Chapter 2

Minimal Surfaces

Since the last century, the name *minimal surfaces* has been applied to surfaces of vanishing mean curvature, because the condition

$$H = 0$$

will necessarily be satisfied by surfaces which minimize area within a given boundary configuration. This was implicitly proved by Lagrange for nonparametric surfaces in 1760, and then by Meusnier in 1776 who used the analytic expression for the mean curvature and determined two minimal surfaces, the catenoid and the helicoid. (The notion of mean curvature was introduced by Young [1] and Laplace [1], but usually it is ascribed to Sophie Germain [1].) In Section 2.1 we shall derive an expression for the first variation of area with respect to general variations of a given surface. From this expression we obtain the equation $H = 0$ as necessary condition for stationary surfaces of the area functional, and we also demonstrate that solutions of the free boundary problem meet their supporting surfaces at a right angle.

In Section 2.2, we particularly investigate nonparametric surfaces, and we state the *minimal surface equation* in divergence and nondivergence form which has to be satisfied by the height function. Finally we prove that, for a nonparametric minimal surface X, the 1-form $N \wedge dX$ is closed. In Section 2.3 it is shown that a nonparametric minimal surface $X(x, y) = (x, y, z(x, y))$ has a real analytic height function $z(x, y)$ and, moreover, that X can be conformally mapped onto some planar domain. This conformal mapping can be constructed explicitly if the domain of definition Ω of the surface X is convex.

Thereafter we prove in Section 2.4 the celebrated Bernstein theorem for nonparametric minimal surfaces and also a quantitative local version of this theorem which was discovered by E. Heinz. Then we show in Section 2.5 that every regular surface $X : \Omega \to \mathbb{R}^3$ satisfies the equation

$$\Delta_X X = 2HN$$

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and, therefore, minimal surfaces are characterized by the equation

$$\Delta_X X = 0.$$

If X is given by conformal parameters, this relation is equivalent to

$$\Delta X = 0.$$

This observation is used in Section 2.6 to enlarge the class of minimal surfaces. We can now admit surfaces with isolated singularities by defining minimal surfaces as harmonic mappings $X : \Omega \to \mathbb{R}^3$ that are given in conformal parameters.

In Section 2.7 we derive a formula for the mean curvature of surfaces that are defined by implicit equations. This relation is used in the last part of the chapter to demonstrate that a minimal surface provides a minimum of area if it can be embedded into a field of minimal surfaces. Finally, an expression for the second variation of area is given, and we comment on the question when a given minimal surface can be embedded into such a field.

2.1 First Variation of Area. Minimal Surfaces

Let $X : \bar{\Omega} \to \mathbb{R}^3$ be a regular surface of class C^2 with its spherical image $N : \bar{\Omega} \to \mathbb{R}^3$ defined by

$$N = \frac{1}{\mathcal{W}} X_u \wedge X_v, \quad \mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{g},$$

and denote by $g_{\alpha\beta}$ and $b_{\alpha\beta}$ (or $\mathcal{E}, \mathcal{F}, \mathcal{G}$ and $\mathcal{L}, \mathcal{M}, \mathcal{N}$, respectively) the coefficients of its first and second fundamental forms. Moreover, H stands for the mean curvature of X. We write $w = (u, v)$, $u^1 = u$, $u^2 = v$, and $X_{,\alpha} = \frac{\partial}{\partial u^\alpha} X$; $\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of the second kind for X introduced in Section 1.3.

We now consider a *variation of* X, that is, a mapping

$$Z : \bar{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3, \quad \varepsilon_0 > 0,$$

of class C^2, with the property that

$$Z(w, 0) = X(w) \quad \text{for all } w \in \bar{\Omega}.$$

This map will be interpreted as a family of surfaces $Z(w, \varepsilon)$, $w \in \bar{\Omega}$, which vary X, and in which X is embedded.

By Taylor expansion, we can write

(1) $$Z(w, \varepsilon) = X(w) + \varepsilon Y(w) + \varepsilon^2 R(w, \varepsilon)$$

with a continuous remainder term $\varepsilon^2 R(w,\varepsilon)$ of square order, i.e. $R(w,\varepsilon) = O(1)$ as $\varepsilon \to 0$. The vector field

$$Y(w) = \frac{\partial}{\partial\varepsilon} Z(w,\varepsilon)\bigg|_{\varepsilon=0} \in C^1(\bar{\Omega}, \mathbb{R}^3)$$

is called *the first variation of the family of surfaces* $\mathrm{Z}(\cdot,\varepsilon)$.

We can write

$$Y(w) = \eta^\beta(w) X_{,\beta}(w) + \lambda(w) N(w) \tag{2}$$

with functions η^1, η^2, λ of class $C^1(\bar{\Omega})$. Then

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon[\eta^\beta_{,\alpha} X_{,\beta} + \eta^\beta X_{,\alpha\beta} + \lambda_{,\alpha} N + \lambda N_{,\alpha}] + \varepsilon^2 R_{,\alpha}.$$

By virtue of the Gauss equations

$$X_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma} + b_{\alpha\beta} N$$

and the Weingarten equations

$$N_{,\alpha} = -b^\beta_\alpha X_{,\beta}, \quad b^\beta_\alpha = b_{\alpha\gamma} g^{\beta\gamma},$$

we obtain that

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon[\xi^\gamma_\alpha X_{,\gamma} + \nu_\alpha N] + \varepsilon^2 R_{,\alpha} \tag{3}$$

where we have set:

$$\begin{aligned} \xi^\gamma_\alpha &= \eta^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\beta}\eta^\beta - b_{\alpha\beta} g^{\beta\gamma}\lambda, \\ \nu_\alpha &= b_{\alpha\beta}\eta^\beta + \lambda_{,\alpha}. \end{aligned} \tag{4}$$

Then, indicating the ε^2-terms by $\cdots$, we find

$$\begin{aligned} |Z_u|^2 &= \mathcal{E} + 2\varepsilon(\xi^1_1 \mathcal{E} + \xi^2_1 \mathcal{F}) + \cdots, \\ |Z_v|^2 &= \mathcal{G} + 2\varepsilon(\xi^1_2 \mathcal{F} + \xi^2_2 \mathcal{G}) + \cdots, \\ \langle Z_u, Z_v\rangle &= \mathcal{F} + \varepsilon[\xi^1_2 \mathcal{E} + (\xi^1_1 + \xi^2_2)\mathcal{F} + \xi^2_1 \mathcal{G}] + \cdots, \end{aligned}$$

whence

$$|Z_u|^2 |Z_v|^2 - \langle Z_u, Z_v\rangle^2 = \mathcal{W}^2[1 + 2\varepsilon(\xi^1_1 + \xi^2_2) + \cdots].$$

We, moreover, have

$$\xi^1_1 + \xi^2_2 = \eta^1_u + \eta^2_v - \lambda b_{\alpha\beta} g^{\alpha\beta} + \Gamma^\alpha_{\alpha\beta}\eta^\beta.$$

Since

$$b_{\alpha\beta} g^{\alpha\beta} = 2H, \quad \Gamma^\alpha_{\alpha\beta} = \frac{1}{2g} g_{,\beta} = \frac{1}{\mathcal{W}} \mathcal{W}_{,\beta}$$

(see formulas (42) of Section 1.2 and (12) of Section 1.3), we infer that

$$\xi_1^1 + \xi_2^2 = \frac{1}{\mathcal{W}}\{(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v\} - 2H\lambda.$$

On account of $\sqrt{1+x} = 1 + x/2 + O(x^2)$ for $|x| \ll 1$, we see that

$$(|Z_u|^2|Z_v|^2 - \langle Z_u, Z_v\rangle^2)^{1/2} = \mathcal{W} + \varepsilon[(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v - 2H\mathcal{W}\lambda] + \cdots .$$

Then we can conclude that the *first variation*

$$\delta A_\Omega(X,Y) := \frac{d}{d\varepsilon}A_\Omega(Z(\cdot,\varepsilon))\Big|_{\varepsilon=0} \tag{5}$$

of the area functional $A_\Omega(X)$ *on* Ω *at* X *in the direction of a vector field* $Y = \eta^\alpha X_\alpha + \lambda N$ *is given by*

$$\delta A_\Omega(X,Y) = \int_\Omega [(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v - 2H\mathcal{W}\lambda]\,du\,dv. \tag{6}$$

Performing an integration by parts, it follows that

$$\delta A_\Omega(X,Y) = \int_{\partial\Omega} \mathcal{W}(\eta^1\,dv - \eta^2\,du) - 2\int_\Omega \lambda H\mathcal{W}\,du\,dv. \tag{7}$$

This, in particular, implies that

$$\begin{aligned}\delta A_\Omega(X,Y) &= -2\int_X \langle Y,N\rangle H\mathcal{W}\,du\,dv\\ &= -2\int_X \langle Y,N\rangle H\,dA\end{aligned} \tag{8}$$

for all $Y \in C_c^\infty(\Omega,\mathbb{R}^3)$. Since $\lambda = \langle Y,N\rangle$ can be chosen as an arbitrary function of class $C_c^\infty(\Omega)$, the fundamental theorem of the calculus of variations yields:

Theorem 1. *The first variation* $\delta A_\Omega(X,Y)$ *of* A_Ω *at* X *vanishes for all vector fields* $Y \in C_c^\infty(\Omega,\mathbb{R}^3)$ *if and only if the mean curvature* H *of* X *is identically zero.*

In other words, the (regular) stationary points of the area functional—and, in particular, its (regular) minimizers—are exactly the surfaces of zero mean curvature. For this reason, *a regular (i.e. immersed) surface* $X : \Omega \to \mathbb{R}^3$ *of class* C^2 *is usually called a minimal surface if its mean curvature function* H *satisfies*

$$H = 0. \tag{9}$$

We shall later broaden the class of minimal surfaces in order to allow also surfaces with isolated singularities, but then we use conformal parameters u, v.

Let us now formulate a more geometric expression for the first variation of the area. Note that

$$\lambda \mathcal{W} = \langle Y, N \rangle \mathcal{W} = \langle Y, X_u \wedge X_v \rangle = [Y, X_u, X_v] \ (:= \det(Y, X_u, X_v))$$

and, for $Y = \eta^1 X_u + \eta^2 X_v + \lambda N$, we obtain

$$[Y, N, dX] = \eta^1 [X_u, N, X_v \, dv] + \eta^2 [X_v, N, X_u \, du] = \mathcal{W} \, \{\eta^2 \, du - \eta^1 \, dv\}.$$

Hence, formula (7) implies that

$$(10) \qquad -\delta A_\Omega(X, Y) = \int_{\partial \Omega} [Y, N, dX] + 2 \int_\Omega H [Y, X_u, X_v] \, du \, dv.$$

Let $\omega(\mathrm{s})$ be a representation of $\partial \Omega$ in terms of the parameter of arc length s of the boundary $X|_{\partial\Omega}$. Then $c(s) := X(\omega(s))$ is a representation of the boundary of X. Moreover, let $\mathfrak{Y}(s) := Y(\omega(s)), \mathfrak{N}(s) := N(\omega(s))$. Then

$$[Y, N, dX] \circ \omega = \langle \mathfrak{Y}, \mathfrak{N} \wedge \boldsymbol{t} \rangle \, ds = \langle \mathfrak{Y}, \boldsymbol{s} \rangle \, ds$$

where $\boldsymbol{s}$ is the side normal of the boundary curve c of the surface X. Hence we get

$$(11) \qquad -\delta A_\Omega(X, Y) = \int_{\partial X} \langle \mathfrak{Y}, \boldsymbol{s} \rangle \, ds + 2 \int_X \langle Y, N \rangle H \, dA.$$

In particular,

$$(12) \qquad \delta A_\Omega(X, \lambda N) = -2 \int_X \lambda H \, dA$$

and

$$(13) \qquad 2H = -\delta A_\Omega(X, N) / A_\Omega(X) \quad \text{if } H = \text{const}.$$

In other words, for surfaces of constant mean curvature H, the expression $-2H$ is just the relative change of the area of the surface with respect to normal variations.

Moreover, we have

$$(14) \qquad \delta A_\Omega(X, Y) = -\int_{\partial X} \langle \mathfrak{Y}, \boldsymbol{s} \rangle \, ds \quad \text{if } H = 0,$$

and we obtain the following

Proposition. *If $X : \bar{\Omega} \to \mathbb{R}^3$ is a minimal surface, then the equation*

$$\delta A_\Omega(X, Y) = 0$$

holds for all $Y \in C^1(\bar{\Omega}, \mathbb{R}^3)$ which are orthogonal to the side normal of the boundary ∂X (that is, $\langle \mathfrak{Y}, \boldsymbol{s} \rangle = 0$ on ∂X).

Furthermore, if we assume that

$$\frac{d}{d\varepsilon} A_\Omega(Z(\cdot,\varepsilon))\Big|_{\varepsilon=0} = 0 \tag{15}$$

holds for all variations $Z(\cdot,\varepsilon)$ of X whose boundary values lie on some *supporting manifold* $S \subset \mathbb{R}^3$ of dimension two, then it follows that

(16) $\delta A_\Omega(X,Y) = 0$ *holds for all* $Y \in C^1(\bar{\Omega},\mathbb{R}^3)$, *the boundary values of which at* $\partial\Omega$ *are tangential to* S.

From this equation, we firstly infer that X is a minimal surface, and secondly, by once again applying the fundamental theorem of the calculus of variations, we obtain from equation (16) that the side normal of ∂X meets S everywhere at a right angle. This means that X intersects S perpendicularly. Thus we have proved:

Theorem 2. *Suppose that* (15) *holds for all variations* $Z(\cdot,\varepsilon)$ *of* X *with boundary on some supporting surface* S. *Then* X *is a minimal surface which meets* S *orthogonally at its boundary* ∂X.

A minimal surface as in Theorem 2 will be called *a stationary surface to the supporting manifold* S, or *solution of the free boundary problem for* S. The study of such free boundary problems will be emphasized in Section 4.6 and particularly in Vols. 2 and 3. In short, if we consider stationary surfaces in boundary configurations which, in part, consist of fixed curves Γ and, in addition, of free surfaces S (called *support surfaces*), then we deal with minimal surfaces that meet S perpendicularly.

2.2 Nonparametric Minimal Surfaces

We shall now consider surfaces which are given in *nonparametric form*, that is, as graph of a function $z = z(x,y)$ on some domain Ω of $\mathbb{R}^2$. Such a surface can be described by the special parameter representation

$$X(x,y) = (x,y,z(x,y)), \quad (x,y) \in \Omega.$$

(In this case, the parameters are usually denoted by x and y instead of u and v.)

We shall assume that the function $z(x,y)$ is at least of class C^2. Introducing the time-honored abbreviations

$$p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy} \tag{1}$$

we compute that

(2) $$\mathcal{E} = 1 + p^2, \quad \mathcal{F} = pq, \quad \mathcal{G} = 1 + q^2, \\ \mathcal{W}^{\,2} = 1 + p^2 + q^2, \quad N = (\xi, \eta, \zeta),$$

where

(3) $$\xi = -p/\sqrt{1 + p^2 + q^2}, \quad \eta = -q/\sqrt{1 + p^2 + q^2}, \\ \zeta = 1/\sqrt{1 + p^2 + q^2}.$$

Moreover,

(4) $$\mathcal{L} = r/\sqrt{1 + p^2 + q^2}, \quad \mathcal{M} = s/\sqrt{1 + p^2 + q^2}, \\ \mathcal{N} = t/\sqrt{1 + p^2 + q^2},$$

whence finally

(5) $$H = \frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{2(1 + p^2 + q^2)^{3/2}},$$

(6) $$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

Therefore, the equation $H = 0$ is equivalent to the nonlinear second order differential equation

(7) $$(1 + q^2)r - 2pqs + (1 + p^2)t = 0,$$

the so-called *minimal surface equation*. It is necessary and sufficient for a surface $z = z(x, y)$ to be a minimal surface.

For nonparametric surfaces $X(x, y) = (x, y, z(x, y))$ the area functional $A_\Omega(X)$ takes the form

(8) $$A_\Omega(X) = \int_\Omega \sqrt{1 + p^2 + q^2}\, dx\, dy.$$

By Theorem 1 of Section 2.1, a nonparametric minimal surface X, defined by the function $z = z(x, y)$, satisfies $\delta A_\Omega(X, Y) = 0$ for all $Y \in C_c^\infty(\Omega, \mathbb{R}^3)$. In particular for $Y = (0, 0, \zeta)$, $\zeta \in C_c^\infty(\Omega)$, we obtain that

$$\int_\Omega \left(\frac{p}{\mathcal{W}} \zeta_x + \frac{q}{\mathcal{W}} \zeta_y \right) dx\, dy = 0,$$

and the fundamental lemma of the calculus of variations yields the Euler equation

(9) $$\left\{ \frac{p}{\sqrt{1 + p^2 + q^2}} \right\}_x + \left\{ \frac{q}{\sqrt{1 + p^2 + q^2}} \right\}_y = 0$$

for the functional

$$\mathcal{A}(z) := \int_{\Omega} \sqrt{1 + p^2 + q^2}\, dx\, dy. \tag{10}$$

Equation (9) can equivalently by written as

$$\operatorname{div} \frac{\nabla z}{\sqrt{1 + |\nabla z|^2}} = 0. \tag{11}$$

This relation will be called *the minimal surface equation in divergence form.*

Actually equations (7) and (11) are equivalent. In fact, by means of a straight-forward computation we infer from (5) that

$$\operatorname{div}(\mathcal{W}^{-1} \nabla z) = 2H$$

holds true for any nonparametric surface $z = z(x, y)$. This equation also implies that *any nonparametric surface $X(x, y) = (x, y, z(x, y))$ described by the function $z(x, y)$ is a minimal surface if and only if the* 1*-form*

$$\gamma = -(p/\mathcal{W})\, dy + (q/\mathcal{W})\, dx$$

is a closed differential form on Ω, that is, if and only if

$$\gamma = -dc$$

with some function $c \in C^2(\Omega)$ provided that the domain Ω is simply connected.

There is actually a stronger version of this result which permits a remarkable geometric interpretation. For this purpose, we introduce the differential form

$$N \wedge dX = (\alpha, \beta, \gamma) \tag{12}$$

with the components

$$\alpha = \eta\, dz - \zeta\, dy, \quad \beta = \zeta\, dx - \xi\, dz, \quad \gamma = \xi\, dy - \eta\, dx. \tag{13}$$

Inserting

$$dz = p\, dx + q\, dy, \quad \xi = -p/\mathcal{W}, \quad \eta = -q/\mathcal{W}, \quad \zeta = 1/\mathcal{W},$$

one obtains

$$\begin{aligned} \alpha &= -\frac{pq}{\mathcal{W}}\, dx - \frac{1 + q^2}{\mathcal{W}}\, dy, \\ \beta &= \frac{1 + p^2}{\mathcal{W}}\, dx + \frac{pq}{\mathcal{W}}\, dy, \\ \gamma &= \frac{q}{\mathcal{W}}\, dx - \frac{p}{\mathcal{W}}\, dy. \end{aligned} \tag{14}$$

Let us introduce the differential expression

$$T := (1 + q^2)r - 2pqs + (1 + p^2)t.$$

Then a straight forward computation shows

$$\begin{aligned}
\left(-\frac{pq}{\mathcal{W}}\right)_y - \left(-\frac{1+q^2}{\mathcal{W}}\right)_x &= -\frac{p}{\mathcal{W}^3}T, \\
\left(\frac{1+p^2}{\mathcal{W}}\right)_y - \left(\frac{pq}{\mathcal{W}}\right)_x &= -\frac{q}{\mathcal{W}^3}T, \\
\left(\frac{q}{\mathcal{W}}\right)_y - \left(\frac{-p}{\mathcal{W}}\right)_x &= \frac{1}{\mathcal{W}^3}T,
\end{aligned}$$

that is,

(15) $$d\alpha = 2Hp\,dx\,dy, \quad d\beta = 2Hq\,dx\,dy, \quad d\gamma = -2H\,dx\,dy,$$

whence

(16) $$d(N \wedge dX) = -2H\mathcal{W}N\,dx\,dy$$

or equivalently

(16′) $$d(N \wedge dX) = -2HN\,dA,$$

where dA denotes the area element $\mathcal{W}\,dx\,dy$.

Thus we have proved the following

Theorem 1. *A nonparametric surface $X(x,y) = (x,y,z(x,y))$, described by a function $z = z(x,y)$ of class C^2 on a simply connected domain Ω of $\mathbb{R}^2$, with the Gauss map $N = (\xi,\eta,\zeta)$ is a minimal surface if and only if the vector-valued differential form $N \wedge dX$ is a total differential, i.e. if and only if there is a mapping $X^* \in C^2(\Omega,\mathbb{R}^3)$ such that*

(17) $$-dX^* = N \wedge dX.$$

If we write

(18) $$X^* = (a,b,c), \quad N \wedge dX = (\alpha,\beta,\gamma),$$

equation (17) is equivalent to

(19) $$-da = \alpha, \quad -db = \beta, \quad -dc = \gamma.$$

This remarkable theorem will be used to prove that each C^2-solution of the minimal surface equation (7) or (11), respectively, is in fact real analytic, and that it can be mapped conformally onto a planar domain provided that its domain of definition Ω is convex. This will be shown in the next section.

We finally note that nonparametric surfaces, besides being interesting in their own right, serve as useful tools for deriving identities between differential invariants of general surfaces. In fact, locally each regular C^2-surface $X(u, v)$ can, after a suitable rotation of the Cartesian coordinate system in $\mathbb{R}^3$, be written in the nonparametric form stated before. In other words, by a suitable coordinate transformation $w = \varphi(x, y)$ we can pass from $X(w)$ to a strictly equivalent surface $Z(x, y) = X(\varphi(x, y))$ which is of type $(x, y, z(x, y))$ if we have chosen appropriate Cartesian coordinates in $\mathbb{R}^2$. It is evident that for such a representation $Z(x, y)$ many differential expressions have a fairly simple form, and therefore it will be much easier than in the general case to recognize identities. Switching back to the original representation $X(u, v)$, these identities are equally well established provided that the terms involved are known to be invariant with respect to parameter changes.

2.3 Conformal Representation and Analyticity of Nonparametric Minimal Surfaces

Let $X(x, y) = (x, y, z(x, y))$ be a nonparametric minimal surface of class C^2 defined on an open convex set Ω of $\mathbb{R}^2$. We will show that $z(x, y)$ is real analytic and that $X(x, y)$ can be mapped conformally onto some planar domain.

By the Theorem 1 of Section 2.2, there exists a function $a \in C^2(\Omega)$ such that

(1) $$da = \frac{pq}{\mathcal{W}}\,dx + \frac{1+q^2}{\mathcal{W}}\,dy,$$

where $p = z_x, q = z_y$, and $\mathcal{W} = \sqrt{1 + p^2 + q^2}$.

Then we consider the mapping $\varphi : \Omega \to \mathbb{R}^2$ defined by $\varphi(x, y) = (x, a(x, y))$ which can be expressed by $(x, y) \mapsto (u, v)$ or by the pair of equations

(2) $$u = x, \quad v = a(x, y).$$

Since $a_y = \mathcal{W}^{-1} \cdot (1 + q^2) > 0$ and Ω is convex, the mapping φ is one-to-one, and its Jacobian J_φ satisfies

$$J_\varphi = \frac{\partial(u, v)}{\partial(x, y)} = a_y > 0.$$

Hence φ is a C^2-diffeomorphism which maps Ω onto some domain Ω^* of $\mathbb{R}^2$. Its inverse $\psi : \Omega^* \to \Omega$ of class C^2 is given by

(3) $$x = u, \quad y = f(u, v)$$

with some function $f \in C^2(\Omega^*)$. Since $D\psi(u,v) = [D\varphi(x,y)]^{-1}$, we then obtain

$$\begin{pmatrix} 1 & 0 \\ f_u & f_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_x & a_y \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -a_x/a_y & 1/a_y \end{pmatrix}.$$

On account of (1), we infer that

$$f_u = -\frac{pq}{1+q^2}, \quad f_v = \frac{\mathcal{W}}{1+q^2}, \tag{4}$$

where the arguments u, v and x, y in (4) are related to each other by (3). Next, we transform the function $z(x,y)$ to the new variables u, v and set

$$g(u,v) := z(u, f(u,v)) \tag{5}$$

and

$$Z(u,v) := (u, f(u,v), g(u,v)) = X(\psi(u,v)). \tag{6}$$

Then the differentials dx, dy, $dz = p\,dx + q\,dy$ of the functions $x = u$, $y = f(u,v)$, $z = g(u,v)$ turn out to be

$$\begin{aligned} dx &= du, \\ dy &= df = -\frac{pq}{1+q^2}\,du + \frac{\mathcal{W}}{1+q^2}\,dv, \\ dz &= dg = \frac{p}{1+q^2}\,du + \frac{q\mathcal{W}}{1+q^2}\,dv. \end{aligned} \tag{7}$$

These equations yield the conformality relations

$$|Z_u|^2 = |Z_v|^2 = \frac{1+p^2+q^2}{1+q^2}, \quad \langle Z_u, Z_v \rangle = 0 \tag{8}$$

for the surface $Z = X \circ \psi$ which is strictly equivalent to the nonparametric surface $X(x,y)$.

For the following, we use the two other equations of Section 2.2, (19):

$$-db = \beta, \quad -dc = \gamma,$$

which state that

$$\begin{aligned} db &= -\frac{1+p^2}{\mathcal{W}}\,dx - \frac{pq}{\mathcal{W}}\,dy \\ dc &= -\frac{q}{\mathcal{W}}\,dx + \frac{p}{\mathcal{W}}\,dy. \end{aligned} \tag{9}$$

We introduce a surface

$$Z^*(u,v) = (v, f^*(u,v), g^*(u,v)) \tag{10}$$

for $(u,v) \in \Omega^*$, the components of which are defined by

$$\begin{aligned} f^*(u,v) &:= b(u, f(u,v)), \\ g^*(u,v) &:= c(u, f(u,v)). \end{aligned} \tag{11}$$

It follows from (4) and (9) that

$$\begin{aligned} df^* &= -\frac{\mathcal{W}}{1+q^2}\,du - \frac{pq}{1+q^2}\,dv, \\ dg^* &= -\frac{q\mathcal{W}}{1+q^2}\,du + \frac{p}{1+q^2}\,dv. \end{aligned} \tag{12}$$

Comparing (7) and (12), we see that Z and Z^* satisfy the Cauchy–Riemann equations

$$Z_u = Z^*_v, \quad Z_v = -Z^*_u \tag{13}$$

on Ω^*, which are equivalent to

$$\begin{aligned} f_u = f^*_v, &\quad f_v = -f^*_u, \\ g_u = g^*_v, &\quad g_v = -g^*_u. \end{aligned} \tag{14}$$

Thus $f + if^*$ and $g + ig^*$ are holomorphic functions of the variable $w = u + iv$, and their real and imaginary parts f, g and f^*, g^*, respectively, are harmonic and therefore real analytic functions on Ω^*. It follows from (3) that $\psi : \Omega^* \to \Omega$ is real analytic, and then the same holds for the inverse mapping $\varphi : \Omega \to \Omega^*$. On the other hand, we infer from (5) that

$$z(x,y) = g(\varphi(x,y)) = g(x, a(x,y)), \tag{15}$$

whence $z(x,y)$ is seen to be real analytic on Ω.

Let us collect the results that are so far proved.

Theorem 1. *If $z \in C^2(\Omega)$ is a solution of the minimal surface equation* (7) *or* (11) *of Section* 2.2 *in the domain Ω of $\mathbb{R}^2$, then z is real analytic.*

Remark. Although we have proved this result only for convex domains, the general statement holds as well because we have only to show that z is real analytic on every ball $B_r(c)$ contained in Ω, and this has been proved.

Theorem 2. *Let $X(x,y) = (x, y, z(x,y))$ be a nonparametric minimal surface of class C^2 defined on some convex domain Ω of $\mathbb{R}^2$. Then there exists a real analytic diffeomorphism $\varphi : \Omega \to \Omega^*$ of Ω onto some simply connected domain*

Ω^*, with a real analytic inverse $\psi : \Omega^* \to \Omega$, such that $Z(u,v) = X(\psi(u,v))$ satisfies the conformality conditions

$$|Z_u|^2 = |Z_v|^2, \quad \langle Z_u, Z_v \rangle = 0.$$

The diffeomorphism φ can be chosen as

$$u = x, \quad v = a(x,y)$$

where $a(x,y)$ is a real analytic function which satisfies

$$a_x = \frac{pq}{\mathcal{W}}, \quad a_y = \frac{1+q^2}{\mathcal{W}}$$

with $p = z_x$, $q = z_y$, $\mathcal{W} = \sqrt{1+p^2+q^2}$. Its inverse ψ is described by

$$x = u, \quad y = f(u,v)$$

where f is a solution of

$$v = a(u, f(u,v)).$$

Finally, there is a surface $X^* = (a,b,c)$ on Ω which satisfies

$$dX^* = -N \wedge dX$$

where N denotes the spherical image of X, and the mapping $\Phi : \Omega^* \to \mathbb{C}^3$ defined by

$$\begin{aligned} \Phi(u+iv) &= Z(u,v) + iZ^*(u,v) \\ &:= X(u, f(u,v)) + iX^*(u, f(u,v)) \end{aligned}$$

is a holomorphic function of the complex variable $w = u + iv$.

As we have already noted in Section 1.4, every regular surface of class $C^{1,\alpha}$ can be mapped conformally onto some plane domain, irrespective of its mean curvature and its way of definition. But the previous reasoning shows that, in the case of nonparametric minimal surfaces, it is not necessary to apply Lichtenstein's mapping theorem. For such surfaces $X(x,y) = (x,y,z(x,y))$ defined on a convex domain Ω, the conformal mapping $\psi : \Omega^* \to \Omega$ can be explicitly constructed from the function $z(x,y)$. Moreover, if we introduce the line integral

$$X^*(x,y) := -\int_{(x_0,y_0)}^{(x,y)} N \wedge dX$$

for some $(x_0,y_0) \in \Omega$, we have the additional feature that $\Phi = (X + iX^*) \circ \psi$ is a holomorphic map $\Omega^* \to \mathbb{C}^3$.

Let us conclude this section with a geometric observation made by Riemann and Beltrami. By the transformation (2) we have introduced conformal parameters u, v on a given nonparametric minimal surface $X(x, y)$, in such a way that the coordinate lines $u = \text{const}$ are planar curves that can be generated by intersecting the given surface by the family of parallel planes $x = \text{const}$.

Conversely, if a regular minimal surface X is intersected by a family of parallel planes P none of which is tangent to the given surface, and if each point of X is met by some P, then the intersection lines of these planes with the minimal surface form a family of curves on the surface which locally belong to a net of conformal parameters u, v on the surface.

In fact, picking any sufficiently small piece of X, we can introduce Cartesian coordinates of $\mathbb{R}^3$ in such a way that the planes P are given as coordinate planes $x = \text{const}$, and that this piece can be written as a nonparametric surface $(x, y, z(x, y))$ over some domain Ω contained in the plane $z = 0$. Then the assertion follows from the previous result.

2.4 Bernstein's Theorem

In this section we want to prove Bernstein's celebrated theorem that every solution of the minimal surface equation defined on the whole plane must be an affine linear function.

To this end we consider an arbitrary nonparametric minimal surface $X(x^1, x^2) = (x^1, x^2, z(x^1, x^2))$ defined on a convex domain Ω of $\mathbb{R}^2$. Its height function $z(x^1, x^2)$ which is supposed to be of class C^2 on Ω will then automatically be real analytic. The coefficients of the first fundamental form of X are given by $g_{\alpha\beta} = \delta_{\alpha\beta} + z_{,\alpha} \cdot z_{,\beta}$. Let $\mathcal{W}^2 = g = \det(g_{\alpha\beta})$, and set

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta}/\mathcal{W}. \tag{1}$$

We have $\det(\bar{g}_{\alpha\beta}) = 1$ and

$$(\bar{g}^{\alpha\beta}) := (\bar{g}_{\alpha\beta})^{-1} = \begin{pmatrix} \bar{g}_{22} & -\bar{g}_{12} \\ -\bar{g}_{21} & \bar{g}_{11} \end{pmatrix}.$$

Since $z(x^1, x^2)$ is a solution of the minimal surface equation, there exist real analytic functions $\tau^\alpha(x^1, x^2), \alpha = 1, 2$, on Ω such that

$$d\tau^\alpha = \bar{g}_{\alpha\beta} dx^\beta, \quad \alpha = 1, 2. \tag{2}$$

(This follows from the equations (14) and (19) of Section 2.2, setting $\tau^1 = -b$ and $\tau^2 = a$.) We use these functions to define a real analytic mapping $\psi : \Omega \to \mathbb{R}^2$ by setting $\sigma = \psi(x) := x + \tau(x)$ or, in components,

$$\begin{aligned} \sigma^1 &= x^1 + \tau^1(x^1, x^2), \\ \sigma^2 &= x^2 + \tau^2(x^1, x^2). \end{aligned} \tag{3}$$

Since $B = D\tau = (\tau^\alpha_{,\beta}) = (\bar{g}_{\alpha\beta})$, the matrix B is symmetric and positive definite and we infer that, for arbitrary $x = (x^1, x^2)$ and $y = (y^1, y^2) \in \Omega$,

$$\langle x - y, \tau(x) - \tau(y) \rangle \geq 0.$$

Then it follows that

$$\begin{aligned} |\psi(x) - \psi(y)|^2 &= |x-y|^2 + |\tau(x) - \tau(y)|^2 + 2\langle x - y, \tau(x) - \tau(y)\rangle \\ &\geq |x-y|^2 \end{aligned}$$

or

$$|\psi(x) - \psi(y)| \geq |x - y|. \tag{4}$$

Therefore ψ maps Ω in a 1–1 way onto $\Omega^* := \psi(\Omega)$. Moreover,

$$\begin{aligned} \rho &:= \det\left(\frac{\partial \psi^\alpha}{\partial x^\beta}\right) = 2 + \bar{g}_{11} + \bar{g}_{22} \\ &= 2 + \mathcal{W} + 1/\mathcal{W} \geq 2, \end{aligned} \tag{5}$$

and thus $\psi : \Omega \to \Omega^*$ is a diffeomorphism. Now we define a second mapping $h(\sigma) = (h^1(\sigma), h^2(\sigma))$ for $\sigma \in \Omega^*$ by

$$\begin{aligned} h^1(\sigma) &= x^1 - \tau^1(x) \\ h^2(\sigma) &= -x^2 + \tau^2(x) \end{aligned} \quad \text{where } \sigma = \psi(x). \tag{6}$$

From the chain rule and from

$$\begin{aligned} \left(\frac{\partial \psi^\alpha}{\partial x^\beta}\right)^{-1} &= \begin{pmatrix} 1 + \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & 1 + \bar{g}_{22} \end{pmatrix}^{-1} \\ &= \frac{1}{2 + \mathcal{W} + 1/\mathcal{W}} \begin{pmatrix} 1 + \bar{g}_{22} & -\bar{g}_{12} \\ -\bar{g}_{21} & 1 + \bar{g}_{11} \end{pmatrix} \end{aligned} \tag{6'}$$

it follows that the derivative $Dh(\sigma)$ of $h(\sigma)$ is given by

$$\left(\frac{\partial h^\alpha}{\partial \sigma^\beta}\right) = \frac{1}{2 + \mathcal{W} + 1/\mathcal{W}} \begin{pmatrix} \bar{g}_{22} - \bar{g}_{11} & -2\bar{g}_{12} \\ 2\bar{g}_{21} & \bar{g}_{22} - \bar{g}_{11} \end{pmatrix} \circ \psi^{-1} \tag{7}$$

or

$$\left(\frac{\partial h^\alpha}{\partial \sigma^\beta}\right) = \frac{1}{(\mathcal{W} + 1)^2} \begin{pmatrix} g_{22} - g_{11} & -2g_{12} \\ 2g_{21} & g_{22} - g_{11} \end{pmatrix} \circ \psi^{-1}. \tag{8}$$

This shows that

$$H(\sigma) := h^1(\sigma) + ih^2(\sigma) \tag{9}$$

is a holomorphic function of $\sigma = \sigma^1 + i\sigma^2$ in Ω^* with the complex derivative

$$H'(\sigma) = \frac{q^2 - p^2 + 2ipq}{(\mathcal{W}+1)^2} = \left(\frac{ip+q}{1+\mathcal{W}}\right)^2 \tag{10}$$

where in the expressions $p = z_{,1}$, $q = z_{,2}$, and $\mathcal{W} = \sqrt{1+p^2+q^2}$ on the right-hand side one has to replace x by $\psi^{-1}(\sigma)$. We finally note that

$$|H'(\sigma)| = \frac{p^2+q^2}{(1+\mathcal{W})^{\ 2}} < \left(\frac{\mathcal{W}}{1+\mathcal{W}}\right)^2 < 1. \tag{11}$$

The image $\Omega^* = \psi(\Omega)$ of the convex set Ω clearly is a simply connected domain. If Ω is the whole plane $\mathbb{R}^2 \hat{=} \mathbb{C}$, then one can infer from (4) that also $\Omega^* = \mathbb{C}$. Then, by Liouville's theorem and by (11), the entire function $H'(\sigma)$ must be constant. Thus, for $\mu := p/(1+\mathcal{W})$, $\nu := q/(1+\mathcal{W}\ )$, we infer that

$$\mu^2 - \nu^2 = c_1, \quad 2\mu\nu = c_2$$

for appropriate constants c_1 and c_2, whence

$$\mu^2 + \nu^2 = \sqrt{c_1^2 + c_2^2}.$$

This shows that the continuous functions μ and ν must be constant, and that there exists a constant $c \geq 0$ such that

$$p^2 + q^2 = c(1 + \sqrt{1+p^2+q^2})^2$$

which implies $p^2 + q^2 = \text{const}$, and therefore

$$p = \alpha_1 \quad \text{and} \quad q = \alpha_2$$

for some numbers α_1 and α_2, that is

$$z(x^1, x^2) = \alpha_0 + \alpha_1 x^1 + \alpha_2 x^2. \tag{12}$$

Thus a nonparametric minimal surface $X(x^1, x^2)$ which is defined on all of $\mathbb{R}^2$ has to be a plane. But this is the assertion of *Bernstein's theorem* from 1916 which we will state as

Theorem 1. *Every C^2-solution of the minimal surface equation on $\mathbb{R}^2$ has to be an affine linear function.*

In order to exploit the previous formulas more thoroughly, we introduce the function

$$F(\sigma) := \frac{p}{1+\mathcal{W}} - i\frac{q}{1+\mathcal{W}} \tag{13}$$

of $\sigma = \sigma^1 + i\sigma^2$. Here and in the sequel, x has to be replaced by $\psi^{-1}(\sigma)$ so that, as in (10), (11), and (13), the right-hand sides are to be understood as functions of σ. (We omit to write the composition by ψ^{-1}, because the formulas would then become rather cumbersome. For instance formula (13) should correctly have been written as

$$F := \left(\frac{p - iq}{1 + \mathcal{W}}\right) \circ \psi^{-1}.$$

We think that the reader will have no difficulties with our sloppy but more suggestive notation.)

Comparing (10) with (13), we see that

$$H' = (iF)^2. \tag{14}$$

Since H is holomorphic on Ω^*, we infer that also F is a holomorphic function. Furthermore,

$$|H'| = |F|^2 = \frac{p^2 + q^2}{(1 + \mathcal{W}\ )^2} = \frac{\mathcal{W}\ ^2 - 1}{(1 + \mathcal{W}\ )^2} = \frac{\mathcal{W} - 1}{\mathcal{W} + 1} \tag{15}$$

whence

$$1 + |F|^2 = \frac{2\mathcal{W}}{\mathcal{W} + 1}$$

and

$$\Lambda := \left(\frac{\mathcal{W}}{\mathcal{W} + 1}\right)^2 = \frac{1}{4}[1 + |F|^2]^2. \tag{16}$$

Let $\gamma_{\mu\nu}(\sigma)$ be the coefficients of the first fundamental form of $Z := X \circ \psi^{-1}$. By the chain rule, we have

$$\gamma_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \sigma^\mu} \frac{\partial x^\beta}{\partial \sigma^\nu} = \mathcal{W} \bar{g}_{\alpha\beta} \frac{\partial x^\alpha}{\partial \sigma^\mu} \frac{\partial x^\beta}{\partial \sigma^\nu}.$$

By (5) and (6′), we obtain

$$\left(\frac{\partial x^\alpha}{\partial \sigma^\mu}\right) = \frac{1}{\rho}(\delta^{\alpha\mu} + \bar{g}^{\alpha\mu})$$

and therefore

$$\begin{aligned} \gamma_{\mu\nu} &= \frac{\mathcal{W}}{\rho^2}(\bar{g}_{\mu\beta} + \delta_{\mu\beta})(\delta^{\beta\nu} + \bar{g}^{\beta\nu}) \\ &= \frac{\mathcal{W}}{\rho^2}(\bar{g}_{\mu\nu} + \delta_{\mu\nu} + \delta^{\mu\nu} + \bar{g}^{\mu\nu}). \end{aligned}$$

On account of (5), we arrive at

$$(\gamma_{\mu\nu}) = \frac{\mathcal{W}}{\rho^2}\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} = \frac{\mathcal{W}}{\rho}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho = 2 + \mathcal{W} + 1/\mathcal{W} = \frac{(\mathcal{W}+1)^2}{\mathcal{W}}, \quad \text{or} \quad \frac{\mathcal{W}}{\rho} = \Lambda,$$

whence

$$\gamma_{\mu\nu} = \Lambda \delta_{\mu\nu}. \tag{17}$$

Hence $Z = X \circ \psi^{-1}$ is represented by conformal parameters. By virtue of the theorema egregium (cf. Section 1.3, (32)), the Gauss curvature $\mathcal{K}(\sigma)$ of $Z(\sigma)$ is given by

$$\mathcal{K} = -\frac{1}{2\Lambda} \Delta \log \Lambda$$

or, equivalently,

$$\mathcal{K} = -\frac{\Lambda \Delta \Lambda - |\nabla \Lambda|^2}{2\Lambda^3}. \tag{18}$$

To simplify the computations, we set $\alpha = \operatorname{Re} F, \beta = \operatorname{Im} F$. Then it follows that

$$\begin{aligned}
&F = \alpha + i\beta, \quad |F|^2 = \alpha^2 + \beta^2, \\
&\Lambda = \tfrac{1}{4}\{1 + \alpha^2 + \beta^2\}^2, \\
&\alpha_{\sigma^1} = \beta_{\sigma^2}, \quad \alpha_{\sigma^2} = -\beta_{\sigma^1}, \quad \Delta\alpha = 0, \quad \Delta\beta = 0.
\end{aligned}$$

From these formulas, we derive

$$\begin{aligned}
\Lambda_{\sigma^1}^2 + \Lambda_{\sigma^2}^2 &= \{1 + \alpha^2 + \beta^2\}^2 [(\alpha\alpha_{\sigma^1} + \beta\beta_{\sigma^1})^2 + (\alpha\alpha_{\sigma^2} + \beta\beta_{\sigma^2})^2] \\
&= \{1 + \alpha^2 + \beta^2\}^2 [\alpha^2(\alpha_{\sigma^1}^2 + \alpha_{\sigma^2}^2) + \beta^2(\beta_{\sigma^1}^2 + \beta_{\sigma^2}^2)] \\
&= 4\Lambda |F|^2 |F'|^2
\end{aligned}$$

and

$$\begin{aligned}
\Delta\Lambda &= 2\sum_{\nu=1}^{2} (\alpha\alpha_{\sigma^\nu} + \beta\beta_{\sigma^\nu})^2 + \{1 + \alpha^2 + \beta^2\}(|\nabla\alpha|^2 + |\nabla\beta|^2) \\
&= 2|F|^2|F'|^2 + 2\{1 + |F|^2\}|F'|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
|\nabla\Lambda|^2 &= 4\Lambda |F|^2 |F'|^2, \\
\Delta\Lambda &= 2(1 + 2|F|^2)|F'|^2.
\end{aligned} \tag{19}$$

By inserting these relations in (18), we arrive at the important equation

$$\mathcal{K} = -|F'|^2/\Lambda^2 \tag{20}$$

which, on account of (16), can also be written as

$$\mathcal{K} = -|F'|^2 \left(1 + \frac{1}{\mathcal{W}}\right)^4. \tag{20'}$$

Fix now some disk $B_r(x_0) \subset\subset \Omega$, and set $\sigma_0 := \psi(x_0)$. It follows from (4) that $B_r(\sigma_0) \subset\subset \Omega^*$.

Next we set $c := F(\sigma_0)$. By (11) or (15), we have $|c| < 1$. Thus

$$R(\sigma) := \frac{\sigma - c}{1 - \bar{c}\sigma}$$

defines a conformal mapping of the unit disk $B = B_1(0)$ onto itself which satisfies

$$R(c) = 0 \quad \text{and} \quad R'(c) = \frac{1}{1 - |c|^2} > 1.$$

Secondly we consider the linear mapping

$$L(\sigma) := \sigma_0 + r\sigma$$

of B onto $B_r(\sigma_0)$ that fulfills

$$L(0) = \sigma_0 \quad \text{and} \quad L'(0) = r.$$

Then the composition

$$M := R \circ F \circ L,$$

which can also be described by

$$M(\sigma) = \frac{F(\sigma_0 + r\sigma) - c}{1 - \bar{c}F(\sigma_0 + r\sigma)},$$

is a holomorphic mapping of B into itself since $|F| < 1$, and $M(0) = 0$. On account of Schwarz's lemma it follows that $|M'(0)| \le 1$.

Since $M'(0) = R'(c)F'(\sigma_0)r$ and $R'(c) > 1$, we obtain

$$|F'(\sigma_0)| \le 1/r,$$

and we infer from (20′) that

$$|\mathcal{K}(\sigma_0)| \le \frac{1}{r^2}\left(1 + \frac{1}{\mathcal{W}(x_0)}\right)^4 \le \frac{1}{r^2}(1+1)^4 = \frac{16}{r^2}.$$

The Gauss curvatures K and $\mathcal{K}$ of X and Z, respectively, are related to each other by

$$\mathcal{K} = K \circ \psi^{-1}.$$

Thus we have proved

$$|K(x_0)| \le \frac{16}{r^2}, \tag{21}$$

and we can formulate the following assertion:

Theorem 2. *If a disk of center x_0 and radius r is contained in the domain of definition of a nonparametric minimal surface $X(x^1, x^2) = (x^1, x^2, z(x^1, x^2))$, then its Gauss curvature in x_0 can be estimated by*

$$|K(x_0)| \leq \frac{16}{r^2}. \tag{22}$$

This result, which is due to E. Heinz [1], can be considered as a quantitative and local version of Bernstein's theorem that follows from Theorem 2 if we let $r \to \infty$.

2.5 Two Characterizations of Minimal Surfaces

We shall prove two results that more or less were already established in Section 2.2. Yet the formulas to be developed here will shed light on the problem from a different angle.

Theorem 1. *If $X : \Omega \to \mathbb{R}^3$ is a regular surface of class C^2 with mean curvature H and with the spherical map $N : \Omega \to \mathbb{R}^3$, then*

$$\Delta_X X = 2HN, \tag{1}$$

where Δ_X denotes the Laplace–Beltrami operator on the surface X.

This implies the following characterization of minimal surfaces:

Corollary 1. *A regular C^2-surface X is a minimal surface if and only if*

$$\Delta_X X = 0 \tag{2}$$

holds.

Suppose now that the parameters u, v of $X(u, v)$ are conformal. Then $\mathcal{W} = \mathcal{E} = \mathcal{G}$, and Section 1.5, (17) implies that

$$\Delta_X = \frac{1}{\mathcal{W}} \Delta \tag{3}$$

where Δ denotes the ordinary Laplace operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$. Moreover, we have

$$\mathcal{W}N = X_u \wedge X_v$$

and therefore

Corollary 2. *If $X(u, v)$ is a regular C^2-surface represented by conformal parameters, then*

$$\Delta X = 2H X_u \wedge X_v. \tag{4}$$

In particular, X is a minimal surface if and only if

$$\Delta X = 0 \tag{5}$$

holds.

Proof of Theorem 1. It obviously suffices to establish (1) in a sufficiently small neighborhood Ω' of every point of Ω. Moreover, because of the invariant character of the expressions on both sides of (1), we only have to verify the assertion for some surface that is strictly equivalent to $X|_\Omega$. Since every regular C^2-surface is locally equivalent (in a strict way) to some nonparametric surface we have convinced ourselves that it suffices to prove (1) for an arbitrary nonparametric surface

$$X(x^1, x^2) = (x^1, x^2, z(x^1, x^2)), \quad (x^1, x^2) \in \Omega.$$

Set as usual,

$$p = z_{,1}, \quad q = z_{,2}, \quad \mathcal{W} = \sqrt{1 + p^2 + q^2}.$$

Then the Gauss equations

$$X_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma} + b_{\alpha\beta} N$$

of Section 1.3 take the form

$$\begin{pmatrix} 0 \\ 0 \\ z_{,\alpha\beta} \end{pmatrix} = \Gamma^1_{\alpha\beta} \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} + \Gamma^2_{\alpha\beta} \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} + b_{\alpha\beta} \begin{pmatrix} -p/\mathcal{W} \\ -q/\mathcal{W} \\ 1/\mathcal{W} \end{pmatrix}$$

whence

$$\begin{aligned} &\Gamma^1_{\alpha\beta} = b_{\alpha\beta} \frac{p}{\mathcal{W}}, \quad \Gamma^2_{\alpha\beta} = b_{\alpha\beta} \frac{q}{\mathcal{W}}, \\ &z_{,\alpha\beta} = \Gamma^1_{\alpha\beta} p + \Gamma^2_{\alpha\beta} q + \frac{b_{\alpha\beta}}{\mathcal{W}}, \end{aligned} \tag{6}$$

and this implies

$$z_{,\alpha\beta} = b_{\alpha\beta} \mathcal{W}. \tag{7}$$

Since

$$2H = b_{\alpha\beta} g^{\alpha\beta}$$

(see (42) of Section 1.2), it follows that

$$g^{\alpha\beta} \Gamma^1_{\alpha\beta} = \frac{2H}{\mathcal{W}} p, \quad g^{\alpha\beta} \Gamma^2_{\alpha\beta} = \frac{2H}{\mathcal{W}} q, \quad g^{\alpha\beta} z_{,\alpha\beta} = 2H\mathcal{W}. \tag{8}$$

On account of Section 1.5, (19), we have

$$\Delta_X f = g^{\alpha\beta} [f_{,\alpha\beta} - \Gamma^\gamma_{\alpha\beta} f_{,\gamma}]$$

for an arbitrary function $f \in C^2(\Omega)$. Then, by virtue of (8),

$$\Delta_X f = g^{\alpha\beta} f_{,\alpha\beta} - \frac{2H}{\mathcal{W}} z_{,\gamma} f_{,\gamma}. \tag{9}$$

Specializing this formula to the functions $f(x^1, x^2) = x^1$, x^2, and $z(x^1, x^2)$ respectively, we obtain

$$(10) \qquad \Delta_X x^1 = 2H\left(-\frac{p}{\mathcal{W}}\right), \quad \Delta_X x^2 = 2H\left(-\frac{q}{\mathcal{W}}\right), \quad \Delta_X z = 2H\left(\frac{1}{\mathcal{W}}\right),$$

and this is equivalent to

$$\Delta_X X = 2HN.$$

This completes the proof of Theorem 1. □

The following result relates the Beltrami operators Δ_X and Δ_N of a minimal surface X and its Gauss map N to each other.

Proposition. *If N is the Gauss map of a minimal surface X, then*

$$(11) \qquad \Delta_X = |K|\Delta_N.$$

Proof. Since $\mathrm{I}_N = \mathrm{III}_X$ and (Section 1.2, (26))

$$K\mathrm{I}_X - 2H\mathrm{II}_X + \mathrm{III}_X = 0,$$

it follows from $H = 0$ that $K \le 0$ and

$$(12) \qquad \mathrm{I}_N = -K\mathrm{I}_X = |K|\mathrm{I}_X.$$

Hence, if X is represented conformally, then the same holds for N, and we infer from (12) and from relation (17) of Section 1.5 that

$$\Delta_X = \frac{1}{\mathcal{W}}\,\Delta, \quad \Delta_N = \frac{1}{|K|\mathcal{W}}\,\Delta$$

whence

$$\Delta_X = |K|\Delta_N.$$

Since both sides are invariant expressions with respect to parameter changes, we conclude on account of Section 2.3, Theorem 2, the general validity of (11). □

Now we turn to the second characterization of minimal surface which follows from

Theorem 2. *Let $X(u,v)$ be a regular surface of class C^2 defined on some domain Ω of $\mathbb{R}^2$, and let $N : \Omega \to \mathbb{R}^3$ be its Gauss map. Then*

$$(13) \qquad N_v \wedge X_u - N_u \wedge X_v = 2H\mathcal{W}N$$

and

$$(14) \qquad (N \wedge X_u)_v - (N \wedge X_v)_u = 2H\mathcal{W}N.$$

Proof. Set $u^1 = u$ and $u^2 = v$. Then the Weingarten equations

$$N_{,\alpha} = -b_\alpha^\beta X_{,\beta} = -g^{\gamma\beta} b_{\alpha\gamma} X_{,\beta}$$

imply

$$\begin{aligned} N_{,2} \wedge X_{,1} &= -g^{\gamma 2} b_{2\gamma} X_{,2} \wedge X_{,1}, \\ N_{,1} \wedge X_{,2} &= -g^{\gamma 1} b_{1\gamma} X_{,1} \wedge X_{,2} \end{aligned}$$

whence

$$N_{,2} \wedge X_{,1} - N_{,1} \wedge X_{,2} = \{g^{\gamma 1} b_{1\gamma} + g^{\gamma 2} b_{2\gamma}\} X_{,1} \wedge X_{,2}.$$

Since

$$g^{\gamma\alpha} b_{\alpha\gamma} = 2H, \quad X_u \wedge X_v = \mathcal{W} N,$$

the relation (13) is established, and (14) is a direct consequence of (13). □

Because of

$$N \wedge dX = N \wedge X_u \, du + N \wedge X_v \, dv$$

equation (14) is equivalent to

$$d(N \wedge dX) = -2H\mathcal{W}N \, du \, dv = -2HN \, dA. \tag{15}$$

This implies

Corollary 3. *A regular C^2-surface $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if the differential form $N \wedge dX$ is closed, that is,*

$$d(N \wedge dX) = 0. \tag{16}$$

If Ω is a simply connected domain, condition (16) is equivalent to the statement that

$$\Psi(u, v) := \int_{(u_0, v_0)}^{(u,v)} N \wedge dX$$

is a path-independent line integral.

Remark. Since formula (16) is invariant with respect to parameter changes and has only to be proved locally, it follows as well from Section 2.2, (17).

2.6 Parametric Surfaces in Conformal Parameters. Conformal Representation of Minimal Surfaces. General Definition of Minimal Surfaces

Now we will provide another proof of the result stated in Corollary 2 of Section 2.5 which is particularly simple because it uses only a minimum of differential geometric formulas.

Theorem 1. *Let $X(u,v)$ be a regular surface of class $C^2(\Omega,\mathbb{R}^3)$ given by conformal parameters u and v, that is,*

$$|X_u|^2 = |X_v|^2 \quad \textit{and} \quad \langle X_u, X_v\rangle = 0. \tag{1}$$

Then necessary and sufficient for a real-valued function $H(u,v)$ to represent the mean curvature of the surface X is that Rellich's equation

$$\Delta X = 2H X_u \wedge X_v \tag{2}$$

holds in Ω.

In particular, $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if

$$\Delta X = 0. \tag{3}$$

Proof. The equation (1) can be written as

$$\Lambda := \mathcal{E} = \mathcal{G} = \mathcal{W}, \quad \mathcal{F} = 0 \quad \text{in } \Omega.$$

According to Section 1.3, (31), the mean curvature is simply

$$H = \frac{1}{2\Lambda}(\mathcal{L} + \mathcal{N}).$$

Recalling that $\mathcal{L} = \langle X_{uu}, N\rangle$, $\mathcal{N} = \langle X_{vv}, N\rangle$, we obtain

$$\langle \Delta X, N\rangle = \langle X_{uu} + X_{vv}, N\rangle = 2\Lambda H. \tag{4}$$

On the other hand, differentiating (1) with respect to u and v yields

$$\begin{aligned}\langle X_u, X_{uu}\rangle &= \langle X_v, X_{vu}\rangle, \quad \langle X_{uu}, X_v\rangle + \langle X_u, X_{uv}\rangle = 0,\\ \langle X_v, X_{vv}\rangle &= \langle X_u, X_{uv}\rangle, \quad \langle X_{vv}, X_u\rangle + \langle X_v, X_{uv}\rangle = 0\end{aligned}$$

and therefore

$$\langle \Delta X, X_u\rangle = 0 \quad \text{and} \quad \langle \Delta X, X_v\rangle = 0. \tag{5}$$

In other words, ΔX is proportional to N.

Since $|N| = 1$, it follows from (4) that

$$\Delta X = 2\Lambda H N,$$

and, by virtue of $\Lambda N = \mathcal{W} N = X_u \wedge X_v$, we arrive at (2), and the theorem is proved. □

The previous theorem provides another approach to the general formula

$$\Delta_X X = 2HN \tag{6}$$

proved in Theorem 1 of Section 2.5. One only has to show that an arbitrary regular surface X of class C^2 is locally strictly equivalent to a surface represented by conformal parameters. Then (6) follows from (2) by an invariance reasoning.

The possibility to introduce conformal parameters on an arbitrary regular C^2-surface is expressed in Lichtenstein's theorem, stated in Section 1.4. For minimal surfaces ($H = 0$) we have—independently of the general Lichtenstein theorem—given two different proofs that it is possible to introduce conformal parameters in the small (cf. Sections 2.3 and 2.4). Thus the equation

$$\Delta_X X = 0, \tag{7}$$

which characterizes minimal surfaces, is independently verified.

In order to transform a regular minimal surface globally to conformal parameters, one can combine Theorem 2 of Section 2.3 with the uniformization theorem proved by Koebe and Poincaré. We cannot give the proof of this celebrated theorem. Instead, in Section 4.11 we shall present a *variational proof* of Lichtenstein's theorem which is based on the solution of a Plateau-type problem. Here we merely state the global version of Theorem 2 in Section 2.3:

Theorem 2. *Every regular surface $X : \Omega \to \mathbb{R}^3$ of class C^2, whether minimal or not, is strictly equivalent to a surface represented by conformal parameters.*

Still it should be noted that the following discussion will not rest on unfortified ground since existence proofs for minimal surfaces that will be given later yield the existence of minimal surfaces represented by conformal parameters.

While the equations (6) and (7) only make sense for regular surfaces, the equations (2) and (3) can also be formulated for surfaces with $\mathcal{W} = 0$. This enables us to give a definition of minimal surfaces that includes surfaces with isolated singularities, called branch points, that will be studied in the next chapter.

Definition 1. *A nonconstant surface $X : \Omega \to \mathbb{R}^3$ of class C^2 is said to be a* **minimal surface** *if it satisfies the conformality relations*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{1}$$

as well as the equation

$$\Delta X = 0 \tag{3}$$

on Ω.

If equation (3) is replaced by (2), we speak of a surface with the mean curvature function $H(u, v)$.

By (1) and (3) we can also define minimal surfaces $X : \Omega \to \mathbb{R}^n$ in $\mathbb{R}^n$, $n \geq 2$, and most of the results in these notes will be true independently of the dimension n. For convenience we shall ordinarily restrict ourselves to the case $n = 3$, and we shall only occasionally touch minimal surfaces $X : \Omega \to M$ in an n-dimensional Riemannian manifold M. If, in local coordinates $x^1, \dots, x^n$ on M, the line element ds^2 of M has the form

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k \tag{8}$$

(summation with respect to Latin indices $i, k, \dots$ from 1 to n), then a surface $X \in C^2(\Omega, M)$ is said to be a minimal surface in M, if its local components $x^1(u, v), \dots, x^n(u, v)$ satisfy

$$g_{ik}(X)x^i_u x^k_u = g_{ik}(X)x^i_v x^k_v, \quad g_{ik}(X)x^i_u x^k_v = 0 \tag{9}$$

and

$$\Delta x^l + \Gamma^l_{ik}(X)(x^i_u x^k_u + x^i_v x^k_v) = 0 \tag{10}$$

where Γ^l_{ik} are the Christoffel symbols of second kind with respect to ds^2.

Although there will be no systematic treatment of (9) and (10) in our notes, these equations will turn up when we replace Cartesian coordinates by general curvilinear coordinates in $\mathbb{R}^3$ which will be essential for the investigation of the boundary behavior of minimal surfaces.

2.7 A Formula for the Mean Curvature

Let us consider a family $\{\mathfrak{F}_c\}_{c \in J}$ of regular C^2-surfaces $\mathfrak{F}_c$ which are embedded in $\mathbb{R}^3$, implying that none of these surfaces has selfcuttings or selftangencies. We also assume that the family depends in a C^2-way on the parameter c.

A set $\mathcal{S}$ of $\mathbb{R}^3$ is said to be *simply covered by the surfaces of the family* $\{\mathfrak{F}_c\}$ if each point $x = (x^1, x^2, x^3)$ of $\mathcal{S}$ is contained in exactly one of the surfaces.

Consider now a domain G in $\mathbb{R}^3$ whose closure $\bar{G}$ is simply covered by a family of C^2-surfaces $\mathfrak{F}_c$ in the sense that there is a function $S \in C^2(\bar{G})$ with $\nabla S(x) \neq 0$ for all $x \in \bar{G}$, such that the *leaves* $\mathfrak{F}_c$ of the *foliation* $\{\mathfrak{F}_c\}$ can be described as its level surfaces

$$\mathfrak{F}_c = \{x \in \bar{G} : S(x) = c\}. \tag{1}$$

Then

$$Q(x) := |\nabla S(x)|^{-1} \cdot \nabla S(x) \tag{2}$$

defines a field $Q \in C^1(\bar{G}, \mathbb{R}^3)$ of unit vectors that is orthogonal to all surfaces $\mathfrak{F}_c$; it is called the *normal field* of the foliation $\{\mathfrak{F}_c\}$.

Theorem 1. *If G is a domain in $\mathbb{R}^3$, and if S is a function of class $C^2(\bar{G})$ such that $\nabla S(x) \neq 0$ on $\bar{G}$, then the mean curvature $H(x)$ of the level surface*

$$\mathfrak{F}_c = \{x \in \bar{G} : S(x) = c\}$$

passing through $x \in \bar{G}$ *is given by the equation*

$$\text{(3)} \qquad \operatorname{div} Q(x) = -2H(x)$$

where $Q(x) = |\nabla S(x)|^{-1} \cdot \nabla S(x)$ *denotes the normal field of the foliation* $\{\mathfrak{F}_c\}$.

Proof. Pick some point $x_0 \in G$, some $r > 0$ with $B_r(x_0) \subset\subset G$, and let x_0 be contained in $\mathfrak{F}_{c_0}$. For $\mathfrak{F} := \bar{B}_r(x_0) \cap \mathfrak{F}_{c_0}$ we then choose a regular C^2-parametrization $X(w), w \in \bar{\Omega}$ such that its surface normal $N(w) = N_X(w)$ satisfies

$$N(w) = Q(X(w)) \quad \text{for all } w \in \bar{\Omega}.$$

We can also achieve that $x_0 = X(w_0)$ for some $w_0 \in \Omega$.

For some sufficiently small $\varepsilon_0 > 0$ we define the *normal variation*

$$Z(w, \varepsilon) = X(w) + \varepsilon N(w), \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0],$$

of the surface $\mathfrak{F}$ represented by $X(w)$. Let $\mathfrak{S}_\varepsilon$ be the surface with the parameter representation $Z(\cdot, \varepsilon)$, and denote by $\mathfrak{C}_\varepsilon$ the *collar*

$$\{X(w) + \lambda N(w) : w \in \partial\Omega, 0 \le \lambda \le \varepsilon\}.$$

The two caps $\mathfrak{F}$ and $\mathfrak{S}_\varepsilon$ together with the collar $\mathfrak{C}_\varepsilon$ bound a domain U_ε in $\mathbb{R}^3$ over which we will integrate $\operatorname{div} Q$. Performing an integration by parts, we obtain

$$\int_{U_\varepsilon} \operatorname{div} Q \, dX = \int_{\partial U_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle \, dA$$

where $\bar{N}_\varepsilon$ denotes the exterior normal of ∂U_ε. Note that

$$\bar{N}_\varepsilon = -N = -Q \quad \text{on } \mathfrak{F}.$$

By virtue of Taylor's theorem, we infer that

$$\langle Q, \bar{N}_\varepsilon \rangle = O(\varepsilon) \quad \text{on } \mathfrak{C}_\varepsilon$$

whence

$$\int_{\mathfrak{C}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle dA = O(\varepsilon^2).$$

If we apply formula (13) of Section 2.5, we obtain for $Z(w, \varepsilon) = X(w) + \varepsilon N(w)$ the relations

$$\begin{aligned} Z_u \wedge Z_v &= X_u \wedge X_v + \varepsilon\{X_u \wedge N_v + N_u \wedge X_v\} + \varepsilon^2\{N_u \wedge N_v\} \\ &= \mathcal{W}N - \varepsilon 2H\mathcal{W}N + \varepsilon^2 N_u \wedge N_v, \end{aligned}$$

and, by $N = Q \circ X$, it follows that

$$\begin{aligned}\int_{\mathfrak{S}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle dA &= \int_\Omega \langle Q(X+\varepsilon N), Z_u \wedge Z_v \rangle \, du \, dv \\ &= \int_\Omega \langle Q(X+\varepsilon N), \mathcal{W}N - \varepsilon 2H\mathcal{W}N + \varepsilon^2 N_u \wedge N_v \rangle \, du \, dv \\ &= \int_{\mathfrak{F}} dA - \varepsilon \int_{\mathfrak{F}} 2H \, dA \\ &\quad + \int_\Omega \langle Q(X+\varepsilon N) - Q(X), \mathcal{W}N - \varepsilon 2H\mathcal{W}N \rangle \, du \, dv + O(\varepsilon^2).\end{aligned}$$

The relations $|Q(x)| = 1$ and $N = Q \circ X$ imply that

$$\langle Q(X(w) + \varepsilon N(w)) - Q(X(w)), N(w) \rangle = O(\varepsilon^2).$$

Thus we obtain from the previous computation that

$$\int_{\mathfrak{S}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle \, dA = \int_{\mathfrak{F}} dA - \varepsilon \int_{\mathfrak{F}} 2H \, dA + O(\varepsilon^2).$$

Since

$$\int_{U_\varepsilon} \operatorname{div} Q \, dX = \int_{\mathfrak{S}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle \, dA - \int_{\mathfrak{F}} dA + \int_{\mathfrak{C}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle \, dA,$$

it follows that

$$\frac{1}{\varepsilon} \int_{U_\varepsilon} \operatorname{div} Q \, dX = -\int_{\mathfrak{F}} 2H \, dA + O(\varepsilon).$$

As $\varepsilon \to +0$, we arrive at the equation

$$\int_{\mathfrak{F}} \operatorname{div} Q \, dA = -2 \int_{\mathfrak{F}} H \, dA.$$

Here $\mathfrak{F}$ stands for $\mathfrak{F}_{c_0} \cap \bar{B}_r(x_0)$. Dividing both sides by $\int_{\mathfrak{F}} dA$, and letting r tend to zero, we arrive at

$$\operatorname{div} Q(x_0) = -2H(x_0).$$

Since x_0 was chosen as an arbitrary point of G, and since both sides of this equation are continuous functions on $\bar{G}$, we finally obtain

$$\operatorname{div} Q(x) = -2H(x)$$

for all $x \in \bar{G}$ which proves that theorem. □

Remark 1. With $D_i = \frac{\partial}{\partial x^i}$ and $Q = (Q_1, Q_2, Q_3)$ we can write

$$\operatorname{div} Q = D_i Q_i = D_i \left\{ \frac{S_{x^i}}{\sqrt{S_{x^k} S_{x^k}}} \right\} = \frac{S_{x^i x^i}}{\sqrt{S_{x^k} S_{x^k}}} - S_{x^i x^k} \frac{S_{x^i} S_{x^k}}{\{S_{x^l} S_{x^l}\}^{3/2}}.$$

If we introduce the Hessian

$$h_S(\xi, \eta) = S_{x^i x^k} \xi^i \xi^k$$

and the Laplacian

$$\Delta S = S_{x^i x^i} = (D_1^2 + D_2^2 + D_3^2) S$$

we can write

$$\operatorname{div} Q = \frac{1}{|\nabla S|} \left\{ \Delta S - \frac{1}{|\nabla S|^2} h_S(\nabla S, \nabla S) \right\}.$$

Thus (3) can be written as

(4) $$H = \frac{1}{2|\nabla S|} \left\{ \frac{1}{|\nabla S|^2}\ h_S(\nabla S, \nabla S) - \Delta S \right\}.$$

This and related formulas for curvature quantities can also be derived by the technique of covariant differentiation applied to manifolds which are implicitly defined. This has in detail been carried out by P. Dombrowski [1].

Remark 2. Consider the nonparametric surface which is given as graph of a function $\psi(x, y), (x, y) \in \bar{\Omega} \subset \mathbb{R}^2$. We can embed $z = \psi(x, y)$ into the family of surfaces

$$z = \psi(x, y) + c$$

which simply cover $\bar{G} := \bar{\Omega} \times \mathbb{R}$. They are the level surfaces

$$S(x, y, z) = c$$

of the function $S(x, y, z) := z - \psi(x, y)$, for which we obtain

$$Q(x, y, z) = \frac{1}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \cdot (-\psi_x, -\psi_y, 1)$$

whence

$$\operatorname{div} Q = -\left\{ \frac{\psi_x}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \right\}_x - \left\{ \frac{\psi_y}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \right\}_y .$$

Thus in this particular case equation (3) takes the form

(5) $$\operatorname{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} = 2H$$

which is equivalent to formula (5) of Section 2.2. If $H = 0$, we obtain the minimal surface equation in divergence form (see Section 2.2, (11)):

(6) $$\operatorname{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} = 0.$$

From the theorem proved above we obtain the following

Corollary 1. *If $\bar{G}$ is simply covered by a foliation of minimal surfaces $\mathfrak{F}_c$ which are the level surfaces of a function $S \in C^2(\bar{G})$ with $\nabla S(x) \neq 0$ on $\bar{G}$, then the normal field $Q = |\nabla S|^{-1} \cdot \nabla S$ of this foliation is divergence-free, that is, the equation*

$$\operatorname{div} Q = 0 \tag{7}$$

holds on $\bar{G}$.

2.8 Absolute and Relative Minima of Area

We begin with a result of the Weierstrass field theory for minimal surfaces which, in a somewhat different form, was developed by H.A. Schwarz.

Lemma 1. *Suppose that G is a domain in $\mathbb{R}^3$ and that $Q \in C^1(\bar{G}, \mathbb{R}^3)$ is a vector field on $\mathbb{R}^3$ with the properties that*

$$|Q(x)| = 1 \quad \text{and} \quad \operatorname{div} Q(x) = 0 \quad \text{in } G. \tag{1}$$

Moreover, let $\mathfrak{F}$ be a regular C^1-surface embedded in G whose surface normal $N_{\mathfrak{F}}$ coincides on $\mathfrak{F}$ with the vector field Q. Then, for every regular C^1-surface $\mathfrak{S}$ that is contained in G and has the same boundary as $\mathfrak{F}$, we have

$$\int_{\mathfrak{F}} dA \leq \int_{\mathfrak{S}} dA. \tag{2}$$

Proof. Let us first assume that the surfaces $\mathfrak{F}$ and $\mathfrak{S}$ bound a domain U whose exterior surface normal on $\mathfrak{F}$ points in the opposite direction of $Q|_{\mathfrak{F}} = N_{\mathfrak{F}}$. Then we infer from Gauss's theorem that

$$\begin{aligned} \int_U \operatorname{div} Q \, dX &= \int_{\partial U} \langle Q, N_{\partial U} \rangle \, dA \\ &= \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}} \rangle \, dA - \int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}} \rangle \, dA. \end{aligned} \tag{3}$$

Because of (1), the left hand side is vanishing, and therefore

$$\int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}} \rangle \, dA = \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}} \rangle \, dA. \tag{4}$$

On account of

$$\langle Q, N_{\mathfrak{F}} \rangle = 1$$

and of

$$\langle Q, N_{\mathfrak{S}} \rangle \leq |Q||N_{\mathfrak{S}}| = 1,$$

we obtain

$$\int_{\mathfrak{F}} dA \le \int_{\mathfrak{S}} dA. \tag{5}$$

If $\mathfrak{S}$ is a general surface as stated in the theorem, the same result holds. This can be proved in essentially the same way by applying the calculus of differential forms and the general Stokes theorem for 1-forms (see, for instance, F. Warner [1]). □

Remark. It is easy to see that the equality sign in (5) holds if and only if $\mathfrak{F}$ and $\mathfrak{S}$ are strictly equivalent.

Lemma 2. *Let the assumptions of Lemma 1 be satisfied, with the following alteration: The boundaries $\partial\mathfrak{F}$ and $\partial\mathfrak{S}$ of $\mathfrak{F}$ and $\mathfrak{S}$ are not necessarily the same but lie on a surface T which is tangent to the vector field Q (that is, $Q(x)$ is a tangent vector to T at every point $x \in T$), and are supposed to be homologous to each other:*

$$\partial\mathfrak{F} \sim \partial\mathfrak{S} \quad \text{on } T.$$

Then the inequality (5) is still satisfied.

Proof. Let us choose a surface $\mathfrak{C} \subset T$ such that $\partial\mathfrak{C} = \partial\mathfrak{S} \setminus \partial\mathfrak{F}$. Applying Gauss's theorem, we obtain

$$\int_U \operatorname{div} Q\, dX = \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}}\rangle\, dA - \int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}}\rangle\, dA + \int_{\mathfrak{C}} \langle Q, N_{\mathfrak{C}}\rangle\, dA.$$

Since

$$\int_{\mathfrak{C}} \langle Q, N_{\mathfrak{C}}\rangle\, dA = 0$$

we arrive once again at (4), from where the proof proceeds as before. □

By combining Lemma 1 or Lemma 2 with the corollary stated in Section 2.7, we obtain the following

Theorem 1. *A C^2-family of regular, embedded C^2-surfaces $\mathfrak{F}_c$ which cover a domain G simply is a family of minimal surfaces if and only if its normal field is divergence-free. Such a foliation by minimal surfaces is area minimizing in the following sense*:

(i) *Let $\mathfrak{F}$ be a piece of some of the minimal leaves $\mathfrak{F}_c$, with $\mathfrak{F} \subset\subset G$. Then we have*

$$\int_{\mathfrak{F}} dA \le \int_{\mathfrak{S}} dA \tag{6}$$

for each regular C^1-surface $\mathfrak{S}$ contained in G with $\partial\mathfrak{S} = \partial\mathfrak{F}$.

(ii) *Let T be a surface in G which, in all of its points, is tangent to the normal field of the minimal foliation, and suppose that T cuts out of each leaf $\mathfrak{F}_c$ some piece $\mathfrak{F}_c^*$ whose boundary $\partial\mathfrak{F}_c^*$ lies on T. Then we have*:

$$\int_{\mathfrak{F}_{c_1}^*} dA = \int_{\mathfrak{F}_{c_2}^*} dA \tag{7}$$

for all admissible parameter values c_1 and c_2, and secondly,

$$\int_{\mathfrak{F}_c} dA \leq \int_{\mathfrak{S}} dA \tag{8}$$

for all regular C^1-surfaces $\mathfrak{S}$ contained in G whose boundaries $\partial\mathfrak{S}$ are homologous to $\partial\mathfrak{F}_c$ on T.

The identity (7) is the minimal surface version of A. Kneser's *transversality theorem.*

The integral $\int_{\mathfrak{F}} \langle Q, N\rangle\, dA$ appearing in the previous reasoning, is the so-called *Hilbert's independent integral* associated with the area functional $\int_{\mathfrak{F}} dA$. If we express $\mathfrak{F}$ by its representation $X(u,v)$, $(u,v) \in \bar{\Omega}$, Hilbert's independent integral takes the form

$$\int_{\Omega} \langle Q(X), X_u \wedge X_v\rangle\, du\, dv. \tag{9}$$

The aforestated results can be summarized as follows:

A regular embedded minimal surface $\mathfrak{F}$ yields a relative minimum of area among all surfaces having the same boundary as $\mathfrak{F}$, if it can be embedded in a foliation (*or field*) *of minimal surfaces in the sense described before. In fact, $\mathfrak{F}$ is an absolute minimum of area among all surfaces with the same boundary which lie in the domain covered by the field.*

Not every minimal surface will have minimal area among all surfaces having the same boundary. It is, in fact, not difficult to find examples of non-minimizing surfaces of vanishing mean curvature. Yet the result just proved shows that a minimal surface yields a relative minimum of area if it can be embedded into a field of minimal surfaces. Thus we ask the question:

When can a minimal surface be embedded in a field of minimal surfaces?

An answer to this question was given by H.A. Schwarz. He proved that *each interior piece of a given regular embedded minimal surface X can be embedded in a field of minimal surfaces if the first eigenvalue of the second variation of the area functional at X is positive.*

Presently we will not prove this result, but refer to Chapter 5 of this volume and also to Volume 1 of Schwarz's collected papers [2] as well as to Chapter I, Section 6, pp. 86–110 of Nitsche's lectures [28] where several examples and further applications are discussed.

However, we shall at least derive an expression for the *second variation of area* $\delta^2 A(X,Y)$ of a regular C^2-surface $X : \bar{\Omega} \to \mathbb{R}^3$ with respect to *normal variations* $Y = \varphi N$.

Here Ω is assumed to be a bounded domain in $\mathbb{R}^2$, and φ is supposed to be of class $C^1(\bar{\Omega})$. Let

$$Z := X + \varepsilon Y, \quad Y = \varphi N. \tag{10}$$

Then

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon\varphi N_{,\alpha} + \varepsilon\varphi_{,\alpha} N,$$

whence

$$\zeta_{\alpha\beta} := \langle Z_{,\alpha}, Z_{,\beta}\rangle = \langle X_{,\alpha}, X_{,\beta}\rangle + 2\varphi\varepsilon\langle X_{,\alpha}, N_{,\beta}\rangle + \varphi^2\varepsilon^2\langle N_{,\alpha} N_{,\beta}\rangle + \varphi_{,\alpha}\varphi_{,\beta}\varepsilon^2$$

and therefore

$$\zeta_{\alpha\beta} = g_{\alpha\beta} - \varepsilon 2\varphi b_{\alpha\beta} + \varepsilon^2\{\varphi^2 c_{\alpha\beta} + \varphi_{,\alpha}\varphi_{,\beta}\}. \tag{11}$$

Then

$$\begin{aligned} \det(\zeta_{\alpha\beta}) &= \zeta_{11}\zeta_{22} - \zeta_{12}\zeta_{21} \\ &= g[1 - \varepsilon 2\varphi g^{\alpha\beta} b_{\alpha\beta} + \varepsilon^2\{g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + \varphi^2 g^{\alpha\beta} c_{\alpha\beta} + 4\varphi^2 b/g\}] + O(\varepsilon^3), \end{aligned}$$

where $g = \det(g_{\alpha\beta})$ and $b = \det(b_{\alpha\beta})$.

From

$$K\mathrm{I} - 2H\mathrm{II} + \mathrm{III} = 0$$

we infer the analogous relation for the corresponding bilinear forms whence

$$K g_{\alpha\beta} - 2H b_{\alpha\beta} + c_{\alpha\beta} = 0 \tag{12}$$

or

$$c_{\alpha\beta} = 2H b_{\alpha\beta} - K g_{\alpha\beta}. \tag{13}$$

Because of

$$g^{\alpha\beta} b_{\alpha\beta} = 2H, \quad g^{\alpha\beta} g_{\alpha\beta} = 2, \quad b = Kg$$

we infer that

$$\det(\zeta_{\alpha\beta}) = g[1 - \varepsilon 4\varphi H + \varepsilon^2\{|\nabla_X\varphi|^2 + \varphi^2(4H^2 + 2K)\}] + O(\varepsilon^3). \tag{14}$$

Moreover,

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \quad \text{for } |x| \ll 1,$$

and therefore

$$\sqrt{1 + \varepsilon\alpha + \varepsilon^2\beta} = 1 + \frac{\alpha}{2}\varepsilon + \left(\frac{\beta}{2} - \frac{\alpha^2}{8}\right)\varepsilon^2 + O(\varepsilon^3)$$

for $|\varepsilon| \ll 1$. Thus we see that

$$\sqrt{\det(\zeta_{\alpha\beta})} = \sqrt{g}[1 - \varepsilon 2\varphi H + \varepsilon^2\{\tfrac{1}{2}|\nabla_X\varphi|^2 + K\varphi^2 + 2H^2\varphi^2 - 2H^2\varphi^2\}] + O(\varepsilon^3)$$

or

$$\sqrt{\det(\zeta_{\alpha\beta})} = \sqrt{g}[1 - \varepsilon 2\varphi H + \varepsilon^2\{\tfrac{1}{2}|\nabla_X\varphi|^2 + K\varphi^2\}] + O(\varepsilon^3)\}. \tag{15}$$

From this expansion, we derive for the second variation

$$\delta^2 A_\Omega(X,Y) := \frac{d^2}{d\varepsilon^2}A(X+\varepsilon Y)\bigg|_{\varepsilon=0} \tag{16}$$

of X in the normal direction $Y = \varphi N$ the formula

$$\delta^2 A_\Omega(X,Y) = \int_\Omega \{|\nabla_X\varphi|^2 + 2K\varphi^2\}\, dA \tag{17}$$

which can be considered as a quadratic form on the Sobolev space $H_2^1(\Omega)$.

We restrict

$$J(\varphi) := \delta^2 A_\Omega(X, \varphi N) \tag{18}$$

to the Sobolev space $\mathring{H}_2^1(\Omega)$ of functions $\varphi \in H_2^1(\Omega)$ with (generalized) boundary values zero on $\partial\Omega$.

Consider the isoperimetric problem

$$J(\varphi) \to \min \quad \textit{for } \varphi \in \mathring{H}_2^1(\Omega) \quad \textit{with } \int_\Omega \varphi^2 dA = 1. \tag{19}$$

Its solution satisfies

$$\begin{aligned} -\Delta_X\varphi + 2K\varphi &= \mu\varphi \quad \textit{in } \Omega, \\ \varphi &= 0 \qquad \textit{on } \partial\Omega \end{aligned} \tag{20}$$

where μ is the smallest real number, for which a nontrivial solution φ of these two equations exists; in other words, $\mu = J(\varphi)$ is the smallest eigenvalue of the operator $-\Delta_X + 2K$ on Ω with respect to zero boundary values.

In the sequel we shall often write $\delta^2 A(X,\varphi)$ instead of $\delta^2 A(X, \varphi N)$.

2.9 Scholia

1 References to the Literature on Nonparametric Minimal Surfaces

The modern theory of the nonparametric minimal surface equation and of related equations begins with the celebrated papers of S. Bernstein [1–4] and with the work of Korn [1,2] and Müntz [1]. The central problem of interest concerning nonparametric minimal surfaces was at that time the solution of Plateau's problem. A new attack on this problem was started by Müntz [2] in 1925 which, however, proved to be faulty (see Radó [11], and also Müntz [3]). The final solution of Plateau's problem in the context of nonparametric minimal surfaces in two dimensions was achieved by Haar in his pioneering paper [3]. Important supplements were given by Radó; see Haar [5], Radó [2,8,15]. In his survey [21], Radó gave a lucid presentation of the development until 1933.

After 1945, many new and surprising results on two-dimensional nonparametric minimal surfaces were found. In particular we mention the work of Bers, Finn, Heinz, E. Hopf, Jörgens and J.C.C. Nitsche. A beautiful and very complete presentation of the whole theory of two-dimensional nonparametric minimal surfaces can be found in Nitsche's treatise [28]; for an updated version see [37]. Certain aspects of the theory based on the work of Sauvigny are presented in Chapters 5 and 7 of this volume.

Even more astounding is the development of the theory of n-dimensional nonparametric minimal surfaces which is to a large extent described in the monographs of Gilbarg and Trudinger [1], Giusti [4], and Massari and Miranda [1]. Finn's treatise [11] leads the reader into the fascinating field of free boundary problems connected with the phenomenon of capillarity.

The theory of nonparametric minimal surfaces of codimension $m > 1$ was initiated by Osserman [11]. Here many new problems arise as was shown by Lawson and Osserman [1]. Osserman proved:

Let $\mathcal{M}$ be an n-dimensional submanifold in $\mathbb{R}^{n+p}$ which is the graph of a function $f \in C^2(\Omega, \mathbb{R}^p)$, $\Omega \subset \mathbb{R}^n$. Let $\gamma_{\alpha\beta}(x) := \delta_{\alpha\beta} + f^i_{x^\alpha}(x) f^i_{x^\beta}(x)$ be the metric tensor of $\mathcal{M}$, $\gamma := \det(\gamma_{\alpha\beta})$ and $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. Then $\mathcal{M}$ is a minimal submanifold of $\mathbb{R}^{n+p}$ if and only if

$$\frac{1}{\sqrt{\gamma}} D_\beta\{\sqrt{\gamma}\gamma^{\alpha\beta} D_\alpha f^i\} = 0, \quad 1 \le i \le p, \tag{1}$$

that is, if and only if the coordinate functions f^i of the mapping f are harmonic with respect to the metric of $\mathcal{M}$. Equivalently we can write

$$\Delta_{\mathcal{M}} f = 0. \tag{2}$$

The equations (1) imply that

$$D_\alpha\{\sqrt{\gamma}\gamma^{\alpha\beta}\} = 0, \quad 1 \le \beta \le n. \tag{3}$$

Therefore the equations (1) are equivalent to the system

$$\gamma^{\alpha\beta} D_\alpha D_\beta f^i = 0, \quad 1 \le i \le p. \tag{4}$$

Morrey [4] proved that any weak solution $f \in C^1(\Omega, \mathbb{R}^p)$, $\Omega \subset \mathbb{R}^n$, of (4) is real analytic. On the other hand, Lawson and Osserman [1] found for $n = 4$, $p = 3$ an example of a Lipschitz continuous weak solution of (1) which is not of class C^1. Furthermore, if Ω is the unit ball in $\mathbb{R}^4$, they discovered a quadratic polynomial $\varphi : \partial\Omega \to \mathbb{R}^3$ which cannot be extended to a mapping

$$f \in C^0(\bar{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$$

solving (1) in Ω. Harvey and Lawson [4] later proved that the singular solution of (1) found by Lawson and Osserman is, in fact, area-minimizing with respect to its boundary values.

Moreover, Lawson and Osserman [1] pointed out that, differently from the case of codimension $p = 1$, the solutions of (1) are no longer uniquely determined by their boundary values. Even if $n = 2$ and Ω is the unit disk, there is a real analytic map $\varphi : \partial\Omega \to \mathbb{R}^2$ to which there correspond three distinct solutions u of (1) in $\bar{\Omega}$ satisfying $u|_{\partial\Omega} = \varphi$.

2 Bernstein's Theorem

Bernstein's theorem is one of the most fascinating results in the theory of nonlinear elliptic differential equations. First published in 1916, it has attracted time and again the attention of analysis since the German translation of Bernstein's paper [4] appeared in 1927. Much later, a gap was discovered in Bernstein's original proof which succeedingly was closed by E.J. Mickle [1] and E. Hopf [3].

A discussion of various ramifications and generalizations of Bernstein's theorem can be found in Osserman [5], Nitsche [28], Giusti [4], Gilbarg and Trudinger [1], Hildebrandt [14,17]. The results presented in Sections 2.2–2.5 are essentially taken from the work of Radó, Nitsche and Heinz.

We mention that for nonparametric n-dimensional minimal surfaces of codimension one Bernstein's theorem holds true if $n \le 7$, whereas Bombieri, de Giorgi, and Giusti [1] derived from the Simons cone $C = \{x \in \mathbb{R}^8 : x = (y, z),\ y, z \in \mathbb{R}^4 \text{ and } |y|^2 = |z|^2\}$ an example which shows that Bernstein's theorem becomes false if $n \ge 8$. A slight error in their reasoning was pointed out by Luckhaus who also saw how it can be removed (cf. Dierkes [5]).

Another major achievement was the paper of Schoen, Simon, and Yau [1] who proved a generalization of Heinz's estimate (22) stated in Theorem 2 of Section 2.4 to all dimensions $n \le 5$, thereby obtaining another proof for Bernstein's theorem in dimensions $n \le 5$. Improvements of this work were made by Simon [1,4]. We present some of these results in Vol. 3, Chapter 3.

A Bernstein theorem in arbitrary dimension and codimension was proved by Hildebrandt, Jost, and Widman [1]:

If $f : \mathbb{R}^n \to \mathbb{R}^p$ is an entire solution of the minimal surface system (1) (*i.e., a solution on all of* $\mathbb{R}^n$) *such that $\sqrt{\gamma(x)} \le \beta_0$ on $\mathbb{R}^n$ where β_0 is a number* *satisfying* $\beta_0 < \cos^{-m}(\pi/2\sqrt{2m})$ *and* $m := \min\{n,p\} \geq 2$*, then* f *is linear, and therefore its graph represents an affine n-plane in* $\mathbb{R}^{n+p}$.

In this theorem $\gamma(x)$ denotes the function $\det(\gamma_{\alpha\beta}(x))$ where $\gamma_{\alpha\beta}(x) = \delta_{\alpha\beta} + f^i_{x^\alpha}(x) f^i_{x^\beta}(x)$. Note that a better result holds true if $m = 1$. A related result was proved by Fischer-Colbrie [1]. In Vol. 3 we present a fairly comprehensive presentation of Bernstein-type theorems.

3 Stable Minimal Surfaces

It is a rather difficult problem to decide whether a given specific minimal surface spanned by a closed curve Γ is actually area minimizing, that is, whether it is an absolute or at least relative minimizer of the area functional among all surfaces of the same topological type bounded by Γ. Suppose that the minimal surface $X : \Omega \to \mathbb{R}^3$ is defined on a bounded domain Ω of $\mathbb{R}^2$. Then it is easy to see that the condition

$$\delta^2 A_\Omega(X, \varphi) \geq 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^3) \tag{5}$$

is necessary for any relative minimizer X within Γ. Let $\lambda_1(\Omega)$ be the smallest eigenvalue of the *second-variation operator* $-\Delta_X + 2K$ on Ω with respect to zero boundary values. Then, by a classical result of the calculus of variations, X is a relative minimizer of area with respect to the C^1-topology if X is a regular minimal surface of class $C^2(\bar{\Omega}, \mathbb{R}^3)$ satisfying

$$\lambda_1(\Omega) > 0. \tag{6}$$

A minimal surface $X : \Omega \to \mathbb{R}^3$ defined on a parameter domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary is said to be *strictly stable* if it is of class $C^2(\bar{\Omega}, \mathbb{R}^3)$, regular (i.e. free of branch points) on $\bar{\Omega}$ and satisfies $\lambda_1(\Omega) > 0$. If $\lambda_1(\Omega) \geq 0$, the surface X is called *stable*.

In certain situations one can show that a stable minimal surface can be embedded in a field, that is, it can be viewed as a leaf of a suitable foliation in $\mathbb{R}^3$ whose leaves are all minimal surfaces. Then we obtain that such a stable surface actually is a relative minimizer of area with respect to the C^0-topology. Such a field construction plays an essential role in the proof of Nitsche's uniqueness theorem (see Section 4.9, and, for details, Sections 5.6 and 5.7).

Barbosa and do Carmo [1] proved that any immersed minimal surface $X : \Omega \to \mathbb{R}^3$ is strictly stable if the image $N(\Omega)$ of Ω under the Gauss map $N : \Omega \to S^2$ corresponding to X has area less than 2π. Later these authors showed in [4] that the assumption $\int K \, dA < \frac{4}{3}\pi$ implies strict stability of any immersed minimal surface $X : \Omega \to \mathbb{R}^n$, for an arbitrary $n \geq 3$, if Ω is simply connected.

Stable minimal surfaces are an important subclass in the set of all minimal surfaces. Roughly speaking, we can view strictly stable minimal surfaces as those surfaces of mean curvature zero that can experimentally be realized by soap films. In some respect they behave like nonparametric minimal surfaces. For instance, R. Schoen [2] proved an analogue of Heinz's estimate (22) stated in Section 2.4 for stable surfaces which, in turn, implies Bernstein's theorem for such surfaces. Moreover, Schoen's estimate also yields an earlier result of do Carmo and Peng [1] and of Fischer-Colbrie and R. Schoen [1], namely that a complete stable minimal surface in $\mathbb{R}^3$ has to be a plane.

For a fairly detailed discussion of the second variation of area and of stable minimal surfaces we refer to Chapter 5 as well as to Nitsche [28], pp. 86–109, and for an updated version to Nitsche [37], pp. 90–116. There the reader will also find a survey of the fundamental contributions of H.A. Schwarz to this problem which are mainly contained in his Festschrift for the 70th birthday of Weierstrass (cf. Schwarz [2], vol. 1, pp. 223–269).

4 Foliations by Minimal Surfaces

In Section 2.8 as well as in Subsection 3 of these Scholia we saw that any leaf of a foliation by minimal surfaces is area minimizing. This is the basic content of Weierstrass's approach to the calculus of variations. Its main ingredients are the Weierstrass field construction (that is, the embedding of a given minimal surface into a field consisting of a foliation with minimal leaves) and Hilbert's independent integral. The method presented in Section 2.8 furnishes a simplification of the original form of the independent integral stated by Schwarz. This simplified version is based on the calculus of differential forms and provides a flexible and important tool in differential geometry which is very easy to handle. For applications and further results we refer to the basic work of Harvey and Lawson [3,4] and of Lawlor and Morgan [1].

Other contributions on foliations by minimal submanifolds of a given Riemannian manifold are due to Haefliger [1], Rummler [1], and Sullivan [1].

In the Sections 5.6 and 5.7 we discuss field constructions for immersed minimal surfaces that are not embedded. They are the geometric basis for Tomi's finiteness theorem and Nitsche's uniqueness theorem.