Chapter 5

Stable Minimal- and H-Surfaces

Solving Plateau's problem in the preceding chapter we concentrated our attention to a solution X of this problem, and we somewhat neglected its Gauss mapping N, the surface normal of X. However, the mapping N turns out to be continuous even in case of a branched solution X, and so it is seen to be a real analytic surface of constant mean curvature one. As it will be very useful to study the pair (X, N) together and not X alone, we are invited to enlarge our spectrum and to investigate directly surfaces of prescribed mean curvature. This will enable us in Chapter 7 to solve the *nonparametric equation of prescribed mean curvature* via the solution of Plateau's problem for parametric surfaces of prescribed mean curvature. Using and extending the ideas presented in Chapter 4, this more general Plateau problem for H-surfaces will be solved in Vol. 2, Chapter 4. In order to shorten the presentation of this chapter we shall strongly rely on the treatise of F. Sauvigny [16] as well as on Vol. 2. Especially the control of the boundary regularity will be indispensable for our considerations.

In Section 5.1 we derive the basic equation for the Gauss map N of an H-surface $X : B \to \mathbb{R}^3$ and prove that N is a classical—and in particular continuous solution of this equation. In Section 5.2 we study a substitute for the Weingarten mapping S introduced in Section 1.2, namely *Bonnet's mapping* $R : T_wX \to T_wX$, which leads to the definition of *Bonnet's surface* $Y : B \to \mathbb{R}^3$ for a constant mean curvature surface (= cmc-surface). This surface again is a cmc-surface of mean curvature one provided that not all points of X are umbilical points, and it might give more information than N if properly exploited.

The *stability* of H-surfaces is discussed in Section 5.3 by means of the second variation $\delta^2 F(X, \varphi N)$, $\varphi \in C_c^\infty(B)$, of a functional F defined by

$$(1) \qquad F(X) = A(X) + 2V(X), \quad V(X) := \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv$$

where the associated vector field $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the equation

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(2) $$\operatorname{div} Q(x) = H(x).$$

Although Q is not uniquely determined by H, the stability condition depends only on H: X is *stable* if

(3) $$\int_B |\nabla\varphi|^2 \, du \, dv \geq \mu \int_B p\varphi^2 \, du \, dv \quad \text{for all } \varphi \in C_c^\infty(B)$$

and $\mu = 2$ with

(4) $$p := \Lambda[2H^2(X) - K - \langle H_x(X), N\rangle], \quad \Lambda := |X_u \wedge X_v|,$$

while *strict stability* of X means that (3) holds true with $\mu > 2$.

The central result of this section states that the stability of X together with the monotonicity condition

(5) $$\frac{\partial H}{\partial e} = \langle H_x, e\rangle \geq 0 \quad \text{for some } e \in S^2$$

and the boundary condition $\langle N, e\rangle > 0$ on ∂B implies $\langle N, e\rangle > 0$ on $\overline{B}$.

In Section 5.4 a kind of converse is proved for immersed cmc-surfaces satisfying $\langle N, e\rangle > 0$ on $\overline{B}$ for some $e \in S^2$ as they prove to be strictly stable. Furthermore a cmc-surface X is strictly stable if its density function $p = 2H^2 - K$ satisfies

(6) $$\int_B (2H^2 - K)\Lambda \, du \, dv < 2\pi.$$

For minimal surfaces ($H = 0,\ K \leq 0$) this condition means

(7) $$\int_B |K| \, dA < 2\pi.$$

Finally Gulliver's estimate

(8) $$A(X) \leq \frac{2\mu}{2\mu - 1}\pi r^2$$

is established for any μ-stable, immersed cmc-surface $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$, $\mu > 1/2$, representing a geodesic disk $K_r(x_0)$ of radius r. This leads to the curvature estimate

(9) $$\kappa_1^2(0) + \kappa_2^2(0) \leq c(h_0) r^{-2}$$

with a universal constant $c(h_0)$ proved in Theorems 1 and 2 of Section 5.5. This estimate holds for all stable, immersed cmc-surfaces X with $X(0) = 0$ and $|H| \leq h_0$ which represent a geodesic disk $K_r(0)$ of radius r around the origin. The estimate (9) implies a "Bernstein-type" result that was first stated by do Carmo and Peng [1] and by Fischer-Colbrie and Schoen [1].

In Section 5.6 the uniqueness theorem of J.C.C. Nitsche is proved, after establishing the perturbation equation for a field embedding and constructing the field immersion of a strictly stable immersed minimal surface that can be slightly extended beyond its boundary.

5.1 H-Surfaces and Their Normals

In Theorem 1 of Section 2.6 we have seen that a regular (i.e. immersed) surface $X \in C^2(\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^2$, that satisfies the conformality relations

$$(1) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

is a surface of mean curvature $\mathcal{H}(u,v)$ at $(u,v) \in \Omega$ if and only if X satisfies *Rellich's equation*

$$(2) \qquad \Delta X = 2\mathcal{H} X_u \wedge X_v.$$

Suppose now that $H(x)$ is a prescribed scalar-valued function of $x \in \mathbb{R}^3$ which is of class $C^{0,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$. Then a C^2-solution X of (2) with $\mathcal{H} := H \circ X$ and satisfying (1) will be called a *surface of prescribed mean curvature H in* $\mathbb{R}^3$. As for minimal surfaces we will also consider *branched surfaces* of this kind, i.e. we allow points $w \in \Omega$ where the function Λ, defined by

$$(3) \qquad \Lambda := |X_u|^2 = |X_v|^2 = |X_u \wedge X_v| = \mathcal{W},$$

is vanishing. Such points are again called **branch points** of X. As usual we write $H(X)$ for the composed function $H \circ X$. Summarizing we give the following

Definition 1. *A nonconstant solution $X \in C^2(\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^2$, of*

$$(4) \qquad \Delta X = 2H(X) X_u \wedge X_v,$$

satisfying the conformality relations (1), *will be called an H-surface. We speak of an immersed H-surface X if Λ given by*

$$|dX|^2 = \Lambda \cdot (du^2 + dv^2)$$

satisfies

$$(5) \qquad \Lambda(u,v) > 0 \quad \textit{for all points } (u,v) \in \Omega.$$

If $H(x) \equiv const$ we may address X as a constant H-surface; *also the notation cmc-surface is common.*

All notions of these definitions pertain to the class $C^2(\overline{\Omega}, \mathbb{R}^3)$. Usually we shall investigate *disk-type H-surfaces*, i.e. the parameter domain Ω will in most cases be the unit disk

$$B := \{w = (u,v) \in \mathbb{R}^2 : |w| < 1\},$$

and often the complex notation $w = u + iv \in \mathbb{C}$ is used.

Remark 1. Suppose that $H \in C^{r,\alpha}(\mathbb{R}^3)$, $r \geq 0$, $\alpha \in (0,1)$. Then any solution $X \in C^2(\Omega, \mathbb{R}^3)$ of (4) is of class $C^{r+2,\alpha}(\Omega, \mathbb{R}^3)$. This result follows from elliptic theory; see e.g. Sauvigny [16], Chapter IX, §4.

Remark 2. Let $H \in C^{0,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$, and suppose that X is an H-surface of class $C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ such that $X(\partial B)$ lies on a regular Jordan curve Γ of class $C^{2,\alpha}$. Then $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$. If $H \in C^{r-2,\alpha}(\mathbb{R}^3)$ and $\Gamma \in C^{r,\alpha}$, $r \geq 2$, it follows that $X \in C^{r,\alpha}(\overline{B}, \mathbb{R}^3)$. For a proof see Vol. 2, Section 7.3.

Remark 3. Let X be an H-surface of class $C^{2,\alpha}(B, \mathbb{R}^3)$ or $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ respectively. Then, for each point $w_0 \in B$ or $\overline{B}$, there is a vector $A \in \mathbb{C}^3$ with $A \neq 0$ and $\langle A, A \rangle = 0$, and a nonnegative integer $n = n(w_0)$ such that

$$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n) \quad \text{as } w \to w_0. \tag{6}$$

A proof of this fact by means of the Hartman-Wintner technique is given in Vol. 2, Section 2.10 (using Section 3.1 of Vol. 2). Another proof can be found in Sauvigny [16], Chapter XII, §10 which is based on the theory of "generalized analytic functions" (see Sauvigny [16], Chapter IV). The point w_0 is a branch point of X if and only if $n(w_0) \geq 1$, and $n(w_0) \geq 1$ is called the **order of the branch point** $w_0 \in B$ (or $\overline{B}$ respectively). The point w_0 is a regular point of X if and only if $n(w_0) = 0$.

Formula (6) implies that branch points w_0 of an H-surface $X \in C^{2,\alpha}(B, \mathbb{R}^3)$ or $\in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ are isolated in B or $\overline{B}$ respectively. In the first case there are at most finitely many branch points in any $\Omega \subset\subset B$, and in the second case there are at most finitely many branch points $w_1, \ldots, w_{k+\ell} \in \overline{B}$, say, $w_1, \ldots, w_k \in B$ and $w_{k+1}, \ldots, w_{k+\ell} \in \partial B$. The points $w_1, \ldots, w_k$ are the **inner branch points** of X, and $w_{k+1}, \ldots, w_{k+\ell}$ the **boundary branch points** of the H-surface X.

The first fundamental form ds^2 of an H-surface X is given by

$$\begin{aligned} ds^2 &= \langle dX, dX \rangle = \Lambda(du^2 + dv^2) \\ &= 2\langle X_w, X_{\overline{w}} \rangle (du^2 + dv^2) = 2|X_w|^2(du^2 + dv^2). \end{aligned} \tag{7}$$

The **set of regular points of** $X \in C^2(\overline{B}, \mathbb{R}^3)$, denoted by $\overline{B}'$, is given by

$$\overline{B}' = \{w \in \overline{B} \colon \Lambda(w) > 0\} = \overline{B} \setminus \{w_1, \ldots, w_{k+\ell}\}. \tag{8}$$

An important tool to cope with branch points analytically is the subsequent

Proposition 1. *There exists a sequence* $\{\chi_n\}$ *of functions* $\chi_n \in C_c^\infty(\overline{B}')$ *with* $0 \leq \chi_n \leq 1$ *satisfying*

$$\lim_{n\to\infty} \chi_n(w) = 1 \quad \textit{for all } w \in \overline{B}' \quad \textit{and} \quad \lim_{n\to\infty} D(\chi_n) = 0. \tag{9}$$

Proof. For $0 < r < R$ we define the functions $\phi(w)$ by $\phi(w) := 1$ for $|w| \le r$, $\phi(w) := 0$ for $|w| \ge R$, and

$$\phi(w) := \frac{\log|w| - \log R}{\log r - \log R} \quad \text{for } r < |w| < R.$$

Then $0 \le \phi \le 1$ and, for any $R > 0$,

$$\int_{\mathbb{R}^2} |\nabla\phi|^2 \, du \, dv = \frac{2\pi}{\log(R/r)} \to 0 \quad \text{as } r \to +0.$$

By mollifying these functions we can construct a sequence $\{\phi_n\}$ of functions $\phi_n \in C^\infty(\mathbb{R}^2)$ with $0 \le \phi_n \le 1$ such that $\phi_n(w) = 0$ for $|w| \ge R_n$, $R_n \to 0$, $\phi_n(w) = 1$ for $|w| \le r_n$, $0 < r_n < R_n$, and $\int_{\mathbb{R}^2} |\nabla\phi_n|^2 \, du \, dv \to 0$ as $n \to \infty$. Furthermore we have $\phi_n(w) \to 0$ for all $w \ne 0$.

Finally we define $\chi_n \in C^\infty(\overline{B})$ for $n \in \mathbb{N}$ by

$$\chi_n(w) := \prod_{\nu=1}^{k+\ell} [1 - \phi_n(w - w_\nu)], \quad w \in \overline{B}.$$

Obviously the sequence $\{\chi_n\}$ possesses the desired properties. □

Now we consider the normal N of an H-surface X near a branch point $w_0 \in \overline{B}$. A straight-forward calculation yields

$$N = \Lambda^{-1} X_u \wedge X_v = \frac{-i}{\langle X_w, X_{\overline{w}}\rangle} X_w \wedge X_{\overline{w}}. \tag{10}$$

Inserting the asymptotic expansion (6) with $A = a - ib$, $a, b \in \mathbb{R}^3$, $|a| = |b|$, $\langle a, b\rangle = 0$, we obtain

$$N(w) \to |a|^{-2} a \wedge b \in S^2 \quad \text{for } w \in \overline{B}' \text{ with } w \to w_0. \tag{11}$$

Therefore the normal N can be extended continuously from $\overline{B}'$ into the branch points of X, i.e. $N \in C^0(\overline{B}, \mathbb{R}^3)$ with $N(\overline{B}) \subset S^2$. Furthermore, N is of class $C^{2,\alpha}$ on B' and $C^{1,\alpha}$ on $\overline{B}'$. F. Sauvigny [1,2] proved that N is even of class $C^{2,\alpha}$ on B and established the following

Theorem 1. *The normal N to an H-surface $X \in C^{3,\alpha}(B, \mathbb{R}^3)$ is of class $C^{2,\alpha}(B, \mathbb{R}^3)$ and satisfies the differential equation*

$$\Delta N + 2pN = -2\Lambda H_x(X) \tag{12}$$

with

$$p := 2\Lambda H^2(X) - \Lambda K - \Lambda\langle H_x(X), N\rangle. \tag{13}$$

For the term ΛK involving the Gaussian curvature K of X we have

$$\Lambda K \in C^{1,\alpha}(B).$$

Proof. (i) Equation (12) with (13) is derived on $B' := B \setminus \{w_1, \dots, w_k\}$ in Sauvigny [16], Chapter XII, §9 (see Proposition 2). If the reader wants to check it, he finds the necessary formulae from classical differential geometry in Chapter 1 above, particularly in Section 1.3.

Using the Weingarten equations one obtains on $\overline{B}'$:

$$
(14)\qquad
\begin{aligned}
|\nabla N|^2 &= |N_u|^2 + |N_v|^2 = \Lambda^{-1}(\mathcal{L}^2 + 2\mathcal{M}^2 + \mathcal{N}^2) \\
&= \Lambda^{-1}[(\mathcal{L} + \mathcal{N})^2 - 2(\mathcal{L}\mathcal{N} - \mathcal{M}^2)] \\
&= 4\Lambda H^2(X) - 2\Lambda K = 2[2\Lambda H^2(X) - \Lambda K].
\end{aligned}
$$

Invoking the evident orthogonal expansion

$$
(15)\qquad \Lambda H_x(X) = \langle H_x(X), X_u \rangle X_u + \langle H_x(X), X_v \rangle X_v + \Lambda \langle H_x(X), N \rangle N
$$

we transform the differential equation (12) into the following equivalent form:

$$
(16)\qquad \Delta N + N|\nabla N|^2 + f(X, \nabla X) = 0 \quad \text{in } B',
$$

with

$$
(17)\qquad f(X, \nabla X) := 2[\langle H_x(X), X_u \rangle X_u + \langle H_x(X), X_v \rangle X_v].
$$

(ii) By the Gauss–Bonnet theorem (see Vol. 2, Section 2.11, Theorem 1 and in particular Remark 2) it follows that $\int_B |K|\, dA = \int_B \Lambda |K|\, du\, dv$ is finite. Then (14) implies that

$$
(18)\qquad \int_B |\nabla N|^2\, du\, dv < \infty.
$$

With the aid of a "smoothing sequence" $\{\chi_n\}$ from Proposition 1 we now derive a weak differential equation for N in B, using (16). To this end we choose an arbitrary test function $\phi \in C_c^\infty(B, \mathbb{R}^3)$, multiply (16) by $\phi \cdot \chi_n$, and perform an integration by parts. Then

$$
(19)\qquad
\begin{aligned}
&\int_B \langle \nabla N, \nabla(\phi \cdot \chi_n) \rangle\, du\, dv \\
&\quad = \int_B \langle N, \phi \rangle |\nabla N|^2 \chi_n\, du\, dv + \int_B \langle f(X, \nabla X), \phi \rangle \chi_n\, du\, dv.
\end{aligned}
$$

In this identity we want to let n tend to infinity. First we consider the left-hand side; we have

$$
\begin{aligned}
&\int_B \langle \nabla N, \nabla(\phi \cdot \chi_n) \rangle\, du\, dv \\
&\quad = \int_B \langle \nabla N, \nabla \phi \rangle \chi_n\, du\, dv + \int_B \langle \nabla N, \phi \nabla \chi_n \rangle\, du\, dv,
\end{aligned}
$$

and Schwarz's inequality yields

$$\begin{aligned}&\left|\int_B \langle \nabla N, \phi \nabla \chi_n \rangle \, du \, dv\right| \\ &\quad \le \left\{\int_B |\nabla N|^2 \, du \, dv\right\}^{1/2} \left\{\int_B |\phi \nabla \chi_n|^2 \, du \, dv\right\}^{1/2} \\ &\quad \le 2 \sup_B |\phi| \sqrt{D(N)} \sqrt{D(\chi_n)} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

on account of (18) and $D(\chi_n) \to 0$. Furthermore, $\chi_n(w) \to 1$ on B'. Then (18) and Lebesgue's convergence theorem imply

$$\int_B \langle \nabla N, \nabla \phi \rangle \chi_n \, du \, dv \to \int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv$$

whence

$$\int_B \langle \nabla N, \nabla (\phi \cdot \chi_n) \rangle \, du \, dv \to \int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv.$$

For the same reason the right-hand side of (19) tends to

$$\int_B \langle N, \phi \rangle |\nabla N|^2 \, du \, dv + \int_B \langle f(X, \nabla X), \phi \rangle \, du \, dv$$

as $n \to \infty$, and so we infer from (19) that

$$\int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv = \int_B \{\langle N, \phi \rangle |\nabla N|^2 + \langle f(X, \nabla X), \phi \rangle\} \, du \, dv. \tag{20}$$

Since N is already known to be continuous on B (and even on $\overline{B}$), and $f(X, \nabla X) \in C^\alpha(B, \mathbb{R}^3)$, a regularity result by Ladyzhenskaya and Uraltseva [1,2] implies $N \in C^{2,\alpha}(B, \mathbb{R}^3)$; for a simple proof of this fact see F. Tomi [1].

Finally, equation (14) leads to

$$-\Lambda K = \tfrac{1}{2} |\nabla N|^2 - 2\Lambda H^2(X), \tag{21}$$

and therefore $\Lambda K \in C^{1,\alpha}(B)$. □

Remark 4. Although we know that $N \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{2,\alpha}(\overline{B} \setminus \Sigma', \mathbb{R}^3)$, $\Sigma' = \{w_{k+1}, \dots, w_{k+\ell}\}$ = set of boundary branch points, we do not know whether $\lim_{w \to w'} \nabla N(w)$ or even $\lim_{w \to w'} \nabla^2 N(w)$ exist for $w' \in \Sigma'$. An answer to this question seems to be complicated but valuable.

5.2 Bonnet's Mapping and Bonnet's Surface

In this section we briefly want to discuss *Bonnet's fundamental form* associated with any H-surface, and the *Bonnet surface* associated with a cmc-surface. The Bonnet surface provides valuable information on the umbilical points of a cmc-surface and can serve as a useful substitute for the Gauss mapping N. It might prove to be useful in further investigations.

For the notations to be used in the sequel we refer to Sections 1.1 and 1.2, and also to the brief introduction to the differential-geometric formulae given in Sauvigny [16], Chapter XI, §1.

Let $S(w) : T_w X \to T_w X$ be the *Weingarten mapping* associated with an arbitrary H-surface $X : \overline{B} \to \mathbb{R}^3$. At each regular point $w \in \overline{B}$ (i.e. for $w \in \overline{B}'$) this mapping is a selfadjoint linear mapping of the tangent space $T_w X$ of X corresponding to w (or, less precisely, the tangent space of the surface X at the point $X(w)$). Secondly, let

$$I(w) : T_w X \to T_w X \quad \text{with } I(w)V = V \text{ for } V \in T_w X$$

be the identity on $T_w X$.

Definition 1. *Let $X : \overline{B} \to \mathbb{R}^3$ be an H-surface of class $C^{2,\alpha}$. Then, for any regular point $w \in \overline{B}$ of X, we define the* **Bonnet mapping**

$$R(w) : T_w X \to T_w X$$

by

$$R(w) := H(X(w))I(w) - S(w). \tag{1}$$

Remark 1. Clearly the Bonnet mapping $R(w)$ is a selfadjoint linear operator on $T_w X$ with the two eigenvalues $\lambda_1(w)$ and $\lambda_2(w)$, given by

$$\lambda_1(w) = H(X(w)) - \kappa_1(w), \quad \lambda_2(w) = H(X(w)) - \kappa_2(w),$$

where $\kappa_1(w)$ and $\kappa_2(w)$ are the principal curvatures of X at $w \in \overline{B}'$. Since $2H(X(w)) = \kappa_1(w) + \kappa_2(w)$, we obtain

$$\lambda_1(w) = \tfrac{1}{2}[\kappa_2(w) - \kappa_1(w)], \quad \lambda_2(w) = \tfrac{1}{2}[\kappa_1(w) - \kappa_2(w)]. \tag{2}$$

Therefore the Bonnet mapping has a vanishing trace,

$$\operatorname{tr} R(w) = 0 \quad \text{for all } w \in \overline{B}', \tag{3}$$

and from

$$\det R = \lambda_1 \lambda_2 = -\tfrac{1}{4}[\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2] = -\left[\tfrac{1}{4}(\kappa_1 + \kappa_2)^2 - \kappa_1\kappa_2\right]$$

it follows for $w \in \overline{B}'$ that

$$\det R(w) = -[H^2(X(w)) - K(w)] = -\lambda_1^2(w) = -\lambda_2^2(w). \tag{4}$$

Since $\lambda_1(w) = -\lambda_2(w)$, the Bonnet map $R(w)$ is either indefinite or the zero mapping. Clearly $R(w) = 0$ if and only if $\kappa_1(w) = \kappa_2(w)$, that is, $R(w)$ vanishes exactly at the umbilical points $w \in \overline{B}'$ of the H-surface X. Furthermore, $R^*R = \lambda_1^2 I$ since $\lambda_1^2 = \lambda_2^2$, and so we obtain the fundamental identity

$$(5) \qquad R^*(w)R(w) = [H^2(X(w)) - K(w)]I(w) \quad \text{for all } w \in \overline{B}'$$

with

$$(6) \qquad H^2(X(w)) - K(w) = \lambda_1^2(w) = \lambda_2^2(w) \geq 0.$$

Since $SX_u = -N_u$, $SX_v = -N_v$, one obtains

$$RX_u = N_u + H(X)X_u, \quad RX_v = N_v + H(X)X_v.$$

Set $M := (X_u, X_v)$ and multiply (5) from the right by M and from the left by M^*. Then the right-hand side becomes

$$[H^2(X) - K] \cdot \begin{pmatrix} |X_u|^2 & \langle X_u, X_v \rangle \\ \langle X_u, X_v \rangle & |X_v|^2 \end{pmatrix} = \Lambda[H^2(X) - K] \cdot I$$

whereas the left-hand side becomes

$$M^*R^*RM = \begin{pmatrix} \mu & \tau \\ \tau & \nu \end{pmatrix}$$

with

$$(7) \qquad \begin{aligned} &\mu := |N_u + H(X)X_u|^2, \quad \nu := |N_v + H(X)X_v|^2, \\ &\tau := \langle N_u + H(X)X_u, N_v + H(X)X_v \rangle. \end{aligned}$$

Thus

$$(8) \qquad \mu = \nu = \Lambda \cdot [H^2(X) - K], \quad \tau = 0.$$

Definition 2. *With any H-surface $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ we associate the quadratic form*

$$(9) \qquad d\sigma^2 = \langle N_{u^\alpha} + H(X)X_{u^\alpha}, N_{u^\beta} + H(X)X_{u^\beta} \rangle \, du^\alpha \, du^\beta,$$

$u^1 := u$, $u^2 := v$, which is called **Bonnet's fundamental form**.

The formulae (7)–(9) imply that Bonnet's fundamental form is conformal to the first fundamental form $|dX|^2 = \Lambda(du^2 + dv^2)$. More precisely, since $X, N \in C^1(B, \mathbb{R}^3)$, we obtain:

Theorem 1. *Bonnet's fundamental form $d\sigma^2$ of an H-surface X with the Gauss curvature K can be written as*

$$(10) \qquad d\sigma^2 = \Lambda[H^2(X) - K](du^2 + dv^2), \quad \Lambda = |X_u|^2.$$

This quadratic form is positive semidefinite and vanishes exactly at those points $w \in B$ which are either umbilical or branch points of X.

Definition 3. *With any H-surface $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ of constant mean curvature H and any vector $Y_0 \in \mathbb{R}^3$ we associate a new surface, the* **Bonnet surface** $Y \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{2,\alpha}(B, \mathbb{R}^3)$ *of X, which is defined by*

(11) $$Y(w) := N(w) + HX(w) + Y_0 \quad \text{for } w \in \overline{B}.$$

As a consequence of Theorem 1 we obtain the following result:

Corollary 1. *The Bonnet surface $Y = N + HX + Y_0$ of any cmc-surface $X \in C^2(B, \mathbb{R}^3)$ satisfies*

(12) $$|dY|^2 = \Lambda(H^2 - K)(du^2 + dv^2) = d\sigma^2 \quad \text{in } B$$

whence in particular

(13) $$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0 \quad \text{in } B$$

and

(14) $$|Y_u \wedge Y_v| = \Lambda \cdot (H^2 - K).$$

Remark 2. The Bonnet surface Y of a cmc-surface X degenerates exactly on the set Σ of branch points of X in B and the set Σ^* of umbilical points of X in B. Whereas the points of Σ are isolated, the set Σ^* might have nonisolated points. Even $\operatorname{int} \Sigma^*$ can be nonvoid as in the case of a planar surface or a spherical cap. Note however that, by regularity theory, each cmc-surface X is real analytic in B, and so $\operatorname{int} \Sigma^* \neq \emptyset$ implies $H^2 - K(w) \equiv 0$ on $B \setminus \Sigma$, i.e. all points $w \in B \setminus \Sigma$ are umbilical points. This implies that X is either planar or a spherical surface.

Theorem 2. *The Bonnet surface Y of a cmc-surface X is either a constant mapping or a cmc-surface of mean curvature one. In the first case all points of X are umbilical, i.e. X is either planar or a spherical surface, while in the second case X has only isolated umbilical points in B, and the normal $\tilde{N}$ of Y coincides with $-N$ where N is the normal of X.*

Proof. On account of Theorem 1 in Section 5.1 we have

$$\Delta N = -4\Lambda H^2 N + 2\Lambda K N \quad \text{in } B,$$

and furthermore

$$\Delta X = 2H\Lambda N \quad \text{in } B.$$

This implies

(15) $$\Delta Y = -2\Lambda(H^2 - K)N \quad \text{in } B.$$

In addition, the relations (4) and (6) imply

$$\det R(w) = -(H^2 - K(w)) \le 0.$$

Thus, by (4) and $Y_u = RX_u$, $Y_v = RX_v$ it follows that

$$Y_u \wedge Y_v = -(H^2 - K) X_u \wedge X_v$$

whence

$$Y_u \wedge Y_v = -\Lambda(H^2 - K)N \quad \text{in } B. \tag{16}$$

From (15) and (16) one finally infers

$$\Delta Y = 2Y_u \wedge Y_v \quad \text{in } B, \tag{17}$$

and the formulae (13) of Corollary 1 state that

$$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0 \quad \text{in } B.$$

Then the Hartman–Wintner theorem states that either (i) $Y(w) \equiv \text{const}$ in B, or (ii) $Y(w)$ is nowhere locally constant in B, and the branch points of Y are isolated. In case (i) the surface is either planar or spherical, while in case (ii) the surface X has at most isolated umbilical points, and Y is a cmc-surface of mean curvature one. Moreover, in this case the surface normal $\tilde{N}$ of Y is defined by

$$\tilde{N} := \frac{1}{|Y_u \wedge Y_v|}(Y_u \wedge Y_v) \quad \text{on } B \setminus (\Sigma \cup \Sigma^*)$$

and can be extended continuously to all of B.

Note also that (14) and (16) imply

$$Y_u \wedge Y_v = -|Y_u \wedge Y_v| N,$$

whence $\tilde{N} = -N$ on B. □

Remark 3. For any cmc-surface X, its Bonnet surface Y "realizes" the Bonnet fundamental form $d\sigma^2$ of X via the formula (12). For an H-surface X with variable H one cannot expect to find a similar realization of its $d\sigma^2$ since the set of umbilical points of X might be very general.

Remark 4. For a cmc-surface X with $H \neq 0$, the associated Bonnet surface Y provides a suitable substitute for the Gauss map N of X.

Remark 5. Let Y be the Bonnet map of a cmc-surface X with $H \neq 0$, and set

$$Z := X + \frac{1}{H} N. \tag{18}$$

Then $Y = HZ$, and it follows that

$$\Delta Z = 2H Z_u \wedge Z_v, \quad |Z_u|^2 = |Z_v|^2, \quad \langle Z_u, Z_v \rangle = 0.$$

Therefore we obtain O. Bonnet's result that (18) *defines a second H-surface parallel to X*, except for a spherical X when Z reduces to a point.

5.3 The Second Variation of F for H-Surfaces and Their Stability

As already mentioned in No. 3 of the *Supplementary Results* to Section 4.5, H-surfaces are closely related to certain functionals $E := D + 2V$ that generalize Dirichlet's integral D. In fact if H is a given scalar function on $\mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field on $\mathbb{R}^3$,

$$Q(x) = (Q^1(x), Q^2(x), Q^3(x)), \quad x = (x^1, x^2, x^3) \in \mathbb{R}^3,$$

such that

$$\text{div } Q = H, \quad \text{i.e.} \quad Q^1_{x^1} + Q^2_{x^2} + Q^3_{x^3} = H, \tag{1}$$

then any H-surface $X : B \to \mathbb{R}^3$ is a stationary point of the functional $E = D + 2V$ where

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv$$

is the *Dirichlet integral* of X and V denotes a *volume integral* defined by

$$V(X) = \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv. \tag{2}$$

Introducing the trilinear product

$$[a, b, c] = \det(a, b, c) = a \cdot (b \wedge c) = b \cdot (c \wedge a) = c \cdot (a \wedge b)$$

we can write V as

$$V(X) = \int_B [Q(X), X_u, X_v] \, du \, dv. \tag{3}$$

In Vol. 2, Chapter 4, we shall construct H-surfaces within a prescribed boundary contour Γ by minimizing the functional[1]

$$E(X) := \int_B \left\{ \frac{1}{2} |\nabla X|^2 + 2[Q(X), X_u, X_v] \right\} du \, dv \tag{4}$$

in a subset of the class $\mathcal{C}(\Gamma)$ defined in Section 4.2.

Closely related to $E = D + 2V$ is the functional $F := A + 2V$ where A is the usual area functional

$$A(X) = \int_B |X_u \wedge X_v| \, du \, dv = \int_B \sqrt{|X_u|^2 |X_v|^2 - \langle X_u, X_v \rangle^2} \, du \, dv,$$

that is,

$$F(X) := \int_B \{ |X_u \wedge X_v| + 2[Q(X), X_u, X_v] \} \, du \, dv. \tag{5}$$

[1] However, we shall write $E = D + V$ which changes (1) to $\text{div}\, Q = 2H$.

We have

$$F(X) \le E(X) \tag{6}$$

and the equality sign holds if and only if

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Clearly, $V(X)$ (and therefore also $E(X)$ and $F(X)$) are well-defined if $X \in H_2^1(B, \mathbb{R}^3)$ and either $\sup_{\mathbb{R}^3} |Q| < \infty$ or $X \in L^\infty(B, \mathbb{R}^3)$.

In Sections 2.1 and 2.8 we have already derived the first variation $\delta A(X, Y)$ and the second variation $\delta^2 A(X, Y)$ of a regular C^2-surface $X : B \to \mathbb{R}^3$ in normal direction $Y = \varphi N$, $\varphi \in C_c^\infty(B)$, N being the normal of X. Now we want to admit also branched surfaces X; for the sake of simplicity we assume that X is an H-surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, $0 < \alpha < 1$, that is regular (i.e. immersed) in $\overline{B}'$ as in 5.1, $\overline{B}' = \overline{B} \setminus \{\text{branch points of } X\}$. For an arbitrary test function $\varphi \in C_c^\infty(B')$ with $B' = \overline{B}' \cap B$ we consider the *normal variation* $Z : \overline{B} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^3$, $\epsilon_0 > 0$, which is defined by

$$Z(w, t) := X(w) + t\varphi(w) N(w), \quad w \in B, \ |t| < \epsilon_0, \tag{7}$$

where N is the normal of X. From formula (15) in Section 2.8 we obtain the following expansion at all regular points $w \in B$ of X:

$$\begin{aligned} &|Z_u(w,t) \wedge Z_v(w,t)| \\ &\quad = \Lambda(w) - 2t\Lambda(w) H(X(w)) \varphi(w) \\ &\qquad + \tfrac{1}{2} t^2 [|\nabla \varphi(w)|^2 + 2\Lambda(w) K(w) \varphi^2(w)] + O(w, t^3) \end{aligned} \tag{8}$$

where $\Lambda = |X_u|^2$ and K is the Gauss curvature of X. The error term $O(w, t^3)$ vanishes outside of $\operatorname{supp} \varphi$, and we have

$$|O(w, t^3)| \le \text{const} \cdot t^3 \quad \text{for all } w \in \operatorname{supp} \varphi \subset\subset B'. \tag{9}$$

For $\varphi \in C_c^\infty(B')$, this implies

$$\frac{d}{dt} A(Z(\cdot, t)) \bigg|_{t=0} = -\int_B 2\Lambda H(X) \varphi \, du \, dv \tag{10}$$

and

$$\frac{d^2}{dt^2} A(Z(\cdot, t)) \bigg|_{t=0} = \int_B \{ |\nabla \varphi|^2 + 2\Lambda K \varphi^2 \} \, du \, dv. \tag{11}$$

By Theorem 1 of 5.1, the right-hand sides of (10) and (11) can continuously be extended onto functions $\varphi \in C_c^\infty(B)$. Therefore we take (10) and (11) as definitions of the first two derivatives of $A(Z(\cdot, t))$ at $t = 0$, i.e. for $\delta A(X, \varphi N)$ and $\delta^2 A(X, \varphi N)$, if $\varphi \in C_c^\infty(B)$. In order to compute $\frac{d}{dt} V(Z(\cdot, t))|_{t=0}$ and

$\frac{d^2}{dt^2}V(Z(\cdot,t))|_{t=0}$ for $\varphi \in C_c^\infty(B)$, we introduce $P(w,t) := Q(Z(w,t))$, $w \in B$, $|t| < \epsilon_0$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}[P, Z_u, Z_v] &= [P_t, Z_u, Z_v] + [P, (\varphi N)_u, Z_v] + [P, Z_u, (\varphi N)_v] \\
&= [P_t, Z_u, Z_v] + [P, \varphi N, Z_v]_u + [P, Z_u, \varphi N]_v \\
&\quad - [P_u, Z_t, Z_v] - [P_v, Z_u, Z_t] \\
&= [P_u, Z_v, Z_t] + [P_v, Z_t, Z_u] + [P_t, Z_u, Z_v] \\
&\quad + \{[P, \varphi N, Z_v]_u + [P, Z_u, \varphi N]_v\} \\
&= [Q_x(Z)Z_u, Z_v, Z_t] + [Q_x(Z)Z_v, Z_t, Z_u] + [Q_x(Z)Z_t, Z_u, Z_v] \\
&\quad + \{\dots\} \\
&= [Z_u, Z_v, Z_t] \cdot (\text{div } Q)(Z) + \{\dots\} \\
&= [Z_u, Z_v, Z_t] \cdot H(Z) + \{\dots\}.
\end{aligned}$$

The divergence theorem implies $\int_B \{\dots\}\, du\, dv = 0$ since $\operatorname{supp}\varphi \subset B$, and so

$$\frac{d}{dt}V(Z) = \int_B H(Z) Z_t \cdot (Z_u \wedge Z_v)\, du\, dv. \tag{12}$$

We have

$$\begin{aligned}
Z_u \wedge Z_v &= (X_u + t\varphi_u N + t\varphi N_u) \wedge (X_v + t\varphi_v N + t\varphi N_v) \\
&= X_u \wedge X_v + t\{\varphi_v X_u \wedge N + \varphi_u N \wedge X_v + \varphi(X_u \wedge N_v + N_u \wedge X_v)\} \\
&\quad + t^2(\varphi\varphi_u N \wedge N_v + \varphi\varphi_v N_u \wedge N + \varphi^2 N_u \wedge N_v).
\end{aligned}$$

Multiplication by $Z_t = \varphi N$ yields

$$Z_t \cdot (Z_u \wedge Z_v) = \Lambda\varphi - 2\Lambda H(X)\varphi^2 t + \Lambda K \varphi^3 t^2.$$

Then formula (12) and Theorem 1 of 5.1 imply

$$\frac{d}{dt}V(Z) = \int_B \{\Lambda H(Z)\varphi - 2\Lambda H(X)H(Z)\varphi^2 t + \Lambda K H(Z)\varphi^3 t^2\}\, du\, dv. \tag{13}$$

Therefore

$$\left.\frac{d}{dt}V(Z)\right|_{t=0} = \int_B \Lambda H(X)\varphi\, du\, dv. \tag{14}$$

Furthermore we infer from (13) that

$$\left.\frac{d^2 V(Z)}{dt^2}\right|_{t=0} = \int_B \{\langle H_x(X), N\rangle - 2H^2(X)\}\varphi^2 \Lambda\, du\, dv. \tag{15}$$

Since $F = A + 2V$, we infer from (10) and (14) that

$$\frac{d}{dt}F(Z(\cdot,t))\bigg|_{t=0} = \int_B \{-2\Lambda H(X)\varphi + 2\Lambda H(X)\varphi\}\,du\,dv = 0,$$

and from (11) and (15) that

$$\frac{d^2}{dt^2}F(Z(\cdot,t))\bigg|_{t=0} = \int_B \{|\nabla\varphi|^2 + 2\Lambda(K - 2H^2(X) + \langle H_x(X), N\rangle)\varphi^2\}\,du\,dv.$$

Set

$$\delta F(X,\varphi N) := \frac{d}{dt}F(Z)\bigg|_{t=0}, \quad \delta^2 F(X,\varphi N) := \frac{d^2}{dt^2}F(Z)\bigg|_{t=0}. \tag{16}$$

Thus we have proved:

Theorem 1. *The first variation $\delta F(X,\varphi N)$ of $F = A + 2V$ with $\operatorname{div} Q = H$ at an H-surface $X \in C^{3,\alpha}(\overline{B},\mathbb{R}^3)$ in the normal direction $Y = \varphi N$ with $\varphi \in C_c^\infty(B)$ vanishes, and for the second variation $\delta^2 F(X,\varphi N)$ we have*

$$\delta^2 F(X,\varphi N) = \int_B \{|\nabla\varphi|^2 - 2p\varphi^2\}\,du\,dv, \tag{17}$$

where the density function p associated with X is defined by

$$p := \Lambda \cdot [2H^2(X) - K - \langle H_x(X), N\rangle]. \tag{18}$$

If $Q \in C^{2,\alpha}(\mathbb{R}^3,\mathbb{R}^3)$, then $p \in C^{0,\alpha}(B)$. Note that p is the same function as in Section 5.1, Theorem 1, formula (13).

Definition 1. *An H-surface $X \in C^{3,\alpha}(\overline{B},\mathbb{R}^3)$ is called* **stable** *if it satisfies the* **stability inequality**

$$\delta^2 F(X,\varphi N) \ge 0 \quad \textit{for all } \varphi \in C_c^\infty(B) \tag{19}$$

which can be written as

$$\int_B |\nabla\varphi|^2\,du\,dv \ge 2\int_B p\varphi^2\,du\,dv \quad \textit{for all } \varphi \in C_c^\infty(B). \tag{20}$$

Remark 1. By means of Proposition 1 in 5.1 it follows easily that the stability condition (19) is equivalent to

$$\delta^2 F(X,\varphi N) \ge 0 \quad \text{for all } \varphi \in C_c^\infty(B')$$

where $B' := B \setminus \{\text{branch points of } X\}$.

We note also that it suffices to assume $X \in C^{3,\alpha}(B,\mathbb{R}^3)$ in Definition 1 and in Theorem 1 since we only consider φ with compact support in B.

Remark 2. When an H-surface is **nonstable** we can find some $\varphi \in C_c^\infty(B)$ such that $\delta^2 F(X, \varphi N) < 0$. Obviously, global and local minimizers of F are stable.

On the other hand, an H-surface is said to be **unstable**, if it does not constitute a strong local minimum of F, i.e. in any $C^0(\overline{B}, \mathbb{R}^3)$-neighborhood of X one can find a surface $\tilde{X}$ with $F(\tilde{X}) < F(X)$. A nonstable surface is necessarily unstable while the converse need not be true.

In the next section we shall define the notions **μ-stable** for $\mu > 0$ and **strictly stable** ($\mu > 2$).

Remark 3. The vector field Q is not uniquely determined by the equation $\operatorname{div} Q = H$, neither is the functional F. Nevertheless the notions "stable and nonstable" are uniquely defined since in $\delta^2 F(X, \varphi N)$ only the expressions H and H_x enter. *Occasionally one prefers the notation $\delta^2 F(X, \varphi)$ which means the same as $\delta^2 F(X, \varphi N)$.*

The following central result for stable H-surfaces was found by F. Sauvigny [1,2]. It is used to prove that under certain assumptions a stable surface is in fact "nonparametric", that is, a graph of a function which is defined on a domain of the x^1, x^2-plane.

Theorem 2. *Suppose that the prescribed mean curvature $H(x) = H(x^1, x^2, x^3)$ is of class $C^{1,\alpha}(\mathbb{R}^3)$ and satisfies the monotonicity condition*

$$H_{x^3}(x) \geq 0 \quad \text{for } x \in \mathbb{R}^3. \tag{21}$$

Furthermore let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, $0 < \alpha < 1$, be a stable H-surface the normal $N = (N^1, N^2, N^3)$ of which satisfies the boundary condition

$$N^3(w) > 0 \quad \text{for all } w \in \partial B. \tag{22}$$

Then it follows that $N^3(w) > 0$ for all $w \in \overline{B}$.

Proof. Let $e_3 = (0, 0, 1)$ be the unit vector in x^3-direction and set

$$f := N^3 = \langle N, e_3 \rangle. \tag{23}$$

Multiplying both sides of equation (12) in 5.1 by e_3 and noting $-2\Lambda H_{x^3}(X) \leq 0$, it follows

$$\Delta f + 2pf \leq 0 \quad \text{in } B. \tag{24}$$

Since, by assumption, $f(w) > 0$ for $w \in \partial B$ holds true, Proposition 1 below yields $f(w) > 0$ for all $w \in \overline{B}$. □

Remark 4. The geometrical content of Theorem 2 is the following: If a stable H-surface constitutes a positively oriented, branched graph over the x^1, x^2-plane at the boundary, then the same property holds true in the interior.

Now we establish the result that was used in the proof of Theorem 2. It is of independent interest; a similar reasoning will be applied when we treat partially free boundary value problems for minimal surfaces (cf. Vol. 3, Section 3).

Proposition 1. *Suppose that* $p \in C^{0,\alpha}(B)$ *satisfies the* **stability inequality**

$$\int_B |\nabla\varphi|^2\,du\,dv \geq 2\int_B p\varphi^2\,du\,dv \quad \textit{for all } \varphi \in C_c^\infty(B) \tag{25}$$

and let $f \in C^0(\overline{B}) \cap C^2(B)$ *be a solution of the boundary value problem*

$$\Delta f + 2pf \leq 0 \quad \textit{in } B, \qquad f(w) > 0 \quad \textit{on } \partial B. \tag{26}$$

Then one has $f(w) > 0$ *for all* $w \in \overline{B}$.

Remark 5. The assertion would already follow from (26) alone if one had $p(w) \leq 0$ on B, as one could apply the maximum principle. The gist of Proposition 1 is that the assumption $p \leq 0$ can be replaced by (25). Note that even for minimal surfaces one has $p = -\Lambda K \geq 0$.

Proof of Proposition 1. We first show that $f(w) \geq 0$ on B. To this end we consider the "negative part" f^- of f, defined by

$$f^-(w) := \min\{f(w), 0\} \quad \text{for } w \in \overline{B},$$

which is of the class $H_2^1(B)$ with compact support in B and satisfies

$$\nabla f^-(w) = \begin{cases} 0 & \text{for almost all } w \in B \text{ with } f(w) \geq 0, \\ \nabla f(w) & \text{for all } w \in B \text{ with } f(w) < 0. \end{cases}$$

Then

$$\int_B |\nabla f^-|^2\,du\,dv = -\int_B f^- \Delta f\,du\,dv$$

on account of a generalized version of the divergence theorem (see e.g. Sauvigny [16], Chapter VIII, §9, Propositions 1 and 2), and by (26):

$$-\int_B f^-\Delta f\,du\,dv \leq 2\int_B pff^-\,du\,dv = 2\int_B p|f^-|^2\,du\,dv.$$

Therefore,

$$\int_B |\nabla f^-|^2\,du\,dv \leq 2\int_B p|f^-|^2\,du\,dv. \tag{27}$$

Next, with $\psi \in C_c^\infty(B)$, we insert $\varphi := f^- + \epsilon\psi \in \mathring{H}_2^1(B)$ into (25), which even holds for test functions of class $\mathring{H}_2^1(B)$, $|\epsilon| \leq \epsilon_0,\ \epsilon_0 > 0$, thus obtaining

$$\begin{aligned} &\int_B |\nabla f^-|^2\,du\,dv + 2\epsilon\int_B \nabla f^- \cdot \nabla\psi\,du\,dv + \epsilon^2\int_B |\nabla\psi|^2\,du\,dv \\ &\quad \geq 2\int_B p|f^-|^2\,du\,dv + 4\epsilon\int_B pf^-\psi\,du\,dv + 2\epsilon^2\int_B p\psi^2\,du\,dv. \end{aligned}$$

With the aid of (27) we arrive at

$$2\epsilon \int_B (\nabla f^- \cdot \nabla\psi - 2pf^-\psi)\, du\, dv + \epsilon^2 \int_B (|\nabla\psi|^2 - 2p\psi^2)\, du\, dv \geq 0$$

for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, whence we obtain the weak differential equation

$$(28) \qquad \int_B (\nabla f^- \cdot \nabla\psi - 2pf^-\psi)\, du\, dv = 0 \quad \text{for all } \psi \in C_c^\infty(B).$$

Applying Moser's inequality (see Gilbarg–Trudinger [1], or Sauvigny [16], Chapter X, §5, Theorem 1) we infer from $f^-(w) \equiv 0$ near ∂B, that $f^-(w) \equiv 0$ in $\overline{B}$, and therefore $f(w) \geq 0$.

Finally, (26) implies

$$\int_B (\nabla f \cdot \nabla\varphi - 2pf\varphi)\, du\, dv \geq 0 \quad \text{for all } \varphi \in C_c^\infty(B) \text{ with } \varphi \geq 0,$$

and we have $f \geq 0$. Invoking once more Moser's inequality (see loc. cit. above) and recalling the assumption $f(w) > 0$ on ∂B we arrive at the desired inequality $f(w) > 0$ for $w \in \overline{B}$. □

5.4 On μ-Stable Immersions of Constant Mean Curvature

The density function p associated with an H-surface X might even change its sign if $H(x)$ is variable. This phenomenon is excluded for constant H since in this case

$$(1) \qquad p = \Lambda \cdot (2H^2 - K) = \tfrac{1}{2}\Lambda \cdot (\kappa_1^2 + \kappa_2^2) \geq 0.$$

Assumption. *In this section we consider immersed cmc-surfaces* $X : \overline{B} \to \mathbb{R}^3$ *of class* $C^{2,\alpha}$, *i.e.*

$$X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3), \quad 0 < \alpha < 1, \quad \textit{and} \quad \Lambda(w) > 0 \quad \textit{on } \overline{B}.$$

Then X is real analytic on B, $H \equiv \text{const}$, $K \in C^{0,\alpha}(\overline{B})$ and the density function p associated with X is of class $C^{0,\alpha}(\overline{B})$; in particular, p is continuous up to the boundary ∂B.

Definition 1. *An immersed cmc-surface X is called* μ-**stable** *with $\mu > 0$ if*

$$(2) \qquad \int_B |\nabla\varphi|^2\, du\, dv \geq \mu \int_B p\varphi^2\, du\, dv \quad \textit{for all } \varphi \in C_c^\infty(B)$$

holds true; *if even $\mu > 2$, the surface X is said to be* **strictly stable**.

Remark 1. Since $p \in L^\infty(B)$, relation (2) is equivalent to

$$\int_B |\nabla\varphi|^2 \, du \, dv \geq \mu \int_B p\varphi^2 \, du \, dv \quad \text{for all } \varphi \in \mathring{H}_2^1(B).$$

Remark 2. The 2-stable, immersed cmc-surfaces X are stable in the sense of Section 5.3.

Let us begin with the following instructive result which for minimal surfaces is due to H.A. Schwarz.

Theorem 1. *If the immersed cmc-surface X with the surface normal $N = (N^1, N^2, N^3)$ satisfies*

$$N^3(w) > 0 \quad \textit{for all } w \in \overline{B}, \tag{3}$$

then X is strictly stable.

Proof. We solve the variational problem

$$D(\varphi) \to \min \quad \text{in } \left\{ \varphi \in \mathring{H}_2^1(B) \colon \int_B p\varphi^2 \, du \, dv = 1 \right\}. \tag{4}$$

Its solution φ_0 is an eigenfunction to the least eigenvalue $\mu > 0$ of the eigenvalue problem

$$-\Delta\varphi_0 = \mu p \varphi_0 \quad \text{in } B, \quad \varphi_0 = 0 \quad \text{on } \partial B, \tag{5}$$

where $\varphi_0 \in \mathring{H}_2^1(B)$. Elliptic theory yields $\varphi_0 \in C^{2,\alpha}(\overline{B})$.

Let $e_3 := (0, 0, 1)$ and set $\psi := N^3 = \langle N, e_3 \rangle \in C^{1,\alpha}(\overline{B})$. The function ψ is real analytic on B and satisfies $\psi(w) > 0$ on $\overline{B}$. In order to compare the eigenfunction φ_0 with the auxiliary function ψ, we first note that ψ is a solution of

$$-\Delta\psi = 2p\psi \quad \text{in } B, \tag{6}$$

taking equation (12) of Section 5.1 into account. Obviously we can find a number $\lambda \in \mathbb{R}$ such that the further auxiliary function

$$\chi := \psi + \lambda\varphi_0 \tag{7}$$

satisfies

$$\chi \geq 0 \quad \text{in } B, \quad \chi > 0 \quad \text{on } \partial B, \quad \chi(w_0) = 0 \quad \text{for at least one } w_0 \in B. \tag{8}$$

From

$$\begin{aligned} -\Delta\chi &= -\Delta\psi - \lambda\Delta\varphi_0 = 2p\psi + \mu p\lambda\varphi_0 \\ &= (2 - \mu)p\psi + \mu p \cdot (\psi + \lambda\varphi_0) \end{aligned}$$

we infer

$$\Delta\chi + \mu p\chi = (\mu - 2)p\psi. \tag{9}$$

Suppose now that X were not strictly stable. Then we had $\mu \le 2$, and (9) would yield the differential inequality

$$\Delta\chi + \mu p\chi \le 0 \quad \text{in } B. \tag{10}$$

Applying the same reasoning as in the proof of Proposition 1 of Section 5.3 we infer $\chi(w) \equiv 0$ in B, which evidently contradicts (8). Therefore X must be strictly stable. □

The following profound result will be used in Section 5.6 to prove a uniqueness result for Plateau's problem.

Theorem 2. *Let X be an immersed cmc-surface whose density function $p = (2H^2 - K)\Lambda$ satisfies*

$$\int_B (2H^2 - K)\Lambda \, du\, dv < 2\pi. \tag{11}$$

Then X is strictly stable.

Proof. (i) On the northern hemisphere $S_r^+ := \{x \in \mathbb{R}^3 : |x| = r,\ x^3 > 0\}$ of radius r with the area $2\pi r^2$ we consider the eigenvalue problem for the Laplace–Beltrami operator with zero boundary values on the equator $\partial S_r^+ = \{x \in \mathbb{R}^3 : |x| = r,\ x^3 = 0\}$. The least eigenvalue $\lambda_1(S_r^+)$ can explicitly be determined as

$$\lambda_1(S_r^+) = 2/r^2 \tag{12}$$

in the following way: Via stereographic projection we construct a conformal mapping

$$Z : \overline{B} \to \overline{S}_r^+ \quad \text{with } Z(\partial B) = \partial S_r^+, \tag{13}$$

which is necessarily a cmc-surface of mean curvature $1/r$. Then the auxiliary function $\varphi := Z^3$ satisfies

$$\varphi > 0 \quad \text{in } B \quad \text{and} \quad \varphi = 0 \quad \text{on } \partial B. \tag{14}$$

From the system

$$\Delta Z = \frac{2}{r} Z_u \wedge Z_v = -\frac{2}{r^2}|Z_u|^2 Z \quad \text{in } B,$$

which is satisfied by the $\frac{1}{r}$-surface Z,we obtain the equation

$$-\Delta\varphi = \frac{2}{r^2}\varphi \quad \text{on } B \tag{15}$$

where Δ is the Laplace–Beltrami operator Δ_Z on S_r^+. From (14) and (15) we infer that $\lambda_1(S_r^+) = \frac{2}{r^2}$, as stated in (12).

(ii) Now we invoke Theorem 2 from Section 5.2. Accordingly the Bonnet surface $Y = N + HX$ associated with X is either a constant surface or else a cmc-surface of mean curvature one. Moreover we have $Y(w) \equiv \text{const}$ on $\overline{B}$ if and only if $H^2 - K(w) \equiv 0$ on $\overline{B}$. In this case it follows trivially for all $r_1 > 0$ that

$$\int_B |\nabla\phi|^2 \, du\, dv \geq \frac{2}{r_1^2}\int_B (H^2 - K)\Lambda\phi^2 \, du\, dv \quad \text{for all } \phi \in C_c^\infty(B).$$

If $H^2 - K(w) \not\equiv 0$ on $\overline{B}$ it makes sense to study the eigenvalue problem for the Laplace–Beltrami operator on the surface Y with respect to zero boundary values. Its smallest eigenvalue

$$\lambda_1(|dY|^2) = \inf\left\{2D(\phi)\colon \phi \in \mathring{H}_2^1(B) \text{ with } \int_B (H^2 - K)\Lambda\phi^2 \, du\, dv = 1\right\}$$

can be compared with that of all surfaces of equal area, whose Gaussian curvature is bounded from above by a constant greater than or equal to one. The smallest eigenvalue is assumed on the spherical cap $S_{r_1}^+$ of radius $r_1 > 0$ with the area $\int_B (H^2 - K)\Lambda \, du\, dv = 2\pi r_1^2$. This yields the estimate

$$\int_B |\nabla\phi|^2 \, du\, dv \geq \frac{2}{r_1^2}\int_B (H^2 - K)\Lambda\phi^2 \, du\, dv \quad \text{for all } \phi \in C_c^\infty(B). \tag{16}$$

Consequently X is strictly stable if $H = 0$.

(iii) In case that $H \neq 0$ we additionally consider the cmc-surface $\tilde{Y} := HX$ with the area $\int_B H^2\Lambda \, du\, dv = 2\pi r_2^2$. By the same arguments as in (ii) we find that

$$\int_B |\nabla\phi|^2 \, du\, dv \geq \frac{2}{r_2^2}\int_B H^2\Lambda\phi^2 \, du\, dv \quad \text{for all } \phi \in C_c^\infty(B) \tag{17}$$

is valid.

(iv) From (11) we infer

$$2\pi > \int_B (2H^2 - K)\Lambda \, du\, dv = 2\pi(r_1^2 + r_2^2)$$

whence $0 < r_1^2 < r_2^2 < 1$. Addition of (16) and (17) yields

$$(r_1^2 + r_2^2)\int_B |\nabla\phi|^2 \, du\, dv \geq 2\int_B (2H^2 - K)\Lambda\phi^2 \, du\, dv \quad \text{for all } \phi \in C_c^\infty(B).$$

Thus the H-surface X is μ-stable with the value

$$\mu := \frac{2}{r_1^2 + r_2^2} > 2. \qquad \square$$

Remark 3. The reasoning used in part (ii) of the preceding proof depends on isoperimetric inequalities and symmetrization techniques in the class of surfaces with bounded Gaussian curvature from above. For the methods that cope with branch points in these surfaces we refer to the paper by Barbosa and do Carmo [4], especially Proposition (3.13) in Section 3. Here the authors prove the following result: Let p be a nonnegative C^2-function on B vanishing only at isolated points, and denote by λ_1 the first eigenvalue of the problem

$$\Delta f + \lambda p f = 0 \quad \text{in } B, \quad f \in \mathring{H}_2^1(B).$$

Furthermore, suppose that the Gaussian curvature $\hat{K}$ of the manifold $(B, d\sigma^2)$ with the singular metric $d\sigma^2 = p\, ds^2$, $ds^2 = du^2 + dv^2$ the standard metric on B, satisfies $\hat{K} \leq K_0$ for some constant $K_0 \in [0, \infty)$. Then we have the inequality $\lambda_1 \geq \tilde{\lambda}_1(B_0)$ where B_0 denotes a geodesic disk in the 2-dimensional space of constant Gaussian curvature K_0, and $\tilde{\lambda}_1(B_0)$ is the smallest eigenvalue of the Laplace–Beltrami operator on B_0 corresponding to this metric.

Remark 4. For minimal surfaces it is advantageous to operate with the normal N whose image might yield a multiple covering on the sphere. In this case the original condition of Barbosa and do Carmo [1], namely that the spherical image $N(B)$ be contained in a spherical domain of area less than 2π, is considerably weaker than the inequality (11).

Remark 5. With the aid of H. Hopf's quadratic differential, H. Ruchert [1] established the above result alternatively without using the Bonnet surface.

The following area estimate constitutes the first step to prove a curvature estimate and subsequent Bernstein results for stable minimal surfaces. The estimate to be presented here even pertains to nonstable H-surfaces. Applied to geodesic disks of radius r on complete minimal submanifolds we see that their areas grow at most quadratically in r as $r \to \infty$.

Theorem 3 (R. Gulliver [15]). *Let $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ be an immersed, μ-stable cmc-surface with $\mu > \frac{1}{2}$, and suppose that $X(\overline{B}) = K_r(x_0)$, where $K_r(x_0)$ denotes a geodesic disk of radius r and center $x_0 := X(0)$ as described in (19)–(21) below. Then we have the estimate*

$$A(X) \leq \frac{2\mu}{2\mu - 1} \pi r^2 \tag{18}$$

for the area of X.

Proof. (i) We represent the geodesic disk $K_r(x_0) = X(\overline{B})$ with respect to geodesic polar coordinates ρ, φ by the mapping

$$Z : [0, r] \times [0, 2\pi] \to \mathbb{R}^3, \quad Z(0,0) = x_0, \\ X(\overline{B}) = \{Z(\rho, \varphi) \colon 0 \le \rho \le r,\ 0 \le \varphi \le 2\pi\}, \tag{19}$$

with the first fundamental form

$$ds^2 = |dZ|^2 = d\rho^2 + P(\rho, \varphi)\, d\varphi^2 \tag{20}$$

(i.e. $|Z_\rho|^2 = 1$, $\langle Z_\rho, Z_\varphi \rangle = 0$, $|Z_\varphi|^2 = P$). Here the function $P(\rho, \varphi) > 0$ in $(0, r] \times [0, 2\pi)$ satisfies the asymptotic conditions

$$\lim_{\rho \to +0} P(\rho, \varphi) = 0 \quad \text{and} \quad \lim_{\rho \to +0} \frac{\partial}{\partial \rho} \sqrt{P(\rho, \varphi)} = 1 \quad \text{for } 0 \le \varphi \le 2\pi. \tag{21}$$

According to Minding's formula for the geodesic curvature $\kappa_g(\rho, \varphi)$ of the curve $\Gamma_\rho := \{Z(\rho, \varphi) \colon 0 \le \varphi < 2\pi\}$ we obtain

$$\frac{\partial}{\partial \rho} \sqrt{P(\rho, \varphi)} = \kappa_g(\rho, \varphi) \sqrt{P(\rho, \varphi)} \quad \text{for } 0 < \rho \le r,\ 0 \le \varphi < 2\pi, \tag{22}$$

cf. Section 1.3, and W. Blaschke [1], §83, formula (127).

(ii) We introduce the length of Γ_ρ by

$$L(\rho) := \int_0^{2\pi} \sqrt{P(\rho, \varphi)}\, d\varphi, \quad 0 < \rho \le r. \tag{23}$$

Differentiating $L(\rho)$ with the aid of (22) and applying the Gauss–Bonnet theorem, we obtain

$$\begin{aligned} L'(\rho) &= \int_0^{2\pi} \kappa_g(\rho, \varphi) \sqrt{P(\rho, \varphi)}\, d\varphi \\ &= 2\pi - \int_0^{\rho} \int_0^{2\pi} K(\tau, \varphi) \sqrt{P(\tau, \varphi)}\, d\tau\, d\varphi \end{aligned} \tag{24}$$

and consequently

$$L''(\rho) = - \int_0^{2\pi} K(\rho, \varphi) \sqrt{P(\rho, \varphi)}\, d\varphi. \tag{25}$$

(iii) In order to apply the stability condition (2) we choose the test function $\varphi(w)$ as

$$\varphi(w) = \eta(\rho) := 1 - \rho/r \quad \text{for } 0 \le \rho \le r \text{ if } w \leftrightarrow (\rho, \varphi).$$

By (23) we have

$$\int_0^r |\eta'(\rho)|^2 L(\rho)\, d\rho = \int_0^r \int_0^{2\pi} |\eta'(\rho)|^2 \sqrt{P(\rho,\varphi)}\, d\rho\, d\varphi =: J.$$

Now we use the invariant first Beltrami operator

$$\|\nabla\phi\|^2 := (\mathcal{E}\mathcal{G} - \mathcal{F}^2)^{-1}(\mathcal{G}\phi_u^2 - 2\mathcal{F}\phi_u\phi_v + \mathcal{E}\phi_v^2)$$

for the metric $ds^2 = \mathcal{E}\, du^2 + 2\mathcal{F}\, du\, dv + \mathcal{G}\, dv^2$. Especially for the geodesic metric $ds^2 = d\sigma^2 + P(\rho,\varphi)\, d\varphi^2$ the stability inequality (2) yields

$$\begin{aligned} J = \int_0^r \int_0^{2\pi} \frac{P\eta_\rho^2 + 1\cdot\eta_\varphi^2}{P}\sqrt{P}\, d\rho\, d\varphi &\geq \mu \int_0^r \int_0^{2\pi} (2H^2 - K)\eta^2 \sqrt{P}\, d\varphi\, d\rho \\ &\geq -\mu \int_0^r \eta^2 \left(\int_0^{2\pi} K\sqrt{P}\, d\varphi \right) d\rho. \end{aligned}$$

Taking (25) into account, we arrive at

$$\begin{aligned} J &\geq \mu \int_0^r L''(\rho)\eta^2(\rho)\, d\rho \\ &= \mu[L'(\rho)\eta^2(\rho)]_{+0}^r - 2\mu \int_0^r L'(\rho)\eta(\rho)\eta'(\rho)\, d\rho \end{aligned}$$

after an integration by parts. Since $\eta(0) = 1$, $\eta(r) = 0$, and $L'(+0) = 2\pi$, it follows that

$$J \geq -2\pi\mu - 2\mu \int_0^r L'\eta\eta'\, d\rho$$

and

$$\int_0^r L'\eta\eta'\, d\rho = [L\eta\eta']_{+0}^r - \int_0^r [L(\eta')^2 + L\eta\eta'']\, d\rho = -\int_0^r L(\eta')^2\, d\rho$$

since $\eta'' = 0$, $L(+0) = 0$, and $\eta(r) = 0$. Thus we obtain

$$\int_0^r L(\eta')^2\, d\rho \geq -2\pi\mu + 2\mu \int_0^r L(\eta')^2\, d\rho$$

whence, by $\eta'(\rho) = -\frac{1}{r}$ it follows that

$$\frac{1}{r^2}\int_0^r L(\rho)\, d\rho \leq \frac{2\pi\mu}{2\mu - 1}$$

and finally

$$A(X) = A(Z) = \int_0^r \int_0^{2\pi} \sqrt{P(\rho,\varphi)}\, d\rho\, d\varphi \leq \frac{2\pi\mu}{2\mu - 1} r^2. \qquad \square$$

5.5 Curvature Estimates for Stable and Immersed cmc-Surfaces

The basic result of this section is the following

Theorem 1 (F. Sauvigny [7,8]). *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be a stable, immersed cmc-surface with $X(0) = 0$ whose mean curvature H is bounded by a constant $h_0 \geq 0$, i.e. $|H| \leq h_0$. Suppose also that X represents a geodesic disk $K_1(0)$ of radius 1 about $X(0) = 0$ such that $X(\overline{B}) = K_1(0)$. Furthermore let κ_1 and κ_2 be the principal curvatures of X. Then there is a universal constant $c(h_0)$ depending only on the parameter value h_0 such that*

$$\kappa_1^2(0) + \kappa_2^2(0) \leq c(h_0). \tag{1}$$

Proof. (i) Since X is 2-stable, Gulliver's estimate yields

$$\int_B |\nabla X|^2 \, du \, dv = 2A(X) \leq \frac{8\pi}{3} \tag{2}$$

(cf. Section 5.4, Theorem 3). In order to effectively use the Courant–Lebesgue lemma, we fix the number

$$\nu_0 := \tfrac{1}{3} \exp\left(-\tfrac{32}{3}\pi^2\right) \in \left(0, \tfrac{1}{3}\right) \tag{3}$$

and claim the following

Preliminary Statement. *There exists a point $w_* = \rho_0 e^{i\varphi_0} \in B$ with $|w_*| \leq 1 - 3\nu_0$ such that the radial derivative X_ρ of X satisfies*

$$|X_\rho(w_*)| \geq \lambda_0 := \tfrac{1}{2} \cdot (1 - 3\nu_0)^{-1} > 0. \tag{4}$$

To verify this claim, we introduce the set $\Gamma(B)$ of continuous and piecewise regular curves $\gamma : [0,1] \to \overline{B}$ with $\gamma(0) = 0$ and $\gamma(1) \in \partial B$. From the properties of the geodesic disk $K_1(0) = X(\overline{B})$ we infer

$$\inf_{\gamma \in \Gamma(B)} \int_0^1 \left| \frac{d}{dt} X(\gamma(t)) \right| dt = 1. \tag{5}$$

We fix a point $w_1 \in \partial B$, and set $\delta := 3\nu_0$. By (2) and the Courant–Lebesgue lemma there is a number $\delta^* \in (\delta, \sqrt{\delta})$ such that for

$$C_{\delta^*}(w) := \{w \in B \colon |w - w_1| = \delta^*\}$$

we can estimate

$$\int_{C_{\delta^*}(w_1)} |dX| \leq 2\left\{\pi \cdot \left(\frac{8}{3}\pi\right) \frac{1}{\log \frac{1}{\delta}}\right\}^{\frac{1}{2}} = 2\left\{\frac{8}{3}\pi^2 \cdot \frac{1}{\frac{32}{3}\pi^2}\right\}^{\frac{1}{2}} = 1. \tag{6}$$

Denote by $\gamma_1 : [0, 1-\delta^*] \to B$ the path

$$\gamma_1(t) := tw_1, \quad 0 \le t \le 1-\delta^*,$$

from the origin to the point $w_2 := (1-\delta^*)w_1$ on the circle $\partial B_{1-\delta^*}(0)$. For $\epsilon = \pm 1$ we additionally consider the paths

$$\gamma_2(t) := w_1 + (w_2 - w_1)e^{i\epsilon t}, \quad 0 \le t \le t_2(\delta^*),$$

leading within B on the circle $C_{\delta^*}(w_1)$ from w_2 to the boundary ∂B. On account of (6) we conclude that either for $\epsilon = 1$ or for $\epsilon = -1$ the inequality

$$\int_0^{t_2(\delta^*)} \left| \frac{d}{dt} X(\gamma_2(t)) \right| dt \le \frac{1}{2} \tag{7}$$

holds true. We combine γ_1 and γ_2 to a path $\gamma \in \Gamma(B)$. By means of (5) and (7) it follows that

$$\begin{aligned} 1 \le \int_0^1 |d(X \circ \gamma)| &= \int_0^{1-\delta^*} \left| \frac{d}{dt} X(\gamma_1) \right| dt + \int_0^{t_2(\delta^*)} \left| \frac{d}{dt} X(\gamma_2) \right| dt \\ &\le \int_0^{1-\delta^*} \left| \frac{d}{dt} X(\gamma_1) \right| dt + \frac{1}{2}. \end{aligned}$$

Hence there is a value $t^* \in [0, 1-\delta^*]$ such that

$$\left| \frac{d}{dt} X(\gamma_1(t^*)) \right| \ge \frac{1}{2(1-\delta)}.$$

This proves the desired "preliminary statement".

(ii) Now we choose a test function $\varphi \in C_c^\infty(B)$ with $\varphi(w) \equiv 1$ for $|w| \le 1-\nu_0$ and $|\nabla\varphi| \le 2/\nu_0$ in B, which will be inserted into the stability condition. By formula (14) of 5.1 we also have

$$\tfrac{1}{2} |\nabla N|^2 = (2H^2 - K)\Lambda,$$

and so

$$\begin{aligned} \int_{|w| \le 1-\nu_0} |\nabla N|^2 \, du \, dv &= 2 \int_{|w| \le 1-\nu_0} (2H^2 - K)\Lambda \, du \, dv \\ &\le 2 \int_B (2H^2 - K)\Lambda\varphi^2 \, du \, dv \\ &\le \int_B |\nabla\varphi|^2 \, du \, dv \le 4\pi\nu_0^{-2}. \end{aligned}$$

Hence we have found the *universal bound*

$$\int_{B_{1-\nu_0}(0)} |\nabla N|^2 \, du \, dv \le 4\pi\nu_0^{-2} \tag{8}$$

for the energy of the unit normal N of X on the disk $B_{1-\nu_0}(0)$ of radius $1-\nu_0$ about the origin.

With the aid of the Courant–Lebesgue lemma we then find a universal constant δ_1 with $0 < \delta_1 < \sqrt{\delta_1} \le 2\nu_0$, such that to each point $w_0 \in B_{1-3\nu_0}(0)$ there exists a radius $\delta^* = \delta^*(w_0, X) \in (\delta_1, \sqrt{\delta_1})$ satisfying

$$(9) \qquad \int_{C_{\delta^*}(w_0)} |dN| \le \pi \quad \text{for } C_{\delta^*}(w_0) := \{w \in B\colon |w - w_0| = \delta^*\}.$$

From this we infer the following result: There is a universal constant $\tau > 0$ with the property that for any $w_0 \in B_{1-3\nu_0}(0)$ there exists a "pole vector" $e_0 = e_0(w_0) \in S^2$ such that

$$\langle N(w), e\rangle > 0 \quad \text{for all } w \in C_{\delta^*}(w_0) \text{ and all } e \in S^2 \text{ with } |e - e_0| \le \tau.$$

Then one derives from Theorem 2 of Section 5.3 the basic

Auxiliary Statement. *There is a universal constant τ with the property that for any $w_0 \in B_{1-3\nu_0}(0)$ there is a "pole vector" $e_0 \in S^2$ such that*

$$(10) \quad \langle N(w), e\rangle > 0 \quad \textit{for all } w \in B_{\delta_1}(w_0) \textit{ and all } e \in S^2 \textit{ with } |e - e_0| \le \tau.$$

(iii) The auxiliary statement means geometrically that $B_{\delta_1}(w_0)$ is mapped by N into a geodesic disk on S^2, i.e. into a spherical cap, with a universal geodesic radius smaller than $\pi/2$ (= geodesic radius of a hemisphere), and that the center of this disk depends on the point $w_0 \in B_{1-3\nu_0}(0)$. Therefore the set $N(B_{\delta_1}(w_0))$ is contained in a closed 3-dimensional ball of a fixed radius $M \in (0, 1)$. Especially at the origin we find a vector $N_0 \in \mathbb{R}^3$, such that

$$(11) \qquad |N(w) - N_0| \le M \quad \text{for all } w \in B_{\delta_1}(0)$$

holds true with a universal constant $M \in (0, 1)$.

Furthermore the formulae (16) and (17) of Section 5.1 imply that

$$(12) \qquad \Delta N = -N|\nabla N|^2 \quad \text{in } B.$$

From the gradient estimate of E. Heinz we infer that there is an a priori constant $c_1 > 0$ such that

$$(13) \qquad |\nabla N(0)| \le c_1$$

holds true (cf. Vol. 2, Section 2.2, Proposition 1, or F. Sauvigny [16], Chapter XII, §2, Theorem 1).

(iv) For an arbitrary point $w_0 \in B_{1-3\nu_0}(0)$ we can achieve that

$$(14) \qquad X(w_0) = 0 \quad \text{and} \quad e_0 = e_3 := (0, 0, 1)$$

applying a suitable translation and rotation in $\mathbb{R}^3$. Consider the planar mapping $f : B_{\delta_1}(w_0) \to \mathbb{R}^2$ defined by

$$f(w) := (X^1(w), X^2(w)), \quad w \in B_{\delta_1}(w_0). \tag{15}$$

By the "auxiliary statement" its Jacobian J_f satisfies

$$J_f := \frac{\partial(X^1, X^2)}{\partial(u,v)} > 0 \quad \text{in } B_{\delta_1}(w_0), \tag{16}$$

and (2) implies

$$\int_{B_{\delta_1}(w_0)} |\nabla f|^2 \, du \, dv \le \frac{8\pi}{3}. \tag{17}$$

From $X_w \cdot X_w = 0$ it follows that $|\nabla X^3|^2 \le |\nabla f|^2$ whence

$$\tfrac{1}{2}|\nabla X|^2 \le |\nabla f|^2 \le |\nabla X|^2. \tag{18}$$

Thus any bound on $|\nabla f|$ is equivalent to a bound on $|\nabla X|$. Furthermore

$$|\Delta f| \le |\Delta X| = 2|H| \cdot |X_u \wedge X_v| \le h_0 |\nabla X|^2,$$

and so we infer from (18) that

$$|\Delta f| \le 2h_0 |\nabla f|^2 \quad \text{in } B_{\delta_1}(w_0). \tag{19}$$

With the aid of the Courant–Lebesgue lemma we obtain a further universal constant δ_2 with $0 < \delta_2 < \sqrt{\delta_2} \le \delta_1$ and an "individual" constant $\delta^{**} = \delta^{**}(w_0, X) \in (\delta_2, \sqrt{\delta_2})$ satisfying

$$4h_0 \int_{C_{\delta^{**}}(w_0)} |df| \le 1 \quad \text{for } C_{\delta^{**}}(w_0) := \{w \in B \colon |w - w_0| = \delta^{**}\}. \tag{20}$$

Therefore, $f(C_{\delta^{**}}(w_0))$ is contained in a closed plane disk of radius $(8h_0)^{-1}$. Since f has a positive Jacobian J_f in $B_{\delta_1}(w_0)$ and $\overline{B}_{\delta^{**}}(w_0) \subset B_{\delta_1}(w_0)$, the mapping f is not allowed to protrude from this disk. Taking $f(w_0) = 0$ into account, we arrive at the inequality

$$|f(w)| \le \frac{1}{4h_0} \quad \text{for all } w \in B_{\delta_2}(w_0). \tag{21}$$

(v) We set $\nu := \frac{1}{2}\delta_2$; then $\nu \in (0, \nu_0)$. Recalling that $f : B_{2\nu}(w_0) \to \mathbb{R}^2$ provides an open mapping of $\overline{B}_{2\nu}(w_0)$ onto its image which satisfies

$$|\Delta f| \le 2h_0 |\nabla f|^2 \quad \text{and} \quad |f| \le \frac{1}{4h_0} \quad \text{on } \overline{B}_{2\nu}(w_0),$$

we are now in the position to apply an inequality of E. Heinz that is based on the theory of pseudoholomorphic functions (see Sauvigny [16], Chapter XII, §5, Theorem 2). Thus we obtain a priori constants $c'(h_0)$ and $c''(h_0)$ with $0 < c' \le c''$ such that

(22) $\quad c'(h_0)|\nabla f(w_0)|^5 \le |\nabla f(w)| \le c''(h_0)|\nabla f(w_0)|^{\frac{1}{5}} \quad \text{for all } w \in \overline{B}_\nu(w_0).$

Furthermore, by virtue of (18), the surface element $\Lambda = \frac{1}{2}|\nabla X|^2$ of X satisfies

$$\tfrac{1}{2}|\nabla f|^2 \le \Lambda \le |\nabla f|^2 \quad \text{in } B_{2\nu}(w_0),$$

and so (22) yields the following

Intermediate Statement. *There exists a universal constant $\Theta = \Theta(h_0) \in (0,1)$ such that the surface element Λ satisfies the distortion estimate*

(23) $$\Theta(h_0)\Lambda^5(w) \le \Lambda(w_0) \quad \text{for all } w \in B_\nu(w_0)$$

holds true for any $w_0 \in B_{1-3\nu_0}(0)$.

(vi) In order to estimate $\Lambda(0)$ from below, we apply the "preliminary statement" and pick a point $w_* \in B_{1-3\nu_0}(0)$ satisfying

(24) $$\Lambda(w_*) \ge \lambda_0^2 > 0,$$

cf. (4). Then we choose $n \in \mathbb{N}$ in such a way that

$$1 - 3\nu \le n\nu < 1 - 2\nu$$

is fulfilled and introduce the points

$$w_j := \frac{j}{n} w_* \quad \text{for } j = 0, 1, \dots, n.$$

Then we have

$$|w_j| = \frac{j}{n}|w_*| \le |w_*| \le 1 - 3\nu_0 \quad \text{for } j = 0, 1, \dots, n$$

and

$$|w_{j+1} - w_j| = \frac{1}{n}|w_*| \le \frac{1 - 3\nu_0}{n} \le \frac{1 - 3\nu}{n} \le \nu \quad \text{for } j = 0, 1, \dots, n-1.$$

Applying repeatedly (23) and (24) we obtain

$$\begin{aligned} \Lambda(0) &= \Lambda(w_0) \ge \Theta\Lambda^5(w_1) \ge \Theta^{1+5}\Lambda^{5^2}(w_2) \ge \Theta^{1+5+5^2}\Lambda^{5^3}(w_3) \\ &\ge \cdots \ge \Theta^{1+5+5^2+\cdots+5^{n-1}}\Lambda^{5^n}(w_n) \ge \Theta^{5^n}\lambda_0^{2\cdot 5^n} =: c_2(h_0) \end{aligned}$$

that is,

(25) $$\Lambda(0) \ge c_2(h_0).$$

(vii) We have

$$\kappa_1^2 + \kappa_2^2 = 4H^2 - 2K = \frac{|\nabla N|^2}{\Lambda}$$

on account of formula (14) in Section 5.1. Setting

$$c(h_0) := c_1^2(h_0)c_2^{-1}(h_0),$$

the estimates (13) and (25) yield the desired inequality

$$\kappa_1^2(0) + \kappa_2^2(0) \le c(h_0). \qquad \square$$

By a scaling argument we immediately obtain the interesting

Theorem 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ denote a stable, immersed minimal surface representing a geodesic disk $K_r(x_0)$ of radius $r > 0$ centered at $x_0 := X(0)$, briefly: $X(\overline{B}) = K_r(x_0)$. Then the principal curvatures κ_1 and κ_2 of X satisfy*

$$\kappa_1^2(0) + \kappa_2^2(0) \le \frac{c(0)}{r^2} \tag{26}$$

where $c(0)$ is the universal constant $c(h_0)$ of Theorem 1 for $h_0 = 0$.

Proof. Consider the scaled minimal surface

$$Y := \frac{1}{r}(X - x_0), \quad r > 0.$$

The normals of X and Y coincide whereas the Weingarten mapping $\tilde{S}$ of Y differs from the Weingarten mapping S of X by the factor r. Hence the principal curvatures of Y are $r\kappa_1$ and $r\kappa_2$ if κ_1, κ_2 are the principal curvatures of X, and $Y(\overline{B}) = K_1(0)$. Then formula (1) of Theorem 1 yields

$$r^2(\kappa_1^2(0) + \kappa_2^2(0)) \le c(0),$$

which is the desired estimate (26). $\square$

As a corollary of Theorem 2 we obtain the following "Bernstein-type" result proved by do Carmo and Peng [1] and Fischer-Colbrie and Schoen [1].

Theorem 3. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^3$ represent a regular and embedded minimal surface which is geodesically complete and stable (that is, stable on each geodesic disk). Then Y represents a plane in $\mathbb{R}^3$.*

Proof. The set $\mathcal{M} := Y(\mathbb{R}^2)$ is a complete Riemannian manifold of dimension two, the Gauss curvature of which is nonpositive. A theorem by Hadamard implies that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2$. Thus, for each $r > 0$ and for any point $x_0 \in \mathcal{M}$, there is a geodesic disk $K_r(x_0)$ on $\mathcal{M}$ about the center x_0. If Y is not already conformal, then we introduce conformal parameters on $K_r(x_0)$, obtaining a harmonic mapping X from $\overline{B}$ onto $K_r(x_0)$ such that $X(0) = x_0$. By Theorem 2, the principal curvatures κ_1 and κ_2 of $\mathcal{M}$ at x_0, i.e. the principal curvatures of X at $w = 0$, are zero, if we let r tend to ∞. Since x_0 can be chosen arbitrarily on $\mathcal{M}$, it follows that Y represents a plane in $\mathbb{R}^3$. $\square$

5.6 Nitsche's Uniqueness Theorem and Field-Immersions

In this section we prove a uniqueness theorem, due to J.C.C. Nitsche [26], for minimal surfaces solving Plateau's problem. This result was already stated in Section 4.9.

Proposition 1. *Let* $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ *denote an immersed minimal surface with the normal* N*. For any function* $\zeta \in C^{3,\alpha}(\overline{B})$ *we consider the* varied surface $Y : \overline{B} \to \mathbb{R}^3$ *defined by*

$$Y := X + \zeta N. \tag{1}$$

Then Y *represents an immersed, but not necessarily conformal, surface of zero mean curvature if and only if* ζ *satisfies the* perturbation equation

$$L\zeta = \Phi(\zeta) \quad \text{in } B \tag{2}$$

where L *denotes the* Schwarzian operator L *associated with the minimal surface* X*, which is defined by*

$$L\zeta := -\Delta\zeta + 2\Lambda K\zeta. \tag{3}$$

Here Λ *is the area element of* X*, and* K *is its Gaussian curvature. The right-hand side* Φ *in* (2) *is a sum of homogeneous terms of second till fifth order in* $\zeta, \nabla\zeta$ *and* $\nabla^2\zeta$*, the coefficient-functions of which depend on* $X, \nabla X, \nabla^2 X, \nabla^3 X$ *and on* $1/\Lambda$*. Furthermore, there is a constant* $c_1 > 0$ *depending only on* $\|X\|_{C^{3,\alpha}(\overline{B},\mathbb{R}^3)}$ *and* $\|1/\Lambda\|_{C^0(\overline{B})}$ *such that* Φ *satisfies*

$$\|\Phi(\zeta) - \Phi(\eta)\|_{C^{0,\alpha}(\overline{B})} \le c_1 [\|\zeta\|_{C^{2,\alpha}(\overline{B})} + \|\eta\|_{C^{2,\alpha}(\overline{B})}] \|\zeta - \eta\|_{C^{2,\alpha}(\overline{B})} \tag{4}$$

for all $\zeta, \eta \in C^{2,\alpha}(\overline{B})$ *whose* $C^{2,\alpha}(\overline{B})$*-norms are bounded by one.*

Proof. Differentiating (1) we obtain

$$Y_u = X_u + \zeta_u N + \zeta N_u, \quad Y_v = X_v + \zeta_v N + \zeta N_v \tag{5}$$

and

$$\begin{aligned} Y_{uu} &= X_{uu} + \zeta_{uu} N + 2\zeta_u N_u + \zeta N_{uu}, \\ Y_{uv} &= X_{uv} + \zeta_{uv} N + (\zeta_u N_v + \zeta_v N_u) + \zeta N_{uv}, \\ Y_{vv} &= X_{vv} + \zeta_{vv} N + 2\zeta_v N_v + \zeta N_{vv}. \end{aligned} \tag{6}$$

We write the first fundamental form of Y as

$$\langle dY, dY \rangle = \mathcal{E}^* \, du^2 + 2\mathcal{F}^* \, du \, dv + \mathcal{G}^* dv^2 \tag{7}$$

and, using (5), evaluate $\mathcal{E}^*, \mathcal{F}^*, \mathcal{G}^*$. Recall that the coefficients of the second fundamental form of X are given by

$$\begin{aligned}\mathcal{L} &= -\langle X_u, N_u\rangle = \langle X_{uu}, N\rangle, \quad \mathcal{N} = -\langle X_v, N_v\rangle = \langle X_{vv}, N\rangle,\\ \mathcal{M} &= -\langle X_u, N_v\rangle = -\langle X_v, N_u\rangle = \langle X_{uv}, N\rangle.\end{aligned}$$

From (5) we infer

$$\frac{\mathcal{E}^*}{\Lambda} = 1 - 2\zeta\frac{\mathcal{L}}{\Lambda} + \cdots, \quad \frac{\mathcal{F}^*}{\Lambda} = -2\zeta\frac{\mathcal{M}}{\Lambda} + \cdots, \quad \frac{\mathcal{G}^*}{\Lambda} = 1 - 2\zeta\frac{\mathcal{N}}{\Lambda} + \cdots \tag{8}$$

where $+\cdots$ stands for terms which are quadratic in $\zeta, \zeta_u, \ldots, \zeta_{vv}$.

The surface Y has zero mean curvature if and only if

$$0 = \left\langle \frac{\mathcal{E}^*}{\Lambda}Y_{vv} - 2\frac{\mathcal{F}^*}{\Lambda}Y_{uv} + \frac{\mathcal{G}^*}{\Lambda}Y_{uu},\ \frac{1}{\Lambda}(Y_u \wedge Y_v)\right\rangle. \tag{9}$$

From (5), (6), and (8) we obtain the differential equation (2) with the Schwarzian operator L and a right-hand side Φ that has the properties described. Let us sketch the necessary computations: We write (9) as

$$\text{Linear expression in } \zeta, \zeta_u, \ldots, \zeta_{vv} + \Phi(\zeta) = 0$$

where $\Phi(\zeta)$ consists of all nonlinear ζ-terms. Now $\Phi(\zeta)$ is a polynomial of fifth degree in $\zeta, \zeta_u, \ldots, \zeta_{vv}$ with coefficients depending on $1/\Lambda, X, \nabla X, \nabla^2 X, N, \nabla N, \nabla^2 N$. Obviously we can estimate these coefficients in the $C^{0,\alpha}$-norm using a bound for $\|X\|_{C^{3,\alpha}}$ and $\|1/\Lambda\|_{C^{0,\alpha}}$ on $\overline{B}$. The terms of $\Phi(\zeta)$ are at least quadratic in ζ and its derivatives up to second order.

Furthermore,

$$\frac{1}{\Lambda}Y_u \wedge Y_v = N + (\text{terms in } \zeta, \zeta_u, \ldots, \zeta_{vv} \text{ of at least first order}), \tag{10}$$

and (6) and (8) imply

$$\begin{aligned}&\frac{\mathcal{E}^*}{\Lambda}Y_{vv} - 2\frac{\mathcal{F}^*}{\Lambda}Y_{uv} + \frac{\mathcal{G}^*}{\Lambda}Y_{uu}\\ &\quad = \Delta X + \Delta\zeta \cdot N + 2\zeta_u N_u + 2\zeta_v N_v + \zeta\Delta N\\ &\qquad - 2\frac{\zeta}{\Lambda}(\mathcal{L}X_{vv} - 2\mathcal{M}X_{uv} + \mathcal{N}X_{uu}) + \cdots\\ &\quad = (\Delta\zeta + 2\Lambda K\zeta)N + 2(\zeta_u N_u + \zeta_v N_v)\\ &\qquad - 2\frac{\zeta}{\Lambda}(\mathcal{L}X_{vv} - 2\mathcal{M}X_{uv} + \mathcal{N}X_{uu}) + \cdots\end{aligned} \tag{11}$$

where $+\cdots$ stands again for terms that are quadratic in $\zeta, \ldots, \zeta_{vv}$.

Here we have used the equation $\Delta N = 2\Lambda K N$, cf. (11) and (12) of 5.1. From (9), (10) and (11) it follows that

$$0 = \Delta\zeta + 2\Lambda K\zeta - 2\frac{\zeta}{\Lambda}(\mathcal{L}\mathcal{N} - 2\mathcal{M}^2 + \mathcal{N}\mathcal{L}) + \cdots.$$

Furthermore, by formula (32) of Section 1.3 we have

$$\Lambda^2 K = \mathcal{L}\mathcal{N} - \mathcal{M}^2$$

and so we arrive at

$$0 = \Delta\zeta - 2\Lambda K\zeta + \cdots .$$

Therefore equation (9) is equivalent to

$$-\Delta\zeta + 2\Lambda K\zeta = \Phi(\zeta) \tag{12}$$

as it was claimed.

The nonlinearity $\Phi(\zeta)$ consists of finitely many terms of the form

$$a(X)\partial^{i_1}\zeta \dots \partial^{i_k}\zeta$$

with $2 \le k \le 5$ where ∂^{i_ℓ} denotes a partial derivative of order i_ℓ with $0 \le i_\ell \le 2$, and $\|a(X)\|_{C^\alpha(\overline{B})}$ can be estimated by $\|X\|_{C^{3,\alpha}(\overline{B},\mathbb{R}^3)}$ and $\|1/\Lambda\|_{C^0(\overline{B})}$. We leave it as an easy exercise to the reader to verify the "condition of contraction" (4). □

With the aid of Schauder's theory we will now show the fundamental result that a strictly stable, immersed minimal surface $X : \overline{B} \to \mathbb{R}^3$ can be embedded into a field of surfaces of zero mean curvature provided that X is extendable beyond ∂B.

Proposition 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be a strictly stable, immersed minimal surface which can be extended as a minimal surface to a larger disk $\Omega := B_{1+\delta}(0)$ with $\delta > 0$. Then there is a one-parameter family*

$$Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$$

of zero mean curvature surfaces $Z(\cdot, t)$ (not necessarily conformally parametrized) which is of class $C^{2,\alpha}(\overline{B} \times [-t_0, t_0], \mathbb{R}^3)$, $|t| \le t_0$, and has the following properties:

(a) $Z(w, 0) = X(w)$ *for* $w \in \overline{B}$;
(b) $J_Z := \frac{\partial(Z^1,Z^2,Z^3)}{\partial(u,v,t)} > 0$ *on* $\overline{B} \times [-t_0, t_0]$;
(c) *If $N^*(\cdot, t)$ denotes the normal to the surface $Z(\cdot, t)$, one has*

$$Z_t(w, t) = \rho(w, t)N^*(w, t) \quad \textit{for } w \in \overline{B} \textit{ and } |t| < t_0$$

with $\rho(w, t) > 0$ on $\overline{B} \times [-t_0, t_0]$.

Definition 1. *A mapping Z as described in Proposition 2 is called a* **field immersion** *of the minimal surface X.*

Proof of Proposition 2. (i) It is easily seen that also the extension $X : \Omega \to \mathbb{R}^3$ is strictly stable and immersed for $0 < \delta \ll 1$. The strict stability of this extension implies that the boundary value problem

$$L\zeta = 0 \quad \text{in } \Omega, \quad \zeta = 0 \quad \text{on } \partial\Omega$$

has only the trivial solution $\zeta(w) \equiv 0$ on $\overline{\Omega}$. Then Schauder's theory implies that there is a uniquely determined solution $\xi \in C^{2,\alpha}(\overline{\Omega})$ of the boundary value problem

$$L\xi = 0 \quad \text{in } \Omega, \quad \xi = 1 \quad \text{on } \partial\Omega \tag{13}$$

(see e.g. Sauvigny [16], Chapter IX, §6, Theorem 5).

By virtue of Proposition 1 in 5.3 it follows that $\xi(w) > 0$ for all $w \in \overline{\Omega}$. Set

$$C_0^{2,\alpha}(\overline{\Omega}) := \{\eta \in C^{2,\alpha}(\overline{\Omega}) \colon \eta(w) = 0 \text{ for all } w \in \partial\Omega\}$$

and note that the operator

$$L_0 := L|_{C_0^{2,\alpha}(\overline{\Omega})} : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}) \tag{14}$$

is an invertible mapping satisfying

$$\|L_0^{-1} f\|_{2,\alpha} \le c_2 \|f\|_\alpha \quad \text{for all } f \in C^{0,\alpha}(\overline{\Omega}) \tag{15}$$

with an a priori constant $c_2 > 0$. Here we have used the abbreviating notation

$$\|\cdot\|_{m,\alpha} := \|\cdot\|_{C^{m,\alpha}(\overline{\Omega})}.$$

Finally set

$$c_3 := \|\xi\|_{2,\alpha}.$$

(ii) For sufficiently small $t_1 > 0$ we now want to solve the nonlinear Dirichlet problem

$$L\zeta(\cdot, t) = \Phi(\zeta(\cdot, t)) \quad \text{in } \Omega, \quad \zeta(\cdot, t) = t \quad \text{on } \partial\Omega \tag{16}$$

by $\zeta(\cdot, t) \in C^{2,\alpha}(\overline{\Omega})$ and for parameter values with $|t| \le t_1$. To this end we make the "Ansatz"

$$\zeta(w, t) := \eta(w, t) + t\xi(w) \quad \text{for } w \in \overline{\Omega}, \quad |t| < t_1, \tag{17}$$

where $\eta(\cdot, t) \in C_0^{2,\alpha}(\overline{\Omega})$ is to be determined as solution of

$$L\eta(\cdot, t) = \Phi(\eta(\cdot, t) + t\xi) \quad \text{in } \Omega. \tag{18}$$

This is equivalent to finding a solution $\eta(\cdot, t) \in C_0^{2,\alpha}(\overline{\Omega})$ of the fixed point equation

$$\eta(\cdot, t) = L_0^{-1} \Phi^t(\eta(\cdot, t)) \quad \text{with } \Phi^t(\zeta) := \Phi(\zeta + t\xi). \tag{19}$$

In order to solve (19) by Banach's fixed point theorem we introduce the balls

$$\mathcal{B}(t) := \{\zeta \in C_0^{2,\alpha}(\overline{\Omega}) \colon \|\zeta\|_{2,\alpha} \le |t|^{3/2}\}$$

with $|t| \le t_1 \ll 1$. For $\zeta \in \mathcal{B}(t)$ and $|t| \ll 1$ we have $\|\zeta + t\xi\|_{2,\alpha} \le 1$. Then by (15) and (4) ($B$ replaced with Ω) we obtain for $\zeta \in \mathcal{B}(t)$ that

$$\begin{aligned} \|L_0^{-1}\Phi^t(\zeta)\|_{2,\alpha} &\le c_2\|\Phi^t(\zeta)\|_\alpha = c_2\|\Phi(\zeta + t\xi)\|_\alpha \\ &\le c_2c_1\|\zeta + t\xi\|^2_{2,\alpha} \le 2c_1c_2\{\|\zeta\|^2_{2,\alpha} + t^2\|\xi\|^2_{2,\alpha}\} \\ &\le 2c_1c_2\{|t|^3 + c_3^2t^2\} = 2c_1c_2\{|t|^{3/2} + c_3^2|t|^{1/2}\}|t|^{3/2}. \end{aligned}$$

For $|t| \le t_1 \ll 1$ it follows that

$$\|L_0^{-1}\Phi^t(\zeta)\|_{2,\alpha} \le |t|^{3/2},$$

and the operator $L_0^{-1}\Phi^t$ maps $\mathcal{B}(t)$ into itself.

Secondly, for $\zeta, \eta \in \mathcal{B}(t)$ with $|t| \ll 1$ we have $\|\zeta + t\xi\|_{2,\alpha} \le 1$ and also $\|\eta + t\xi\|_{2,\alpha} \le 1$, whence

$$\begin{aligned} &\|L_0^{-1}\Phi^t(\zeta) - L_0^{-1}\Phi^t(\eta)\|_{2,\alpha} = \|L_0^{-1}(\Phi^t(\zeta) - \Phi^t(\eta))\|_{2,\alpha} \\ &\le c_2\|\Phi^t(\zeta) - \Phi^t(\eta)\|_\alpha = c_2\|\Phi(\zeta + t\xi) - \Phi(\zeta + t\eta)\|_\alpha \\ &\le c_2c_1|t| \cdot \|\zeta - \eta\|_{2,\alpha} \le \tfrac{1}{2}\|\zeta - \eta\|_{2,\alpha} \quad \text{for } |t| \ll 1, \end{aligned}$$

and so

$$\begin{aligned} &\|L_0^{-1}\Phi^t(\zeta) - L_0^{-1}\Phi^t(\eta)\|_{2,\alpha} \le \tfrac{1}{2}\|\zeta - \eta\|_{2,\alpha} \\ &\quad \text{for } \zeta, \eta \in \mathcal{B}(t) \text{ and } |t| < t_1 \text{ with } 0 < t_1 \ll 1. \end{aligned}$$

Therefore the mapping $L_0^{-1}\Phi^t : \mathcal{B}(t) \to \mathcal{B}(t)$ is contracting, and so it possesses a uniquely determined fixed point $\eta(\cdot, t) \in C_0^{2,\alpha}(\overline{\Omega})$ for $0 < |t| < t_1$ with $0 < t_1 \ll 1$, and for $t = 0$ we have $L_0^{-1}\Phi^0(0) = 0$, i.e. $\eta(\cdot, 0) = 0$. A slight modification of the proof shows that $\eta(\cdot, t)$ is differentiable with respect to t and that even $\eta \in C^{2,\alpha}(\overline{\Omega} \times [-t_1, t_1])$ holds true (see e.g. Giaquinta and Hildebrandt [1], vol. 1, Chapter 6). Moreover, the choice of $\mathcal{B}(t)$ shows that

$$\eta_t(w, 0) = 0 \quad \text{for } w \in \overline{\Omega},$$

and so the superposition (17) yields a solution $\zeta \in C^2(\overline{\Omega} \times [-t_1, t_1])$ of (16), satisfying

$$\zeta_t(w, 0) = \xi(w) > 0 \quad \text{for all } w \in \overline{\Omega}.$$

For $0 < t_1 \ll 1$ we then obtain

$$\zeta_t(w, t) > 0 \quad \text{for } w \in \overline{\Omega} \text{ and } |t| \le t_1. \tag{20}$$

Hence the family $Y : \overline{\Omega} \times [-t_1, t_1] \to \mathbb{R}^3$ constitutes a field of surfaces

$$Y(\cdot, t) = X + \zeta(\cdot, t)N \tag{21}$$

of zero mean curvature surfaces in $\mathbb{R}^3$ with $Y(\cdot, 0) = X$. Finally by reparametrizing Y via their orthogonal trajectories we obtain for some $t_0 \in [0, t_1]$ a family

(22) $$Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$$

of zero mean curvature surfaces $Z(\cdot, t)$ satisfying (a)–(c). (The reparametrization is left to the reader as an exercise in ordinary differential equations.) $\square$

From Section 2.8 we already know that those minimal surfaces that can be embedded into a foliation of simply covering surfaces of zero mean curvature furnish a relative minimum of the area functional. Now we are confronted with the more intricate problem to prove a similar property for immersed minimal surfaces that can be embedded into a field of surfaces with $H = 0$ that might have selfintersections.

Let Γ be an oriented Jordan curve in $\mathbb{R}^3$, and denote by $\mathcal{C}(\Gamma)$ the class of surfaces $X \in H_2^1(B, \mathbb{R}^3)$ bounded by Γ in the sense of Section 4.2. Set

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3),$$

and define

$$\overline{\mathcal{C}}^*(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon X(w_j) = Q_j,\ j = 1, 2, 3\}$$

where Q_1, Q_2, Q_3 are three fixed points on Γ and $w_j = \exp(\frac{2\pi i}{3} j)$, $j = 1, 2, 3$.

Proposition 3. *Let $X \in \mathcal{C}(\Gamma)$ be a minimal surface that satisfies the assumptions of Proposition 2. Then there is a number $\epsilon = \epsilon(X) > 0$ such that*

$$D(X) < D(Y) \quad \textit{for all } Y \in \overline{\mathcal{C}}^*(\Gamma) \textit{ with } 0 < \sup_B |Y(w) - X(w)| < \epsilon.$$

Proof. (i) We embed X in a field immersion $Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$ as described in Proposition 2 and consider the corresponding surface element

$$\mathcal{W}(w, t) := |Z_u(w, t) \wedge Z_v(w, t)| = [N^*(w, t), Z_u(w, t), Z_v(w, t)].$$

Using the orthogonality condition and (c) we obtain

$$\begin{aligned} \mathcal{W}_t &= [N_t^*, Z_u, Z_v] + [N^*, Z_{tu}, Z_v] + [N^*, Z_u, Z_{tv}] \\ &= 0 + [N^*, (\rho N^*)_u, Z_v] + [N^*, Z_u, (\rho N^*)_v] \\ &= \rho\{[N^*, N_u^*, Z_v] + [N^*, Z_u, N_v^*]\} \\ &= \rho\langle N^*, N_u^* \wedge Z_v + Z_u \wedge N_v^*\rangle. \end{aligned}$$

Furthermore, Theorem 2 of Section 2.5 yields

$$N_u^* \wedge Z_v + Z_u \wedge N_v^* = 0.$$

Thus we have

(23) $$\mathcal{W}_t(w, t) = 0 \quad \text{for } w \in \overline{B} \text{ and } |t| \le t_0.$$

(ii) The field immersion of Proposition 2 is constructed even on a larger disk Ω_0 with $B \subset\subset \Omega_0 \subset\subset \Omega$. This implies that $Z : \overline{\Omega}_0 \times [-t_0, t_0] \to \mathbb{R}^3$ furnishes a local diffeomorphism provided that $0 < t_0 \ll 1$, but globally the inverse of Z need not exist. For compactness reasons the local inverse Z^{-1} is defined on domains of uniform size. Consequently there is an $\epsilon = \epsilon(X) > 0$ such that all admissible $Y \in \overline{\mathcal{C}}^*(\Gamma)$ with $\sup_B |Y - X| < \epsilon$ can be written as

$$Y(w) = Z(f(w), \tau(w)), \quad w \in \overline{B}, \tag{24}$$

with a continuous mapping f from $\overline{B}$ into $\mathbb{R}^2$ and a continuous function $\tau : \overline{B} \to \mathbb{R}$ such that

$$f \text{ maps } \partial B \text{ monotonically onto itself}, \quad f(w_j) = w_j, \quad j = 1, 2, 3, \tag{25}$$

and $f|_{\partial B}$ is positive-oriented with respect to B, and

$$\tau(w) = 0 \quad \text{for } w \in \partial B \quad \text{and} \quad |\tau(w)| \le t_0 \quad \text{on } \overline{B}. \tag{26}$$

On account of Dirichlet's principle, harmonic functions are unique minimizers of D for given boundary values; thus it suffices to consider Y, f, τ that are real analytic on B and of class C^0 on $\overline{B}$.

(iii) Assume for the moment that f furnishes a diffeomorphism from B onto B with the inverse g, and set $\sigma := \tau \circ g$ as well as

$$\tilde{Y}(w) := Y(g(w)) = Z(w, \sigma(w)). \tag{27}$$

Then it follows that

$$\begin{aligned} \tilde{Y}_u(w) \wedge \tilde{Y}_v(w) &= Y_u(g(w)) \wedge Y_v(g(w)) J_g(w) \\ &= [(Z_u + Z_t\sigma_u) \wedge (Z_v + Z_t\sigma_v)](w, \sigma(w)) \\ &= [Z_u \wedge Z_v + \sigma_u Z_t \wedge Z_v + \sigma_v Z_u \wedge Z_t](w, \sigma(w)). \end{aligned}$$

Multiplication by $N^*(w, \sigma(w))$ yields by virtue of (23) that

$$\begin{aligned} &\langle N^*(w, \sigma(w)), Y_u(g(w)) \wedge Y_v(g(w))\rangle J_g(w) \\ &\quad = \mathcal{W}(w, \sigma(w)) = \mathcal{W}(w, 0) = |X_u(w) \wedge X_v(w)|. \end{aligned}$$

Integration over B then leads to the *Schwarz comparison formula*:

$$\int_B \langle N^*(f(w), \tau(w)), Y_u(w) \wedge Y_v(w)\rangle \, du\, dv = \int_B |X_u \wedge X_v| \, du\, dv. \tag{28}$$

(As demonstrated in Section 2.8, this formula can be seen as a precursor of Hilbert's independent integral.) The relation (28) implies $A(Y) \ge A(X)$, and the equality sign holds if and only if $Y_u \wedge Y_v$ points in the direction of $N^*(f, \tau)$. This implies $Y = X$, and the result is proved.

(iv) In the sequel we have to verify this result even if f is not a global diffeomorphism of B onto B. We have to deal with the possibility that $f(B)$

might "overshoot" B, and that $f(B)$ could cover B in several layers. We note first that, in general, the critical values of the real analytic mapping $f : B \to B$ constitute a Lebesgue null set $\mathcal{N}$ in $\mathbb{R}^2$ (see Sauvigny [16], Chapter III, §4). Combining this observation with arguments using the winding number, we come to the following

Conclusion. There exist sequences $\{G_\ell\}$ and $\{H_\ell\}$ of subdomains of B such that

$$f_\ell := f|_{G_\ell} : G_\ell \to \mathbb{R}^2 \quad \text{is a positively oriented diffeomorphism from } G_\ell \text{ onto } H_\ell;\ \tau_\ell := \tau|_{G_\ell}; \tag{29}$$

and

$$G_\ell \cap G_k = \emptyset, \quad H_\ell \cap H_k = \emptyset \quad \text{for } \ell \neq k; \qquad B \Big\backslash \bigcup_{\ell=1}^{\infty} H_\ell \text{ is a Lebesgue null set in } \mathbb{R}^2. \tag{30}$$

One obtains the G_ℓ and H_ℓ as follows: For $z_0 \in B \setminus \mathcal{N}$ one has pre-images $w_0, w_0', w_0'', \dots$ such that $J_f(w_0) \neq 0$, $J_f(w_0') \neq 0$, $J_f(w_0'') \neq 0, \dots$. Since $f|_{\partial B}$ is positive oriented, at least one of these numbers has to be positive, say, $J_f(w_0) > 0$. Then there is a neighborhood G of w_0 such that $f|_G$ is a positively oriented diffeomorphism of G onto a neighborhood H of z_0. A repeated application of this argument leads to the selection of diffeomorphisms $f_\ell : G_\ell \to H_\ell$ with the properties (29) and (30). Note that $B \setminus \bigcup G_\ell$ might have positive measure. This means that we have omitted multiple coverings of B by $f(B)$ as well as parts of B that are mapped onto $f(B) \setminus (B)$.

When we apply the arguments of part (iii) to these individual diffeomorphisms, the Schwarz comparison formula (28) implies

$$\begin{aligned} D(Y) \geq A(Y) &= \int_B |Y_u \wedge Y_v| \, du \, dv \\ &\geq \sum_{\ell=1}^{\infty} \int_{G_\ell} |Y_u \wedge Y_v| \, du \, dv \\ &\geq \sum_{\ell=1}^{\infty} \int_{G_\ell} \langle N^*(f_\ell, \tau_\ell), Y_u \wedge Y_v \rangle \, du \, dv \\ &= \sum_{\ell=1}^{\infty} \int_{H_\ell} |X_u \wedge X_v| \, du \, dv = A(X) = D(X). \end{aligned} \tag{31}$$

(v) If $D(X) = D(Y)$ then all inequalities in (31) turn into equalities, and so we have in particular that $\mathcal{S} := B \setminus \bigcup G_\ell$ is a two-dimensional null set. Otherwise we would have $\nabla Y(w) = 0$ for $w \in \mathcal{S}$ with meas $\mathcal{S} > 0$, and therefore $\nabla Y(w) \equiv 0$ on B since Y is real analytic. This implies $Y(w) \equiv$ const on B, a contradiction to $Y \in \mathcal{C}(\Gamma)$. Thus we obtain $J_f(w) = \frac{\partial(f^1,f^2)}{\partial(u,v)} > 0$ a.e. on B. Furthermore, the vectors

$$Y_u \wedge Y_v = Z_u(f_\ell,\tau_\ell) \wedge Z_v(f_\ell,\tau_\ell)\frac{\partial(f_\ell^1,f_\ell^2)}{\partial(u,v)} + Z_v(f_\ell,\tau_\ell) \wedge Z_t(f_\ell,\tau_\ell)\frac{\partial(f_\ell^2,\tau_\ell)}{\partial(u,v)} + Z_t(f_\ell,\tau_\ell) \wedge Z_u(f_\ell,\tau_\ell)\frac{\partial(\tau_\ell,f_\ell^1)}{\partial(u,v)}$$

have to point into the direction of the normals $N^*(f_\ell,\tau_\ell)$ on G_ℓ. Thus the two determinants

$$\frac{\partial(f_\ell^2,\tau_\ell)}{\partial(u,v)} \quad \text{and} \quad \frac{\partial(\tau_\ell,f_\ell^1)}{\partial(u,v)}$$

have to vanish on G_ℓ for $\ell = 1, 2, \dots$, since $Z_v \wedge Z_t$ and $Z_t \wedge Z_u$ are two linearly independent vectors perpendicular to N^*, and so we obtain

$$\nabla\tau = 0 \quad \text{in } \bigcup_{\ell=1}^{\infty} G_\ell$$

because of $J_f = \frac{\partial(f^1,f^2)}{\partial(u,v)} > 0$ on G_ℓ. Since meas $\mathcal{S} = 0$ we obtain $\nabla\tau(w) \equiv 0$ in B whence $\tau(w) \equiv$ const in $\overline{B}$, and $\tau = 0$ on ∂B yields $\tau(w) = 0$ on $\overline{B}$. Thus we arrive at

$$Y(w) = Z(f(w),0) = X(f(w)) \quad \text{for } w \in \overline{B}. \tag{32}$$

(vi) We have found: Any $Y \in \overline{\mathcal{C}}^*(\Gamma)$ with $\max_{\overline{B}} |Y - X| < \epsilon \ll 1$ satisfies $D(Y) \ge D(X)$, and $D(Y) = D(X)$ if and only if $Y = X \circ f$. It follows that $D(Y) \le D(\tilde{Y})$ for all $\tilde{Y} \in \overline{\mathcal{C}}^*(\Gamma)$ with $\max_{\overline{B}} |\tilde{Y} - Y| < \tilde{\epsilon}$ for some $\tilde{\epsilon}$ with $0 < \tilde{\epsilon} \ll 1$. This implies that Y is conformally parametrized in the sense that $Y_w \cdot Y_w = 0$. Hence it follows from (32) that f is a mapping from $\overline{B}$ onto $\overline{B}$ which is conformal (in the generalized sense) in B, monotonic on ∂B with $f(\partial B) = \partial B$ and $f(w_j) = w_j$, $j = 1, 2, 3$. We conclude that $f(w) \equiv w$ on $\overline{B}$ and therefore $Y(w) = X(w)$ on $\overline{B}$. Thus we have proved

$$D(X) < D(Y) \quad \text{for } 0 < \sup_B |X - Y| < \epsilon. \qquad \square$$

We are now prepared to prove the following

Theorem 1 (J.C.C. Nitsche). *Let Γ be a real analytic, regular Jordan curve with a total curvature $\kappa(\Lambda)$ less or equal to 4π. Then there is exactly one disk-type minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$, i.e. exactly one solution $X \in \overline{\mathcal{C}}^*(\Gamma)$ solving Plateau's problem to the contour Γ. This solution is free of branch points up to the boundary and can be continued analytically across Γ as a minimal surface.*

Proof. (i) If Γ lies in the plane E, any minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ is contained in this plane as well, due to the convex hull property, and so it reduces to a strictly conformal or anticonformal mapping from B onto the interior of Γ in E which is uniquely determined by the three-point condition $X(w_j) = Q_j$, $j = 1, 2, 3$ (cf. Section 4.11). By the asymptotic expansion of X_w at $w_0 \in \partial B$ it turns out that there are no boundary branch points of X, because otherwise $X(\overline{B})$ would overshoot Γ into $\mathbb{R}^2 \setminus \overline{\Omega}$, where Ω is the interior domain of Γ. Thus the assertion of the theorem holds in this case even without the assumption $\kappa(\Gamma) \le 4\pi$; actually it would suffice that $\Gamma \in C^{2,\alpha}$ (or even $\Gamma \in C^{1,\alpha}$) in order to prove the uniqueness of $X \in \overline{\mathcal{C}}^*(\Gamma)$.

(ii) Thus from now on we assume that Γ *is nonplanar*. By H. Lewy's regularity theorem [5] we know that any minimal surface $X \in \overline{\mathcal{C}}(\Gamma)$ can be continued analytically across Γ onto a larger disk $\Omega := B_{1+\delta}(0)$, cf. Vol. 2, Section 2.8. Furthermore, by the Gauss–Bonnet formula established in Vol. 2, Section 2.11, we have the following: Let $w_1, \dots, w_k \in B$ and $w_{k+1}, \dots, w_{k+\ell} \in \partial B$ be the finitely many branch points of a minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ with the orders $\nu_1, \dots, \nu_k$ and $\nu_{k+1}, \dots, \nu_{k+\ell}$ respectively (see Vol. 2, Section 2.10), $\nu_j \in \mathbb{N}$. Then

(33) $$0 \le \sum_{j=1}^{k} \nu_j + \frac{1}{2} \sum_{j=k+1}^{k+\ell} \nu_j = \frac{1}{2\pi} \left\{ \int_B K\Lambda \, du \, dv + \int_\Gamma \kappa_g \, ds - 2\pi \right\}.$$

In (iii) we shall see that the integral of the geodesic curvature κ_g of Γ on X is bounded by the total curvature of Γ, i.e.

(34) $$\int_\Gamma \kappa_g \, ds \le \int_\Gamma \kappa \, ds =: \kappa(\Gamma),$$

and by assumption we have $\kappa(\Gamma) \le 4\pi$. Therefore we obtain

(35) $$\int_\Gamma \kappa_g \, ds - 2\pi \le 2\pi.$$

The Gaussian curvature K of X satisfies $K = \kappa_1 \kappa_2 = -\kappa_1^2 \le 0$ in $B' = B \setminus \{w_1, \dots, w_k\}$ since $0 = 2H = \kappa_1 + \kappa_2$. If we had $K(w) \equiv 0$ in B', it would follow that the Weingarten mapping of X were everywhere trivial in B', i.e. $N(w) \equiv \text{const}$ in B' and then in B. This would imply that X and thus Γ were planar, which is excluded by the assumption above. Therefore $K(w) \not\equiv 0$ on B', and consequently

(36) $$\int_B K\Lambda \, du \, dv < 0.$$

From (33), (35), and (36) it follows that

(37) $$0 \le \sum_{j=1}^{k} \nu_j + \frac{1}{2} \sum_{j=k+1}^{k+\ell} \nu_j < 1.$$

Furthermore the orders of the boundary branch points $w_{k+1}, \dots, w_{k+\ell}$ have to be even because of the monotonicity of the mapping $X|_{\partial B}$ (see Vol. 2, Section 2.10). Therefore (37) implies

$$\nu_j = 0 \quad \text{for } j = 1, \dots, k+\ell, \tag{38}$$

i.e. any minimal surface $X \in \mathcal{C}(\Gamma)$ with $\kappa(\Gamma) \le 4\pi$ is an immersion of $\overline{B}$ into $\mathbb{R}^3$, and (33) reduces to the classical Gauss–Bonnet theorem

$$-\int_B K\Lambda \, du\, dv = \int_\Gamma \kappa_g \, ds - 2\pi \tag{39}$$

for an immersed minimal surface X.

(iii) Let us parametrize Γ by the arc length parameter $s \in [0, L]$, $L =$ length of Γ, setting

$$Y(s) := X(\cos\varphi(s), \sin\varphi(s)), \quad 0 \le s \le L,$$

satisfying $|Y'(s)| \equiv 1$ for $0 \le s \le L$. Furthermore set

$$Z(s) := N(\cos\varphi(s), \sin\varphi(s)), \quad 0 \le s \le L.$$

Then $\kappa(s) = |Y''(s)|$ is the curvature of the arc Γ at the point $Y(s)$, and

$$\kappa(\Gamma) = \int_0^L \kappa(s)\, ds = \int_0^L |Y''(s)|\, ds$$

is the total curvature of Γ. Since the geodesic curvature satisfies

$$|\kappa_g(s)| = |[Y''(s), Z(s), Y'(s)]|$$

and the normal curvature κ_n of Y fulfills

$$|\kappa_n(s)| = |\langle Z(s), Y''(s)\rangle|,$$

we have the decomposition

$$\kappa^2(s) = \kappa_g^2(s) + \kappa_n^2(s)$$

whence indeed

$$|\kappa_g(s)| \le \kappa(s) \quad \text{for } 0 \le s \le L.$$

For $\int_0^L \kappa_g\, ds = \int_0^L \kappa\, ds$, then $\kappa_n(s) \equiv 0$ for $0 \le s \le L$. Note that with $w(s) := (\cos\varphi(s), \sin\varphi(s))$ we obtain

$$Y' = [X_u(w)(-\sin\varphi) + X_v(w)\cos\varphi]\varphi'$$

and

$$Y'' = [X_{uu}(w)(\sin^2\varphi) - 2X_{uv}(w)\sin\varphi\cos\varphi + X_{vv}(w)\cos^2\varphi]|\varphi'|^2 + \cdots$$

where $+\cdots$ stands for the neglected tangential terms.

From $\kappa_n(s) \equiv 0$ and $|\kappa_n(s)| = |\langle Z(s), Y''(s)\rangle|$ as well as $X_{uu} = -X_{vv}$ we infer that

$$
\begin{aligned}
(40) \qquad 0 &= \langle Z, X_{uu}(w)[\cos^2\varphi - \sin^2\varphi] + 2X_{uv}(w)\sin\varphi\cos\varphi\rangle \\
&= \mathcal{L}(w)[\cos^2\varphi - \sin^2\varphi] + 2\mathcal{M}\sin\varphi\cos\varphi \\
&= \operatorname{Re}\,\{[\mathcal{L}(w) - i\mathcal{M}(w)](\cos\varphi + i\sin\varphi)^2\}.
\end{aligned}
$$

In Section 1.3 we have seen via the Codazzi equations that

$$f(w) := [\mathcal{L}(w) - i\mathcal{M}(w)]w^2, \quad w \in \overline{B},$$

is holomorphic on B. Then (40) implies $\operatorname{Re} f|_{\partial B} = 0$ and therefore $\operatorname{Re} f(w) \equiv 0$ in $\overline{B}$, whence $f(w) \equiv \text{const}$ in $\overline{B}$. Since $f(0) = 0$ it follows that $f(w) \equiv 0$ in $\overline{B}$. Thus we arrive at

$$\mathcal{L}(w) \equiv 0, \quad \mathcal{M}(w) \equiv 0, \quad \mathcal{N}(w) \equiv 0 \quad \text{in } \overline{B}$$

whence $K(w) \equiv 0$ in B which contradicts (36). Thus $\kappa_n(s) \equiv 0$ is impossible, and (35) is strengthened into

$$\int_\Gamma \kappa_g\, ds < 4\pi.$$

Combining this with (39) it follows that

$$(41) \qquad -\int_B K\Lambda\, du\, dv < 2\pi.$$

Because of Theorem 2 in Section 5.4 we infer from (41) that X is strictly stable. According to Proposition 2 we can therefore embed X into a field immersion of minimal surfaces, and so Proposition 3 implies that any minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ furnishes a strict relative minimum for Dirichlet's integral D in $\overline{\mathcal{C}}^*(\Gamma)$.

Suppose now that two different minimal surfaces X_1 and X_2 existed in $\overline{\mathcal{C}}^*(\Gamma)$. Then both would furnish a strict relative minimum of D in $\overline{\mathcal{C}}^*(\Gamma)$. Then by Courant's "Mountain Pass Lemma", to be presented in the next chapter, there would exist a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which were unstable in the sense that it were not a local minimizer of D (cf. Theorem 2 in Section 6.7). The existence of such a surface X_3 is impossible as we have seen above, and so there cannot be two different minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$. However there is always one minimal surface X in $\overline{\mathcal{C}}^*(\Gamma)$, which proves the theorem. □

Remark 1. The unique solution in Nitsche's theorem actually is not only immersed, but even *embedded*, according to the following remarkable result due to T. Ekholm, B. White, and D. Wienholtz [1]:

Theorem 2. *Let Γ be a closed Jordan curve in $\mathbb{R}^n$ with total curvature $\leq 4\pi$, and let $X : \overline{B} \to \mathbb{R}^n$ be a minimal surface in $\mathcal{C}(\Gamma)$. Then X is embedded up to and including the boundary, with no interior branch points.*

In fact Theorem 2 even holds for minimal surfaces $X : M \to \mathbb{R}^n$ defined on a compact 2-manifold M with boundary ∂M which is mapped homeomorphically onto Γ.

5.7 Some Finiteness Results for Plateau's Problem

For Plateau's problem the most challenging question is: "How many minimal surfaces of the type of the disk, or of general topological type, are bounded by a preassigned 'well-behaved' closed Jordan curve Γ?" The Courant–Levy examples (cf. No. 4 of Section 4.15) show that Γ may bound infinitely many solutions even if it is regular and smooth *except for one point*. Thus a reasonable answer can only be expected if we interpret the attribute "well-behaved" in a suitably restricted way, say as *regular and real analytic*, or as *regular and of class C^k for some $k \geq 1$*, or as *piecewise linear* (i.e. Γ is a *polygon*). Moreover it is interesting to find *upper or lower bounds* for the number of solutions bounded by a well-behaved contour Γ. However, even the decision whether or not a well-behaved Γ spans only finitely many disk-type minimal surfaces is still open.

We shall prove in this section that stable, immersed surfaces of the type of the disk bounded by a real analytic, regular contour Γ are isolated; hence only finitely many of them can be bounded by such a Γ. The possibility to estimate quantitatively a suitable neighborhood, where no further solution exists, seems to be out of reach. The pioneering contribution towards isolatedness of stable solutions for Plateau's problem is due to F. Tomi [6].

We begin our discussion with the following *local uniqueness theorem* that is already contained in the considerations of the last section. To this end we need the *perturbation equation* $L\zeta = \Phi(\zeta)$ defined in 5.6, Proposition 1, which is associated with a given immersed minimal surface X. We have the following result:

Proposition 1. *Let X be an immersed, strictly stable minimal surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with the normal N. Then there is a number $\epsilon(X) > 0$ such that all solutions $\zeta \in C^{2,\alpha}(\overline{B})$ of*

$$L(\zeta) = \Phi(\zeta) \quad \text{in } B, \quad \zeta = 0 \quad \text{on } \partial B, \tag{1}$$

satisfying $|\zeta(w)| < \epsilon(X)$ for all $w \in B$, are identically zero. Consequently, if $Y \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ is an immersed zero mean curvature surface with $Y(w) = X(w)$ for $w \in \partial B$ and

$$|Y(w) - X(w)| < \epsilon(X) \quad \text{for } w \in B \tag{2}$$

which can be written as

$$Y(w) = X(w) + \zeta(w)N(w) \quad \textit{for } w \in \overline{B}, \tag{3}$$

ζ *as above, then* $Y = X$.

Proof. In Proposition 2 of Section 5.6 we have constructed a one-parameter family $\eta \in C^{2,\alpha}(\overline{B} \times [-t_0, t_0])$ of functions $\eta(w,t)$, $w \in \overline{B}$, $t \in [-t_0, t_0]$, $t_0 > 0$, solving

$$L\eta(\cdot, t) = \Phi(\eta(\cdot, t)) \quad \text{in } B, \quad \eta(\cdot, t) = t \quad \text{on } \partial B, \tag{4}$$

such that the family of surfaces

$$Z(w,t) := X(w) + \eta(w,t)N(w), \quad w \in \overline{B}, \ |t| \le t_0 \tag{5}$$

yields a field immersion of X, as $\eta(w, 0) \equiv 0$ on $\overline{B}$. (Note that in 5.6 the function η was called ζ.) Then there is a number $\epsilon = \epsilon(X) > 0$ such that any Y of the form (3) with $\zeta \in C_0^{2,\alpha}(\overline{B})$ satisfying $L\zeta = \Phi(\zeta)$ and $|\zeta(w)| < \epsilon(X)$ for $w \in \overline{B}$ is covered by the field (5). Then we can write

$$\zeta(w) = \eta(w, \tau(w)) \quad \text{for } w \in \overline{B} \tag{6}$$

where the "height function" τ is of class $C^{2,\alpha}(\overline{B})$ and satisfies $|\tau(w)| \le t_0$ for $w \in \overline{B}$ as well as

$$\tau(w) = 0 \quad \text{on } \partial B. \tag{7}$$

Now we prove $\tau(w) \equiv 0$ on $\overline{B}$ which is turn implies

$$\zeta(w) = \eta(w, 0) \equiv 0 \quad \text{on } \overline{B}$$

whence $Y = X$.

In fact, suppose that $\tau(w) \not\equiv 0$. Then there is a point $w_0 \in B$ such that $\tau(w_0) = t_0$ with

$$|t_0| = \max\{|\tau(w)| \colon w \in \overline{B}\} > 0.$$

Then the minimal immersion Y of the form (3), satisfying (1) and (2), touches the minimal immersion $Z(\cdot, t_0)$ at the interior point $x_0 := X(w_0)$. We represent both Y and $Z(\cdot, t)$ locally as minimal graphs over the same plane in a neighborhood of x_0. Applying the maximum principle to the difference of the two equations for these graphs we conclude that the two graphs coincide. Repeating this reasoning, a continuity argument yields $Y(w) \equiv Z(w, t_0)$ for $w \in \overline{B}$, whence $\zeta(w) \equiv \eta(w, t_0)$ for all $w \in \overline{B}$, and therefore

$$\zeta(w) = t_0 \quad \text{for all } w \in \partial B. \tag{8}$$

Since $t_0 \ne 0$, this contradicts the assumption $\zeta|_{\partial B} = 0$, and so we have verified $\tau(w) \equiv 0$ on $\overline{B}$. □

Next we modify the reasoning used to prove Proposition 2 of Section 5.6. This will lead to the following central result due to F. Tomi [6] and J.C.C. Nitsche [26].

Proposition 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be an immersed, stable minimal surface, and suppose that $\{\zeta_j\}$ is a sequence of functions $\zeta_j \in C^{2,\alpha}(\overline{B})$ satisfying*

$$L\zeta_j = \Phi(\zeta_j) \quad \text{in } B, \quad \zeta_j = 0 \quad \text{on } \partial B \tag{9}$$

and

$$0 < \|\zeta_j\|_{2,\alpha} \to 0 \quad \text{for } j \to \infty \tag{10}$$

(where $\|\cdot\|_{2,\alpha}$ is the $C^{2,\alpha}(\overline{B})$-norm). Then X is weakly stable, and there exists a real analytic one-parameter family

$$\zeta : \overline{B} \times [-t_0, t_0] \to \mathbb{R}, \quad t_0 > 0,$$

of solutions $\zeta(\cdot, t) \in C^{2,\alpha}(\overline{B})$ of

$$L\zeta(\cdot, t) = \Phi(\zeta(\cdot, t)) \quad \text{in } B, \quad \zeta(w, t) = 0 \quad \text{for } w \in \partial B, \tag{11}$$

$|t| \le t_0$, satisfying

$$\frac{\partial}{\partial t}\zeta(w, t)\bigg|_{t=0} > 0 \quad \text{for all } w \in B. \tag{12}$$

Proof. Since the stable minimal immersion is not isolated, we infer from Proposition 1 that X is only "weakly stable" in the sense that the Schwarzian operator L has zero as its lowest eigenvalue with respect to zero boundary values. Equivalently this means: There exists a function $\xi \in C^{2,\alpha}(\overline{B})$ satisfying

$$L\xi = 0 \quad \text{in } B, \quad \xi|_{\partial B} = 0, \quad \xi(w) > 0 \quad \text{for all } w \in B. \tag{13}$$

Consider the closed subspace

$$\tilde{\mathfrak{B}} := \left\{ \eta \in C_0^{2,\alpha}(\overline{B}) \colon \int_B \xi\eta \, du \, dv = 0 \right\}$$

of the Banach space $(C_0^{2,\alpha}(\overline{B}), \|\cdot\|_{2,\alpha})$, as well as the restriction

$$\tilde{L} := L|_{\tilde{\mathfrak{B}}} : \tilde{\mathfrak{B}} \to C^{0,\alpha}(\overline{B}). \tag{14}$$

Next we define the "projection" $\tilde{\Phi}^t$ by

$$\tilde{\Phi}^t(\eta) := \Phi(t\xi + \eta) - \left\{ \int_B \xi\Phi(t\xi + \eta) \, du \, dv \right\} \xi, \quad \eta \in \tilde{\mathfrak{B}} \tag{15}$$

for all $t \in \mathbb{R}$. Similarly as in the proof of Propositions 1 and 2 in Section 5.6 one can show that, for any t with $|t| \le t_0$ and $0 < t_0 \ll 1$ there is exactly one solution $\eta(\cdot, t) \in \tilde{\mathcal{B}}$ of

$$\tilde{L}(\eta(\cdot, t)) = \tilde{\Phi}^t(\eta(\cdot, t)) \quad \text{in } B, \tag{16}$$

and the structure of the right-hand side in (16) yields a real analytic dependence of $\eta(\cdot, t)$ on the parameter $t \in [-t_0, t_0]$. From the assumptions (9) and (10) we infer the representations

$$\zeta_j = t_j \zeta + \eta(\cdot, t_j) \tag{17}$$

with

$$t_j \to 0 \quad \text{as } j \to \infty \quad \text{and} \quad t_j \ne 0 \tag{18}$$

for $j \gg 1$. Define the real analytic function $\psi : (-t_0, t_0) \to \mathbb{R}$ by

$$\psi(t) := \int_B \Phi(t\xi(u, v) + \eta(u, v, t))\xi(u, v)\, du\, dv. \tag{19}$$

With the aid of (9), (17), (13), (16) and (15) we obtain for $j \gg 1$ that

$$\begin{aligned} \Phi(\zeta_j) &= L(\zeta_j) = L(\eta(\cdot, t_j)) = \tilde{L}(\eta(\cdot, t_j)) \\ &= \tilde{\Phi}^{t_j}(\eta(\cdot, t_j)) = \Phi(t_j\xi + \eta(\cdot, t_j)) - \psi(t_j)\xi \\ &= \Phi(\zeta_j) - \psi(t_j)\xi. \end{aligned}$$

This implies

$$\psi(t_j) = 0 \quad \text{for } j \gg 1,\ t_j \to 0;$$

hence the real analytic function ψ satisfies

$$\psi(t) \equiv 0 \quad \text{on } (-t_0, t_0).$$

Finally we infer from (16) that

$$L\zeta(\cdot, t) = \Phi(\zeta(\cdot, t)) \quad \text{in } B \quad \text{for } |t| \le t_0$$

with the family of functions

$$\zeta(w, t) := t\xi(w) + \eta(w, t), \quad w \in \overline{B},\ |t| \le t_0$$

satisfying

$$\frac{\partial}{\partial t}\xi(w, 0) = \xi(w) + \frac{\partial}{\partial t}\eta(w, 0) > 0 \quad \text{for } w \in B$$

since

$$\frac{\partial}{\partial t}\eta(w, 0) = 0 \quad \text{for all } w \in \overline{B}$$

(see part (ii) of the proof of Proposition 2 in Section 5.6). □

The Propositions 1 and 2 motivate the following

Definition 1. *An immersed minimal surface* $X \in \mathcal{C}(\Gamma)$ *is called* **weakly stable** *if it is stable, but not strictly stable.*

Remark 1. Let λ_1 be the smallest eigenvalue of the Schwarzian operator $L = -\Delta + 2\Lambda K$ of X on B with respect to zero boundary values, i.e. the smallest number $\lambda \in \mathbb{R}$ such that the boundary value problem

$$L\zeta = \lambda\zeta \quad \text{in } B, \quad \zeta = 0 \quad \text{on } \partial B$$

possesses a nontrivial solution ζ. It is well known that λ_1 is simple and that each eigenfunction ζ corresponding to λ_1 satisfies $\zeta(w) \neq 0$ for all $w \in B$. Thus the eigenspace to λ_1 is one-dimensional and will be spanned by an eigenfunction ζ satisfying $\zeta(w) > 0$ for all $w \in B$. Hence we have:

X is stable if and only if $\lambda_1 \geq 0$, weakly stable if and only if $\lambda_1 = 0$, strictly stable if and only if $\lambda_1 > 0$, nonstable if and only if $\lambda_1 < 0$.

Furthermore we have:

1. X is weakly stable if and only if there is a $\zeta \in C^{2,\alpha}(\overline{B})$ with $L\zeta = 0$ in B, $\zeta = 0$ on ∂B, and $\zeta(w) > 0$ for $w \in B$.
2. X is strictly stable if there is a $\zeta \in C^{2,\alpha}(\overline{B})$ with $L\zeta = 0$ in B and $\zeta > 0$ on $\overline{B}$.
3. X is nonstable if there is a subdomain Ω of B with $\Omega \neq B$ such that $B \setminus \overline{\Omega}$ is non-empty and "$L\zeta = 0$ in Ω" possesses a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ with $\zeta = 0$ on $\partial\Omega$ and $\zeta > 0$ on Ω.

Remark 2. Tomi's original proof of Proposition 2 did not use the "perturbation equation" $L\zeta = \Phi(\zeta)$, but was based on a real analytic version of the implicit function theorem in Banach spaces.

Proposition 3. *With the family $\zeta(\cdot, t)$, $|t| \leq t_0$ from Proposition 2 we define the real analytic one-parameter family of immersions*

$$Y(\cdot, t) := X + \zeta(\cdot, t)N, \quad |t| \leq t_0, \tag{20}$$

from $\overline{B}$ into $\mathbb{R}^3$ which have mean curvature zero and satisfy

$$Y(w, t) = X(w) \quad \textit{for } w \in \partial B$$

and

$$|Y_t(w, t)| = |\zeta_t(w, t)| > 0 \quad \textit{for } w \in B \textit{ and } |t| \leq t_0.$$

Furthermore, all surfaces $Y(\cdot, t)$ have the same area, i.e.

$$A(Y(\cdot, t)) \equiv \text{const} \quad \textit{for } |t| \leq t_0. \tag{21}$$

Proof. As at the end of the proof of Proposition 2 in Section 5.6, formula (22), we reparametrize the surfaces $Y(\cdot, t)$ via their orthogonal trajectories and obtain (possibly for some smaller $t_0 > 0$) a family $Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$ of zero mean curvature surfaces $Z(\cdot, t)$, whose area elements $\mathcal{W}(u, v, t) := |Z_u(u, v, t) \wedge Z_v(u, v, t)|$ satisfy

$$\frac{\partial}{\partial t}\mathcal{W}(u, v, t) \equiv 0 \quad \text{on } \overline{B} \times [-t_0, t_0] \tag{22}$$

(cf. the proof of formula (23) in Section 5.6). This implies

$$A(Z(\cdot t)) \equiv \text{const} \quad \text{for } |t| \le t_0.$$

Since $Z(\cdot, t)$ is a reparametrization of $Y(\cdot, t)$, it follows that

$$A(Y(\cdot, t)) = A(Z(\cdot, t)) \quad \text{for } |t| \le t_0,$$

which in conjunction with the preceding identity implies (21). □

The above lense-shaped field of zero mean curvature surfaces $Y(\cdot, t)$ defined by (20) is defined in a similar way as a field of conjugate geodesics. This motivates

Definition 2. *A family $Y(\cdot, t) = X + \zeta(\cdot, t)N$, $|t| \le t_0$, of zero mean curvature immersions $\overline{B} \to \mathbb{R}^3$ and of constant area $A(Y(\cdot, t))$, as described in Propositions 2 and 3, is called* **conjugate field for** X. *We also say: X is embedded in the conjugate field $\{Y(\cdot, t)\}_{|t| \le t_0}$.*

This leads to the question whether a minimal immersion that is sufficiently close to a surface X and has the properties required in Propositions 2 and 3, can be "covered" by a conjugate field for X. This might not be the case if we interpret "close" in the sense of the $C^0(\overline{B}, \mathbb{R}^3)$-norm. However, this property can be proved if we understand "close" in the $C^{2,\alpha}$-sense. This is a consequence of the following result due to R. Böhme and F. Tomi [1], §3, pp. 15–20. For the convenience of the reader we shall provide a proof.

Proposition 4. *Let $\{X_j\}$ be a sequence of immersions $\overline{B} \to \mathbb{R}^3$ with $X_j \in \mathcal{C}(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R})$ and*

$$\lim_{j \to \infty} \|X_j - X\|_{C^{3,\beta}(\overline{B}, \mathbb{R}^3)} = 0 \quad \textit{for } \beta \in (0, \alpha),$$

where the limit X is also an immersion $\overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$. Then there are reparametrizations $Y_j \in \mathcal{C}(\Gamma) \cap C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ of X_j which can be expressed as "generalized graphs above X" in the form

$$Y_j = X + \zeta_j N \quad \textit{with } \zeta_j \in C_0^{2,\alpha}(\overline{B})$$

and

$$\|\zeta_j\|_{2,\beta} \to 0 \quad \textit{as } j \to \infty \quad \textit{for } 0 < \beta < \alpha.$$

Proof. First we continue X to an immersion of class $C^{3,\alpha}(\Omega, \mathbb{R}^3)$ for some Ω with $\overline{B} \subset \Omega$ and consider the family of surfaces

$$Z(w,t) := X(w) + tN(w), \quad w \in \Omega,\ |t| < \epsilon,$$

for some ϵ with $0 < \epsilon \ll 1$. Then $Z \in C^{2,\alpha}(\Omega \times (-\epsilon,\epsilon), \mathbb{R}^3)$, and the Jacobian J_Z of Z is everywhere positive on $\Omega \times (-\epsilon,\epsilon)$. Thus Z is an open mapping of $\Omega \times (-\epsilon,\epsilon)$ into $\mathbb{R}^3$, and Z can locally be inverted. In conjunction with a monodromy argument it follows that, for $j \gg 1$, each X_j can be represented in the form

$$X_j(w) = Z(f_j(w), z_j(w)), \quad w \in \overline{B},$$

with a mapping $f_j : \overline{B} \to \mathbb{R}^3$ of the class $C^{2,\alpha}(\overline{B}, \mathbb{R}^2)$ such that $f_j|_{\partial B}$ maps ∂B monotonically onto itself, and a height function $z_j \in C_0^{2,\alpha}(\overline{B})$.

Setting $f(w) := w$ and $z(w) := 0$ for $w \in \overline{B}$ we can write

$$X(w) = Z(f(w), z(w)).$$

Then we infer from $X_j \to X$ in $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ and the fact that the local inverse of Z is of class $C^{2,\alpha}$:

$$f_j \to f \quad \text{in } C^{2,\alpha}(\overline{B}, \mathbb{R}^2), \quad z_j \to z = 0 \quad \text{in } C^{2,\alpha}(\overline{B}).$$

Since $f(w) \equiv w$ on $\overline{B}$, the mappings f_j satisfy

$$J_{f_j}(w) > 0 \quad \text{on } \overline{B} \quad \text{for } j \gg 1,$$

and so every $f_j|_B$ is an open mapping of B into $\mathbb{R}^2$, $j \gg 1$. Since $f_j \in C^0(\overline{B}, \mathbb{R}^2)$ and $f_j|_{\partial B}$ is a homeomorphism of ∂B onto ∂B, we infer $f_j(\overline{B}) = \overline{B}$ for $j \gg 1$; therefore the f_j are $C^{2,\alpha}$-diffeomorphisms of $\overline{B}$ onto $\overline{B}$ for $j \gg 1$. Setting $\zeta_j := z_j \circ f_j^{-1} \in C_0^{2,\alpha}(\overline{B})$ and $Y_j := X_j \circ f_j^{-1} = Z(\mathrm{id}_{\overline{B}}, \zeta_j)$ we obtain

$$Y_j(w) = X_j(f_j(w)) = X(w) + \zeta_j(w)N(w) \quad \text{for } w \in \overline{B},\ j \gg 1,$$

with

$$\|\zeta_j\|_{C^{2,\alpha}(\overline{B})} \to 0 \quad \text{as } j \to \infty.$$

□

Remark 3. Let us interpret the preceding results in a geometric way. Proposition 1 states that a strictly stable, immersed $X \in \mathcal{C}(\Gamma)$ can be embedded into a field $\{Z(\cdot,t)\}_{|t|\le t_0}$ of minimal immersions such that every immersion $Y \in \mathcal{C}(\Gamma)$ with the mean curvature zero, given in the "normal form" $Y = X + \zeta N$ with $\zeta \in C_0^{2,\alpha}(\overline{B})$ satisfying $L\zeta = \Phi(\zeta)$, coincides with X if it is sufficiently close to X in the $C^0(\overline{B}, \mathbb{R}^3)$-norm. This means: *A strictly stable minimal immersion is isolated with respect to the C^0-norm compared with normal variations $Y = X + \zeta N$, ζ as above.*

Yet it is not clear whether every minimal immersions $\tilde{X} \in \mathcal{C}(\Gamma)$ that is C^0-close to X has a normal-form representation Y; but, by Proposition 4, such a reparametrization can be achieved if $\tilde{X}$ is $C^{2,\alpha}$-close to X. Thus we obtain: *Any strictly stable minimal immersion $X \in \mathcal{C}(\Gamma)$ is isolated in the $C^{2,\alpha}$-norm among all minimal immersions of class $\mathcal{C}(\Gamma)$.*

If, however, the stable immersion $X \in \mathcal{C}(\Gamma)$ is the $C^{2,\alpha}$-limit of stable immersions $X_j \in \mathcal{C}(\Gamma)$ with $X_j \neq X$, then X is weakly stable and can be embedded into a conjugate field $\{Y(\cdot,t)\}_{|t|\le t_0}$ which forms a regular, real analytic curve in $C^{2,\alpha}(B,\mathbb{R}^3)$.

Now we turn to Tomi's "finiteness result". We recall some definitions and formulate a compactness result.

The class $\mathcal{C}^*(\Gamma)$ consists of those $X \in \mathcal{C}(\Gamma)$ which satisfy a preassigned three-point condition $*$, and $\overline{\mathcal{C}}^*(\Gamma) := \mathcal{C}^*(\Gamma) \cap C^0(\overline{B},\mathbb{R}^3)$. For $X \in \mathcal{C}(\Gamma)$ the area $A(X)$ and Dirichlet's integral $D(X)$ are

$$A(X) = \int_B |X_u \wedge X_v|\,du\,dv, \quad D(X) = \frac{1}{2}\int_B |\nabla X|^2\,du\,dv.$$

We know that

$$a(\Gamma) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}^*(\Gamma)} A = \inf_{\overline{\mathcal{C}}^*(\Gamma)} D.$$

Proposition 5. *For any $\Gamma \in C^{k,\alpha}$ there is a constant $c(\Gamma,k,\alpha,*)$ such that each minimal surface $X \in \mathcal{C}^*(\Gamma)$ is of class $C^{k,\alpha}(\overline{B},\mathbb{R}^3)$ and satisfies*

$$\|X\|_{C^{k,\alpha}(\overline{B},\mathbb{R}^3)} \le c(\Gamma,k,\alpha,*), \quad k \in \mathbb{N},\ \alpha \in (0,1),$$

where $c(\Gamma,k,\alpha,*)$ is a constant which depends only on $\Gamma,k,\alpha,*$. Hence, from any sequence of minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$, we can extract a subsequence $X_{j_\nu} \to X$ in $C^{k,\beta}(\overline{B},\mathbb{R}^3)$ as $\nu \to \infty$ for any $\beta \in (0,\alpha)$, where $X \in \mathcal{C}^*(\Gamma) \cap C^{k,\alpha}(\overline{B},\mathbb{R}^3)$ is a minimal surface.

Proof. See Vol. 2, Chapter 2. □

Theorem 1 (F. Tomi [6]). *Let Γ be a closed Jordan curve in $\mathbb{R}^3$ of class $C^{3,\alpha}$, and suppose that every minimal surface X of class $\mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ is an immersion of $\overline{B}$ into $\mathbb{R}^3$, i.e. X be free both of interior and boundary branch points. Then Γ spans only finitely many minimal surfaces $X \in \mathcal{C}^*(\Gamma)$ which satisfy $A(X) = a(\Gamma)$, i.e. which are area minimizing in $\mathcal{C}(\Gamma)$.*

This immediately implies the following

Corollary 1. *If all minimal surfaces $X \in \mathcal{C}(\Gamma)$ with $\Gamma \in C^{3,\alpha}$ are immersed up to the boundary, i.e. have no branch points on $\overline{B}$, then there are only finitely many minimal surfaces $X \in \mathcal{C}^*(\Gamma)$ with $A(X) = a(\Gamma)$.*

Remark 4. In Section 4.9 we have exhibited conditions on Γ which ensure that any $X \in \mathcal{C}(\Gamma)$ is free of branch points, in which case Corollary 1 can be applied.

Remark 5. By the papers by R. Osserman, H.W. Alt, R. Gulliver, and Gulliver/Osserman/Royden it follows that any minimal surface $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ is free of interior branch points. Furthermore, R. Gulliver and F.D. Lesley [1] have stated that, in addition, every $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ has no boundary branch point if Γ is a regular, real analytic Jordan curve. This result implies

Corollary 2. *If Γ is a regular, real analytic, closed Jordan curve, then there exist only finitely many $X \in \mathcal{C}^*(\Gamma)$ with $A(X) = a(\Gamma)$, and all of them are immersions.*

Proof of Theorem 1. Suppose that Γ bounds infinitely many X with $A(X) = a(\Gamma)$. By Proposition 5 there is a sequence $\{X_j\}$ of minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$ with $A(X_j) = a(\Gamma)$ and

$$0 < \|X_j - X\|_{C^{3,\beta}(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } j \to 0$$

for $\beta \in (0, \alpha)$, and the limit X is a minimal surface of class $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $A(X) = a(\Gamma)$.

By Propositions 2, 3, 4 we embed X into a conjugate field, with α replaced by $\beta \in (0, \alpha)$, i.e. there is a regular, real analytic curve $\{Y(\cdot, t)\}_{|t| \le t_0}$ with $Y(\cdot, 0) = X$ which lies in the level set

$$\mathcal{M}_c(\Gamma) := \{X \in \mathcal{C}^*(\Gamma) \colon D(X) = A(X) = c\}, \quad c := a(\Gamma).$$

We equip $\mathcal{M}_c(\Gamma)$ with the $C^2(\overline{B}, \mathbb{R}^3)$-norm and denote by $\mathcal{K}_c$ the closed, connected component of $\mathcal{M}_c(\Gamma)$ containing X. A continuity argument combined with the above reasoning yields: Through every $X_0 \in \mathcal{K}_c$ there is a real analytic, regular curve $\{Y(\cdot, t)\}_{|t| \le t_0}$ contained in $\mathcal{K}_c$ such that $Y(\cdot, 0) = X_0$.

Consider now the volume functional V on the "block" $\mathcal{K}_c$ which is defined by

$$V(X) := \frac{1}{3} \int_B [X, X_u, X_v] \, du \, dv. \tag{23}$$

Since $\mathcal{K}_c$ is a compact subset of $C^2(\overline{B}, \mathbb{R}^3)$ and V is continuous on $\mathcal{K}_c$, there is an $X_0 \in \mathcal{K}_c$ such that

$$V(X_0) = \max_{\mathcal{K}_c} V.$$

Let $\{Y(\cdot, t)\}_{|t| \le t_0}$ be a regular, real analytic arc with $Y(\cdot, 0) = X_0$. Then

$$\frac{d}{dt} V(Y(\cdot, t))\Big|_{t=0} = 0. \tag{24}$$

On the other hand we have

$$Y(\cdot, t) := X_0 + \zeta(\cdot, t) N_0,$$

N_0 = normal of X_0, with $\xi = \zeta_t(\cdot, 0), \xi(w) > 0$ on B, and $Y_t(\cdot, 0) = \xi N_0$. Set

$$\Lambda_0 := |X_{0,u} \wedge X_{0,v}| = |X_{0,u}|^2$$

with $\Lambda_0(w) > 0$ on $\overline{B}$. The computations in 5.3 show that

$$\frac{d}{dt} V(Y(\cdot, t))\Big|_{t=0} = \int_B \Lambda_0(w)\xi(w)\, du\, dv > 0 \tag{25}$$

if we take div $\frac{1}{3}x = 1$ into account. Clearly, (25) contradicts (24), and so the theorem is proved. □

Now we want to generalize Theorem 1 to stable solutions of Plateau's problem. So far we can carry out this program only for some special classes of boundaries, e.g. for *extreme curves.*

Definition 3. *A closed Jordan curve Γ in $\mathbb{R}^3$ is called* **extreme** *if for any point P of Γ there is a plane of support, that is, a plane Π such that Γ lies on one side of Π but is not completely contained in Π.*

Clearly, *Γ is extreme if and only if it lies on the boundary of a convex body. Equivalently we have*: *Γ is extreme if and only if it lies on the boundary of its convex hull.*

Proposition 6 (Compactness property of stable minimal immersions). *Let Γ be a closed regular Jordan curve of class $C^{3,\alpha}$ which is extreme, and suppose that $\{X_j\}$ is a sequence of stable minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$ free of branch points on $\overline{B}$. Then*:

(i) *We can extract a subsequence $\{X_{j_\nu}\}$ converging in $C^{3,\beta}(\overline{B}, \mathbb{R}^3)$ with $\beta \in (0, \alpha)$ to a minimal surface $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$.*
(ii) *The limit surface X is a stable minimal immersion of $\overline{B}$ into $\mathbb{R}^3$.*

Proof. Statement (i) follows from Proposition 5, and the limit X of the X_{j_ν} has no branch points on ∂B since Γ is extreme. It remains to prove that X is stable and has no branch points in B. To this end we consider the Gauss curvature K_ν and the surface element $\Lambda_\nu = |D_u X_{j_\nu}|^2$ of X_{j_ν} as well as the normal $N_\nu : \overline{B} \to S^2 \subset \mathbb{R}^3$ of X_{j_ν}. We have

$$\tfrac{1}{2}|\nabla N_\nu|^2 = -\Lambda_\nu K_\nu,$$

and the Gauss–Bonnet formula yields

$$-\int_B \Lambda_\nu K_\nu du\, dv = \int_{\partial B} (\kappa_g)_\nu\, ds - 2\pi \le \kappa(\Gamma) - 2\pi.$$

Thus the total curvature $\kappa(\Gamma)$ of Γ estimates the Dirichlet integrals $D(N_\nu) = \frac{1}{2}\int_B |\nabla N_\nu|^2\, du\, dv$ of the normals N_ν by

$$D(N_\nu) \le \kappa(\Gamma) - 2\pi \quad \text{for all } \nu \in \mathbb{N}. \tag{26}$$

Furthermore the isoperimetric inequality yields

$$D(X_{j_\nu}) \le \frac{1}{4\pi} L^2(\Gamma) \quad \text{for all } \nu \in \mathbb{N}. \tag{27}$$

Thus by the reasoning in (i), (ii), (iii) of the proof of Theorem 1 in Section 5.5 we conclude: For any $B' \subset\subset B$ there is a constant $c(B') > 0$ such that

$$|\nabla N_\nu(w)| \le c(B') \quad \text{for all } w \in B'. \tag{28}$$

Since $|N_\nu| \le 1$ we may assume that the subsequence $\{j_\nu\}$ also satisfies

$$N_\nu(w) \rightrightarrows N(w) \quad \text{for } w \in \overline{B} \text{ and for any } B' \subset\subset B. \tag{29}$$

(Actually it suffices to apply merely (i) and (ii) of the proof quoted above since in this way we already obtain a uniform modulus of continuity of the N_ν on any $B' \subset\subset B$.)

Suppose now that X had an interior branch point $w_0 \in B$; we may assume that $w_1 = 0$, $X(0) = 0$, $N(0) = e_3 = (0,0,1)$. Then the associated planar mapping $f : B \to \mathbb{C}$ with

$$f(w) := X^1(w) + iX^2(w), \quad w \in B,$$

has the asymptotic expansion

$$f(w) = aw^n + o(|w|^{n+1}) \quad \text{as } w \to 0, \quad a \in \mathbb{C} \setminus \{0\}$$

where $n \ge 2$. Thus the winding number $i(f,0)$ of f about $w = 0$ is at least 2. On the other hand the planar mappings $f_\nu : B \to \mathbb{C}$ associated with X_{j_ν},

$$f_\nu(w) := X^1_{j_\nu}(w) + iX^2_{j_\nu}, \quad w \in B,$$

satisfy $f_\nu(w) \rightrightarrows f(w)$ for $|w| \ll 1$ as well as

$$f_\nu(w) = a_\nu w + o(|w|^2) \quad \text{for } |w| \le \delta,\ 0 < \delta \ll 1,\ a_\nu \in \mathbb{C} \setminus \{0\},\ \nu \gg 1,$$

since $X_{j_\nu}(w) \rightrightarrows X(w)$ and $N_\nu(w) \rightrightarrows N(w)$ as $\nu \to \infty$ for $|w| \le \delta$ with $0 < \delta \ll 1$. Hence the winding numbers $i(f_\nu, 0)$ of f_ν about 0 satisfy $i(f_\nu, 0) = 1$ for $\nu \gg 1$. Since $i(f_\nu, 0) \to i(f, 0)$ as $\nu \to \infty$, we obtain $i(f,0) = 1$, a contradiction to $i(f,0) \ge 2$. Thus X has no branch points in $\overline{B}$.

Then we conclude

$$\Lambda_\nu(w) K_\nu(w) \to \Lambda(w) K(w) \quad \text{as } \nu \to \infty \quad \text{for } w \in \overline{B}, \tag{30}$$

and Lebesgue's theorem on dominated convergence yields the stability of X. □

When we combine the reasoning in the proof of Theorem 1 with Proposition 6, we obtain

Theorem 2. *An extreme, regular Jordan contour $\Gamma \in C^{3,\alpha}$ bounds at most finitely many stable minimal immersions $\overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$.*

Remark 6. The central reason why we can carry over the proof of Theorem 1 to the situation considered in Theorem 2 is the observation stated in Proposition 3 that all elements $Y(\cdot, t)$ of the regular, real analytic family (20) have the same area $A(Y(\cdot, t))$.

Remark 7. The arguments used for the proof of Theorem 2 and the subsequent Theorem 3 are based on Sauvigny's paper [10].

Remark 8. The same reasoning holds true if we replace the assumption that *Γ be extreme* by the property: *No minimal surface $X \in \mathcal{C}(\Gamma)$ has a boundary branch point on ∂B.*

In this context, J.C.C. Nitsche [31] has proved the following result:

Proposition 7. *Let Γ be a closed, regular, real analytic Jordan curve in $\mathbb{R}^3$ with the property that there is a straight line in $\mathbb{R}^3$ such that no plane through this line intersects Γ in more than two distinct points. Then every solution of Plateau's problem for Γ is free of branch points.*

We now present a modified version of the 6π-*finiteness theorem* by J.C.C. Nitsche [31] which considers also nonstable solutions of Plateau's problem.

Proposition 8. *Let $\Gamma \in C^{3,\alpha}$ be a closed, regular, extreme Jordan curve in $\mathbb{R}^3$ with a total curve $\kappa(\Gamma)$ less than 6π. Then from any sequence $\{X_j\}$ of minimal immersions $X_j : \overline{B} \to \mathbb{R}^3$ we can extract a subsequence $\{X_{j_\nu}\}$ converging in $C^{3,\beta}(\overline{B}, \mathbb{R}^3)$ for $0 < \beta < \alpha$ to a minimal immersion $X : \overline{B} \to \mathbb{R}^3$ of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$.*

Proof. We copy the reasoning used for proving Proposition 6, but we have to replace the stability condition with the subsequent *curvatura-integra condition* to achieve a uniform modulus of continuity for the normals N_ν of the converging subsequence $\{X_{j_\nu}\}$ of $\{X_j\}$. The estimate (26) yields

$$(31) \qquad D(N_\nu) = A(N_\nu) \le \kappa(\Gamma) - 2\pi =: \omega \quad \text{with } 0 \le \omega < 4\pi.$$

If $\omega = 0$ then $N_\nu = \text{const}$ for all $\nu \in \mathbb{N}$, and thus the N_ν are certainly uniformly continuous. Hence we can assume that

$$0 < \omega < 4\pi.$$

With the aid of the Courant–Lebesgue lemma we obtain a universal radius $\rho > 0$ such that N_ν maps the circle $\partial B_\rho(w_0)$ contained in B into a spherical cap on S^2 with a "sufficiently small" geodesic radius. Since $N_\nu(w) \not\equiv \text{const}$ on $\overline{B}$, it follows that $N_\nu : B \to S^2$ is an open mapping (because the composition $\sigma \circ N_\nu$ with a stereographic projection $\sigma : S^2 \to \mathbb{C}$ is locally holomorphic, see Section 3.3). We then conclude that all spherical images $N_\nu(B_\rho(w_0))$ have to remain within this cap. Otherwise $N_\nu(B_\rho(w_0))$ would entirely cover the complementary cap, in contradiction to the integral condition (31). Thus we obtain a modulus of continuity for the mappings $N_\nu, \nu \in \mathbb{N}$, in the interior of B. □

Now we present the following version of Nitsche's 6π-theorem:

Theorem 3. *Let $\Gamma \in C^{3,\alpha}$ be a closed, regular, extreme Jordan curve of the total curvature $\kappa(\Gamma) < 6\pi$. Then there exist only finitely many minimal immersions $X : \overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$.*

Proof. If there were infinitely many minimal immersions, Proposition 8 would yield a sequence of distinct minimal immersions $X_j : \overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$ which converge in $C^{2,\beta}(\overline{B}, \mathbb{R}^3)$, $0 < \beta < \alpha$, to some minimal immersion $X \in \mathcal{C}^*(\Gamma)$ of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with

$$\Lambda(w) := \tfrac{1}{2}|\nabla X(w)|^2 > 0 \quad \text{in } \overline{B}$$

and

$$-\int_B K\Lambda\, du\, dv \le \omega < 4\pi. \tag{32}$$

By virtue of Proposition 4 we can represent the surfaces X_j as graphs Y_j over X in the form

$$Y_j(w) = X(w) + \zeta_j(w) N(w) \quad \text{for } w \in \overline{B} \tag{33}$$
$$\text{with } \zeta_j \in C_0^{2,\alpha}(\overline{B}) \text{ and } \|\zeta_j\|_{2,\beta} \to 0 \text{ for } 0 < \beta < \alpha.$$

For $j \gg 1$ the ζ_j are solutions of

$$L\zeta_j = \Phi(\zeta_j) \quad \text{in } B \quad \text{with } \zeta_j = 0 \text{ on } \partial B, \tag{34}$$

where L is the Schwarzian operator for X. If $\lambda = 0$ were not an eigenvalue of L with respect to the boundary condition $\zeta = 0$ on ∂B, we would obtain

$$\zeta_j = L_0^{-1}\Phi(\zeta_j), \quad j \in \mathbb{N},$$

with $L_0 := L|_{C_0^{2,\beta}(\overline{B})}$. Since $L_0^{-1}\phi$ is contracting (see Proposition 2 of Section 5.6) we obtain a contradiction to the property $\|\zeta_j\|_{2,\beta} \to 0$ as $j \to \infty$. Thus $\lambda = 0$ is an eigenvalue of L.

If $\lambda = 0$ is the smallest eigenvalue of L then X is stable, and the arguments used in the proofs of the Theorems 1 and 2 lead to a contradiction.

Now we show that $\lambda = 0$ has to be the smallest eigenvalue of L. Otherwise there is a $\xi \in C_0^{2,\beta}(\overline{B})$ with

$$L\xi = 0 \quad \text{in } B$$

with

$$\int_B \xi(w) \cdot \xi_1(w)\, dw = 0,$$

where ξ_1 is an eigenfunction to the smallest eigenvalue of L, i.e.

$$L\xi_1 = \lambda_1 \xi_1 \quad \text{in } B, \quad \xi_1 = 0 \quad \text{on } \partial B,$$

satisfying $\xi_1(w) > 0$ in B. Then there are two disjoint and nonempty open subsets Ω_1 and Ω_2 of $\{w \in B : \xi(w) \neq 0\}$ such that

$$(35) \qquad L\xi = 0 \quad \text{in } \Omega_j, \quad \xi = 0 \quad \text{on } \partial\Omega_j, \quad \xi(w) \neq 0 \quad \text{on } \Omega_j \quad \text{for } j = 1, 2.$$

Condition (32) implies that one of the domains Ω_j, say Ω_1, has the property

$$(36) \qquad -\int_{\Omega_1} K\Lambda\, du\, dv < 2\pi.$$

In virtue of the stability theorem by Barbosa–do Carmo (see Section 5.4), property (36) implies that $X|_{\Omega_1}$ is strictly stable, which is a contradiction to (35) for $j = 1$.

Therefore, Γ bounds only finitely many minimal immersions of class $\mathcal{C}^*(\Gamma)$. □

Remark 9. The last theorem remains true under the weaker assumption $\kappa(\Gamma) \leq 6\pi$. To cover the case $\kappa(\Gamma) = 6\pi$ we refer to the proof of Theorem 1 in Section 5.6, estimating the total geodesic curvature by the total curvature.

Remark 10. It would be desirable to establish Theorem 3 for real analytic contours, renouncing the assumption that Γ be extreme. Nitsche's 6π-theorem in [31] states finiteness under the assumption that Γ be real analytic and that no minimal surface $X \in \mathcal{C}(\Gamma)$ has a branch point on $\overline{B}$. We also hint at the work of Beeson [3–5].

5.8 Scholia

H.A. Schwarz initiated the study of the second variation of area for immersed minimal surfaces in his celebrated memoir *Ueber ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung* (1885), dedicated to K. Weierstrass on occasion of his seventieth birthday (cf. Schwarz [2], Vol. I, pp. 223–269). The main purpose of that paper is to establish a criterion whether or not a given minimal surface furnishes a relative minimum of area among all surfaces bounded by the same contour. As Schwarz showed, a minimal surface is a local minimizer if it can be embedded in a field, i.e. a one-parameter foliation, of minimal surfaces. We have described this idea in Sections 2.7 and 2.8. When is such an embedding possible? To decide this question, Schwarz considered the spherical image Ω of the given surface and introduced the *Schwarz operator* L on this image. The desired embedding is possible if the equation $L\zeta = 0$ in Ω possesses a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ which is positive on $\overline{\Omega}$. In this connection, Schwarz connected the study of the operator $\mathfrak{L} = \Delta + p$, $p > 0$, with the minimum problem for the Rayleigh quotient $J_1(\zeta)/J_0(\zeta)$, where

$$J_1(\zeta) := \int_\Omega |\nabla\zeta|^2 \, dx \, dy, \quad J_0(\zeta) := \int_\Omega p\zeta^2 \, dx \, dy.$$

This led him to the minimum characterization of the smallest eigenvalue for Δ and $\mathfrak{L}$ respectively, which can be considered as the beginning of Hilbert's theory of eigenvalue problems in the form that later was developed by Courant. In this paper one also finds *Schwarz's inequality* (see Schwarz [2], Vol. I, p. 251) in the form

$$\left| \int_\Omega \varphi\psi \, dx \, dy \right| \le \sqrt{\int_\Omega \varphi^2 \, dx \, dy} \sqrt{\int_\Omega \psi^2 \, dx \, dy}.$$

These ideas were generalized by L. Lichtenstein, and later by many other mathematicians to study the corresponding minimum problem for general multiple integrals in the calculus of variations; see Giaquinta and Hildebrandt [1], Vol. 1, Chapter 6, in particular Section 4.

J.C.C. Nitsche [26] revived Schwarz's field construction to prove the celebrated uniqueness theorem presented in Section 5.6. Basic ingredients of Nitsche's proof are the results of Chapter 6 concerning the existence of unstable minimal surfaces, obtained by the mountain-pass lemma, and the stability theorem of J.L. Barbosa and M. do Carmo [1].

We also note that the renewed interesting and flourishing study of stable minimal surfaces was, in fact, initiated by the work of Barbosa and do Carmo.

A very careful and comprehensive description of results connected with the second variation of surface area and stable minimal surfaces can be found in J.C.C. Nitsche's treatises [28] and [37], §§98–119; in particular a lucid presentation of Schwarz's approach is given.

In Sections 5.1–5.5 we essentially followed the work of F. Sauvigny [1, 2,7–11]. We also mention prior work by R. Schoen [2], who generalized the fundamental curvature estimate by E. Heinz [1], presented in Section 2.4, to minimal immersions $X : B \to N$ in a three-dimensional oriented Riemannian manifold N. A special case of his Theorem 3 is the following result: *Let $M = X(B)$ be an immersed, stable surface in $\mathbb{R}^3$ which compactly contains a geodesic ball $B_{R_0}(P_0)$ for some $P_0 \in M$ and some $r_0 > 0$. Then there is an absolute constant $c > 0$ such that the second fundamental form A of M at P_0 is estimated by $|A|^2(P_0) \le cr_0^{-2}$.* The corresponding analogue for cmc-surfaces, due to F. Sauvigny [7,8], is given in Section 5.5, see Theorems 1 and 2. We also refer the reader to the interesting work of S. Fröhlich [1–5] on curvature estimates for immersions of mean curvature type, even with higher codimensions, where the notion of μ-stable extremals appears.

Nitsche's uniqueness result had a predecessor in an unpublished paper by R. Schneider, who formulated the following beautiful theorem (1968): *A closed polygon in* $\mathbb{R}^3$ *with a total curvature* $\kappa(\Gamma) < 4\pi$ *bounds only one disk-type minimal surface.* Moreover, he conjectured that every Jordan curve with a total curvature less than 4π spans only one disk-type minimal surface, and for any $\epsilon > 0$ he gave an example of a curve Γ with $\kappa(\Gamma) < 4\pi + \epsilon$ bounding at least two disk-type minimal immersions.

Schneider's Example (1968): Consider the minimal surface $X : \overline{\Omega} \to \mathbb{R}^3$ defined by

$$X(u,v) := (-v \sin u, v \cos u, u), \quad \Omega := \{(u,v) \colon |u| < \alpha\pi, |v| < R\}$$

for $R > 0$ and $0 < \alpha < 1$, which is part of the helicoid given by the equation $x + y \tan z = 0$. The boundary Γ of X consists of two straight segments Γ_1, Γ_2 and two parts Γ_3, Γ_4 of helices meeting Γ_1 and Γ_2 perpendicularly. The total curvature of Γ_3 as well of Γ_4 is $2\pi\alpha R(1+R^2)^{-\frac{1}{2}}$. Adding the contributions of the four corners of Γ, one obtains

$$\kappa(\Gamma) = 2\pi[1 + 2\alpha R(1 + R^2)^{-\frac{1}{2}}].$$

Given $\epsilon \in (0, 2\pi)$ we choose $\alpha \in (\frac{1}{2}, 1)$ as $\alpha := \frac{1}{2} + \frac{\epsilon}{4\pi}$. Then

$$\kappa(\Gamma) = 2\pi + (2\pi + \epsilon)R(1 + R^2)^{-\frac{1}{2}}.$$

The right-hand side is an increasing function of $R \in [0, \infty)$ which tends to 1 as $R \to \infty$, thus $\kappa(\Gamma) < 4\pi + \epsilon$ for any $R > 0$. On the other hand, Schwarz showed in 1872 (see [1]; and [2], Vol. 1, pp. 161–163) that for $\alpha \in (\frac{1}{2}, 1)$ there is a value $R_0(\alpha) \in (\sqrt{3}, \infty)$ with $R_0(\alpha) \to \infty$ as $\alpha \to \frac{1}{2} + 0$, $R_0(\alpha) \to \sqrt{3}$ as $\alpha \to 1 - 0$, such that $X \circ \tau$ does not furnish a relative minimum of area in $\mathcal{C}(\Gamma)$ if $R \in (R_0(\alpha), \infty)$, where τ is a conformal mapping of the unit disk B onto Ω. We know however that there is a minimal surface $\tilde{X} \in \mathcal{C}(\Gamma)$ which minimizes area in $\mathcal{C}(\Gamma)$. This surface is an immersion for the following reason. Since Γ lies on the boundary of a compact convex set K, $\tilde{X}$ cannot have any boundary branch points. Furthermore, through every point of K there is a plane which intersects Γ only in two points. Hence no minimal surface of class $\mathcal{C}(\Gamma)$ has an interior branch point (see Radó [21], p. 35). *Thus* Γ *bounds at least two regular* (i.e. immersed) *minimal surfaces.* We recall that Böhme [6] later on showed that for any $\epsilon > 0$ and any $N \in \mathbb{N}$ there is a real analytic Jordan curve Γ with $\kappa(\Gamma) < 4\pi + \epsilon$ which bounds at least N disk-type minimal surfaces.

Schneider's paper was not published since it depended on fragmentary results by Marx and Shiffman (cf. Marx [1]) which in 1968 were considered to be unproved (Oberwolfach meeting on the "Calculus of Variations"). This desideratum stimulated E. Heinz to write a series of fundamental papers (cf. Heinz [19–24]) which rigorously dealt with the asymptotic behaviour of minimal surfaces in corners and led to the theory of quasi-minimal surfaces. Some of Heinz's results are described in the Scholia to Chapter 6. Using these results, F. Sauvigny [3–5] developed a theory of the second variation of the area for minimal surfaces bounded by polygons, and he rediscovered Schneider's unpublished result, thereby also establishing an analog for $\mathbb{R}^p$, $p > 3$. In addition, the "finiteness question" for certain polygonal boundaries was answered affirmatively by R. Jakob [9,10], building on Heinz's results.

We also mention a paper by H. Ruchert [2] where Nitsche's uniqueness theorem is carried over to "small" surfaces of constant mean curvature.

In a fundamental paper by R. Böhme and F. Tomi [1], the structure of the set of solutions to Plateau's problems was analyzed with the aid of semianalytic sets. This in turn led to F. Tomi's seminal paper [6] about the finiteness of the number of absolute minimizers for Plateau's problem.

J.C.C. Nitsche proved the 6π-finiteness theorem in his paper [31]. The isolatedness of cmc-immersions solving the corresponding Plateau problem was investigated by F. Sauvigny [10]. His ideas are used in Section 5.7, especially for the compactness results concerning minimal immersions.