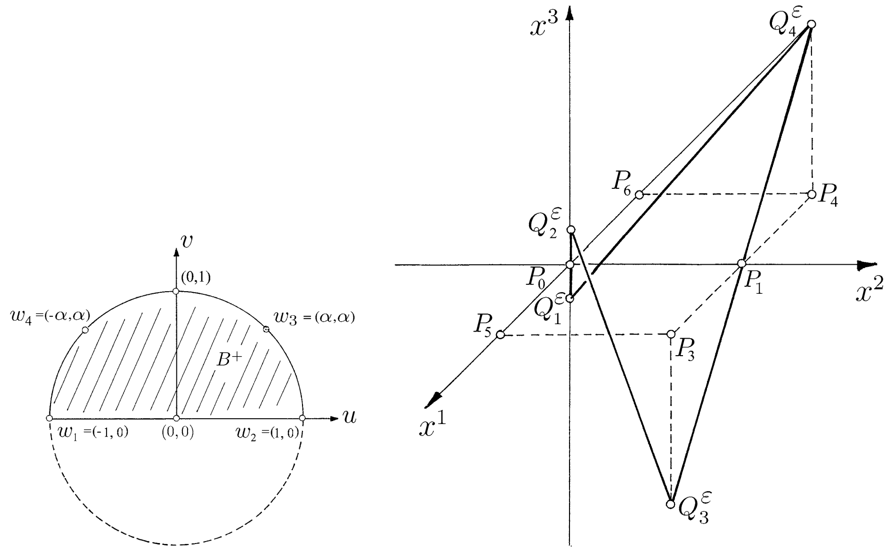

Fig. 1. Lewerenz curve

and $Z_\epsilon \in \mathcal{C}_\epsilon$ immediately yields

$$Z^2_\varepsilon(0,0) = 0, \quad Z^2_\varepsilon(0,1) = 1,$$

that is

$$Z_\varepsilon(0,0) = P_0 := (0,0,0), \quad Z_\varepsilon(0,1) = P_1 := (0,1,0).$$

Fix some ε' with $0 < \varepsilon' \ll 1$. By an elementary construction we can find surfaces $Y_\varepsilon \in \mathcal{C}_\varepsilon$ such that $D_{B^+}(Y_\varepsilon) \le \text{const}$ for $0 < \varepsilon < \varepsilon'$. Then $D_{B^+}(Z_\varepsilon) \le \text{const}$ for $0 < \varepsilon < \varepsilon'$, and by the Courant–Lebesgue Lemma there exists a sequence of numbers $\varepsilon_j \in (0,1)$ with $\varepsilon_j \to 0$ such that the harmonic mappings Z_{ε_j} converge uniformly on $\overline{B^+}$ to some $Z_0 \in C^0(\overline{B^+}, \mathbb{R}^3) \cap H^1_2(B^+, \mathbb{R}^3)$ which maps ∂B^+ monotonically onto the polygon Γ_0 with the three vertices P_0, Q_3, Q_4 such that

$$Z_0(u,0) = P_0 \quad \text{for } -1 \le u \le 1, \quad Z_0(w_3) = Q_3, \quad Z_0(w_4) = Q_4$$

and

$$Z(0,1) = P_1.$$

Now we reflect Z^3_ε and Z^3_0 symmetrically at the u-axis, setting

$$\zeta_\varepsilon(u,v) := \begin{cases} Z^3_\varepsilon(u,v) & \text{for } v \ge 0, \\ Z^3_\varepsilon(u,-v) & \text{for } v \le 0, \end{cases} \qquad \zeta_0(u,v) := \begin{cases} Z^3_0(u,v) & \text{for } v \ge 0, \\ Z^3_0(u,-v) & \text{for } v \le 0, \end{cases}$$

where $(u,v) \in \overline{B}$ and $B := \{(u,v) \colon u^2 + v^2 < 1\}$.

Then we consider the functions $h_\varepsilon, h_0 \in H_2^1(B) \cap C^0(\overline{B}) \cap C^2(B)$ which are harmonic in B and satisfy

$$h_\varepsilon|_{\partial B} = \zeta_\varepsilon|_{\partial B}, \quad h_0|_{\partial B} = \zeta_0|_{\partial B}.$$

By Dirichlet's principle,

$$D_B(h_\varepsilon) \le D_B(\zeta_\varepsilon) \quad \text{and} \quad D_B(h_0) \le D_B(\zeta_0),$$

and the equality sign holds if and only if $h_\varepsilon = \zeta_\varepsilon$ and $h_0 = \zeta_0$ respectively. The symmetry of ζ_ε and ζ_0 implies

$$D_B(\zeta_\varepsilon) = 2D_{B^+}(\zeta_\varepsilon) \quad \text{and} \quad D_B(\zeta_0) = 2D_{B^+}(\zeta_0).$$

Let $w^* = (u, -v)$ be the mirror point of $w = (u, v)$. Then

$$\zeta_\varepsilon(w) = \zeta_\varepsilon(w^*) \quad \text{and} \quad \zeta_0(w) = \zeta_0(w^*) \quad \text{for any } w \in \partial B.$$

Set

$$h_\varepsilon^*(w) := h_\varepsilon(w^*) \quad \text{and} \quad h_0^*(w) := h_0(w^*) \quad \text{for } w \in \overline{B}.$$

Then h_ε^*, h_0^* are continuous on $\overline{B}$, harmonic in B, and $h_\varepsilon^*|_{\partial B} = h_\varepsilon|_{\partial B}$, $h_0^*|_{\partial B} = h_0|_{\partial B}$. The maximum principle implies $h_\varepsilon^* = h_\varepsilon$ and $h_0^* = h_0$, and so

$$h_\varepsilon(u, -v) = h_\varepsilon(u, v), \quad h_0(u, -v) = h_0(u, v) \quad \text{for } (u, v) \in \overline{B}.$$

This in turn yields

$$D_B(h_\varepsilon) = 2D_{B^+}(h_\varepsilon), \quad D_B(h_0) = 2D_{B^+}(h_0),$$

therefore

$$D_{B^+}(h_\varepsilon) \le D_{B^+}(\zeta_\varepsilon), \quad D_{B^+}(h_0) \le D_{B^+}(\zeta_0),$$

and equality occurs if and only if $h_\varepsilon = \zeta_\varepsilon$ and $h_0 = \zeta_0$ respectively.

Furthermore, $Z_\varepsilon^3(-u, v) = -Z_\varepsilon^3(u, v)$ for $(u, v) \in \overline{B}$ yields

$$h_\varepsilon(-u, v) = -h_\varepsilon(u, v) \quad \text{and} \quad h_0(-u, v) = -h_0(u, v) \quad \text{for } (u, v) \in \partial B.$$

Then an analogous reasoning furnishes

$$h_0(-u, v) = -h_0(u, v) \quad \text{for } (u, v) \in \overline{B},$$

and so

$$h_0(0, v) = 0 \quad \text{for all } v \text{ with } |v| \le 1.$$

Moreover, Z_0 maps the quarter circle $\{(\cos\theta, \sin\theta)\colon 0 \le \theta \le \frac{\pi}{2}\}$ onto the polygonal subarc $P_0 Q_3 P_1$ of Γ_0, whence $h_0(u, v) \le 0$ on the boundary of the semidisk $S^+ := \{(u, v) \in B\colon u > 0\}$. The maximum principle yields $h_0(u, v) < 0$ in S^+, in particular

$$h_0(u, 0) < 0 \quad \text{for } 0 < u < 1,$$

and similarly

$$h_0(u, 0) > 0 \quad \text{for } -1 < u < 0.$$

Since $Z_{\varepsilon_j} \rightrightarrows Z_0$ on $\overline{B^+}$, it follows that $\zeta_{\varepsilon_j}|_{\partial B} \rightrightarrows \zeta_0|_{\partial B}$, and therefore $h_{\varepsilon_j} \rightrightarrows h_0$ on $\overline{B}$. Hence there are numbers $\varepsilon_0 > 0$ and $u_1^+, u_2^+, u_1^-, u_1^-$ such that $0 < u_1^+ < u_2^+ < 1$, $-1 < u_1^- < u_2^- < 0$, and

$$h_{\varepsilon_0}(u, 0) < 0 \quad \text{for } u_1^+ < u < u_2^+, \quad h_{\varepsilon_0}(u, 0) > 0 \quad \text{for } u_1^- < u < u_2^-.$$

Now we define a new harmonic mapping X_{ε_0} by

$$X_{\varepsilon_0} := (Z^1_{\varepsilon_0}, Z^2_{\varepsilon_0}, h_{\varepsilon_0}|_{\overline{B}^+})$$

which satisfies

$$X_{\varepsilon_0}(w_1) = Q_1^{\varepsilon_0}, \quad X_{\varepsilon_0}(w_2) = Q_2^{\varepsilon_0}, \quad X_{\varepsilon_0}(0, 0) = P_0,$$

hence

$$h_{\varepsilon_0}(-1, 0) = -\varepsilon_0, \quad h_{\varepsilon_0}(1, 0) = \varepsilon_0, \quad h_{\varepsilon_0}(0, 0) = 0.$$

Therefore X_{ε_0} is not monotonic on $\{(u, 0)\colon -1 \le u \le 1\}$. Thus h_{ε_0} does not coincide with ζ_{ε_0}; hence $D_{B^+}(h_{\varepsilon_0}) < D_{B^+}(\zeta_{\varepsilon_0})$ and therefore

$$D_{B^+}(X_{\varepsilon_0}) < D_{B^+}(Z_{\varepsilon_0}).$$

We also note that X_ε is an admissible mapping for Shiffman's variational problem since it maps the subarcs of ∂B^+ between w_j and w_{j+1} into the straight lines through $Q_j^{\varepsilon_0}$ and $Q_{j+1}^{\varepsilon_0}$ (with $w_{j+4} := w_j$, $Q_{j+4}^{\varepsilon_0} := Q_j^{\varepsilon_0}$). Hence, for w_1, w_2, w_3, w_4 and Γ_{ε_0} the "Marx–Shiffman minimizer" furnishes a smaller value for D_{B^+} than the "Courant minimizer", and consequently $\Theta^*(t) < \Theta(t)$ for some $t \in T$ if we return to our original notation.

By a slight modification of the preceding reasoning one can construct "Lewerenz examples" Γ with more than four vertices. □

The elementary results that we so far have proved are taken from Lewerenz [1] and Heinz [19]. The following work is much more profound and rests on classical results by H. Poincaré, L. Schlesinger [1–4], and J. Plemelj [1] about the Riemann–Hilbert problem. Here we can only present the statements of Heinz's principal theorems without any proof.

The main result of [19] is

Theorem 1. *For* $t^* = (t^1, \dots, t^{N+3}) \in T^* := \{t^* \in \mathbb{R}^{N+3}\colon t^1 < \cdots < t^{N+3} < t^1 + 2\pi\}$ *we set* $X(u, v, t^*) := Z^*(t^*)(w)$, $w = (u, v)$, *where* $Z^*(t^*)$ *is the quasi-minimal surface (i.e. the Marx–Shiffman mapping), bounded by the polygon* Γ, *that belongs to* $t^* \in T^*$ *(see Definition 1 and Remark 1). Let* $t_0^* \in (t_0^1, \dots, t_0^{N+3}) \in T^*$, $\mathring{w}_k := \exp(i t_0^k)$ *for* $k = 1, \dots, N+3$ *and* $\mathring{w} = \mathring{u} + i\mathring{v} \mathrel{\hat{=}} (\mathring{u}, \mathring{v}) \in \overline{B}$ *with* $\mathring{w} \ne \mathring{w}_1, \dots, \mathring{w}_{N+3}$. *Then, in a sufficiently small neighborhood of* $(\mathring{u}, \mathring{v}, t_0^*)$, *the mapping* $X(u, v, t^*)$ *possesses a convergent power series expansion.*

In [21] Heinz also allowed the corners $Q_1, \dots, Q_{N+3}$ of Γ to vary under the assumption that none of the angles at $\mathring{Q}_1, \dots, \mathring{Q}_{N+3}$ of $\mathring{\Gamma}$ will be 90°. Then it turns out that $X(u, v, t^*, Q)$ can be expanded in a convergent power series of the variables (u, v, t^*, Q) near $(\mathring{u}, \mathring{v}, t_0^*, \mathring{Q})$ where $Q := (Q_1, \dots, Q_{N+3}), \mathring{Q} := (\mathring{Q}_1, \dots, \mathring{Q}_{N+3}), X(w_k, t^*, Q) = Q_k$ for $w_k = \exp(it^k)$ and $X(\mathring{w}_k, t_0^*, \mathring{Q}_k) = \mathring{Q}_k$ for $k = 1, \dots, N+3$.

In [20] (and with simplified proofs in [23]) Heinz proved analyticity of the Shiffman function $\Theta^*(t)$ in $t \in T$:

Theorem 2. *For $t \in T$ set $X = X(\cdot, t) = Z^*(t)$. Then one has:*

(i) *In $B \times T$ the mapping X satisfies*

$$w^2 X_w(w) \cdot X_w(w) = \frac{i}{8\pi} \sum_{k=1}^{N+3} R_k(t) \frac{w_k + w}{w_k - w}, \quad w_k := \exp(it^k),$$

where the $R_k(t)$ are real analytic in $t \in T$ and satisfy

$$\sum_{k=1}^{N+3} R_k(t) = 0 \quad \text{and} \quad \sum_{k=1}^{N+3} w_k R_k(t) = 0.$$

(ii) *$\Theta^*(t)$ is real analytic in T, and*

$$\frac{\partial \Theta^*(t)}{\partial t^k} = R_k(t) \quad \text{for } k = 1, \dots, N.$$

(iii) *$X(\cdot, t)$ is minimal surface (i.e. $\Delta X(\cdot, t) = 0$ and $X_w(\cdot, t) \cdot X_w(\cdot, t) = 0$ in B) if and only if $\nabla \Theta^*(t) = 0$.*

According to Heinz [21], also the function $\Theta^*(t, Q) = D(X(\cdot, t, Q))$ is real analytic in $t \in T$ and $Q \in \mathbb{R}^{3(N+3)}$ if we avoid angles of 90° at the vertices $\mathring{Q}_k$ of the polygon $\mathring{\Gamma}$ that is to be varied; in fact, it suffices that the angle at one of the vertices $\mathring{Q}_k$ is different from 90° (see Heinz [21], pp. 33–34).

In order to formulate further results it will be convenient to use the following notations:

$$\mathcal{M}(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon \Delta X = 0, X_w \cdot X_w = 0\}$$

is the class of disk-type minimal surfaces $X : B \to \mathbb{R}^3$ bounded by Γ, and $\mathcal{M}^*(\Gamma)$ is the subclass

$$\mathcal{M}^*(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon \Delta X = 0, X_w \cdot X_w = 0\}$$

of $X \in \mathcal{M}(\Gamma)$ satisfying the three-point condition

$$(*) \qquad X(w_k) = Q_k \quad \text{for } k = N+1, N+2, N+3$$

and $w_{N+\nu} := \exp(it^{N+\nu})$, $1 \le \nu \le 3$, and $t^{N+\nu} = \frac{\pi}{2}(1+\nu)$. As always in the present context, Γ is a polygon in $\mathbb{R}^3$ with the "true vertices" $Q_1, Q_2, \dots, Q_{N+3}$. By $\mathcal{S}(Q)$ we denote the class of quasi-minimal surfaces $X = Z^*(t)$ with "$\partial X \subset \Gamma_1' \cup \Gamma_2' \cup \dots \cup \Gamma_{N+3}'$", precisely speaking:

$$\mathcal{S}(Q) := \{X : X = Z^*(t) \text{ for some } t \in T\}, \quad Q := (Q_1, \dots, Q_{N+3}),$$

where $T = \{t = (t^1, \dots, t^N) : 0 < t^1 < \dots < t^N < \pi\}$ and $Z^*(t)$ is the Marx–Shiffman mapping for $t \in T$, i.e. the minimizer of D in the class $U^*(t)$ of surfaces $Y \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with $Y(\gamma_k) \subset \Gamma_k' = Q_k + \Gamma_k$, $k = 1, \dots, N+3$. In particular, the elements $X \in \mathcal{S}(Q)$ satisfy the same 3-point condition $(*)$ as the elements $X \in \mathcal{M}^*(\Gamma)$, and $X(w_k) = Q_k$ for $k = 1, \dots, N+3$.

Finally we denote by $\mathcal{S}_k(Q)$ the class of quasi-minimal surfaces $X \in \mathcal{S}(Q)$ which are minimal surfaces, i.e.:

$$\begin{aligned} \mathcal{S}_{\mathcal{M}}(Q) &= \{X \in \mathcal{S}(Q) : X_w \cdot X_w = 0\} \\ &= \{X : X = Z^*(t) \text{ for some } t \in T \text{ with } \nabla \Theta^*(t) = 0\}. \end{aligned}$$

Then

$$\mathcal{M}^*(\Gamma) \subset \mathcal{S}_{\mathcal{M}}(Q) \subset \mathcal{S}(Q).$$

Now we want to define the notion of a *branch point* of a quasi-minimal surface and of its *branch point order*.

Proposition 3 (E. Heinz [19], Satz 2; [22], pp. 549–550; [23], pp. 385–386). *Let $X = X(\cdot, t) = Z^*(t) \in \mathcal{S}(Q)$, $t \in T$, $V := \{w_1, \dots, w_{N+3}\}$. Then for any $\zeta \in \overline{B}$ there exist $A \in \mathbb{C}^3$ with $A \ne 0$, $\nu \in \mathbb{Z}$ with $\nu \ge 0$, and $\alpha \in (-1, 0]$ such that $X_w = \frac{1}{2}(X_u - iX_v)$ has the asymptotic representation*

$$X_w(w,t) = A \cdot (w - \zeta)^{\nu+\alpha} + o(|w-\zeta|^{\nu+\alpha}) \quad \text{for } w \in \overline{B}, w \to \zeta. \tag{14}$$

Moreover, $\alpha = 0$ if $\zeta \notin V$.

This expansion is uniquely determined.

Definition 2. *One calls $\zeta \in \overline{B}$ a* **branch point** *of $X \in \mathcal{S}(Q)$ if $\nu > 0$, and $\nu = \nu(\zeta)$ is said to be the* **order** *of the branch point ζ. If $\zeta \in \overline{B}$ is not a branch point, we set $\nu(\zeta) = 0$. Clearly the set $\Sigma(X)$ of branch points $\zeta \in \overline{B}$ is finite, and for any $\zeta \in \overline{B}$ we have:*

$$\zeta \in \Sigma(X) \quad \text{if and only if} \quad |X_w(w,\tau)| \to 0 \quad \text{as } w \to \zeta, \ w \in \overline{B}.$$

The **total order of branch points of** *X will be called $\kappa(X)$; it is defined as*

$$\kappa(X) := \sum_{\zeta \in B} \nu(\zeta) + \frac{1}{2} \sum_{\zeta \in \partial B} \nu(\zeta). \tag{15}$$

In order to estimate $\kappa(X)$ for $X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$, we need one more definition:

Definition 3. *For $X \in X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one defines the* **Schwarz operator** $S = S^Y : \mathrm{dom}(S) \to L_2(B)$ *by* $S := -\Delta + 2KE$ *on its domain*

$$\mathrm{dom}\, S = \{\varphi \in \overset{\circ}{H}{}_2^1(B) \cap C^2(B) \colon S\varphi \in L_2(B)\},$$

where $E := |X_u|^2$, *and K is the Gauss curvature of X. By* $\ker S$ *we denote the kernel of S,*

$$\ker S := \{\varphi \in \mathrm{dom}\, S \colon S\varphi = 0\}.$$

Realizing that for any critical point t of Θ^* the pairing $\langle Y(\cdot,t), \cdot\rangle$ with the unit normal field $Y(\cdot,t) := |X_u \wedge X_v|^{-1}(X_u \wedge X_v)(\cdot,t)$ maps the vector space

$$V^t := \left\{ \sum_{k=1}^N c^k X_{t^k}(\cdot,t) \colon c = (c^1,\ldots,c^N) \in \ker D^2\Theta^*(t) \right\}$$

onto the kernel of $S^{X(\cdot,t)}$ with

$$\dim\bigl\{\ker(\langle X(\cdot,t),\cdot\rangle)\bigr\} = 2\kappa(X(\cdot,t)) - \#\bigl\{e^{it^\ell} \in \Sigma(X(\cdot,t)) \colon 1 \le \ell \le N\bigr\}.$$

E. Heinz (cf. [22], p. 563, Satz 3) obtained the following fundamental result:

Theorem 3. *For $X = X(\cdot,t) = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$, $t \in T$, with the Schwarz operator $S = S^X$ one has*

$$\dim\ker S^X + \mathrm{rank}\, \nabla^2\Theta^*(t) + 2\kappa(X) = N, \tag{16}$$

where $\nabla^2\Theta^*(t)$ denotes then Hessian matrix of Θ^* at t:

$$\nabla^2\Theta^*(t) = \left(\frac{\partial^2\Theta^*(t)}{\partial t^j\,\partial t^k}\right)_{j,k=1,\ldots,N}.$$

Corollary 3. *For $X = X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one has $\kappa(X) \le \frac{N}{2}$, and $\kappa(X) = \frac{N}{2}$ if and only if $\lambda = 0$ is not an eigenvalue of S^X and $\Theta^*_{t^j t^k}(t) = 0$ for $1 \le j,k \le N$.*

Corollary 4. *For $X = X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$, the Hessian matrix $\nabla^2\Theta^*(t)$ is invertible if $\lambda = 0$ is not an eigenvalue of S^X and X has no branch points in $\overline{B}$.*

What can one say about $\kappa(X)$ if $X = X(\cdot,t)$ merely is an element of $\mathcal{S}(Q)$, but not a minimal surface? E. Heinz [24] has found that in this case one still has

$$\kappa(X) \le \frac{N}{2}. \tag{17}$$

Furthermore, for any $N \in \mathbb{N}$ he constructed a closed simple polygon Γ in $\mathbb{R}^3$ with $N+3$ vertices $Q_1, \dots, Q_{N+3}$ and a surface $X = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ such that $\kappa(X) = \frac{N}{2}$. Then $\nabla \Theta^*(t) = 0$ and $\nabla^2 \Theta^*(t) = 0$.

There is a much sharper estimate for the total branch point order $\kappa(X)$ of a quasi-minimal surface $X = X(\cdot, t) \in \mathcal{S}(Q)$, due to F. Sauvigny [6], p. 300:

Let α_k be the angle $\alpha \in (-1, 0]$ appearing in the expansion (14) of $X_w(w,t)$ at the point $\zeta = w_k := \exp(it^k) \in \partial B$, $k = 1, 2, \dots, N+3$.

The Heinz expansion (14) can be written as

$$X_w(w,t) = (w-\zeta)^{\nu+\alpha} g(w) \quad \text{as } w \to \zeta,$$

with $g(\zeta) \neq 0$. This expansion has the companion

$$X_{ww}(w,t) = \frac{\beta}{w-\zeta} X_w(w,t) + (w-\zeta)^\beta g_w(w), \quad \beta := \nu + \alpha > -1.$$

The "normal component" Y^* of any $Y \in \mathbb{C}^3$ is

$$Y^* := Y - |X_w(w,t)|^{-2} \langle Y, X_{\overline{w}}(w,t) \rangle X_w(w,t).$$

Thus

$$X^*_{ww}(w) = (w-\zeta)^\beta g^*_w(w),$$

and consequently the curvature function

$$\Psi(w) := 2|X_w(w,t)|^{-2} \cdot |X^*_{ww}(w,t)| \tag{18}$$

is integrable on B. If $X_w \cdot X_w = 0$ (i.e. $X \in \mathcal{S}_{\mathcal{M}}(Q)$) one finds that $\Psi = -E \cdot K$, hence $\int_B \Psi \, du \, dv$ in this case is the total curvature $\int_X |K| \, dA = -\int_B EK \, du \, dv$ of X. Now we can formulate Sauvigny's result:

Theorem 4. *For any quasi-minimal surface $X = X(\cdot, t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one has*

$$2\pi \cdot [1 + \kappa(X)] = \pi \sum_{k=1}^{N+3} |\alpha_k| + \int_B \Psi \, du \, dv. \tag{19}$$

This is the analog of formula (19) in Section 2.11 of Vol. 2.

For $X = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one can relate the second variation $\delta^2 A(X,Y)$ of the area A at X in normal direction $Y = \lambda \mathcal{W}^{-1}(X_u \wedge X_v)$, $\lambda \in C^1_0(B)$, $\mathcal{W} = |X_u \wedge X_v|$, to the Hessian matrix $\nabla^2 \Theta(t)$:

Theorem 5 (F. Sauvigny [5], pp. 180–181). *If $X = X(\cdot, t) = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ and $\delta^2 A(X,Y) \geq 0$ for all normal directions Y then $\nabla^2 \Theta^*(t)$ is positive semidefinite. If X has no branch points in $\overline{B}$ and X is* **strictly stable**, *that is, if*

$$\int_B (|\nabla \varphi|^2 + 2EK\zeta^2) \, du \, dv > 0$$

for all $\varphi \in \mathring{H}^1_2(B) \cap C^0(\overline{B})$, then $\nabla^2 \Theta^*(t)$ is positive definite.

In his paper [4], Sauvigny was even able to show that *the Morse index* $m(X)$ *of a mapping* $X = X(\cdot, t) \in \mathcal{S}_{\mathcal{M}}(Q)$, i.e. the number of negative eigenvalues of the Schwarz operator, *agrees with the Morse index of* $\nabla^2 \Theta^*(t)$, i.e. with the number of negative eigenvalues of this symmetric $N \times N$-matrix (cf. [2], p. 186, Theorem 3; a weaker version of this result was already formulated by I. Marx [1], without proof). Furthermore Sauvigny in [4] generalized Heinz's identity (16) (which only holds for surfaces in $\mathbb{R}^3$) to an inequality for surfaces and polygons in $\mathbb{R}^p$, namely,

$$\dim \ker \nabla^2 \Theta^*(t) \le \dim \ker S^X + 2\kappa(X). \tag{20}$$

For this purpose we note that most results discussed in this subsection can be carried over from $\mathbb{R}^3$ to $\mathbb{R}^p$ with $p > 3$, except for Theorem 3 and for the addendum to Theorem 1 (cf. Heinz [21]), which are restricted to $\mathbb{R}^3$.

We finally mention several other results for surfaces $X \in \mathcal{M}^*(\Gamma)$ bounded by polygons Γ:

Theorem 6 (F. Sauvigny [3]). *If Γ is an extreme, simple polygon of total curvature $k(\Gamma) < 4\pi$ then Γ bounds exactly one minimal surface, i.e. $\#\mathcal{M}^*(\Gamma) = 1$. This surface has no branch points in $\overline{B}$.*

As usual Γ is called *extreme* if it lies on the boundary of a compact convex set.

The *total curvature* $k(\Gamma)$ is defined as the sum $\eta_1 + \eta_2 + \cdots + \eta_{N+3}$ of the unoriented angles $\eta_k :\equiv \sphericalangle(\xi_{k-1}, \xi_k) \in (0, \pi)$.

The result above can be generalized to $\mathbb{R}^p$ with $p > 3$ if one replaces the assumption $k(\Gamma) < 4\pi$ by the stronger condition $k(\Gamma) < \frac{10\pi}{3}$ (cf. Sauvigny [3]). At last we quote three finiteness results:

Theorem 7 (R. Jakob [6–8]). *Let Γ be a simple, closed, extreme polygon Γ in $\mathbb{R}^3$. Then every immersed, stable minimal surface spanning Γ is an isolated point of $\mathcal{M}^*(\Gamma)$. In particular, $\mathcal{M}^*(\Gamma)$ contains only finitely many stable minimal surfaces without branch points.*

This result can be generalized in the following way:

Theorem 8 (R. Jakob [9]). *Let Γ be a simple, closed, extreme polygon in $\mathbb{R}^3$ whose angles at the corners are different from $\frac{\pi}{2}$. Then there exists a neighborhood $\mathcal{N}(\Gamma)$ of Γ in $\mathbb{R}^3$ and an integer $\mu(\Gamma)$ such that the number of immersed, stable minimal surfaces in $\mathcal{M}^*(\Gamma')$ is bounded by $\mu(\Gamma)$ for any simple closed polygon Γ' which is contained in $\mathcal{N}(\Gamma)$ and has as many vertices as Γ.*

Recently, R. Jakob [10] has also obtained the following generalization of Theorem 7:

Theorem 9 (R. Jakob [10]). *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed polygon having the following two properties: Firstly it has to bound only minimal surfaces* *without boundary branch points, and secondly its total curvature, i.e. the sum of the exterior angles $\{\eta_k\}$ at its $N+3$ vertices, has to be smaller than 6π. Then every immersed minimal surface spanning Γ is an isolated point of the space $\mathcal{M}^*(\Gamma)$ of all disk-type minimal surfaces spanning Γ, and in particular Γ can bound only finitely many immersed minimal surfaces of disk-type.*

Sketch of the proof: At first we prove that any immersed $X^* \in \mathcal{M}^*(\Gamma)$ is an isolated point of $\mathcal{M}^*(\Gamma)$ with respect to the $\|\cdot\|_{C^0(\bar{B})}$-norm. Hence we assume the contrary, i.e. the existence of some immersed minimal surface X^* and of some sequence $\{X_j\} \subset \mathcal{M}^*(\Gamma)$ satisfying $\|X_j - X^*\|_{C^0(\bar{B})} \to 0$. Now Heinz's formula (16) in Theorem 3 implies that the Schwarz operator S^X (cf. Definition 3) of any immersed minimal surface X which is a non-isolated point of $\mathcal{M}^*(\Gamma)$ must have a non-trivial kernel. There are two possibilities: Either 0 is the smallest eigenvalue of S^{X^*}, or 0 is the nth eigenvalue of S^{X^*} for some $n > 1$. In the first case X^* is a stable immersed minimal surface and therefore an isolated point of $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$ by Theorem 1.1 in Jakob [10]. Hence, only the second case can hold true here. Now let u_n be some eigenfunction of S^{X^*} corresponding to the nth eigenvalue $\lambda_n = 0$, for some $n > 1$. In this case it is known (cf. Theorem 2.3 in Jakob [10]) that the zero set of u_n is not empty and subdivides B into at least 2 disjoint nodal domains. Sauvigny's Gauss–Bonnet formula (19) in Theorem 4 yields especially under the requirement that the total curvature $\sum_{k=1}^{N+3} \eta_k$ of Γ, as defined below Theorem 6, is smaller than 6π:

$$2\pi - \int_B KE\,du\,dv = \pi \sum_{k=1}^{N+3} |\alpha_k| = \sum_{k=1}^{N+3} \eta_k < 6\pi, \tag{21}$$

thus $\int_B |KE|\,du\,dv < 4\pi$ for every immersed minimal surface $X = X(\cdot, t) \in \mathcal{M}^*(\Gamma)$. Here we have used the fact that the exterior angles η_k at the vertices of Γ coincide with the angles $-\pi\alpha_k$ of X, where α_k appears in the exponent of the leading summand in the asymptotic expansion (14) of $X_w(\cdot, t)$ about each e^{it_k} respectively, on account of the fact that particularly the points $\{e^{it_k}\}_{k=1,\dots,N+3}$ are not branch points of the immersed surface X. Thus there is at least one nodal domain D of u_n such that

$$\int_D |(KE)^*|\,du\,dv < 2\pi. \tag{22}$$

Now again by Theorem 2.3 in Jakob [10] there are two possibilities: (i) ∂D is a finite, disjoint union of piecewise analytic, closed Jordan curves, or (ii) ∂D is piecewise real analytic about each of its points with the exception of at most finitely many points $e^{it_{k_j}}$, for some subcollection $\{k_j\} \subset \{1, \dots, N+3\}$, about each of which ∂D fails to be representable as a graph of a Lipschitz-continuous function. We shall first examine case (i): The stability theorem of Barbosa and do Carmo [4], in its version for minimal surfaces with polygonal boundaries (cf. Theorem 2.4 in Jakob [10]), guarantees that the restriction $X^*|_D$ of X^* has to be even strictly stable, i.e. one has

$$\lambda_{\min}(S^{X^*|_D}) > 0 \tag{23}$$

for the smallest eigenvalue $\lambda_{\min}(S^{X^*|_D})$ of the Schwarz operator assigned to the restricted surface $X^*|_D$. But on the other hand, since there holds $S^{X^*}(u_n) = 0$ on B, the restriction $u_n|_D$ satisfies in particular

$$S^{X^*|_D}(u_n|_D) = 0 \quad \text{on } D$$

and is moreover of class $\mathring{H}_2^1(D) \cap C^\omega(D)$, because $u_n|_D$ vanishes identically on the piecewise real analytic boundary of D and is continuous on $\bar{B}$. Hence u_n is an eigenfunction of $S^{X^*|_D}$ corresponding to the eigenvalue 0, in contradiction to (23). In case (ii) the argument is slightly more involved. Firstly one has to prove that on the considered domain D there still exists some function $\phi^* \in S\mathring{H}_2^1(D)$ which minimizes the quadratic form

$$J^{X^*|_D}(\phi) := \int_D \{|\nabla\phi|^2 + 2(KE)^*\phi^2\}\, du\, dv,$$

assigned to $S^{X^*|_D}$, on the $L^2(D)$-sphere

$$S\mathring{H}_2^1(D) := \{\phi \in \mathring{H}_2^1(D) \colon \|\phi\|_{L^2(D)} = 1\}$$

of $\mathring{H}_2^1(D)$, and which satisfies

$$J^{X^*|_D}(\phi^*) = \lambda_{\min}(S^{X^*|_D}). \tag{24}$$

Moreover we need the following pointwise estimate of $|KE|$ due to Heinz (see (3.3) in Heinz [22] or (26) in Jakob [9]) about each of the points $e^{it_1}, \dots, e^{it_{N+3}}$:

$$|KE|(w) \le \text{const}(X, \Gamma)|w - e^{it_k}|^{-2+\alpha} \quad \text{for all } w \in \bar{B} \cap B_\delta(e^{it_k}) \setminus \{e^{it_k}\} \tag{25}$$

for $\delta < \frac{1}{2}\min_{k=1,\dots,N+3}\{|e^{it_k} - e^{it_{k-1}}|\}$ and for some $\alpha > 0$ depending only on Γ. Now combining (22), (25) and the absolute continuity of the Lebesgue integral we can infer the existence of some sufficiently small $\delta > 0$ such that the enlargement $\tilde{D} := D \cup \bigcup_j (B_\delta(e^{it_{k_j}}) \cap B)$ of D is a finitely connected domain whose boundary is a disjoint union of piecewise real analytic closed Jordan curves and still satisfies

$$\int_{\tilde{D}} |(KE)^*|\, du\, dv < 2\pi. \tag{26}$$

Hence, we can apply the above mentioned stability theorem, i.e. Theorem 2.4 in Jakob [10], to $X^*|_{\tilde{D}}$ and obtain

$$\lambda_{\min}(S^{X^*|_{\tilde{D}}}) > 0. \tag{27}$$

Next we extend ϕ^* onto $\tilde{D}$ by simply setting $\tilde{\phi}^*(w) = 0$ for $w \in \tilde{D} \setminus D$, obtaining $\tilde{\phi}^* \in S\mathring{H}_2^1(\tilde{D})$. Now since $\tilde{D}$ has a piecewise real analytic boundary we know that $\lambda_{\min}(S^{X^*|_{\tilde{D}}}) = \min_{S\mathring{H}_2^1(\tilde{D})} J^{X^*|_{\tilde{D}}}$. Combining this with (24) and (27) we achieve:

$$(28) \qquad \begin{aligned} \lambda_{\min}(S^{X^*|_D}) &= J^{X^*|_D}(\phi^*) = J^{X^*|_{\tilde{D}}}(\tilde{\phi}^*) \geq \min_{S\mathring{H}_2^1(\tilde{D})} J^{X^*|_{\tilde{D}}} \\ &= \lambda_{\min}(S^{X^*|_{\tilde{D}}}) > 0. \end{aligned}$$

But on the other hand we know that $u_n|_D \in H_2^1(D) \cap C^0(\bar{D})$, on account of $u_n \in H_2^1(B) \cap C^0(\bar{B})$, and that $u_n = 0$ on ∂D, from which one can deduce that $u_n|_D \in \mathring{H}_2^1(D)$. Since we also know that $u_n|_D \in C^\omega(D)$ and that $S^{X^*|_D}(u_n|_D) = 0$ on D, we obtain that $u_n|_D$ is an element of the domain of $S^{X^*|_D}$, and thus an eigenfunction of $S^{X^*|_D}$ corresponding to the eigenvalue 0, in contradiction to (28), which proves the first assertion of Theorem 9.

In order to derive from this the "finiteness statement" of Theorem 9, it suffices to show the closedness of the subset $\mathcal{M}_i^*(\Gamma)$ of immersed minimal surfaces within the compact space $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$. Thus let $\{X_j\}$ be a sequence in $\mathcal{M}_i^*(\Gamma)$ converging to some X^* in $\mathcal{M}^*(\Gamma)$. As in (21) we infer the constant value $\int_B |(KE)_j|\, dw \equiv \sum_{k=1}^{N+3} \eta_k - 2\pi < 4\pi$ for every j from Sauvigny's Gauss–Bonnet formula. Now, by Theorem 1 in Sauvigny [10], this is in fact a sufficient condition for the limit minimal surface X^* to be free of interior branch points again. Finally, since X^* is spanned by Γ, it must be free of boundary branch points as well, just by assumption on Γ. Hence X^* is an element of $\mathcal{M}_i^*(\Gamma)$, and consequently $\mathcal{M}_i^*(\Gamma)$ inherits the compactness of $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$.