# Chapter 4

# The Plateau Problem and the Partially Free Boundary Problem

The remainder of this book is essentially devoted to boundary value problems for minimal surfaces. The simplest of such problems was named *Plateau's problem*, in honor of the Belgian physicist J.A.F. Plateau, although it had been formulated much earlier by Lagrange, Meusnier, and other mathematicians. It is the question *of finding a surface of least area spanned by a given closed Jordan curve* $\Gamma$.

In his treatise *Statique expérimentale et théorétique des liquides soumis aux seules forces moléculaires* from 1873, Plateau described a multitude of experiments connected with the phenomenon of capillarity. Among other things, Plateau noted that every contour consisting of a single closed wire, whatever be its geometric form, bounds at least one soap film. Now the mathematical model of a thin wire is a closed Jordan curve of finite length. Moreover, the mathematical objects modeling soap films are two-dimensional surfaces in $\mathbb{R}^3$. To every such surface, the phenomenological theory of capillarity, due to Gauss, attaches a potential energy that is proportional to its surface area. Hence, by Johann Bernoulli's principle of virtual work, *soap films in stable equilibrium correspond to surfaces of minimal area.*

Turning this argument around, it stands to reason that every rectifiable closed Jordan curve bounds at least one surface of least area and that all possible solutions to Plateau's problem can be realized by soap film experiments. However, as R. Courant [15] has remarked, *empirical evidence can never establish mathematical existence—nor can the mathematician's demand for existence be dismissed by the physicist as useless rigor. Only a mathematical existence proof can ensure that the mathematical description of a physical phenomenon is meaningful.*

The mathematical question that we have formulated above as Plateau's problem was a great challenge to mathematicians. It turned out to be a formidable task. During the nineteenth century, Plateau's problem was solved for many special contours $\Gamma$, but a sufficiently general solution was only obtained in 1930 by J. Douglas [11,12] and simultaneously by T. Radó [17,18].

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Grundlehren der mathematischen Wissenschaften 339,
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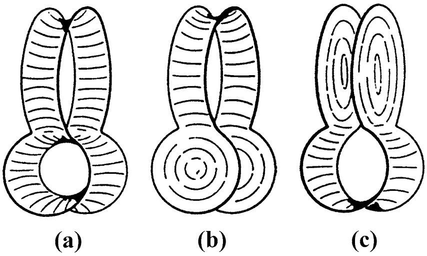



**Fig. 1.** A Jordan contour bounding two disk-type minimal surfaces **(b)**, **(c)** and a minimal surface of genus one **(a)**

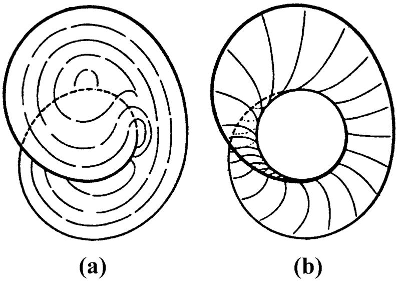



**Fig. 2.** A Jordan curve bounding **(a)** a disk-type minimal surface and **(b)** a minimal Möbius strip

A considerable simplification of their methods was found by R. Courant [4, 5] and, independently, by L. Tonelli [1]. In the present chapter we want to describe the Courant–Tonelli approach to Plateau's problem.

Recall that regular surfaces of least area are minimal surfaces, in the sense that their mean curvature vanishes throughout. Thus we can formulate a somewhat more general version of Plateau's problem: *Given a closed rectifiable Jordan curve $\Gamma$, find a minimal surface spanned by $\Gamma$.* Then the least area problem for $\Gamma$ is more stringent than the Plateau problem: the first question deals with the (absolute or relative) minimizers of area, whereas the second is concerned with the stationary points of the area functional.

Note that for a fixed boundary contour $\Gamma$ the solutions to Plateau's problem are by no means uniquely determined. Moreover, there may exist solutions of different genus within the same boundary curve, and there may exist both orientable and non-orientable minimal surfaces within the same boundary frame. This is illustrated by the minimal surfaces depicted in Fig. 2.

Even if we fix the topological type of the solutions to Plateau's problem, the unique solvability is, in general, not ensured. For instance, Figs. 1 and 4 depict some boundary configurations which can span several minimal surfaces of the topological type of the disk. In Section 4.9, we shall give a survey of what is known about the number of disk-type solutions to Plateau's problem. In the Scholia (Section 4.15) as well as in Chapters 5 and 7, the reader will find more

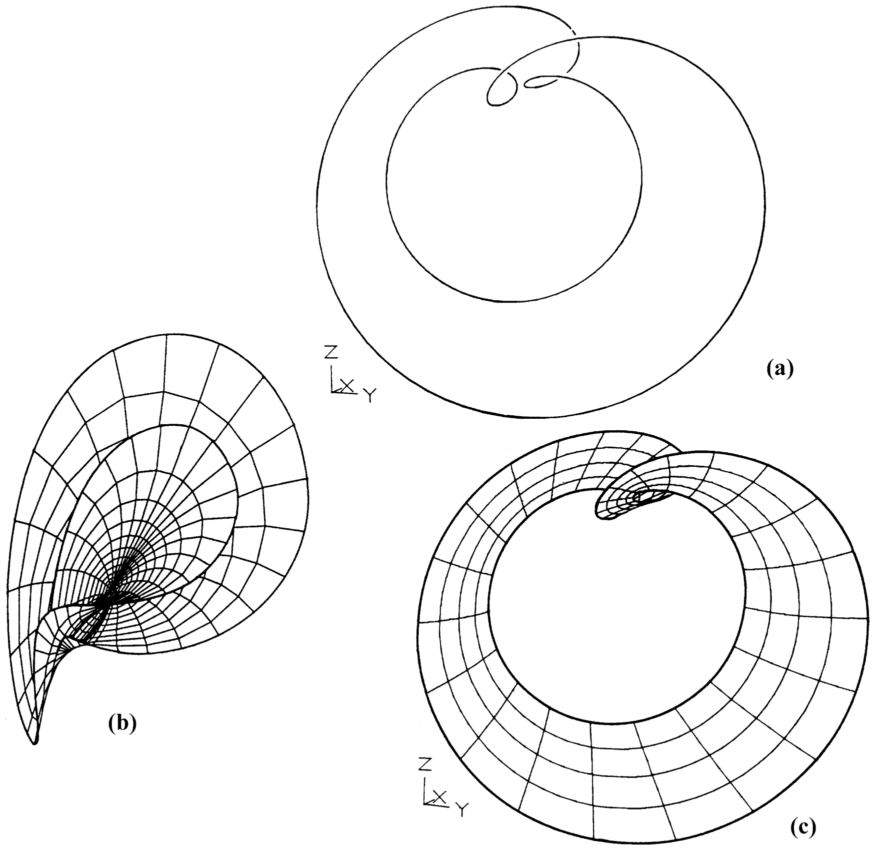



**Fig. 3.** A closed Jordan curve **(a)**, bounding a disk-type minimal surface **(b)**, as well as a Möbius strip **(c)**

examples and further results on the number of solutions of Plateau's problem, and we shall also discuss the question whether solutions are immersed or even embedded.

Other boundary value problems for minimal surfaces will be considered in Chapter 8 and in Vols. 2 and 3. For example, the last chapter of this volume as well as Chapter 4 of Vol. 3 deal with solutions of the *general Plateau problem* (also called *Douglas problem*) where one has to find a minimal surface of possibly higher topological type spanned by a frame consisting of one or several curves.

We begin the present chapter by having a closer look at Plateau's problem. First we compare Dirichlet's integral with the area functional, and we shall explain why it seems to be more profitable to minimize the Dirichlet integral rather than the area. Then, in Section 4.2, we set up Plateau's problem in a form that we shall deal with in Sections 4.3–4.5. In Section 4.2, we

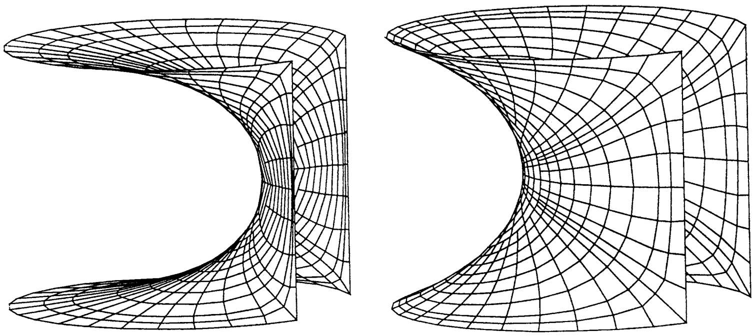


**Fig. 4.** Another Jordan curve spanned by two disk-type minimal surfaces

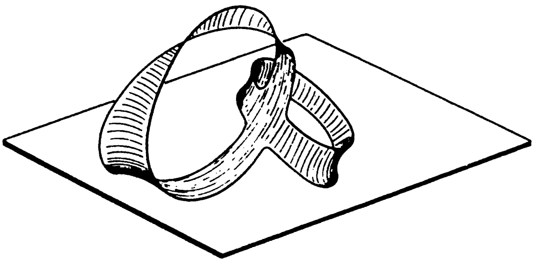


**Fig. 5.** A Jordan curve bounding a one-sided minimal surface of higher topological type

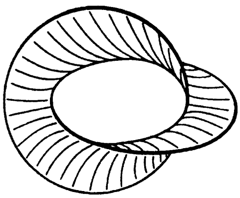


**Fig. 6.** Two interlocked Jordan curves spanned by an annulus-type minimal surface

describe the minimization procedure that will lead to a solution of Plateau's problem, and in Section 4.3, we prove the uniform convergence of a suitably chosen minimizing sequence to a harmonic mapping. This is achieved with the aid of the Courant–Lebesgue lemma proved in Section 4.4. In Section 4.5 we use variations of the independent variables for establishing a variational formula, from which we can derive that the minimizer $X(u,v)$, constructed in Section 4.3, also satisfies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Hence it follows that $X$ actually is a minimal surface solving Plateau's problem for the prescribed boundary curve $\Gamma$. Finally we shall see why $X$ is also

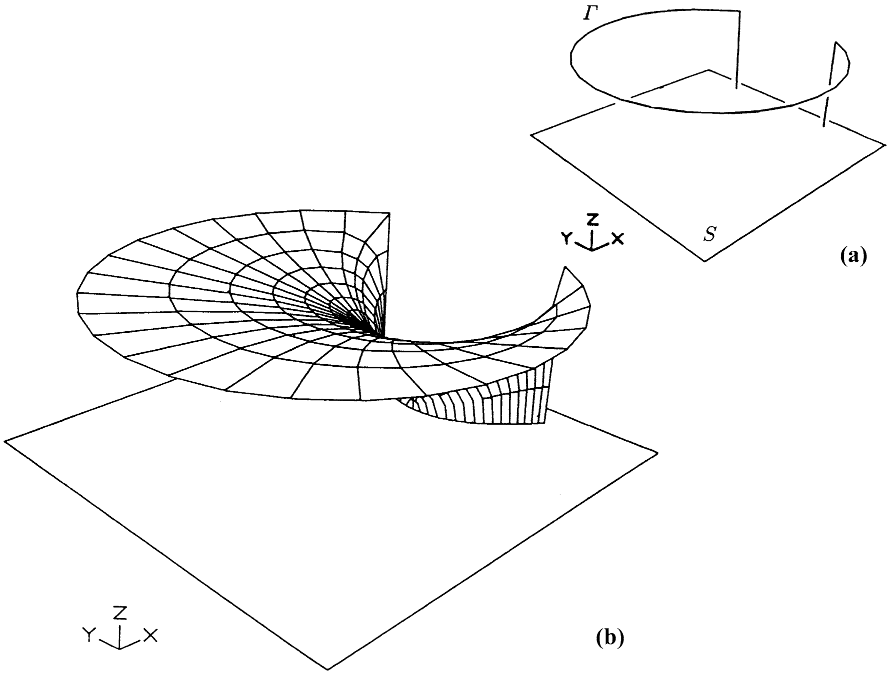



**Fig. 7. (a)** A configuration consisting of a planar surface $S$ and a Jordan arc $\Gamma$. **(b)** Solution of the partially free boundary value problem corresponding to the configuration $\langle \Gamma, S \rangle$, computed by a finite-element method

a solution of the least area problem, using Morrey's lemma on $\epsilon$-conformal mappings. A self-contained proof of this result is presented in Section 4.10; it is described below.

A slight modification of Courant's approach, given in Section 4.6, will lead to the solution of the *partially free boundary problem*.

A few results concerning the boundary behavior of minimal surfaces with rectifiable boundaries are collected in Section 4.7. They will in particular be needed in Chapter 5 of Vol. 2.

Reflection principles for minimal surfaces will be formulated in Section 4.8. Essentially we shall prove again two results from Section 3.4, without using Schwarz's solution to Björling's problem.

In 4.9 we give a survey on some results concerning the uniqueness and nonuniqueness of solutions to Plateau's problem; in particular Radó's uniqueness result is proved. Generalizations of Radó's theorem to free boundary problems are studied in Chapters 1 and 2 of Vol. 3.

Another approach to Plateau's problem, presented in 4.10, proceeds by minimizing the convex combination $(1-\epsilon)A+\epsilon D$ of the area functional $A$ and the Dirichlet integral $D$ for any $\epsilon \in (0,1]$ in $\mathcal{C}(\Gamma)$. It turns out that any minimizer yields a conformally parametrized solution of the problem "$A \to$ min

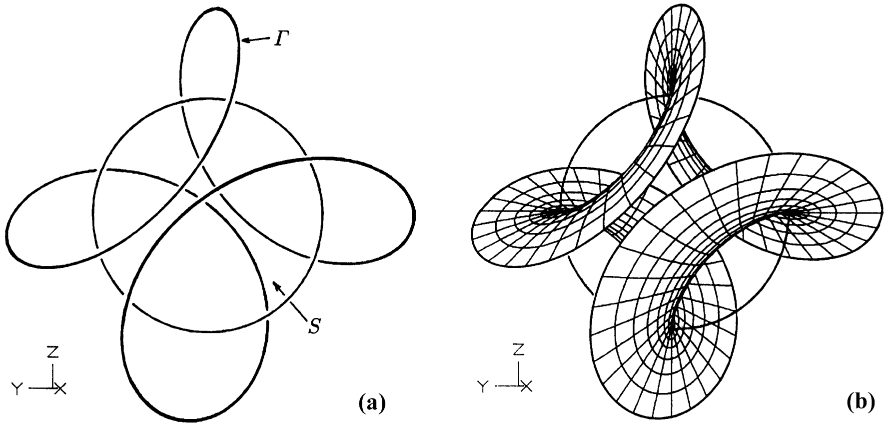



**Fig. 8.** **(a)** A boundary configuration $\langle \Gamma, S \rangle$ consisting of a disk $S$ and of a closed Jordan curve $\Gamma$ disjoint from $S$. **(b)** An annulus-type minimal surface which is stationary in $\langle \Gamma, S \rangle$

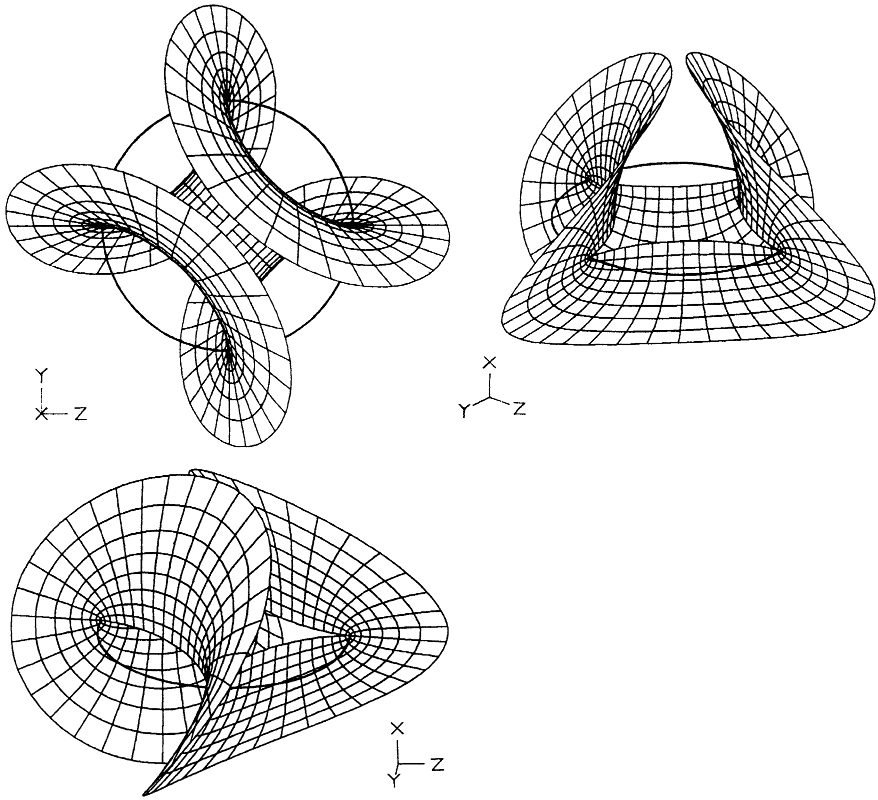



**Fig. 9.** Three more views of the minimal surface described in Fig. 8

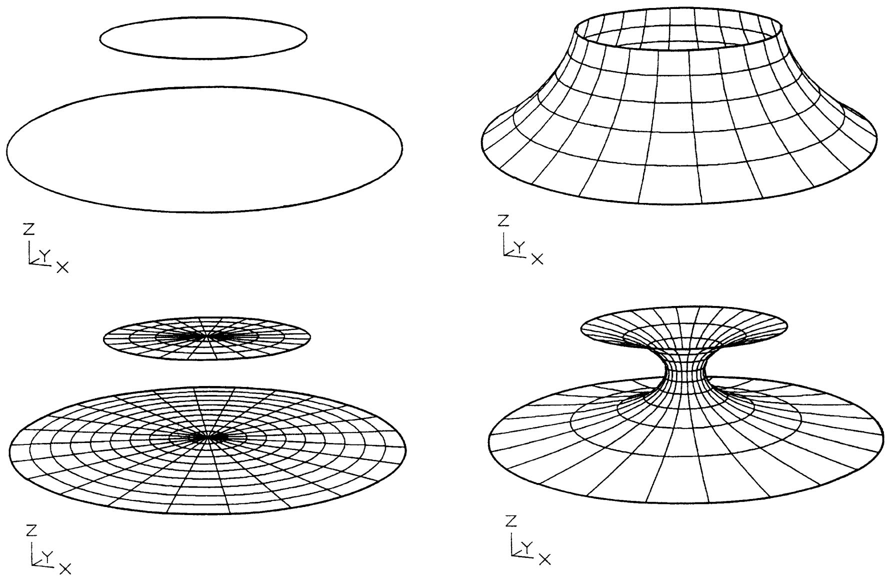



**Fig. 10.** The general Plateau problem consists in finding minimal surfaces spanning several closed Jordan curves. Here we show two parallel coaxial circles bounding three minimal surfaces of rotation

in $\mathcal{C}(\Gamma)$" which also minimizes $D$ in $\mathcal{C}(\Gamma)$. This way we arrive at another proof of Theorem 4 in 4.5 and in particular of the relation (40) in 4.5 stating that $\overline{a}(\Gamma) = \overline{e}(\Gamma)$. This new approach only applies methods developed in the present chapter and completely avoids Morrey's Lemma on $\epsilon$-conformal mappings (see 4.5). Thus no results on quasiconformal mappings nor on conformal representations of surfaces are needed for solving the minimal-area problem. Actually, the underlying idea of 4.10 can be used to obtain conformal representations of surfaces or of two-dimensional Riemannian metrics. This will be carried out in 4.11 where we show that the solution of Plateau's problem for planar contours provides a proof of the *Riemann mapping theorem*. This way we also verify that planar solutions to Plateau's problem are area-minimizing, free of branch points, and uniquely determined (up to a conformal reparametrization).

In a similar manner we derive other mapping theorems such as Lichtenstein's mapping theorem.

Nonrectifiable Jordan curves in $\mathbb{R}^3$ no longer need to bound a disk-type surface of finite Dirichlet integral. Nevertheless J. Douglas proved that any closed Jordan curve in $\mathbb{R}^3$ bounds a continuous disk-type minimal surface. A proof of this fact is presented in Section 4.12.

In Section 4.13 it is proved that every oriented closed, rectifiable Jordan curve bounds a continuous and conformally parametrized disk-type surface of finite area that minimizes an arbitrarily given regular Cartan functional, i.e. a given regular two-dimensional and parameter invariant variational integral $\mathcal{F}(X) = \int_B F(X, X_u \wedge X_v)\, du\, dv$. Here no general regularity theory for the corresponding Euler equation is available; therefore the existence proof is based on a variational method that resembles the technique of Section 4.10.

Thereafter we derive the basic isoperimetric inequality for disk-type minimal surfaces. Generalizations of this inequality are studied in Chapter 6 and in Chapter 4 of Vol. 2.

Finally the Scholia in Section 4.15 give a brief survey of the history of Plateau's problem as well as references to the literature. Moreover some basic results on the nonexistence of branch points for minimizers are described. In addition we discuss the question as to whether a contour bounds embedded solutions, the problem of uniqueness and nonuniqueness, index theorems, generic finiteness, and Morse-theoretic results. These topics will also (and in more detail) be treated in Chapter 6 and in Vol. 3. Thereafter we review some results on solutions to obstacle problems, a detailed presentation of which is given in Chapter 4 of Vol. 2. At last, some results on systems of minimal surfaces are described.

## 4.1 Area Functional Versus Dirichlet Integral

If one tries to formulate and to solve Plateau's problem, cumbersome difficulties may turn up. Among other problems one has to face the fact that there exist mathematical solutions to Plateau's problem which cannot be realized in experiment by soap films. This is, of course, to be expected for merely stationary solutions which are not minimizing, because they correspond to unstable soap films, and these will be destroyed by the tiniest perturbation of the soap lamellae caused by, say, a slight shaking of the boundary frame or by a breath of air.

However it can also happen that (mathematical) solutions of Plateau's problem have branch points, and that they have self-intersections. Both phenomena are unrealistic in the physical sense because Plateau has discovered the following rule for a stable configuration of soap films:

*Three adjacent minimal surfaces of an area-minimizing system of surfaces, corresponding to a stable system of soap films, meet in a smooth line at an angle of* $120°$. *Only four such lines, each being the soul of three soap films, can meet at a common point. At such a vertex, each pair of liquid edges forms an angle* $\varphi$ *of* $109°28'16''$ *or, more precisely, of* $\cos\varphi = -1/3$.

Figure 11 in Section 4.15 shows a system of soap films exhibiting these features.

Solutions of Plateau's problem, which are absolute minimizers of area, cannot have interior branch points according to a result by Osserman–Gulliver–Alt. Their proof of this result is rather difficult and lengthy; thus it will only be sketched in Sections 1.9 and 5.3 of Vol. 2 (see also the Scholia 4.15 of the

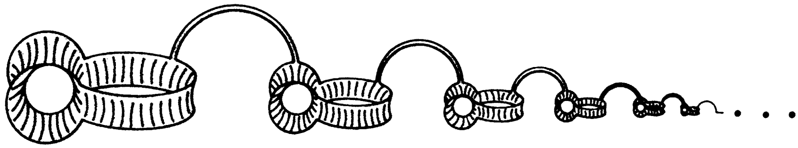


**Fig. 1.** The monster surface: a minimal surface of infinite genus

present chapter and the Scholia 6.7 of Vol. 2). A new approach leading to this result is described in Chapter 6 of Vol. 2.

Yet, despite the absence of branch points for minimizers, self-intersections of (mathematical) solutions are still conceivable, and so far only a few positive results are known, for instance:

*If $\Gamma$ is a closed Jordan curve that lies on a convex surface, then $\Gamma$ bounds a disk-type minimal surface without self-intersections.*

Another positive result, due to Ekholm, White, and Wienholtz [1] is the following:

*If $\Gamma$ is a closed Jordan curve in $\mathbb{R}^3$ with total curvature less or equal to $4\pi$, then any minimal surface—independently of its topological type—is embedded up to and including the boundary, with no interior branch points.*

A brief survey on the existence of embedded solutions of Plateau's problem is given in the Scholia 4.15, Subsection 3.

To solve Plateau's problem we would like to use the classical approach, which consists in minimizing area among surfaces given as mappings from a two-dimensional parameter domain into $\mathbb{R}^3$, this way fixing the topological type of the admissible surfaces. However, as we have already seen, it is by no means clear what the topological type of the surface of least area in a given configuration $\Gamma$ will be. In fact, there may be rectifiable boundaries for which the area-minimizing solution of Plateau's problem is of infinite genus. An example for this phenomenon is depicted in Fig. 1.

Let us now restrict ourselves to surfaces $X \in C^0(\bar{B}, \mathbb{R}^3)$ which are parametrized on the closure of the unit disk $B = \{w \in \mathbb{C} \colon |w| < 1\}$, and which map the circle $\partial B$ topologically onto a prescribed closed Jordan curve $\Gamma$ in $\mathbb{R}^3$. Such a surface is said to be a solution of Plateau's problem for $\Gamma$ if its restriction to $B$ is a minimal surface. Since minimal surfaces are the critical points of the area functional

$$A_B(X) = \int_B |X_u \wedge X_v| \, du \, dv,$$

one is tempted to look for solutions of Plateau's problem by minimizing $A_B(X)$ in the class of all surfaces $X \in C^0(\bar{B}, \mathbb{R}^3)$ mapping $\partial B$ homeomorphically onto $\Gamma$. But this method will produce literally hair-raising solutions. This can be seen as follows. Suppose that $\Gamma$ is a circle in $\mathbb{R}^3$ contained in the $x, y$-plane, say

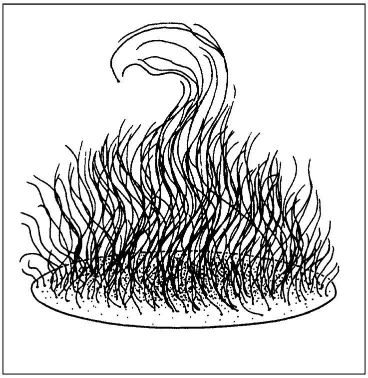


**Fig. 2.** A hairy disk

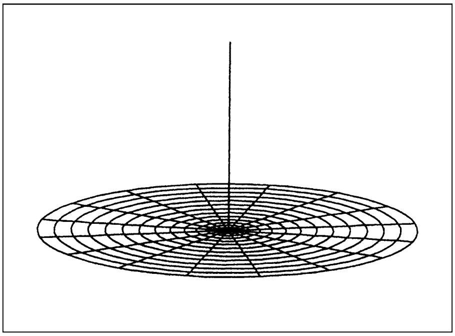


**Fig. 3.** A hair $C^\infty$-grown on a disk

$$\varGamma = \{(x,y,z)\colon x^2+y^2=1, z=0\},$$

and let $K(\varGamma) = \{(x,y,z)\colon x^2+y^2 \le 1, z=0\}$ be the disk which is bounded by $\varGamma$. On account of the maximum principle, the only minimal surfaces $X$ of class $C^0(\bar{B},\mathbb{R}^3)\cap C^2(B,\mathbb{R}^3)$ which map $\partial B$ topologically onto $\varGamma$ and satisfy

$$(1) \qquad \Delta X = 0,$$

$$(2) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0$$

in $B$, are regular conformal mappings of $\bar{B}$ onto $K(\varGamma)$ (cf. Section 4.11).

On the other hand, among the minimizers of the area functional $A_B(X)$, there are mappings $X : \bar{B} \to \mathbb{R}^3$ which parametrize sets $K^*(\varGamma)$ which may be viewed as *hairy disks* bounded by $\varGamma$ (see Fig. 2). They occur as additional, though nonregular, minimizers of $A_B$ since hairs do not contribute to surface area. For example, *let us raise just one hair on the disk* $K(\varGamma)$. To this end, we consider the set

$$K^*(\varGamma) = K(\varGamma) \cup H$$

consisting of the disk $K(\Gamma)$ and the hair

$$H = \{(x, y, z) \colon x = y = 0,\ 0 \le z \le 1\}$$

attached to the center of $K(\Gamma)$. Then $K^*(\Gamma)$ can be parametrized by the following mapping $X(u, v)$ of class $C^\infty(\bar{B}, \mathbb{R}^3)$:

$$x(u, v) = y(u, v) := 0, \quad z(u, v) := \varphi(r) \quad \text{for } 0 \le r \le \tfrac{1}{2},$$

where $r = \sqrt{u^2 + v^2}$, and

$$x(u, v) := \psi(r) \cos\theta, \quad y(u, v) := \psi(r) \sin\theta, \quad z(u, v) := 0 \quad \text{for } \tfrac{1}{2} \le r \le 1.$$

Here, the functions $\varphi(r)$ and $\psi(r)$ are defined by

$$\varphi(r) := \exp 4\left(1 - \frac{1}{1 - 4r^2}\right), \quad \psi(r) := \exp 4\left(\frac{1}{3} - \frac{1}{4r^2 - 1}\right).$$

Note that the surface $X(u, v)$ is irregular for $0 \le r \le \frac{1}{2}$ which is also evident from the fact that the whole disk $B_{1/2} = \{(u, v) \colon u^2 + v^2 < \frac{1}{4}\}$ is mapped into the hair $H$ (cf. Fig. 3).

Consequently, if we would use the variational problem

$$A_B(X) \to \min,$$

we would have to cope with a host of nasty solutions. In order to derive a reasonable solution satisfying equations (1) and (2), we would have to cut off all the hairs from a hairy solution.[1] This is fairly easy in the setting of geometric measure theory since a two-dimensional measure neglects hairs as sets of measure zero, whereas in the context of mappings the regularization of solutions requires quite an elaborate procedure.

In order to avoid this difficulty, we shall proceed similarly as in Riemannian geometry where one studies the one-dimensional Dirichlet instead of the length functional, using the fact that the critical points of Dirichlet's integral are also critical points of the length functional which are parametrized proportionally to the arc length, and vice versa. An analogous relation holds between the stationary surfaces of the two-dimensional *Dirichlet integral*

$$D_B(X) = \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2)\, du\, dv \tag{3}$$

and the area functional $A_B(X)$. This can be seen as follows: For arbitrary vectors $p, q \in \mathbb{R}^3$ we have

$$|p \wedge q| \le |p||q|,$$

[1] When David Hilbert had established Dirichlet's principle, Felix Klein wrote: "Hilbert schneidet den Flächen die Haare ab" (cf. D. Hilbert, Gesammelte Abhandlungen, Vol. 3, p. 409).

and therefore

$$|p \wedge q| \le \tfrac{1}{2}|p|^2 + \tfrac{1}{2}|q|^2. \tag{4}$$

The equality sign in (4) holds if and only if $p \perp q$ and $|p| = |q|$. Suppose now that $X \in C^1(B, \mathbb{R}^3)$ has a finite Dirichlet integral $D_B(X)$. Then we obtain the inequality

$$A_B(X) \le D_B(X), \tag{5}$$

and the equality sign is satisfied if and only if the conformality relations (2) are fulfilled on $B$. In other words, *area functional and Dirichlet integral coincide exactly on the conformally parametrized surfaces $X$, and, in general, the Dirichlet integral furnishes a majorant for the area functional.*

Moreover, every smooth regular surface $X : B \to \mathbb{R}^3$ can, by Lichtenstein's theorem, be reparametrized by a regular change $\tau : B \to B$ of parameters such that $Y := X \circ \tau$ satisfies the conformality relations

$$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0,$$

and we obtain

$$D_B(Y) = A_B(Y) = A_B(X).$$

This observation makes it plausible that, within a class $\mathcal{C}$ of surfaces which is invariant with respect to parameter changes, minimizers of $D_B(X)$ will also be minimizers of $A_B(X)$, and more generally, that stationary points of $D_B(X)$ will be stationary points of $A_B(X)$.

Certainly the class $\mathcal{C}$ defined by Plateau's boundary condition $X : \partial B \to \Gamma$ has this invariance property. Thus we are led to the idea that we should minimize Dirichlet's integral instead of the area functional since we would also obtain a minimizer for $A_B(X)$.

We will presently dispense with putting this idea on solid ground by making the above reasoning rigorous. Instead we shall simply use the following idea: Minimize $D_B(X)$ instead of $A_B(X)$, and justify it a posteriori by proving that, in suitable classes $\mathcal{C}$, the stationary points of $D_B(X)$ are in fact minimal surfaces.

The use of Dirichlet's integral in the minimizing procedure is a advantageous for several reasons:

(i) It is not advisable to carry out the minimization among regular surfaces only, because the class of such surfaces is not closed with respect to uniform convergence of $\bar{B}$ or to $H_2^1(B)$-convergence, and a better convergence of minimizing sequences will be difficult (or even impossible) to obtain. However, if we admit general surfaces for minimization, the hairy monsters will also turn up as minimizers when $A_B(X)$ is minimized. They are excluded if we instead minimize $D_B(X)$.

(ii) Minimizing sequences of $D_B(X)$ have better compactness properties than those of $A_B(X)$.

The basic reason for (i) and (ii) is that the expression $|p|^2 + |q|^2$ only vanishes if $p = 0$ and $q = 0$ holds, whereas $|p \wedge q|$ is zero for any pair of collinear vectors $p$ and $q$. Moreover, $A_B(X)$ is invariant with respect to arbitrary reparametrizations of $X$, while $D_B(X)$ remains unchanged only under conformal parameter transformations.

Keeping these ideas in mind, we will now proceed to formulate a minimum problem, the solution of which will turn out to be a solution of Plateau's problem.

*Notational convention*: Occasionally we shall write $D(X, B)$ and $A(X, B)$ instead of $D_B(X)$ and $A_B(X)$, and, for two mappings $X, Y$, we denote by $D_B(X, Y)$ the polarization of the Dirichlet integral:

$$(6) \quad D_B(X,Y) := \frac{1}{2}\int_B (\langle X_u, Y_u\rangle + \langle X_v, Y_v\rangle)\, du\, dv = \frac{1}{2}\int_B \langle \nabla X, \nabla Y\rangle\, du\, dv.$$

## 4.2 Rigorous Formulation of Plateau's Problem and of the Minimization Process

Set

$$B := \{w \in \mathbb{C}\colon |w| < 1\}$$

and

$$C := \{w \in \mathbb{C}\colon |w| = 1\} = \partial B.$$

A *closed Jordan curve $\Gamma$ in* $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is homeomorphic to $\partial B$. By distinguishing some fixed homeomorphism $\gamma : C \to \Gamma$ from $C$ onto $\Gamma$ we equip $\Gamma$ with an *orientation*, and we say that $\Gamma$ is *oriented* (by $\gamma$).

**Definition 1.** *Given a closed Jordan curve $\Gamma$ in $\mathbb{R}^3$, we say that $X : \bar{B} \to \mathbb{R}^3$ is a solution of Plateau's problem for the boundary contour $\Gamma$ (or: a minimal surface spanned in $\Gamma$) if it fulfills the following three conditions*:

(i) $X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$;

(ii) *The surface $X$ satisfies in $B$ the equations*

$$(1) \qquad \Delta X = 0,$$

$$(2) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0;$$

(iii) *The restriction $X|_C$ of $X$ to the boundary $C$ of the parameter domain $B$ is a homeomorphism of $C$ onto $\Gamma$.*

If it is necessary to be more precise, we shall denote a minimal surface $X$ described in this definition as *disk-type solution of Plateau's problem for the contour $\Gamma$.*

Condition (iii) is equivalent to the assumption that $X|_C$ is a continuous, strictly monotonic (i.e. injective) mapping of $C$ onto $\Gamma$.

Clearly this condition is not closed with respect to uniform convergence on $C$ since uniform limits of strictly monotonic functions can be merely weakly monotonic, that is, they may have arcs of constancy on $C$. To be precise, we give the following

**Definition 2.** *Suppose that $\Gamma$ is a closed Jordan curve in $\mathbb{R}^3$, which is oriented by a homeomorphism $\gamma : C \to \Gamma$ from $C$ onto $\Gamma$. Then a continuous mapping $\varphi : C \to \Gamma$ of $C$ onto $\Gamma$ is said to be weakly monotonic if there is a nondecreasing continuous function $\tau : [0, 2\pi] \to \mathbb{R}$ with $\tau(2\pi) = \tau(0) + 2\pi$ such that*

$$\varphi(e^{i\theta}) = \gamma(e^{i\tau(\theta)}) \quad \text{for } 0 \le \theta \le 2\pi. \tag{3}$$

In other words, $\varphi$ is weakly monotonic if the image points $\varphi(w)$ traverse $\Gamma$ in a constant direction when $w$ moves along $C$ in a constant direction. The image points may stand still but never move backwards if $w$ moves monotonically on $C$, and $\varphi(w)$ moves once around $\Gamma$ if $w$ travels once around $C$.

Introducing the mapping $\mathcal{E} : [0, 2\pi] \to C$ by $\mathcal{E}(\theta) := e^{i\theta}$, we can write (3) as

$$\varphi \circ \mathcal{E} = \gamma \circ \mathcal{E} \circ \tau$$

whence we arrive at

$$\mathcal{E} \circ \tau = \gamma^{-1} \circ \varphi \circ \mathcal{E}. \tag{4}$$

From this formula we obtain at once:

**Lemma 1.** *Let $\{\varphi_n\}$ be a sequence of weakly monotonic, continuous mappings of $C$ onto a closed Jordan curve $\Gamma$, and suppose that the mappings $\varphi_n$ converge uniformly on $C$ to some mapping $\varphi : C \to \mathbb{R}^3$. Then $\varphi$ is a weakly monotonic continuous mapping of $C$ onto $\Gamma$.*

**Remark.** The assertion of Lemma 1 remains true if we assume that the mappings $\psi_n$ are weakly monotonic, continuous mappings of $C$ onto closed Jordan arcs $\Gamma_n$ which converge in the sense of Fréchet to some Jordan arc $\Gamma$. That means, there are homeomorphisms $\gamma_n$ and $\gamma$ of $C$ onto $\Gamma_n$ and $\Gamma$ respectively, such that $\gamma_n$ tends uniformly to $\gamma$ as $n \to \infty$.

Now we want to set up the variational problem that will lead us to a solution of Plateau's problem. First we define the *class* $\mathcal{C}(\Gamma)$ *of admissible functions*. We have exactly two essentially different orientations of $\Gamma$. Correspondingly there will be exactly two possibilities to define $\mathcal{C}(\Gamma)$ if $\Gamma$ is not oriented, while $\mathcal{C}(\Gamma)$ will be uniquely defined for an oriented contour $\Gamma$.

Recall that every function $X \in H_2^1(B, \mathbb{R}^3)$ has a trace $X|_C$ on the boundary $C = \partial B$ which is of class $L_2(C, \mathbb{R}^3)$.

**Definition 3.** *Given a closed Jordan curve $\Gamma$ in $\mathbb{R}^3$, a mapping $X : B \to \mathbb{R}^3$ is said to be of class $\mathcal{C}(\Gamma)$ with respect to a fixed orientation $\gamma : C \to \Gamma$ of $\Gamma$ if $X \in H_2^1(B, \mathbb{R}^3)$ and if its trace $X|_C$ can be represented by a weakly monotonic, continuous mapping $\varphi : C \to \Gamma$ of $C$ onto $\Gamma$ (i.e., every $L_2(C)$-representative of $X|_C$ coincides with $\varphi$ except for a subset of zero 1-dimensional Hausdorff measure).*

Let

$$D(X) = D_B(X) := \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2)\, du\, dv \tag{5}$$

be the Dirichlet integral of a mapping $X \in H_2^1(B, \mathbb{R}^3)$. Then we define the variational problem $\mathcal{P}(\Gamma)$ associated with Plateau's problem for the oriented curve $\Gamma$ as the following task:

*Minimize Dirichlet's integral $D(X)$, defined by (5), in the class $\mathcal{C}(\Gamma)$.*

In other words, setting

$$e(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}(\Gamma)\}, \tag{6}$$

we have to find a surface $X \in \mathcal{C}(\Gamma)$ such that

$$D(X) = e(\Gamma) \tag{7}$$

is satisfied.

In order to solve the minimum problem $\mathcal{P}(\Gamma)$, we shall have to find a minimizing sequence $\{X_n\}$ whose boundary values $X_n|_C$ contain a subsequence which is uniformly convergent on $C$. The selection of such a minimizing sequence will be achieved by the following artifice:

Fix three different points $w_1$, $w_2$, $w_3$ on $C$, an orientation $\gamma : C \to \Gamma$ of $\Gamma$, and three different points $Q_1$, $Q_2$, $Q_3$ on $\Gamma$ such that $\gamma(w_k) = Q_k$, $k = 1, 2, 3$. Let $\mathcal{C}(\Gamma)$ be defined with respect to the orientation $\gamma$ of $\Gamma$, and consider those mappings $X \in \mathcal{C}(\Gamma)$ which satisfy the *three-point condition*

$$X(w_k) = Q_k, \quad k = 1, 2, 3. \tag{8}$$

The set of such mappings $X$ will be denoted by $\mathcal{C}^*(\Gamma)$. Set

$$e^*(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}^*(\Gamma)\}. \tag{9}$$

We clearly have

$$e(\Gamma) \le e^*(\Gamma).$$

Moreover, if $X \in \mathcal{C}(\Gamma)$, then there exist three different points $\zeta_1, \zeta_2, \zeta_3$ on $C$ such that

$$X(\zeta_k) = Q_k, \quad k = 1, 2, 3.$$

Let $\sigma$ be a strictly conformal mapping of $\bar{B}$ onto itself with the property that

$$\sigma(w_k) = \zeta_k, \quad k = 1, 2, 3.$$

Then the mapping $Y := X \circ \sigma$ is of class $\mathcal{C}^*(\Gamma)$ and satisfies $D(Y) = D(X)$, because of the conformal invariance of the Dirichlet integral. Hence we even obtain

$$e(\Gamma) = e^*(\Gamma). \tag{10}$$

Consequently, any solution $X$ of the *restricted minimum problem*

$$\mathcal{P}^*(\Gamma)\colon \quad \textit{Minimize } D(X) \textit{ in the class } \mathcal{C}^*(\Gamma) \tag{11}$$

is also a solution of the original minimum problem $\mathcal{P}(\Gamma)$. Hence we shall try to solve $\mathcal{P}^*(\Gamma)$ instead of $\mathcal{P}(\Gamma)$, in this way obtaining a convenient compactness property of the boundary values of any minimizing sequence, as we shall see.

Before we can start with our minimizing process, one final difficulty remains to be solved. Since $\mathcal{P}^*(\Gamma)$ would not have a solution if $\mathcal{C}^*(\Gamma)$ were empty, let us now study under which circumstances $\mathcal{C}^*(\Gamma)$ or, equivalently, $\mathcal{C}(\Gamma)$ is certainly nonempty.

Let $\varphi : C \to \Gamma$ be a homeomorphism representing $\Gamma$, and let

$$\varphi(e^{i\theta}) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \{A_n \cos n\theta + B_n \sin n\,\theta\} \tag{12}$$

be its Fourier expansion, $A_n, B_n \in \mathbb{R}^3$, which is convergent in $L_2([0, 2\pi], \mathbb{R}^3)$. We can assume that $\varphi$ satisfies the prescribed three-point condition, i.e.,

$$\varphi(w_k) = Q_k, \quad k = 1, 2, 3.$$

Let $\rho, \theta$ be polar coordinates about the origin of the $w$-plane, that is,

$$w = \rho e^{i\theta},$$

and set

$$X(w) := \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta). \tag{13}$$

Since $|A_n|$ and $|B_n|$ are bounded by $2 \sup_C |\varphi|$, the series on the right-hand side converges uniformly on every compact subset of $B$, and a well-known computation shows that its limit is nothing but Poisson's integral for the boundary values $\varphi(e^{i\theta})$, i.e.,

$$X(w) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\psi}) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\theta - \psi)}\, d\psi \tag{14}$$

for $w = \rho e^{i\theta}$, $\rho < 1$. By the classical result of H.A. Schwarz, the mapping $X(w)$ is harmonic in $B$ and satisfies $X(w) \to \varphi(w_0)$ as $w \to w_0$, $w \in B$, for every $w_0 \in \partial B$. Hence $X$ can be extended to a continuous function on $\bar{B}$ with the boundary values $\varphi$ on $C = \partial B$. A straight-forward computation yields

$$D(X) = \frac{\pi}{2} \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2). \tag{15}$$

Consequently the map $X : \bar{B} \to \mathbb{R}^3$ belongs to the class $H_2^1(B, \mathbb{R}^3)$ if and only if

$$\sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2) < \infty. \tag{16}$$

If this is true, then $\mathcal{C}^*(\Gamma)$ is nonempty.

Condition (16) is satisfied if and only if $\phi(\theta) := \varphi(e^{i\theta})$ has half a derivative which is square-integrable. This is, for example, true if the representation $\varphi : C \to \Gamma$ of the Jordan curve $\Gamma$ is Lipschitz continuous. Such a representation of $\Gamma$ exists if and only if $\Gamma$ has finite length. Hence, *for any rectifiable Jordan curve $\Gamma$, neither $\mathcal{C}(\Gamma)$ nor $\mathcal{C}^*(\Gamma)$ are empty.* Note, however, that the rectifiability of $\Gamma$ is only sufficient but not necessary for $\mathcal{C}(\Gamma)$ to be nonempty.

**Remark.** Since $D$ is invariant under strictly conformal as well as under anticonformal mappings of $B$, its infimum $e(\Gamma)$ in $\mathcal{C}(\Gamma)$ is independent of the chosen orientation of $\Gamma$. The same holds for the generalized Dirichlet integral (34) in Section 4.5, whereas the infimum of the integral (36) in 4.5 may depend on the orientation of $\Gamma$, and the same holds for "Cartan functionals", as considered in Section 4.13. Thus for conformally invariant integrals in the general sense, such as $D$, we may neglect the orientation of the boundary contour $\Gamma$; both orientations lead to the same solutions of $\mathcal{P}(\Gamma)$; in the noninvariant cases we might obtain different solutions for opposite orientations.

**Convention.** *It goes without saying that $\mathcal{C}(\Gamma)$ always is defined with respect to a fixed orientation of $\Gamma$.*

## 4.3 Existence Proof, Part I: Solution of the Variational Problem

Let $\Gamma$ be a closed oriented Jordan curve in $\mathbb{R}^3$, and let $\mathcal{C}(\Gamma)$ be the class of admissible surfaces bounded by $\Gamma$ which we have defined in Section 4.2. The aim of this section is to find a solution of the minimum problem

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \text{in the class } \mathcal{C}(\Gamma).$$

We are going to prove the following

**Theorem 1.** *If $\mathcal{C}(\Gamma)$ is nonempty, then the minimum problem $\mathcal{P}(\Gamma)$ has at least one solution which is continuous on $\overline{B}$ and harmonic in $B$. In particular, $\mathcal{P}(\Gamma)$ has such a solution for every rectifiable curve $\Gamma$.*

*Proof.* As we have seen in Section 4.2, the class $\mathcal{C}(\Gamma)$ is nonempty for every closed Jordan curve of finite length. Hence it suffices to prove the first part of the assertion. Recall that we only have to find a solution of

$$\mathcal{P}^*(\Gamma): \quad D(X) \to \min \quad \text{in the class } \mathcal{C}^*(\Gamma),$$

where $\mathcal{C}^*(\Gamma)$ denotes the set of surfaces $X \in \mathcal{C}(\Gamma)$ satisfying a fixed three-point condition

$$X(w_k) = Q_k, \quad k = 1, 2, 3. \tag{1}$$

Here, $w_1, w_2, w_3$ are three different points on $C = \partial B$, and $Q_1, Q_2, Q_3$ denote three different points on $\Gamma$.

Choose a sequence $\{X_n\}$ of mappings $X_n \in \mathcal{C}^*(\Gamma)$ such that

$$\lim_{n\to\infty} D(X_n) = e^*(\Gamma) \tag{2}$$

holds. We can assume without loss of generality that $X_n$ is a surface of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which satisfies

$$\Delta X_n = 0 \quad \text{in } B,$$

$n = 1, 2, 3, \ldots$. (Otherwise we replace $X_n$ by the solution $Z_n$ of the boundary value problem

$$\begin{aligned} \Delta Z_n &= 0 \quad \text{in } B, \\ Z_n &= X_n \quad \text{on } C \end{aligned}$$

which is continuous on $\bar{B}$ and of class $C^2(B, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$. It is well known that this problem has exactly one solution. This solution minimizes $D(X)$ among all $X \in H_2^1(B, \mathbb{R}^3)$ with $X - X_n \in \mathring{H}_2^1(B, \mathbb{R}^3)$. Consequently, $D(Z_n) \le D(X_n)$, and by construction we have $Z_n \in \mathcal{C}^*(\Gamma)$ whence $e^*(\Gamma) \le D(Z_n)$. Thus we obtain

$$e^*(\Gamma) \le D(Z_n) \le D(X_n) \to e^*(\Gamma),$$

and therefore

$$\lim_{n\to\infty} D(Z_n) = e^*(\Gamma).$$

Hence we have found a minimizing sequence $\{Z_n\}$ for $\mathcal{P}^*(\Gamma)$ consisting of harmonic mappings $Z_n$ which are continuous on $\bar{B}$.)

We now claim that the boundary values $X_{n|C}$ of the terms of any minimizing sequence $\{X_n\}$ for $\mathcal{P}^*(\Gamma)$ are equicontinuous on $C$. The key to this crucial result is the so-called Courant–Lebesgue lemma. We defer its proof to the next section so as not to interrupt our reasoning.

**Courant–Lebesgue lemma.** *Let $X$ be of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$ and suppose that*

$$D(X) \le M \tag{3}$$

*for some $M$ with $0 \le M < \infty$. Then, for every $z_0 \in C$ and for each $\delta \in (0,1)$, there exists a number $\rho \in (\delta, \sqrt{\delta})$ such that the distance of the images $X(z), X(z')$ of the two intersection points $z$ and $z'$ of $C$ with the circle $\partial B_\rho(z_0)$ can be estimated by*

$$|X(z) - X(z')| \le \left\{ \frac{4M\pi}{\log 1/\delta} \right\}^{1/2}. \tag{4}$$

This lemma will be applied as follows: Since $\Gamma$ is the topological image of $C$, there exists, for every $\varepsilon > 0$, a number $\lambda(\varepsilon) > 0$ with the following property:

*Any pair of points $P, Q \in \Gamma$ with*

$$0 < |P - Q| < \lambda(\varepsilon) \tag{5}$$

*decomposes $\Gamma$ into two arcs $\Gamma_1(P,Q)$ and $\Gamma_2(P,Q)$ such that*

$$\operatorname{diam} \Gamma_1(P,Q) < \varepsilon \tag{6}$$

*holds*. Hence, if $0 < \varepsilon < \varepsilon_0 := \min_{j \ne k} |Q_j - Q_k|$, then $\Gamma_1(P,Q)$ can contain at most one of the points $Q_j$ appearing in the three-point condition (1).

Let now $X$ be an arbitrary mapping in $\mathcal{C}^*(\Gamma)$ that fulfills the assumptions of the Courant–Lebesgue lemma, and let $\delta_0 \in (0,1)$ be a fixed number with

$$2\sqrt{\delta_0} < \min_{j \ne k} |w_j - w_k| \tag{7}$$

where $w_1, w_2, w_3$ appear in (1).

For an arbitrary $\varepsilon \in (0, \varepsilon_0)$, we choose some number $\delta = \delta(\varepsilon) > 0$ such that

$$\left\{ \frac{4\pi M}{\log 1/\delta} \right\}^{1/2} < \lambda(\varepsilon) \tag{8}$$

and

$$\delta < \delta_0. \tag{9}$$

Consider an arbitrary point $z_0$ on $C$, and let $\rho \in (\delta, \sqrt{\delta})$ be some number such that the images $P := X(z), Q := X(z')$ of the two intersection points $z, z'$ of $C$ and $\partial B_\rho(z_0)$ satisfy

$$|P - Q| \le \left\{ \frac{4M\pi}{\log 1/\delta} \right\}^{1/2}.$$

Then we infer from (8) that $|P - Q| < \lambda(\varepsilon)$, whence

$$\operatorname{diam} \mathit{\Gamma}_1(P, Q) < \varepsilon$$

holds on account of (6). Because of $\varepsilon < \varepsilon_0$ the arc $\mathit{\Gamma}_1(P, Q)$ contains at most one of the points $Q_j$. On the other hand, it follows from $X \in \mathcal{C}^*(\mathit{\Gamma})$ and from (1), (7), (9) that $X(C \cap \overline{B_\rho(z_0)})$ contains at most one of the points $Q_j$ and must therefore coincide with the arc $\mathit{\Gamma}_1(P, Q)$:

$$\mathit{\Gamma}_1(P, Q) = X(C \cap \overline{B_\rho(z_0)}).$$

Consequently we have

$$|X(w) - X(w')| < \varepsilon \quad \text{for all } w, w' \in C \cap B_\rho(z_0).$$

This implies

$$(10) \qquad |X(w) - X(w')| < \varepsilon \quad \text{for all } w, w' \in C \text{ with } |w - w'| < \delta.$$

Consider now the minimizing sequence $\{X_n\}$. By (2), there is some number $M > 0$ such that

$$D(X_n) \le M$$

holds for all $n \in \mathbb{N}$. Thus we can apply (10) to $X = X_n, n = 1, 2, \ldots,$ and we conclude that the functions $X_n|_C$ are equicontinuous. Moreover, we infer from $X_n(C) = \mathit{\Gamma}$ that the functions $X_n|_C$ are uniformly bounded. Hence, by the theorem of Arzelà–Ascoli, we can assume that the $X_n|_C$ tend to some mapping $\varphi \in C^0(C, \mathbb{R}^3)$ as $n \to \infty$, uniformly on $C$, and that $\varphi$ is a weakly monotonic mapping of $C$ onto $\mathit{\Gamma}$. Since the functions $X_n$ are continuous on $\bar{B}$ and harmonic in $B$, it follows that $X_n$ tends uniformly on $\bar{B}$ to some function $X$, which is continuous on $\bar{B}$, harmonic in $B$, satisfies (1), and has the boundary values $\varphi$. Consequently, $X$ is of class $\mathcal{C}^*(\mathit{\Gamma})$, and therefore

$$e^*(\mathit{\Gamma}) \le D(X).$$

Moreover, a classical result for harmonic functions implies that grad $X_n$ tends to grad $X$ as $n \to \infty$, uniformly on every $B' \subset\subset B$, whence

$$\lim_{n\to\infty} D_{B'}(X_n) = D_{B'}(X)$$

and therefore

$$\liminf_{n\to\infty} D_B(X_n) \ge D_{B'}(X) \quad \text{if } B' \subset\subset B.$$

Thus we finally obtain

$$e^*(\mathit{\Gamma}) = \lim_{n\to\infty} D(X_n) \ge D(X) \ge e^*(\mathit{\Gamma}),$$

or

$$D(X) = e^*(\mathit{\Gamma}).$$

Therefore $X \in \mathcal{C}^*(\mathit{\Gamma})$ is a minimizer of the Dirichlet integral $D(X)$ within the class $\mathcal{C}(\mathit{\Gamma})$. □

In the previous theorem we have obtained at least one harmonic minimizer of $D(X)$ in the class $\mathcal{C}(\Gamma)$. Now we want to show that every solution of $\mathcal{P}(\Gamma)$ is a harmonic mapping. In fact, we have

**Theorem 2.** *Every minimizer $X$ of the Dirichlet integral within the class $\mathcal{C}(\Gamma)$ is continuous in $\bar{B}$ and harmonic in $B$.*

*Proof.* Let $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ be an arbitrary test function of class $C_c^\infty(B, \mathbb{R}^3)$. Then we have $X + \varepsilon\varphi \in \mathcal{C}(\Gamma)$ for every $\varepsilon \in \mathbb{R}$. On account of the minimum property of $X$, the quadratic polynomial

$$f(\varepsilon) := D(X + \varepsilon\varphi) = D(X) + 2\varepsilon D(X, \varphi) + \varepsilon^2 D(\varphi), \quad \varepsilon \in \mathbb{R},$$

has an absolute minimum at $\varepsilon = 0$, whence $f'(0) = 0$, or

$$D(X, \varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(B, \mathbb{R}^3). \tag{11}$$

By a classical result for harmonic functions (Weyl's lemma), we obtain from (11) that $X$ is harmonic in $B$. Since $X \in H_2^1(B, \mathbb{R}^3)$ and $X|_C \in C^0(C, \mathbb{R}^3)$, it also follows that $X \in C^0(\bar{B}, \mathbb{R}^3)$. □

By the same reasoning that led to Theorem 1, we also obtain the following results (cf. Section 4.2, Lemma 1):

**Theorem 3.** *Let $\{\Gamma_n\}$ be a sequence of closed (oriented) Jordan curves in $\mathbb{R}^3$ which converge in the sense of Fréchet to some closed (oriented) Jordan curve $\Gamma$ (notation: $\Gamma_n \to \Gamma$ as $n \to \infty$), and let $\{X_n\}$ be a sequence of mappings $X_n \in \mathcal{C}(\Gamma_n)$ with uniformly bounded Dirichlet integral, i.e.,*

$$D(X_n) \le M, \quad n \in \mathbb{N}. \tag{12}$$

*Then their boundary values $\varphi_n := X_n|_C$ are equicontinuous if they satisfy a uniform three-point condition*

$$\varphi_n(w_j) = Q_j^{(n)}, \quad j = 1, 2, 3, \tag{13}$$

*with some points $w_j \in C$ and $Q_j^{(n)} \in \Gamma_n$, $j = 1, 2, 3$, such that $\lim_{n\to\infty} Q_j^{(n)} = Q_j$ holds, where $Q_1, Q_2, Q_3$ denote three different points on the limit curve $\Gamma$.*

*If, moreover, the mappings $X_n$ are continuous on $\bar{B}$ and harmonic in $B$, then we can extract a subsequence $\{X_{n_p}\}$ that converges uniformly on $\bar{B}$ to some mapping $X \in \mathcal{C}(\Gamma)$ which is continuous on $\bar{B}$ and harmonic in $B$.*

**Remark.** For minimal surfaces $X_n$, the isoperimetric inequality (cf. Section 4.14) implies that

$$D(X_n) \le \frac{1}{4\pi} L^2(\Gamma_n) \tag{14}$$

holds, where $L(\Gamma_n)$ denotes the lengths of the curves $\Gamma_n$. Hence condition (12) is satisfied by every sequence of minimal surfaces $X_n \in \mathcal{C}(\Gamma_n)$, $n = 1, 2, \dots$, spanned by closed Jordan curves $\Gamma_n$ of uniformly bounded lengths,

$$L(\Gamma_n) \le l \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

**Theorem 4.** *Let $\Gamma, \Gamma_1, \Gamma_2, \ldots$ be closed (oriented) Jordan curves in $\mathbb{R}^3$ with $\Gamma_n \to \Gamma$ as $n \to \infty$ (Fréchet convergence) and $\lim_{n\to\infty} e(\Gamma_n) = e(\Gamma)$. Furthermore, let $X_n \in \mathcal{C}(\Gamma_n)$ be a sequence of solutions for $\mathcal{P}(\Gamma_n)$ whose boundary values $\varphi_n = X_n|_C$ satisfy a uniform three-point condition such as in Theorem 3. Then we can extract a subsequence $\{X_{n_p}\}$ which converges uniformly on $\bar{B}$ to some solution $X$ of $\mathcal{P}(\Gamma)$ as $p \to \infty$, and*

$$\lim_{n\to\infty} D(X_n) = D(X). \tag{16}$$

## 4.4 The Courant–Lebesgue Lemma

We now want to supply a proof for the Courant–Lebesgue lemma that was used in the previous section. In fact, this lemma will be an immediate consequence of the next proposition.

Let us introduce the following notations:

$$B := \{w \colon |w| < 1\}, \quad C := \partial B,$$

$$S_r(z_0) := B \cap B_r(z_0), \quad C_r(z_0) := \bar{B} \cap \partial B_r(z_0).$$

If $z_0 \in C$, then we can write

$$C_r(z_0) = \{z_0 + re^{i\theta} \colon \theta_1(r) \le \theta \le \theta_2(r)\}$$

with

$$0 < \theta_2(r) - \theta_1(r) < \pi.$$

**Proposition 1.** *Suppose that $X$ is of class $C^0(\bar{B}, \mathbb{R}^n) \cap C^1(B, \mathbb{R}^n)$, $n \in \mathbb{N}$, and satisfies $D(X) < \infty$. Let $z_0$ be any point on $C$, and set $Z(r,\theta) := X(z_0 + re^{i\theta})$ where $r, \theta$ denote polar coordinates about $z_0$. Then, for every $\delta \in (0, R^2)$, $0 < R < 1$, there is a number $\rho \in (\delta, \sqrt{\delta})$ such that, for every pair $\theta, \theta'$ with $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, we obtain the estimate*

$$\int_\theta^{\theta'} |Z_\theta(\rho, \theta)|\, d\theta \le \eta(\delta, R)|\theta - \theta'|^{1/2} \tag{1}$$

*with*

$$\eta(\delta, R) := \left\{ \frac{2}{\log(1/\delta)} \int_{S_R(z_0)} |\nabla X|^2 \, du\, dv \right\}^{1/2}, \tag{2}$$

*and in particular*

$$|Z(\rho, \theta) - Z(\rho, \theta')| \le \eta(\delta, R)|\theta - \theta'|^{1/2}. \tag{3}$$

**Remark.** The assumption $z_0 \in C$ is not essential as we shall see from the proof. We shall leave it to the reader to formulate a corresponding result in other situations.

We begin the proof of Proposition 1 by verifying the following

**Lemma 1.** *Let* $X$ *satisfy the assumptions of Proposition* 1, *and set*

$$Z(r, \theta) := X(z_0 + re^{i\theta}), \quad z_0 \in C,$$

*and*

$$p(r) := \int_{\theta_1(r)}^{\theta_2(r)} |Z_\theta(r, \theta)|^2 \, d\theta. \tag{4}$$

*Moreover, let* $\mathcal{I}$ *be a measurable subset of* $(0, 1)$, *and suppose that both*

$$0 < \int_{\mathcal{I}} \frac{dr}{r} < \infty \quad \textit{and} \quad \int_{\mathcal{I}} \frac{p(r)}{r} \, dr \le M < \infty \tag{5}$$

*are satisfied. Then the set* $\mathcal{I}_M := \{\rho \in \mathcal{I} \colon p(\rho) \int_{\mathcal{I}} \frac{dr}{r} \le M\}$ *has a positive* 1*-dimensional Lebesgue measure,*

$$\mathcal{L}^1(\mathcal{I}_M) > 0, \tag{6}$$

*and for every* $\rho \in \mathcal{I}_M$ *and all* $\theta, \theta'$ *with* $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$ *we obtain the inequality*

$$\int_{\theta}^{\theta'} |Z_\theta(\rho, \theta)| \, d\theta \le \left\{ M \Big/ \int_{\mathcal{I}} \frac{dr}{r} \right\}^{1/2} |\theta - \theta'|^{1/2}. \tag{7}$$

*Proof.* (i) If $\mathcal{L}^1(\mathcal{I}_M) = 0$, then we would obtain

$$p(\rho) > M \Big/ \int_{\mathcal{I}} \frac{dr}{r} \quad \text{for almost all } \rho \in \mathcal{I}.$$

Multiplying by $1/\rho$, and integrating over $\mathcal{I}$ with respect to $\rho$, we would arrive at the inequality

$$\int_{\mathcal{I}} \frac{p(\rho)}{\rho} \, d\rho > M$$

which is a contradiction to (5). Hence we see that $\mathcal{L}^1(\mathcal{I}_M) > 0$.

(ii) Let $\rho \in \mathcal{I}_M$ and $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$. Then it follows that

$$\begin{aligned} \int_{\theta}^{\theta'} |Z_\theta(\rho, \theta)| \, d\theta &\le \left\{ \int_{\theta}^{\theta'} |Z_\theta(\rho, \theta)|^2 \, d\theta \right\}^{1/2} |\theta - \theta'|^{1/2} \\ &\le \left\{ M \Big/ \int_{\mathcal{I}} \frac{dr}{r} \right\}^{1/2} |\theta - \theta'|^{1/2}. \end{aligned}$$

□

*Proof of Proposition 1.* Let $p(r)$ be the function defined by (4). Then we obtain

$$\begin{aligned}\int_0^r \frac{p(\rho)}{\rho}\,d\rho &\le \int_0^r \int_{\theta_1(\rho)}^{\theta_2(\rho)} \left\{|Z_\rho(\rho,\theta)|^2 + \frac{1}{\rho^2}|Z_\theta(\rho,\theta)|^2\right\}\rho\,d\theta\,d\rho \\ &= 2D(X, S_r(z_0)).\end{aligned}$$

For $M := 2D(X, S_R(z_0))$ and $\mathcal{I} = (\delta, \sqrt{\delta})$, we infer from Lemma 1 that there is some $\rho$ with $\delta < \rho < \sqrt{\delta} \le R$ such that

$$\begin{aligned}\int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta &\le \left\{M \Big/ \int_\delta^{\sqrt{\delta}} \frac{dr}{r}\right\}^{1/2} |\theta - \theta'|^{1/2} \\ &= \left\{4D(X, S_R(z_0)) \frac{1}{\log 1/\delta}\right\}^{1/2} |\theta - \theta'|^{1/2} \\ &= \eta(\delta, R)|\theta - \theta'|^{1/2},\end{aligned}$$

and from

$$Z(\rho,\theta') - Z(\rho,\theta) = \int_\theta^{\theta'} Z_\theta(\rho,\theta)\,d\theta$$

we infer that

$$|Z(\rho,\theta') - Z(\rho,\theta)| \le \int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta \le \eta(\theta, R)|\theta - \theta'|^{1/2}. \qquad \square$$

There is a generalization of Proposition 1 which holds for functions $X(w)$ of class $H_2^1(B, \mathbb{R}^n)$; see e.g. Morrey [8], Theorem 3.1.2(g). Recall the following property of such functions:

If $Z(r,\theta) := X(z_0 + re^{i\theta})$ is the transformation of $X$ into polar coordinates $r, \theta$ about some point $z_0 \in C$, then there is representation of $Z$, again denoted by $Z$, such that $Z(r,\theta)$ is absolutely continuous with respect to $\theta$ for almost all $r \in (0,2)$, and that $Z(r,\theta)$ is absolutely continuous with respect to $r \in (r_0, 2)$, for any $r_0 > 0$ and for almost all $\theta$. Moreover, the partial derivatives $Z_r, Z_\theta$ of $Z$ with respect to $r$ and $\theta$ coincide almost everywhere on $\{(r,\theta)\colon 0 < r < 2, \theta_1(r) < \theta < \theta_2(r)\}$ with the corresponding distributional derivatives. Consequently, the function

$$p(r) = \int_{\theta_1(r)}^{\theta_2(r)} |Z_\theta(r,\theta)|^2\,d\theta$$

is defined for almost all $r \in (0,2)$. Moreover, $p(r)$ is measurable on $(0,2)$, and $\int_0^2 \frac{p(r)}{r}\,dr < \infty$. Instead of Lemma 1, we now obtain

**Lemma 2.** *Let* $\mathcal{I}$ *be a measurable subset of* $(0,1)$ *such that*

$$0 < \int_{\mathcal{I}} \frac{dr}{r} < \infty \quad \text{and} \quad \int_{\mathcal{I}} \frac{p(r)}{r}\,dr \le M < \infty.$$

*Then the set* $\mathcal{I}_M := \{\rho \in \mathcal{I}\colon p(\rho)\int_{\mathcal{I}} \frac{dr}{r} \le M\}$ *satisfies* $\mathcal{L}^1(\mathcal{I}_M) > 0$, *and for almost all* $\rho \in \mathcal{I}_M$ *and all* $\theta, \theta'$ *with* $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$ *we obtain*

$$\int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta \le \left\{ M \Big/ \int_{\mathcal{J}} \frac{dr}{r} \right\}^{1/2} |\theta-\theta'|^{1/2}.$$

Consequently we arrive at the following analogue of Proposition 1:

**Proposition 2.** *Every $X \in H_2^1(B,\mathbb{R}^n)$ possesses a representative $Z(r,\theta)$ of $X(z_0+re^{i\theta}), z_0 \in C$, which is absolutely continuous with respect to $\theta$ for a.a. $r \in (0,2)$ and which has the following property*:

*For every $\delta \in (0,R^2), 0 < R < 1$, there is a measurable subset $\mathcal{J}$ of the interval $(\delta,\sqrt{\delta})$ with $\mathcal{L}^1(\mathcal{J}) > 0$ such that*

$$|Z(\rho,\theta) - Z(\rho,\theta')| \le \int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta \le \eta(\delta,R)|\theta-\theta'|^{1/2}$$

*holds for a.a $\rho \in \mathcal{J}$ and $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, where*

$$\eta(\delta,R) := \left\{ \frac{4}{\log 1/\delta} D(X, S_R(z_0)) \right\}^{1/2}.$$

This and other versions of the Courant–Lebesgue lemma are quite useful for many purposes, in particular for the treatment of free boundary value problems.

## 4.5 Existence Proof, Part II: Conformality of Minimizers of the Dirichlet Integral

In this section, we want to prove that the solutions $X(u,v)$ of the minimum problem $\mathcal{P}(\Gamma)$ satisfy the conformality relations

$$(1) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0.$$

To this end, we exploit the minimum property of $X$ by changing the independent variables $u,v$ in direction of arbitrarily prescribed vector fields $\lambda(u,v) = (\mu(u,v),\nu(u,v))$ on $\bar{B}$. Such variations of $X$ will be called *inner variations*.

In order to make this variational technique precise, we start with an arbitrary vector field $\lambda = (\mu,\nu)$ on $\bar{B}$ which is of class $C^1(\bar{B},\mathbb{R}^2)$. Without restriction we can assume that $\lambda$ is defined on all of $\mathbb{R}^2$ and is of class $C^1(\mathbb{R}^2,\mathbb{R}^2)$. With $\lambda$ we associate some 1-parameter family of mappings $\tau_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies

$$(2) \qquad \tau_\varepsilon(w) = \tau(w,\varepsilon) = w - \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

$w = (u,v)$. For instance, we could take $\tau_\varepsilon(w) = w - \varepsilon\lambda(w)$. The function $\tau(w,\varepsilon)$ is of class $C^1$ on $\mathbb{R}^2 \times \mathbb{R}$. Choose some open set $B_0$ with $B \subset\subset B_0$. Then it is easy to see that $\tau_\varepsilon : B_0 \to \tau_\varepsilon(B_0)$ furnishes an orientation preserving

$C^1$-diffeomorphism of $B_0$ onto its image $\tau_\varepsilon(B_0)$ provided that $|\varepsilon| < \varepsilon_0$, for some sufficiently small $\varepsilon_0 > 0$, because $\tau_\varepsilon(w)$ is just a perturbation of the identity map $\tau_0(w) = w$.

Clearly the inverse mappings $\sigma_\varepsilon = \tau_\varepsilon^{-1}$ exist on a common domain of definition $\Omega$ satisfying $B_\varepsilon^* \subset\subset \Omega \subset\subset B_0$, where we have set $B_\varepsilon^* := \tau_\varepsilon(B)$. We write $\omega = \tau_\varepsilon(w) = \tau(w, \varepsilon)$ and $w = \sigma_\varepsilon(\omega) = \sigma(\omega, \varepsilon)$. The function $\sigma(\omega, \varepsilon)$ is of class $C^1$ on $\Omega \times (-\varepsilon_0, \varepsilon_0)$ and satisfies both

$$\sigma(\omega, \varepsilon) = \omega + \varepsilon\lambda(\omega) + o(\varepsilon) \tag{3}$$

and

$$\tau(\sigma(\omega, \varepsilon), \varepsilon) = \omega \tag{4}$$

for all $(\omega, \varepsilon) \in \Omega \times (-\varepsilon_0, \varepsilon_0)$.

Restricting the region of definition of $\tau_\varepsilon = \tau(\cdot, \varepsilon)$ and $\sigma_\varepsilon = \sigma(\cdot, \varepsilon)$ to $\bar{B}$ and $\bar{B}_\varepsilon^*$, respectively, the mapping $\tau_\varepsilon$ is a diffeomorphism of $\bar{B}$ onto $\bar{B}_\varepsilon^*$, with the inverse $\sigma_\varepsilon$, and we have in particular

$$B_0^* = B, \quad \sigma(w, 0) = w, \quad \frac{\partial}{\partial\varepsilon}\sigma(w, \varepsilon)\bigg|_{\varepsilon=0} = \lambda(w) \quad \text{for } w \in \bar{B}. \tag{5}$$

Moreover, the Jacobian of the mapping $\tau_\varepsilon(w)$ is given by

$$\det D\tau_\varepsilon = \begin{vmatrix} 1 - \varepsilon\mu_u + o(\varepsilon) & -\varepsilon\mu_v + o(\varepsilon) \\ -\varepsilon\nu_u + o(\varepsilon) & 1 - \varepsilon\nu_v + o(\varepsilon) \end{vmatrix} = 1 - \varepsilon(\mu_u + \nu_v) + o(\varepsilon)$$

whence

$$\frac{\partial}{\partial\varepsilon} \det D\tau_\varepsilon\bigg|_{\varepsilon=0} = -(\mu_u + \nu_v) = -\operatorname{div} \lambda. \tag{6}$$

Consider now an arbitrary function $X \in C^1(\bar{B}, \mathbb{R}^3)$. We embed $X$ into the family of functions

$$Z_\varepsilon := X \circ \sigma_\varepsilon, \quad \sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}, \tag{7}$$

which are obtained from $X$ by the inner variations $\sigma_\varepsilon$. Let us compute the rate of change of the Dirichlet integral $D(Z_\varepsilon, B_\varepsilon^*)$ at $\varepsilon = 0$. Since we later may want to carry out the same computation for other variational integrals $\mathcal{F}(X)$ of the type

$$\mathcal{F}_B(X) = \mathcal{F}(X, B) := \int_B F(X, X_u, X_v)\, du\, dv \tag{8}$$

with a $C^1$-Lagrangian $F(x, p, q)$, we shall compute the derivative $f'(0)$ of the function

$$f(\varepsilon) := \mathcal{F}(Z_\varepsilon, B^*_\varepsilon) = \int_{B^*_\varepsilon} F\left(Z_\varepsilon, \frac{\partial}{\partial\alpha} Z_\varepsilon, \frac{\partial}{\partial\beta} Z_\varepsilon\right) d\alpha\, d\beta \tag{9}$$

where we have set $w = (u, v), \omega = (\alpha, \beta)$. By applying the transformation theorem to this integral and to the mapping $\tau_\varepsilon : \bar{B} \to \overline{B^*_\varepsilon}$, we obtain

$$f(\varepsilon) = \int_B F\left(X, \left(\frac{\partial}{\partial\alpha} Z_\varepsilon\right) \circ \tau_\varepsilon, \left(\frac{\partial}{\partial\beta} Z_\varepsilon\right) \circ \tau_\varepsilon\right) |\det D\tau_\varepsilon|\, du\, dv. \tag{10}$$

Set

$$\sigma_\varepsilon(\omega) = \sigma_\varepsilon(\alpha, \beta) = (\sigma^1_\varepsilon(\alpha, \beta), \sigma^2_\varepsilon(\alpha, \beta)).$$

From

$$Z_\varepsilon(\alpha, \beta) = X(\sigma^1_\varepsilon(\alpha, \beta), \sigma^2_\varepsilon(\alpha, \beta))$$

we infer that

$$\begin{aligned}
\frac{\partial}{\partial\alpha} Z_\varepsilon(\alpha, \beta) &= X_u(\sigma_\varepsilon(\omega)) \frac{\partial\sigma^1_\varepsilon}{\partial\alpha}(\omega) + X_v(\sigma_\varepsilon(\omega)) \frac{\partial\sigma^2_\varepsilon}{\partial\alpha}(\omega), \\
\frac{\partial}{\partial\beta} Z_\varepsilon(\alpha, \beta) &= X_u(\sigma_\varepsilon(\omega)) \frac{\partial\sigma^1_\varepsilon}{\partial\beta}(\omega) + X_v(\sigma_\varepsilon(\omega)) \frac{\partial\sigma^2_\varepsilon}{\partial\beta}(\omega).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\partial}{\partial\alpha} Z_\varepsilon\right)(\tau_\varepsilon(w)) &= X_u(w) \frac{\partial\sigma^1_\varepsilon}{\partial\alpha}(\tau_\varepsilon(w)) + X_v(w) \frac{\partial\sigma^2_\varepsilon}{\partial\alpha}(\tau_\varepsilon(w)), \\
\left(\frac{\partial}{\partial\beta} Z_\varepsilon\right)(\tau_\varepsilon(w)) &= X_u(w) \frac{\partial\sigma^1_\varepsilon}{\partial\beta}(\tau_\varepsilon(w)) + X_v(w) \frac{\partial\sigma^2_\varepsilon}{\partial\beta}(\tau_\varepsilon(w)).
\end{aligned} \tag{11}$$

Moreover, we have

$$\left.\begin{aligned}
\sigma^1_\varepsilon(\alpha, \beta) &= \alpha + \varepsilon\mu(\alpha, \beta) + o(\varepsilon) \\
\sigma^2_\varepsilon(\alpha, \beta) &= \beta + \varepsilon\nu(\alpha, \beta) + o(\varepsilon)
\end{aligned}\right. \quad \text{as } \varepsilon \to 0,$$

and therefore

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\sigma^1_\varepsilon(\alpha, \beta) &= 1 + \varepsilon \frac{\partial}{\partial\alpha}\mu(\alpha, \beta) + o(\varepsilon), \\
\frac{\partial}{\partial\beta}\sigma^1_\varepsilon(\alpha, \beta) &= \varepsilon \frac{\partial}{\partial\beta}\mu(\alpha, \beta) + o(\varepsilon), \\
\frac{\partial}{\partial\alpha}\sigma^2_\varepsilon(\alpha, \beta) &= \varepsilon \frac{\partial}{\partial\alpha}\nu(\alpha, \beta) + o(\varepsilon), \\
\frac{\partial}{\partial\beta}\sigma^2_\varepsilon(\alpha, \beta) &= 1 + \varepsilon \frac{\partial}{\partial\beta}\nu(\alpha, \beta) + o(\varepsilon).
\end{aligned} \tag{12}$$

Replacing $\alpha$ and $\beta$ by

$$\alpha = u - \varepsilon\mu(u, v) + o(\varepsilon), \quad \beta = v - \varepsilon\nu(u, v) + o(\varepsilon),$$

differentiating (12) with respect to $\varepsilon$, and setting $\varepsilon = 0$, we arrive at

$$\frac{\partial}{\partial\varepsilon}\frac{\partial\sigma_\varepsilon^1}{\partial\alpha}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \mu_u(u,v), \quad \frac{\partial}{\partial\varepsilon}\frac{\partial\sigma_\varepsilon^1}{\partial\beta}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \mu_v(u,v),$$
$$\frac{\partial}{\partial\varepsilon}\frac{\partial\sigma_\varepsilon^2}{\partial\alpha}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \nu_u(u,v), \quad \frac{\partial}{\partial\varepsilon}\frac{\partial\sigma_\varepsilon^2}{\partial\beta}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \nu_v(u,v).$$

On account of (11), we then conclude that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial}{\partial\alpha}Z_\varepsilon\right)(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} &= X_u(w)\mu_u(w) + X_v(w)\nu_u(w),\\ \frac{\partial}{\partial\varepsilon}\left(\frac{\partial}{\partial\beta}Z_\varepsilon\right)(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} &= X_u(w)\mu_v(w) + X_v(w)\nu_v(w).\end{aligned} \tag{13}$$

Combining formulas (6) and (10)–(13), we finally obtain

$$\begin{aligned} f'(0) = \int_B \big\{&\langle F_p(X,X_u,X_v), X_u\mu_u + X_v\nu_u\rangle \\ &+ \langle F_q(X,X_u,X_v), X_u\mu_v + X_v\nu_v\rangle \\ &- F(X,X_u,X_v)[\mu_u+\nu_v]\big\}\,du\,dv.\end{aligned} \tag{14}$$

Following Giaquinta and Hildebrandt [1], we denote $\partial\mathcal{F}_B(X,\lambda) := f'(0)$ as (first) *inner variation of the functional $\mathcal{F}_B$ at $X$ in direction of the vector field $\lambda = (\mu,\nu)$, that is,*

$$\begin{aligned}\partial\mathcal{F}_B(X,\lambda) := \int_B \{&\langle F_p, X_u\mu_u + X_v\nu_u\rangle + \langle F_q, X_u\mu_v + X_v\nu_v\rangle \\ &- F[\mu_u + \nu_v]\}\,du\,dv\end{aligned} \tag{15}$$

where the arguments of $F, F_p, F_q$ are to be taken as $X, X_u, X_v$.

Collecting the previous results, we obtain the following

**Proposition 1.** *If $\{\tau_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ is a $C^1$-family of $C^1$-diffeomorphisms $\tau_\varepsilon : \bar{B} \to \overline{B_\varepsilon^*}$ with the inverses $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$, such that $B_0^* = B$ holds and that $\sigma(w,\varepsilon) := \sigma_\varepsilon(w)$ satisfies*

$$\sigma(w,0) = w, \quad \frac{\partial\sigma}{\partial\varepsilon}(w,0) = \lambda(w), \quad \textit{and} \quad \lambda \in C^1(\bar{B},\mathbb{R}^2), \tag{16}$$

*then, for every $X \in C^1(\bar{B},\mathbb{R}^3)$, we obtain*

$$\frac{d}{d\varepsilon}\mathcal{F}(X\circ\sigma_\varepsilon, B_\varepsilon^*)\bigg|_{\varepsilon=0} = \partial\mathcal{F}_B(X,\lambda) \tag{17}$$

*where $\partial\mathcal{F}_B(X,\lambda)$ is defined by (15).*

*Moreover, given any vector field $\lambda \in C^1(\bar{B},\mathbb{R}^2)$, we can find a 1-parameter family of diffeomorphisms $\sigma_\varepsilon$ with the above stated properties and, in particular, with the property (16).*

Let us now consider two important cases:

**Examples 1.** For the Dirichlet integral

$$D(X) = \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2)\, du\, dv$$

and for any vector field $\lambda = (\mu, \nu) \in C^1(\bar{B}, \mathbb{R}^2)$, the first inner variation $\partial D(X, \lambda)$ is given by

$$2\partial D(X,\lambda) = \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv, \tag{18}$$

where $a$ and $b$ denote the functions

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v\rangle. \tag{19}$$

Note that the expression $\partial D(X, \lambda)$ is not only defined for surfaces $X \in C^1(\bar{B}, \mathbb{R}^3)$, but also for surfaces $X \in H_2^1(B, \mathbb{R}^3)$. In fact, a closer inspection of the previous computations yields the following result:

**Proposition 2.** *If $\mathcal{F}_B(X) = D(X)$, then the assertion of Proposition 1 holds for every $X \in H_2^1(B, \mathbb{R}^3)$, and the inner variation $\partial D(X, \lambda)$ of the Dirichlet integral at $X$ in direction of any $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$ is given by formulas (18) and (19).*

**Examples 2.** For the generalized Dirichlet integral

$$E(X) = \frac{1}{2}\int_B g_{jk}(X)\{X_u^j X_u^k + X_v^j X_v^k\}\, du\, dv \tag{20}$$

and for any $\lambda = (\mu, \nu) \in C^1(\bar{B}, \mathbb{R}^2)$, we obtain

$$2\partial E(X,\lambda) = \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv$$

with

$$\begin{aligned} a &:= g_{jk}(X) X_u^j X_u^k - g_{jk}(X) X_v^j X_v^k, \\ b &:= 2 g_{jk}(X) X_u^j X_v^k. \end{aligned} \tag{21}$$

Again we can prove a generalization of Proposition 1 which is similar to Proposition 2 and holds for $E$ and $X \in H_2^1(B, \mathbb{R}^3)$.

Now we are in a position to prove the main results of this section.

**Theorem 1.** *Let $X(u, v)$ be a surface of class $H_2^1(B, \mathbb{R}^3)$ such that*

$$\partial D(X, \lambda) = 0 \quad \textit{for all } \lambda \in C^1(\bar{B}, \mathbb{R}^2) \tag{22}$$

*is satisfied. Then $X$ fulfills the conformality relations (1) a.e. in $B$. Conversely, if (1) holds a.e. in $B$ for some $X \in H_2^1(B, \mathbb{R}^3)$, then the relation (22) is satisfied.*

*Proof.* Choose arbitrary functions $\rho, \sigma \in C_c^\infty(B)$ and determine functions $h, k \in C^\infty(\bar{B})$ with

$$\begin{aligned} \Delta h = \rho, \quad \Delta k &= \sigma \quad \text{on } B, \\ h = 0, \qquad k &= 0 \quad \text{on } \partial B. \end{aligned}$$

(This is possible on account of well known results of potential theory, cf. Gilbarg and Trudinger [1].)

Then the functions

$$\mu := h_u + k_v, \quad \nu := -h_v + k_u$$

are of class $C^\infty(\bar{B})$ and satisfy

$$\mu_u - \nu_v = \rho, \quad \mu_v + \nu_u = \sigma.$$

We now infer from assumption (22) in conjunction with (18) and (19) that

$$\int_B \{a\rho + b\sigma\}\, du\, dv = 0$$

holds for all $\rho, \sigma \in C_c^\infty(B)$. By the fundamental lemma of the calculus of variations we conclude that

$$a = 0 \quad \text{and} \quad b = 0$$

a.e. on $B$.

It is a trivial conclusion from (18) and (19) that, conversely, the conformality relations (1) imply (22). □

**Corollary 1.** *If $X \in H_2^1(B, \mathbb{R}^3)$ is harmonic in $B$ and satisfies (22), then $X$ is a minimal surface.*

**Theorem 2.** *Every solution $X$ of the variational problem*

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \textit{in the class } \mathcal{C}(\Gamma)$$

*is of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and satisfies*

$$\Delta X = 0,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

*in $B$, that is, $X$ is a minimal surface.*

*Proof.* By virtue of Section 4.3, Theorem 2, we only have to verify the conformality relations. Let $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$ be a family of inner variations as described in Proposition 1, and set $Z_\varepsilon := X \circ \sigma_\varepsilon$, where $X$ is a minimizer of $D_B(\cdot)$ in the class $\mathcal{C}(\Gamma)$. Clearly we have $Z_\varepsilon \in H_2^1(B_\varepsilon^*, \mathbb{R}^3)$. Since $\bar{B}$ and $\overline{B_\varepsilon^*}$ are diffeomorphic, $|\varepsilon| < \varepsilon_0$, there is a conformal mapping $\kappa_\varepsilon : B \to B_\varepsilon^*$ of $B$ onto

$B_\varepsilon^*$, by virtue of Riemann's mapping theorem. Moreover, a classical result in function theory yields that $\kappa_\varepsilon$ can be extended to a homeomorphism of $\bar{B}$ onto $\overline{B_\varepsilon^*}$ since $\partial B_\varepsilon^*$ is a Jordan curve. It follows that $Y_\varepsilon := Z_\varepsilon \circ \kappa_\varepsilon$ is of class $\mathcal{C}(\Gamma)$, whence

$$D(X, B) \le D(Y_\varepsilon, B) \quad \text{for } |\varepsilon| < \varepsilon_0, \tag{23}$$

because of the minimum property of $X$.

A straightforward computation shows that the Dirichlet integral is invariant with respect to conformal mappings. Therefore we have

$$D(Y_\varepsilon, B) = D(Z_\varepsilon \circ \kappa_\varepsilon, B) = D(Z_\varepsilon, B_\varepsilon^*),$$

and in conjunction with (23), we arrive at

$$D(X, B) \le D(Z_\varepsilon, B_\varepsilon^*), \quad |\varepsilon| < \varepsilon_0. \tag{24}$$

Set $f(\varepsilon) := D(Z_\varepsilon, B_\varepsilon^*)$ and note that $X = Z_0$. Then we can write (24) as

$$f(0) \le f(\varepsilon), \quad |\varepsilon| < \varepsilon_0,$$

and we obtain

$$0 = f'(0) = \partial D(X, \lambda)$$

for every $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$, on account of (9) and of Proposition 2. Then the conformality relations (1) are a consequence of Theorem 1. □

**Theorem 3.** *Every solution of $\mathcal{P}(\Gamma)$ and, more generally, every minimal surface of class $\mathcal{C}(\Gamma)$ yields a topological mapping of $C$ onto $\Gamma$.*

*Proof.* Let $X \in \mathcal{C}(\Gamma)$ be continuous in $\bar{B}$, harmonic in $B$, and suppose that (1) holds in $B$. It suffices to prove that $X$ provides a one-to-one mapping of $C$ onto $\Gamma$. Suppose that this were not true. Since $X|_C$ is weakly monotonic, we could then find an arc $C_0 = \{e^{i\theta} \colon \theta_1 < \theta < \theta_2\}$ which is mapped onto a single point $P \in \mathbb{R}^3$:

$$X(e^{i\theta}) = P \quad \text{for all } \theta \in (\theta_1, \theta_2). \tag{25}$$

By Schwarz's reflection principle we could extend $X(w)$ as a harmonic mapping across $C_0$. Differentiating (25) in the tangential direction, we would then obtain

$$\frac{\partial}{\partial \theta} X(e^{i\theta}) = 0$$

and, applying the conformality relations, it would follow that grad $X$ vanishes identically on $C_0$. This would imply grad $X \equiv 0$ on $B$, or $X(w) \equiv P$, a contradiction to $X \in \mathcal{C}(\Gamma)$. □

Combining Theorems 1–3 with the results of Section 4.3, we have found the following

**Main Theorem.** *Let $\Gamma$ be a closed curve in $\mathbb{R}^3$ and suppose that $\mathcal{C}(\Gamma)$ is nonempty. Then the minimum problem*

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \textit{in the class } \mathcal{C}(\Gamma)$$

*has at least one solution. Every solution $X$ of $\mathcal{P}(\Gamma)$ is continuous on $\bar{B}$, harmonic in $B$, satisfies the conformality relations (1) in $B$, and maps $C$ topologically onto $\Gamma$. In particular, every closed rectifiable curve $\Gamma$ in $\mathbb{R}^3$ spans at least one minimal surface of the type of the disk.*

Obviously the proof of Theorem 3 does not use the fact that $D(X) < \infty$, and so we have also

**Corollary 2.** *Let $X : B \to \mathbb{R}^3$ be a minimal surface which is continuous on $\overline{B}$ and maps $C = \partial B$ in a weakly monotonic way onto $\Gamma$ (as defined in 4.2, Definition 2). Then $X$ yields a homeomorphism from $C$ onto $\Gamma$.*

**Supplementary Remarks.**

1. For the proof of Theorem 2 we have used the Riemann mapping theorem. This can be avoided as we shall presently see. The advantage of this different proof is that the Main Theorem above can be used to provide an independent approach to Riemann's mapping theorem; see Section 4.11. Let us use the complex notation $w = u + iv$, and consider the variations

(26) $$\omega = \tau_\varepsilon(w) = we^{i\varepsilon\varphi(r,\theta)}$$

with $\varphi(r,\theta) = \psi(w), w = re^{i\theta}$, where $\psi(u,v)$ denotes an arbitrary function of class $C^1(\bar{B})$. Writing

(27) $$\tau_\varepsilon(w) = w - \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

we obtain

(28) $$\lambda(w) = \mu(u,v) + i\nu(u,v) = -iw\varphi(r,\theta).$$

Clearly, the mappings $\tau_\varepsilon$ define diffeomorphisms of $\bar{B}$ onto itself, provided that $|\varepsilon|$ is sufficiently small. Hence, if we set $\sigma_\varepsilon := \tau_\varepsilon^{-1}$ and $Z_\varepsilon := X \circ \sigma_\varepsilon$ for some solution $X$ of $\mathcal{P}(\Gamma)$, then the functions $Z_\varepsilon$ are of class $\mathcal{C}(\Gamma)$, and we obtain

$$D(Z_\varepsilon) \geq D(X) \quad \text{for } |\varepsilon| \ll 1.$$

As in the proof of Theorem 2, we now conclude that

(29) $$\int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv = 0$$

holds for

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v\rangle.$$

One easily verifies the Cauchy–Riemann equations

$$a_u = -b_v, \quad a_v = b_u$$

in $B$, using the relation $\Delta X = 0$ which holds for every solution $X$ of $\mathcal{P}(\Gamma)$. Consequently the mapping $\Phi : B \to \mathbb{C}$ defined by $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u + iv \in B$, and only this fact is used in the sequel. Suppose first that we had $X \in C^1(\bar{B}, \mathbb{R}^3)$. Then, by employing $\Delta a = \Delta b = 0$, we could transform the left-hand side of (29) into a line integral over $C = \partial B$, thus obtaining

$$\text{(30)} \qquad \operatorname{Im} \int_C \lambda(w)\Phi(w)\, dw = 0.$$

On account of (28), we then arrive at

$$\text{(31)} \qquad \operatorname{Im} \int_0^{2\pi} \varphi(1,\theta) w^2 \Phi(w)\, d\theta = 0, \quad w = e^{i\theta}.$$

Let $H(r,\theta) := \operatorname{Im} w^2\Phi(w)$, $w = re^{i\theta}$, and choose

$$\text{(32)} \qquad \varphi(r,\theta) := \zeta(r;\rho) K(r,\theta;\rho,\theta')$$

where $w' = \rho e^{i\theta'}$ is some fixed point in $B, \zeta(r;\rho)$ is a function of class $C^\infty(\mathbb{R})$ with respect to $r$ which satisfies $\zeta(r;\rho) = 0$ for $0 \le r \le \rho'$, and $\zeta(r;\rho) = 1$ for $\rho'' < r$, where the numbers $\rho', \rho''$ satisfy $\rho < \rho' < \rho'' < 1$, and $K$ denotes the Poisson kernel for the disk $B_r(0)$:

$$K(r,\theta;\rho,\theta') := \frac{1}{2\pi} \frac{r^2 - \rho^2}{r^2 - 2\rho r\cos(\theta - \theta') + \rho^2}.$$

Then we infer from (31) that

$$\int_0^{2\pi} K(1,\theta;\rho,\theta') H(1,\theta)\, d\theta = 0,$$

and Poisson's formula yields

$$H(\rho,\theta') = 0$$

for every $\rho \in (0,1)$ and $0 \le \theta' \le 2\pi$, or $\operatorname{Im} w^2\Phi(w) = 0$. Hence, $\operatorname{Re} w^2\Phi(w)$ is constant in $B$, whence

$$w^2\Phi(w) \equiv c$$

or

$$\Phi(w) \equiv \frac{c}{w^2}.$$

Since $\Phi(w)$ is holomorphic in $B$, we infer that $c = 0$ or $\Phi(w) \equiv 0$, that is, $a = 0$ and $b = 0$.

In general, however, we only know that $X \in C^1(B, \mathbb{R}^3)$. Thus we have to modify our proof slightly. Let

$$B_R := \{w\colon |w| < R\}, \quad C_R := \partial B_R, \quad 0 < R < 1.$$

Then we infer from (29) that

$$\int_{B_R} [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv \to 0 \quad \text{as } R \to 1 - 0.$$

Performing the same integration by parts as before, we obtain instead of (30) the relation

$$\operatorname{Im} \int_{C_R} \lambda(w)\Phi(w)\, dw \to 0 \quad \text{as } R \to 1 - 0$$

whence

$$\lim_{R\to 1-0} \operatorname{Im} \int_0^{2\pi} \varphi(R,\theta) w^2 \Phi(w)\, d\theta = 0, \quad w = \mathrm{Re}^{i\theta}\,. \tag{33}$$

If we choose $\varphi(r,\theta)$ as in (32) and assume that $\rho < \rho' < \rho'' < R < 1$, then Poisson's formula yields

$$\operatorname{Im} \int_0^{2\pi} \varphi(R,\theta) w^2 \Phi(w)\, d\theta = \int_0^{2\pi} K(R,\theta;\rho,\theta') H(R,\theta)\, d\theta = H(\rho,\theta')$$

and (33) implies $\lim_{R\to 1-0} H(\rho,\theta') = 0$, or $H(\rho,\theta') = 0$. The rest of the proof is the same as before.

2. Results that are similar to Theorems 1–3 can be obtained for the generalized Dirichlet integral

$$E_B(X) = \frac{1}{2} \int_B g_{jk}(X)\{X_u^j X_u^k + X_v^j X_v^k\}\, du\, dv, \tag{34}$$

where $X = (X^1, X^2, \ldots, X^n)$. The conformality relations for the minimizers of $E_B(X)$ in $\mathcal{C}(\Gamma)$, which will replace (1), are now of the form

$$g_{jk}(X) X_u^j X_u^k = g_{jk}(X) X_v^j X_v^k, \quad g_{jk}(X) X_u^j X_v^k = 0. \tag{35}$$

Using the complex notation $w = u + iv$, we can express (35) by the single complex equation

$$g_{jk}(X) X_w^j X_w^k = 0.$$

3. Other functionals $\mathcal{F}_B(X)$ which can be handled in the same way as $D_B(X)$ or $E_B(X)$ are expressions of the type

$$\mathcal{F}_B(X) = E_B(X) + V_B(X) \tag{36}$$

where $V(X)$ is invariant with respect to diffeomorphisms of the parameter domain $B$ which have a positive Jacobian. In fact, if $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$ is such a family of diffeomorphisms from $\overline{B_\varepsilon^*}$ onto $\bar{B}$, then the property

$$V_B(X) = V_{B_\varepsilon^*}(X \circ \sigma_\varepsilon)$$

implies that

$$\mathcal{F}_{B_\varepsilon^*}(X \circ \sigma_\varepsilon) - \mathcal{F}_B(X) = E_{B_\varepsilon^*}(X \circ \sigma_\varepsilon) - E_B(X).$$

Hence, a minimum property of $X$ with respect to $\mathcal{F}_B$ can be translated into a minimum property with respect to $E$, and we are in the previously considered situation. Under suitable assumptions we shall therefore obtain the conformality relations (35).

If, for instance, $V_B(X)$ denotes a volume functional of the type

(37) $$V_B(X) = \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv$$

where $Q = (Q^1, Q^2, Q^3)$ is a $C^1$-vector field defined on $\mathbb{R}^3$ and $X = (X^1, X^2, X^3)$, then the Euler equations of the functional $\mathcal{F}_B(X) = E_B(X) + V_B(X)$ are given by

(38) $$\Delta X^l + \Gamma^l_{jk}(X)[X^j_u X^k_u + X^j_v X^k_v] = \operatorname{div} Q(X)[X_u \wedge X_v]_m g^{lm}(X).$$

Here $(g_{jk}(x))$ is assumed to be a positive definite $3\times3$-matrix, and $(g^{jk}(x))$ denotes its inverse. Moreover, $\Gamma_{jkl}$ and $\Gamma^k_{jl}$ denote the Christoffel symbols of first and second kind:

$$\begin{aligned} \Gamma_{jkl} &= \tfrac{1}{2}\{g_{jk,l} + g_{kl,j} - g_{jl,k}\}, \\ \Gamma^l_{jk} &= g^{lm}\Gamma_{jmk} \end{aligned}$$

where $g_{jk,l}$ stands for the derivative $g_{jk,x^l}$. Finally, we have used the notation

$$\operatorname{div} Q = Q^1_{x^1} + Q^2_{x^2} + Q^3_{x^3}.$$

If $X$ is conformal, then the equations (38) express that $X$ is a surface of mean curvature

(39) $$H(X) = \frac{1}{2\sqrt{g(X)}} \operatorname{div} Q(X), \quad g := \det(g_{jk}),$$

in the Riemannian manifold $(\mathbb{R}^3, ds^2)$ with the line element $ds^2 = g_{jk}(x)\, dx^j \, dx^k$. In Chapter 4 of Vol. 2 we give a survey on results concerning the Plateau problem for functionals $\mathcal{F} = D + V$ and present some of the proofs. The Plateau problem for the general definite parametric integral (= Cartan functional) $\mathcal{F}$ is treated in Section 4.13.

So far we have proved that every closed rectifiable curve $\Gamma$ in $\mathbb{R}^3$ bounds at least one minimal surface $X$ of class $\mathcal{C}(\Gamma)$, and this solution of the Plateau problem has been obtained by minimization of the Dirichlet integral among all (disk-type) surfaces of class $\mathcal{C}(\Gamma)$. Since any minimizer $X$ is automatically continuous on $\bar{B}$, the solution of Plateau's problem can as well be achieved by minimizing $D(X)$ within the class

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3).$$

Although every minimizer $X$ satisfies

$$D(X) = A(X),$$

it is by no means clear that a minimizer of the Dirichlet integral in $\bar{\mathcal{C}}(\Gamma)$ also minimizes the area functional among all surfaces in $\bar{\mathcal{C}}(\Gamma)$. For this we need to know that

$$\bar{a}(\Gamma) = \bar{e}(\Gamma), \tag{40}$$

where $\bar{a}(\Gamma)$ and $\bar{e}(\Gamma)$ denote the infimum of $A(X)$ and $D(X)$ respectively, among all $X \in \overline{\mathcal{C}}(\Gamma)$. However, the inequality

$$A(X) \le D(X)$$

only implies that

$$\bar{a}(\Gamma) \le \bar{e}(\Gamma).$$

In fact, the proof of the equality sign is not a trivial matter. Usually it is based on the fact proved by Carathéodory that polyhedral surfaces can be represented conformally (in the generalized sense). Equivalently one can apply a basic result on "$\varepsilon$-conformal mappings" due to C.B. Morrey which is derived by means of quasiconformal mappings; a somewhat weaker version was already stated by T. Radó [21]. We only quote Morrey's lemma without proving it, because we shall later present a self-contained proof of (40) that uses only fairly elementary tools (see Section 4.10). Roughly speaking, Morrey's lemma says that one can introduce nearly conformal parameters on every reasonable surface $X$. To be precise, we need the following

**Lemma on $\varepsilon$-conformal mappings.** *Let $X$ be a mapping $\overline{B} \to \mathbb{R}^3$ of class $C^0(\bar{B}, \mathbb{R}^3) \cap H_2^1\,(B, \mathbb{R}^3)$. Then, for every $\varepsilon > 0$, there exists a homeomorphism $\tau_\varepsilon$ of $\bar{B}$ onto itself which is of class $H_2^1$ on $\bar{B}$ and satisfies both*

$$Z_\varepsilon := X \circ \tau_\varepsilon \in C^0(\bar{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$$

*and*

$$D(Z_\varepsilon) \le A(X) + \varepsilon.$$

(For a proof, we refer to Morrey [1], pp. 141–143, and [3], pp. 814–815.)

Let us turn to the *proof of* (40): Let $X$ be an arbitrary surface in $\overline{\mathcal{C}}(\Gamma)$. Then, by Morrey's lemma, we can find homeomorphisms $\tau_n$ of $\bar{B}$ onto itself such that $Z_n := X \circ \tau_n \in \overline{\mathcal{C}}(\Gamma)$ and

$$D(Z_n) \le A(X) + \frac{1}{n}, \quad n = 1, 2, \ldots .$$

Since

$$\bar{e}(\Gamma) \le D(Z_n) \quad \text{for all } n \in \mathbb{N},$$

we obtain

$$\bar{e}(\Gamma) \le A(X)$$

and therefore

$$\bar{e}(\Gamma) \le \bar{a}(\Gamma).$$

Thus the relation (40) is proved.

We notice that (40) implies the conformality relations (40). In fact, if $X$ minimizes the Dirichlet integral in $\mathcal{C}(\Gamma)$, then $X \in C^0(\bar{B}, \mathbb{R}^3)$, and (40) yields $A(X) = D(X)$. As we have observed in Section 4.1, this equality can only hold if (1) is satisfied. □

Thus we have proved:

**Theorem 4.** *Every solution $X \in \mathcal{C}(\Gamma)$ of the minimum problem $\mathcal{P}(\Gamma)$ is a surface of least area in $\mathcal{C}(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$.*

Another, completely self-contained proof of this result will be given in Section 4.10, which even shows

(41) $$e(\Gamma) = \overline{e}(\Gamma) = a(\Gamma) = \overline{a}(\Gamma),$$

where $a(\Gamma)$ and $\overline{a}(\Gamma)$ denote the infima of $A$ over $\mathcal{C}(\Gamma)$ and $\overline{\mathcal{C}}(\Gamma)$ respectively, while $e(\Gamma)$ and $\overline{e}(\Gamma)$ are the corresponding infima of $D$. Note that the relation $e(\Gamma) = \overline{e}(\Gamma)$ follows from Theorem 2 whereas $a(\Gamma) = \overline{a}(\Gamma)$ is not immediately obvious.

## 4.6 Variant of the Existence Proof. The Partially Free Boundary Problem

In this section we want to give another existence proof for the minimum problem $\mathcal{P}(\Gamma)$ which is of a more functional-analytic nature and can easily be modified to handle other boundary value problems for minimal surfaces, for instance, the partially free problem. The Courant–Lebesgue lemma will once again play an essential role. We shall use it in the following form:

**Proposition 1.** *Let $\Gamma$ be a closed (oriented) Jordan curve in $\mathbb{R}^3$, and let $\mathcal{C}^*(\Gamma)$ be the class of surfaces bounded by $\Gamma$ and normalized by a fixed three-point condition as defined in Section 4.2. Then $\mathcal{C}^*(\Gamma)$ is a weakly sequentially closed subset of $H_2^1(B, \mathbb{R}^3)$.*

*Proof.* Let $\{X_n\}$ be a sequence of surfaces $X_n \in \mathcal{C}^*(\Gamma)$ which converge weakly in $H_2^1(B, \mathbb{R}^3)$ to some element $X \in H_2^1(B, \mathbb{R}^3)$. Then the norms of $X_n$ are uniformly bounded,

$$(1) \qquad |X_n|_{H_2^1(B)} \le c, \quad n = 1, 2, \dots,$$

and Rellich's theorem yields both

$$|X_n - X|_{L_2(B)} \to 0 \quad \text{as } n \to \infty$$

and

$$(2) \qquad |\phi_n - \phi|_{L_2(C)} \to 0 \quad \text{as } n \to \infty$$

where $\phi_n$ and $\phi$ denote the $L_2$(C)-traces of $X_n$ and $X$ on $C$.

By (1) and Theorem 3 in Section 4.3, the functions $\phi_n$, $n \in \mathbb{N}$, are equicontinuous on $C$, and $\phi_n(C) = \Gamma$ implies

$$(3) \qquad \sup_C |\phi_n| \le \text{const}, \quad n = 1, 2, \dots.$$

Thus the functions $\phi_n$ are compact in $C^0(C, \mathbb{R}^3)$, and we can extract a subsequence $\{\phi_{n_p}\}$ which converges uniformly on $C$ to some $\phi' \in C^0(C, \mathbb{R}^3)$ as $p \to \infty$. From (2) we infer that $\phi' = \phi$, and a well-known reasoning yields that $\{\phi_n\}$ itself converges to $\phi$ as $n \to \infty$. Moreover, Lemma 1 of Section 4.2 implies that $\phi$ is a weakly monotonic mapping of $C$ onto $\Gamma$ which satisfies the same three-point condition as the $\phi_n$. Consequently, $X$ is contained in $\mathcal{C}^*(\Gamma)$, and the assertion is proved. □

Now we shall give a new proof of the following result:

**Theorem 1.** *The minimum problem $\mathcal{P}(\Gamma)$ has at least one solution. Any solution of $\mathcal{P}(\Gamma)$ is harmonic in $B$, continuous on $\overline{B}$, and satisfies*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \textit{in } B.$$

*Proof.* We proceed in four steps:

(i) *First we show that there is a minimizing sequence $\{X_n\}$ for $\mathcal{P}(\Gamma)$, $X_n \in \mathcal{C}^*(\Gamma)$, which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$.*

In fact, let $\{X_n\}$ be a sequence of surface $X_n \in \mathcal{C}^*(\Gamma)$ which satisfy

$$(4) \qquad \lim_{n \to \infty} D(X_n) = e(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}(\Gamma)\}.$$

Then we have

$$D(X_n) \le \text{const}, \quad n = 1, 2, \ldots, \tag{5}$$

and the boundary values $\phi_n := X_n|_C$ satisfy (3). A suitable variant of Poincaré's inequality, together with (3) and (5), yields

$$|X_n|_{H_2^1(B)} \le \text{const}, \quad n = 1, 2, \ldots. \tag{6}$$

In Hilbert space, any closed ball is weakly sequentially compact. Thus there is a subsequence $\{X_{n_p}\}$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$, and clearly

$$\lim_{p \to \infty} D(X_{n_p}) = e(\Gamma).$$

Renumbering the $X_{n_p}$ and writing $X_n$ instead of $X_{n_p}$, the assertion (i) is proved.

(ii) *The Dirichlet integral is weakly lower semicontinuous in* $H_2^1(B, \mathbb{R}^3)$. To verify this, we consider any sequence of elements $X_1, X_2, \ldots \in H_2^1(B, \mathbb{R}^3)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$. Since

$$\mathcal{F}(Z) := D(X, Z)$$

is a bounded linear functional on $H_2^1(B, \mathbb{R}^3)$, we obtain

$$\lim_{n \to \infty} \mathcal{F}(X_n) = \mathcal{F}(X),$$

and therefore

$$\begin{aligned} D(X_n) &= D(X - X_n) + 2D(X, X_n) - D(X) \\ &\ge 2D(X, X_n) - D(X) \to D(X). \end{aligned}$$

That is,

$$\liminf_{n \to \infty} D(X_n) \ge D(X), \tag{7}$$

and (ii) is verified.

(iii) *The set* $\mathcal{C}^*(\Gamma)$ *is a weakly (sequentially) closed subset of* $H_2^1(B, \mathbb{R}^3)$.

This assertion is the statement of Proposition 1.

Combining (i)–(iii), we obtain that $X$ is a solution of $\mathcal{P}(\Gamma)$. In fact, (i) and (ii) imply

$$D(X) \le \lim_{n \to \infty} D(X_n) = e(\Gamma),$$

and (i) and (iii) yield $X \in \mathcal{C}^*(\Gamma)$, whence

$$e(\Gamma) \le D(X),$$

and therefore

$$D(X) = e(\Gamma).$$

(iv) Finally, if $X$ is a solution of $\mathcal{P}(\Gamma)$, it follows from Weyl's lemma that $X$ is harmonic in $B$, and then a well-known reasoning yields that $X$ is continuous on $\overline{B}$. The conformality relations for $X$ were derived in the previous section. □

Let us apply this method to another boundary value problem for minimal surfaces, the *semi-free* (or: *partially free*) *problem*.

Consider a boundary configuration $\langle \Gamma, S \rangle$ consisting of a closed set $S$ in $\mathbb{R}^3$ (e.g., a smooth surface $S$ with or without boundary, or something more exotic, see Figs. 4–7), and a Jordan curve $\Gamma$ the endpoints $P_1$ and $P_2$ of which lie on $S, P_1 \neq P_2$, but all other points of $\Gamma$ are disjoint from $S$.

Let us denote the arcs of $\partial B$ lying in the half-planes $\{\operatorname{Im} w \geq 0\}$ and $\{\operatorname{Im} w \leq 0\}$ by $C$ and $I$ respectively. The class $\mathcal{C}(\Gamma, S)$ of admissible surfaces for the semi-free problem is the set of all maps $X \in H_2^1(B, \mathbb{R}^3)$ whose $L_2$-traces on $C$ and $I$ satisfy

(i) $X(w) \in S$ for $\mathcal{H}^1$-almost all $w \in I$;
(ii) $X|_C$ maps $C$ continuously and in a weakly monotonic way onto $\Gamma$ such that $X(1) = P_1$ and $X(-1) = P_2$.

We orient $\Gamma$ and $\mathcal{C}(\Gamma, S)$ by taking $P_1$ as the initial point and $P_2$ as the endpoint of $\Gamma$.

The corresponding variational problem $\mathcal{P}(\Gamma, S)$ reads:

$$D(X) \to \min \quad \text{in the class } \mathcal{C}(\Gamma, S).$$

Again, as in the study of the Plateau problem, it is desirable to introduce a three-point-condition. Since we have already fixed the images of two boundary points, the image of only one more point needs to be prescribed: Let $P_3$ be some point of $\Gamma$ different from $P_1$ and $P_2$, and let $\mathcal{C}^*(\Gamma, S)$ denote the class of all those surfaces $X \in \mathcal{C}(\Gamma, S)$ mapping $i = \sqrt{-1} \in C$ to $P_3$. The corresponding variational problem $\mathcal{P}^*(\Gamma, S)$ then requires:

$$D(X) \to \min \quad \text{in } \mathcal{C}^*(\Gamma, S).$$

**Theorem 2.** *If $\mathcal{C}(\Gamma, S)$ is nonempty, then there exists a solution of the minimum problem $\mathcal{P}(\Gamma, S)$. Moreover, every solution $X$ of $\mathcal{P}(\Gamma, S)$ is of class $C^0(B \cup C', \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ for every arc $C'$ contained in the interior of $C$, and satisfies both*

$$\Delta X = 0 \quad \textit{in } B$$

*and*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

*in $B$. Finally, the class $\mathcal{C}(\Gamma, S)$ is nonempty if $\Gamma$ is rectifiable and if there exists a rectifiable arc in $S$ which connects $P_1$ and $P_2$.*

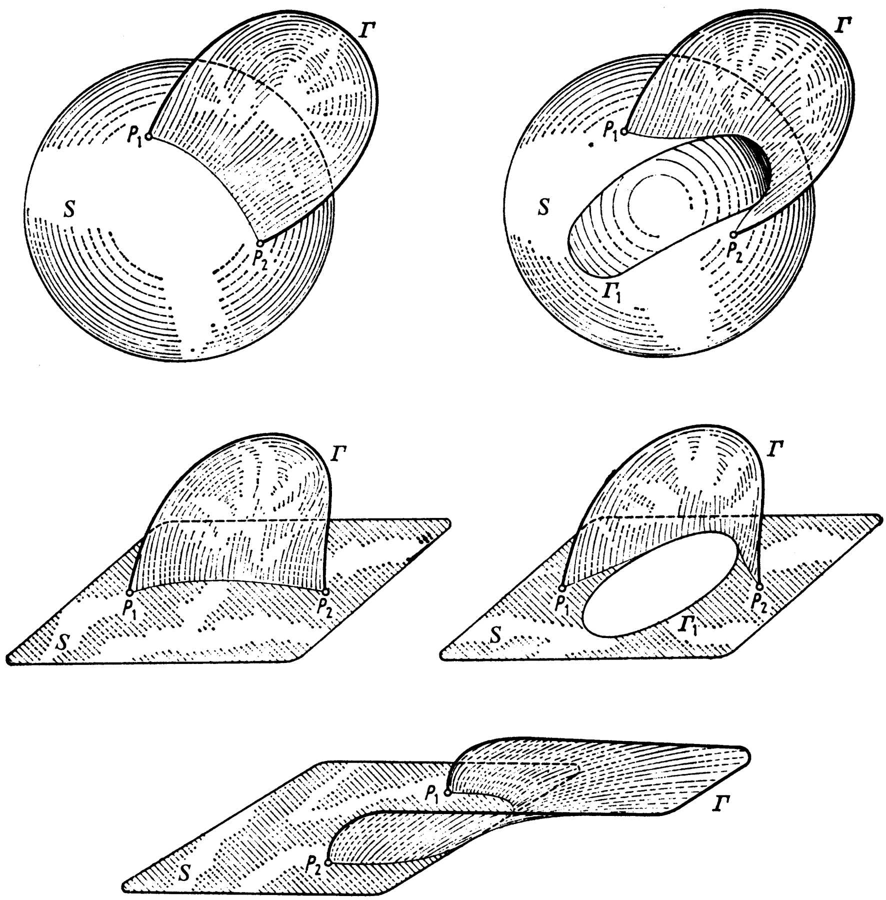



**Fig. 1.** Partially free problems and area minimizing solutions. From S. Hildebrandt and J.C.C. Nitsche [3]

*Proof.* The existence of a minimizer can be established more or less in the same way as for Theorem 1. The steps (i), (ii) and (iv) can be carried out in the same manner, whereas (iii) is to be replaced by:

(iii′) *The class* $\mathcal{C}^*(\Gamma, S)$ *is closed with respect to weak convergence of sequences in* $H_2^1(B, \mathbb{R}^3)$.

In fact, if $\{X_n\}$ is a sequence of surfaces $X_n \in \mathcal{C}^*(\Gamma, S)$ which converge weakly in $H_2^1(B, \mathbb{R}^3)$ to some element $X \in H_2^1(B, \mathbb{R}^3)$, then the norms of $X_n$ are uniformly bounded, and we have

$$\lim_{n\to\infty} |\phi_n - \phi|_{L_2(\partial B)} = 0$$

for $\phi = X|_{\partial B}, \phi_n = X_n|_{\partial B}$. Hence there is a subsequence $\{\phi_{n_p}\}$ such that

$$\phi_{n_p}(w) \to \phi(w) \quad \text{as } p \to \infty,$$

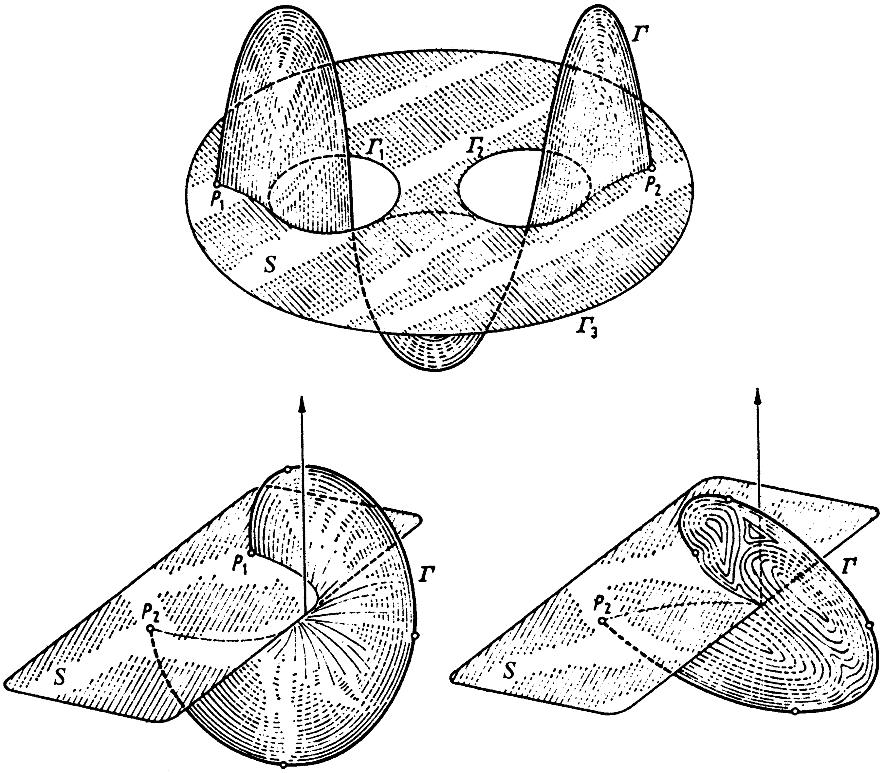



**Fig. 2.** Other partially free problems and area minimizing solutions. From S. Hildebrandt and J.C.C. Nitsche [3]

for $\mathcal{H}^1$-almost all $w \in \partial B$. Since

$$X_n(w) \in S \quad \text{for } \mathcal{H}^1\text{-almost all } w \in I,$$

we thus obtain that also

$$X(w) \in S \quad \text{for } \mathcal{H}^1\text{-almost all } w \in I.$$

Furthermore, a similar reasoning as in the proof of Proposition 1 yields that $X|_C$ maps $C$ continuously and weakly monotonically onto $\Gamma$ and satisfies the 3-point condition

$$X(1) = P_1, \quad X(i) = P_3, \quad X(-1) = P_2,$$

that is, $X \in \mathcal{C}^*(\Gamma, S)$.

In fact, all we have to prove is that the mappings $\phi_n|_C$ are equicontinuous on $C$. By the Courant–Lebesgue lemma, the $\phi_n$ are equicontinuous on every closed subarc $C'$ lying in the interior of $C$. Thus we have to investigate how the

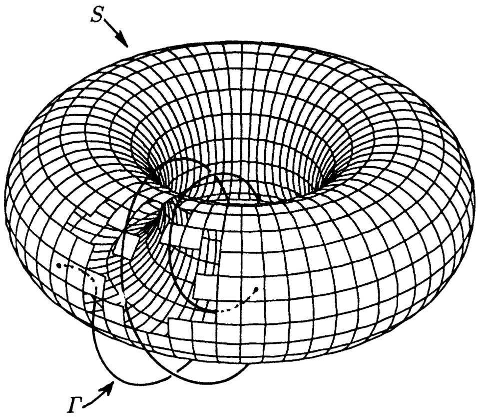



**Fig. 3.** An irregular support surface for the semifree boundary problem

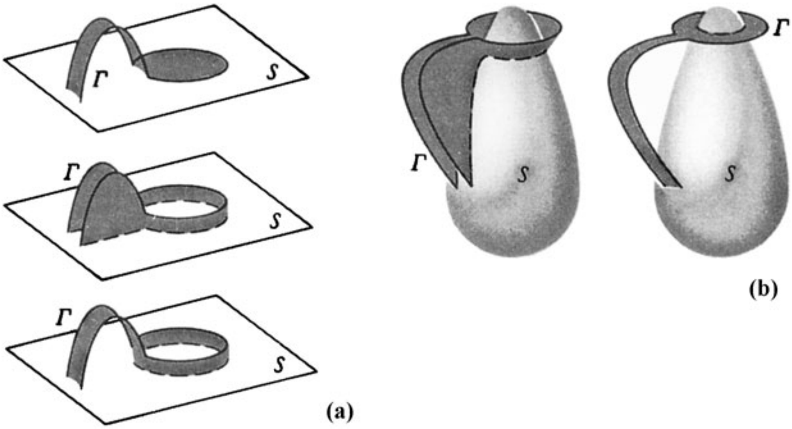



**Fig. 4.** Partially free problems can have several solutions

functions $\phi_n(e^{i\theta}), 0 \le \theta \le \pi$, behave for $\theta \to +0$ or $\theta \to \pi - 0$. To this end we use the assumption that $\Gamma$ and $S$ have only the points $P_1$ and $P_2$ in common. Let $\Gamma_1$ and $\Gamma_2$ be the subarcs of $\Gamma$ with the endpoints $P_1, P_3$ and $P_2, P_3$ respectively. We conclude that, for every $\varepsilon > 0$, there is a number $\Delta(\varepsilon) > 0$ such that $|P - P_1| < \varepsilon$ holds true for every $P \in \Gamma_1$ with $\text{dist}(P, S) < \Delta(\varepsilon)$, and that $|P_2 - P| < \varepsilon$ is fulfilled for every $P \in \Gamma_2$ with $\text{dist}(P, S) < \Delta(\varepsilon)$.

Moreover, applying the Courant–Lebesgue lemma to the surfaces $X_n$ (or, to be precise, Proposition 2 of Section 4.4 to $X = X_n$ and $z_0 = \pm 1$), we obtain sequences $\{w'_n\}, \{w''_n\}$ of points $w'_n, w''_n \in C$ with $w'_n \to 1, w''_n \to -1$ as $n \to \infty$ such that $\text{dist}(X_n(w'_n), S) \to 0$, $\text{dist}(X_n(w''_n), S) \to 0, X_n(w'_n) \in \Gamma_1, X_n(w''_n) \in \Gamma_2$. As each $X_n$ furnishes a weakly monotonic map of $C$ onto $\Gamma$, this implies the equicontinuity of the mappings $X_n$ on $C$. □

## 4.7 Boundary Behavior of Minimal Surfaces with Rectifiable Boundaries

So far we have considered (disk-type) minimal surfaces $X$ of class $\mathcal{C}(\Gamma)$. They have continuous boundary values on $C = \partial B$ which are continuously assumed by $X(w), w \in B$, as $w$ tends to some boundary point. In this section we want to prove that the first derivatives of $X$ assume boundary values of class $L_1(C)$ on $C$ if $\Gamma$ is rectifiable, and that we can establish a general formula for integration by parts.

Throughout this section we shall only make the following

**General assumption.** *Let $X : \bar{B} \to \mathbb{R}^3$ be a surface of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which has boundary values of finite variation, i.e.,*

$$L(X) := \int_C |dX| < \infty, \tag{1}$$

*and which satisfies in $B$ the equations $X(w) \not\equiv \text{const}$ and*

$$\Delta X = 0, \tag{2}$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{3}$$

*Moreover let $X^*$ be any adjoint minimal surface to $X$, defined by the Cauchy–Riemann equations*

$$X_u = X_v^*, \quad X_v = -X_u^* \quad \textit{in } B. \tag{4}$$

Clearly, the *general assumption* will be satisfied by the solutions of Plateau's problem in $\mathcal{C}(\Gamma)$, but it will be fulfilled in many other situations as well.

The main goal of this section is the following

**Theorem 1.** *If the minimal surface $X$ satisfies the general assumption and if $X^*$ is an adjoint surface to $X$, then we have*:

(i) *$X^*$ admits a continuous extension to all of $\bar{B}$, and the boundary values $X^*|_C$ are likewise rectifiable and satisfy*

$$\int_C |dX| = \int_C |dX^*|. \tag{5}$$

(ii) *The boundary values $X|_C$ and $X^*|_C$ are absolutely continuous functions on $C$.*

(iii) *Set $X(r,\theta) := X(re^{i\theta})$ and $X^*(r,\theta) := X^*(re^{i\theta})$. Then the partial derivatives $X_r(r,\theta), X_\theta(r,\theta), X_r^*(r,\theta), X_\theta^*(r,\theta)$, considered as periodic functions of $\theta \in [0, 2\pi]$, tend to limits in $L_1([0,2\pi], \mathbb{R}^3)$ as $r$ increases to $1$, both in the $L_1$-norm on $[0,2\pi]$ and pointwise almost everywhere on $[0,2\pi]$. The limits of $X_\theta$ and $X_\theta^*$ coincide a.e. on $\partial B$ with the pointwise derivatives of the boundary values $X(e^{i\theta})$ and $X^*(e^{i\theta})$. Moreover, these derivatives vanish at most on a subset of $C$ of $1$-dimensional Hausdorff measure zero.*

An essential step in the proof of the theorem is the following

**Proposition 1.** *The function* $\lambda(r)$, $0 \le r \le 1$, *defined by*

$$\lambda(r) := L(X|_{C_r}) = \int_{C_r} |dX|, \quad C_r := \{re^{i\theta} : 0 \le \theta \le 2\pi\},$$

*increases monotonically and is bounded from above by* $L(X|_C)$. *Consequently, we also have* $\lambda(1) = \lim_{r\to 1-0} \lambda(r)$.

*Proof.* We have to show that, if $0 \le r < R \le 1$, then $\lambda(r) \le \lambda(R)$. It suffices, however, to assume that $R = 1$, because the general case will then follow by considering the minimal surface $X(\frac{w}{R}) : B_R \to \mathbb{R}^3$.

Since $X$ is continuous on $\bar{B}$, Poisson's formula yields that $X(r,\theta) := X(re^{i\theta})$ satisfies

$$X(r,\theta) = \int_0^{2\pi} K(r, \varphi - \theta) X(1,\varphi)\, d\varphi, \tag{6}$$

where

$$K(r,\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2} = \frac{1}{2\pi} \frac{1 - |w|^2}{|1 - w|^2}, \quad \text{if } w = re^{i\alpha}.$$

Hence

$$\begin{aligned} X_\theta(r,\theta) &= \int_0^{2\pi} K_\theta(r,\varphi-\theta) X(1,\varphi)\, d\varphi = -\int_0^{2\pi} K_\varphi(r,\varphi-\theta) X(1,\varphi)\, d\varphi \\ &= \int_0^{2\pi} K(r,\varphi-\theta)\, dX(1,\varphi). \end{aligned}$$

The integration by parts is justified since the total variation of $X|_{\partial B}$, i.e. the length of $X|_{\partial B}$, is finite (cf. Natanson [1], Chapter VIII). In addition, $K(r,\alpha)$ is positive throughout; thus

$$|X_\theta(r,\theta)| \le \int_0^{2\pi} K(r,\varphi-\theta) |\, dX(1,\varphi)|,$$

whence

$$\begin{aligned} \lambda(r) &= \int_0^{2\pi} |X_\theta(r,\theta)|\, d\theta \\ &\le \int_0^{2\pi}\int_0^{2\pi} K(r,\varphi-\theta)\, d\theta |\, dX(1,\varphi)| \le \lambda(1) \end{aligned}$$

because of

$$\int_0^{2\pi} K(r,\alpha)\, d\alpha = 1.$$

As $\lambda(r)$ is lower semicontinuous, we obtain $\lambda(r) \to \lambda(1)$ as $r \to 1-0$. □

(Note that neither here nor in the proof of the next result the conformality relations (3) are used.)

**Proposition 2.** *If we write* $X(r,\theta) := X(re^{i\theta})$, *then we obtain*

$$\int_0^1 |X_r(r,\theta)|\,dr \le \frac{1}{2}\int_C |dX| \tag{7}$$

*for every* $\theta \in [0, 2\pi]$.

*Proof.* It suffices to prove the inequality for $\theta = 0$. Applying (6) and an integration by parts, we obtain

$$X(r,0) = \int_0^{2\pi} K(r,\varphi)X(1,\varphi)\,d\varphi = X(1,0) - \int_0^{2\pi} h(r,\varphi)\,dX(1,\varphi)$$

where

$$h(r,\varphi) := \int_0^{\varphi} K(r,\alpha)\,d\alpha = \frac{\varphi}{2\pi} + \frac{1}{2\pi i}\log\frac{1-\bar{w}}{1-w}, \quad w = re^{i\varphi}.$$

Then it follows that

$$X_r(r,0) = -\int_0^{2\pi} h_r(r,\varphi)\,dX(1,\varphi)$$

and therefore

$$|X_r(r,0)| \le \int_0^{\pi} h_r(r,\varphi)|\,dX(1,\varphi)| - \int_{\pi}^{2\pi} h_r(r,\varphi)|\,dX(1,\varphi)|$$

since

$$h_r(r,\varphi) = \frac{1}{\pi}\frac{\sin\varphi}{1 - 2r\cos\varphi + r^2}$$

is positive for $0 < \varphi < \pi$ and negative for $\pi < \varphi < 2\pi$. Consequently,

$$\int_0^1 |X_r(r,0)|\,dr \le \int_0^{\pi} \{h(1,\varphi) - h(0,\varphi)\}|\,dX(1,\varphi)| + \int_{\pi}^{2\pi} \{h(0,\varphi) - h(1,\varphi)\}|\,dX(1,\varphi)|.$$

As $|h(1,\varphi) - h(0,\varphi)| \le \frac{1}{2}$, we arrive at the desired inequality.

**Proposition 3.** *The conjugate surface* $X^*$ *can be extended continuously to* $\bar{B}$. *Moreover, both* $X$ *and* $X^*$ *are contained in* $H_2^1(B,\mathbb{R}^3)$, *and we obtain*

$$\int_C |dX| = \int_C |dX^*|, \quad D_B(X) = D_B(X^*), \tag{8}$$

*and*

$$\int_0^1 |X_r^*(r,\theta)|\,dr \le \frac{1}{2}\int_C |dX^*|. \tag{9}$$

*Proof.* (i) Similar to Proposition 2, we have used the notations $X(r,\theta) = X(re^{i\theta})$ and $X^*(r,\theta) = X^*(re^{i\theta})$. The Cauchy–Riemann equations read

$$rX_r = X^*_\theta, \quad rX^*_r = -X_\theta, \tag{10}$$

and the conformality relations are equivalent to

$$r^2|X_r|^2 = |X_\theta|^2, \quad \langle X_r, X_\theta\rangle = 0. \tag{11}$$

Therefore we also have

$$|X_r| = |X^*_r|, \quad |X_\theta| = |X^*_\theta|, \tag{12}$$

and it follows that

$$\begin{aligned} |X^*(r_2,\theta) - X^*(r_1,\theta)| &\le \int_{r_1}^{r_2} |X^*_r(r,\theta)|\, dr \\ &= \int_{r_1}^{r_2} |X_r(r,\theta)|\, dr \le \frac{1}{2}\int_C |dX| \end{aligned} \tag{13}$$

for $0 < r_1 < r_2 < 1$, on account of Proposition 2. Hence,

$$\int_0^1 |X^*_r(r,\theta)|\, dr < \infty,$$

and the convergence of this integral implies that $\lim_{r\to 1-0} X^*(r,\theta)$ exists for all $\theta \in [0, 2\pi]$.

Consider now points $w_j = e^{i\theta_j}, 0 \le j \le n$, on $C$ with

$$0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi.$$

Then

$$\begin{aligned} \sum_{j=1}^n |X^*(w_j) - X^*(w_{j-1})| &= \lim_{r\to 1} \sum_{j=1}^n |X^*(rw_j) - X^*(rw_{j-1})| \\ &\le \lim_{r\to 1} \int_0^{2\pi} |X^*_\theta(r,\theta)|\, d\theta \\ &= \lim_{r\to 1} \int_0^{2\pi} |X_\theta(r,\theta)|\, d\theta = \int_C |dX|, \end{aligned}$$

and we infer that

$$\int_0^{2\pi} |dX^*(1,\theta)| \le \int_0^{2\pi} |dX(1,\theta)| < \infty.$$

In other words, $X^*(1,\theta)$ is a function of bounded variation for $0 \le \theta \le 2\pi$.

(ii) From $X \in C^0(\bar{B}, \mathbb{R}^3)$ we infer that $\sup_B |X| < \infty$. Moreover, (13) implies

$$|X^*(w)| \le |X^*(0)| + |X^*(w) - X^*(0)| \le |X^*(0)| + \frac{1}{2}\int_C |dX|$$

whence also $\sup_B |X^*| < \infty$.

Moreover, the Cauchy–Riemann equations (4) yield

$$D_B(X) = D_B(X^*).$$

Hence, in order to prove, $X, X^* \in H_2^1(B, \mathbb{R}^3)$, we only have to verify that $D_B(X) < \infty$. Let $B_R = \{re^{i\theta} : 0 \le r < R\}$. Then an integration by parts leads to

$$\begin{aligned}
\int_{B_R} |\nabla X|^2 \, du \, dv &= \int_{\partial B_R} \langle X, X_r \rangle \, ds \le \int_{\partial B_R} |X||X_r| \, ds \\
&= \int_0^{2\pi} |X(R,\theta)||X_\theta(R,\theta)| \, d\theta \\
&\le \sup_B |X| \cdot \int_0^{2\pi} |X_\theta(R,\theta)| \, d\theta \\
&\le \sup_B |X| \cdot \int_C |dX|,
\end{aligned}$$

and for $R \to 1-0$ we obtain

$$\int_B |\nabla X|^2 \, du \, dv \le \sup_B |X| \cdot \int_C |dX| < \infty. \tag{14}$$

(iii) As we have shown that $X^*(1,\theta)$ is a function of bounded variation with respect to $\theta$, these boundary values can have only denumerably many discontinuities, and, for every $\theta_0 \in \mathbb{R}$, both one-sided limits

$$\lim_{\theta \to \theta_0 - 0} X^*(1,\theta), \qquad \lim_{\theta \to \theta_0 + 0} X^*(1,\theta)$$

exist. Because of $D(X^*) < \infty$ and of the Courant–Lebesgue lemma (cf. Section 4.4, Proposition 2), we then conclude that $\lim_{\theta \to \theta_0} X^*(1,\theta)$ exists for all $\theta_0 \in \mathbb{R}$, and therefore $X^*(1,\theta)$ depends continuously on $\theta$. Hence we can apply Proposition 2 to $X^*$ instead of $X$, and we then obtain

$$\int_0^1 |X_r^*(r,\theta)| \, dr \le \frac{1}{2}\int_C |dX^*|.$$

Finally, Proposition 1, applied to both $X$ and $X^*$, yields

$$\begin{aligned}
\lim_{r \to 1-0} \int_0^{2\pi} |X_\theta(r,\theta)| \, d\theta &= \int_C |dX|, \\
\lim_{r \to 1-0} \int_0^{2\pi} |X_\theta^*(r,\theta)| \, d\theta &= \int_C |dX^*|,
\end{aligned}$$

and both limits coincide because of (12), whence

$$\int_C |dX| = \int_C |dX^*|.$$

□

Now we turn to the

*Proof of Theorem 1.* Let us introduce the holomorphic function $f : B \to \mathbb{C}^3$ by

$$f(w) := X(w) + iX^*(w)$$

with the complex derivative $f'(w)$. By the conformality relations (3), we infer that

$$|f'(w)| = \sqrt{2}|X_r(w)| = r^{-1}\sqrt{2}\,|X_\theta(w)|, \quad w = re^{i\theta},$$

and Proposition 1 implies that the (increasing) Hardy function

$$\mu(r) := \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|\,d\theta = \frac{1}{\pi r\sqrt{2}}\int_0^{2\pi} |X_\theta(re^{i\theta})|\,d\theta$$

of $f'(w)$ satisfies

$$\lim_{r\to 1-0} \mu(r) \le \int_C |dX| < \infty.$$

Thus the holomorphic function $f'(w)$, $w \in B$, belongs to the Hardy class $\mathcal{H}_1$, and a well known theorem by F. Riesz [1] ensures the existence of a function $g(\theta)$ of class $L_1([0, 2\pi], \mathbb{C}^3)$ such that both

$$\lim_{r\to 1-0} \int_0^{2\pi} |f'(re^{i\theta}) - g(\theta)|\,d\theta = 0$$

and

$$\lim_{r\to 1-0} f'(re^{i\theta}) = g(\theta) \quad \text{a.e. on } [0, 2\pi].$$

If we write

$$f(re^{i\theta}) = X(r,\theta) + iX^*(r,\theta),$$

we see that

$$\begin{aligned} X_r(r,\theta) + iX_r^*(r,\theta) &= \frac{\partial}{\partial r} f(re^{i\theta}) = e^{i\theta} f'(re^{i\theta}) \to e^{i\theta} g(\theta), \\ X_\theta(r,\theta) + iX_\theta^*(r,\theta) &= \frac{\partial}{\partial \theta} f(re^{i\theta}) = ire^{i\theta} f'(re^{i\theta}) \to ie^{i\theta} g(\theta) \end{aligned} \tag{15}$$

as $r \to 1-0$.

For any $r \in (0,1)$ and for $0 \le \theta_1 \le \theta_2 \le 2\pi$, we can write

$$f(re^{i\theta_2}) - f(re^{i\theta_1}) = \int_{\theta_1}^{\theta_2} ire^{i\theta} f'(re^{i\theta})\,d\theta.$$

Letting $r \to 1-0$, it follows that

$$f(e^{i\theta_2}) - f(e^{i\theta_1}) = \int_{\theta_1}^{\theta_2} ie^{i\theta} g(\theta)\,d\theta$$

for any $\theta_1, \theta_2 \in [0, 2\pi]$, where $g \in L_1([0, 2\pi], \mathbb{C}^3)$. This implies that

$$f(e^{i\theta}) = X(1,\theta) + iX^*(1,\theta)$$

is an absolutely continuous function of $\theta \in [0, 2\pi]$ whose derivative

$$X_\theta(1,\theta) + iX_\theta^*(1,\theta) = \frac{\partial}{\partial\theta} f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta}) \tag{16}$$

exists a.e. on $[0, 2\pi]$. Comparing formulas (15) and (16), we obtain the assertions of (ii) and (iii), except for the fact that $X_\theta(1,\theta) \neq 0$ and $X_\theta^*(1,\theta) \neq 0$ a.e. on $[0, 2\pi]$.

Taking $X(w) \not\equiv \text{const}$ on $B$ into account, it follows that $f(w) \not\equiv \text{const}$ on $B$, and a well known theorem by F. and M. Riesz [1] implies that the boundary values of $f'(w)$ can only vanish on a subset of $C$ of measure zero.

Finally the assertion of (i) follows from Proposition 3. □

Notice that the proof of Proposition 3 and in particular formula (14) yield the following result:

**Proposition 4.** *If the minimal surface $X$ is contained in a ball*

$$K_R(P_0) := \{P \in \mathbb{R}^3 \colon |P - P_0| \le R\}$$

*of radius $R$, then*

$$A_B(X) = D_B(X) \le R/2 \cdot L(X|_C). \tag{17}$$

Local versions of Theorem 1 are of course available. For instance, one has

**Theorem 1′.** *If $C'$ is an open subarc of $C = \partial B$, and if $X \in H_2^1(B, \mathbb{R}^3)$ is a minimal surface in $B$ which is continuous and has rectifiable boundary values on $C'$, i.e.,*

$$L(X|_{C'}) = \int_{C'} |dX| < \infty,$$

*then $X|_{C''}$ is absolutely continuous on any subarc $C'' \subset\subset C'$, and the tangential derivative $X_\theta$ of $X|_{C''}$ is nonzero a.e. on $C''$.*

The proof can be reduced to the previous case by using the Courant–Lebesgue lemma (see Proposition 2 of Section 4.4) and suitable conformal reparametrizations.

**Theorem 2 (Integration by parts).** *If the minimal surface $X$ satisfies the general assumption of this section, and if $Y$ is an arbitrary function of class $L_\infty \cap H_2^1(B, \mathbb{R}^3)$, then we have*

$$\int_B \langle \nabla X, \nabla Y \rangle \, du \, dv = \int_{\partial B} \left\langle Y, \frac{\partial}{\partial \nu} X \right\rangle ds \tag{18}$$

*where the line integral on the right-hand side is to be taken with positive orientation of $\partial B$, and $\frac{\partial}{\partial\nu} X$ denotes the normal derivative of $X$ with respect to the exterior normal $\nu$ to $\partial B$.*

**Remark.** An analogous result holds if the minimal surface $X$ is parametrized on an arbitrary parameter domain $B$ with piecewise smooth boundary.

*Proof of the theorem.* Let $0 < R < 1$ and $B_R = \{w\colon |w| < R\}$. Since

$$X \in C^1(\overline{B}_R, \mathbb{R}^3),$$

we have the classical formula

$$\int_{B_R} \langle \nabla X, \nabla Y \rangle \, du \, dv = \int_{\partial B_R} \langle X_r, Y \rangle \, ds.$$

Letting $R \to 1 - 0$, the left-hand side obviously tends to $\int_B \langle \nabla X, \nabla Y \rangle \, du \, dv$, whereas the right-hand side converges to $\int_C \langle X_r, Y \rangle \, ds$, on account of Theorem 1 and of Lebesgue's theorem on dominated convergence. □

## 4.8 Reflection Principles

In this section $\Omega$ denotes a domain in the complex plane which is symmetric with respect to the real axis, i.e., $w \in \Omega$ if and only if $\overline{w} \in \Omega$. Set

$$\begin{aligned} \Omega^+ &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w > 0\}, \\ \Omega^- &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w < 0\}, \\ \mathfrak{I} &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w = 0\}, \end{aligned}$$

where $\mathfrak{I}$ is an open subset of $\mathbb{R}$.

We want to prove two reflection principles for minimal surfaces which generalize the well known reflection principle for harmonic functions due to H.A. Schwarz.

**Theorem 1.** *Suppose that $X$ is of class $C^0(\Omega^+ \cup \mathfrak{I}, \mathbb{R}^3) \cap C^2(\Omega^+, \mathbb{R}^3)$ and satisfies both*

(1) $$\Delta X = 0$$

*and*

(2) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

*in $\Omega^+$. Assume also that $X$ maps $\mathfrak{I}$ into a straight line $L_0$. Then $X$ can be extended across $\mathfrak{I}$ onto all of $\Omega$ by reflection in $L_0$, and the extended surface $X$ satisfies (1) and (2) on $\Omega$. To be precise, the extension of $X$ to $\Omega^-$ is defined by*

$$X(w) := (X(\overline{w}))^*, \quad w \in \Omega^-,$$

*where, for $P \in \mathbb{R}^3$, we denote by $P^*$ the reflection image of $P$ in $L_0$.*

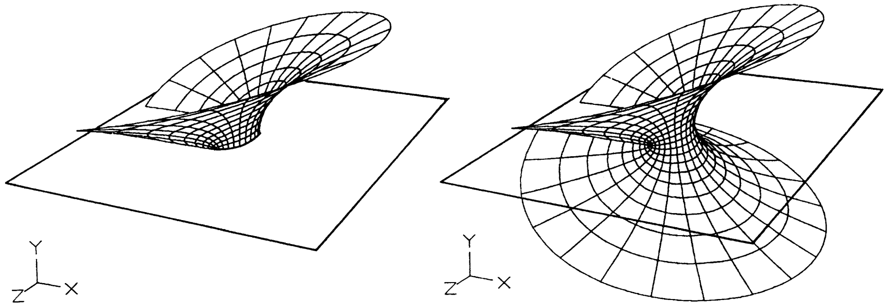



**Fig. 1.** Reflection of a minimal surface in a plane (Catalan's surface)

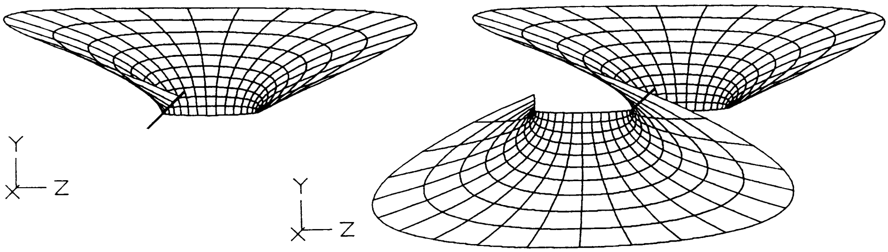



**Fig. 2.** Reflection of a minimal surface in a straight line (Catalan's surface)

**Theorem 2.** *Suppose that $X$ is of class $C^1(\Omega^+ \cup \mathfrak{I}, \mathbb{R}^3) \cap C^2(\Omega^+, \mathbb{R}^3)$ and satisfies both* (1) *and* (2) *in $\Omega^+$. Assume also that $X$ maps $\mathfrak{I}$ into a plane $S$ such that $X$ is perpendicular to $S$ along $\mathfrak{I}$ (i.e., $X_v(w) \perp S$ for all $w \in \mathfrak{I}$). Then $X$ can be extended across $\mathfrak{I}$ as a minimal surface on all of $\Omega$ if we reflect $X$ in $S$. To be precise, the extension of $X$ to $\Omega^-$ is defined by*

$$X(w) := (X(\overline{w}))^*, \quad w \in \Omega^-,$$

*where $P^*$ denotes the mirror image in $S$ of any point $P \in \mathbb{R}^3$.*

Note that these two reflection principles are more or less the same as those formulated in Section 3.4, only that we a priori require less regularity than before. To solve Björling's problem, we needed real analyticity of $X$ along $\mathfrak{I}$ whereas here it suffices to assume $X \in C^0$ and $X \in C^1$ respectively along $\mathfrak{I}$. In fact, we shall prove that, under the assumptions of Theorems 1 and 2, $X$ must be real analytic on $\mathfrak{I}$. Thus both theorems provide special cases of boundary regularity results. In Chapter 2 of Vol. 2 we shall treat the question of boundary regularity of minimal surfaces in some generality.

*Proof of Theorem 1.* Let us introduce Cartesian coordinates $x, y, z$ in $\mathbb{R}^3$ such that $L_0$ becomes the $z$-axis, and set $X(w) = (x(w), y(w), z(w))$, $w = (u, v) = u + iv$, $\bar{w} = (u, -v) = u - iv$. Then we have

$$(3) \qquad x(w) = 0 \quad \text{and} \quad y(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Now, by Schwarz's reflection principle, we can extend $x$ and $y$ as harmonic functions to all of $\Omega$ if we set

$$(4) \qquad x(w) := -x(\overline{w}) \quad \text{and} \quad y(w) := -y(\overline{w}) \quad \text{for } w \in \Omega^-.$$

Moreover (3) implies that

$$(5) \qquad x_u(w) = 0, \quad y_u(w) = 0 \quad \text{for } w \in \mathfrak{I}.$$

Let $\{w_n\}$ be some sequence of points $w_n \in \Omega^+$ such that $w_n \to w_0 \in \mathfrak{I}$ as $n \to \infty$. We obtain from (2), (5) and $x, y \in C^\infty(\Omega)$ that

$$(6) \qquad \lim_{n\to\infty} z_u(w_n) z_v(w_n) = 0,$$

and from

$$\begin{aligned} z_u^2(w_n) &= |X_u(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \\ &= |X_v(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \\ &\geq |z_v(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \end{aligned}$$

together with (5) and (6) we infer that

$$(7) \qquad \lim_{n\to\infty} z_v(w_n) = 0$$

for all sequences $\{w_n\}$, $w_n \in \Omega^+$, with $w_n \to w_0 \in \mathfrak{I}$. Hence the harmonic function $z_v(w)$, $w \in \Omega^+$, is continuous on $\Omega^+ \cup \mathfrak{I}$ and satisfies

$$(8) \qquad z_v(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Schwarz's reflection principle yields that we can extend $z(w)$ as harmonic function across $\mathfrak{I}$ to $\Omega$ by setting

$$(9) \qquad z(w) := z(\overline{w}) \quad \text{for } w \in \Omega^-.$$

Then $X$ is harmonic in $\Omega$, and formulas (4) and (9) together with (2) show that $X$ fulfills the conformality relations on all of $\Omega$. □

*Proof of Theorem 2.* We now introduce Cartesian coordinates $x, y, z$ in $\mathbb{R}^3$ such that $S$ is described by the equation $z = 0$. Then the minimal surface

$$X(w) = (x(w), y(w), z(w))$$

satisfies

$$(10) \qquad z(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Moreover, $X_u(w)$ is tangential to $S$ for all $w \in \mathcal{I}$, and $X_v$ is perpendicular to $X_u$. Since we have assumed that $X(w)$ meets $S$ along $\mathcal{I}$ at a right angle, it follows that $X_v(w)$ is orthogonal to the two vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0)$$

spanning $S$, for all $w \in \mathcal{I}$, whence we conclude that

$$(11) \qquad x_v(w) = 0, \quad y_v(w) = 0 \quad \text{for all } w \in \mathcal{I}.$$

Applying Schwarz's reflection principle for harmonic functions, we infer from (10) and (11) that $x(w), y(w), z(w)$ can be continued to $\Omega$ as harmonic functions, by setting

$$(12) \qquad x(w) = x(\overline{w}), \quad y(w) = y(\overline{w}), \quad z(w) = -z(\overline{w}) \quad \text{for } w \in \Omega^-.$$

One easily checks that the harmonic vector $X(w)$, $w \in \Omega$, satisfies the conformality relations on all of $\Omega$. □

Recently, Choe [4] proved that a minimal surface can also be analytically extended across a plane $S$ if it meets this plane at a constant angle $\theta$ with $0 < \theta < \pi$, and the extension is again carried out by reflection in $S$.

## 4.9 Uniqueness and Nonuniqueness Questions

How many minimal surfaces can be spanned in a given closed Jordan curve? The answer to this question is not known in general, not even if we fix the topological type of the solutions of Plateau's problem. As we have considered only disk-type minimal surfaces, we want to consider the more modest question of:

*How many minimal surfaces of the type of the disk can be spanned in a given closed Jordan curve $\Gamma$?*

The situation would be simple if we could prove that $\Gamma$ bounds only one disk-type minimal surface $X \in \mathcal{C}(\Gamma)$ (up to reparametrizations $X \circ \tau$ of $X$ by conformal mappings $\tau : B \to B$ of the parameter domain $B$ onto itself; such reparametrizations would not be counted as different from $X$ and could be excluded by fixing a three-point condition for the surfaces $X \in \mathcal{C}(\Gamma)$ which are prospective solutions; in other words: *Uniqueness of the solution of Plateau's problem in $\mathcal{C}(\Gamma)$ actually means 'uniqueness in $\mathcal{C}^*(\Gamma)$'*).

However, examples (cf. Figs. 1 and 4 of the Introduction) warn us not to expect uniqueness for disk-type solutions of Plateau's problem. Thus we may ask whether additional geometric conditions for $\Gamma$ are known which ensure this uniqueness. Essentially, we know three results:

1. **Theorem of Radó** [16]: *If $\Gamma$ has a one-to-one parallel projection onto a planar convex curve $\gamma$, then $\Gamma$ bounds at most one disk-type minimal surface.*

This result will be proved in the sequel (cf. Theorem 1). By the same technique, Radó [20] was able to ensure uniqueness in the case of $\Gamma$ admitting a one-to-one *central projection* onto a planar convex curve $\gamma$. (See also Nitsche [28], pp. 360–362.)

For the sake of completeness we mention a result of Tromba [3] which looks like a corollary to Radó's theorem but, at closer inspection, turns out not to be included. Actually it is proved in a completely different way.

**Tromba's observation.** *If $\Gamma$ is $C^2$-close to a planar curve $\gamma$ of class $C^2$, then $\Gamma$ bounds a unique minimal surface of the type of the disk.*

2. **Theorem of Nitsche** [26]: *If $\Gamma$ is regular, real analytic and has a total curvature less than or equal to $4\pi$, then $\Gamma$ bounds only one disk-type minimal surface.*

A proof of this result is given in Section 5.6. It is based on a "field embedding". We shall establish this by using a technique due to H.A. Schwarz, modified by J.C.C. Nitsche. The third uniqueness theorem, due to F. Sauvigny, is described in Section 7.2.

For polygonal $\Gamma$ of total curvature less than $4\pi$, this result was earlier conjectured by R. Schneider [2] whose sketch of a proof contained some of the ideas used in Nitsche's proof.

In general, however, nothing is known about the number of solutions of Plateau's problem which are of class $\mathcal{C}(\Gamma)$. Actually, the situation seems to be rather unpromising on account of the following remarkable result due to Böhme [6]:

*For each positive integer $N$ and for each $\varepsilon > 0$, there exists a regular real analytic Jordan curve $\Gamma$ in $\mathbb{R}^3$ with total curvature less than $4\pi + \varepsilon$ which bounds at least $N$ minimal surfaces of class $\mathcal{C}^*(\Gamma)$, i.e., of the type of the disk.*

One does not even know whether the number of solutions $X \in \mathcal{C}(\Gamma)$ of Plateau's Problem for the curve $\Gamma$ is finite or not. There are suggestive examples of P. Levy [2] and Courant [15] which indicate that there might be rectifiable Jordan curves $\Gamma$ bounding non-denumerably many minimal surfaces. The validity of these examples, however, depends strictly on the validity of the *strong bridge theorem* which recently was rigorously proved by B. White. For the construction principle of the Levy–Courant examples and for comments on the bridge principle we refer the reader to the Scholia.

Whatever may be the case, we have two satisfactory partial answers to the *finiteness question*:

1. **Theorem of Böhme–Tromba** [1]. *Generically, the number of disk-type solutions of Plateau's problem is finite.*

For a proof of this result, see Vol. 3.

2. **Theorem of Tomi** [6]. *There are only finitely many disk-type solutions of Plateau's problem which are absolute minimizers of the area functional in* $\mathcal{C}^*(\Gamma)$ *provided that* $\Gamma$ *is a regular real-analytic Jordan curve.*

3. **Theorem of Nitsche** [31]. *If the regular contour* $\Gamma \in C^{3,\alpha}$ *is either extreme or real analytic and has a total curvature of less than* $6\pi$*, then there exist only finitely many immersions* $X : \overline{B} \to \mathbb{R}^3$ *of class* $\mathcal{C}^*(\Gamma)$.

A proof of the results 2 and 3 can be found in Section 5.7.

Doubtless, the *number-of-solutions problem* is the most exciting and most challenging question that can be raised in connection with Plateau's problem.

Now we want to discuss Radó's result.

**Theorem 1.** *If* $\Gamma$ *possesses a one-to-one parallel projection onto a plane convex Jordan curve* $\gamma$*, then* $\Gamma$ *bounds at most one minimal surface except, of course, for conformal reparametrizations. It has no branch points, and it admits a non-parametric representation.*

As an example, let us consider an arbitrary quadrilateral $\Gamma$ in $\mathbb{R}^3$. By this we mean a Jordan curve which is a polygon with four edges and four vertices. If $\Gamma$ is a planar curve, then it bounds exactly one (planar) minimal surface on account of the maximum principle. On the other hand it is easy to verify that any nonplanar quadrilateral admits a one-to-one orthogonal projection onto a convex plane quadrilateral. Applying Radó's theorem, we then obtain:

*Every quadrilateral bounds a uniquely determined minimal surface of the type of the disk.*

For the proof of Theorem 1 we need the following

**Lemma 1 (Monodromy principle).** *Let* $\Omega$ *be a simply connected, bounded domain in* $\mathbb{C}$ *and let* $f \in C^0(\bar{\Omega}, \mathbb{C}) \cap C^1(\Omega, \mathbb{C})$ *be a mapping whose Jacobian* $\det Df$ *vanishes nowhere in* $\Omega$ *so that* $f$ *is an open mapping of* $\Omega$ *onto the domain* $\Omega' = f(\Omega)$*. Then* $f$ *is injective if at least one of the following conditions is satisfied*:

(i) $f$ *maps* $\partial\Omega$ *into a closed Jordan curve* $\gamma$ *in* $\mathbb{C}$;
(ii) $f$ *maps* $\Omega$ *into a simply connected domain* $\hat{\Omega}$ *and* $\partial\Omega$ *into* $\partial\hat{\Omega}$.

*Proof.* First we will show that $\partial\Omega' \subset f(\partial\Omega)$. In fact, for an arbitrary point $z \in \partial\Omega'$ we can find a sequence of points $z_n \in \Omega'$ converging to $z$, and another sequence of points $w_n \in \Omega$ such that $f(w_n) = z_n$ and $w_n \to w$ for some $w \in \bar{\Omega}$. Since $f$ is continuous on $\bar{\Omega}$, we obtain $f(w) = z$, and this implies $w \in \partial\Omega$ as the mapping $f$ is open. Thus we have proved that $\partial\Omega' \subset f(\partial\Omega)$.

Let us now assume that (i) holds true. Then $\mathbb{C} \setminus \gamma$ consists of two components, the simply connected interior $\hat{\Omega}$ of $\gamma$, and the unbounded exterior $\tilde{\Omega}$.

Because of

$$\partial\Omega' \subset f(\partial\Omega) \subset \gamma = \partial\tilde{\Omega}$$

we obtain

$$\overline{\Omega'} \cap \tilde{\Omega} = \Omega' \cap \tilde{\Omega},$$

and this implies $\Omega' \cap \tilde{\Omega} = \emptyset$, whence $f(\Omega) = \Omega' \subset \hat{\Omega}$ and $f(\partial\Omega) \subset \gamma = \partial\hat{\Omega}$. Thus we are in the situation described by (ii). Let us now consider this case. Repeating the previous reasoning, we get $\overline{\Omega'} \cap \hat{\Omega} = \Omega' \cap \hat{\Omega}$, and we conclude that $f(\Omega) = \Omega' = \hat{\Omega}$, and therefore $\partial f(\Omega) = \partial\hat{\Omega}$.

The injectivity of $f$ can now be justified by a standard monodromy argument. Suppose that two points $w_1$ and $w_2$ in $\Omega$ were mapped onto the same image point $z \in \hat{\Omega}$. Any arc $\alpha$ in $\Omega$ joining $w_1$ and $w_2$ will be mapped by $f$ onto a closed curve $\beta$ in $\hat{\Omega}$ since $f(w_1) = f(w_2) = z$. By some homotopy in the simply connected domain $\hat{\Omega}$ we can shrink $\beta$ to the point $z$. Since $f$ is a local diffeomorphism in $\Omega$, each curve of the homotopy is the image of an arc in $\Omega$ which joins $w_1$ and $w_2$, and this curve must be closed as soon as its image lies in a sufficiently small neighborhood of $z$. □

For a more detailed proof of the monodromy principle in $\mathbb{C}$ we refer the reader to a suitable text book of complex analysis such as Ahlfors [5] or Bieberbach [2]. (More general versions of this principle in algebraic topology can for instance be found in Greenberg [1].)

The next two lemmata contain the essential ideas needed for the proof of the theorem.

We shall again encounter the reasoning employed in the proof of the following lemma in Chapters 1 and 2 of Vol. 3 where similar uniqueness theorems for surfaces with semifree boundaries will be proved.

**Lemma 2 (Radó's lemma).** *If $f : \bar{B} \to \mathbb{R}$ is harmonic in $B$, continuous on $\bar{B}$, not identically zero, and if its derivatives of orders $0, 1, \dots, m$ vanish at some point $w_0 \in B$, then $f$ changes its sign on $\partial B$ at least $2(m+1)$ times.*

*Proof.* The function $f$ is the real part of a holomorphic function $F : B \to \mathbb{C}$ whose power series expansion close to $w_0$ is given by

$$F(w) = i\beta_0 + a_n(w - w_0)^n + O(|w - w_0|^{n+1})$$

for $|w - w_0| \to 0$, where $n \geq m + 1, a_n \neq 0$, and $\beta_0$ is real. Consequently the set $\{w \in B \colon f(w) = 0\}$ divides a neighborhood of $w_0$ into $2n$ open sectors $\sigma_1, \sigma_2, \dots, \sigma_{2n}$ by means of $2n$ analytic arcs emanating from $w_0$ such that $f$ is positive on $\sigma_1, \sigma_3, \dots, \sigma_{n-1}$ and negative on $\sigma_2, \sigma_4, \dots, \sigma_{2n}$, cf. Fig. 1.

The set $\{w \in B \colon f(w) \neq 0\}$ is open, therefore it has at most denumerably many connected components. Let $Q_1, Q_2, \dots, Q_{2n}$ be the components containing the sectors $\sigma_1, \sigma_2, \dots, \sigma_{2n}$ respectively. We claim that no two of them coincide.

Suppose for example that $Q_{2k} = Q_{2l}, k \neq l$. Then we can construct a (piece-wise linear) closed Jordan curve $\gamma$ starting at $w_0$, running first through

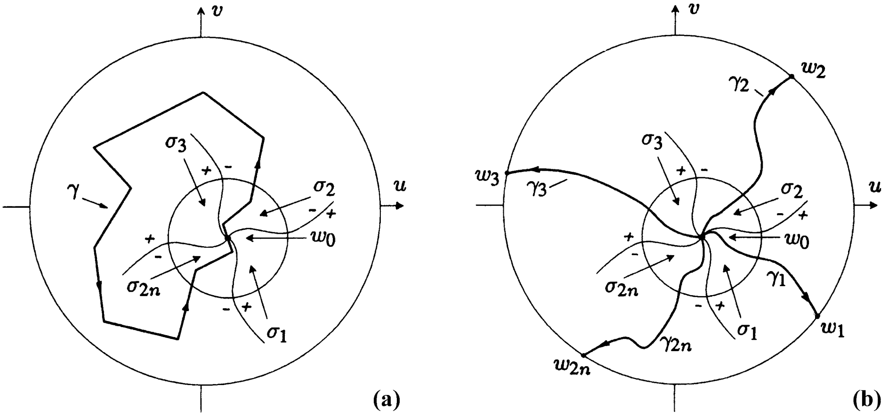



**Fig. 1.** Rado's lemma: Sectors

the sector $\sigma_{2k}$ and finally traversing $\sigma_{2l}$ before it returns to $w_0$. Now either the sector $\sigma_{2k-1}$ or $\sigma_{2k+1}$ belongs to the bounded component $\Omega$ of $\mathbb{C} \setminus \gamma$. The function $f$ is non-positive on $\gamma = \partial\Omega$ but positive on $\sigma_{2k-1}$ and on $\sigma_{2k+1}$. The maximum principle applied to the harmonic function $f : \Omega \to \mathbb{R}$ yields the desired contradiction, and the remaining cases are excluded similarly.

Another application of the maximum principle shows that each of the components $Q_j$, $j = 1, \ldots, 2n$, has a boundary point $w_j \in \partial Q_j$ lying on $\partial B$ such that $f(w_j)$ is positive for $j = 1, 3, \ldots, 2n-1$ and negative for $j = 2, 4, \ldots, 2n$. Moreover, for any of these $w_j$ we can construct a path $\gamma_j$ in $Q_j$ starting in the sector $\sigma_j$ and ending at $w_j$. Since these paths $\gamma_j$ do not intersect, the pattern of the points $w_j$ on $\partial B$ reflects the one of sectors $\sigma_j$ close to $w_0$. Thus between any $w_j$ and its successor $w_{j+1}$ the continuous function $f|_{\partial B}$ has a zero. □

The third lemma is a variant of Lemma 2 and a consequence of the monodromy principle. The conclusions are the same, but the assumptions are different. This result is known as **Kneser's lemma** (cf. T. Radó [5], H. Kneser [1]).

**Lemma 3.** *Suppose that $\varphi : \bar{B} \to \mathbb{R}^2$ is a transformation which is harmonic in $B$, continuous in $\bar{B}$, and which maps $\partial B$ in a weakly monotonic manner onto the boundary $\partial\Omega$ of a convex domain $\Omega \subset \mathbb{R}^2$. Then $\varphi$ is a diffeomorphism from $B$ onto $\Omega$. If in addition $\varphi : \partial B \to \partial\Omega$ is a homeomorphism, then so is $\varphi : \bar{B} \to \bar{\Omega}$.*

*Proof.* This lemma will follow immediately from the monodromy principle, if we can show that the Jacobian $\det D\varphi$ of the transformation $\varphi$ has no zeros in $B$.

First of all, the maximum principle for harmonic functions implies that $\varphi(B)$ lies in $\Omega$. Now, if $\det D\varphi(w_0) = 0$ for some $w_0 \in B$, then the rows of the

Jacobi matrix $D\varphi(w_0)$ are linearly dependent, i.e., there are real constants $a$ and $b$, at least one of which is nonzero and such that

$$ax_u(w_0) + by_u(w_0) = 0$$

and

$$ax_v(w_0) + by_v(w_0) = 0$$

where $x$ and $y$ are the real and the imaginary parts of $\varphi$ respectively. Moreover there is a real number $c$ such that

$$ax(w_0) + by(w_0) + c = 0,$$

which means that $\varphi(w_0)$ lies on the straight line

$$L(w_0) = \{x + iy\colon ax + by + c = 0\}$$

which intersects $\partial\Omega$ in exactly two points $P_1$ and $P_2$. Let us now inspect the harmonic function

$$f(w) = ax(w) + by(w) + c$$

which is continuous on $\bar{B}$. Note that $\varphi$ maps $\partial B$ onto $\partial\Omega$. Then, for any $w \in \partial B$, the function $f(w)$ vanishes if and only if $\varphi(w)$ lies on the intersection of $\partial\Omega$ with the straight line $L(w_0)$. Furthermore, since $\varphi$ maps $\partial B$ in a weakly monotonic manner onto $\partial\Omega$, the pre-image

$$\varphi^{-1}\{P_1, P_2\} = \varphi^{-1}(\partial\Omega \cap L(w_0)) = f^{-1}_{|\partial B}\{0\}$$

consists of two closed connected subarcs of $\partial B$. On the other hand, Radó's lemma implies that $f$ has at least four zeros in $\partial B$ which are separated by points where $f$ does *not* vanish. This contradiction shows that the assumption $\det D\varphi(w_0) = 0$ is impossible. □

Now we turn to the

*Proof of Theorem 1.* After a rotation of coordinates we may suppose that the parallel projection mentioned in the theorem is the orthogonal projection onto the $xy$-plane. Then $\Gamma$ possesses a 1–1 orthogonal projection $\overline{\gamma}$ which is a convex curve contained in the plane $\{z = 0\}$. Replacing $\gamma$ by $\overline{\gamma}$, we may assume that $\Gamma$ lies as a graph above a plane convex curve $\gamma$ which is contained in the plane $\{z = 0\}$. Therefore the preceding lemma shows that the first two components $x$ and $y$ of any minimal surface $X = (x, y, z) \in \mathcal{C}(\Gamma)$ which solves Plateau's problem for $\Gamma$ determine a diffeomorphism $\varphi$ from $B$ onto the convex domain $\Omega$ enclosed by $\gamma$. Denoting the inverse of $\varphi$ by $(u(x, y), v(x, y))$, the function

$$Z(x, y) := z(u(x, y), v(x, y))$$

defines a nonparametric representation of the surface $X(B)$. Of course, $X$ has no branch points since its first two components define a diffeomorphism. Consequently $X(B)$ is a regular embedded surface whose mean curvature vanishes.

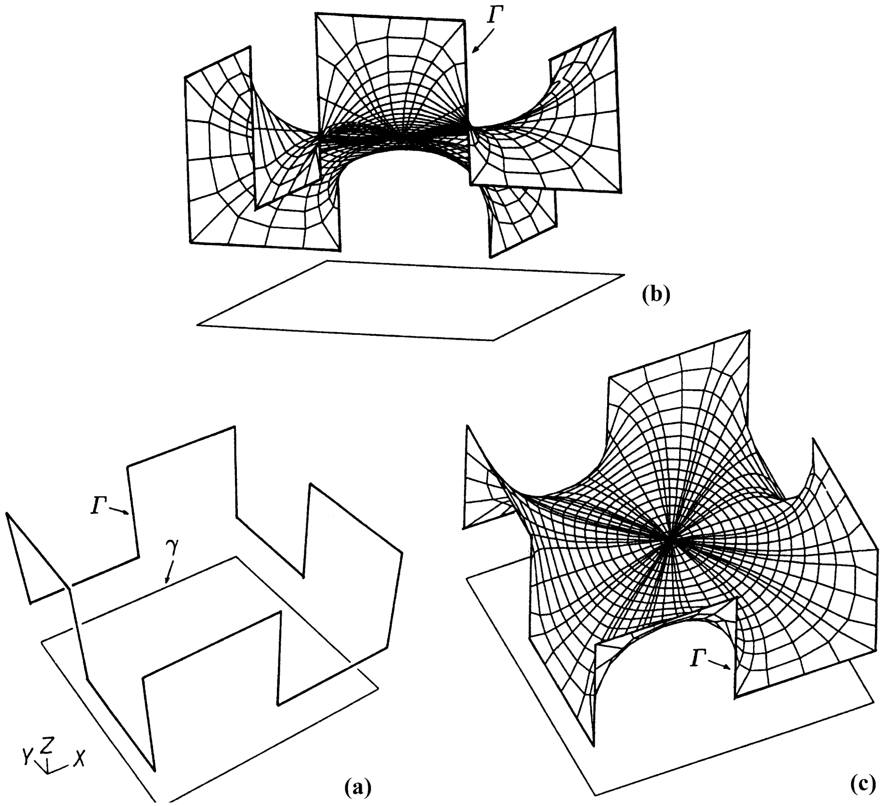



**Fig. 2.** Rado's uniqueness theorem (strengthened): The Jordan curve $\Gamma$ is a generalized graph over a plane convex curve, the square shown in **(a)**. The solution to Plateau's problem for $\Gamma$ is therefore unique **(b)**, **(c)**

Thus, as we have seen in Section 2.2, $Z(x, y)$ is a solution of the minimal surface equation with bounded, but not necessarily continuous boundary values.

Suppose now that $X$ and $\hat{X}$ are two solutions of Plateau's problem for $\Gamma$, and denote their corresponding non-parametric representations by $Z(x, y)$ and $\hat{Z}(x, y)$ respectively. If the projection of $\Gamma$ onto $\gamma$ is one-to-one, then $\varphi$ is a homeomorphism from $\bar{B}$ onto $\bar{\Omega}$. Consequently, since $X$ and $\hat{X}$ are continuous on $\bar{B}$, the functions $Z$ and $\hat{Z}$ are continuous on $\bar{\Omega}$, and so is the difference $Z - \hat{Z}$, which vanishes on $\partial\Omega$. Moreover $Z - \hat{Z}$ satisfies a second order linear equation in $\Omega$ for which the maximum principle holds true (cf. Gilbarg and Trudinger [1], p. 208). This implies that $Z$ and $\hat{Z}$ coincide in $\bar{\Omega}$ so that we have in particular $X(B) = \hat{X}(B)$.

Now since $X$ and $\hat{X}$ are conformal and invertible, $X^{-1} \circ \hat{X}$ is a conformal mapping from $B$ onto itself. Thus a three-point-condition guarantees that $X$ is equal to $\hat{X}$. $\square$

**Remark to Theorem 1.** The uniqueness result of Theorem 1 remains true under somewhat weaker assumptions on $\Gamma$. Instead of requiring the existence of a 1-to-1 parallel projection of $\Gamma$ onto a plane convex curve $\gamma$, we can allow vertical segments for $\Gamma$ which are mapped onto single points of $\gamma$. For the proof of this more general fact one needs a sharpening of the maximum principle provided by Nitsche [11]; see also Nitsche [28], §401 and §586. This reasoning is essentially based on an extension of Theorem 6 in Section 7.3.

Concluding this section, we want to draw some further results from Radó's lemma.

**Theorem 2.** *If $w_0 \in B$ is an interior branch point of the minimal surface $X \in \mathcal{C}(\Gamma)$, then each plane $\Pi$ through the point $X(w_0)$ intersects $\Gamma$ in at least four distinct points.*

*Proof.* Let $\nu \in S^2$ be a vector normal to $\Pi$. Then we have

$$\Pi = \{x \in \mathbb{R}^3 \colon \langle x - X(w_0), \nu \rangle = 0\}.$$

Consider the function $f : \overline{B} \to \mathbb{R}^3$ defined by

$$f(w) := \langle X(w) - X(w_0), \nu \rangle, \quad w \in \overline{B},$$

which is continuous on $\overline{B}$, harmonic in $B$, and satisfies

$$f(w_0) = 0, \quad f_u(w_0) = 0, \quad f_v(w_0) = 0.$$

On account of Lemma 2 it follows that $f$ has at least four zeros on $\partial B$. □

**Corollary 1.** *If there is a straight line $\mathcal{L}$ in $\mathbb{R}^3$ such that each plane $\Pi$ through $\mathcal{L}$ intersects $\Gamma$ in at most three points, then any minimal surface $X \in \mathcal{C}(\Gamma)$ is free of interior branch points.*

An immediate consequence of this result is

**Corollary 2.** *A minimal surface $X \in \mathcal{C}(\Gamma)$ has no interior branch points if $\Gamma$ possesses a one-to-one parallel or central projection onto a star-shaped planar curve.*

## 4.10 Another Solution of Plateau's Problem by Minimizing Area

In this section we want to present a solution of the *minimal area problem* for disk-type surfaces which is obtained by minimizing the functional $A^\epsilon := (1-\epsilon)A + \epsilon D$ in the class $\mathcal{C}(\Gamma)$. This will lead to a direct solution of the *simultaneous problem* of finding a minimal surface of class $\mathcal{C}(\Gamma)$ that minimizes both the area functional

$$A(X) = \int_B |X_u \wedge X_v| \, du \, dv$$

and the Dirichlet integral

$$D(X) = \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2) \, du \, dv$$

among all admissible surfaces $X \in \mathcal{C}(\Gamma)$. Thereby we obtain another proof of Theorem 4 and of relation (40) in Section 4.5.

We begin by recalling that $D$ is (sequentially) weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$; cf. 4.6. It turns out that $A$ has the same property:

**Lemma 1.** *Let $\{X_n\}$ be a sequence in $H_2^1(B, \mathbb{R}^3)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$. Then*

$$A(X) \le \liminf_{n \to \infty} A(X_n). \tag{1}$$

*Proof.* First we note the identity

$$A(Z) = \sup \left\{ \int_B \phi \cdot (Z_u \wedge Z_v) \, du \, dv \colon \phi \in C_c^\infty(B, \mathbb{R}^3), \ |\phi| \le 1 \right\} \tag{2}$$

which holds for any $Z \in H_2^1(B, \mathbb{R}^3)$; it can easily be verified.

We claim that for proving (1) it suffices to show

$$\lim_{n \to \infty} \int_B \phi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv = \int_B \phi \cdot (X_u \wedge X_v) \, du \, dv \tag{3}$$

for any $\phi \in C_c^\infty(B, \mathbb{R}^3)$ satisfying $|\phi| \le 1$. In fact, equations (2) and (3) imply

$$\begin{aligned} \int_B \phi \cdot (X_u \wedge X_v) \, du \, dv &= \lim_{n \to \infty} \int_B \phi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv \\ &\le \liminf_{n \to \infty} \left[ \sup \left\{ \int_B \psi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv \colon \psi \in C_c^\infty(B, \mathbb{R}^3), \ |\psi| \le 1 \right\} \right] \\ &= \liminf_{n \to \infty} A(X_n). \end{aligned}$$

Taking the supremum over all $\phi$ in $C_c^\infty(B, \mathbb{R}^3)$ with $|\phi| \le 1$ we then arrive at (1).

Thus it suffices to verify (3). Let $Z$ be of class $C^2(B, \mathbb{R}^3)$; then for $\phi \in C_c^\infty(B, \mathbb{R}^3)$ an integration by parts yields

$$\int_B \phi \cdot (Z_u \wedge Z_v) \, du \, dv = -\frac{1}{2} \int_B \left[ \phi_u \cdot (Z \wedge Z_v) + \phi_v \cdot (Z_u \wedge Z) \right] du \, dv. \tag{4}$$

Using a suitable approximation device, this identity can as well be established for arbitrary $Z \in H_2^1(B, \mathbb{R}^3)$.

Suppose now that $X_n \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. By Rellich's theorem we obtain $X_n \to X$ in $L_2(B, \mathbb{R}^3)$, and so (3) follows from (4). □

Next we define the functionals $A^\epsilon : H_2^1(B, \mathbb{R}^3) \to \mathbb{R}$ by

$$A^\epsilon := (1-\epsilon)A + \epsilon D, \quad 0 \le \epsilon \le 1.$$

Since $A$ and $D$ are weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$ also $A^\epsilon$ has this property, i.e. we have

**Lemma 2.** *If $X_n \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$ then*

$$A^\epsilon(X) \le \liminf_{n\to\infty} A^\epsilon(X_n)$$

*for any $\epsilon \in [0,1]$.*

Our goal is now to find a conformally parametrized minimizer of $A$ in $\mathcal{C}(\Gamma)$. As $A$ is a somewhat singular functional we take a detour by first considering the *modified variational problem*

(5) $$A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma)$$

for an arbitrary $\epsilon \in (0,1]$. As $A^\epsilon$ is conformally invariant we can find a minimizing sequence $\{X_n\}$ for $A^\epsilon$ in $\mathcal{C}(\Gamma)$ that satisfies a fixed three-point condition, i.e.

$$A^\epsilon(X_n) \to \alpha(\epsilon) := \inf_{\mathcal{C}(\Gamma)} A^\epsilon = \inf_{\mathcal{C}^*(\Gamma)} A^\epsilon$$

and $X_n \in \mathcal{C}^*(\Gamma)$ if we use the notation of 4.3. Then

$$(1-\epsilon)A(X_n) + \epsilon D(X_n) = A^\epsilon(X_n) \le \alpha(\epsilon) + 1 \quad \text{for } n \gg 1,$$

whence

$$D(X_n) \le \text{const} \quad \text{for all } n \in \mathbb{N}.$$

Now we can proceed as in the proof of Theorem 1 in 4.6: We obtain a subsequence $\{X_{n_p}\}$ of $\{X_n\}$ that tends weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X^\epsilon$ which is contained in $\mathcal{C}^*(\Gamma)$ as this set is weakly (sequentially) closed in $H_2^1(B, \mathbb{R}^3)$. It follows that

$$\alpha(\epsilon) \le A^\epsilon(X^\epsilon) \le \lim_{p\to\infty} A^\epsilon(X_{n_p}) = \alpha(\epsilon),$$

and so $A^\epsilon(X^\epsilon) = \alpha(\epsilon)$. Thus, for any $\epsilon > 0$, we have found a minimizer $X^\epsilon \in \mathcal{C}^*(\Gamma)$ of $A^\epsilon$ in $\mathcal{C}(\Gamma)$. As in 4.5 this minimum property implies

(6) $$\partial A^\epsilon(X^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1(\overline{B}, \mathbb{R}^2).$$

Since $A$ is parameter invariant it follows that

$$\partial A^\epsilon(X^\epsilon, \lambda) = \epsilon \partial D(X^\epsilon, \lambda),$$

and so we obtain

(7) $$\partial D(X^\epsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1(\overline{B}, \mathbb{R}^2).$$

By Theorem 1 of 4.5 we see that $X^\epsilon$ satisfies the conformality relations

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad \langle X_u^\epsilon, X_v^\epsilon\rangle = 0. \tag{8}$$

Before we proceed we remark the following: In proving relation (6) we have used the *Riemann mapping theorem*. This can be avoided by using the method presented in the Supplementary Remark 1 of 4.5: Using vector fields $\lambda \in C^1(\overline{B}, \mathbb{R}^3)$ such that $\lambda(w)$ is tangential to $\partial B$ for any $w \in \partial B$ we construct diffeomorphisms $\tau_\epsilon$ of $\overline{B}$ onto itself which are of the form $\tau_\epsilon(w) = w - \epsilon\lambda(w) + o(\epsilon)$ as $\epsilon \to 0$. Let us denote the class of these vector fields by $C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$. Then we arrive at

$$\partial A^\epsilon(X^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$$

without employing the Riemann mapping theorem. This leads to

$$\partial D(X^\epsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

and by the formulae derived in Example 1 of 4.5 we arrive at

$$\int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv = 0 \quad \text{for all } \lambda = (\mu, \nu) \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2) \tag{9}$$

where

$$a := |X_u^\epsilon|^2 - |X_v^\epsilon|^2, \quad b := 2\langle X_u^\epsilon, X_v^\epsilon\rangle.$$

We claim that $a, b$ satisfy the Cauchy–Riemann equations

$$a_u = -b_v, \quad a_v = b_u \quad \text{on } B \tag{10}$$

whence $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u + iv$ in $B$. Since we do not yet know that $X^\epsilon$ is harmonic in $B$, we cannot derive (10) as in the Supplementary Remark 1 of 4.5. Instead we apply (9) to vector fields $\lambda$ of the form $\lambda = \mathcal{S}_\delta \eta$ with $\eta = (\eta^1, \eta^2) \in C_c^\infty(B', \mathbb{R}^2)$ with $B' \subset\subset B$, where $\mathcal{S}_\delta$ is a smoothing operator with a symmetric kernel $k_\delta$, $0 < \delta \ll 1$, i.e. $\mathcal{S}_\delta \eta = k_\delta * \eta$. Set

$$a^\delta := \mathcal{S}_\delta a, \quad b^\delta := \mathcal{S}_\delta b.$$

Then we obtain

$$\begin{aligned}
0 &= \int_B \big\{a[(\mathcal{S}_\delta\eta^1)_u - (\mathcal{S}_\delta\eta^2)_v] + b[(\mathcal{S}_\delta\eta^1)_v + (\mathcal{S}_\delta\eta^2)_u]\big\}\, du\, dv \\
&= \int_B \big\{a[\mathcal{S}_\delta(\eta^1_u) - \mathcal{S}_\delta(\eta^2_v)] + b[\mathcal{S}_\delta(\eta^1_v) + \mathcal{S}_\delta(\eta^2_u)]\big\}\, du\, dv \\
&= \int_B \big\{a^\delta(\eta^1_u - \eta^2_v) + b^\delta(\eta^1_v + \eta^2_u)\big\}\, du\, dv \\
&= \int_B \big\{-(a^\delta_u + b^\delta_v)\eta^1 + (a^\delta_v - b^\delta_u)\eta^2\big\}\, du\, dv
\end{aligned}$$

since $\mathcal{S}_\delta$ commutes with $\partial/\partial u$ and $\partial/\partial v$ and

$$\int_B f \cdot \mathcal{S}_\delta \varphi \, du \, dv = \int_B \mathcal{S}_\delta f \cdot \varphi \, du \, dv$$

for $f \in L_1(B)$ and $\varphi \in C_c^\infty(B')$, $B' \subset\subset B$, $0 < \delta \ll 1$. By the fundamental theorem of the calculus of variations it follows that

$$a_u^\delta + b_v^\delta = 0 \quad \text{and} \quad a_v^\delta - b_u^\delta = 0 \quad \text{in } B' \subset\subset B.$$

In other words: For any fixed $B' \subset\subset B$ the function $\Phi^\delta(w) := a^\delta(u,v) - ib^\delta(u,v)$ is holomorphic for $w = u+iv \in B'$ if $0 < \delta < \delta_0(B')$ where $\delta_0(B') > 0$ is a sufficiently small number depending on $B'$. Since

$$\|a - a^\delta\|_{L_1(B')} \to 0 \quad \text{and} \quad \|b - b^\delta\|_{L_1(B')} \to 0 \quad \text{as } \delta \to +0$$

we obtain

$$\int_{B'} |\Phi - \Phi^\delta| \, du \, dv \to 0 \quad \text{as } \delta \to +0.$$

Since the $L_1$-limit of holomorphic functions is holomorphic we infer that $\Phi$ is holomorphic in $B' \subset\subset B$, and so it is holomorphic in $B$. Thus we have verified (10), and from now on we can proceed as in the Supplementary Remark 1 of 4.5 obtaining $\Phi(w) \equiv 0$ in $B$, i.e. $a(u,v) \equiv 0$ and $b(u,v) \equiv 0$ on $B$. Therefore we have verified the conformality relations

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad \langle X_u^\epsilon, X_v^\epsilon \rangle = 0 \quad \text{in } B$$

for any $\epsilon \in (0,1]$, which imply $A(X^\epsilon) = D(X^\epsilon)$, and we obtain

$$A^\epsilon(X^\epsilon) = A(X^\epsilon) = D(X^\epsilon) \quad \text{for } 0 < \epsilon \le 1.$$

On the other hand we infer from $A \le D$ and the minimum property of $X^\epsilon$ that

$$A^\epsilon(X^\epsilon) \le A^\epsilon(X) = (1-\epsilon)A(X) + \epsilon D(X) \le D(X)$$

holds for any $X \in \mathcal{C}(\Gamma)$ and any $\epsilon \in (0,1]$. Choosing $X = X^{\epsilon'}$ we arrive at

$$D(X^\epsilon) \le D(X^{\epsilon'}) \quad \text{for any } \epsilon, \epsilon' \in (0,1]$$

whence

$$(11) \qquad D(X^\epsilon) = A(X^\epsilon) = A^\epsilon(X^\epsilon) \equiv \text{const} =: c \quad \text{for } 0 < \epsilon \le 1.$$

Set

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then, for arbitrary $Z \in \mathcal{C}(\Gamma)$ and any $\epsilon, \epsilon' \in (0,1]$ we obtain

$$a(\Gamma) \le A(X^\epsilon) = A^\epsilon(X^\epsilon) = A^{\epsilon'}(X^{\epsilon'}) \le A^{\epsilon'}(Z)$$

and

$$e(\Gamma) \le D(X^\epsilon) = A^\epsilon(X^\epsilon) \le A^\epsilon(Z) \le D(Z).$$

Letting $\epsilon' \to +0$ the first set of inequalities yields

$$a(\Gamma) \le A(X^\epsilon) \le A(Z),$$

and the second furnishes

$$e(\Gamma) \le D(X^\epsilon) \le D(Z)$$

for all $Z \in \mathcal{C}(\Gamma)$. This implies

$$a(\Gamma) \le A(X^\epsilon) \le a(\Gamma) \quad \text{and} \quad e(\Gamma) \le D(X^\epsilon) \le e(\Gamma)$$

whence

$$a(\Gamma) = A(X^\epsilon) = D(X^\epsilon) = e(\Gamma) \quad \text{for all } \epsilon \in (0, 1].$$

Set $\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$ and

$$\overline{a}(\Gamma) = \inf_{\overline{\mathcal{C}}(\Gamma)} A, \quad \overline{e}(\Gamma) := \inf_{\overline{\mathcal{C}}(\Gamma)} D.$$

Then we know that every minimizer $X$ of $D$ in $\mathcal{C}(\Gamma)$ lies in $\overline{\mathcal{C}}(\Gamma)$, and so

$$a(\Gamma) \le \overline{a}(\Gamma) \le A(X) \le D(X) = e(\Gamma) = a(\Gamma)$$

and

$$e(\Gamma) \le \overline{e}(\Gamma) \le D(X) = e(\Gamma).$$

Thus we have $a(\Gamma) = \overline{a}(\Gamma) = A(X) = D(X) = e(\Gamma) = \overline{e}(\Gamma)$. In addition, every conformally parametrized minimizer $X$ of $A$ in $\mathcal{C}(\Gamma)$ satisfies $a(\Gamma) = A(X) = D(X)$. So we have proved

**Theorem 1.** *For any rectifiable curve $\Gamma$ in $\mathbb{R}^3$ one has*

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D, \tag{12}$$

*and any minimizer of Dirichlet's integral in $\mathcal{C}(\Gamma)$ is simultaneously a minimizer of area in $\mathcal{C}(\Gamma)$, and conversely every conformally parametrized minimizer of area in $\mathcal{C}(\Gamma)$ is as well a minimizer of Dirichlet's integral in $\mathcal{C}(\Gamma)$.*

**Remark 1.** Starting from (11) we alternatively could have argued in the following way: Applying the reasoning of 4.6 we obtain a sequence of positive numbers $\epsilon_j$ with $\epsilon_j \to 0$ and an $X \in \mathcal{C}^*(\Gamma)$ such that $X^{\epsilon_j} \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. Then

$$\begin{aligned} a(\Gamma) &\le A(X) \le \liminf_{j\to\infty} A(X^{\epsilon_j}) = \lim_{\epsilon\to 0} A^\epsilon(X^\epsilon) = c \\ &\le \lim_{\epsilon\to 0} A^\epsilon(Z) = A(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma) \end{aligned}$$

and so $a(\Gamma) \le A(X) \le a(\Gamma)$, i.e. $A(X) = a(\Gamma)$. Therefore the weak limit $X$ of the $X^{\epsilon_j}$ is a minimizer of $A$ in $\mathcal{C}(\Gamma)$. By (11) and the minimum property of $X^\epsilon$ we have

$$c = A^\epsilon(X^\epsilon) \le A^\epsilon(X) \quad \text{for } 0 < \epsilon \le 1,$$

and by $\epsilon \to +0$ we get

$$c \le A(X) \le D(X) \le \liminf_{j\to\infty} D(X^{\epsilon_j}) = c,$$

and so $c = A(X) = D(X)$, which implies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0$$

as well as $a(\Gamma) = e(\Gamma)$.

In other words, one can—by the detour via $A^\epsilon$—solve the variational problem "$A \to$ min in $\mathcal{C}(\Gamma)$" thereby simultaneously solving the problem "$D \to$ min in $\mathcal{C}(\Gamma)$" by a minimal surface $X \in \mathcal{C}(\Gamma)$.

**Remark 2.** We have proved Theorem 1 without using Riemann's mapping theorem. Therefore it is no circulus vitiosus if we try to prove this theorem by using the solution of Plateau's problem that is provided by Theorem 1. This idea will be carried out in the next section.

## 4.11 The Mapping Theorems of Riemann and Lichtenstein

First we want to show that the solution of Plateau's problem applied to planar curves provides a proof of **Riemann's mapping theorem**, which states the following:

*Suppose that $\Omega$ is a simply connected domain in $\mathbb{C}$ bounded by a closed Jordan curve $\Gamma$. Then there is a homeomorphism $\varphi$ from $\overline{\Omega}$ onto $\overline{B}$ which is holomorphic in $\Omega$ and provides a conformal mapping of $\Omega$ onto $B$, i.e. $\varphi'(z) \ne 0$ for all $z \in \Omega$.*

We prove an equivalent assertion:

**Theorem 1.** *Let $\Omega$ be a simply connected domain in $\mathbb{C}$ bounded by a closed Jordan curve $\Gamma$. Then there exists a homeomorphism $f$ from $\overline{B}$ onto $\overline{\Omega}$, $B := \{w \in \mathbb{C} : |w| < 1\}$, which is holomorphic in $B$ and satisfies $f'(w) \ne 0$ for all $w \in B$.*

*Proof.* (i) Firstly we prove the assertion under the additional assumption that the contour $\Gamma$ is rectifiable. We identify $\mathbb{C}$ with the $x^1, x^2$-plane $\mathbb{R}^2$ and consider a minimal surface $X = (X^1, X^2, X^3)$ of class $\mathcal{C}(\Gamma)$ which is continuous on $\overline{B}$. Since $\Gamma$ lies in the $x^1, x^2$-plane we obtain $X^3(w) \equiv 0$ on account of the maximum principle. Thus the conformality relations for $w = u + iv \in B$ read as

$$(1) \qquad |X^1_u|^2 + |X^2_u|^2 = |X^1_v|^2 + |X^2_v|^2,$$

$$(2) \qquad X^1_u X^1_v + X^2_u X^2_v = 0.$$

Equation (2) implies that

$$(3) \qquad X^1_v = -\lambda X^2_u, \quad X^2_v = \lambda X^1_u$$

holds for some function $\lambda : B \to \mathbb{R}$, and (1) yields that $\lambda(u, v) = \pm 1$ on $B \setminus \Sigma$ where $\Sigma$ denotes the set of branch points of $X$ in $B$. On $\Sigma$ equation (3) is satisfied for any choice of $\lambda$. Since the points of $\Sigma$ are isolated in $B$ it follows that either $\lambda(u, v) \equiv 1$ or $\lambda(u, v) \equiv -1$. In the first case we set $\lambda(u, v) := 1$ on $\Sigma$, and $\lambda(u, v) := -1$ in the second. Thus either $X^1, X^2$ or $X^1, -X^2$ satisfy the Cauchy–Riemann equations on $B$. By applying the reflection $z = x^1 + ix^2 \mapsto \overline{z} = x^1 - ix^2$ we can assume that the equations

$$(4) \qquad X^1_u = X^2_v, \quad X^1_v = -X^2_u$$

hold in $B$, and so $f(w) := X^1(u, v) + iX^2(u, v)$, $w = u + iv$, is holomorphic in $B$ and continuous on $\overline{B}$. Furthermore, $f|_{\partial B}$ yields a homeomorphism from $\partial B$ onto $\Gamma$. Therefore the loop $\varphi : [0, 2\pi] \to \mathbb{C}$ defined by $\varphi(t) := f(e^{it})$ has the winding numbers

$$(5) \qquad W(\varphi, z) := W(\varphi - z) = \begin{cases} 1 & \text{for } z \in \Omega, \\ 0 & \text{for } z \in \mathbb{C} \setminus \Omega. \end{cases}$$

For $0 < r < 1$ and $\varphi_r(t) := f(re^{it})$ we have

$$\max_{[0,2\pi]} |\varphi(t) - \varphi_r(t)| \to 0 \quad \text{as } r \to 1 - 0.$$

Hence for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(t) - \varphi_r(t)| < \epsilon \quad \text{for all } t \in [0, 2\pi], \text{ provided that } 1 - \delta < r < 1.$$

Then for any $z \in \mathbb{C}$ with $\text{dist}(z, \Gamma) > \epsilon$ we obtain

$$(6) \qquad W(\varphi_r, z) = W(\varphi, z).$$

Since $\varphi_r$ is real analytic we on the other hand have

$$(7) \qquad W(\varphi_r, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\dot{\varphi}_r(t)}{\varphi_r(t) - z}\, dt.$$

This equation can be written as

$$(8) \qquad \begin{aligned} W(\varphi_r, z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(re^{it})}{f(re^{it}) - z} ire^{it}\, dt \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{f'(w)}{f(w) - z}\, dz \end{aligned}$$

where $C_r$ denotes the positively oriented circle $\{re^{it}: 0 \le t \le 2\pi\}$ bounding the disk $B_r(0) := \{w \in \mathbb{C}: |w| < r\}$. By Rouché's formula we know that

$$\frac{1}{2\pi i}\int_{C_r} \frac{f'(w)}{f(w)-z}\,dz = n(f, B_r(0), z) \tag{9}$$

where $n(f, B_r(0), z)$ is the number of zeros of the function $f - z$ in $B_r(0)$ counted with respect to their multiplicities. From (5)–(9) we infer the following: For any $z \in \mathbb{C} \setminus \Gamma$ there is a $\delta \in (0,1)$ such that

$$n(f, B_r(0), z) = \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{if } z \notin \overline{\Omega}, \end{cases} \quad \text{provided that } 1-\delta < r < 1.$$

This implies for $z \in \mathbb{C} \setminus \Gamma$ that

$$n(f, B, z) = \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

In other words, the equation $f(w) = z$ has no solution $w \in B$ if $z \in \mathbb{C} \setminus \overline{\Omega}$, and exactly one solution $w \in B$ if $z \in \Omega$; this solution is a zero of order 1 for the function $f - z$. Thus $f$ yields a 1–1 mapping of $B$ onto $\Omega$ such that $f'(w) \ne 0$ for all $w \in B$, i.e. $f$ is a conformal mapping from $B$ onto $\Omega$. Moreover, $f$ maps $\partial B$ one-to-one onto $\Gamma$ (see 4.5, Theorem 3), and so $f$ provides a bijective mapping of $\overline{B}$ onto $\overline{\Omega}$. Since $f$ is continuous on $\overline{B}$ it finally follows that $f$ is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and so the assertion is proved in case that $\Gamma$ is rectifiable.

(ii) If $\Gamma$ is not rectifiable we choose a sequence of rectifiable Jordan curves $\Gamma_j$ that converge to $\Gamma$ in the sense of Fréchet as $j \to \infty$. Let $\Omega_j$ be the bounded component of $\mathbb{C} \setminus \Gamma_j$. On account of (i) there is for every $j \in \mathbb{N}$ a homeomorphism of $\overline{B}$ onto $\overline{\Omega}_j$ which maps $B$ conformally onto $\Omega_j$.

Now we proceed as in the proof of Theorem 3 in Section 4.3. Since we did only sketch this proof we shall now fill in the details for the convenience of the reader.

We can assume that the $f_j$ satisfy three-point conditions

$$f_j(w_k) = z_{k,j}, \quad k = 1,2,3,\ j \in \mathbb{N},$$

where $w_1, w_2, w_3$ are three different points on $\partial B$, and $z_{1,j}, z_{2,j}, z_{3,j}$ are three different points on $\Gamma_j$ converging to three different points $z_1, z_2, z_3$ on $\Gamma$ : $z_{k,j} \to z_k$ as $j \to \infty$.

Any pair of points $P_j$, $Q_j$ on $\Gamma_j$ divides $\Gamma_j$ into two subarcs $\Gamma_j'$ and $\Gamma_j''$. There is a $\sigma_0 > 0$ such that one of the two arcs contains at most one of the three points $z_{1,j}, z_{2,j}, z_{3,j}$ if $|P_j - Q_j| < \sigma_0$; let this arc be $\Gamma_j'$. Since $\Gamma_j \to \Gamma$ in the sense of Fréchet, there is a uniform estimate of the moduli of continuity of the Jordan curves $\Gamma_j$, i.e.: For every $\epsilon > 0$ there is a number $\sigma(\epsilon)$ with

$0 < \sigma(\epsilon) < \sigma_0$ such that $\operatorname{diam} \Gamma'_j < \epsilon$ holds for any "short" subarc $\Gamma'_j$ of $\Gamma_j$ provided that its endpoints $P_j, Q_j$ satisfy $|P_j - Q_j| < \sigma(\epsilon)$.

Moreover, there is a constant $M > 0$ such that $\operatorname{meas} \Omega_j \le M$ for all $j \in \mathbb{N}$. This implies

$$D(f_j) = A(f_j) = \operatorname{meas} \Omega_j \le M \quad \text{for all } j \in \mathbb{N}.$$

For $0 < r < 1$ and $w_0 \in \partial B$ we define the two-gon

$$S_r(w_0) := B \cap B_r(w_0)$$

which is bounded by the two closed circular arcs $C'_r$ and $C''_r$ with common endpoints $\zeta'_r$ and $\zeta''_r$ on $\partial B$ and $C''_r \subset \partial B$. By the Courant–Lebesgue lemma we obtain: For every $\delta \in (0,1)$ there is a number $\rho_j \in (\delta, \sqrt{\delta})$ such that the oscillation of $f_j$ on $C_{\rho_j}$ is estimated by

$$\operatorname{osc}(f_j, C'_{\rho_j}) \le \left\{ \frac{8\pi M}{\log 1/\delta} \right\}^{1/2} \quad \text{for all } w_0 \in \partial B.$$

For a given $\epsilon > 0$ we can find a number $\tau(\epsilon) > 0$ such that for $0 < \delta < \tau(\epsilon)$ the arc $C''_{\sqrt{\delta}}$ contains at most one of the points $z_k$ (and so $f_j$ maps $C''_{\rho_j}$ onto the short arc $\Gamma''_j$ with the endpoints $f_j(\zeta'_{\rho_j})$ and $f_j(\zeta''_{\rho_j})$), and secondly that

$$\operatorname{osc}(f_j, C'_{\rho_j}(w_0)) < \sigma(\epsilon).$$

It follows that

$$\operatorname{osc}(f_j, C''_{\rho_j}(w_0)) < \epsilon \quad \text{for all } w_0 \in \partial B \text{ and } j \in \mathbb{N}.$$

Since $f_j$ maps $\partial B$ homeomorphically onto $\Gamma$, we conclude that

$$\operatorname{osc}(f_j, C''_{\delta}(w_0)) < \epsilon \quad \text{for all } w_0 \in \partial B \text{ and } j \in \mathbb{N},$$

provided that $0 < \delta < \tau(\epsilon)$. Furthermore $f_j(\partial B) = \Gamma_j \to \Gamma$ implies

$$\max_{\partial B} |f_j| \le \text{const} \quad \text{for all } j \in \mathbb{N},$$

and so $\{f_j|_{\partial B}\}$ is compact in $C^0(\partial B, \mathbb{C})$ equipped with the sup-norm on $\partial B$. Thus, after renumbering, we may assume that $\{f_j|_{\partial B}\}$ converges uniformly on $\partial B$ to some continuous function. By virtue of Weierstrass's theorem we obtain $f_j \rightrightarrows f$ for some $f \in C^0(\overline{B}, \mathbb{C})$, and $f \in \mathcal{C}^*(\Gamma)$ as $f_j \in \mathcal{C}^*(\Gamma_j)$ and $\Gamma_j \to \Gamma$, where the $*$ denotes the corresponding three-point conditions $f_j(w_k) = z_{k,j}$ and $f(w_k) = z_k$ with $z_{k,j} \to z_k$ as $j \to \infty$. The uniform limit of holomorphic functions is holomorphic. Therefore $f$ is holomorphic in $B$, continuous on $\overline{B}$, and non-constant as $f$ is of class $\mathcal{C}^*(\Gamma)$. By a theorem of Hurwitz the uniform limit of injective holomorphic maps is injective, provided that this limit is nonconstant. Consequently the holomorphic mapping $f|_B$ is injective, and so it maps $B$ conformally onto the open set $f(B)$ and $\partial B$ continuously and weakly monotonically onto $\Gamma$. Since $f$ is open it follows that $f(B)$ is the inner domain $\Omega$ of the Jordan contour $\Gamma$, and Theorem 3 of 4.5 yields that $f|_{\partial B}$ maps $\partial B$ one-to-one onto $\Gamma$. Thus $f$ is a continuous bijective mapping from $\overline{B}$ onto $\overline{\Omega}$, and so it is a homeomorphism. $\square$

**Remark 1.** A lucid presentation of the properties of the winding number can be found in Sauvigny [15], Vol. 1, III.1.

**Remark 2.** The mapping $f$ in Theorem 1 is essentially unique. In fact, if $f_1$ and $f_2$ are two (strictly) conformal mappings of $B$ onto $\Omega$ then $f_1^{-1} \circ f_2$ is a (strictly) conformally automorphism $\tau$ of $B$, i.e.

$$f_2 = f_1 \circ \tau \quad \text{with } \tau(w) = e^{i\varphi} \frac{w-a}{1-\overline{a}w}, \; a \in B, \; 0 \le \varphi < 2\pi.$$

**Remark 3.** We now want to sketch another proof of Theorem 1 for a rectifiable contour which in essence describes the approach to proving Lichtenstein's theorem that will follow next. So let us return to the mapping $f := X^1 + iX^2$ which we can assume to be holomorphic in $B$. Moreover, $f$ is continuous on $\overline{B}$, and $f|_{\partial B}$ provides a homeomorphism from $\partial B$ onto $\Gamma$. Hence $f(w) \not\equiv \text{const}$ on $B$, and therefore $f$ is an open mapping from $B$ onto the open set $f(B)$. Furthermore, $f(\overline{B})$ is compact, and we conclude that $f(\partial B) = \partial f(B) = \Gamma$ and $\Omega = \operatorname{int} f(\overline{B}) = f(B)$. The set of zeros of $f'$ in $B$ coincides with the set $\Sigma$ of branch points of $X$ in $B$. We claim that $\Sigma$ is empty and $f$ is univalent in $B$. In fact if $f'(w_0) = 0$ and $z_0 := f(w_0)$ for some $w_0 \in B$ it follows by Rouché's theorem in connection with Theorem 1 of 4.7 that for any $z \in \Omega$ the function $f(w) - z$ has at least two zeros $w_1$ and $w_2$ in $B$, except if $z$ is the image of a branch point. Since $\Sigma$ is at most denumerable it follows that

$$N(f, B, z) \ge 2 \quad \text{for almost all } z \in \Omega,$$

where $N(f, B, z)$ denotes the number of different solutions $w \in B$ for the equation $f(w) = z$ with $z \in \Omega$. Since the area of $X$ is given by

$$A(X) = \int_B |f'(w)|^2 \, du \, dv$$

the area formula yields for $z = x^1 + ix^2$ that

$$A(X) = \int_\Omega N(f, B, z) \, dx^1 \, dx^2 \ge 2 \operatorname{meas} \Omega. \tag{10}$$

Moreover we may assume that $X$ minimizes $A$, taking Theorem 1 of 4.10 into account. Since $f(B) = \Omega$, inequality (10) contradicts the minimizing property of $X$, and so we obtain $N(f, B, z) = 1$ for all $z \in \Omega$. Consequently $f|_B$ is injective and $\Sigma$ is empty. Now one concludes as before that $f$ is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$ that maps $B$ conformally onto $\Omega$.

Now we shall use the approach of Section 4.10, combined with the ideas described in the preceding Remark 3 to give a proof of *Lichtenstein's theorem* (cf. 1.4). As we shall nowhere base the reasoning of this book onto this result we may use some regularity results that are proved only later in Chapters 7 and 8.

Let $B$ again be the standard unit disk $\{w \in \mathbb{R}^2 : |w| < 1\}$, $w = (u, v)$, equipped with the Euclidean metric

$$ds_e^2 := du^2 + dv^2,$$

and $\Omega$ be a simply connected, open set in $\mathbb{R}^2$, bounded by a closed rectifiable Jordan curve $\Gamma$. We assume that $\overline{\Omega}$ carries a Riemannian metric

$$ds^2 := g_{jk}(x)\, dx^j\, dx^k, \quad x = (x^1, x^2).$$

For mappings $\tau \in H_2^1(B, \mathbb{R}^2)$ we define the "Gauss functions" $\mathcal{E}(\tau), \mathcal{F}(\tau)$, $\mathcal{G}(\tau) : B \to \mathbb{R}$ by

$$\mathcal{E}(\tau) := g_{jk}(\tau)\tau_u^j \tau_u^k, \quad \mathcal{G}(\tau) := g_{jk}(\tau)\tau_v^j\, \tau_v^k, \quad \mathcal{F}(\tau) := g_{jk}(\tau)\tau_u^j\, \tau_v^k.$$

We call $\tau$ *weakly conformal* if $\tau$ satisfies the *conformality relations*

$$\mathcal{E}(\tau) = \mathcal{G}(\tau), \quad \mathcal{F}(\tau) = 0. \tag{11}$$

**Definition 1.** *A* **conformal mapping** *from $\overline{B}$ onto $\Omega$ is a diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$ satisfying the conformality relations* (11).

The pull-back $\tau^* \, ds^2$ of $ds^2$ by a diffeomorphism $\tau : \overline{B} \to \overline{\Omega}$ from $\overline{\Omega}$ to $\overline{B}$ is given by

$$\tau^* \, ds^2 = \mathcal{E}(\tau)\, du^2 + 2\mathcal{F}(\tau)\, du\, dv + \mathcal{G}(\tau)\, dv^2.$$

For a conformal mapping $\tau : \overline{B} \to \overline{\Omega}$ we have

$$\lambda := \mathcal{E}(\tau) = \mathcal{G}(\tau) > 0 \quad \text{on } \overline{B}$$

and

$$\tau^* \, ds^2 = \lambda(u, v) \cdot (du^2 + dv^2).$$

It follows from (11) that the components $\tau^1, \tau^2$ of a conformal mapping $\tau(u, v) = (\tau^1(u, v), \tau^2(u, v))$, satisfy the *Beltrami equations*

$$\begin{aligned} \sqrt{g(\tau)}\tau_v^1 &= -\rho[g_{12}(\tau)\tau_u^1 + g_{22}(\tau)\tau_u^2], \\ \sqrt{g(\tau)}\tau_v^2 &= \rho[g_{11}(\tau)\tau_u^1 + g_{12}(\tau)\tau_u^2] \end{aligned} \tag{12}$$

where

$$g(x) := \det(g_{jk}(x))$$

and either $\rho(u, v) \equiv 1$ or $\rho(u, v) \equiv -1$. From (12) it follows that

$$\sqrt{g(\tau)}\,\det D\tau = \rho\,\mathcal{E}(\tau).$$

Thus $\tau$ is orientation preserving or reversing if $\rho = 1$ or $\rho = -1$ respectively. The Riemannian analogue of the area functional is

$$A(\tau) := \int_B \sqrt{\mathcal{E}(\tau)\mathcal{G}(\tau) - \mathcal{F}^2(\tau)}\,du\,dv = \int_B \sqrt{g(\tau)}|\det D\tau|\,du\,dv$$

and the corresponding Dirichlet integral is defined as

$$D(\tau) := \frac{1}{2}\int_B [\mathcal{E}(\tau) + \mathcal{G}(\tau)]\,du\,dv.$$

We now state the following global version of **Lichtenstein's theorem**:

**Theorem 2.** *Suppose that $\Gamma \in C^{m,\alpha}$ and $g_{jk} \in C^{m-1,\alpha}(\overline{\Omega})$ for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there is a conformal mapping $\tau$ from $\overline{B}$ onto $\overline{\Omega}$ which is of class $C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$.*

*Proof.* We extend $(g_{jk})$ to all of $\mathbb{R}^2$, in such a way that $g_{jk}(x) = \delta_{jk}$ for $|x| \gg 1$ and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$. Then there are numbers $0 < m_1 \le m_2$ such that

$$m_1|\xi|^2 \le g_{jk}(x)\xi^j\xi^k \le m_2|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^2.$$

For any $\tau \in H_2^1(B, \mathbb{R}^2)$ the functions $\mathcal{E}(\tau)$, $\mathcal{F}(\tau)$, $\mathcal{G}(\tau)$ are of class $L_1(B)$, and so $A$ and $D$ are well-defined on $H^{1,2}(B, \mathbb{R}^2)$. Analogous to Definition 3 in 4.2 we define $\mathcal{C}(\Gamma)$ as the class of mappings $\tau \in H_2^1(B, \mathbb{R}^2)$ whose trace $\tau|_{\partial B}$ can be represented by a weakly monotonic, continuous mapping from $\partial B$ onto $\Gamma$, and $\mathcal{C}^*(\Gamma)$ is the subclass of mappings $\tau \in \mathcal{C}(\Gamma)$ satisfying a fixed three-point condition.

Now we define the functionals $A^\epsilon : H_2^1(B, \mathbb{R}^2) \to \mathbb{R}$ by

$$A^\epsilon := (1-\epsilon)A + \epsilon D, \quad 0 \le \epsilon \le 1.$$

As in 4.10 we have the following lower semicontinuity property: *If $\tau_n \rightharpoonup \tau$ in $H_2^1(B, \mathbb{R}^2)$ then*

$$A^\epsilon(\tau) \le \liminf_{n\to\infty} A^\epsilon(\tau_n) \quad \text{for any } \epsilon \in [0,1].$$

Unfortunately the simple proof of Lemma 1 in Section 4.10 does not seem to work in the present situation; therefore we refer the reader to the general lower semicontinuity theorem in Acerbi and Fusco [1] which contains the above stated property as a special case.

Consider the variational problem "$A^\epsilon \to$ min in $\mathcal{C}(\Gamma)$" for an arbitrary $\epsilon \in (0,1]$. By the same reasoning as in 4.10 we see that there is a minimizer $\tau^\epsilon \in \mathcal{C}^*(\Gamma)$ satisfying

$$\partial A^\epsilon(\tau^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$$

and so

$$\partial D(\tau^\epsilon, \lambda) = 0 \quad \text{for all } \lambda = (\mu, \nu) \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

where

$$\begin{aligned} \partial D(\tau^\epsilon, \lambda) &= \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv, \\ a &:= \mathcal{E}(\tau^\epsilon) - \mathcal{G}(\tau^\epsilon), \quad b := 2\mathcal{F}(\tau^\epsilon). \end{aligned}$$

It follows that $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u+iv$ in $B$, and then $\Phi(w) \equiv 0$ using the Supplementary Remark 1 of 4.5; see 4.10. Thus we have

$$\mathcal{E}(\tau^\epsilon) = \mathcal{G}(\tau^\epsilon), \quad \mathcal{F}(\tau^\epsilon) = 0 \quad \text{for any } \epsilon \in (0,1]$$

whence $A(\tau^\epsilon) = D(\tau^\epsilon)$ and so

$$A^\epsilon(\tau^\epsilon) = A(\tau^\epsilon) = D(\tau^\epsilon) \quad \text{for } 0 < \epsilon \le 1.$$

On the other hand we infer from $A \le D$ and the minimum property of $\tau^\epsilon$ that

$$A^\epsilon(\tau^\epsilon) \le A^\epsilon(\tau) = (1-\epsilon)A(\tau) + \epsilon D(\tau) \le D(\tau)$$

holds for any $\tau \in \mathcal{C}(\Gamma)$ and $0 < \epsilon \le 1$. Choosing $\tau = \tau^{\epsilon'}$ we obtain

$$D(\tau^\epsilon) \le D(\tau^{\epsilon'}) \quad \text{for all } \epsilon, \epsilon' \in (0,1],$$

and so

$$D(\tau^\epsilon) = A(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) \equiv \text{const} =: c \quad \text{for } 0 < \epsilon \le 1.$$

Set

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then, for arbitrary $\tau \in \mathcal{C}(\Gamma)$ and $\epsilon, \epsilon' \in (0,1]$, we have

$$\begin{aligned} a(\Gamma) &\le A(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) = A^{\epsilon'}(\tau^{\epsilon'}) \le A^{\epsilon'}(\tau), \\ e(\Gamma) &\le D(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) \le A^\epsilon(\tau) \le D(\tau). \end{aligned}$$

Letting $\epsilon' \to +0$ we arrive at

$$a(\Gamma) \le A(\tau^\epsilon) \le A(\tau), \quad e(\Gamma) \le D(\tau^\epsilon) \le D(\tau) \quad \text{for any } \tau \in \mathcal{C}(\Gamma),$$

which implies

$$a(\Gamma) \le A(\tau^\epsilon) \le a(\Gamma), \quad e(\Gamma) \le D(\tau^\epsilon) \le e(\Gamma)$$

whence

$$a(\Gamma) = A(\tau^\epsilon) = D(\tau^\epsilon) = e(\Gamma) \quad \text{for all } \epsilon \in (0,1].$$

In particular we obtain

$$a(\Gamma) = A(\tau) = D(\tau) = e(\Gamma)$$

for $\tau := \tau^1$, that is, the minimizer $\tau$ of Dirichlet's integral $D$ in $\mathcal{C}(\Gamma)$ minimizes also the area functional $A$ in $\mathcal{C}(\Gamma)$. From $D(\tau) = e(\Gamma)$ it follows that $\tau$ is a minimal surface in the two-dimensional Riemannian manifold $(\mathbb{R}^2, ds^2)$, provided that $m \geq 2$ and $\alpha \in (0,1)$, and $\tau \in C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$; in particular, $\tau$ satisfies (11). Furthermore, if $w_0$ is a branch point of $\tau$, i.e. $\mathcal{E}(\tau)(w_0) = 0$, then there is an $a \in \mathbb{C}^2$, $a \neq 0$, and a number $\nu \in \mathbb{N}$ such that the Wirtinger derivative $\tau_w$ has the expansion

$$\tau_w(w) = a(w - w_0)^\nu + o(|w - w_0|)^\nu) \quad \text{as } w \to w_0.$$

These results are derived in Chapters 2 and 3 of Vol. 2 for the Euclidean case. In the Riemannian case the statements at the boundary are verified in the same way, and the interior results are even easier to prove than the boundary results. (We also refer to Morrey [8], Chapter 9; Tomi [1], and Heinz and Hildebrandt [1].) Integrating the asymptotic expansion of $\tau_w$ we obtain for $0 < |x - \tau(w_0)| \ll 1$ and $x \in \mathbb{R}^2$ that the indicatrix

$$N(\tau, \overline{B}, x) := \#\{w \in \overline{B}, \tau(w) = x\}$$

satisfies

$$N(\tau, \overline{B}, x) \geq \begin{cases} 2 & \text{if } w_0 \in B, \\ 1 & \text{if } w_0 \in \partial B \end{cases} \tag{13}$$

in case that $w_0$ is a branch point of $\tau$.

Since $\tau$ maps $\partial B$ weakly monotonically and continuously onto $\Gamma$ and $\tau \in C^0(\overline{B}, \mathbb{R}^2)$, a topological argument yields $\overline{\Omega} \subset \tau(\overline{B})$. Therefore we also have

$$N(\tau, \overline{B}, x) \geq 1 \quad \text{for all } x \in \overline{\Omega}. \tag{14}$$

Let $\tau_0$ be a conformal mapping of $B$ onto $\Omega$, $\tau_0 \in \mathcal{C}(\Gamma)$. Then

$$A(\tau_0) = \int_\Omega \sqrt{g(x)}\, dx^1\, dx^2 = \int_{\overline{\Omega}} \sqrt{g(x)}\, dx^1\, dx^2$$

since $\mathcal{L}^2$-meas $\Gamma = 0$ for a rectifiable curve $\Gamma$. Since $\tau$ minimizes $A$ in $\mathcal{C}(\Gamma)$ we obtain

$$A(\tau) \leq A(\tau_0),$$

and the *area formula* yields

$$A(\tau) = \int_{\mathbb{R}^2} N(\tau, \overline{B}, x) \sqrt{g(x)}\, dx^1\, dx^2.$$

Thus,

$$\int_{\mathbb{R}^2} N(\tau, \overline{B}, x)\sqrt{g(x)}\,dx^1\,dx^2 \le \int_{\Omega} \sqrt{g(x)}\,dx^1\,dx^2. \tag{15}$$

On account of (13)–(15) it firstly follows that $\tau$ has no branch points on $\overline{B}$, whence $\nabla\tau(w) \neq 0$ for all $w \in \overline{B}$. Thus $\tau|_{\partial B}$ is 1–1, and so it yields a homeomorphism from $\partial B$ onto $\Gamma$. Secondly, $\tau|_B$ is open; hence it follows from (14) and (15) that $N(\tau, \overline{B}, x) = 1$ for $x \in \overline{\Omega}$ and $N(\tau, \overline{B}, x) = 0$ for $x \in \mathbb{R}^2 \setminus \overline{\Omega}$. Consequently $\tau$ is a conformal mapping from $\overline{B}$ onto $\overline{\Omega}$ satisfying the Beltrami equations (12).

If we merely assume $\Gamma \in C^{1,\alpha}$ and $g_{jk} \in C^{0,\alpha}$, $\tau$ turns out to be a conformal mapping of class $C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$ from $\overline{B}$ onto $\overline{\Omega}$. This one obtains from the preceding result ($m \ge 2$) by approximating $\Gamma$ and $g_{jk}$ by $C^\infty$-data $\Gamma_n$, $g^n_{jk}$, and applying a priori estimates for the corresponding mappings $\tau_n$ and their inverses $\tau_n^{-1}$ which satisfy similar Beltrami equations as the $\tau_n$ (see e.g. Schulz [1], Chapter 6; Jost [17], Chapter 3; or Morrey [8], pp. 373–374). □

A slight modification of the preceding reasoning combined with a suitable approximation argument yields the following analog of Theorem 1:

**Theorem 3.** *If $\Gamma$ is a closed Jordan curve with the inner domain $\Omega$ and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$ for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$, then there is a homeomorphism $\tau$ from $\overline{B}$ onto $\overline{\Omega}$ which yields a conformal mapping of class $C^{m,\alpha}(B, \mathbb{R}^2)$ from $B$ onto $\Omega$.*

As a corollary of Theorem 2 we obtain the following version of the original Lichtenstein theorem:

**Theorem 4.** *If $X : \overline{B} \to \mathbb{R}^n$, $n \ge 2$, is an immersed surface of class $C^{m,\alpha}$, $m \in \mathbb{N}$, $\alpha \in (0,1)$, then there exists an equivalent representation $Y = X \circ \tau$ which is conformally parametrized, i.e. $|Y_u|^2 = |Y_v|^2$, $\langle Y_u, Y_v\rangle = 0$.*

*Proof.* $X(x^1, x^2)$ with $(x^1, x^2) \in \overline{B}$ induces on $\overline{B}$ the Riemannian metric $ds^2 = g_{jk}(x)\,dx^j\,dx^k$ with

$$g_{jk} := \langle X_{x^j}, X_{x^k}\rangle \in C^{m-1,\alpha}(\overline{B}).$$

If we now determine a conformal mapping $\tau$ from $(\overline{B}, ds_e)$ onto $(\overline{B}, ds)$ as in Theorem 2 then $Y := X \circ \tau$ has the desired property. □

## 4.12 Solution of Plateau's Problem for Nonrectifiable Boundaries

A general closed Jordan curve $\Gamma$ need not bound any surface $X : B \to \mathbb{R}^3$ with a finite Dirichlet integral. In fact, $\mathcal{C}(\Gamma)$ is nonempty if and only if $\Gamma$ possesses a representation of class $H_2^{1/2}([0, 2\pi], \mathbb{R}^3)$. Nevertheless J. Douglas proved that every closed Jordan curve $\Gamma$ in $\mathbb{R}^3$ spans a continuous disk-like minimal surface. To see this we approximate $\Gamma$ by sequences of rectifiable $\Gamma_n$, each of which bounds a minimal surface $X_n$ of finite area. There is a subsequence $\{X_{n_p}\}$ that uniformly converges to a minimal surface $X \in C^2(B, \mathbb{R}^3)$ on every $\Omega' \subset\subset B$. Yet it is not obvious that the limit $X$ is continuous and that it maps $\partial B$ onto $\Gamma$ in the sense of 4.2, Definition 2. Namely, as $A(X_n)$ may tend to infinity, one cannot derive a uniform bound for the moduli of continuity of the boundary values $X_n|_{\partial B}$ by means of the Courant–Lebesgue lemma, and so, at first, it only follows that $X|_{\partial B}$ yields a weakly monotonic mapping from $\partial B$ into $\Gamma$ which might have denumerably many jump discontinuities. The crucial part of the proof consists in showing that these discontinuities do not appear.

We use a result on sequences of monotonic functions that in essence is due to Helly; a proof can be derived from A. Wintner [1].

**Lemma 1.** *Let $\{\tau_n\}$ be a sequence of increasing functions $\tau_n \in C^0(\mathbb{R})$ with $\tau_n(0) = 0$ and $\tau_n(\theta + 2\pi) = \tau_n(\theta)$. Then there is a function $\tau : \mathbb{R} \to \mathbb{R}$ and a subsequence $\{\tau_{n_k}\}$ with the following properties*:

(i) *$\tau$ is nondecreasing and continuous except for at most denumerably many jump discontinuities.*

(ii) *If $\tau$ is continuous at $\theta$ then $\tau_{n_k}(\theta) \to \tau(\theta)$ as $k \to \infty$.*

(iii) *Because of* (i) *the one-sided limits $\tau(\theta_0 - 0)$ and $\tau(\theta_0 + 0)$ exist at any $\theta_0 \in \mathbb{R}$, and we can redefine $\tau$ by $\tau(\theta_0) := \frac{1}{2}[\tau(\theta_0 - 0) + \tau(\theta_0 + 0)]$ without changing* (i) *and* (ii). *Set*

$$\sigma(\theta_0) := \tfrac{1}{2}[\tau(\theta_0 + 0) - \tau(\theta_0 - 0)].$$

(iv) *For any $\delta > 0$ there exist numbers $\eta(\delta) > 0$ and $N_0(\delta) \in \mathbb{N}$ such that for all $\theta_0 \in \mathbb{R}$ the following holds*:

$$\begin{aligned} |\tau(\theta) - \tau(\theta_0)| &\le \sigma(\theta_0) + \delta \quad \text{if } |\theta - \theta_0| < \eta, \\ |\tau_{n_k}(\theta) - \tau(\theta_0)| &\le \sigma(\theta_0) + \delta \quad \text{if } |\theta - \theta_0| < \eta \text{ and } k > N_0. \end{aligned}$$

Now we can state the main result.

**Theorem 1.** *For any closed Jordan curve $\Gamma$ in $\mathbb{R}^3$ there is a minimal surface $X : B \to \mathbb{R}^3$ of class $C^0(\overline{B}, \mathbb{R}^3)$ which maps $\partial B$ homeomorphically onto $\Gamma$.*

*Proof.* Let $\Gamma$ be represented by $\gamma \in C^0(\mathbb{R}, \mathbb{R}^3)$ which is monotonic and $2\pi$-periodic such that $\Gamma = \gamma([0, 2\pi])$. We approximate $\Gamma$ by rectifiable Jordan curves $\Gamma_n$ (say, by simple closed polygons) with continuous, monotonic, $2\pi$-periodic representations $\gamma_n : \mathbb{R} \to \mathbb{R}^3$, $\Gamma_n = \gamma_n([0, 2\pi])$, such that $\gamma_n$ converges uniformly to $\gamma : \gamma_n(t) \rightrightarrows \gamma(t)$ on $\mathbb{R}$ as $n \to \infty$. For any $n$ there is a minimal surface $X_n \in \mathcal{C}(\Gamma_n) \cap C^0(\overline{B}, \mathbb{R}^3)$ that maps $\partial B$ homeomorphically onto $\Gamma_n$. If we choose the orientation of $\Gamma_n$ appropriately and require that $X_n(e^{i\theta})$ respects this orientation, we can write

(1) $$X_n(e^{i\theta}) = \gamma_n(\tau_n(\theta)) \quad \text{for } \theta \in \mathbb{R} \text{ and } n \in \mathbb{N}$$

where the $\tau_n$ are increasing functions of class $C^0(\mathbb{R})$ with $\tau_n(\theta+2\pi) = \tau_n(\theta)+2\pi$. As one can impose an arbitrarily chosen three-point condition on any $X_n$ we may also assume that $\tau_n(0) = 0$, $\tau_n(1) = 1$, $\tau_n(2) = 2$ for any $n \in \mathbb{N}$. Passing to a suitable subsequence of $\{X_n\}$ and renumbering it we obtain $X_n(w) \rightrightarrows X(w)$ on $\Omega \subset\subset B$ where $X : B \to \mathbb{R}^3$ is a minimal surface. On account of Lemma 1 we can furthermore assume that there is a nondecreasing, possibly discontinuous function $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau_n(\theta) \to \tau(\theta)$ as $n \to \infty$, provided that $\tau$ is continuous at $\theta$, and for any $\delta > 0$ there are numbers $\eta(\delta) > 0$ and $N_0(\delta) \in \mathbb{N}$ such that

(2) $$|\tau(\theta) - \tau(\theta_0)| \le \sigma(\theta_0) + \delta \quad \text{for } |\theta - \theta_0| < \eta(\delta)$$

and

(3) $$|\tau_n(\theta) - \tau(\theta_0)| \le \sigma(\theta_0) + \delta \quad \text{for } |\theta - \theta_0| < \eta(\delta) \text{ and } n > N_0(\delta)$$

where $\sigma(\theta_0) := \frac{1}{2}[\tau(\theta_0 + 0) - \tau(\theta_0 - 0)]$ and $\tau$ is redefined as

$$\tau(\theta_0) = \tfrac{1}{2}[\tau(\theta_0 + 0) + \tau(\theta_0 - 0)].$$

First we will prove that

(4) $$\lim_{w \to w_0} X(w) = \gamma(\tau(\theta_0)) \quad \text{for } w_0 = e^{i\theta_0} \in \partial B,$$

provided that $\tau$ is continuous at $\theta_0$. So let us assume that

(5) $$\sigma(\theta_0) = 0$$

for some fixed $\theta_0$, and choose some $\epsilon > 0$. Since $\gamma$ is uniformly continuous on $\mathbb{R}$ there is some $\delta_1(\epsilon) > 0$ such that

(6) $$|\gamma(t) - \gamma(t')| < \epsilon \quad \text{for } |t - t'| < \delta_1(\epsilon).$$

Because of $\gamma_n(t) \rightrightarrows \gamma(t)$ on $\mathbb{R}$ there is an $N_1(\epsilon) \in \mathbb{N}$ such that

(7) $$|\gamma(t) - \gamma_n(t)| < \epsilon \quad \text{for } n > N_1(\epsilon) \text{ and all } t \in \mathbb{R}.$$

Furthermore, by (3) and (5) we obtain

(8) $$|\tau_n(\theta_0) - \tau(\theta_0)| < \delta_1(\epsilon) \quad \text{for } n > N_0(\delta_1(\epsilon)).$$

On account of (6)–(8) and

$$\begin{aligned}\left|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\right| \le{}& \left|\gamma(\tau(\theta_0)) - \gamma(\tau_n(\theta_0))\right| + \left|\gamma(\tau_n(\theta_0)) - \gamma_n(\tau_n(\theta_0))\right| \\ &+ \left|\gamma_n(\tau_n(\theta_0)) - X_n(re^{i\theta})\right|\end{aligned}$$

we see that

$$\text{(9)} \qquad \big|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\big| < 2\epsilon + \big|\gamma_n(\tau_n(\theta_0)) - X_n(re^{i\theta})\big| \quad \text{for } n > N(\epsilon) := \max\big\{N_1(\epsilon), N_0(\delta_1(\epsilon))\big\}.$$

For $0 \le r < 1$ Poisson's integral formula and (1) yield

$$\text{(10)} \qquad \begin{aligned} &\big|X_n(re^{i\theta}) - \gamma_n(\tau_n(\theta_0))\big| \\ &\le \int_{\theta_0-\pi}^{\theta_0+\pi} K(r, \varphi - \theta)\big|\gamma_n(\tau_n(\varphi)) - \gamma_n(\tau_n(\theta_0))\big|\, d\varphi \\ &= \int_{I_1} \cdots + \int_{I_2} \cdots \end{aligned}$$

with $I_1 := \{\varphi\colon |\varphi - \theta_0| < \eta(\delta_1(\epsilon))\}$, $I_2 := [\theta_0 - \pi, \theta_0 + \pi] \setminus I_1$, and

$$K(r, \alpha) := \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}.$$

On account of (3) and (6)–(8) we obtain for $n > N(\epsilon)$, $|\varphi - \theta_0| < \eta(\delta_1(\epsilon))$ and $|\theta - \theta_0| < \eta(\delta_1(\epsilon))$ that

$$\begin{aligned} \big|\gamma_n(\tau_n(\varphi)) - \gamma_n(\tau_n(\theta_0))\big| &\le \big|\gamma(\tau_n(\varphi)) - \gamma_n(\tau_n(\varphi))\big| + \big|\gamma(\tau_n(\theta_0)) - \gamma_n(\tau_n(\theta_0))\big| \\ &\quad + \big|\gamma(\tau_n(\varphi)) - \gamma(\tau(\theta_0))\big| + \big|\gamma(\tau(\theta_0)) - \gamma(\tau_n(\theta_0))\big| \\ &< \epsilon + \epsilon + \epsilon + \epsilon = 4\epsilon, \end{aligned}$$

whence

$$\text{(11)} \qquad \left|\int_{I_1} \cdots\right| < 4\epsilon \int_{I_1} K(r, \varphi - \theta)\, d\varphi \le 4\epsilon \int_0^{2\pi} K(r, \alpha)\, d\alpha = 4\epsilon.$$

Because of $\gamma_n \rightrightarrows \gamma$ there is a constant $c_0$ such that

$$|\gamma_n(t)| + |\gamma(t)| \le c_0 \quad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{N},$$

and so

$$\left|\int_{I_2} \cdots\right| \le c_0 \int_{I_2} K(r, \varphi - \theta)\, d\theta.$$

Hence there is a constant $\delta_2(\epsilon) > 0$ such that

$$\text{(12)} \qquad \left|\int_{I_2} \cdots\right| < \epsilon \quad \text{for } 0 < 1 - r < \delta_2(\epsilon), \quad |\theta - \theta_0| < \delta_3(\epsilon) := \tfrac{1}{2}\eta(\delta_1(\epsilon)) \text{ and all } n \in \mathbb{N}.$$

By (9)–(12) it follows that

$$\text{(13)} \qquad \big|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\big| < 2\epsilon + 4\epsilon + \epsilon = 7\epsilon \quad \text{for } n > N(\epsilon),\ 0 < 1 - r < \delta_2(\epsilon) \text{ and } |\theta - \theta_0| < \delta_3(\epsilon).$$

With $n \to \infty$ we obtain

$$(14) \quad \left|\gamma(\tau(\theta_0)) - X(re^{i\theta})\right| \le 7\epsilon \quad \text{for } 0 < 1 - r < \delta_2(\epsilon) \text{ and } |\theta - \theta_0| < \delta_3(\epsilon).$$

This implies

$$\lim_{w \to w_0} X(w) = \gamma(\tau(\theta_0)) \quad \text{for } w_0 = e^{i\theta_0},$$

provided that $\tau(\theta)$ is continuous at $\theta = \theta_0$.

Now we want to show by reductio ad absurdum that $\tau$ is everywhere continuous. To this end, suppose that $\tau$ is discontinuous at $\theta_0$; then it is no loss of generality if we assume that $\theta_0 = 0$. Set $\tau^+ := \tau(+0)$, $\tau^- := \tau(-0)$, and $X^+ := \gamma(\tau^+)$, $X^- := \gamma(\tau^-)$. Since $\tau^+ = \lim_{\theta \to +0} \tau(\theta)$ we have: For any $\delta > 0$ there is a number $\eta^*(\delta) > 0$ such that

$$\tau^+ \le \tau(\theta) \le \tau^+ + \delta/4 \quad \text{for } 0 < \theta < \eta^*(\delta).$$

By virtue of (3) we may therefore even assume that

$$|\tau_n(\theta) - \tau^+| < \delta \quad \text{for } 0 < \theta < \eta^*(\delta) \text{ and } n > N_0(\delta).$$

Choose some $\epsilon > 0$ and set $\delta := \delta_1(\epsilon)$. Then the same reasoning as before yields for $\delta_4(\epsilon) := \eta^*(\delta_1(\epsilon))$ the following: For any point $\tilde{w} = e^{i\varphi}$ with $0 < \varphi < \delta_4(\epsilon)$ there is an open neighborhood $U(\varphi)$ of $\tilde{w}$ in $B$ such that

$$(15) \qquad |X^+ - X(w)| < 7\epsilon \quad \text{for } w \in U(\varphi),\ 0 < \varphi < \delta_4(\epsilon),$$

and correspondingly we can achieve

$$(16) \qquad |X^- - X(w)| < 7\epsilon \quad \text{for } w \in U(\varphi), -\delta_4(\epsilon) < \varphi < 0.$$

Now we are going to derive a contradiction to the assumption $\tau^+ \neq \tau^-$ by proving that $X^+ \neq X^-$ is impossible. To this end we consider the conformal automorphisms $f_a$ of $\overline{B}$ which are defined by

$$z = f_a(w) := \frac{w - a}{1 - aw} \quad \text{with } a \in \mathbb{R},\ 0 < a < 1.$$

We have $f_a(1) = 1$, $f_a(0) = -a$, $f_a(-1) = -1$ and $\overline{f_a(w)} = f_a(\overline{w})$ whence $f_a(\mathbb{R}) = \mathbb{R}$ and $f_a(C^+) = C^+$, $f_a(C^-) = C^-$ for $C^+ := \{w \in \partial B\colon\ \operatorname{Im} w > 0\}$, $C^- := \{w \in \partial B\colon\ \operatorname{Im} w < 0\}$. Moreover,

$$\lim_{a \to 1-0} f_a(w) = -1 \quad \text{for any } w \in \overline{B} \setminus \{1\}.$$

Hence, for $a \in (0,1)$ sufficiently close to 1, we see that $f_a$ maps the arc $C_0^+ := \{e^{i\varphi}\colon 0 < \varphi < \delta_4(\epsilon)\}$ onto an arc $f_a(C_0^+)$ which contains $C_1^+ := \{e^{i\psi}\colon 1 \le \psi \le 2\}$ in its interior. Then $C_0^- := \{e^{i\varphi}\colon -\delta_4(\epsilon) < \varphi < 0\}$ is mapped onto $f_a(C_0^-)$ which contains $C_1^- := \{e^{i\psi}\colon -2 \le \psi \le -1\}$ in its interior. Set

$$Y(z) := X(f_a^{-1}(z)) \quad \text{for } z \in \overline{B}.$$

Clearly, $Y|_B$ is again a minimal surface. Let

$$U^+ := \bigcup_{0<\varphi<\delta_4(\epsilon)} U(\varphi), \quad U^- := \bigcup_{-\delta_4(\epsilon)<\varphi<0} U(\varphi).$$

By the choice of $a$, the image $f_a(U^+)$ covers a whole strip $\Sigma^+(\epsilon)$ along $C_1^+$ in $B$, and $f_a(U^-)$ covers a strip $\Sigma^-(\epsilon)$ along $C_1^-$ in $B$ where $\Sigma^+(\epsilon)$ and $\Sigma^-(\epsilon)$ are of the form

$$\Sigma^+(\epsilon) = \{\rho e^{i\psi} : 1 - \delta_5(\epsilon) < \rho < 1, 1 \le \psi \le 2\},$$
$$\Sigma^-(\epsilon) = \{\rho e^{i\psi} : 1 - \delta_5(\epsilon) < \rho < 1, -2 \le \psi \le -1\},$$

and $\delta_5(\epsilon)$ is some positive number depending on $\epsilon > 0$. Then we infer from (15) and (16) that

(17) $|Y(z) - X^+| < 7\epsilon \quad \text{for } z \in \Sigma^+(\epsilon), \quad |Y(z) - X^-| < 7\epsilon \quad \text{for } z \in \Sigma^-(\epsilon).$

We choose a sequence of numbers $\epsilon_j > 0$ with $\epsilon_j \to 0$, thereafter a sequence of radii $\rho_j$ with $1 - \delta_5(\epsilon_j) < \rho_j < 1$, and then we set $Z_j(z) := Y(\rho_j z)$ for $z \in \overline{B}$. The mappings $Z_j$ are minimal surfaces of class $C^0(\overline{B}, \mathbb{R}^3)$ which satisfy

(18)
$$|Z_j(e^{i\psi}) - X^+| < 7\epsilon \quad \text{for } 1 \le \psi \le 2,$$
$$|Z_j(e^{i\psi}) - X^-| < 7\epsilon \quad \text{for } -2 \le \psi \le -1.$$

Moreover,

(19) $|Z_j(e^{i\psi}) - X^+|, |Z_j(e^{i\psi}) - X^-| \le c_0 \quad \text{for } \psi \in \mathbb{R} \text{ and } j \in \mathbb{N}.$

From Poisson's integral formula we get

$$|Z_j(re^{i\theta}) - X^+| \le \int_0^{2\pi} K(r, \psi - \theta)|Z_j(e^{i\psi}) - X^+|\, d\psi$$
$$= \int_{|\psi-\theta|<\frac14} \cdots + \int_{\frac14 \le |\psi-\theta| \le \pi} \cdots .$$

If we restrict $\theta$ by $\frac54 \le \theta \le \frac74$, then for $|\psi - \theta| < \frac14$ we have $1 < \psi < 2$, and so it follows from (18) and (19) that

$$|Z_j(re^{i\theta}) - X^+| < 7\epsilon_j + c_0 p(r) \quad \text{if } \tfrac54 \le \theta \le \tfrac74$$

with

$$p(r) := \int_{\frac14 \le |\alpha| \le \pi} K(r, \alpha)\, d\alpha \to 0 \quad \text{as } r \to 1 - 0.$$

Because of $Z_j(z) = Y(\rho_j z)$ we then obtain for $j \to \infty$ that

$$|Y(re^{i\theta}) - X^+| \le c_0 p(r) \quad \text{for } 0 < r < 1,\ \tfrac{5}{4} < \theta < \tfrac{7}{4},$$

and similarly,

$$|Y(re^{i\theta}) - X^-| \le c_0 p(r) \quad \text{for } 0 < r < 1,\ -\tfrac{7}{4} < \theta < -\tfrac{5}{4}.$$

Thus $Y$ assumes the constant boundary values $X^+$ on $C_2^+ := \{e^{i\theta} \colon \frac{5}{4} < \theta < \frac{7}{4}\}$ and the constant boundary values $X^-$ on $C_2^- := \{e^{i\theta} \colon -\frac{7}{4} < \theta < -\frac{5}{4}\}$. By the reasoning used in the proof of Theorem 3 in Section 4.5 it follows that $Y(z) \equiv \text{const}$ on $B \cup C_2^+ \cup C_2^-$ which is a contradiction to $Y|_{C_2^+} = X^+$, $Y|_{C_2^-} = X^-$, $X^+ \ne X^-$.

Therefore $\tau$ is continuous on $\mathbb{R}$, and so $X$ is continuous on $\overline{B}$ and yields a weakly monotonic mapping from $\partial B$ onto $\Gamma$. By virtue of Corollary 2 in Section 4.5 we see that $X|_{\partial B}$ is a homeomorphism from $\partial B$ onto $\Gamma$. □

**Remark 1.** Another proof of Theorem 1 can be found in Nitsche's treatise [28], pp. 269–271. The above proof is a slight modification of the approach used by H. Werner [2], which also works for surfaces $X$ of constant mean curvature $H$ provided that $|X| \le 1$ and $|H| < \frac{1}{2}$. The general case $|H| \le 1$ is apparently not yet treated. Similarly Theorem 1 has not been carried over to surfaces of prescribed variable mean curvature $H(x)$ or to minimal surfaces in a Riemannian manifold.

## 4.13 Plateau's Problem for Cartan Functionals

Now we want to solve Plateau's problem for regular Cartan functionals. Here a *Cartan functional* means a two-dimensional variational integral

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du\, dv \tag{1}$$

with a continuous Lagrangian $F(x, z)$, $(x, z) \in \mathbb{R}^3 \times \mathbb{R}^3$, that is positively homogeneous of first degree in $z$, i.e.

$$F(x, tz) = tF(x, z) \quad \text{for } t > 0 \text{ and } (x, z) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{H}$$

As before we assume that $B$ is the unit disk $\{w = (u, v) \colon u^2 + v^2 < 1\}$ in $\mathbb{R}^2$.

A Cartan functional $\mathcal{F}$ is said to be *regular* if its Lagrangian $F(x, z)$ is *definite* and *weakly elliptic*. The first assumption means that there are constants $m_1$, $m_2$ with $0 < m_1 \le m_2$ such that

$$m_1 \le F(x, z) \le m_2 \quad \text{for } (x, z) \in \mathbb{R}^3 \times S^2$$

with $S^2 := \{z \in \mathbb{R}^3 \colon |z| = 1\}$. Because of (H) the *assumption of definiteness* means that

(D) $$m_1|z| \le F(x,z) \le m_2|z| \quad \text{for } (x,z) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Secondly, *weak ellipticity* of $F(x,z)$ is defined as *convexity* of $F(x,z)$ in $z$ for any $x \in \mathbb{R}^3$, i.e. we assume

(C) $$F(x, tz_1 + (1-t)z_2) \le tF(x,z_1) + (1-t)F(x,z_2) \quad \text{for } t \in [0,1] \text{ and } x, z_1, z_2 \in \mathbb{R}^3.$$

Because of (D), a regular Cartan functional $\mathcal{F}$, given by (1), is well-defined for any $X \in H_2^1(B,\mathbb{R}^3)$, and by (H) it follows that $\mathcal{F}(X \circ \tau) = \mathcal{F}(X)$ for any orientation preserving $C^1$-diffeomorphism of $\overline{B}$ onto itself, i.e. a Cartan functional is a *parameter invariant (two-dimensional) variational integral*. The notation "Cartan functional" is derived from Elie Cartan's memoir [1] where he introduced a geometry based on an "angular metric" that is defined by means of an integral (1) as

$$ds^2 = g_{jk}\,dx^j\,dx^k, \quad (g_{jk}) = (g^{jk})^{-1}, \quad g^{jk} := a^{-1/2}a^{jk},$$
$$a := \det(a^{jk}), \quad a^{jk} := \left(\frac{1}{2}F^2\right)_{z^j z^k} = FF_{z^j z^k} + F_{z^j}F_{z^k}.$$

This is a generalization of Finsler's geometry which is based on one-dimensional integrals $\mathcal{F}(X) = \int_0^1 F(X,\dot{X})\,dt$ with a Lagrangian $F(x,z)$ satisfying (H), (D), and (C).

Note that an $F$ satisfying (H) and (D) cannot be of class $C^1(\mathbb{R}^3 \times \mathbb{R}^3)$, but it may very well be of class $C^s$ on $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$. The prototype of a regular Cartan functional is the area integral $A(X) = \int_B |X_u \wedge X_v|\,du\,dv$ with the Lagrangian $F(x,z) = |z|$. If $F \in C^2(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}))$ and $F(x,z)$ is convex in $z$, then $F_{zz}(x,z) \ge 0$ for $z \ne 0$, but we never have $F_{zz}(x,z) > 0$ since Euler's relation implies $F_{zz}(x,z)z = 0$ because of (H). Thus the best we can hope for is: $F_{zz}(x,z) > 0$ on $\{z\}^\perp$, which is equivalent to

$$\zeta \cdot |z| F_{zz}(x,z)\zeta \ge \lambda[|\zeta|^2 - |z|^{-2}(\zeta \cdot z)^2] \quad \text{for } z \ne 0$$

and some constant $\lambda > 0$, i.e. to $F_\lambda(x,z) := F(x,z) - \lambda|z|$ being convex in $z$.

Let $\Gamma$ be a closed, rectifiable Jordan curve in $\mathbb{R}^3$ which is oriented, and denote by $\mathcal{C}(\Gamma)$ the class of surfaces $X \in H_2^1(B,\mathbb{R}^3)$ bounded by $\Gamma$ (see Section 4.2, Definitions 2 and 3); then $\mathcal{C}(\Gamma)$ is nonempty. We want to solve the variational problem

(2) $$\mathcal{F} \to \min \quad \text{in } \mathcal{C}(\Gamma),$$

which we denote as *Plateau problem for the Cartan functional* $\mathcal{F}$. This will be achieved by a method that is similar to the reasoning used in Section 4.10 for solving the problem "$A \to \min$ in $\mathcal{C}(\Gamma)$".

**Theorem 1.** *For any regular Cartan functional* (1) *the minimum problem* (2) *has a solution $X \in \mathcal{C}(\Gamma)$ which is conformally parametrized in the sense that*

(3) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

*Proof.* Instead of (2) we first consider the modified minimum problems

$$\mathcal{F}^\varepsilon \to \min \quad \text{in } \mathcal{C}(\Gamma), \tag{4}$$

$\varepsilon \in (0,1]$, where the auxiliary functionals $\mathcal{F}^\varepsilon$ are defined for $0 < \varepsilon \leq 1$ as

$$\mathcal{F}^\varepsilon := \mathcal{F} + \varepsilon D. \tag{5}$$

We can write

$$\mathcal{F}^\varepsilon(X) = \int_B f^\varepsilon(X, \nabla X)\, du\, dv$$

where the Lagrangian

$$f^\varepsilon(x,p) := F(x, p_1 \wedge p_2) + \frac{\varepsilon}{2}\,|p|^2$$

is *polyconvex* in $p = (p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and satisfies

$$m_1|p_1 \wedge p_2| + \frac{\varepsilon}{2}\,|p|^2 \leq f^\varepsilon(x,p) \leq \frac{1}{2}(m_2 + \varepsilon)|p|^2$$

because of $2|p_1 \wedge p_2| \leq |p_1|^2 + |p_2|^2$. By a theorem of Acerbi and Fusco [1] the functional $\mathcal{F}^\varepsilon$ is (sequentially) weakly lower semicontinuous (w.l.s.) on $H_2^1(B, \mathbb{R}^3)$. Let $X_j \in \mathcal{C}(\Gamma)$ be a minimizing sequence for the problem (4), i.e.

$$\mathcal{F}^\varepsilon(X_j) \to d(\varepsilon) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F}^\epsilon.$$

We can assume that all $X_j$ satisfy a uniform three-point condition $X_j(w_k) = Q_k$, $k = 1, 2, 3$, with $w_k \in \partial B$ and $Q_k \in \Gamma$, i.e. $X_j \in \mathcal{C}^*(\Gamma)$ in the sense of Section 4.2. From (5) we infer

$$D(X_j) \leq \varepsilon^{-1}\mathcal{F}^\epsilon(X_j) \leq \text{const} \quad \text{for all } j \in \mathbb{N} \text{ and fixed } \varepsilon > 0,$$

and the "boundary values" (= Sobolev traces) $\phi_j$ of $X_j$ on $\partial B$ satisfy $\sup_{\partial B} |\phi_j| \leq \text{const}$. Then a suitable variant of Sobolev's inequality yields

$$|X_j|_{H_2^1(B,\mathbb{R}^3)} \leq \text{const} \quad \text{for all } j \in \mathbb{N}.$$

Passing to an appropriate subsequence of $\{X_j\}$ which (by renumbering) is again called $\{X_j\}$ we obtain $X_j \rightharpoonup X^\varepsilon$ in $H_2^1(B, \mathbb{R}^3)$ for some $X^\varepsilon \in H_2^1(B, \mathbb{R}^3)$ whence

$$\mathcal{F}^\varepsilon(X^\varepsilon) \leq \lim \mathcal{F}^\varepsilon(X_j) = d(\varepsilon).$$

On the other hand $\mathcal{C}^*(\Gamma)$ is a weakly sequentially closed subset of $H_2^1(B, \mathbb{R}^3)$ (cf. 4.6, Proposition 1), and so $X^\varepsilon \in \mathcal{C}^*(\Gamma)$, whence $d(\varepsilon) \leq \mathcal{F}^\varepsilon(X^\varepsilon)$. This implies

$$\mathcal{F}^\varepsilon(X^\varepsilon) = d(\varepsilon), \tag{6}$$

i.e. $X^\varepsilon$ is a solution of (4). As in Section 4.10 we obtain

$$\partial \mathcal{F}^\varepsilon(X^\varepsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

therefore

$$\partial D(X^\varepsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

and now the reasoning of 4.10 yields

$$|X^\varepsilon_u|^2 = |X^\varepsilon_v|^2, \quad \langle X^\varepsilon_u, X^\varepsilon_v\rangle = 0. \tag{7}$$

This is equivalent to

$$A(X^\varepsilon) = D(X^\varepsilon) \quad \text{for } \varepsilon \in (0,1]. \tag{8}$$

On the other hand assumption (D) implies $m_1 A \le \mathcal{F}$, and so we infer from (5) and (8) that

$$(m_1 + \varepsilon) D(X^\varepsilon) \le \mathcal{F}^\varepsilon(X^\varepsilon).$$

Furthermore,

$$\mathcal{F}^\varepsilon \le (m_2 + \epsilon) D$$

by $\mathcal{F} \le m_2 A$ and $A \le D$, and we also have

$$\mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma)$$

on account of (6). Consequently,

$$(m_1 + \varepsilon) D(X^\varepsilon) \le (m_2 + \varepsilon) D(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma).$$

Since

$$\frac{m_2 + \varepsilon}{m_1 + \varepsilon} < \frac{m_2}{m_1} \quad \text{for any } \epsilon > 0,$$

we arrive at

$$D(X^\varepsilon) \le (m_2/m_1) \cdot e(\Gamma) \quad \text{for all } \varepsilon \in (0,1] \tag{9}$$

with

$$e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} \mathcal{D},$$

and by the same reasoning as above it follows that

$$|X^\varepsilon|_{H^1_2(B,\mathbb{R}^3)} \le \text{const} \quad \text{for all } \varepsilon \in (0,1].$$

Hence there is an $X \in \mathcal{C}^*(\Gamma)$ and a sequence of numbers $\varepsilon_j > 0$ with $\varepsilon_j \to 0$ such that $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$. Since also $\mathcal{F}$ is sequentially w.l.s. by Acerbi and Fusco [1], it follows that

$$d(0) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F} \le \mathcal{F}(X) \le \liminf_{j\to\infty} \mathcal{F}(X^{\varepsilon_j}).$$

As $d(\varepsilon)$ is nondecreasing, $\lim_{\varepsilon\to+0} d(\varepsilon)$ exists, and by

$$d(\varepsilon) = \mathcal{F}^\varepsilon(X^\varepsilon) = \mathcal{F}(X^\varepsilon) + \varepsilon D(X^\varepsilon)$$

we infer from (9) that

$$\lim_{\varepsilon\to+0} d(\varepsilon) = \lim_{\varepsilon\to+0} \mathcal{F}^\varepsilon(X^\varepsilon) = \lim_{\varepsilon\to+0} \mathcal{F}(X^\varepsilon).$$

Thus we have

$$d(0) \le \mathcal{F}(X) \le \lim_{\varepsilon\to+0} d(\varepsilon). \tag{10}$$

On the other hand,

$$d(\varepsilon) = \mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(Z) = \mathcal{F}(Z) + \varepsilon D(Z) \quad \text{for any } Z \in \mathcal{C}(\varGamma).$$

Hence $\lim_{\varepsilon\to+0} d(\varepsilon) \le \mathcal{F}(Z)$, and so $\lim_{\varepsilon\to+0} d(\varepsilon) \le d(0)$. By virtue of (10) we arrive at

$$\mathcal{F}(X) = d(0) := \inf_{\mathcal{C}(\varGamma)} \mathcal{F},$$

and so $X \in \mathcal{C}(\varGamma)$ is a solution of (2), i.e. a minimizer of $\mathcal{F}$ in $\mathcal{C}(\varGamma)$.

We still have to prove (3) which does not immediately follow from (7) because the $X^{\varepsilon_j}$ merely converge weakly to $X$ in $H_2^1(B, \mathbb{R}^3)$. However (3) follows from (7) as soon as we have proved the strong convergence $X^{\varepsilon_j} \to X$ in $H_2^1(B, \mathbb{R}^3)$. For this it suffices to prove

$$\lim_{\varepsilon_j\to 0} D(X^{\varepsilon_j}) = D(X), \tag{11}$$

which will be verified as follows: Since $X^\varepsilon$ minimizes $\mathcal{F}^\varepsilon$ in $\mathcal{C}(\varGamma)$, we have $\mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(X)$, i.e.

$$\mathcal{F}(X^\varepsilon) + \varepsilon D(X^\varepsilon) \le \mathcal{F}(X) + \varepsilon D(X),$$

and we also have

$$\mathcal{F}(X) \le \mathcal{F}(X^\varepsilon)$$

as $X$ minimizes $\mathcal{F}$ in $\mathcal{C}(\varGamma)$. Therefore

$$\varepsilon D(X^\varepsilon) \le \varepsilon D(X), \quad \varepsilon \in (0, 1],$$

and so

$$D(X^\varepsilon) \le D(X) \quad \text{for } \varepsilon \in (0, 1],$$

whence

$$\limsup_{j\to\infty} D(X^{\varepsilon_j}) \le D(X).$$

On the other hand, $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$ implies

$$D(X) \le \liminf_{j\to\infty} D(X^{\varepsilon_j})$$

and so we obtain (11). □

**Theorem 2.** *Every minimizer $X$ of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ that satisfies (3) is Hölder continuous in $B$ and continuous on $\overline{B}$.*

*Proof.* Fix some $w_0 \in B$ and set $R := 1 - |w_0| > 0$. For $0 < r < R$ we define $H \in H_2^1(B_r(w_0), \mathbb{R}^3)$ as the solution of

$$\Delta H = 0 \quad \text{in } B_r(w_0), \quad H - X \in \mathring{H}_2^1(B_r(w_0), \mathbb{R}^3),$$

and then we set $Y(w) := H(w)$ for $w \in B_r(w_0)$ and $Y(w) := X(w)$ for $w \in B \setminus B_r(w_0)$. Since $Y \in \mathcal{C}(\Gamma)$ it follows that

$$\mathcal{F}(X) \leq \mathcal{F}(Y),$$

whence

$$\mathcal{F}_{B_r(w_0)}(X) \leq \mathcal{F}_{B_r(w_0)}(Y).$$

Here and in the following the index $B_r(w_0)$ means that the corresponding integrals are to be taken over the set $B_r(w_0)$. By (D) and (3) we have

$$m_1 D_{B_r(w_0)}(X) = m_1 A_{B_r(w_0)}(X) \leq \mathcal{F}_{B_r(w_0)}(X),$$

and (3) together with $A \leq D$ and $Y = H$ on $B_r(w_0)$ yields

$$\mathcal{F}_{B_r(w_0)}(Y) = \mathcal{F}_{B_r(w_0)}(H) \leq m_2 A_{B_r(w_0)}(H) \leq m_2 D_{B_r(w_0)}(H).$$

Thus

$$D_{B_r(w_0)}(X) \leq \frac{m_2}{m_1} D_{B_r(w_0)}(H),$$

that is,

$$\text{(12)} \qquad \Phi(r) := \int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \leq \frac{m_2}{m_1} \int_{B_r(w_0)} |\nabla H|^2 \, du \, dv.$$

Let us introduce polar coordinates $\rho$, $\theta$ around $w_0$ by $w = w_0 + \rho e^{i\theta}$; we write

$$X(w) = X(w_0 + \rho e^{i\theta}) =: X^*(\rho, \theta).$$

Then

$$\Phi(r) = \int_0^r \int_0^{2\pi} \{|X_\rho^*(\rho, \theta)|^2 + \rho^{-2} |X_\theta^*(\rho, \theta)|^2\} \rho \, d\rho \, d\theta.$$

Since

$$|X_\rho^*|^2 = \rho^{-2} |X_\theta^*|^2, \quad \langle X_\rho^*, X_\theta^* \rangle = 0$$

we have

$$\Phi(r) = 2 \int_0^r \rho^{-1} \left( \int_0^{2\pi} |X_\theta^*(\rho, \theta)|^2 \, d\theta \right) d\rho.$$

We can find a representative $X^*(\rho, \theta)$ that is absolutely continuous in $\theta$ for almost all $\rho \in (0, R)$ and $\int_0^{2\pi} |X_\theta^*(\rho, \theta)|^2 \, d\theta < \infty$ for these $\rho$. The function $\Phi(r)$ is absolutely continuous on $[0, R]$, and

$$\Phi'(r) = 2r^{-1} \int_0^{2\pi} |X_\theta^*(r,\theta)|^2 \, d\theta \quad \text{for } r \in (0,R) \setminus \mathcal{N}$$

where $\mathcal{N}$ is a one-dimensional null set.

Furthermore we have

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv \le \int_0^{2\pi} |X_\theta^*(r,\theta)|^2 \, d\theta$$

(see e.g. Vol. 2, Section 2.5, (18)), and so

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv \le \frac{1}{2} r\Phi'(r) \quad \text{for } r \in (0,R) \setminus \mathcal{N}.$$

By virtue of (12) we arrive at

$$\Phi(r) \le \frac{1}{2} \frac{m_2}{m_1} r\Phi'(r) \quad \text{a.e. on } (0,R).$$

Setting $\mu := m_1/m_2$ we have

$$2\mu\Phi(r) \le r\Phi'(r) \quad \text{a.e. on } (0,R)$$

whence

$$\Phi(r) \le (r/R)^{2\mu}\Phi(R) \quad \text{for } r \in (0,R), \tag{13}$$

and then it follows that $X \in C^{0,\mu}(B,\mathbb{R}^3)$ on account of Morrey's "Dirichlet growth theorem" (see Morrey [8], p. 79).

It remains to prove that $X \in C^0(\overline{B},\mathbb{R}^3)$. To this end we introduce polar coordinates $\rho$, $\vartheta$ around the origin and write

$$X(w) = X(\rho e^{i\vartheta}) =: X^*(\rho,\vartheta).$$

Set

$$\varepsilon(X,h) := \int_{1-2h}^1 \int_0^{2\pi} \left[|X_\rho^*(\rho,\vartheta)|^2 + |X_\vartheta^*(\rho,\vartheta)|^2\right] d\rho \, d\vartheta$$

for $0 < h < 1/4$; then

$$\frac{1}{2}\varepsilon(X,h) \le \int_{1-2h}^1 \int_0^{2\pi} (|X_\rho^*|^2 + \rho^{-2}|X_\vartheta^*|^2)\rho \, d\rho \, d\vartheta \le 2\varepsilon(X,h).$$

It follows as in Morrey [8], Theorem 3.5.2, that there is a number $c_0(\mu)$ depending only on $\mu$ such that

$$|X^*(1-h,\theta) - X^*(1-h,\theta')| \le c_0(\mu)\varepsilon(X,h)h^{-\mu}|\theta-\theta'|^\mu \le c_0(\mu)\varepsilon(X,h)$$

for all $\theta' \in \mathbb{R}$ with $|\theta - \theta'| \le h < 1/4$.

Let $\xi(\theta)$ be the continuous Sobolev trace $X^*(1,\theta)$ of $X^*$ on $\rho = 1$, and set

$$\omega(\xi, h) := \sup\{|\xi(\theta') - \xi(\theta'')| \colon \theta', \theta'' \in \mathbb{R}, |\theta' - \theta''| < h\}.$$

Then $\omega(\xi, h) \to 0$ as $h \to +0$.

Furthermore, for any $\theta \in \mathbb{R}$ there is a $\theta_1$ with $|\theta - \theta_1| \le h$ such that $X(\rho, \theta_1)$ is absolutely continuous in $\rho \in [1/2, 1]$, $X_\rho(\cdot, \theta_1) \in L_2([1/2, 1], \mathbb{R}^3)$ and

$$|X^*(\rho, \theta_1) - \xi(\theta_1)| \to 0 \quad \text{as } \rho \to 1 - 0$$

as well as

$$\int_{1-h}^{1} |X^*_\rho(\rho, \theta_1)|^2 d\rho \le h^{-1} \varepsilon^2(X, h).$$

It follows that

$$\begin{aligned} |\xi(\theta_1) - X^*(1-h, \theta_1)| &\le \int_{1-h}^{1} |X^*_\rho(\rho, \theta_1)| \, d\rho \\ &\le \sqrt{h} \cdot \left\{ \int_{1-h}^{1} |X^*_\rho(\rho, \theta_1)|^2 \, d\rho \right\}^{1/2} \le \varepsilon(X, h). \end{aligned}$$

Given $\theta_0$ and $\theta$ with $|\theta - \theta_0| \le h' < 1/4$ we choose $\theta_1$ as above. Because of

$$\begin{aligned} &|X^*(1-h, \theta) - \xi(\theta_0)| \\ &\quad \le |X^*(1-h, \theta) - X^*(1-h, \theta_1)| + |X^*(1-h, \theta_1) - \xi(\theta_1)| \\ &\qquad + |\xi(\theta_1) - \xi(\theta)| + |\xi(\theta) - \xi(\theta_0)| \end{aligned}$$

we then obtain

$$|X^*(1-h, \theta) - \xi(\theta_0)| \le [1 + c_0(\mu)]\varepsilon(X, \theta, h) + \omega(\xi, h) + \omega(\xi, h').$$

This proves $X^*(\rho, \theta) \to \xi(\theta_0)$ as $\rho \to 1-0$ and $\theta \to \theta_0$. Hence $X \in C^0(\overline{B}, \mathbb{R}^3)$. □

**Remark 1.** So far no general results concerning higher regularity of solutions to (2) are known. For a special class of Cartan functionals it was proved that the minimizers $X$ of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ satisfy $X \in H^2_2(B, \mathbb{R}^3) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ for some $\alpha \in (0,1)$ provided that $F \in C^2$ on $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ and $\Gamma \in C^4$; see Hildebrandt and von der Mosel [1–7].

## 4.14 Isoperimetric Inequalities

Now we want to derive the isoperimetric inequality for disk-type surfaces $X : B \to \mathbb{R}^3$ of class $C^1(\overline{B}, \mathbb{R}^3)$ or, more generally, for $X \in H^1_2(B, \mathbb{R}^3)$ with the parameter domain

$$B = \{w \in \mathbb{C} \colon |w| < 1\},$$

the boundary of which is given by

$$C = \partial B = \{w \in \mathbb{C}\colon |w| = 1\}.$$

Recall that any $X \in H_2^1(B, \mathbb{R}^3)$ has boundary values $X|_C$ of class $L_2(C, \mathbb{R}^3)$. Denote by $L(X)$ the length of the boundary trace $X|_C$, i.e.,

$$L(X) = L(X|_C) := \int_C |dX|.$$

We recall a result that, essentially, has been proved in Section 4.7.

**Lemma 1.** (i) *Let $X : B \to \mathbb{R}^3$ be a minimal surface with a finite Dirichlet integral*

$$D(X) = \frac{1}{2}\int_B |\nabla X|^2\, du\, dv$$

*and with boundary values $X|_C$ of finite total variation*

$$L(X) = \int_C |dX|.$$

*Then $X$ is of class $H_2^1(B, \mathbb{R}^3)$ and has a continuous extension to $\overline{B}$, i.e., $X \in C^0(\overline{B}, \mathbb{R}^3)$. Moreover, the boundary values $X|_C$ are of class $H_1^1(C, \mathbb{R}^3)$. Setting $X(r, \theta) := X(re^{i\theta})$, we obtain that, for any $r \in (0, 1]$, the function $X_\theta(r, \theta)$ vanishes at most on a set of $\theta$-values of one-dimensional Hausdorff measure zero, and that the limits*

$$\lim_{r\to 1-0} X_r(r, \theta) \quad \textit{and} \quad \lim_{r\to 1-0} X_\theta(r, \theta)$$

*exist, and that*

$$\frac{\partial}{\partial\theta} X(1, \theta) = \lim_{r\to 1-0} X_\theta(r, \theta) \quad \textit{a.e. on } [0, 2\pi]$$

*holds true. Finally, setting $X_r(1, \theta) := \lim_{r\to 1-0} X_r(r, \theta)$, it follows that*

$$\int_B \langle \nabla X, \nabla \phi\rangle\, du\, dv = \int_C \langle X_r, \phi\rangle\, d\theta \tag{1}$$

*is satisfied for all $\phi \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$. Moreover, we have*

$$\lim_{r\to 1-0} \int_0^{2\pi} |X_\theta(r, \theta)| r\, d\theta = \int_0^{2\pi} |dX(1, \theta)|. \tag{2}$$

(ii) *If $X : B \to \mathbb{R}^3$ is a minimal surface with a continuous extension to $\overline{B}$ such that $L(X) := \int_C |dX| < \infty$, then we still have (2).*

*Proof.* Since $L(X) < \infty$, the finiteness of $D(X)$ is equivalent to the relation $X \in H_2^1(B, \mathbb{R}^3)$, on account of Poincaré's inequality. Hence $X$ has an $L_2(C)$-trace on the boundary $C$ of $\partial B$ which, by assumption, has a finite total variation $\int_C |dX|$. Consequently, the two one-sided limits

$$\lim_{\theta \to \theta_0 - 0} X(1, \theta) \quad \text{and} \quad \lim_{\theta \to \theta_0 + 0} X(1, \theta)$$

exist for every $\theta_0 \in \mathbb{R}$. In conjunction with the Courant–Lebesgue lemma, we obtain that $X(1, \theta)$ is a continuous function of $\theta \in \mathbb{R}$ whence $X \in C^0(\overline{B}, \mathbb{R}^3)$ (cf. Section 4.7, part (iii) of the proof of Proposition 3). The rest of the proof follows from Theorems 1 and 2 in Section 4.7. □

**Lemma 2 (Wirtinger's inequality).** *Let $Z : \mathbb{R} \to \mathbb{R}^3$ be an absolutely continuous function that is periodic with the period $L > 0$ and has the mean value*

$$P := \frac{1}{L} \int_0^L Z(t)\, dt. \tag{3}$$

*Then we obtain*

$$\int_0^L |Z(t) - P|^2\, dt \le \left(\frac{L}{2\pi}\right)^2 \int_0^L |\dot{Z}(t)|^2\, dt, \tag{4}$$

*and the equality sign holds if and only if there are constant vectors $A_1$ and $B_1$ in $\mathbb{R}^3$ such that*

$$Z(t) = P + A_1 \cos\left(\frac{2\pi}{L} t\right) + B_1 \sin\left(\frac{2\pi}{L} t\right) \tag{5}$$

*holds for all $t \in \mathbb{R}$.*

*Proof.* We first assume that $L = 2\pi$ and $\int_0^{2\pi} |\dot{Z}|^2\, dt < \infty$. Then we have the expansions

$$\begin{aligned} Z(t) &= P + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt), \\ \dot{Z}(t) &= \sum_{n=1}^{\infty} n(B_n \cos nt - A_n \sin nt) \end{aligned}$$

of $Z$ and $\dot{Z}$ into Fourier series with $A_n, B_n \in \mathbb{R}^3$, and

$$\begin{aligned} \int_0^{2\pi} |Z - P|^2\, dt &= \pi \sum_{n=1}^{\infty} (|A_n|^2 + |B_n|^2), \\ \int_0^{2\pi} |\dot{Z}|^2\, dt &= \pi \sum_{n=1}^{\infty} n^2 (|A_n|^2 + |B_n|^2). \end{aligned} \tag{6}$$

Consequently it follows that

$$\int_0^{2\pi} |Z - P|^2 \, dt \le \int_0^{2\pi} |\dot{Z}|^2 \, dt, \tag{7}$$

and the equality sign holds if and only if all coefficients $A_n$ and $B_n$ vanish for $n > 1$. Thus we have verified the assertion under the two additional hypotheses. If $\int_0^{2\pi} |\dot{Z}|^2 \, dt = \infty$, the statement of the lemma is trivially satisfied, and the general case $L > 0$ can be reduced to the case $L = 2\pi$ by the scaling transformation $t \mapsto (2\pi/L)t$. □

Now we shall state the isoperimetric inequality for minimal surfaces in its simplest form.

**Theorem 1.** *Let $X \in C^2(B, \mathbb{R}^3)$ with $B = \{w \colon |w| < 1\}$ be a minimal surface, i.e. $X$ be nonconstant and satisfy*

$$\Delta X = 0,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

*Assume also that $X$ is either of class $H_2^1(B, \mathbb{R}^3)$ or of class $C^0(\overline{B}, \mathbb{R}^3)$, and that $L(X) = \int_C |dX| < \infty$. Then $D(X)$ is finite, and we have*

$$D(X) \le \frac{1}{4\pi} L^2(X). \tag{8}$$

*Moreover, the equality sign holds if and only if $X : B \to \mathbb{R}^3$ represents a (simply covered) disk.*

**Remark 1.** Note that for every minimal surface $X : B \to \mathbb{R}^3$ the area functional $A(X)$ coincides with the Dirichlet integral $D(X)$. Thus (8) can equivalently be written as

$$A(X) \le \frac{1}{4\pi} L^2(X). \tag{8'}$$

*Proof of Theorem 1.* (i) Assume first that $X$ is of class $H_2^1(B, \mathbb{R}^3)$, and that $P$ is a constant vector in $\mathbb{R}^3$. Because of $L(X) < \infty$, the boundary values $X|_C$ are bounded whence $X$ is of class $L_\infty(B, \mathbb{R}^3)$ (this follows from the maximum principle in conjunction with a suitable approximation device). Thus we can apply formula (1) to $\phi = X - P$, obtaining

$$\begin{aligned} &\int_B \langle \nabla X, \nabla X \rangle \, du \, dv \\ &\quad = \int_B \langle \nabla X, \nabla (X - P) \rangle \, du \, dv \\ &\quad = \int_C \langle X_r, X - P \rangle \, d\theta \le \int_C |X_r||X - P| \, d\theta \\ &\quad = \int_C |X_\theta||X - P| \, d\theta = \int_0^{2\pi} |X_\theta(1, \theta)||X(1, \theta) - P| \, d\theta. \end{aligned} \tag{9}$$

Introducing $s = \sigma(\theta)$ by

$$\sigma(\theta) := \int_0^\theta |X_\theta(1,\theta)|\,d\theta,$$

we obtain that $\sigma(\theta)$ is a strictly increasing and absolutely continuous function of $\theta$, and $\dot\sigma(\theta) = |X_\theta(1,\theta)| > 0$ a.e. on $\mathbb{R}$. Hence $\sigma : \mathbb{R} \to \mathbb{R}$ has a continuous inverse $\tau : \mathbb{R} \to \mathbb{R}$. Let us introduce the reparametrization

$$Z(s) := X(1,\tau(s)), \quad s \in \mathbb{R},$$

of the curve $X(1,\theta)$, $\theta \in \mathbb{R}$. Then, for any $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, the numbers $\theta_1 := \tau(s_1)$, $\theta_2 := \tau(s_2)$ satisfy $\theta_1 < \theta_2$ and

$$(10) \qquad \int_{s_1}^{s_2} |dZ| = \int_{\theta_1}^{\theta_2} |dX| = \sigma(\theta_2) - \sigma(\theta_1) = s_2 - s_1,$$

whence

$$|Z(s_2) - Z(s_1)| \le s_2 - s_1.$$

Consequently, the mapping $Z : \mathbb{R} \to \mathbb{R}^3$ is Lipschitz continuous and therefore also absolutely continuous, and we obtain from (10) that

$$(11) \qquad \int_{s_1}^{s_2} |Z'(s)|\,ds = s_2 - s_1$$

($' = \frac{d}{ds}$), whence

$$(12) \qquad |Z'(s)| = 1 \quad \text{a.e. on } \mathbb{R}.$$

In other words, the curve $Z(s)$ is the reparametrization of $X(1,\theta)$ with respect to the parameter $s$ of its arc length.

As the mapping $\sigma : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, it maps null sets onto null sets, and we derive from

$$\frac{\tau(s_2) - \tau(s_1)}{s_2 - s_1} = \frac{1}{\frac{\sigma(\theta_2) - \sigma(\theta_1)}{\theta_2 - \theta_1}}$$

and from $\dot\sigma(\theta) > 0$ a.e. on $\mathbb{R}$ that

$$(13) \qquad \tau'(s) = \frac{1}{\dot\sigma(\tau(s))} > 0 \quad \text{a.e. on } \mathbb{R}.$$

On account of

$$\dot\sigma(\theta) = |X_\theta(1,\theta)| \quad \text{a.e. on } \mathbb{R}$$

it then follows that

$$(14) \qquad |X_\theta(1,\tau(s))| \frac{d\tau}{ds}(s) = 1 \quad \text{a.e. on } \mathbb{R},$$

and thus we obtain

$$(15)\qquad \int_0^{2\pi} |X_\theta(1,\theta)||X(1,\theta)-P|\,d\theta = \int_0^L |Z(s)-P|\,ds.$$

We now infer from (9) and (15) that

$$(16)\qquad \int_B \langle \nabla X, \nabla X\rangle\,du\,dv \le \int_0^L |Z(s)-P|\,ds.$$

By Schwarz's inequality, we have

$$(17)\qquad \int_0^L |Z(s)-P|\,ds \le \sqrt{L}\left\{\int_0^L |Z(s)-P|^2\,ds\right\}^{1/2},$$

and Wirtinger's inequality (4) together with (12) implies that

$$(18)\qquad \left\{\int_0^L |Z(s)-P|^2\,ds\right\}^{1/2} \le \frac{L^{3/2}}{2\pi}$$

if we choose $P$ as the barycenter of the closed curve $Z:[0,L]\to\mathbb{R}^3$, i.e., if

$$P := \frac{1}{L}\int_0^L Z(s)\,ds.$$

By virtue of (16)–(18), we arrive at

$$(19)\qquad \int_B |\nabla X|^2\,du\,dv \le \frac{1}{2\pi}L^2$$

which is equivalent to the desired inequality (8).

Suppose that equality holds true in (8) or, equivalently, in (19). Then equality must also hold in Wirtinger's inequality (18), and by Lemma 2 we infer

$$Z(s) = P + A_1\cos\left(\frac{2\pi}{L}s\right) + B_1\sin\left(\frac{2\pi}{L}s\right).$$

Set $R := L/(2\pi)$ and $\varphi = s/R$. Because of $|Z'(s)|\equiv 1$, we obtain

$$R^2 = |A_1|^2\sin^2\varphi + |B_1|^2\cos^2\varphi - 2\langle A_1, B_1\rangle \sin\varphi\cos\varphi.$$

Choosing $\varphi = 0$ or $\frac{\pi}{2}$, respectively, it follows that

$$|A_1| = |B_1| = R,$$

and therefore

$$\langle A_1, B_1\rangle = 0.$$

Then the pair of vectors $E_1, E_2 \in \mathbb{R}^3$, defined by

$$E_1 := \frac{1}{R} A_1, \quad E_2 := \frac{1}{R} B_1,$$

is orthonormal, and we have

$$Z(R\varphi) = P + R\{E_1 \cos\varphi + E_2 \sin\varphi\}.$$

Consequently $Z(R\varphi)$, $0 \le \varphi \le 2\pi$, describes a simply covered circle of radius $R$, centered at $P$, and the same holds true for the curve $X(1, \theta)$ with $0 \le \theta \le 2\pi$. Hence $X : \overline{B} \to \mathbb{R}^3$ represents a (simply covered) disk of radius $R$, centered at $P$. This can be seen as follows: We may assume that the circle $\Gamma := \{X(1, \theta) \colon 0 \le \theta \le 2\pi\}$ lies in the $x, y$-plane and is given by

$$\Gamma = \{(x, y, z) \colon x^2 + y^2 = R^2,\ z = 0\}.$$

Then the maximum principle implies that $X$ has the form

$$X = (X^1, X^2, 0) \quad \text{with } |X^1(w)|^2 + |X^2(w)|^2 \le R^2 \text{ for } w \in \overline{B}$$

since $\Delta X^3 = 0$ and $\Delta(|X^1|^2 + |X^2|^2) \ge 0$. Using the conformality relation it follows that either $f(w) = X^1(w) + iX^2(w)$ or $\overline{f(w)}$ is holomorphic and, in fact, conformal on $B$ (for details, we refer to the proof of Theorem 1 in Section 4.11).

Conversely, if $X : \overline{B} \to \mathbb{R}^3$ represents a simply covered disk, then the equality sign holds true in ($8'$) and, therefore also in (8).

Thus the assertion of the theorem is proved under the assumption that $X \in H_2^1(B, \mathbb{R}^3)$.

(ii) Suppose now that $X$ is of class $C^0(\overline{B}, \mathbb{R}^3)$. Then we introduce nonconstant minimal surfaces $X_k : B \to \mathbb{R}^3$ of class $C^\infty(\overline{B}, \mathbb{R}^3)$ by defining

$$X_k(w) := X(r_k w) \quad \text{for } |w| \le 1,\ r_k := \frac{k}{k+1}.$$

We can apply (i) to each of the surfaces $X_k$, thus obtaining

$$4\pi D(X_k) \le \left\{ \int_0^{2\pi} |dX_k(1, \theta)| \right\}^2. \tag{20}$$

For $k \to \infty$, we have $r_k \to 1 - 0$, $D(X_k) \to D(X)$, and part (ii) of Lemma 1 yields

$$\lim_{k \to \infty} \int_0^{2\pi} |dX_k(1, \theta)| = \int_0^{2\pi} |dX(1, \theta)|.$$

Thus we infer from (20) that

$$4\pi D(X) \le L^2(X)$$

which implies in particular that $X$ is of class $H_2^1(B, \mathbb{R}^3)$. For the rest of the proof, we can now proceed as in part (i). □

If the boundary of a minimal surface $X$ is very long in comparison to its "diameter", then another estimate of $A(X) = D(X)$ might be better which depends only linearly on the length $L(X)$ of the boundary of $X$. We call this estimate *the linear isoperimetric inequality*. It reads as follows:

**Theorem 2.** *Let $X$ be a nonconstant minimal surface with the parameter domain $B = \{w \colon |w| < 1\}$, and assume that $X$ is either continuous on $\overline{B}$ or of class $H_2^1(B, \mathbb{R}^3)$. Moreover, suppose that the length $L(X) = \int_C |dX|$ of its boundary is finite, and let $\mathcal{K}_R(P)$ be the smallest ball in $\mathbb{R}^3$ containing $X(\partial B)$ and therefore also $X(\overline{B})$. Then we have*

$$D(X) \le \tfrac{1}{2} R L(X). \tag{21}$$

*Equality holds in (21) if and only if $X(B)$ is a plane disk.*

*Proof.* By Theorem 1 it follows that $D(X) < \infty$ and $X \in H_2^1(B, \mathbb{R}^3)$, and formula (9) implies

$$2D(X) \le \int_C |X_\theta||X - P| \, d\theta \le RL(X) \tag{22}$$

whence we obtain (21).

Suppose now that

$$D(X) = \tfrac{1}{2} R L(X). \tag{23}$$

Then we infer from (9) and (22) that

$$\int_C \langle X_r, X - P \rangle \, d\theta = \int_C |X_r||X - P| \, d\theta$$

is satisfied; consequently we have

$$\langle X_r, X - P \rangle = |X_r||X - P|$$

a.e. on $C$, that is, the two vectors $X_r$ and $X - P$ are collinear a.e. on $C$.

Secondly we infer from (22) and (23) that

$$|X - P| = R \quad \text{a.e. on } C.$$

Hence the $H_1^1$-curve $\Sigma$ defined by $X : C \to \mathbb{R}^3$ lies on the sphere $S_R(P)$ of radius $R$ centered at $P$, and the side normal $X_r$ of the minimal surface $X$ at $\Sigma$ is proportional to the radius vector $X - P$. Thus $X_r(1, \theta)$ is perpendicular to $S_R(P)$ for almost all $\theta \in [0, 2\pi]$. Hence the surface $X$ meets the sphere $S_R(P)$ orthogonally a.e. along $\Sigma$. As in the proof of Theorem 1 in Section 5.4 we can show that $X$ is a stationary surface with a free boundary on $S_R(P)$ and that $X$ can be viewed as a stationary point of Dirichlet's integral in the class $\mathcal{C}(S_R(P))$. By Theorems 1 and 2 of Vol. 2, Section 2.8, the surface $X$ is real analytic on the closure $\overline{B}$ of $B$. Then it follows from the Theorem in Vol. 2, Section 1.7 that $X(\overline{B})$ is a plane disk.

Conversely, if $X : B \to \mathbb{R}^3$ represents a plane disk, then (23) is fulfilled. □

A more general version of the isoperimetric inequality (8′) will be proved in Vol. 2, Section 6.5. We also refer to Section 6.4 of this volume where the isoperimetric inequality of Morse–Tompkins for harmonic surfaces is derived.

## 4.15 Scholia

### *1 Historical Remarks and References to the Literature*

Although Plateau's problem is one of the classical questions in geometry and analysis, progress in solving it was very slow. The problem was already formulated by Lagrange in his *Essai d'une nouvelle méthode* ... [1]: *trouver la surface qui est la moindre de toutes celles qui ont un même périmètre donné*, but neither he nor Euler were able to solve the question. It seemed even difficult to find solutions of the minimal surface equation, not to speak of the corresponding boundary value problem. In the early 19th century, Gergonne [1] drew the attention of his contemporaries again to this and related boundary value problems, but still Jacobi was unable to tackle them. *In his Lectures on the Calculus of Variations* at Königsberg, 1837/38, he said: *Es haben sich in der neuesten Zeit die ausgezeichnetsten Mathematiker wie Poisson und Gauß mit der Auffindung der Variation des Doppelintegrals beschäftigt, die wegen der willkürlichen Funktionen unendliche Schwierigkeiten macht. Dennoch wird man durch ganz gewöhnliche Aufgaben darauf geführt, z.B. durch das Problem: unter allen Oberflächen, die durch ein schiefes Viereck im Raum gelegt werden können, diejenige anzugeben, welche den kleinsten Flächeninhalt hat. Es ist mir nicht bekannt, daß schon irgend jemand daran gedacht hätte, die zweite Variation solcher Doppelintegrale zu untersuchen; auch habe ich, trotz vieler Mühe, nur erkannt, daß der Gegenstand zu den allerschwierigsten gehört.*

The problem mentioned by Jacobi, namely to span a minimal surface in a general quadrilateral of $\mathbb{R}^3$, was first solved by H.A. Schwarz and, independently and at about the same time, by Riemann. Riemann's paper appeared posthumously in 1867, the same year that Schwarz's prize-essay was sent to the Berlin Academy. Later on, Plateau's problem was solved for other polygonal boundaries and, more generally, also free and partially free boundary problems for so-called Schwarzian chains were tackled. In particular, we mention the work of Weierstraß [4], Tallquist [2], and Neovius [1–5]. An outline of the techniques used by these authors can be found in the treatise of Bianchi [1]; a very extensive presentation is given in volume 1 of Darboux's *Leçons* [1].

The first general existence proof for the nonparametric Plateau problem was given by A. Haar [3] in 1927, with important supplements by Radó concerning the regularity of minimizers. The contributions of Haar and Radó were major mathematical achievements; for the first time, the program envisioned by Hilbert in his problems 19 and 20 had been carried out for a fundamental variational problem with nonlinear Euler equations.

A first solution of the Plateau problem for a general contour was published by R. Garnier [2] in 1928. By a limit procedure he obtained a solution for unknotted and piecewise smooth Jordan curves from a penetrating analysis of the Plateau problem for polygonal boundaries. However, Garnier's long paper was apparently seldom read if it was read at all (see Nitsche [28], p. 251),[2] and it was soon superseded by the convincing proofs of J. Douglas [11,12] and T. Radó [17,18] published about 1930. Douglas began to publish on Plateau's problem in 1927, and he announced a solution as early as 1929 (see Douglas [5] and [17]). Still, his first papers were apparently not convincing to everyone (see Constance Reid [1], pp. 173–174), and the long list of Douglas's announcements prior to 1931 might indicate that Douglas himself did not think he had found the best possible presentation, see Douglas [1–11].

Douglas based his approach to Plateau's problem on the functional

$$A_0(X) := \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|\mathcal{X}(\theta)-\mathcal{X}(\varphi)|^2}{4\sin^2\frac{1}{2}(\theta-\varphi)}\,d\theta\,d\varphi,$$

$\mathcal{X}(\theta) := X(e^{i\theta}) = X(\cos\theta, \sin\theta)$, which, for harmonic mappings

$$X : B = \{w\colon |w| < 1\} \to \mathbb{R}^N,$$

coincides with Dirichlet's integral $D(X)$ (cf. Section 6.4). The Douglas functional $A_0(X)$ has certain advantages as it only takes the boundary values $\mathcal{X}(\theta)$ of a harmonic mapping $X : B \to \mathbb{R}^N$ into account, but the Dirichlet integral is more natural and easier to handle. In the case of the general Plateau problem, the Dirichlet integral can still be used while the Douglas functional has to be replaced by a rather unwieldy expression, and also for free boundary problems the Dirichlet integral seems to be the natural tool.

Radó's method to attack Plateau's problem is much closer to the approach used in the present chapter than the method of Douglas. Radó runs through several approximation steps. First he treats the case of a polygonal boundary $\Gamma$ where one can find a sequence of polyhedra $P_n$ whose areas approach the infimum $a(\Gamma)$ of areas of surfaces within $\Gamma$. As polyhedra admit conformal representations $Z_n : B \to \mathbb{R}^3$, the Dirichlet integrals of these representations approach the infimum value $e(\Gamma)$ of Dirichlet's integral for surfaces $X : B \to \mathbb{R}^3$ within $\Gamma$, and we have $e(\Gamma) = a(\Gamma)$. Replacing the $Z_n$ by harmonic maps $X_n : B \to \mathbb{R}^3$ with the same boundary values as $Z_n$, we obtain $D(X_n) \to e(\Gamma) = a(\Gamma)$ as $n \to \infty$. A standard selection theorem for harmonic maps implies that we can extract a subsequence from $\{X_n\}$, again denoted by $\{X_n\}$, which converges uniformly on any $B' \subset\subset B$ to some harmonic map $X : B \to \mathbb{R}^3$, and whose derivatives converge uniformly on $B' \subset\subset B$ to the derivatives of $X$. Then we obtain

$$\int_{B'} (|D_u X_n| - |D_v X_n|)^2\,du\,dv \to \int_{B'} (|D_u X| - |D_v X|)^2\,du\,dv,$$
$$\int_{B'} |\langle D_u X_n, D_v X_n\rangle|\,du\,dv \to \int_{B'} |\langle D_u X, D_v X\rangle|\,du\,dv$$

[2] However, note the recent work of L. Desideri; cf. p. 364.

as $n \to \infty$. On the other hand, the choice of the $X_n$ together with a simple estimation yields that the integrals on the left-hand side tend to zero as $n \to \infty$ since the $X_n$ are approximate solutions for the Plateau problem to $\Gamma$. This implies that $X$ is a minimal surface, i.e., a harmonic map satisfying the conformality conditions $|X_u| = |X_v|$, $\langle X_u, X_v \rangle = 0$. Moreover, a sophisticated *approximation theorem* yields that $X$ is continuous on $\bar{B}$, and that $X|_{\partial B}$ gives a parametrization of $\Gamma$. Thus Plateau's problem is solved for polygons.

In the next step, a rectifiable curve $\Gamma$ is approximated by polygons $\Gamma_n$ in the sense of Fréchet. Solving the Plateau problem for any of the $\Gamma_n$ by a minimal surface $X_n$, another application of the approximation theorem together with a suitable compactness result for sequences of harmonic maps yields a solution of Plateau's problem minimizing area.

An admirably clear and short presentation of the results of Haar, Douglas and Radó is given in the report [21] by Radó.

We note that the methods of Douglas and Radó yield area-minimizing minimal surfaces spanned into $\Gamma$ if $a(\Gamma) < \infty$ whereas Garnier's solutions might only be stationary. Moreover, Douglas was able to solve Plateau's problem even in the case when $a(\Gamma) = \infty$. The essential simplification achieved in the proofs of R. Courant [4] and L. Tonelli [1] presented in this chapter follows from the Courant–Lebesgue lemma which is also of use in many other situations. The method of deriving the conformality conditions by a variation of the independent variables is due to Radó (cf. [21], pp. 87–89). The efficient variational formula generalizing Radó's idea was stated by Courant [15].

Another solution of Plateau's problem was found by McShane [1,2] in 1933 who directly attacked the problem of minimizing area. Using ideas of Lebesgue he showed: (i) One can find a minimizing sequence of Lebesgue monotone surfaces. (ii) Each of these surfaces can be replaced by a (weakly) conformally parametrized Lebesgue monotone surface. (iii) The minimizing sequence obtained by (i), (ii) is compact in $C^0(\overline{B}, \mathbb{R}^3)$. A detailed presentation of McShane's approach is given in Nitsche [28], pp. 414–430.

The approach of Section 4.10 is due to S. Hildebrandt and H. von der Mosel [1–7]; it leads to another solution of Plateau's problem by minimizing area. Contrary to all other methods this approach does not use any results on conformal or quasiconformal reparametrizations of a given surface such as the theorems of Lichtenstein or of Carathéodory, and so it establishes an *elementary proof* of the fact that the minimizers of Dirichlet's integral in the class of disk-type surfaces bounded by a given rectifiable Jordan contour are as well area minimizing. This was thought to be impossible; see Courant [15], pp. 116–118. Moreover, a modification of the method is used in 4.11 to derive the *global Lichtenstein theorem* by a variational method (cf. Hildebrandt and von der Mosel [6,7]). Another variational proof of this theorem was earlier given by J. Jost [6] and [17], rectifying the original approach by C.B. Morrey (see [8], Chapter 9) which contains a gap.

The partially free problem was originally treated by Courant using some of the ideas described in Chapter 1 of Vol. 2. The simplified version of Section 4.6 is due to Morrey [8]. Courant's original approach can be studied in his monograph [15] and also in Nitsche's treatise [28].

A solution of Plateau's problem for minimal surfaces $X : B \to M$ in Riemannian manifolds $M$ of great generality was given by C.B. Morrey [3], with later supplements by L. Lemaire [1] and J. Jost [6]. Extensions to surfaces of constant or prescribed mean curvature $H$ (=$H$-surfaces) are due to E. Heinz [2], H. Werner [1,2], S. Hildebrandt [4–10], H. Wente [1–5], K. Steffen [1–6], R. Gulliver [1,3], Gulliver and Spruck [1,2], Hildebrandt and Kaul [1], Brezis and Coron [1–4], M. Struwe [5,7,11,12,14], J. Jost [17], U. Dierkes [2], and Duzaar and Steffen [6,7]. The presentation given by Morrey in Chapter 9 of his treatise [8] is not quite correct but can be rectified. This was carried out by Jost in his paper [6] and also in his monograph [17] where one finds a complete theory of two-dimensional geometric variational problems comprising the theory of conformal and harmonic mappings, Teichmüller theory, minimal surfaces of disk-type as well as of higher topological type, Plateau's problem, and free boundary problems. We also refer to Sections 4.10–4.13 above.

Detailed presentations of the results concerning Plateau's problem can be found in the survey of Radó [21], Courant's monograph [15] and, most complete of all, in Nitsche's *Lectures* [28,37]. Beautiful recent surveys, also covering results on $H$-surfaces, were written by M. Struwe [11] and J. Jost [17].

In solving Plateau's problem, it is essential that $\Gamma$ is a Jordan curve, i.e., a continuous embedding of the unit circle $S^1$ into $\mathbb{R}^3$, in other words, that $\Gamma$ is not allowed to have selfintersections. Nevertheless one can pose the problem of minimizing area among surfaces bounded by a rectifiable closed curves $\Gamma$ with selfintersections if one enlarges the notion of admissible surfaces. For example, if $\Gamma$ is the "figure eight" in $\mathbb{R}^2$, it bounds a surface of minimal area that splits into two minimal disks. Still one can write it as a continuous mapping. Using the Lebesgue notion of area, J. Hass [2] proved:

*Any closed rectifiable curve $\Gamma$ in $\mathbb{R}^3$ bounds a "disk of least area" which is a smooth immersion away from the boundary.* This means: There is a mapping $X \in C^0(\overline{B}, \mathbb{R}^3)$ of the unit disk $\overline{B} \subset \mathbb{R}^2$ into $\mathbb{R}^3$ bounded by $\Gamma$ such that $X$ yields a smooth immersion of $B \setminus X^{-1}(\Gamma)$. The phrase "$X$ is bounded by $\Gamma$" means: If $\gamma : S^1 \to \mathbb{R}^3$ is a continuous representation of $\Gamma$ and $\gamma' = X|_{\partial B}$, then $\gamma'$ is a continuous mapping $S^1 \to \mathbb{R}^3$ with $d_F(\gamma, \gamma') = 0$ where $d_F(\gamma, \gamma')$ is the "Fréchet distance" of $\gamma$ and $\gamma'$, i.e.

$$d_F(\gamma, \gamma') = \inf\Big\{\sup_{S^1} |\gamma - \gamma' \circ \varphi| : \varphi \in \mathrm{Hom}(S^1)\Big\},$$

where $\mathrm{Hom}(S^1)$ denotes the set of homeomorphisms $\varphi : S^1 \to S^1$. Here the splitting phenomenon is expressed by the fact that $X^{-1}(\Gamma)$ can be larger than $\partial B$, i.e. $B \cap X^{-1}(\Gamma)$ can be nonempty.

Another approach to the splitting (or bubbling) problem is contained in the work of E. Kuwert [5–7], operating with Dirichlet's integral; cf. Vol. 2, Scholia to Chapter 1.

*2 Branch Points*

In Section 3.2, Proposition 1 we have derived asymptotic expansions for minimal surfaces $X : B \to \mathbb{R}^3$ and their complex derivative $X_w$. Analogous expansions can be established at boundary branch points as we shall see in Section 2.10 of Vol. 2. The basic tool for proving such asymptotic formulas is a method due to Hartman and Wintner which is described in Chapter 3 of Vol. 2.

It was a long-standing question whether the area-minimizing solution of Plateau's problem obtained by Douglas and Radó is a *regular* surface, that is, an immersion. This was eventually confirmed in a series of papers by R. Osserman [12], R. Gulliver [2], H.W. Alt [1,2], and Gulliver, Osserman, and Royden [1]. The break-through was achieved by Osserman [12] who, by an ingenious idea, was able to rule out the existence of *true branch points* for minimizers. A true branch point of a minimal surface $X : B \to \mathbb{R}^3$ is characterized by the fact that there are several geometrically different sheets of the surface lying over the tangent plane at $w_0$. These sheets intersect transversally along smooth curves in $\mathbb{R}^3$ emanating from $X(w_0)$. A false (interior) branch point is a singular point $w_0 \in B$ which has a neighborhood $U$ in $B$ such that $X(U)$ turns out to be (the trace of) an embedded surface. In other words, false branch points cannot be detected by looking at the image of a minimal surface; they are just the result of a false parametrization.

Osserman's reasoning did not rule out the existence of false branch points for a Douglas–Radó solution. This second part of the regularity proof was, more or less simultaneously, achieved by Gulliver and Alt in the papers cited above. Another treatment can be found in the paper of Gulliver–Osserman–Royden. It is still an open problem whether there can be branch points at the boundary $\partial B$; however, Gulliver and Lesley [1] indicated that the Douglas–Radó solution is free of boundary branch points if $\Gamma$ is a regular, real-analytic Jordan curve. Thus we now have the following sharpened version of the

**Fundamental existence theorem.** *Every closed rectifiable Jordan curve $\Gamma$ in $\mathbb{R}^3$ bounds an area minimizing surface $X : B \to \mathbb{R}^3$ of the type of the disk, and all solutions of this type are regular surfaces, i.e., they are free of branch points $w_0 \in B$. If $\Gamma$ is regular and real analytic, then they have no branch points on $\partial B$, either.*

So far, all known proofs excluding the existence of branch points of area minimizing solutions of Plateau's problem were quite involved; thus we have abstained from presenting them. However, on two occasions we have used the opportunity to sketch the basic ideas. At the end of Section 5.3 in Vol. 2 we have outlined Osserman's idea of how to exclude true branch points at the boundary, and in Section 1.9 of Vol. 2 we have indicated how false branch points can be excluded.

A. Tromba has recently developed a method to exclude true interior branch points for minimizers of $A$, which is technically simple and applies in many

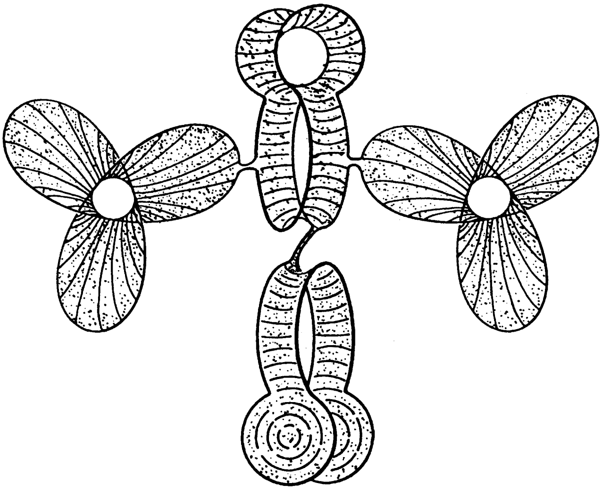


**Fig. 1.** A knotted curve bounding an embedded minimal surface of higher topological type

cases also to weak minimizers of $D$. This approach is presented in Chapter 6 of Vol. 2.

We should like to mention that Gulliver and Alt have ruled out interior branch points for other surfaces such as for minimal surfaces in Riemannian manifolds or for surfaces of prescribed mean curvature which satisfy Plateau-type boundary conditions and minimize a suitable functional. However, all these results only hold in $\mathbb{R}^3$ or, more generally, in a three-dimensional manifold, and they become false if $n \geq 4$, i.e., if the codimension exceeds one. For instance, let $z = x + iy$ and set $X(x,y) = (x, y, \operatorname{Re} z^4, \operatorname{Im} z^4)$. Then $X(z)$, $z \in B_R(0)$, describes a nonparametric minimal surface in $\mathbb{R}^4$ with a singular point at $z = 0$. The surface $S$ given by $X : B_R(0) \to \mathbb{R}^4$ is bounded by a Jordan curve, and a simple differential-form argument similar to the one used in Section 2.8 shows that $S$ is in fact area minimizing. The branch-point result is one of the very few basic results mentioned in our notes which only holds true for codimension-one surfaces. The same remark applies to Nitsche's uniqueness theorem, cf. Section 5.6. We also mention that Steffen and Wente [1] have excluded the existence of branch points for minimizers of Dirichlet's integral (as well as of more general functionals) subject to a volume constraint.

*3 Embedded Solutions of Plateau's Problem*

The absence of branch points does not mean that a minimal surface is free of selfintersections. However, selfintersecting minimal surfaces can never be realized as soap films, i.e., they are unrealistic from the physical point of view. Soap films either appear as surfaces of higher topological type (see Fig. 1), thereby avoiding selfintersections which necessarily have to appear

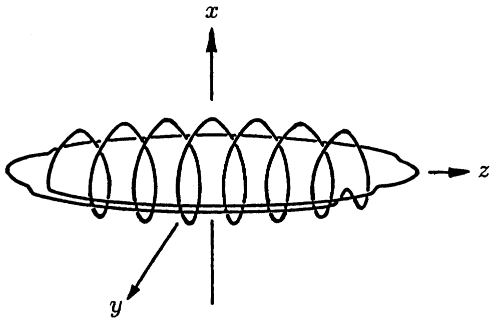



**Fig. 2.** An example of Almgren and Thurston

for disk-type surfaces spanned into knotted curves, or they arrange themselves as systems forming the characteristic 120-degrees angle at their common liquid edges (with a $Y$-shaped cross-cut; see No. 7 of these Scholia), but true selfintersections can never be seen. Thus it is of interest to see whether a given boundary curve $\Gamma$ can be spanned by an embedded minimal disk (i.e., by an injective mapping $X : B \to \mathbb{R}^3$). For topological reasons, this cannot be the case for knotted boundaries $\Gamma$, and we therefore have to look among unknotted curves for promising candidates.

Let us begin with an interesting example of an unknotted closed curve $\Gamma$ described by Almgren and Thurston [1] (see Fig. 2) which can only bound an oriented and embedded surface $S$ lying in the convex hull of $\Gamma$ if $S$ has at least three handles. (By stretching in the $z$-direction with a suitably large factor, one can even achieve that the total curvature of $\Gamma$ does not exceed the value $4\pi + \varepsilon$ where $\varepsilon$ is an arbitrarily given positive number.) Hence no minimal disk spanned by $\Gamma$ can be an embedding since, by the maximum principle, its image in $\mathbb{R}^3$ is necessarily contained in the convex hull of $\Gamma$. Similar constructions lead to boundaries $\Gamma$ spanning only embedded surfaces $S$ with $S \subset$ convex hull of $\Gamma$ if the genus of $S$ is at least $p$ where $p$ is an arbitrarily prescribed positive integer.

Another example, which is simpler than that of Almgren–Thurston, but shows the same phenomenon, was somewhat later given by J.H. Hubbard [1].

Generally speaking, the classical mapping-approach to minimal surfaces pursued in our notes has the disadvantage that one a priori fixes the topological type of the geometric object. Thus it is much more difficult to decide in this setting whether an area minimizing surface is *geometrically* regular. In geometric measure theory this and other disadvantages have been overcome by the introduction of generalized objects called currents and varifolds.

Simply speaking, an $n$-current $T \in \mathcal{D}_n(U)$ is a continuous linear functional on the space $\mathcal{D}^n(U)$ of $n$-forms with compact support in a domain $U$ of $\mathbb{R}^m$. Then each $n$-dimensional oriented submanifold $M$ of $\mathbb{R}^{n+k}$ (with locally finite $n$-dimensional Hausdorff measure $\mathcal{H}^n$) represents an $n$-current in the following way: Let $\tau_1, \dots, \tau_n$ be an adapted orthonormal frame of the tangent space

$T_xM$, and let $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$ be an orientation on $T_xM$. Then we define the current $[M]$ by

$$[M](\omega) := \int_M \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^n(x)$$

for $\omega \in \mathcal{D}^n(U)$.

Conversely, the currents which are representable by a manifold (or, more precisely, by a rectifiable $n$-varifold with integer multiplicity) are of basic importance. They are called *locally rectifiable* (in the terminology of Federer and Fleming), or they are said to be *integer multiplicity currents* (Simon [8], p. 146). To be precise, $T$ is of integer multiplicity if it is representable as

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \theta(x) \, d\mathcal{H}^n(x), \quad \omega \in \mathcal{D}^n(U),$$

where $M$ is an $\mathcal{H}^n$-measurable, countably $n$-rectifiable subset of $U$, $\theta$ is a locally $\mathcal{H}^n$-integrable positive integer-valued function, and $\xi(x)$ is an $\mathcal{H}^n$-measurable orientation for the approximate tangent space $T_xM$ (see Simon [8] for details). The *mass of a current* $T$ in $U$ is defined as

$$\mathbf{M}_U(T) := \sup\{T(\omega) \colon \|\omega\| \le 1, \ \omega \in \mathcal{D}^n(U)\}.$$

Using the tools of geometric measure theory, Hardt and Simon [1] answered the question of embeddedness in the following way.

**Theorem 1.** *Each closed Jordan curve $\Gamma \subset \mathbb{R}^3$ of class $C^{1,\alpha}$ bounds at least one embedded orientable minimal surface.*

However, note that, because of the semicontinuity of mass with respect to weak convergence, one has no control over the topological type of the minimal surface except for an upper bound on its genus. In fact, in the limit, cancellation of several parts of currents (with opposite orientation) may produce higher connectivity of the minimizing current. On the other hand it seems plausible that under suitable geometric assumptions on $\Gamma$ one might obtain embedded minimal surfaces of prescribed topological type which are bounded by $\Gamma$. In a sequence of papers starting with Gulliver and Spruck [3], the following was proved by Tomi and Tromba [1], Almgren and Simon [1], and Meeks and Yau [3]:

**Theorem 2.** *Let $K$ be a strictly convex body in $\mathbb{R}^3$ whose boundary $\partial K$ is of class $C^2$, and suppose that $\Gamma$ is a closed rectifiable Jordan curve contained in $\partial K$. Then there exists an embedded minimal surface of the type of the disk which is bounded by $\Gamma$.*

All the papers cited above use different methods. Gulliver and Spruck gave the first proof with the additional requirement that the total curvature of $\Gamma$ be not larger than $4\pi$. Tomi and Tromba used methods from global analysis, while Almgren and Simon minimized area in the class of embedded disks, thereby obtaining in the limit a certain varifold which corresponds to the minimal embedded disk. Finally, Meeks and Yau proved that the minimizing surface of the type of the disk is embedded.

Moreover, in their paper [4] Meeks and Yau established a connection between the problem of embeddedness and the problem of uniqueness. First they gave a generalization of Theorem 2.

**Theorem 3.** *Let $M$ be a compact region in $\mathbb{R}^3$ whose boundary is $C^2$-smooth and has nonnegative mean curvature with respect to the inward normal. Secondly, let $\Gamma$ be a closed rectifiable Jordan curve contained in $\partial M$. Then any (in $\mathcal{C}(\Gamma)$) area minimizing minimal surface of disk type which is contained in $M$ and bounded by $\Gamma$ has to be embedded.*

Another result of Meeks and Yau is the following

**Theorem 4.** *Let $X : \Sigma \to M \subset \mathbb{R}^3$ be a minimal surface defined on a compact Riemann surface $\Sigma$ with boundary, and suppose that $M$ satisfies the assumptions of Theorem 3. Assume also that $X|_{\partial\Sigma}$ is a regular smooth embedding of $\partial\Sigma$ into $\partial M$ which decomposes $\partial M$ into components $\Sigma_j$, and that $X|_{\partial\Sigma}$ is homotopically trivial in the component of $M \setminus X(\Sigma)$ which contains $\Sigma_j$. Then each such component $\Gamma := X(\partial\Sigma)$ bounds an embedded stable minimal surface which is disjoint from $X(\Sigma)$ unless $X(\Sigma)$ is an embedded stable disk.*

As a consequence of this result one obtains:

**Theorem 5.** *If $\partial M$ is a $C^2$-surface homeomorphic to $S^2$ and if the mean curvature of $\partial M$ with respect to the inward normal is nonnegative, then every smooth Jordan curve $\Gamma$ on $\partial M$ either bounds at least two distinct embedded minimal disks in $M$, or the only immersed minimal surface $X : \Sigma \to \mathbb{R}^3$ bounded by $\Gamma$ (with no restriction on the genus of $\Sigma$) is a uniquely determined stable, embedded minimal surface of the type of the disk.*

Suppose that $\Gamma$ is a regular, real analytic, closed Jordan curve which lies on the boundary $\partial K$ of a convex body, and suppose that the total curvature of $\Gamma$ is less than $4\pi$. Then Theorem 5 in conjunction with Nitsche's uniqueness theorem implies that the only minimal surface $X : \Sigma \to \mathbb{R}^3$ is a unique area minimizing disk. Clearly, this result is a considerable refinement of Nitsche's uniqueness theorem. It is unknown whether one can omit the assumption that $\Gamma$ lies on a convex surface. (*Remark*: Meeks and Yau indicate that in the above conclusion $\partial K$ need not be smooth.)

Further work in this connection has been done by F.H. Lin [3].

We also mention the fundamental paper by Ekholm, White, and Wienholtz [1] on the embeddedness of minimal surfaces. The main result of this article is

**Theorem 6.** *Let $\Gamma$ be a closed Jordan curve in $\mathbb{R}^n$, $n \geq 3$, with total curvature $\leq 4\pi$, and let $X : \Sigma \to \mathbb{R}^n$ be a minimal surface with boundary $\Gamma$ where $\Sigma$ is* *a compact, 2-dimensional* $C^\infty$*-manifold with boundary (i.e.* $X \in C^0(\Sigma, \mathbb{R}^n)$ *is harmonic and conformal in* $\operatorname{int} \Sigma$*, and* $X|_{\partial\Sigma}$ *maps* $\partial\Sigma$ *homeomorphically onto* $\Gamma$*). Then* $X$ *is an embedding of* $M$ *up to and including the boundary, with no interior branch points. If* $\Gamma$ *is regular and of class* $C^{s,\alpha}$*,* $s \geq 1$*,* $0 < \alpha < 1$*, then* $X \in C^{s,\alpha}(\Sigma, \mathbb{R}^n)$ *is smoothly embedded and therefore has no boundary branch points.*

Furthermore, the authors point out that there are closed Jordan curves in $\mathbb{R}^3$ with total curvature $<4\pi$ that bound "minimal Möbius strips", and they make the following interesting

**Conjecture.** *Let* $\Gamma$ *be a smooth, closed Jordan curve in* $\mathbb{R}^3$ *with total curvature* $\leq 4\pi$*. Then, in addition to a unique minimal disk,* $\Gamma$ *bounds either* (i) *no other minimal surface, or* (ii) *exactly one minimal Möbius strip and no other minimal surfaces, or* (iii) *exactly two minimal Möbius strips and no other minimal surfaces.*

Returning to geometric measure theory, we denote by $\mathcal{R}_n^{(\mathrm{loc})}(U)$ for an open set $U$ in $\mathbb{R}^m$ the set of all currents in $U$ which locally are of integer multiplicity.

A fact of central importance concerning the Plateau problem in arbitrary dimensions and codimensions is the following compactness theorem which was first proved by Federer and Fleming [1]:

**Theorem 7.** *If* $T_j \in \mathcal{D}_n(U)$*,* $j = 1, 2, \ldots,$ *is a sequence of integer multiplicity currents with*

$$\sup_{j \geq 1} (\mathbf{M_W}(\mathbf{T}_j) + \mathbf{M}(\partial \mathbf{T}_j)) < \infty \quad \textit{for all } \mathbf{W} \subset\subset \mathbf{U},$$

*then there is a current* $T \in \mathcal{R}_n^{\mathrm{loc}}(U)$ *and a subsequence* $\{T_{j'}\}$ *converging weakly to* $T$ *in* $U$*.*

(The nontrivial part in the proof is to show that the limit is, in fact, of integer multiplicity.)

Employing the lower semicontinuity of mass under weak convergence of currents, one concludes by means of Theorem 7 the following existence result:

**Theorem 8.** *Let* $S \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ *be of integer multiplicity, of compact support* $\operatorname{supp} S$ *and with* $\partial S = 0$*. Then there is a current* $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ *with* $\partial T = S$ *such that* $\operatorname{supp} T$ *is compact and* $\mathbf{M}(T) \leq \mathbf{M}(R)$ *for all* $R \in \mathcal{R}_n(\mathbb{R}^{n+k})$ *with compact support and with* $\partial R = S$*.*

(Here the boundary current $\partial T$ is defined by the relation

$$\partial T(\omega) = T(d\omega) \quad \text{for all } \omega \in \mathcal{D}^{n-1}(U),$$

in analogy with Stokes's theorem.)

The next step is to examine the regularity of a minimizing current. One sets

$$\begin{aligned}\mathrm{Reg}(T) := \{x \in \operatorname{supp} T\colon &\text{ there is a neighborhood } U(x)\\ &\text{such that } \operatorname{supp} T \cap U(x) \text{ is an embedded}\\ &n\text{-dimensional submanifold } M \text{ of } \mathbb{R}^{n+k}\}\end{aligned}$$

and

$$\mathrm{Sing}(T) := \operatorname{supp} T \setminus \mathrm{Reg}(T),$$

to denote the regular and the singular part of the support of $T$ respectively. In codimension one, the following basic regularity result was proved by Fleming [2] ($n-2$), Almgren [1] ($n=3$), Simons [1] ($n=4,5,6$), and Federer [3]:

**Theorem 9.** *Let $U \subset \mathbb{R}^{n+1}$ be open, $T \in \mathcal{R}_n(U)$ with $\mathbf{M}(T) \le \mathbf{M}(R)$ for all $R$ with* $\operatorname{supp}(T \setminus R) \subset\subset U$. *Then* $\mathrm{Sing}(T \cap U)$ *is empty for $n \le 6$, locally finite for $n = 7$, and $\mathcal{H}^{n-7+\alpha}$* $(\mathrm{Sing}(T \cap U)) = 0$ *for all $\alpha > 0$ and $n > 7$.*

Bombieri, de Giorgi, and Giusti [1] proved that the seven-dimensional cone in $\mathbb{R}^8$ given by $\{x \in \mathbb{R}^8\colon x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$ is mass-minimizing which proves the sharpness of Theorem 8.

If the codimension is greater than one, we have the following result of Almgren [6]:

**Theorem 10.** *An $n$-dimensional, area minimizing integer multiplicity current in $\mathbb{R}^{n+k}$ is in the interior a smooth embedded manifold, except for a singular set whose Hausdorff dimension is at most $n-2$.*

This result is again sharp.

Finally the question of boundary regularity in codimension one was completely settled by Hardt and Simon [1]:

**Theorem 11.** *In the setting of Theorem 4, let $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ be area minimizing with an $(n-1)$-dimensional oriented submanifold $S$ of class $C^{1,\alpha}$ as boundary. Then, near $S$, the support of $T$ is an embedded $C^1$-manifold with boundary.*

Note that in Theorem 11 there is no restriction on the dimension $n$.

### *4 More on Uniqueness and Nonuniqueness*

Let us begin with a classical example, Enneper's surface

$$X(w) = \mathrm{Re}\bigg(w - \frac{w^3}{3}, iw + i\frac{w^3}{3}, w^2\bigg), \quad w = u + iv,$$

and define the closed curve $\Gamma_r$ by

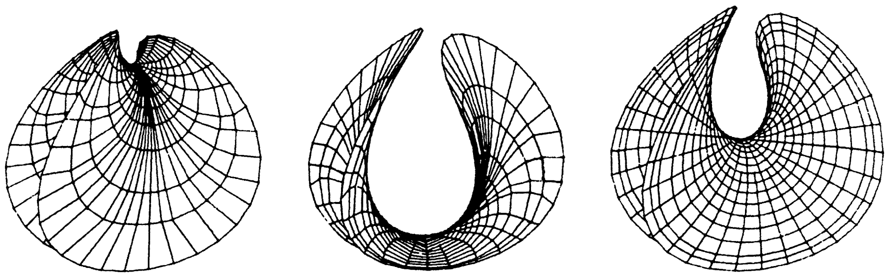



**Fig. 3.** A closed curve bounding a part of Enneper's surface **(c)** as well as two other minimal surfaces of the type of the disk: see **(a), (b)**. Courtesy of O. Wohlrab

$$\varGamma_r := \{X(w)\colon |w| = r\}.$$

Nitsche [14] has proved that $\varGamma_r$ bounds at least two distinct minimal surfaces of the type of the disk provided that $1 < r < \sqrt{3}$, and that it bounds at least three disk-type solutions if $r_0 < r < \sqrt{3}$ where the value of $r_0$ is about 1.681475 (see Fig. 3). For $0 < r < 1/\sqrt{3}$ the orthogonal projection of $\varGamma_r$ onto the $x, y$-plane is convex and one-to-one whence one concludes that $\varGamma_r$ bounds exactly one disk-type surface. By a sharpened version of Nitsche's uniqueness theorem, Ruchert [1] proved uniqueness for $0 < r \le 1$. Thereafter, Beeson and Tromba [1] showed that a bifurcation occurs at $r = 1$ which is of the type of the cusp catastrophe (in Thom's morphogenesis) and that there is a number $\delta_0 > 0$ such that $\varGamma_r$ bounds at least three disk-type surfaces if $1 < r < 1 + \delta_0$. By means of the estimates of Chapter 2 of Vol. 2 one can then show that $\varGamma_r$ bounds *exactly* three disk-type surfaces if $1 < r < 1 + \delta_0$.

The bifurcation of minimal surfaces was also studied in a remarkable paper by Büch [1]. Starting with Weierstrass's representation formula (27) of Section 3.3 he was able to establish conditions on the Weierstrass function $\mathfrak{F}(\omega)$ which imply the appearance of bifurcations of the type of the *fold*, the *cusp*, and of the *swallow tail* (of Thom's list).

Although it is not easy to find curves which bound only one disk-type solution, the opposite problem is complicated as well, namely to verify by a rigorous mathematical proof that a given curve bounds at least two minimal surfaces. Therefore the following result of Quien and Tomi [1] might be of interest:

*There exist Jordan curves $\varGamma$ which are arbitrarily close to a plane and which bound (at least) a given number of geometrically distinct immersed minimal surfaces of the type of the disk.*

Let us outline the *proof*. Suppose that $\varphi : S^1 = \partial B \to \mathbb{R}^2$ is an immersion of the unit circle. We begin by looking at the question as to whether $\varphi$ can be extended to an immersion $f : \bar{B} \to \mathbb{R}^2$ with $f|_{\partial B} = \varphi$ and, if so, how many nonequivalent such extensions will exist (two immersions $f$ and $g$ are

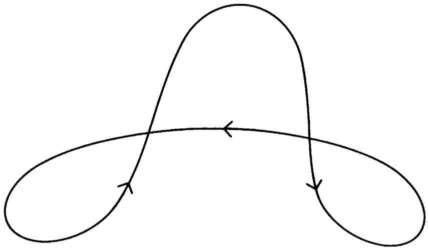
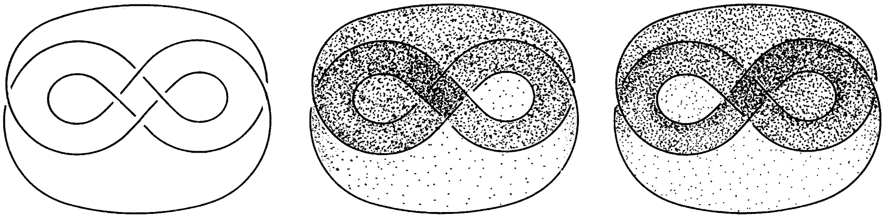
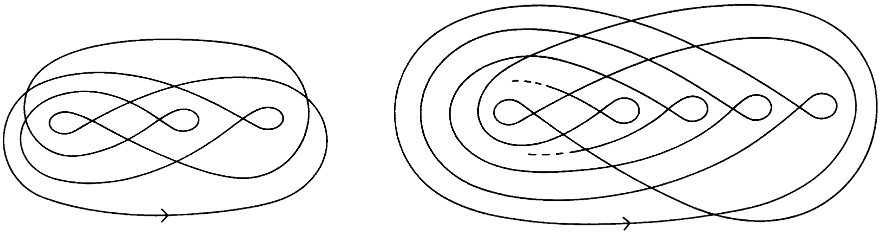


**Fig. 4.** An immersion $\varphi : S^1 \to \mathbb{R}^2$ which cannot be extended as an immersion $f : \bar{B} \to \mathbb{R}^2$ of the disk into $\mathbb{R}^2$



**(a)** **(b)**

**Fig. 5.** **(a)** A Milnor curve $\varphi : S^1 \to \mathbb{R}^2$ and its two extensions $f : \bar{B} \to \mathbb{R}^2$ which are immersions of the disk. **(b)** The leaves of two extensions to Milnor's curve



**(a)** **(b)**

**Fig. 6.** Milnor curves admitting **(a)** three extensions, **(b)** $n$ extensions

equivalent if there is a diffeomorphism $\sigma$ of $\bar{B}$ onto itself such that $f = g \circ \sigma$). For instance, the immersion $\varphi : S^1 \to \mathbb{R}^2$ depicted in Fig. 4 cannot be extended while Fig. 5a depicts an example due to Milnor which allows two extensions, the leaves of which are depicted in Fig. 5b. Then, in Fig. 6 we exhibit a curve with three different extensions which can inductively be improved to a curve $\varphi : S^1 \to \mathbb{R}^2$ allowing $n$ extensions (see Fig. 6b). For a proof of these results we refer to Poénaru [1].

Let us now consider an immersion $\varphi : S^1 \to \mathbb{R}^2$ which allows $n$ different extensions $f$ of class $C^3(\bar{B}, \mathbb{R}^2)$. By the Lichtenstein mapping theorem we can assume that $f(u, v) = (f^1(u, v), f^2(u, v))$ is conformally parametrized, i.e., we have

$$|f_u|^2 = |f_v|^2 =: \Lambda, \quad \langle f_u, f_v \rangle = 0.$$

Next we choose a perturbation function $\psi \in C^{2,\beta}(\partial B)$, $0 < \beta < 1$, such that $F := (f^1, f^2, \psi)$ defines a Jordan curve $F : \partial B \to \mathbb{R}^3$ in $\mathbb{R}^3$. This can be achieved by a function $\psi$ with arbitrarily small $C^2$-norm. Now we consider the class $\mathcal{C}$ of functions

$$Z(u,v) = (f^1(u,v), f^2(u,v), z(u,v)), \quad (u,v) \in B,$$

such that $z \in \text{Lip}(\bar{B})$ and $z|_{\partial B} = \psi|_{\partial B}$. The area of $Z \in \mathcal{C}$ is given by

$$A(z) := \int_B |Z_u \wedge Z_v| \, du \, dv = \int_B \Lambda \sqrt{1 + \Lambda^{-1} |\nabla z|^2} \, du \, dv.$$

This functional is strictly convex whence there can exist at most one stationary point $x(u,v)$ of $A$, and the corresponding surface $X = (f^1, f^2, x)$ would be the absolute minimum of $A$ within $\mathcal{C}$. The Euler equation of $A$ is

$$\mathcal{L}(x) := a^{\alpha\beta} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} + b = 0$$

where we have set

$$\begin{aligned} a^{\alpha\beta} &:= (1 + \Lambda^{-1}) |\nabla x|^2 \delta^{\alpha\beta} - \Lambda^{-1} \frac{\partial x}{\partial u^\alpha} \frac{\partial x}{\partial u^\beta}, \\ b &:= -\frac{1}{2} |\nabla x|^2 \frac{\partial}{\partial u^\alpha} \Lambda^{-1} \frac{\partial x}{\partial u^\alpha}. \end{aligned}$$

For $\Lambda = 1$, the equation $\mathcal{L}(x) = 0$ is the classical minimal surface equation.

We will show that the boundary value problem

$$\mathcal{L}(x) = 0 \quad \text{in } B, \quad x = \psi \quad \text{on } \partial B$$

can be solved for boundary values $\psi$ with a sufficiently small $C^2$-norm. We only have to establish a gradient estimate along $\partial B$ for any solution since then a priori bounds for $x$ and $\nabla x$ follow from standard estimates for scalar problems (cf. Gilbarg and Trudinger [1], Chapters 9 and 14). To derive the desired estimate we consider barrier functions of the type

$$c^{\pm}(w) := \psi(w) \pm \varepsilon(1 - |w|^2), \quad w = u + iv,$$

where $|\psi|_{C^2(\bar{B})} < \varepsilon \le 2/\sqrt{27M}, M := \max_B |\nabla \Lambda^{-1}|$. Then a brief computation will show that $\mathcal{L}(c^-) \ge 0$, and similarly we obtain $\mathcal{L}(c^+) \le 0$. Consequently $\nabla x$ can be estimated along $\partial B$ by means of the maximum principle. This shows that, for every equivalence class $[f]$, we find a minimal immersion $X = (f^1, f^2, x)$ which is bounded by $\Gamma = F(\partial B), F = (f^1, f^2, \psi)$. □

It is still unknown whether a smooth regular Jordan curve can bound infinitely many minimal surfaces of the type of the disk (or, more generally, of

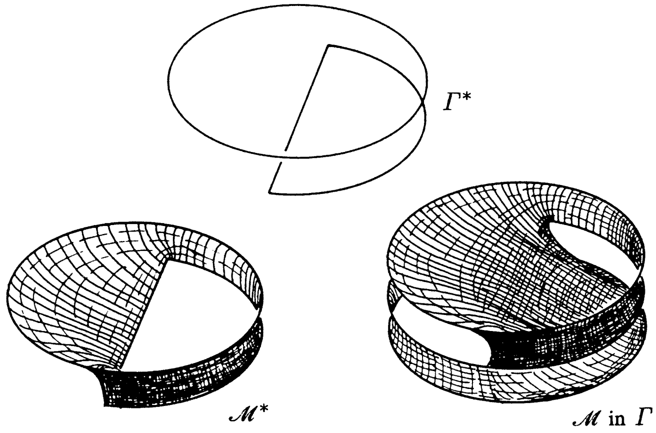



**Fig. 7.** Construction of a boundary configuration $\Gamma$ bounding a one-parameter family of (congruent) minimal surfaces of genus zero. The rotationally symmetric configuration $\Gamma$ consists of three coaxial circles $\Gamma_0, \Gamma_1, \Gamma_{-1}$

the same topological type). Note, however, that one can find boundary configurations consisting of several closed curves which even bound one-parameter families of distinct minimal surfaces of the same topological type. In fact, one can construct rotationally symmetric configurations $\Gamma = \langle \Gamma_1, \Gamma_2, \ldots, \Gamma_n \rangle$ consisting of $n$ coaxial circles $\Gamma_1, \ldots, \Gamma_n$ bounding one-parameter families of solutions. The first example of this kind was given by Morgan [3] for $n = 4$. In the paper [1] of Gulliver and Hildebrandt an example working with three circles is exhibited which will be described below. Note that $n = 3$ is the minimum number of circles for which such examples can be found since R. Schoen [3] proved that, for $n = 2$, each immersed minimal surface bounded by two coaxial circles $\Gamma_1$ and $\Gamma_2$ is either a pair of disks or a piece of a catenoid.

Now we are going to describe the construction of a rotationally symmetric 1-parameter family of minimal surfaces of genus zero which are bounded by three coaxial circles which lie in parallel planes cf. Fig. 7.

To this end we consider a configuration $\Gamma$ consisting of three circles $\Gamma_0, \Gamma_1, \Gamma_{-1}$ described by the equations $x^2 + y^2 = 1$ and $z = 0, \lambda$ and $-\lambda$ respectively, $\lambda > 0$, and a second configuration $\Gamma^*$ which consists of the circle $\Gamma_1$ and another closed curve $\gamma$ that lies in the same plane as $\Gamma_0$, and is formed by the semicircle $\Gamma_0' = \Gamma_0 \cap \{x \geq 0\}$ and by the interval $I = \{x = 0,\ z = 0,\ -1 < y < 1\}$ on the $y$-axis. For small $\lambda$ there is a minimal surface $\mathcal{M}^*$ of the type of an annulus bounded by $\Gamma^*$ (see below). By Schwarz's reflection principle, we can extend $\mathcal{M}^*$ as a minimal surface across the straight segment $I$. For this purpose we rotate $\mathcal{M}^*$ by 180° about the $y$-axis to form a second minimal surface $\mathcal{M}^{**}$. Their union $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is a minimal surface with boundary $\Gamma$ having genus zero. The segment $I$ has become part of the interior of $\mathcal{M}$, and the surface $\mathcal{M}$ can be described by a harmonic mapping

$X : B \to \mathbb{R}^3$ given in conformal coordinates of a triply connected planar domain $B$. Since $\mathcal{M}^*$ is not symmetric under rotations about the $z$-axis, also $\mathcal{M}$ has to be rotationally nonsymmetric.

We still have to find a connected minimal surface $\mathcal{M}^*$ which is bounded by the configuration $\Gamma^*$. By virtue of J. Douglas's theorem (cf. Chapter 8), there exists an area minimizing minimal surface $\mathcal{M}^*$ which is defined on an annulus and has $\Gamma^*$ as boundary, provided that $\lambda$ is small enough. In fact, the existence of Douglas' solution is ascertained under the hypothesis that

$$a(\Gamma^*) < a(\gamma) + a(\Gamma_1), \tag{1}$$

where $a(\Gamma^*)$ is the greatest lower bound of area for surfaces of the type of the annulus with boundary $\Gamma^* = \gamma \cup \Gamma_1$, and $a(\gamma)$ and $a(\Gamma_1)$ are the corresponding lower bounds for disk-type surfaces bounded by $\gamma$ and $\Gamma_1$ respectively. Clearly,

$$a(\gamma) = \pi/2, \quad a(\Gamma_1) = \pi,$$

and $a(\Gamma^*)$ is smaller than the area $A(S)$ of the surface $S$ that consists of the cylinder surface between $\Gamma_0$ and $\Gamma_1$ and of the half-disk $\{x^2 + y^2 \le 1,\ x \le 0,\ z = 0\}$, that is,

$$a(\Gamma^*) < 2\pi\lambda + \pi/2.$$

Thus Douglas's condition (1) is satisfied for $\lambda \le 1/2$. A somewhat more complicated comparison surface $S$, consisting of half of a catenoid, half of a cone, and two triangles shows that even the condition $\lambda \le 0.7$ suffices to ensure the existence of a Douglas solution $\mathcal{M}^*$ within the frame $\Gamma^*$. Moreover, hypothesis (1) implies that the surface $\mathcal{M}^*$ is an immersion (cf. Gulliver [7], Theorem 10.5). By the maximum principle, the interior of $\mathcal{M}^*$ lies between two planes $z = 0$ and $z = \lambda$. Therefore the interior of $\mathcal{M}^*$ does not meet the interior of $\mathcal{M}^{**}$ where $\mathcal{M}^{**}$ is the reflection of $\mathcal{M}^*$ at the $y$-axis. Thus also $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is immersed. Since $\mathcal{M}$ is not rotationally symmetric, we have shown:

*The configuration $\Gamma$ consisting of three coaxial unit circles in parallel planes at a distance $\lambda \le 0.7$ bounds a continuum of congruent immersed minimal surfaces of genus zero.*

We also note that $\mathcal{M}$ cannot have branch points on the boundary since its boundary lies on a strictly convex set, a cylinder (see Section 2 of Vol. 2).

Let us now discuss examples of rectifiable Jordan curves bounding infinitely many minimal surfaces of the type of the disk. Such examples were first described by P. Lévy [2] and R. Courant [15]; they are based on the so-called *bridge-theorem*. This is a very convincing heuristic reasoning which, in essence, amounts to the following (see Fig. 8):

*Let $\Gamma_1$ and $\Gamma_2$ be two disjoint Jordan curves in $\mathbb{R}^3$. Then construct a new Jordan curve $\Gamma$ by connecting $\Gamma_1$ and $\Gamma_2$ by a bridge $\beta$ consisting of two arcs $\gamma_1$ and $\gamma_2$ which look like two parallel lines, and by omitting two pieces of $\Gamma_1$ and $\Gamma_2$. Suppose also that the two arcs $\gamma_1$ and $\gamma_2$ have a small distance $\varepsilon > 0$.*

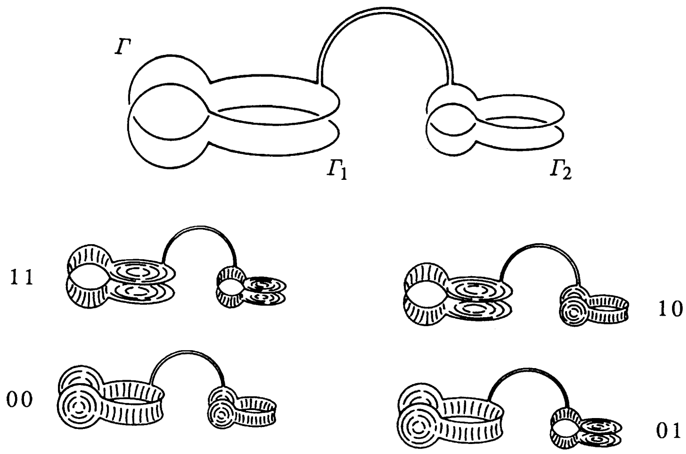



**Fig. 8.** Application of the bridge principle

**Claim.** *If $X_1$ and $X_2$ are two disk-type minimal surfaces bounded by $\Gamma_1$ and $\Gamma_2$ respectively, then there exists a disk-type surface $X$ spanned into $\Gamma$ which is close to the surface $Z$ formed by $X_1, X_2$ and a small strip $\sigma$ spanned into the bridge $\beta$. As $\varepsilon$ tends to zero, the surface $X$ converges to a geometric figure consisting of $X_1, X_2$ and an arc $\gamma$ connecting $\Gamma_1$ and $\Gamma_2$.*

A few remarks might be appropriate:

(i) It is unlikely that the claim holds if $X_1$ and $X_2$ are unstable solutions since a very tiny perturbation of the boundary might completely destroy them. Thus one probably has to assume that $X_1$ and $X_2$ are local minimizers of area within the classes $\mathcal{C}(\Gamma_1)$ and $\mathcal{C}(\Gamma_2)$ respectively, or at least stable minimal surfaces. Even then the assertion might not be true as it stands since it is unknown if minimizers are *isolated* or not. It is conceivable that there exist *blocks* of minimizers, and therefore it might occur that, for $\varepsilon \to 0$, the surface $X$ in the unified contour $\Gamma$ approaches surfaces $\tilde{X}_1$ and $\tilde{X}_2$ in the contours $\Gamma_1$ and $\Gamma_2$ which belong to the same blocks as $X_1$ and $X_2$ but are different from these surfaces.

(ii) Very likely one has to impose restrictions on the positions of $\Gamma_1$ and $\Gamma_2$ if the bridge theorem is to hold. For instance, if $\Gamma_1$ and $\Gamma_2$ are two circles of radius 1 and 2 respectively which have the same center and lie in the same plane $\Pi$, and if $\beta$ is a bridge consisting of two parallel lines joining $\Gamma_1$ and $\Gamma_2$, then there is no bridge-solution $X$ in the joint $\Gamma$. To remove this difficulty we could, for instance, assume that the convex hulls of $\Gamma_1$ and $\Gamma_2$ are disjoint. Another option is to leave suitable freedom in the choice of the bridge and not to insist on a given pair of bridging curves $\gamma_1$ and $\gamma_2$. It might even be necessary to leave freedom for the whole curve $\Gamma$ in the sense that $\Gamma$ should merely be a curve close to the joint formed of $\Gamma_1, \Gamma_2$ and the bridge $\beta$, that is, we might have to wiggle the joint a little bit. Possibly we might also have to smoothen the corners to make the procedure work.

We know of several (published and unpublished) attempts to establish a rigorous version of the bridge theorem. Neither Courant nor Levy indicated how to go about this task. The first paper containing such a proof was written by Courant's student M. Kruskal [1]; however, his reasoning turned out to be incomplete. Another very promising attack was carried out by Meeks and Yau in their paper [4] dealing with the connection between uniqueness and embeddedness on which we have reported in Subsection 3 of these Scholia. However, we are not able to follow all of their arguments, and we think that possibly a more detailed discussion might be needed to establish a good bridge theorem that will imply the existence of curves bounding infinitely many minimal surfaces of the type of the disk. We have to mention that N. Smale [1] gave a satisfactory proof of a bridge principle; however, his result is of no use for the construction of contours bounding many or even infinitely many solutions of Plateau's problem because he constructs $\Gamma$ not only in dependence on $\Gamma_1$ and $\Gamma_2$ but also in dependence on two (stable) minimal surfaces $X_1$ and $X_2$ within $\Gamma_1$ and $\Gamma_2$. That means that, if we pick different surfaces $\tilde{X}_1$ and $\tilde{X}_2$ in $\Gamma_1$ and $\Gamma_2$, N. Smale's construction will lead to another joint $\tilde{\Gamma}$ which, in general, will differ from the joint $\Gamma$. At last the matter was settled by B. White [21,22] who proved fairly general versions of the bridge principle.

Let us now turn to the (heuristic) Levy–Courant construction. We take a contour $\Gamma_1$ which bounds at least two stable disk-type minimal surfaces such as in Fig. 8. Next we consider a sequence $\Gamma_1, \Gamma_2, \Gamma_3, \dots$ of curves of the same kind, selected in such a way that $\Gamma_2$ is half the size of $\Gamma_1$, the curve $\Gamma_3$ is half the size of $\Gamma_2$, and so on (see Fig. 9). Then we join $\Gamma_1$ and $\Gamma_2$ by a bridge $\beta_1, \Gamma_2$ and $\Gamma_3$ by a bridge $\beta_2$ etc. such that a rectifiable Jordan arc $\Gamma$ is formed. Each $\Gamma_j$ spans two stable surfaces which we say to be of type 0 or 1. Pick for each $\Gamma_j$ one of these two numbers. Then we obtain a sequence $A = \{a_j\}$ of digits $a_j = 0$ or 1, and to any such sequence there corresponds a stable disk-type minimal surface $X_A$ bounded by $\Gamma$ which in $\Gamma_j$ is close to a surface of the type $a_j$. Hence $A \neq A'$ implies that $X_A \neq X_{A'}$, and we have found a bijective mapping $\tau : A \to X_A$ of all binary representations of the interval $[0, 1]$ onto the set of geometrically different minimal surfaces bounded by $\Gamma$. In other words, if we are willing to accept a strong bridge principle applying to infinitely many curves, the above reasoning yields the following result (see Fig. 9):

*There exist rectifiable Jordan curves $\Gamma$ which bound nondenumerably many minimal surfaces of the type of the disk.*

In fact, the construction seems to imply that one can choose $\Gamma$ as a regular $C^\infty$-curve except for a single kink.

It would be very interesting to make the Levy–Courant construction precise with the aid of B. White's versions of the bridge principle.

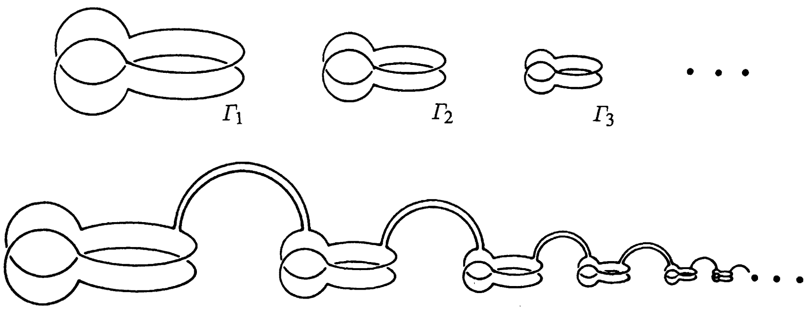



**Fig. 9.** Construction of a curve $\Gamma$ bounding nondenumerably minimal surfaces of the type of the disk

The *finiteness question* is a truly fundamental problem. J.C.C. Nitsche conjectured that every reasonable (i.e., smooth, analytic, ...) curve $\Gamma$ bounds only finitely many minimal surfaces of disk-type. Despite the generic finiteness result of Böhme–Tromba mentioned in Section 4.9, this question is completely open. It would be very desirable to obtain *upper and lower bounds for the number of solutions of Plateau's problem.*

According to J.C.C. Nitsche [31,32], *a regular, real analytic Jordan curve $\Gamma$ bounds only finitely many minimal surfaces of disk-type if its total curvature does not exceed* $6\pi$, *and if every disk-solution for $\Gamma$ is free of branch points* (cf. also Beeson [5]).

Nitsche indicated that instead of $\Gamma \in C^\omega$ the assumption $\Gamma \in C^{3,\alpha}$ is sufficient. A version of the $6\pi$-theorem is proved in Section 5.7 (cf. 5.7, Theorem 3 and Remark 10).

Important contributions to the finiteness problem were also given in the papers [3,4] of M. Beeson. In this context, we mention the papers of R. Böhme [1,5], and of Böhme and Tomi [1] who started to investigate the structure of the space of solutions for Plateau's problem. Major progress in this direction was achieved in Böhme–Tromba's papers [1] and [2] where a fundamental *index theorem* was derived. This index theorem has in the meantime been carried over to various cases of the general Plateau problem (cf. Thiel [1–3], Schüffler [6], Schüffler and Tomi [1], and finally Tomi and Tromba [6]).

In this respect we also have to mention the work on *unstable minimal surfaces in a given contour*. In particular, we refer to the work of Courant which is described in Chapter 6 of his treatise [15], and the generalizations of his work given by E. Heinz [13,14], G. Ströhmer [1–4], and F. Sauvigny [3–6]. In Chapter 6, we present a version of Courant's approach to unstable minimal surfaces that also uses ideas due to E. Heinz. In the Scholia to Chapter 6 as well as in Vol. 3, Chapter 6, further results concerning the existence of unstable minimal surfaces will be described, in particular the work of M. Struwe.

Here we mention the following uniqueness theorem by Sauvigny [3]: *Let $\Gamma$ be a polygon of total curvature less than* $4\pi$ *which lies on the boundary of* *a bounded convex set of* $\mathbb{R}^3$*. Then* $\Gamma$ *bounds exactly one disk-type minimal surface, and this solution is free of branch points up to the boundary.*

Interestingly, Sauvigny could generalize his uniqueness result to $\mathbb{R}^n$, $n \geq 4$, under the assumption that the total curvature of $\Gamma$ is less than $10\pi/3$. This generalization was possible since Sauvigny did not work with a field construction but with the so-called *Courant function* $d(\tau)$ and with the *Marx–Shiffman function* $\theta(\tau)$. The function $d(\tau)$ was introduced by Courant in his monograph [15], pp. 223–236, where it plays an important role in his treatment of unstable minimal surfaces with polygonal boundaries. On the other hand, Heinz in his subsequent basic work [19–24] emphasized the role of the *Marx-Shiffman function* $\theta(\tau)$. The functions $d(\tau)$ and $\theta(\tau)$ are defined as follows. Let $\Gamma$ be a polygon with $N+3$ vertices $e_j$, $1 \leq j \leq N+3$. Consider mappings $X : \bar{B} \to \mathbb{R}^3$ of class $C^\circ(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ such that $X(-1) = e_{N+1}$, $X(-i) = e_{N+2}$, $X(1) = e_{N+3}$ and $X(e^{i\tau_j}) = e_j$, $1 \leq j \leq N$, where $\tau = (\tau_1, \dots, \tau_n)$ is an $N$-tupel of parameter values $\tau_j$ satisfying $0 < \tau_1 < \tau_2 < \cdots \tau_N < \pi$. Let $\mathcal{F}(\tau)$ be the class of such mappings which map the arc $C_k := \{e^{i\theta} : \tau_k \leq \theta \leq \tau_{k+1}\}$ into the straight line $\Gamma_k$ through the points $e_k$ and $e_{k+1}$, whereas $\mathcal{F}'(\tau)$ denotes the subset of mappings $X \in \mathcal{F}(\tau)$ which map $C_k$ weakly monotonically onto the interval $[e_k, e_{k+1}]$ on $\Gamma_k (\tau_j = \tau_k, e_j = e_k$ if $j \equiv k \bmod N+3)$. We set

$$d(\tau) := \inf\{D(X) \colon X \in \mathcal{F}'(\tau)\},$$
$$\theta(\tau) := \inf\{D(X) \colon X \in \mathcal{F}(\tau)\}.$$

Then we clearly have $d(\tau) \geq \theta(\tau)$, and simple examples show that we can have $d(\tau) > \theta(\tau)$ for certain values of $\tau$ (see F. Lewerenz [1]). The function $d(\tau)$ is of class $C^1$, and its critical points correspond bijectively to the solutions of Plateau's problem of disk-type bounded by the polygon $\Gamma$. In this way, Plateau's problem for polygonal boundaries is connected with the critical points of a function of finitely many variables. Unfortunately it is unknown whether $d(\tau)$ is of class $C^2$; therefore Courant's function is not suited to develop a Morse theory. The situation is much better for the function $\theta(\tau)$. Heinz [20,23] proved that $\theta(\tau)$ is real analytic and that its critical points correspond to solutions of a *generalized Plateau problem* for $\Gamma$ (generalized means: the solution $X$ can overshoot the vertices, and we only know that $X(C_k) \subset \Gamma_k$). The *Morse index* of such generalized solutions was computed by Sauvigny [4], by studying the second derivative of the function $\theta$. Note that the two functions $d(\tau)$ and $\theta(\tau)$ are closely connected as they coincide in the critical points of $d(\tau)$.

We have presented some of the results by Courant and Shiffman as well as extensions by Heinz, Sauvigny, and Jakob in Chapter 6 (note that there the functions $d$ and $\theta$ are denoted by $\Theta$ and $\Theta^*$ respectively).

Uniqueness theorems and finiteness questions for minimal surfaces in Riemannian manifolds and for $H$-surfaces were discussed by Ruchert [2], Koiso [1,4,6], and Quien [1].

*5 Index Theorems, Generic Finiteness, and Morse-Theory Results*

In this subsection we sketch some results for minimal surfaces which in Sections 5 and 6 of Vol. 3 are developed in detail.

Let $B$ be the unit disk and $S^1 = \partial B$. For integers $r$ and $s$, $r \geq 2s + 4$, define

$$\mathcal{D} = \mathcal{D}^s = \{u : S^1 \to S^1 : \deg u = 1 \text{ and } u \in H^s(S^1, \mathbb{C})\},$$

where $H^s$ denotes the Sobolev space of $s$-times differentiable functions with values in $\mathbb{C}$; set

$$\mathcal{A} = \{\alpha : S^1 \to \mathbb{R}^n : \alpha \in H^r(S^1, \mathbb{R}^n),\ \alpha \text{ an embedding}\}$$

(i.e. $\alpha$ is one-to-one and $\alpha'(\xi) \neq 0$ for all $\xi \in S^1$), and let the total curvature of $\Gamma^\alpha = \alpha(S^1)$ be bounded by $\pi(s-2)$.

Denote by $\pi : \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ the projection map onto the first factor. A minimal surface $X : \bar{B} \to \mathbb{R}^n$ spanning $\alpha \in \mathcal{A}$ can be viewed as an element of $\mathcal{A} \times \mathcal{D}$, since $X$ is harmonic and therefore determined by its boundary values

$$X|_{S^1} = \alpha \circ u, \quad \text{where } (\alpha, u) \in \mathcal{A} \times \mathcal{D}.$$

The classical approach to minimal surfaces is to understand the set of minimal surfaces spanning a given fixed wire $\alpha$; that is, the set of minimal surfaces in $\pi^{-1}(\alpha)$. The approach of Böhme–Tomi–Tromba is to first understand the structure of the subset of minimal surfaces in the bundle $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ viewed as a fiber bundle over $\mathcal{A}$, and then to attack the question of the set of minimal surfaces in the fiber $\pi^{-1}(\alpha)$ in terms of the singularities of the projection map $\pi$ restricted to a suitable subvariety of $\mathcal{N}$. This is in the spirit of Thom's original approach to unfoldings of singularities.

Let us say that a minimal surface $X \in \mathcal{A} \times \mathcal{D}$ has branching type $(\lambda, \nu)$, $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{Z}^p$, $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{Z}^q$, $\lambda_i, \nu_i \geq 0$ if $X$ has $p$ distinct but arbitrarily located interior branch points $w_1, \ldots, w_p$ in $B$ of integer orders $\lambda_1, \ldots, \lambda_p$ and $q$ distinct boundary branch points $\xi_1, \ldots, \xi_q$ in $S^1$ of (even) integer orders $\nu_1, \ldots, \nu_q$. In a formal sense, the subset $\mathcal{M}$ of minimal surfaces in $\mathcal{N}$ is an algebraic subvariety of $\mathcal{N}$ which is a stratified set, stratified by branching types. To be more precise, let $\mathcal{M}^\lambda_\nu$ denote the minimal surfaces of branching type $(\lambda, \nu)$. Then we have the following index result of Böhme and Tromba [2].

**Index theorem for disk surfaces.** *The set $\mathcal{M}^\lambda_0$ is a $C^{r-s-1}$-submanifold of $\mathcal{N}$, and the restriction $\pi^\lambda$ of $\pi$ to $\mathcal{M}^\lambda_0$ is of class $C^{r-s-1}$. Moreover, $\pi^\lambda$ is a Fredholm map of index $I(\lambda) = 2(2-n)|\lambda| + 2p + 3$, where $|\lambda| = \sum \lambda_i$.*

*Moreover, locally, for $\nu \neq 0$, we have $\mathcal{M}^\lambda_\nu \subset \mathcal{W}^\lambda_\nu$ where $\mathcal{W}^\lambda_\nu$ is a submanifold of $\mathcal{N}$ and where the restriction $\pi^\lambda_\nu$ of $\pi$ to $\mathcal{W}^\lambda_\nu$ is Fredholm of index $I(\lambda, \nu) = 2(2-n)|\lambda| + (2-n)|\nu| + 2p + q + 3$, $|\nu| = \sum \nu_i$. The number 3 comes from the equivariance of the problem under the action of the three dimensional conformal group of the disk.*

Ursula Thiel [3] has shown that if one uses weighted Sobolev spaces as a model, the sets $\mathcal{M}_\nu^\lambda$ can indeed be given a manifold structure with the index of $\pi_\nu^\lambda := \pi|\mathcal{M}_\nu^\lambda$ being $I(\lambda, \nu)$.

These stratification and index results are the basis to prove the generic finiteness and stability of minimal surfaces of the type of the disk as discussed in Böhme and Tromba [2]: There exists an open dense subset $\hat{\mathcal{A}} \subset \mathcal{A}$ such that if $\alpha \in \hat{\mathcal{A}}$, then there exists only a finite number of minimal surfaces bounded by $\alpha$, and these minimal surfaces are stable under perturbations of $\alpha$. If $n > 3$, they are nondegenerate critical points of Dirichlet's integral. The open set $\hat{\mathcal{A}}$ will be the set of regular values of the map $\pi$. Moreover we have the following

**Remark.** If $n > 3$, the minimal surfaces spanning $\alpha \in \hat{\mathcal{A}}$ are all immersed up to the boundary, and if $n = 3$, they are at most simply branched.

Schüffler [1–4,6,8], Schüffler and Tomi [1], and Thiel [1,2] have extended the index theorem in various directions. Tomi and Tromba [6] have obtained an index theorem for higher genus minimal surfaces employing the Teichmüller theory; cf. Vol. 3, Chapters 4 and 5.

Finally, these results are also essential for a Morse theory for disk surfaces.

Let $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ be the bundle over $\mathcal{A}, \alpha \in \mathcal{A}$, and let $\Gamma^\alpha = \alpha(S^1)$ be the image of such an embedding. Consider the manifold of maps $H^s(S^1, \Gamma^\alpha)$. In A. Tromba [5] it is shown that $H^s(S^1, \Gamma^\alpha)$ is a $C^{r-s}$-submanifold of $H^2(S^1, \mathbb{R}^n)$. Let $\mathcal{N}(\alpha)$ denote the component of $H^s(S^1, \Gamma^\alpha)$ determined by $\alpha$. We can identify $\mathcal{N}(\alpha)$ with the set of mappings $X \in C^0(\bar{B}, \mathbb{R}^n)$ which are harmonic in $B$ and whose boundary values $X|_{\partial B}$ yield a parametrization of $\Gamma^\alpha$. Then the Dirichlet functional $E_\alpha : \mathcal{N}(\alpha) \to \mathbb{R}$ is defined by

$$E_\alpha(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv.$$

We know by the index theorem that *there exists an open dense set of contours $\hat{\mathcal{A}} \subset \mathcal{A}$, $\mathcal{A} \subset H^r(S^1, \mathbb{R}^n)$, $n \geq 4$, such that if $\alpha \in \hat{\mathcal{A}}$, there are only a finite number of nondegenerate minimal surfaces $X_1, \ldots, X_m$ spanning $\alpha$. Let $D^2 E_\alpha(X_i) : T_{X_i}\mathcal{N}(\alpha) \times T_{X_i}\mathcal{N}(\alpha) \to \mathbb{R}$ denote the Hessian of Dirichlet's functional at $X_i$, and be $\lambda_i$ the dimension of the maximal subspace on which $D^2 E_\alpha(X_i)$ is negative definite. Then A. Tromba* [11] *proved the Morse equality*

$$\sum_i (-1)^{\lambda_i} = 1. \tag{1}$$

A version of this formula which holds in $\mathbb{R}^3$ was developed by A. Tromba in his papers [10,11]. The theory leading to these results is presented in Chapter 6 of Vol. 3.

The full Morse inequalities in the case $n \geq 4$ were established by Struwe [4], who proved

$$\sum_{\lambda=0}^{l} (-1)^{l-\lambda} m_\lambda \geq (-1)^l$$

and

$$m_0 \geq 1$$

where $m_\lambda$ is the number of minimal surfaces of Morse index $\lambda$.

However, the case $n = 3$ remains open since the generic nondegeneracy assumption is known not to hold (see Böhme and Tromba [2]). Here only Tromba's version of formula (1) is known.

### *6 Obstacle Problems*

The minimization procedure can also be used to solve *obstacle problems*, that is, to find surfaces of minimal area (or of a minimal Dirichlet integral) which are spanning a prescribed boundary configuration and avoid certain open sets (obstacles). In other words, the competing surfaces $X$ of the variational problem are confined to certain closed subsets of $\mathbb{R}^3$ (or, more generally, to closed subsets of the target manifold $M$ of the mappings $X : B \to M$). Problems of this kind were treated by F. Tomi [2–4], S. Hildebrandt [12,13], and Hildebrandt and Kaul [1]. One can also consider obstacle problems where the obstacle is *thin*. (In elasticity theory these problems are called *Signorini problems*.) In the context of minimal surfaces such problems occur naturally if we consider free or partially free boundary problems with a supporting surface $S$. If $S$ has a nonempty boundary, then we can view $S$ as part of a larger surface $S_0$ without boundary, and the part $S_0 \setminus S$ can be considered as an obstacle since the boundary values of the competing surfaces $X$ are confined to $S$. The existence theory for such boundary problems with a thin obstacle can be carried along the lines of Chapters 4 and 5, and no additional difficulties will arise. The boundary behavior of solutions of such problems will be investigated in the Vols. 2 and 3.

Presently we shall confine our attention to *thick obstacles* in $\mathbb{R}^3$ (or $M$) which are to be avoided by the admissible surfaces. To describe some of the results, consider the functionals

$$\mathcal{F}_B(X) := E_B(X) + V_B(X)$$

where

$$E_B(X) := \frac{1}{2}\int_B g_{jk}(X)(X_u^j X_u^k + X_v^j X_v^k)\,du\,dv,$$
$$V_B(X) := \int_B \langle Q(X), X_u \wedge X_v\rangle\,du\,dv$$

that is,

$$\mathcal{F}_B(X) = \int_B e(X, \nabla X)\,du\,dv$$

with the Lagrangian

$$e(x,p) = \tfrac{1}{2} g_{jk}(x)(p_1^j p_1^k + p_2^j p_2^k) + \langle Q(x), p_1 \wedge p_2\rangle$$

where $x \in \mathbb{R}^3$ and $p = (p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Furthermore let $\mathcal{C}^*$ denote one of the classes $\mathcal{C}^*(\Gamma)$ or $\mathcal{C}^*(\Gamma, S)$, i.e., the set of surfaces bounded by $\Gamma$ or $\langle \Gamma, S\rangle$ respectively which are normalized by a three point condition, see Sections 4.2 and 4.6. Suppose that $K \subset \mathbb{R}^3$ is a closed set; then we put $C = C(K, \mathcal{C}^*) := \mathcal{C}^* \cap H_2^1(B, K)$, where $H_2^1(B, K)$ denotes the subset of functions $f \in H_2^1(B, \mathbb{R}^3)$ which map almost all of $B$ into $K$. We consider the variational problem $\mathcal{P}(\mathcal{F}, C)$ given by

$$\mathcal{F} \to \min \quad \text{in } C.$$

**Theorem.** *Suppose that* $Q \in C^0(K, \mathbb{R}^3)$, $g_{ij} \in C^0(K, \mathbb{R})$, $g_{ij} = g_{ji}$, $i, j \in \{1, 2, 3\}$, *and let* $0 < m_0 \le m_1$ *be numbers with the property*

$$m_0|p|^2 \le e(x, p) \le m_1|p|^2 \quad \textit{for all } (x, p) \in K \times \mathbb{R}^6. \tag{1}$$

*Moreover assume that* $K$ *is a closed set in* $\mathbb{R}^3$ *such that* $C = C(K, \mathcal{C}^*)$ *is nonempty. Then the variational problem* $\mathcal{P}(\mathcal{F}, C)$ *has (at least) one solution in* $C(K, \mathcal{C}^*)$.

*Proof.* The following three statements have to be verified:

(i) The class $C(K, \mathcal{C}^*)$ is a weakly closed subset of $H_2^1(B, \mathbb{R}^3)$.
(ii) There exists a minimizing sequence $X_n \in C(K, \mathcal{C}^*)$ for $\mathcal{P}(\mathcal{F}, C)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$.
(iii) The functional $\mathcal{F}_B(\cdot)$ is weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$.

(i)–(iii) immediately imply that $X$ is an element of $C$ furnishing a solution of $\mathcal{P}(\mathcal{F}, C)$; in fact, (iii) yields

$$\mathcal{F}_B(X) \le \liminf_{n\to\infty} \mathcal{F}_B(X_n) = e = \inf_C \mathcal{F}_B,$$

and hence $\mathcal{F}_B(X) = e$.

Property (i) follows from the weak closedness of $\mathcal{C}^*$, from a theorem of Rellich and from the fact that one can extract from any $L_2$-convergent sequence a subsequence which converges pointwise almost everywhere, cf. Theorem 2 in Section 4.6.

Statement (ii) is a consequence of the fact that for any minimizing sequence of surfaces $X_n \in C(K, \mathcal{C}^*)$ there holds an estimate

$$\|X_n\|_{H_2^1(B)} \le \text{const}$$

which follows from the ellipticity condition (1) and from a suitable Poincaré inequality. Finally (iii) is a special case of a general lower semicontinuity result of Serrin, see Morrey [8], Theorem 1.8.2. □

In addition to the preceding theorem we have the following result concerning *conformal parameters*.

**Proposition 12.** *Any solution* $X \in C(K, \mathcal{C}^*)$ *of the variational problem* $\mathcal{P}(\mathcal{F}, C)$ *satisfies almost everywhere in* $B$ *the conformality relations*

$$g_{ij}X_u^i X_u^j = g_{ij}X_v^i X_v^j \quad \textit{and} \quad g_{ij}X_u^i X_v^j = 0.$$

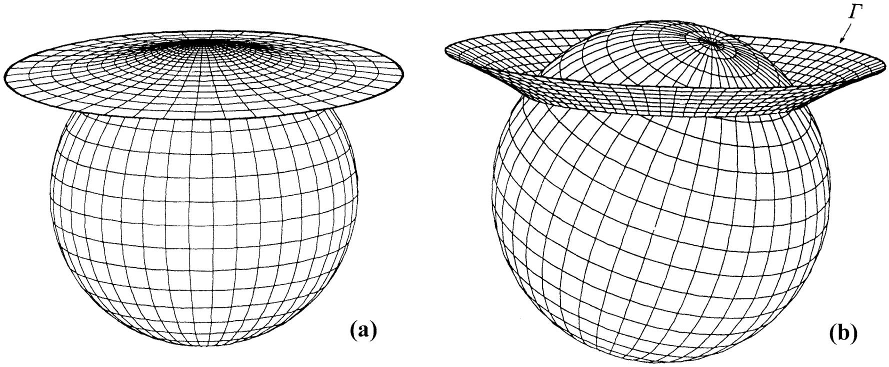



**Fig. 10.** **(a)** The obstacle problem is to find a surface of least Dirichlet integral which is bounded by a Jordan curve $\Gamma$ and which remains outside given solid bodies. The minimizers among the disk-type surfaces are minimal surfaces away from the obstacle, but where they touch it they may have non-zero mean curvature. **(b)** If multiply connected surfaces with free boundaries on the obstacle are also admitted as comparison surfaces, then smaller Dirichlet integrals can be achieved and the minimizers will be perpendicular to the obstacle along their free boundaries

The proof of this result is obtained by a suitable adaptation of the argument given in Sections 4.5 and 4.10.

In the special case where $g_{ij} = \delta_{ij}$ and $Q = 0$ we conclude from the above theorems the existence of a minimal surface $X$ bounded by $\Gamma$ or $\langle \Gamma, S \rangle$ respectively which is spanned over the obstacle $\partial K$. Note that, in general, the coincidence set $\mathcal{T} = \{w \in B \colon X(w) \in \partial K\}$ will be a nonempty subset of $B$. If $\mathcal{T}$ is nonempty, then a soap film corresponding to $X$ touches the surface $\partial K$ of the obstacle. If one allows the film to change its topological type by, say, admitting a number of holes, it can slide down on $\partial K$, thereby reducing its area (see Fig. 10). The corresponding surfaces $X : \Omega \to \mathbb{R}^3$ will then be defined on a multiply connected parameter domain $\Omega \subset \mathbb{C}$ and have free boundaries on $\partial K$. This phenomenon was treated by Tolksdorf in his paper [1] where he proved the existence of a minimum $X$ for the functional

$$\tilde{D}(X) = \int_B |\tilde{\nabla} X|^2 \, du \, dv$$

with

$$\tilde{\nabla} X(u,v) := \begin{cases} \nabla X(u,v) & \text{if } X(u,v) \notin \partial K, \\ 0 & \text{if } X(u,v) \in \partial K \end{cases}$$

in a suitably chosen class of comparison functions. For details we refer the reader to Tolksdorf's paper.

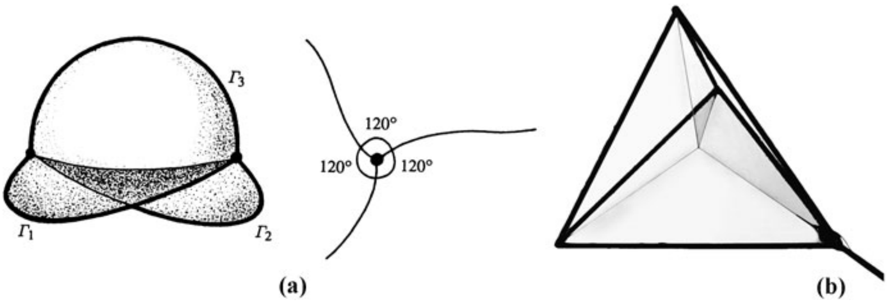



**Fig. 11.** **(a)** Rule 1, **(b)** Rule 2 demonstrated by a system of 6 soap-films in tetrahedron

## *7 Systems of Minimal Surfaces*

Usually one encounters soap films and soap bubbles in the shape of *foam*. Roughly speaking, foam is a system of soap films and soap bubbles which are attached to each other and meet at common *liquid edges*. More than a hundred years ago Plateau observed in experiments that such systems obey two simple rules which he stated in his treatise [1]. Let us formulate these rules simply for systems of minimal surfaces ($H = 0$), neglecting systems of bubbles ($H = \text{const} \neq 0$) and mixed systems.

A *system of minimal surfaces* is a connected set which is a finite union of smooth regular manifolds of zero mean curvature which sit in a given frame and meet each other at free boundary curves called liquid edges. These liquid edges form the singular part of the minimal surface system.

**Rule 1.** *At each liquid edge meet exactly three minimal surfaces of the system, and any two of them enclose an angle of* 120 *degrees.*

**Rule 2.** *Liquid edges can meet at supersingular points p. Each supersingular point is the meeting point of exactly four liquid edges. Any two adjacent edges form an angle* $\varphi = 109°28'16''$ (*precisely speaking,* $\cos\varphi = -1/3$).

These two principles are illustrated by Fig. 11.

The first rigorous proof for the two rules governing systems of minimal surfaces was given by J. Taylor [2] using the means of geometric measure theory. Let us briefly outline her arguments. Consider a system $S$ of minimal surfaces which is bounded by a closed system $\Gamma$ of Jordan arcs $\Gamma_1, \Gamma_2, \ldots$, and assume that $S$ minimizes area within all other systems bounded by $\Gamma$. In the first step, a *monotonicity formula* is employed to prove the existence of tangent cones $T_pS$ at each point $p$ of S. Moreover, it is verified that each tangent cone (which in general is not known to be unique) is again area minimizing for the frame formed by the intersection of the cone with the unit sphere $S^2$ centered at $p$. Such a frame is a system of arcs on $S^2$, and each arc is part of a great circle. At any vertex of such a system only three arcs

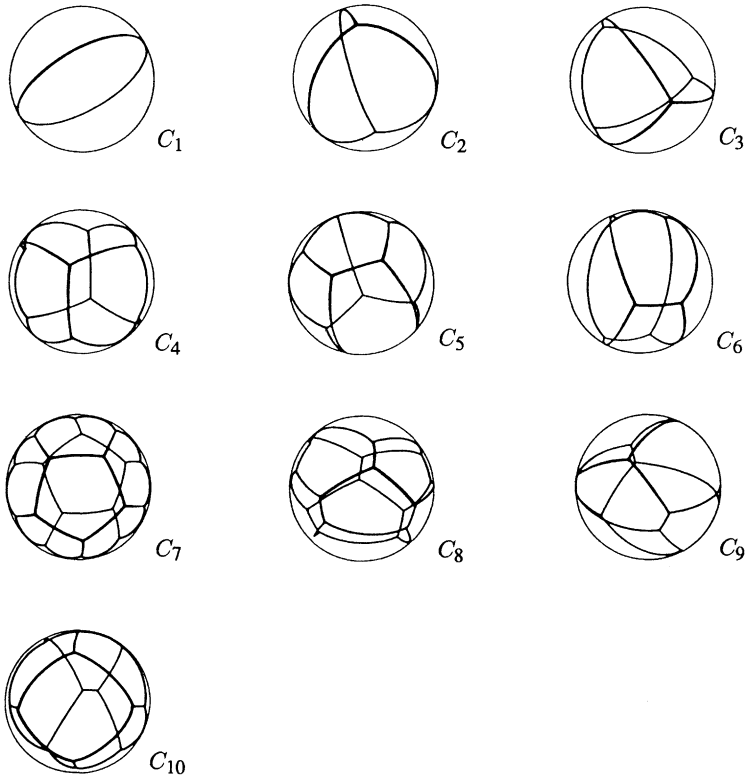



**Fig. 12.** The ten geodesic nets on $S^2$

can run together, and any two of them form an angle of 120°. A frame on $S^2$ with these properties is called a *geodesic net*. Thus, in order to classify all area minimizing cones, one first looks at the simpler question of determining all (equiangular) geodesic nets on $S^2$. It turns out that exactly ten different such nets exist. This classification was already carried out by Lamarle who, however, missed one net. The complete list, depicted in Fig. 12, was given by Heppes [1]. According to Lamarle and Heppes, the ten (equiangular) geodesic nets $C_1, \dots, C_{10}$ can be described as follows:

(a) $C_1$ is a great circle;
(b) $C_2$ consists of three halves of great circles with common endpoints;
(c) $C_3$ is a spherical tetrahedron;
(d) $C_4$ is a spherical cube;
(e) $C_5$ consists of 15 arcs forming the 1-skeleton of the prism over the regular pentagon;
(f) $C_6$ is a prism over a regular triangle and consists of 9 arcs;
(g) $C_7$ is a spherical dodecahedron made of 30 arcs;

(h) $C_8$ consists of 24 arcs forming two regular quadrilaterals and 8 congruent pentagons;
(i) $C_9$ is formed by 18 arcs which determine 4 equal pentagons and 4 equal quadrilaterals;
(j) $C_{10}$ consists of 21 arcs forming three regular quadrilaterals and six congruent pentagons.

Now that we know $C_1, \dots, C_{10}$, the crucial question of determining all area minimizing tangent cones is reduced to the problem of finding out which of the cones over $C_j$ with their vertex at $p$ are area minimizing. Jean Taylor proved that $C_1, C_2$ and $C_3$ are minimizers whereas the cones over $C_4, \dots, C_{10}$ are not even stable. The mathematical proof is rather elaborate whereas the physical demonstration of this fact is easily provided by a soap film experiment which is depicted in Hildebrandt and Tromba [1], pp. 128–129. The pictures show that the area minimizing soap films in $C_1, C_2$ and $C_3$ are cones but not those in $C_4, \dots, C_{10}$. Thus we are led to the following

**Theorem of J. Taylor.** *Let $S$ be a system of minimal surfaces which is bounded by a closed system of Jordan arcs and minimizes area within its boundary. Then the following holds true*:

(i) *At each point $p \in S$ there exists a unique tangent cone which is congruent to one of the cones* (a), (b) *or* (c) *in Fig.* 12.
(ii) *Let $\mathcal{R}(S) := \{p \in S$: the tangent cone to $S$ at $p$ is congruent to* (a)$\}$ *denote the regular part of $S$. Then $\mathcal{R}(S)$ is a two-dimensional manifold in $\mathbb{R}^3$. Each component of $\mathcal{R}(S)$ has mean curvature zero.*
(iii) *Let $\Sigma(S) := \{p \in S$: the tangent cone to $S$ at $p$ is congruent to* (b)$\}$ *denote the set of singular points in $S$. Then $\Sigma(S)$ is a one-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^3$ for some $\alpha \in (0,1)$. There exists a neighborhood $\mathcal{U}(p)$ for each $p \in \Sigma(S)$ and a conformal $C^{1,\alpha}$-diffeomorphism $f$ of $\mathbb{R}^3$ onto itself such that $\mathcal{U} \cap S$ is the image of* (b) *under $f$.*
(iv) *Let $\sigma(S) := \{p \in S$: the tangent cone to $S$ at $p$ is congruent to* (c)$\}$ *denote the set of supersingular points in $S$. Then $\sigma(S)$ consists of isolated points. Furthermore, for each $p \in \sigma(S)$ there exists a neighborhood $\mathcal{U}(p)$ in $\mathbb{R}^3$ and a conformal $C^{1,\alpha}$-diffeomorphism $f$ of $\mathbb{R}^3$ onto itself such that $\mathcal{U}(p) \cap S$ is the image of (c) under $f$.*
(v) *The system $S$ decomposes into $S = \mathcal{R}(S) \cup \Sigma(S) \cup \sigma(S)$.*

For the proof of this result we refer to J. Taylor [2]. The above theorem also extends to systems of surfaces of constant mean curvature as well as to systems of surfaces which are extremals of some functional which is close to the area functional in a suitable sense.

Nitsche [33] proved that the singular part $\Sigma(S)$ is a union of regular $C^\infty$-curves, and Kinderlehrer, Nirenberg, and Spruck [1] even showed that $\Sigma(S)$ is a union of real analytic curves.

It is fairly easy to prove the existence of area minimizing systems in a given frame $\Gamma$ by means of geometric measure theory.

We finally note that, under rather restrictive symmetry assumptions on the boundary $\Gamma$, the existence of area minimizing systems $S$ and the regularity of their singular parts had earlier been established by A. Solomon [1,2].

### *8 Isoperimetric Inequalities*

Historical remarks on this topic and further comments can be found in the Scholia to Chapter 4 of Vol. 2.

### *9 Plateau's Problem for Infinite Contours*

It is also of interest to determine minimal surfaces which are not bounded by one or several loops, but by one or several arcs, finite or infinite. Actually, already Riemann [2] developed a method to construct minimal surfaces which are simply connected and have straight line segments as boundaries. As examples he studied two infinite straight lines which are not contained in a plane ([1], §15), two infinite half lines meeting at a common endpoint and an infinite straight line parallel to the plane of the first two ([1], §16), three pairwise skew lines ([1], §17). As a main idea to solve these three problems as well as Plateau's problem for the skew quadrilateral ([1], §18), Riemann used the fact that the surface normal of a solution maps any straight segment of the boundary onto an arc of a great circle on $S^2$. This work was generalized by E. Neovius [1–5]. In this context we also refer to the treatises of Darboux [1] and Bianchi [1,2].

The problem of determining minimal surfaces with prescribed unbounded contours was anew taken up by López and Wei [1], López and Martín [1], and Ferrer and Martín for unbounded polygonal boundaries. For a fairly general class of unbounded contours the problem was recently solved by F. Tomi [13]. The curves $\Gamma$ considered by Tomi are described by $\Gamma = \xi(\mathbb{R})$, where $\xi$ provides a noncompact proper embedding of $\mathbb{R}$ into $\mathbb{R}^3$ which is piecewise of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and satisfies $\xi(0) = 0$ and $|\xi'(s)| = 1$ as well as the following conditions:

(i) There is a constant $\delta > 0$ such that $|p - q| \le \delta$ for all $p, q$ contained in different components of $\Gamma \setminus \Gamma_1$ where $\Gamma_1$ is the connected components of $\Gamma \cap B_1$ containing 0.

(ii) Let $\gamma(s) := |\xi(s)|^{-1}\xi(s)$ for $s \neq 0$. Then

$$|\langle \gamma(x), \xi'(x)\rangle| \to 1 \quad \text{as } |s| \to \infty.$$

(iii) $\int_{\Gamma\setminus\Gamma_1} |\xi(s)|^{-1}\sqrt{1 - \langle\gamma(s), \xi'(s)\rangle^2}\, ds < \infty.$

Then Tomi's theorem reads as follows:

*There exists a proper mapping* $X \in C^0(\overline{H}, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R})$ *of a closed half-plane* $H$ *in* $\mathbb{R}^2$ *which is an immersed minimal surface on* $H$ *and maps* $\partial H$ *in a strictly monotonic way onto* $\Gamma$.

Tomi's class of admissible curves $\Gamma$ contains all properly embedded curves with polynomial ends. The main idea of the proof is to work with surfaces whose area in a ball of radius $R$ growth at most quadratically in $R$.

### *10 Plateau's Problem for Polygonal Contours*

(Added in Proof, May 2010)

In her recent thesis (Dec. 4, 2009) Laura Desideri [1] has rectified and supplemented Garnier's approach to Plateau's problem for polygonal boundaries. She proved the following beautiful theorem:

*Let* $\Gamma$ *be a polygon in* $\mathbb{R}^3 \cup \{\infty\}$ *with* $n+3$ *sides in "generic position", possibly with one of its vertices lying at infinity. Then* $\Gamma$ *bounds an immersed minimal surface* $X : \mathbb{C}_+ \to \mathbb{R}^3$ *defined on the upper halfplane* $\mathbb{C}_+$ *in the sense that* $\Gamma$ *is the boundary of the image* $X(\mathbb{C}_+)$. *If* $\Gamma$ *has a vertex at infinity, then the immersion* $X$ *has a helicoidal end at this vertex.*

This result contributes also to the problems discussed in No. 2 and No. 9. Furthermore, Desideri has proved an analog of the above theorem for the Plateau problem in Minkowski space.