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MINIMAL SURFACES

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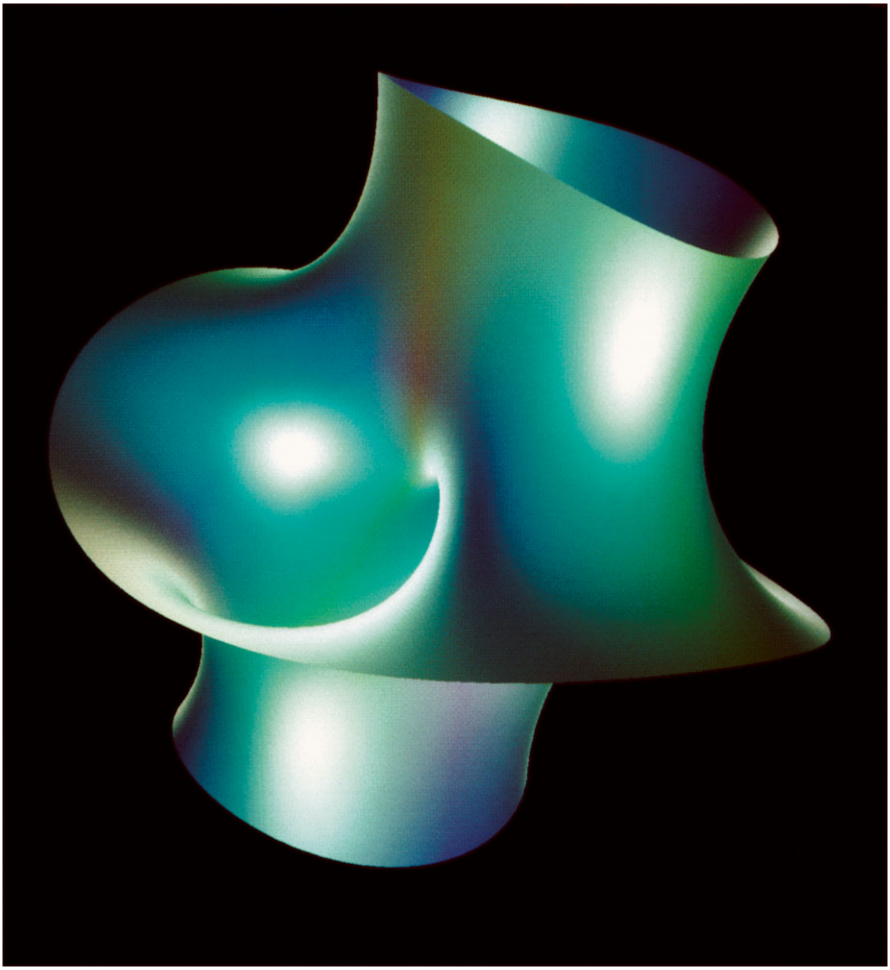

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Minimal Surfaces

With assistance and contributions by A. Küster and R. Jakob

Revised and enlarged 2nd edition

With 139 Figures and 9 Color Plates

Springer

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Preface

This book is the first volume of a treatise on minimal surfaces consisting of altogether three volumes, which can be read and studied independently of each other. The central theme is *boundary value problems for minimal surfaces*, such as Plateau's problem. The present treatise forms a greatly extended version of the monograph *Minimal Surfaces I, II* by U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, published in 1992, which is often cited in the literature as [DHKW]. New coauthors are Friedrich Sauvigny for the first volume and Anthony J. Tromba for the second and third volume.

The topic of the treatise, belonging to differential geometry and the calculus of variations as well as to the theory of partial differential equations and functions of a complex variable, may at a first glance seem rather special. However, we believe that it is both attractive and advantageous to consider mathematical ideas in the light of special problems, even though mathematicians nowadays often tend to prefer the opposite approach, namely to emphasize general theories while relegating specific problems to play the modest role of examples. Both ways to present mathematics are equally valuable and necessary, but the theory of minimal surfaces is a good case for the first approach, to study in some detail examples which are as fascinating as they are important.

Our intention in writing this book is best characterized by a quote from Courant's treatise *Dirichlet's principle* which in several respects has been a model for our work: "*Enlightenment, however, must come from an understanding of motives; live mathematical development springs from specific natural problems which can easily be understood, but whose solutions are difficult and demand new methods of more general significance.*"

One might think that three books are more than enough in order to give a more or less complete presentation of the theory of minimal surfaces, but we failed in many respects. Thus the reader should not expect an encyclopedic treatment of the theory of minimal surfaces, but merely an introduction to the field, followed by a more thorough presentation of certain aspects which relate to boundary value problems. For further study we refer to our extensive bibliography as well as to comments and references in the Scholia attached to each chapter. In particular, we mention the various lecture notes, cited at the beginning of our bibliography, as well as the treatises by Radó [21], Courant [15], Osserman [10], Federer [1], Nitsche [28,37], Giusti [4], Massari and Miranda [1], Struwe [11], Simon [8], Jost [17], and Giaquinta, Modica, and Souček [1].

As Courant remarked, "*in a field which has attracted so many mathematicians it is difficult to achieve a fair accounting of the literature and to appraise the merits of others.*" By adding Scholia to each chapter we have tried to give a sufficiently detailed account of how the theory of minimal surfaces has developed and what are the basic sources of information and inspiration, and we hope that not too many were omitted.

We thank M. Beeson, F. Duzaar, K. Große-Brauckmann, R. Jakob, J. Jost, E. Kuwert, F. Müller, M. Pingen, F. Tomi, H. von der Mosel, and D. Wienholtz for pointing out errors and misprints in [DHKW]. Special thanks we owe to Ruben Jakob who studied and corrected most of the new material added to [DHKW], thereby eliminating numerous mistakes. His assistance was invaluable. Moreover, Chapter 6 of this volume is substantially inspired by his diploma thesis [1]. We also thank Robert Osserman for providing us with Example $\boxed{5}$ in Section 3.7, and Albrecht Küster for his cooperation in writing [DHKW], and for numerous illustrations supplied by him.

We should also like to thank David Hoffman, Hermann Karcher, Konrad Polthier and Meinhard Wohlgemuth for permitting us to use some of their drawings of complete and of periodic minimal surfaces, and Imme Haubitz for allowing us to reproduce some of her drawings of Thomsen surfaces. We are grateful to Klaus Bach, Frei Otto and Eric Pitts for providing us with photographs of various soap film experiments.

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Last but not least we should like to thank our publisher and in particular our very patient editors, Catriona Byrne, Marina Reizakis, and Angela Schulze-Thomin, for their encouragement and support.

Duisburg *Ulrich Dierkes*
Bonn *Stefan Hildebrandt*
Cottbus *Friedrich Sauvigny*

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Introduction

This text on minimal surfaces is arranged in three volumes, which in the sequel will be quoted as Vol. 1 (*Minimal Surfaces*), Vol. 2 (*Regularity of Minimal Surfaces*), and Vol. 3 (*Global Analysis of Minimal Surfaces*). Each volume is divided into two parts.

The present volume is in many ways an introduction to differential geometry and to the classical theory of minimal surfaces, and the first four chapters should more or less be readable for any graduate student. For these the only prerequisites are the elements of vector analysis and some basic knowledge of complex analysis. After an exposition of the basic ideas of the theory of surfaces in three-dimensional Euclidean space given in Chapter 1, we begin Chapter 2 by introducing minimal surfaces as regular surfaces which are stationary points of the area functional. This is equivalent to the fact that the mean curvature H of such a surface vanishes identically. Then we show that any minimal surface can be represented both in an elementary and a geometrically significant way by conformal parameters. In general this representation will only be local. However, invoking the uniformization theorem, we are led to global conformal representations. This reasoning will suggest a new definition of minimal surfaces that includes the old one but is much more convenient: a minimal surface $X(w)$ is defined as a nonconstant harmonic mapping from a parameter domain Ω in the complex plane into $\mathbb{R}^3$ which satisfies the conformality relation $\langle X_w, X_w \rangle = 0$. Note that such a mapping X may have isolated zeros of its derivative $X_w = \frac{1}{2}(X_u - iX_v)$, called *branch points*. Hence a minimal surface in this general sense need not be a regular surface, i.e. an immersion, and therefore one occasionally speaks of a *branched* minimal surface. Mostly we do not use this notation; for us a minimal surface $X : \Omega \to \mathbb{R}^3$ is a harmonic mapping with conformal parameters. If the parameters of X are not conformal, but X is an immersion with $H \equiv 0$, we often speak of a *zero mean curvature surface* (e.g. in Chapters 5 and 7), except if X is also the graph of a real valued function $z(x, y)$, $(x, y) \in \Omega$; then X is said to be a *nonparametric minimal surface*.

Other parts of Chapter 2 are concerned with basic features of nonparametric minimal surfaces such as Bernstein's theorem, stating that entire solutions of the nonparametric minimal surface equation in $\mathbb{R}^2$ have to be planes, and with foliations by one-parameter families of minimal surfaces and their significance in establishing the minimum property. Finally we derive the formula for the second variation of area.

The third chapter deals with the classical theory of minimal surfaces which is in particular connected with the names of Monge, Scherk, Bonnet, Weierstrass, Riemann, Enneper, and Schwarz. First we show that minimal surfaces can be viewed as real parts of holomorphic isotropic curves in $\mathbb{C}^3$. This in turn leads us to representation formulas of minimal surfaces by means of arbitrary meromorphic functions. We shall see how the Gauss map, the second fundamental form and the Gauss curvature of a minimal surface can be computed from such a representation formula. The reader might particularly enjoy Section 3.5 where we present some of the celebrated minimal surfaces, most of which have been known for more than a century, and illustrate them by numerous drawings.

In Section 3.6 we introduce the notion of a global minimal surface and in particular that of a complete minimal surface. The spherical image of complete minimal surfaces is then studied in Section 3.7. We present some results of the work of Osserman–Xavier–Fujimoto which can be viewed as a profound generalization of both Bernstein's theorem and of Picard's theorem in complex analysis which in turn led to Nevanlinna's value distribution theory. In the Scholia we give a brief survey of some of the more recent results on complete and on periodic minimal surfaces. Here the development of the last twenty years has brought many new results which are not at all covered by this chapter. We refer the interested reader to the reports in GTMS (2005), the encyclopaedia article by Hoffman and Karcher, and the survey by Rosenberg (1992).

The second part of the present volume deals with the *existence of minimal surfaces* which are *bounded by prescribed boundary configurations*. In Chapter 4 we treat the simplest problem of this kind, the Plateau problem. This is the question of whether one can find a minimal surface spanning a given closed Jordan curve Γ. We present the celebrated existence theorem of Douglas and Radó in the form described by Courant and Tonelli. A slight variation of their method then leads to solutions of partially free boundary problems. Further sections as well as the Scholia are concerned with Schwarz's reflection principles, obstacle problems, the existence of regular and of embedded minimal surfaces, the isoperimetric inequality, and in particular with the question of whether there can be more than one solution of Plateau's problem.

In Chapter 4 we only use the simplest method to prove existence results which is based on Dirichlet's principle. This is to say, we obtain solutions of a given boundary problem by minimizing Dirichlet's integral within a suitable class $\mathcal{C}(\Gamma)$ of mappings. This method does not give all solutions as it only leads to minimizers and misses the unstable minimal surfaces and even the relative minima. Furthermore we prove that minimizers of Dirichlet's integral in $\mathcal{C}(\Gamma)$ also minimize area in $\mathcal{C}(\Gamma)$. The same method is used to derive Lichtenstein's theorem on conformal representation of regular surfaces and to solve Plateau's problem for regular Cartan functionals. In addition we show that every Jordan curve bounds a minimal surface, even if this surface cannot be obtained by minimizing area.

In Chapter 5 we study *stable minimal surfaces* and *stable surfaces of prescribed mean curvature* ("H-surfaces"). Here the essential tool is the *stability inequality*, which for minimal surfaces expresses the fact that the second variation of the area functional is nonnegative. The basic results of this chapter are curvature estimates, field embeddings, Nitsche's uniqueness theorem, and various "finiteness results", in particular Tomi's theorem. Some of these results are used in Chapter 7 to treat the Dirichlet problem for *nonparametric H-surfaces*. Here we also apply results on the solvability of the Plateau problem for (parametric) H-surfaces, to be proved in Section 4.7 of Vol. 2.

Chapter 6 deals with the existence of unstable minimal surfaces when the *mountain-pass lemma* can be applied. We present Courant's approach to this problem.

Finally, in Chapter 8 we present an introduction to the *general problem of Plateau* that, justifiedly, is often called the *Douglas problem*. This is the question whether a configuration of several nonintersecting closed Jordan curves in $\mathbb{R}^3$ may bound multiply connected minimal surfaces of prescribed Euler characteristic and prescribed character of orientability. In a general form, the Douglas problem will be tackled in Vol. 3. Here we treat only the simplest form of the problem, namely to find a minimal surface bounded by a prescribed configuration which is parametrized on a "schlicht" domain in $\mathbb{C} = \mathbb{R}^2$, precisely, on a k-circle domain in $\mathbb{C}$. As it will be seen in Vol. 2, there is not always a solution; however, we prove the existence of a solution if *Douglas's sufficient condition* is satisfied. This solution is a minimizer both of area and of Dirichlet's integral. For example, Douglas's condition holds if the boundary configuration consists of two linked closed Jordan curves.

In many ways the material of this volume is self-contained; but there are some exceptions. We use a few ideas from Sobolev space theory, and in Chapters 4–8 we also apply basic results from the regularity theory of minimal surfaces which will be established in Vol. 2. In fact, Volume 2 can be regarded as an exercise in regularity theory for nonlinear boundary value problems of elliptic systems. Nevertheless, regularity results are not only an interesting exercise in generalizing classical results on conformal mappings to minimal surfaces and to H-surfaces, but they may also have interesting applications in geometry, for instance in establishing compactness results, index theorems, or geometric inequalities such as estimates on the length of the free trace, or generalized Gauss–Bonnet formulas.

Actually, the notions of *regular curve, regular surface, regularity* are used in an ambiguous way. On the one hand, regularity of a map $X : \overline{\Omega} \to \mathbb{R}^3$ can mean that X is *smooth* and belongs to a class $C^1, C^2, \dots, C^s, C^\infty, C^\omega$, or to a Hölder class $C^{k,\alpha}$, or to a Sobolev class H^k_p. The regularity results obtained in Chapter 2 of Vol. 2 are to be understood in this sense. On the other hand, a map $X : \overline{\Omega} \to \mathbb{R}^3$, viewed as a parameter representation of a surface in $\mathbb{R}^3$, is called *regular* or a *regular surface* or an *immersion* if the Jacobi matrix (X_u, X_v) has rank 2, i.e., if at all of its points the surface has a well-defined tangent space. If $X(w)$, $w = u + iv$, is given in conformal parameters, then the singular (i.e., nonregular) points of X are exactly its branch points w_0, which are characterized by the relation $X_w(w_0) = 0$. In Chapter 3 of Vol. 2 we shall derive asymptotic expansions of minimal surfaces at boundary branch points, which can be seen as a generalization of Taylor's formula to the nonanalytic case. Chapter 1 of Vol. 2 deals with minimal surfaces having *free boundary values*. This is a generalization of the *partially free boundary value problem* studied in Section 4.6 of Vol. 1. Chapter 2 presents the basic results on the boundary behavior of minimal surfaces under Plateau or free boundary conditions, and asymptotic expansions at branch points as well as the general Gauss–Bonnet formula for branched surfaces are derived. In Chapter 3, the Hartmann–Wintner–Heinz technique for obtaining asymptotic expansions is described, together with Dziuk's expansions at singular boundary points.

The second part of Vol. 2 deals with *geometric properties of minimal surfaces and H-surfaces*, furthermore with obstacle problems and the Plateau problem for H-surfaces. As a generalization of the isoperimetric inequality, the *thread problem* for minimal surfaces is studied. The volume ends with a new approach by A. Tromba towards the celebrated result that a minimizer of area in a given contour has no interior branch points.

The first part of Vol. 3 investigates solutions of partially free boundary value problems. Then we study various generalizations of Bernstein's theorem for minimal surfaces. These results, and even more so those of Part II of Vol. 3, are of a global nature. In this second part, a version of the *general Plateau problem* (the "*Douglas problem*") is solved by an approach via Teichmüller theory, and then the fundamental *index theorems* by Böhme and Tromba and by Tomi and Tromba are proved. In the final chapter of Vol. 3 methods from global analysis are applied to Plateau's problem.

The prospective reader will probably find many sections of the present volume elementary, in that they require only basic knowledge of analysis and that the exposition of the principal facts is fairly broad. The presentation of Volumes 2 and 3 is somewhat more advanced although we have tried to develop the necessary facts from potential theory ab ovo. Only a few results of regularity theory will be borrowed from other sources; usually this will be information needed for more refined statements such as higher regularity at the boundary. For asymptotic expansions in corners we rely on some results taken from Vekua's treatise [1,2] and from the work of Dziuk. Part II of Vol. 3 probably requires additional reading since we use results about Riemann surfaces and from Teichmüller theory as well as from Global Analysis.

All the Scholia provide sources of additional information. In particular, we try to give credit to the authorship of the results presented in the main text, and we sketch some of the main lines of the historical development. References to the literature and brief surveys of relevant topics, not treated in our notes, complete the picture.

Our *notation* is essentially the same as in the treatises of Morrey [8] and of Gilbarg and Trudinger [1]. Sobolev spaces are denoted by H_p^k instead of $W^{k,p}$; the definition of the classes C^0, C^k, C^∞, and $C^{k,\alpha}$ is the same as in Gilbarg and Trudinger [1]; C^ω denotes the class of real analytic functions; $C_c^\infty(\Omega)$ stands for the set of C^∞-functions with compact support in Ω. For greater precision we write $C^k(\Omega, \mathbb{R}^3)$ for the class of C^k-mappings $X : \Omega \to \mathbb{R}^3$, whereas the corresponding class of scalar functions is denoted by $C^k(\Omega)$, and likewise we proceed for the other classes of differentiability. Another standard symbol is $B_r(w_0)$ for the disk $\{w = u + iv \in \mathbb{C} \colon |w - w_0| < r\}$ in the complex plane. If formulas become too cumbersome to read, we shall occasionally write $B(w_0, r)$ instead of $B_r(w_0)$. In general we shall deal with minimal surfaces defined on simply connected bounded parameter domains Ω in $\mathbb{C}$ which, by Riemann's mapping theorem, all are conformally equivalent to each other. Hence we can pick a standard representation B for Ω: we take it to be either the unit disk $\{w \colon |w| < 1\}$ or the semidisk $\{w \colon |w| < 1, \operatorname{Im} w > 0\}$. In the first case we write C for ∂B, in the second C will denote the semicircle $\{w \colon |w| = 1, \operatorname{Im} w > 0\}$ while I stands for the interval $\{u \in \mathbb{R} \colon |u| < 1\}$. On some occasions it is convenient to switch several times from one meaning of B to the other. Moreover, some definitions based on one meaning of B have to be transformed mutatis mutandis to the other one. This may sometimes require slight changes but we have refrained from pedantic adjustments which the reader can easily supply himself.

Part I

Introduction to the Geometry of Surfaces and to Minimal Surfaces

Chapter 1

Differential Geometry of Surfaces in Three-Dimensional Euclidean Space

In this chapter we give a brief introduction to the differential geometry of surfaces in three-dimensional Euclidean space. The main purpose of this introduction is to provide the reader with the basic notions of differential geometry and with the essential formulas that will be needed later on.

Section 1.1 discusses the notion of surfaces that is mainly used in these notes. Moreover, the notions of tangent space, surface normal, surface area, equivalent surfaces, as well as tangent and normal vector fields are defined.

In Section 1.2 we consider the spherical image of a surface X and its negative differential, the Weingarten map. This leads to the three fundamental forms on a surface which, in turn, give rise to the definition of the principal curvatures, and of the Gauss curvature and the mean curvature. By means of the orthonormal frame $\{\boldsymbol{t}, \boldsymbol{s}, \mathfrak{N}\}$ along a curve c on X consisting of the tangent $\boldsymbol{t} = \dot{c}$ to the curve, the side normal $\boldsymbol{s}$ and the surface normal $\mathfrak{N}$, the geodesic curvature and the normal curvature of c are defined. This leads to the standard interpretation of the principal curvatures and of the Gauss and mean curvatures. The principal curvatures turn out to be the eigenvalues of the Weingarten map, and the Gauss curvature is interpreted as the ratio $\frac{dA_N}{dA_X}$ of the area element of a surface X and of its spherical image N if we take orientation into account. After defining geodesics, asymptotic curves, and lines of curvature, we note the invariance properties of the various curvature measures.

Section 1.3 begins by stating the Gauss equations for the second derivatives $X_{,\alpha\beta}$ of a surface representation X which lead to the definition of the Christoffel symbols of the first and second kind. It will then be shown that these symbols can be expressed in terms of the coefficients of the first fundamental form whence it follows that the same holds for the Gauss curvature. This is essentially the content of Gauss's celebrated theorema egregium. It is proved by connecting the Gauss curvature with the Riemann curvature tensor. Finally, a general expression for the geodesic curvature of a curve on the surface is computed. In addition, we provide a collection of formulas for

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the Christoffel symbols and for the Gauss, mean, and geodesic curvatures in orthogonal and conformal coordinates which will be useful later.

In Section 1.4 we define conformally equivalent surfaces and conformal parameters, discuss the theorem of Gauss–Lichtenstein that surfaces $X : \Omega \to \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^2$ can be mapped conformally into the plane, and finally we state and prove different versions of the Gauss–Bonnet theorem by employing conformal representations. This approach is particularly well suited for generalizing the Gauss–Bonnet theorem to surfaces with singularities (branch points) as we shall see later.

In Section 1.5 we deal with basic vector analysis on surfaces X. After introducing the covariant differentiation of tangential vector fields, we in particular define the X-gradient $\nabla_X f$ of a scalar function f as a tangential vector field, and the X-divergence of a tangential vector field. Of basic importance is the Laplace–Beltrami operator Δ_X on X, a linear elliptic differential operator which is defined as the divergence of the gradient. After providing an invariant form of the Gauss integration theorem involving the Laplace–Beltrami operator, we interpret the Laplace–Beltrami equation $\Delta_X f = 0$ as the Euler equation of the generalized Dirichlet integral. We close our discussion by defining the covariant derivative $\frac{DV}{dt}$ of a tangential vector field $V(t)$ along a curve $c(t)$. Autoparallel vector fields $V(t)$ are defined by the equation $\frac{DV}{dt} = 0$, and geodesics $c(t)$ on the surface are curves with $\frac{D}{dt}\dot{c} = 0$. This turns out to be equivalent to the fact that such c have zero geodesic curvature so that the new definition of geodesics is equivalent to the one of Section 1.2. Finally geodesics are proved to be the stationary curves of the energy functional and the length functional.

1.1 Surfaces in Euclidean Space

Most of the *surfaces* studied in this book are mappings

$$X : \Omega \to \mathbb{R}^3 \tag{1}$$

from a domain Ω in $\mathbb{R}^2$ into $\mathbb{R}^3$. The points of $\mathbb{R}^2$ are written as $w = (u, v) = (u^1, u^2)$ or, if we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, as $w = u+iv = u^1+iu^2$. Then X maps $w \in \Omega$ onto some image point $X(w) \in \mathbb{R}^3$.

In the following, we shall usually assume that X is at least of class C^3, and that X is *regular* on Ω, except for isolated points. By definition, at all regular points w of Ω, the Jacobian matrix

$$X_*(w) := \nabla X(w) = (X_u(w), X_v(w)) = \left(\frac{\partial X}{\partial u}(w), \frac{\partial X}{\partial v}(w)\right) \tag{2}$$

has the maximal rank two. At such points the *tangent space* $T_w X$ of X corresponding to the parameter value w, defined by

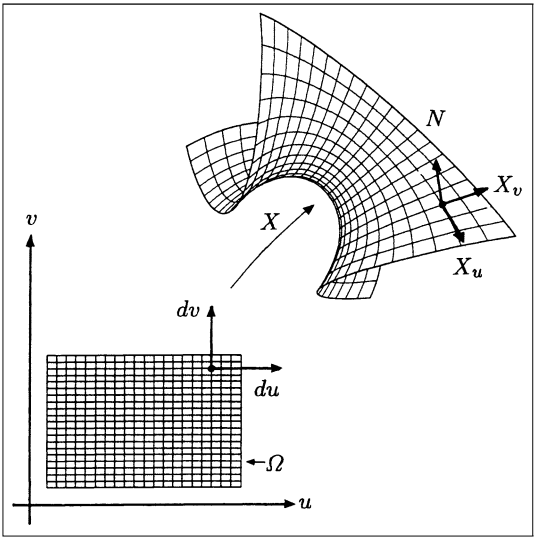


Fig. 1. A parametric surface

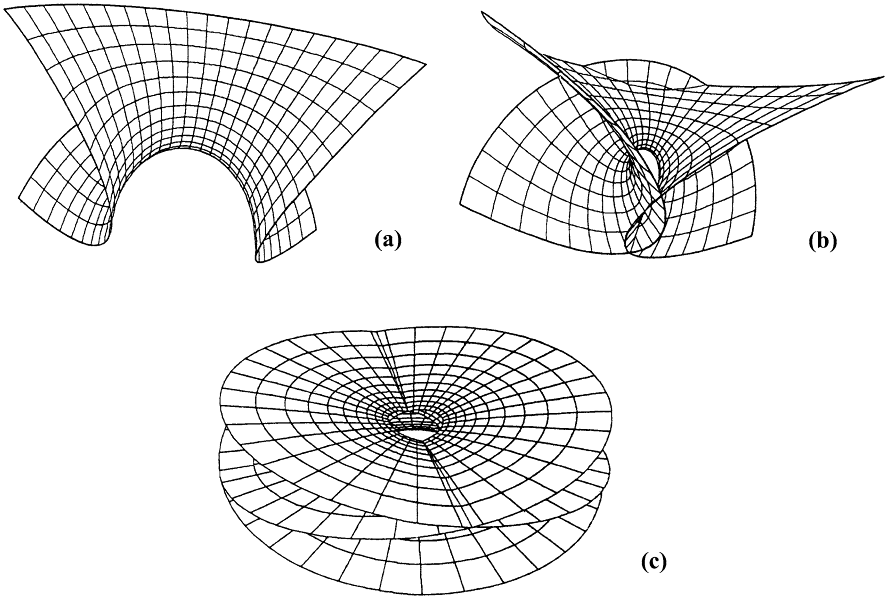


Fig. 2. **(a)** An embedded surface $\mathcal{X} : M \to \mathbb{R}^3$. **(b)** An immersed but not embedded surface $\mathcal{X} : M \to \mathbb{R}^3$. **(c)** A branched covering; the branch point in the center has been removed to provide a better view of the surface. **(a)** and **(b)** are parts of Enneper's surface, **(c)** a part of Henneberg's surface

$$T_w X := X_*(w)(\mathbb{R}^2), \tag{3}$$

is the two-dimensional subspace of $\mathbb{R}^3$ spanned by the linearly independent vectors $X_u(w)$ and $X_v(w)$. Note that we have attached the tangent space to the parameter point w and not to the point $P = X(w)$ on the *trace* $\mathfrak{S} := X(\Omega)$ of the surface since selfintersections of $\mathfrak{S}$ are not excluded. Thus, if $P = X(w) = X(\tilde{w}), w, \tilde{w} \in \Omega$ and $w \neq \tilde{w}, \mathfrak{S}$ could have the two different tangent planes $P + T_w X$ and $P + T_{\tilde{w}} X$; in other words, the tangent space $T_P\mathfrak{S}$ of $\mathfrak{S}$ at P would in general not be well defined, except for embeddings X.

At a regular point w the exterior product $X_u \wedge X_v$ does not vanish, i.e. $\mathcal{W} \neq 0$, where

$$\mathcal{W} = |X_u \wedge X_v| = \sqrt{|X_u|^2 |X_v|^2 - \langle X_u, X_v \rangle^2}. \tag{4}$$

Hence, in a neighborhood of a regular point w, the *normal vector to the surface*,

$$N = \frac{1}{\mathcal{W}} \cdot X_u \wedge X_v, \tag{5}$$

is well defined.

The *area* $A_\Omega(X)$ (*or simply* $A(X)$) *of the surface* $X : \Omega \to \mathbb{R}^3$ is defined as

$$A_\Omega(X) = \int_\Omega |X_u \wedge X_v| \, du \, dv. \tag{6}$$

Introducing the area element dA as

$$dA = \mathcal{W} \, du \, dv = |X_u \wedge X_v| \, du \, dv,$$

we may write

$$A(X) = \int_X dA = \int_\Omega \mathcal{W} \, du \, dv. \tag{6'}$$

Two mappings $X : \Omega \to \mathbb{R}^3$ and $\hat{X} : \hat{\Omega} \to \mathbb{R}^3$ of class $C^s, s \geq 1$, are said to be *equivalent* (strictly speaking: C^s-*equivalent*) if there is a C^s-diffeomorphism $\varphi : \hat{\Omega} \to \Omega$, mapping $\hat{\Omega}$ onto Ω, such that

$$\hat{X} = X \circ \varphi \quad \text{or} \quad \hat{X}(w) = X(\varphi(w)) \quad \text{for } w \in \hat{\Omega}. \tag{7}$$

Let $\varphi(w)$ be given by $\varphi(w) = (\alpha(w), \beta(w))$ and denote the Jacobian of φ by

$$J_\varphi = \det D\varphi = \begin{vmatrix} \alpha_u & \beta_u \\ \alpha_v & \beta_v \end{vmatrix}.$$

If, in addition to (7), also the condition

$$J_\varphi > 0 \tag{8}$$

is satisfied, the mappings X and $\hat{X}$ are called *strictly equivalent.* Since

$$(\hat{X}_u \wedge \hat{X}_v) = J_\varphi \cdot (X_\alpha \wedge X_\beta) \circ \varphi,$$

the transformation theorem for multiple integrals implies that $A_\Omega(X) = A_{\hat{\Omega}}(\hat{X})$, that is, *equivalent surfaces have the same area.* If the surfaces are strictly equivalent, we get

$$|\hat{X}_u \wedge \hat{X}_v|^{-1} \cdot (\hat{X}_v \wedge \hat{X}_v) = |X_\alpha \wedge X_\beta|^{-1} \cdot (X_\alpha \wedge X_\beta) \circ \varphi.$$

Denoting the normal vectors of X and $\hat{X}$ by N and $\hat{N}$, respectively, this implies that

$$\hat{N} = N \circ \varphi. \tag{9}$$

In other words, *the normal vectors of two strictly equivalent surfaces are equivalent.*

Equivalent surfaces can be considered as identical geometric objects. They are said to be *equally* or *oppositely oriented* if $J_\varphi > 0$ or $J_\varphi < 0$, respectively.

A mapping $V : \Omega \to \mathbb{R}^3$ can be interpreted as a *vector field along a surface* $X : \Omega \to \mathbb{R}^3$. The proper geometric picture is to imagine that, for each $w \in \Omega$, the vector $V(w)$ is attached to the point $X(w)$ of the surface. Of particular importance are *tangential* and *normal* vector fields. We say that V is tangential, if

$$V(w) \in T_w X \quad \text{holds for all } w \in \Omega, \tag{10}$$

and we call it normal, if

$$V(w) \perp T_w X \quad \text{for all } w \in \Omega. \tag{11}$$

Clearly, a vector field $V(w)$ is tangential along X if and only if it can be written in the form

$$V(w) = V^1(w) X_u(w) + V^2(w) X_v(w) = V^\alpha(w) X_{u^\alpha}(w) \tag{10$'$}$$

for all $w \in \Omega$ (summation with respect to repeated Greek indices from 1 to 2), and it is normal along X if and only if it is of the form

$$V(w) = \lambda(w) N(w) \quad \text{for all } w \in \Omega, \tag{11$'$}$$

with appropriate functions $V^1(w)$, $V^2(w)$ and $\lambda(w)$ respectively.

To make these definitions precise we have to assume that all points of Ω are regular points of X. If not, we either have to restrict ourselves to sufficiently small neighborhoods Ω' of regular points (instead of Ω), or we must replace Ω by the (open) set Ω_0 of its regular points.

Often a surface $X : \Omega \to \mathbb{R}^3$ can be extended to the closure $\overline{\Omega}$ or at least to $\Omega \cup C$, where C is a subset of $\partial\Omega$. Then the previous definitions may be carried over in an appropriate way.

In most cases we will restrict ourselves to surfaces which are mappings $X : \Omega \to \mathbb{R}^3$. Sometimes, however, we shall have to adopt a more global point of view as, for instance, in Chapter 3. Then by a (*general*) *surface in the three-dimensional Euclidean space* $\mathbb{R}^3$ we mean a mapping

$$\mathcal{X} : M \to \mathbb{R}^3 \tag{12}$$

from a two-dimensional manifold M into $\mathbb{R}^3$. The image of any parameter point $p \in M$ will be denoted by $\mathcal{X}(p)$. Occasionally we shall also write

$$\mathcal{X} = \mathcal{X}(p), \quad p \in M,$$

as symbol for the mapping (12). This is quite convenient although somewhat imprecise. If we have fixed a Cartesian coordinate system in $\mathbb{R}^3$ with coordinates x, y, z, the map $\mathcal{X}(p)$ will be given by a triple of real-valued functions:

$$\mathcal{X}(p) = (x(p), y(p), z(p)).$$

Let ∂M be the (possibly empty) boundary of M and $\mathring{M}$ its interior.

The differentiability properties of a surface $\mathcal{X} : M \to \mathbb{R}^3$ are defined by means of the charts of an atlas $\mathfrak{A}$ of M. If, for example, M is at least of class C^s, $s \geq 1$, then we may say that *the surface* $\mathcal{X}$ *is of class* C^s *up to the boundary* (notation: $\mathcal{X} \in C^s(M, \mathbb{R}^3)$), if for every chart $\varphi : G \to \mathbb{R}^2$ of the atlas $\mathfrak{A}$ the local map or local parametrization

$$X := \mathcal{X} \circ \varphi^{-1} : \varphi(G) \to \mathbb{R}^3$$

of the mapping $\mathcal{X}$ is of class $C^s(\varphi(G), \mathbb{R}^3)$. Such a local map X yields a parameter representation of a patch of the global surface on a planar domain $\Omega = \varphi(G)$ which is of the type discussed at the beginning.

Similarly, the space $C^s(\mathring{M}, \mathbb{R}^3)$ will be defined as the set of surfaces $\mathcal{X}$ that are of class C^s on the interior $\mathring{M}$ of M. Usually we shall only admit surfaces $\mathcal{X} : M \to \mathbb{R}^3$ which are regular (i.e., each local representation X of $\mathcal{X}$ is regular) or have at most isolated singular points $p \in M$. This still leaves us with a fairly general class of geometric objects. For instance, all embedded and even all two-dimensional manifolds immersed in $\mathbb{R}^3$ belong to our general surfaces, be they orientable or not. Moreover, we have also included branched coverings.

The area $A(\mathcal{X})$ of a general surface $\mathcal{X} : M \to \mathbb{R}^3$ can be defined by employing a partition of unity. Vector fields $\mathcal{V} : M \to \mathbb{R}^3$ along $\mathcal{X}$ will be treated by investigation of their parameterizations $V = \mathcal{V} \circ \varphi^{-1}$ along the local representations $X = \mathcal{X} \circ \varphi^{-1}$ of $\mathcal{X}$.

The reader who is not very familiar with manifolds need not worry. Only in Chapter 3 we shall assume more than the most elementary facts about them. Until then we shall only study local surfaces as in (1).

1.2 Gauss Map, Weingarten Map. First, Second and Third Fundamental Form. Mean Curvature and Gauss Curvature

In the following we shall assume that $X : \Omega \to \mathbb{R}^3$ is a regular surface of class C^3.

Let N be its normal field defined by formula (5) of the previous section. Since $|N| = 1$, we can view N as a mapping of Ω into the unit sphere S^2 of $\mathbb{R}^3$,

$$N : \Omega \to S^2 \subset \mathbb{R}^3. \tag{1}$$

This mapping will be called the *normal map*, the *spherical map*, or the *Gauss map* of the surface X. The set $N(\Omega)$ is called the *spherical image* of the surface $X : \Omega \to \mathbb{R}^3$. However, sometimes also the map $N : \Omega \to S^2$ will be called the spherical image of X, following an old custom of geometers.

Fix now some point $w = (u, v) \in \Omega$, and consider the tangential map $N_*(w)$ of N at w, that is,

$$N_*(w) = \nabla N(w) = (N_u(w), N_v(w)). \tag{2}$$

Then the *Weingarten map* $S(w)$ at w is a linear mapping of the tangent space T_wX into itself,

$$S(w) : T_wX \to T_wX, \tag{3}$$

which maps a tangent vector

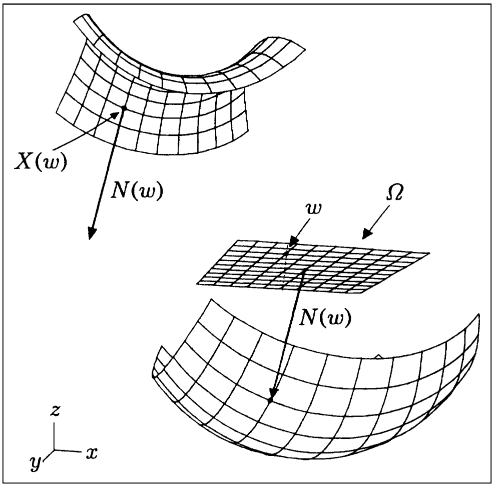


Fig. 1. A surface $X : \Omega \to \mathbb{R}^3$ and its Gauss map $N : \Omega \to S^2$. X parametrizes the part of Enneper's surface corresponding to $\Omega = I \times I, I = [-1/2, 1/2]$

$$V = V^1 X_u(w) + V^2 X_v(w) = V^\alpha X_{u^\alpha}(w)$$

onto the vector

$$S(w)V := -V^1 N_u(w) - V^2 N_v(w) = -V^\alpha N_{u^\alpha}(w). \tag{4}$$

Since $\langle N, N \rangle = 1$, we obtain by differentiation

$$\langle N, N_u \rangle = 0 \quad \text{and} \quad \langle N, N_v \rangle = 0, \tag{5}$$

i.e., the derivatives $N_u(w)$ and $N_v(w)$ are orthogonal to $N(w)$ and must, therefore, be elements of $T_w X$. Hence $S(w)V$ is indeed an element of $T_w X$, as we have stated.

If no misunderstanding is possible, we drop the argument w and write S instead of $S(w)$ for the Weingarten map.

We now claim that *S is a selfadjoint linear mapping* on the tangent space $T_w X$ equipped with the scalar product $\langle V, W \rangle$ of the surrounding Euclidean space $\mathbb{R}^3$. In other words, we have

$$\langle SV, W \rangle = \langle V, SW \rangle \tag{6}$$

for arbitrary tangent vectors $V = V^\alpha X_{u^\alpha}(w)$, $W = W^\beta X_{u^\beta}(w)$. In fact, the equation $\langle N, X_{u^\beta} \rangle = 0$ implies

$$\langle N_{u^\alpha}, X_{u^\beta} \rangle + \langle N, X_{u^\alpha u^\beta} \rangle = 0 \tag{7}$$

whence

$$\langle N_{u^\alpha}, X_{u^\beta} \rangle = \langle N_{u^\beta}, X_{u^\alpha} \rangle, \tag{8}$$

and therefore

$$\begin{aligned} \langle SV, W \rangle &= -\langle N_{u^\alpha} V^\alpha, X_{u^\beta} W^\beta \rangle = -\langle N_{u^\alpha}, X_{u^\beta} \rangle V^\alpha W^\beta \\ &= -\langle X_{u^\alpha}, N_{u^\beta} \rangle V^\alpha W^\beta = -\langle X_{u^\alpha} V^\alpha, N_{u^\beta} W^\beta \rangle = \langle V, SW \rangle. \end{aligned}$$

Thus we can define on $T_w X$ three symmetric bilinear forms

$$\mathrm{I}(V, W) := \langle V, W \rangle, \quad \mathrm{II}(V, W) := \langle SV, W \rangle, \quad \mathrm{III}(V, W) := \langle SV, SW \rangle \tag{9}$$

for all $V, W \in T_w X$, with their corresponding quadratic forms

$$\mathrm{I}(V) := |V|^2, \quad \mathrm{II}(V) := \langle SV, V \rangle, \quad \mathrm{III}(V) := |SV|^2 \tag{10}$$

which are called *first*, *second*, and *third fundamental form* of the surface X at w.

The first fundamental form is also called the *metric form* of X. If it should be necessary to indicate that these forms depend on w, we write $\mathrm{I}_w(V), \mathrm{II}_w(V), \mathrm{III}_w(V)$.

To understand *the geometric meaning of the second fundamental form*, we consider an arbitrary C^3-curve ω in Ω which starts at w, for example:

$$\omega : [0, \varepsilon] \to \Omega, \quad \omega(0) = w, \quad \omega(t) = (\omega^1(t), \omega^2(t)).$$

Then $c := X \circ \omega$ is a C^3-curve on the surface X with initial point $c(0) = X(w)$ and initial velocity

$$\dot{c}(0) = X_{u^\alpha}(w)\dot{\omega}^\alpha(0) \in T_w X.$$

We note that, by definition,

$$|\dot{c}(0)|^2 = \mathrm{I}(\dot{c}(0)).$$

Let us temporarily assume that t is the parameter of arc length s of the curve c; therefore we write $c = c(s), 0 \le s \le l$. Then we have $|\dot{c}(s)| = 1$. Moreover,

$$\boldsymbol{t}(s) = \dot{c}(s)$$

is the unit tangent vector of the curve c,

$$\kappa(s) = |\dot{\boldsymbol{t}}(s)|$$

its *curvature*, and, for $\kappa(s) \neq 0$, its *principal normal* $\boldsymbol{n}(s)$ is uniquely defined by the equation

$$\dot{\boldsymbol{t}}(s) = \kappa(s)\boldsymbol{n}(s).$$

From

$$\boldsymbol{t}(s) = \dot{c}(s) = X_{u^\alpha}(\omega(s))\dot{\omega}^\alpha(s)$$

we infer

$$\dot{\boldsymbol{t}}(s) = \ddot{c}(s) = X_{u^\alpha u^\beta}(\omega(s))\dot{\omega}^\alpha(s)\dot{\omega}^\beta(s) + X_{u^\alpha}(\omega(s))\ddot{\omega}^\alpha(s).$$

By taking the scalar product of $\dot{\boldsymbol{t}}(0)$ with $N = N(w)$ we arrive at

$$\langle N, \dot{\boldsymbol{t}}(0)\rangle = \langle N, X_{u^\alpha u^\beta}(w)\rangle\dot{\omega}^\alpha(0)\dot{\omega}^\beta(0).$$

On account of (7), we obtain

$$\langle N, \dot{\boldsymbol{t}}(0)\rangle = -\langle N_{u^\alpha}(w), X_{u^\beta}(w)\rangle\dot{\omega}^\alpha(0)\dot{\omega}^\beta(0)$$

whence by (4)

$$\langle N, \dot{\boldsymbol{t}}(0)\rangle = \langle S\dot{c}(0), \dot{c}(0)\rangle, \tag{11}$$

that is,

$$\kappa(0)\langle N(w), \boldsymbol{n}(0)\rangle = \langle S\dot{c}(0), \dot{c}(0)\rangle. \tag{12}$$

By defining the surface normal $\mathfrak{N}(s) := N(\omega(s))$ and the *side normal* $\boldsymbol{s}(s) := \mathfrak{N}(s) \wedge \boldsymbol{t}(s)$ along c, we obtain the moving orthonormal frame

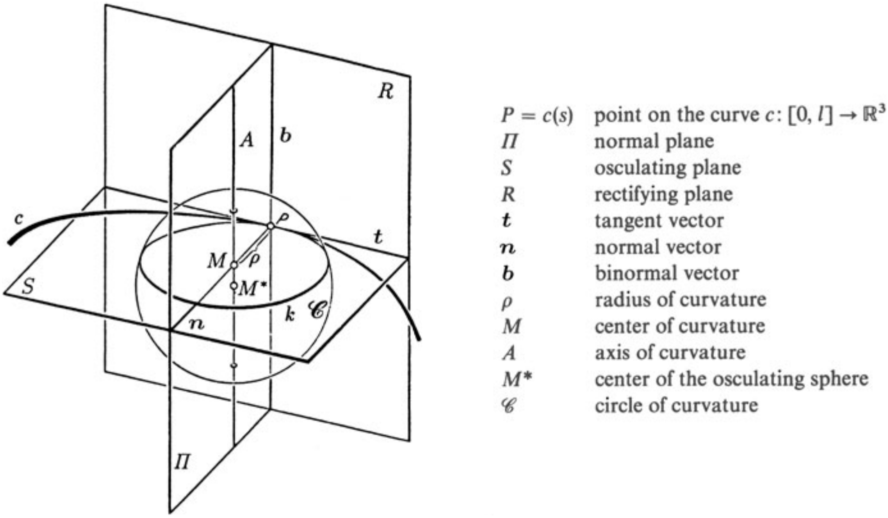


Fig. 2. Normal plane, osculating plane, rectifying plane. To describe a curve $c : [0, l] \to \mathbb{R}^3$ in space satisfying $|\dot{c}| = 1$ we introduce: $\boldsymbol{t} = \dot{c}$ = tangent vector, $\kappa = |\dot{\boldsymbol{t}}|$ = curvature, $\rho = 1/\kappa$ = radius of curvature, $\boldsymbol{n} = \rho\dot{\boldsymbol{t}}$ = normal vector, $\boldsymbol{b} = \boldsymbol{t} \wedge \boldsymbol{n}$ = binormal vector, $M = c + \rho\boldsymbol{n}$ = center of curvature, $\tau = -\langle \boldsymbol{b}', \boldsymbol{n}\rangle = \langle \boldsymbol{n}', \boldsymbol{b}\rangle$ = torsion. The normal plane is spanned by $\boldsymbol{n}$ and $\boldsymbol{b}$, the tangent plane S by $\boldsymbol{t}$ and $\boldsymbol{n}$, and the rectifying plane R by $\boldsymbol{t}$ and $\boldsymbol{b}$. The circle of curvature $\mathcal{C}$ lies in S and has ρ as its radius. Its center is M; $\mathcal{C}$ is the limit of a circle through three points P, P', P'' on the curve as $P', P'' \to P$: The sphere of curvature has the center $M^* = c + \rho\boldsymbol{n} + (\dot{\rho}/\tau)\boldsymbol{b}$ and intersects the tangent plane S in $\mathcal{C}$. It is defined as limit of the sphere determined by four points P, P', P'', P''' on the curve as $P', P'', P''' \to P$. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

$$\{\boldsymbol{t}(s), \boldsymbol{s}(s), \mathfrak{N}(s)\} \tag{13}$$

along the curve $c(s), 0 \le s \le l$, where $\boldsymbol{t}(s)$ and $\boldsymbol{s}(s)$ span the tangent space $T_{\omega(s)}X$, and $\mathfrak{N}(s)$ is orthogonal to $T_{\omega(s)}X$.

From $\langle \boldsymbol{t}, \boldsymbol{t}\rangle = 1$ we infer that $\langle \boldsymbol{t}, \dot{\boldsymbol{t}}\rangle = 0$. Thus $\dot{\boldsymbol{t}}(s)$ is a linear combination of $\boldsymbol{s}(s)$ and $\mathfrak{N}(s)$, and we have functions $\kappa_g(s)$ and $\kappa_n(s)$ such that

$$\frac{d\boldsymbol{t}}{ds} = \kappa_g \boldsymbol{s} + \kappa_n \mathfrak{N}. \tag{14}$$

One calls $\kappa_g(s)$ the *geodesic curvature* and $\kappa_n(s)$ the *normal curvature of the curve* c for the parameter value s. If $\theta(s)$ denotes the angle between $\boldsymbol{n}(s)$ and $\mathfrak{N}(s)$, we have

$$\cos\theta = \langle \boldsymbol{n}, \mathfrak{N}\rangle, \tag{15}$$

and it follows that

$$\begin{aligned} \kappa_g &= \langle \dot{\boldsymbol{t}}, \boldsymbol{s}\rangle = \kappa\langle \boldsymbol{n}, \boldsymbol{s}\rangle = \pm\kappa\sin\theta, \\ \kappa_n &= \langle \dot{\boldsymbol{t}}, \mathfrak{N}\rangle = \kappa\langle \boldsymbol{n}, \mathfrak{N}\rangle = \kappa\cos\theta, \quad \kappa = \sqrt{\kappa_g^2 + \kappa_n^2}. \end{aligned} \tag{16}$$

Equation (12) is therefore equivalent to

$$(17) \qquad \kappa_n(0) = \mathrm{II}_{\omega(0)}(\dot{c}(0)), \quad \dot{c} = \frac{dc}{ds}.$$

If we drop the condition $|\dot{c}(t)| = 1$ and return to an arbitrary parametrization of the curve c, we have $\frac{ds}{dt} = |\dot{c}| = \sqrt{\mathrm{I}(\dot{c})}$, and the chain rule yields

$$\frac{dc}{dt} = \frac{dc}{ds}\frac{ds}{dt} = \frac{dc}{ds}\mathrm{I}^{1/2}(\dot{c}), \quad \dot{c} = \frac{dc}{dt}.$$

We therefore infer from (17), that

$$(18) \qquad \kappa_n = \frac{\mathrm{II}(V)}{\mathrm{I}(V)}$$

where $\mathrm{I}(V)$ and $\mathrm{II}(V)$ are the values of the first and second fundamental forms of X at w, and κ_n is the normal curvature of a curve $c = X \circ \omega : [0, \varepsilon] \to \mathbb{R}^3$ on X with the initial values $c(0) = X(w)$ and $\dot{c}(0) = V$ at $t = 0$. If, in particular, V is a velocity vector of a normal section c of an embedded surface X, the vectors $N(w)$ and $\boldsymbol{n}(0)$ are collinear, i.e., $\kappa_n = \pm\,\kappa$, or

$$(19) \qquad \kappa = \pm\frac{\mathrm{II}(V)}{\mathrm{I}(V)}.$$

In other words, the Rayleigh quotient II/I measures the curvatures κ of all possible normal sections of the surface X at the parameter point w, and the sign of the quotient indicates whether $\boldsymbol{n}(0)$ points in the same direction as $N(w)$ or in the opposite direction, that is, whether the normal section curves towards $N(w)$ or away from it.

Since the Rayleigh quotient II/I has the meaning of a curvature, we call

$$(20) \qquad \begin{aligned} \kappa_1 \;&:=\; \min\left\{\frac{\mathrm{II}(V)}{\mathrm{I}(V)} : V \in T_wX, V \neq 0\right\} \\ &=\; \min\{\mathrm{II}(V) \colon V \in T_wX, \mathrm{I}(V) = 1\}, \end{aligned}$$

$$(21) \qquad \begin{aligned} \kappa_2 \;&:=\; \max\left\{\frac{\mathrm{II}(V)}{\mathrm{I}(V)} : V \in T_wX, V \neq 0\right\} \\ &=\; \max\{\mathrm{II}(V) \colon V \in T_wX, \mathrm{I}(V) = 1\} \end{aligned}$$

the *principal curvatures of the surface X at w*. Note that $\kappa_1 = \kappa_1(w), \kappa_2 = \kappa_2(w)$, and $\mathrm{II}(V) = \mathrm{II}_w(V) = \langle S(w)V, V\rangle$. The numbers $\rho_i = 1/\kappa_i$ are said to be the *principal radii of curvature at w*.

We can find unit vectors $V_1, V_2 \in T_wX$ such that

$$(22) \qquad \kappa_1 = \mathrm{II}(V_1), \quad \kappa_2 = \mathrm{II}(V_2).$$

An elementary reasoning yields that

$$SV_1 = \kappa_1 V_1, \quad SV_2 = \kappa_2 V_2. \tag{23}$$

In fact, we infer from the minimum property (20) that, for all $\varepsilon \in \mathbb{R}$ and all $V \in T_w X$,

$$\mathrm{II}(V_1 + \varepsilon V) \geq \kappa_1 \mathrm{I}(V_1 + \varepsilon V)$$

or

$$\mathrm{II}(V_1) + 2\varepsilon \mathrm{II}(V_1, V) + \varepsilon^2 \mathrm{II}(V) \geq \kappa_1 \mathrm{I}(V_1) + 2\kappa_1 \varepsilon \mathrm{I}(V_1, V) + \kappa_1 \varepsilon^2 \mathrm{I}(V).$$

Since $\mathrm{II}(V_1) = \kappa_1$ and $\mathrm{I}(V_1) = 1$, we arrive at

$$2\varepsilon\{\mathrm{II}(V_1, V) - \kappa_1 \mathrm{I}(V_1, V)\} + \varepsilon^2[\mathrm{II}(V) - \kappa_1 \mathrm{I}(V)] \geq 0$$

whence

$$\mathrm{II}(V_1, V) - \kappa_1 \mathrm{I}(V_1, V) = 0$$

and therefore

$$\langle SV_1, V \rangle = \kappa_1 \langle V_1, V \rangle \quad \text{for all } V \in T_w X.$$

But this is equivalent to the first equation of (23), and similarly the second equation of (23) can be proved.

In other words, the *principal curvatures are the eigenvalues of the Weingarten map* $S : T_w X \to T_w X$. If $\kappa_1 \neq \kappa_2$, we infer from (23) that $\langle V_1, V_2 \rangle = 0$. If $\kappa_1 = \kappa_2 =: \kappa$, the point w is said to be an *umbilical point* of X. In this case, we have

$$SV = \kappa V \quad \text{for all } V \in T_w X.$$

Therefore we may choose V_1 and $V_2 \in T_w X$ such that

$$SV_i = \kappa_i V_i, \quad |V_1| = |V_2| = 1, \quad \langle V_1, V_2 \rangle = 0.$$

Thus the eigenvectors V_1, V_2 of S can always be assumed to be orthogonal. They can be considered as tangent vectors of normal sections of X at w which have the smallest or largest *signed* curvature; thus they are called *principal directions of curvature of* X *at* w.

Of particular geometric importance are the elementary symmetric functions

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1 \kappa_2 \tag{24}$$

of the principal curvatures κ_1, κ_2. One calls $H(w)$ the *mean curvature* and $K(w)$ the *Gauss curvature* of X at w.

Let E be the identity map on $T_w X$. By (23), κ_1 and κ_2 are the roots of the characteristic polynomial

$$p(\kappa) = \det(S - \kappa E)$$

whence

$$p(\kappa) = (\kappa - \kappa_1)(\kappa - \kappa_2) = \kappa^2 - 2H\kappa + K.$$

On account of the Hamilton–Cayley theorem, we arrive at

$$S^2 - 2HS + KE = 0 \quad (= \text{zero map}), \tag{25}$$

and therefore, the identity

$$K\mathrm{I} - 2H\mathrm{II} + \mathrm{III} = 0 \tag{26}$$

holds.

For computations, it is often advantageous to use *coordinates*, namely the coefficients of the first, second, and third fundamental forms:

$$\begin{aligned} g_{\alpha\beta}(w) &:= \langle X_{u^\alpha}(w), X_{u^\beta}(w)\rangle, \\ b_{\alpha\beta}(w) &:= -\langle N_{u^\alpha}(w), X_{u^\beta}(w)\rangle, \\ c_{\alpha\beta}(w) &:= \langle N_{u^\alpha}(w), N_{u^\beta}(w)\rangle. \end{aligned} \tag{27}$$

Obviously,

$$g_{\alpha\beta} = g_{\beta\alpha}, \quad c_{\alpha\beta} = c_{\beta\alpha}. \tag{28}$$

Because of (7) and (8), we also have

$$b_{\alpha\beta} = b_{\beta\alpha} = \langle N, X_{u^\alpha u^\beta}\rangle. \tag{29}$$

Sometimes, the Gaussian notation

$$G := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F} & \mathcal{G} \end{pmatrix}, \quad B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} \tag{30}$$

will be used, i.e.,

$$\begin{aligned} \mathcal{E} &= \langle X_u, X_u\rangle, \quad \mathcal{F} = \langle X_u, X_v\rangle, \quad \mathcal{G} = \langle X_v, X_v\rangle, \\ \mathcal{L} &= -\langle N_u, X_u\rangle = \langle N, X_{uu}\rangle, \quad \mathcal{N} = -\langle N_v, X_v\rangle = \langle N, X_{vv}\rangle, \\ \mathcal{M} &= -\langle N_u, X_v\rangle = -\langle N_v, X_u\rangle = \langle N, X_{uv}\rangle = \langle N, X_{vu}\rangle. \end{aligned} \tag{31}$$

We, moreover, write

$$\begin{aligned} g &:= \det G = g_{11}g_{22} - g_{12}^2, \\ b &:= \det B = b_{11}b_{22} - b_{12}^2. \end{aligned} \tag{32}$$

Then

$$\mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{g} = \sqrt{\det(g_{\alpha\beta})}, \tag{33}$$

and, for $V = V^1 X_u + V^2 X_v = V^\alpha X_{u^\alpha}$,

$$
(34)\quad \begin{aligned}
\mathrm{I}(V) &= g_{\alpha\beta}(w)V^\alpha V^\beta = \mathcal{E}(w)(V^1)^2 + 2\mathcal{F}(w)V^1V^2 + \mathcal{G}(w)(V^2)^2,\\
\mathrm{II}(V) &= b_{\alpha\beta}(w)V^\alpha V^\beta = \mathcal{L}(w)(V^1)^2 + 2\mathcal{M}(w)V^1V^2 + \mathcal{N}(w)(V^2)^2,\\
\mathrm{III}(V) &= c_{\alpha\beta}V^\alpha V^\beta.
\end{aligned}
$$

We also introduce $g^{\alpha\beta}(w)$, setting

$$
(35)\qquad G^{-1} = (g^{\alpha\beta}).
$$

Hence

$$
(36)\qquad g^{\alpha\beta} = g^{\beta\alpha} \quad \text{and} \quad g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma,
$$

where δ_α^γ is the Kronecker symbol, and

$$
(37)\qquad \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \frac{1}{\mathcal{W}^2}\begin{bmatrix} \mathcal{G} & -\mathcal{F} \\ -\mathcal{F} & \mathcal{E} \end{bmatrix}.
$$

As we know (cf. (5)), N_u and N_v are contained in T_wX and, therefore, linear combinations of the two linearly independent vectors X_u and X_v. Hence there are uniquely determined coefficients $a_\alpha^\beta = a_\alpha^\beta(w)$ such that

$$
N_{u^\alpha} = a_\alpha^\beta X_{u^\beta}.
$$

Then

$$
\langle N_{u^\alpha}, X_{u^\gamma}\rangle = a_\alpha^\beta \langle X_{u^\beta}, X_{u^\gamma}\rangle, \quad \text{i.e.,} \quad -b_{\alpha\gamma} = a_\alpha^\beta g_{\beta\gamma}
$$

whence

$$
-b_{\alpha\gamma}g^{\gamma\nu} = a_\alpha^\beta g_{\beta\gamma}g^{\gamma\nu} = a_\alpha^\beta \delta_\beta^\nu = a_\alpha^\nu.
$$

Thus we have found the *Weingarten equations*

$$
(38)\qquad N_{u^\alpha} = -b_\alpha^\beta X_{u^\beta} \quad \text{with } b_\alpha^\beta := b_{\alpha\gamma}g^{\gamma\beta}.
$$

They can also be written in the form

$$
(39)\qquad N_u = aX_u + bX_v, \quad N_v = cX_u + dX_v
$$

with

$$
(40)\qquad \begin{aligned}
-\mathcal{L} &= a\mathcal{E} + b\mathcal{F}, \quad -\mathcal{M} = a\mathcal{F} + b\mathcal{G},\\
-\mathcal{M} &= c\mathcal{E} + d\mathcal{F}, \quad -\mathcal{N} = c\mathcal{F} + d\mathcal{G},
\end{aligned}
$$

or

$$
(41)\qquad \begin{aligned}
a &= \mathcal{W}^{-2}(\mathcal{F}\mathcal{M} - \mathcal{G}\mathcal{L}), \quad b = \mathcal{W}^{-2}(\mathcal{F}\mathcal{L} - \mathcal{E}\mathcal{M}),\\
c &= \mathcal{W}^{-2}(\mathcal{F}\mathcal{N} - \mathcal{G}\mathcal{M}), \quad d = \mathcal{W}^{-2}(\mathcal{F}\mathcal{M} - \mathcal{E}\mathcal{N}).
\end{aligned}
$$

(Note that, in the last three formulas, the coefficient b is not to be confused with the determinant $b = \det(b_{\alpha\beta}) = \mathcal{L}\mathcal{N} - \mathcal{M}^2$!)

Let us now compute H and K in terms of the $g_{\alpha\beta}$ and $b_{\alpha\beta}$. To this end, we remember that the principal curvatures κ_1 and κ_2 are the two eigenvalues of the Weingarten map $S : T_w X \to T_w X$. The eigenvalue equation for S,

$$SV = \kappa V,$$

can equivalently be written as

$$\langle SV, W\rangle = \kappa\langle V, W\rangle \quad \text{for all } W \in T_w X$$

or, by writing $V = V^\alpha X_{u^\alpha}, W = W^\beta X_{u^\beta}$, as

$$b_{\alpha\beta}V^\alpha W^\beta = \kappa g_{\alpha\beta}V^\alpha W^\beta \quad \text{for all } (W^1, W^2) \in \mathbb{R}^2$$

whence

$$b_{\alpha\beta}V^\alpha = \kappa g_{\alpha\beta}V^\alpha.$$

Since $b_{\alpha\beta} = b_{\beta\alpha}$ and $g_{\alpha\beta} = g_{\beta\alpha}$, we infer that the equation $SV = \kappa V$ is equivalent to

$$B\mathcal{V} = \kappa G\mathcal{V} \quad \text{or} \quad G^{-1}B\mathcal{V} = \kappa\mathcal{V}$$

with $\mathcal{V} = (V^1, V^2) \in \mathbb{R}^2$ (to be read as column). Thus κ_1 and κ_2 are the roots of

$$\begin{aligned}
\det(B - \kappa G) &= \begin{vmatrix} b_{11} - \kappa g_{11} & b_{12} - \kappa g_{12} \\ b_{21} - \kappa g_{21} & b_{22} - \kappa g_{22} \end{vmatrix} \\
&= \det(g_{\alpha\beta})\kappa^2 - (b_{11}g_{22} + b_{22}g_{11} \\
&\quad - b_{12}g_{21} - b_{21}g_{12})\kappa + \det(b_{\alpha\beta}) \\
&= \det(g_{\alpha\beta})(\kappa - \kappa_1)(\kappa - \kappa_2) \\
&= \det(g_{\alpha\beta})[\kappa^2 - (\kappa_1 + \kappa_2)\kappa + \kappa_1\kappa_2].
\end{aligned}$$

Hence, by comparing the coefficients of the powers of κ, we obtain

$$\begin{aligned}
\kappa_1\kappa_2 &= \frac{\det B}{\det G} = \det(G^{-1}B) = \det(b_\alpha^\beta), \\
\kappa_1 + \kappa_2 &= \operatorname{trace}(G^{-1}B) = b_{\alpha\beta}g^{\alpha\beta} = b_1^1 + b_2^2.
\end{aligned}$$

In other words, since $\mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{g}$, we have

$$H = \frac{\mathcal{L}\mathcal{G} + \mathcal{N}\mathcal{E} - 2\mathcal{M}\mathcal{F}}{2(\mathcal{E}\mathcal{G} - \mathcal{F}^2)} = \frac{1}{2}b_{\alpha\beta}g^{\alpha\beta} = \frac{1}{2}(b_1^1 + b_2^2), \tag{42}$$

$$K = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \frac{\det(b_{\alpha\beta})}{\det(g_{\alpha\beta})} = \frac{b}{g} = \det(b_\alpha^\beta). \tag{43}$$

Let us now interpret the Gauss curvature K in terms of the Gauss map N of the surface X.

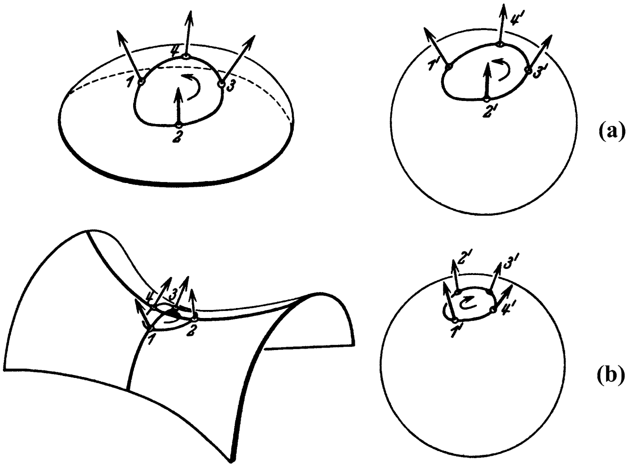


Fig. 3. The Gauss map is **(a)** orientation preserving if $K > 0$, but **(b)** orientation reversing if $K < 0$. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

The Weingarten equations (38) imply that

$$N_{u^1} \wedge N_{u^2} = (b_1^1 b_2^2 - b_2^1 b_1^2) X_{u^1} \wedge X_{u^2}$$

whence

$$N_u \wedge N_v = K(X_u \wedge X_v). \tag{44}$$

Let us fix an ε-neighborhood $\Omega_\varepsilon = \{w \colon |w - w_0| < \varepsilon\}$ of some point $w_0 \in \Omega$. Then

$$\begin{aligned} A_{\Omega_\varepsilon}(X) &= \int_{\Omega_\varepsilon} |X_u \wedge X_v| \, du \, dv, \\ A_{\Omega_\varepsilon}(N) &= \int_{\Omega_\varepsilon} |N_u \wedge N_v| \, du \, dv = \int_{\Omega_\varepsilon} |K||X_u \wedge X_v| \, du \, dv \end{aligned}$$

and therefore

$$|K(w_0)| = \lim_{\varepsilon \to 0} \frac{A_{\Omega_\varepsilon}(N)}{A_{\Omega_\varepsilon}(X)} = \frac{dA_N}{dA_X}. \tag{45}$$

Thus *the absolute value of the Gauss curvature of X at some point $w \in \Omega$ is the ratio of the area elements dA_N and dA_X of the spherical image N of X and of X itself.* Moreover, suppose that $K \neq 0$ in a neighborhood of $w \in \Omega$. On account of (44), the surface normal $\overline{N} = |N_u \wedge N_v|^{-1}(N_u \wedge N_v)$ will there be well defined, and

$$\overline{N} = \begin{cases} N & \text{if } K > 0, \\ -N & \text{if } K < 0. \end{cases} \tag{46}$$

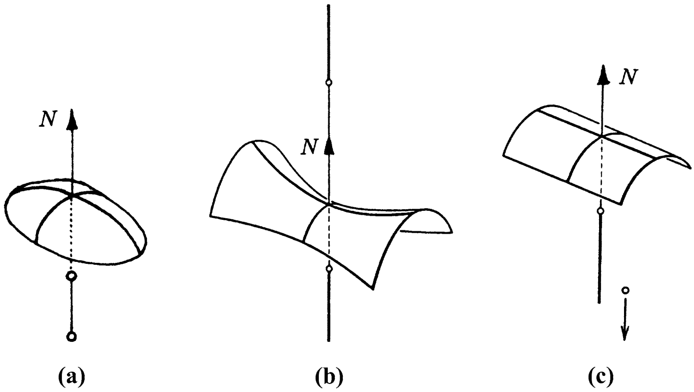


Fig. 4. **(a)** Elliptic point, **(b)** hyperbolic point, **(c)** parabolic point. (The *fat vertical lines* mark the various positions of the curvature centers with regard to different normal planes.) Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

In other words, X *and its spherical image* N *are equally oriented at* w *if* $K(w) > 0$*, but they carry an opposite orientation if* $K(w) < 0$*.*

We continue this section with another geometric interpretation of the second fundamental form II of an *embedded surface* $X : \Omega \to \mathbb{R}^3$. To this end, we fix some point $w \in \Omega$ and consider the (affine) tangent plane Π to X at the point $X(w)$, which may be oriented by $N(w)$; Π divides $\mathbb{R}^3$ into two half spaces $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$ where $N(w)$ is pointing into $\mathbb{R}^3_+$. Let $\delta(Q)$ be the oriented distance of some point Q from Π, that is, $\delta(Q) \geq 0$ if $Q \in \mathbb{R}^3_+, \delta(Q) \leq 0$ if $Q \in \mathbb{R}^3_-$.

Let now $Q = X(w+h), h = (h^1, h^2)$, be a point on the surface X in the neighborhood of $X(w)$. Then, by Taylor's formula,

$$X(w+h) = X(w) + X_{u^\alpha}(w)h^\alpha + \tfrac{1}{2}X_{u^\alpha u^\beta}(w)h^\alpha h^\beta + o(|h|^2)$$

whence

$$\begin{aligned} \delta(Q) &= \langle X(w+h) - X(w), N(w)\rangle \\ &= \tfrac{1}{2}\langle X_{u^\alpha u^\beta}(w), N(w)\rangle h^\alpha h^\beta + o(|h|^2) \end{aligned}$$

that is,

$$\delta(Q) = \tfrac{1}{2}\mathrm{II}(V_h) + o(|h|^2) \quad \text{as } h \to 0 \tag{47}$$

where $V_h = h^1 X_u(w) + h^2 X_v(w), Q = X(w+h)$. That means, $\frac{1}{2}\mathrm{II}(V_h)$ measures—up to an error term of higher than second order—*the height of the surface* X *above the tangent plane* Π *at* $X(w)$*.*

One calls the point $X(w)$ an *elliptic, hyperbolic*, or *parabolic point on the surface* X if $K(w) > 0$, $K(w) < 0$, or $K(w) = 0$. By (47), all points of X locally lie on the same side of the tangent plane Π at $X(w)$ if $K(w) > 0$, while they lie on both sides if $K(w) < 0$. We only must note that $\mathrm{II}_w(V)$ is a definite quadratic form if $K(w) > 0$, but an indefinite form for $K(w) < 0$.

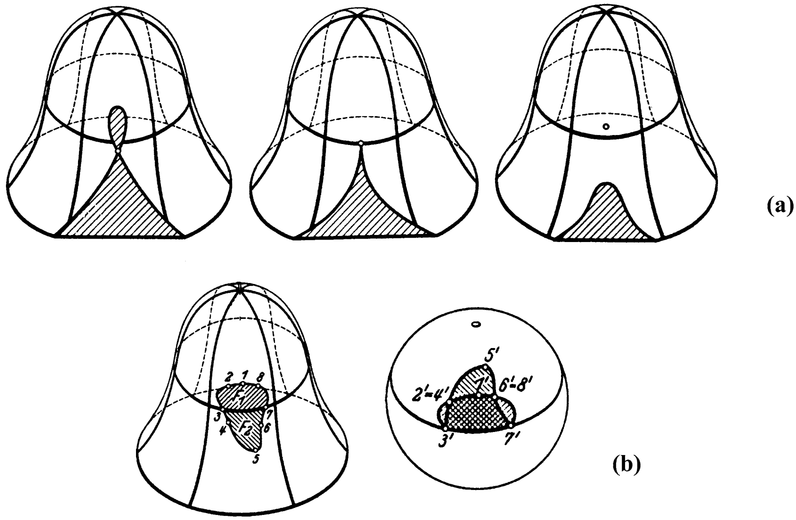


Fig. 5. A bell-shaped surface carrying a closed parabolic line. The domain *above* the line is elliptic, the domain *below* hyperbolic. **(a)** The tangent plane at a hyperbolic point intersects the surface in a loop, and in a cusp at a parabolic point. For an elliptic point P, the intersection set consists of P and of a disconnected smooth curve. **(b)** These two figures depict the spherical image of a closed curve around a parabolic point. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

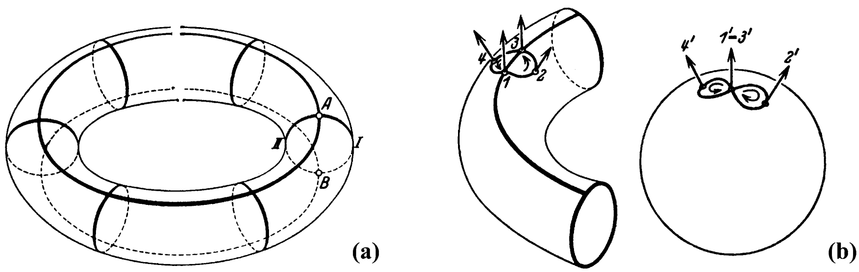


Fig. 6. (a) A regular torus carries two parabolic circles which are mapped to antipodal points of S^2, say, to the north pole and the south pole. These two circles bound two domains on $\mathfrak{F}$, an elliptic and a hyperbolic one, each of which is bijectively mapped onto S^2 punctured at the two poles. **(b)** The spherical image of a curve encircling a parabolic point. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

Consider now a curve $c = X \circ \omega$ on the surface X, that is,

$$c(t) = X(\alpha(t), \beta(t)), \quad t \in I \subset \mathbb{R}.$$

It will be called a *geodesic curve on* X (or briefly: a *geodesic of* X) if its geodesic curvature $\kappa_g(t)$ vanishes for all $t \in I$, and it will be said to be an *asymptotic curve on* X if its normal curvature $\kappa_n(t)$ vanishes everywhere.

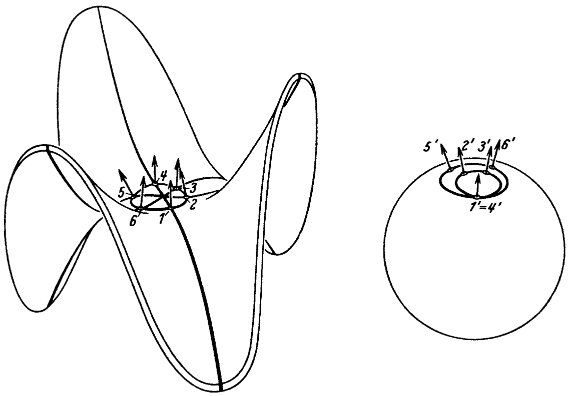


Fig. 7. An isolated parabolic point P on a monkey saddle. The spherical image of a small loop about P encircles twice the image P' of P on S^2, i.e., the spherical image of a monkey saddle is a branched surface over S^2 with P' as branch point. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

For an asymptotic curve $c(t)$, the *osculating plane* (that is, the plane spanned $\boldsymbol{t}(t)$ and $\boldsymbol{n}(t)$) coincides at all values of $t \in I$ with the tangent space $T_{\omega(t)}X$ of the surface X at $\omega(t)$. By (18), we have

$$\kappa_n = \frac{\mathrm{II}_\omega(\dot{c})}{\mathrm{I}_\omega(\dot{c})}.$$

Thus the equation $\kappa_n = 0$ is equivalent to $\mathrm{II}_\omega(\dot{c}) = 0$, or

$$\mathcal{L}(\omega)\dot{\alpha}^2 + 2\mathcal{M}(\omega)\dot{\alpha}\dot{\beta} + \mathcal{N}(\omega)\dot{\beta}^2 = 0. \tag{48}$$

Hence there will be no asymptotic curves on X if we suppose $K > 0$ whereas the assumption $K < 0$ implies that, for every $w \in \Omega$, there exist two asymptotic curves passing through $X(w)$.

Furthermore, $c = X \circ \omega$ will be called a *line of curvature* of X if its velocity vector $\dot{c}(t)$ is proportional to a principal direction of X at $w = \omega(t)$ for all t. Consequently,

$$S(\omega(t))\dot{c}(t) = k(t)\dot{c}(t) \tag{49}$$

holds, where $k(t) = \kappa_1(\omega(t))$ or $\kappa_2(\omega(t))$. But (49) implies

$$-N_u(\omega)\dot{\alpha} - N_v(\omega)\dot{\beta} = k\frac{d}{dt}X(\omega), \tag{50}$$

that is,

$$-\frac{d}{dt}(N \circ \omega) = k\frac{d}{dt}(X \circ \omega). \tag{51}$$

By employing the Weingarten equations (41), we get

$$
\tag{52}
\begin{aligned}
\frac{\mathcal{F}\mathcal{M}-\mathcal{G}\mathcal{L}}{\mathcal{W}^2}\dot\alpha + \frac{\mathcal{F}\mathcal{N}-\mathcal{G}\mathcal{M}}{\mathcal{W}^2}\dot\beta &= -k\dot\alpha,\\
\frac{\mathcal{F}\mathcal{L}-\mathcal{E}\mathcal{M}}{\mathcal{W}^2}\dot\alpha + \frac{\mathcal{F}\mathcal{M}-\mathcal{E}\mathcal{N}}{\mathcal{W}^2}\dot\beta &= -k\dot\beta
\end{aligned}
$$

(in these formulas, $\mathcal{W}, \mathcal{E}, \dots, \mathcal{L}, \dots$ have to be understood as $\mathcal{W}(\omega), \mathcal{E}(\omega), \dots, \mathcal{L}(\omega), \dots$).

We now multiply the first equation with $\dot\beta$, the second with $-\dot\alpha$, and add the resulting equations. Then, after a multiplication by $\mathcal{W}^2$, we arrive at

$$
\tag{53}
(\mathcal{E}\mathcal{M}-\mathcal{F}\mathcal{L})\dot\alpha^2 + (\mathcal{E}\mathcal{N}-\mathcal{G}\mathcal{L})\dot\alpha\dot\beta + (\mathcal{F}\mathcal{N}-\mathcal{G}\mathcal{M})\dot\beta^2 = 0
$$

or

$$
\tag{54}
\begin{vmatrix}
\dot\beta^2 & -\dot\alpha\dot\beta & \dot\alpha^2\\
\mathcal{E} & \mathcal{F} & \mathcal{G}\\
\mathcal{L} & \mathcal{M} & \mathcal{N}
\end{vmatrix} = 0.
$$

We finally want to demonstrate the *invariance properties of the various notions of curvatures* introduced before.

To this end we consider two strictly equivalent surfaces

$$
X : \Omega \to \mathbb{R}^3 \quad \text{and} \quad \hat X : \hat\Omega \to \mathbb{R}^3
$$

which are related to each other by $\hat X = X \circ \varphi$ where $\varphi : \hat\Omega \to \Omega$ is a diffeomorphism, with the inverse $\psi = \varphi^{-1}$, such that $J_\varphi > 0$. Choose now some $w \in \Omega$ and some tangent vector $V = V^\alpha X_{u^\alpha}(w)$ in $T_w X$. We can determine a smooth curve $\omega : [0, \varepsilon] \to \Omega, \varepsilon > 0$, such that

$$
\tag{55}
\omega(0) = w \quad \text{and} \quad \dot\omega^\alpha(0) = V^\alpha, \quad \alpha = 1, 2.
$$

Then the curve $c(t) := X(\omega(t)), 0 \le t \le \varepsilon$, has the initial point $c(0) = X(w)$, and its initial velocity $\dot c(0)$ satisfies

$$
\tag{56}
V = \dot c(0).
$$

Thus the tangent space $T_w X$ is spanned by the tangent vectors $\dot c(0)$ of curves $c(t) = X(\omega(t))$ with property (55).

Let now $S(w)$ be the Weingarten map of $T_w X$ into itself, and set $n(t) := N(\omega(t)), 0 \le t \le \varepsilon$. Then n is a curve on the spherical image $N : \Omega \to \mathbb{R}^3$ of the surface $X : \Omega \to \mathbb{R}^3$, and, by definition of $S(w)$, we obtain

$$
\tag{57}
S(w)V = -\dot n(0).
$$

Yet from (56) and (57) we infer that

$$
\tag{58}
T_w X = T_{\hat w} \hat X \quad \text{and} \quad S(w) = \hat S(\hat w)
$$

where $\hat{w} := \psi(w)$, and $\hat{S}(\hat{w}) : T_{\hat{w}}\hat{X} \to T_{\hat{w}}\hat{X}$ is the Weingarten map for $\hat{X}$ at $\hat{w} \in \hat{\Omega}$.

In fact, by (9) we know that the spherical image $\hat{N} : \hat{\Omega} \to \mathbb{R}^3$ of $\hat{X} : \hat{\Omega} \to \mathbb{R}^3$ is given by $\hat{N} = N \circ \varphi$. Set $\hat{\omega} := \psi \circ \omega, \hat{c} := \hat{X} \circ \hat{\omega}$, and $\hat{n} := \hat{N} \circ \hat{\omega}$. Then $\hat{V} := \frac{d\hat{c}}{dt}(0) \in T_{\hat{w}}\hat{X}$ since $\hat{w} = \hat{\omega}(0)$, and we see as before that

$$\hat{S}(\hat{w})\hat{V} = -\frac{d\hat{n}}{dt}(0).$$

On the other hand, it is easily seen that $c(t) = \hat{c}(t)$ and $n(t) = \hat{n}(t)$ for all $t \in [0, \varepsilon]$ whence $V = \hat{V}$ and $S(w)V = \hat{S}(\hat{w})V$. Thus (58) is proved if we note in addition that the roles of X and $\hat{X}$ can be interchanged.

In other words, *two strictly equivalent surfaces* X *and* $\hat{X}$ *have the same tangent space and the same Weingarten map at corresponding parameter points* $w \in \Omega$ *and* $\hat{w} \in \hat{\Omega}$:

$$T_w X = T_{\psi(w)}\hat{X} \quad \text{and} \quad S(w) = \hat{S}(\psi(w)) \quad \text{for } w \in \Omega \text{ and } \psi = \varphi^{-1}.$$

Thus, if $\varphi : \hat{\Omega} \to \Omega$ *is a* C^3*-diffeomorphism of* $\hat{\Omega}$ *onto* Ω *with* $J_\varphi > 0$ *and if* $\hat{X} = X \circ \varphi$*, then*

$$\hat{N} = N \circ \varphi \quad \text{and} \quad \hat{S} = S \circ \varphi \tag{59}$$

and therefore

$$\hat{\kappa}_1 = \kappa_1 \circ \varphi, \quad \hat{\kappa}_2 = \kappa_2 \circ \varphi, \quad \hat{H} = H \circ \varphi, \quad \hat{K} = K \circ \varphi, \tag{60}$$

where κ_1, κ_2, H, K *and* $\hat{\kappa}_1, \hat{\kappa}_2, \hat{H}, \hat{K}$ *are the principal curvatures, the mean curvature, and the Gauss curvature of* X *and* $\hat{X}$ *respectively.*

Corresponding statements hold for the geodesic curvature κ_g and the normal curvature κ_n of curves on X. If we apply parameter transformations φ with $J_\varphi < 0$ that change the orientation, then

$$\hat{N} = -N \circ \varphi, \quad \hat{S} = -S \circ \varphi, \quad \hat{H} = -H \circ \varphi,$$

but still

$$\hat{K} = K \circ \varphi.$$

Consequently, the sign of K has an intrinsic geometrical meaning but the sign of H has not.

From their definitions we see that the three bilinear forms $\mathrm{I}(U, V), \mathrm{II}(U, V)$, and $\mathrm{III}(U, V)$ are invariantly defined and can, therefore, be interpreted as covariant 2-tensors. Hence, if $g_{\alpha\beta}(w)$ and $\hat{g}_{\alpha\beta}(\hat{w})$ are the coefficients of the first fundamental form I and $\hat{\mathrm{I}}$ of X and $\hat{X} = X \circ \varphi$, respectively, and $w = \varphi(\hat{w}), \hat{w} = (\hat{u}^1, \hat{u}^2)$, then

$$\hat{g}_{\alpha\beta}(\hat{w}) = g_{\gamma\delta}(\varphi(\hat{w}))\frac{\partial \varphi^\gamma}{\partial \hat{u}^\alpha}(\hat{w})\frac{\partial \varphi^\delta}{\partial \hat{u}^\beta}(\hat{w}). \tag{61}$$

In the same way, the coefficients $b_{\alpha\beta}$ and $c_{\alpha\beta}$ of II and III have to be transformed.

1.3 Gauss's Representation Formula, Christoffel Symbols, Gauss–Codazzi Equations, Theorema Egregium, Minding's Formula for the Geodesic Curvature

Let us again assume that $X : \Omega \to \mathbb{R}^3$ is a regular surface of class C^3. We recall that, at each $w \in \Omega$, we have the frame $\{X_u(w), X_v(w), N(w)\}$ consisting of three linearly independent vectors of which $X_u(w)$ and $X_v(w)$ span the tangent space T_wX, whereas $N(w)$ spans the orthogonal complement $(T_wX)^\perp$. Hence we can write

(1) $$X_{u^\alpha u^\beta} = \Gamma^\gamma_{\alpha\beta} X_{u^\gamma} + b_{\alpha\beta} N$$

on Ω with uniquely determined functions $\Gamma^\gamma_{\alpha\beta}(w)$ and $b_{\alpha\beta}(w)$. If we multiply (1) by N, we obtain

$$b_{\alpha\beta} = \langle X_{u^\alpha u^\beta}, N\rangle.$$

Thus the coefficients $b_{\alpha\beta}$ in (1) are in fact the coefficients of the second fundamental form, and our choice of notation in (1) is justified.

The equations (1) are called *Gauss's representation formulas* of the second derivatives of X. They accompany the Weingarten equations

(2) $$N_{u^\alpha} = -b^\beta_\alpha X_{u^\beta}, \quad b^\beta_\alpha = g^{\beta\gamma} b_{\alpha\gamma},$$

that were derived in the previous section.

The coefficients $\Gamma^\gamma_{\alpha\beta}$ are called *Christoffel symbols of second kind*, whereas the functions

(3) $$\Gamma_{\alpha\beta\gamma} := g_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma}$$

are the *Christoffel symbols of first kind*.

Remark. The reader should be warned that, unfortunately, the conventions in differential geometry are not uniquely fixed. Thus a certain care is required if one wants to use formulas from different sources. For instance, some authors write $g_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma} = \Gamma_{\alpha\gamma\beta}$. The classical notations introduced by Christoffel is $\left[{\alpha\beta \atop \gamma}\right]$ and $\left\{{\alpha\beta \atop \gamma}\right\}$ for the Christoffel symbols of first and second kind $\Gamma_{\alpha\gamma\beta}$ and $\Gamma^\gamma_{\alpha\beta}$, whereas Eisenhart [3] writes $[\alpha\beta, \gamma]$ and $\left\{{\gamma \atop \alpha\beta}\right\}$.

From (1) we infer that

$$\langle X_{u^\alpha u^\beta}, X_{u^\gamma}\rangle = \Gamma^\sigma_{\alpha\beta}\langle X_{u^\sigma}, X_{u^\gamma}\rangle = \Gamma^\sigma_{\alpha\beta} g_{\sigma\gamma},$$

whence by (3)

(4) $$\langle X_{u^\alpha u^\beta}, X_{u^\gamma}\rangle = \Gamma_{\alpha\gamma\beta}.$$

This yields the symmetry relations

$$\Gamma_{\alpha\gamma\beta} = \Gamma_{\beta\gamma\alpha}, \quad \Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}. \tag{5}$$

Moreover,

$$\frac{\partial}{\partial u^\gamma} g_{\alpha\beta} = \frac{\partial}{\partial u^\gamma} \langle X_{u^\alpha}, X_{u^\beta} \rangle = \langle X_{u^\alpha u^\gamma}, X_{u^\beta} \rangle + \langle X_{u^\alpha}, X_{u^\beta u^\gamma} \rangle,$$

and therefore

$$g_{\alpha\beta,\gamma} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = g_{\beta\tau}\Gamma^{\tau}_{\alpha\gamma} + g_{\alpha\tau}\Gamma^{\tau}_{\beta\gamma}. \tag{6}$$

Here $g_{\alpha\beta,\gamma}$ stands for $\frac{\partial}{\partial u^\gamma} g_{\alpha\beta}$. (More generally, we sometimes use the notation $f_{,\gamma} = \frac{\partial}{\partial u^\gamma} f$, $f_{,\gamma\alpha} = \frac{\partial^2}{\partial u^\gamma \partial u^\alpha} f$, etc.) Then

$$-g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} = 2\Gamma_{\alpha\gamma\beta},$$

and thus

$$\Gamma_{\alpha\gamma\beta} = \tfrac{1}{2}\{g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}\}. \tag{7}$$

Consequently, both Christoffel symbols can be computed from the coefficients of the first fundamental form.

Furthermore,

$$g_{\alpha\tau} g^{\tau\beta} = \delta^\beta_\alpha$$

implies

$$g_{\alpha\tau,\gamma} g^{\tau\beta} + g_{\alpha\tau} g^{\tau\beta}_{,\gamma} = 0,$$

and multiplication by $g^{\alpha\sigma}$ yields

$$g^{\sigma\beta}_{,\gamma} = -g_{\alpha\tau,\gamma} g^{\tau\beta} g^{\alpha\sigma}, \tag{8}$$

and, on account of (6),

$$g^{\sigma\beta}_{,\gamma} = -\{\Gamma_{\alpha\tau\gamma} + \Gamma_{\tau\alpha\gamma}\} g^{\tau\beta} g^{\alpha\sigma}.$$

Thus

$$g^{\sigma\beta}_{,\gamma} = -g^{\alpha\sigma}\Gamma^{\beta}_{\alpha\gamma} - g^{\tau\beta}\Gamma^{\sigma}_{\tau\gamma}. \tag{9}$$

These equations are the counterpiece to (6).

In order to compute the derivatives $\frac{\partial}{\partial u^\gamma} g$ of the determinant $g = \det(g_{\alpha\beta}) = g_{11}g_{22} - g_{12}^2$, we recall the equations

$$g^{11} = \frac{g_{22}}{g}, \quad g^{22} = \frac{g_{11}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial u^\gamma} g &= g_{11,\gamma} g_{22} + g_{11} g_{22,\gamma} - 2 g_{12} g_{12,\gamma} \\ &= g\{g^{11} g_{11,\gamma} + g^{22} g_{22,\gamma} + 2 g^{12} g_{12,\gamma}\}\end{aligned}$$

or

$$\frac{\partial g}{\partial u^\gamma} = g g^{\alpha\beta} g_{\alpha\beta,\gamma}. \tag{10}$$

Together with (6), it follows that

$$\begin{aligned}\frac{\partial}{\partial u^\gamma} g &= g g^{\alpha\beta}\{\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}\} \\ &= g g^{\alpha\beta}\{g_{\beta\tau}\Gamma^\tau_{\alpha\gamma} + g_{\alpha\tau}\Gamma^\tau_{\beta\gamma}\} \\ &= g\{\Gamma^\alpha_{\alpha\gamma} + \Gamma^\beta_{\beta\gamma}\},\end{aligned}$$

and therefore

$$g_{u^\gamma} = 2g\Gamma^\alpha_{\alpha\gamma}. \tag{11}$$

It follows that

$$\Gamma^\alpha_{\alpha\gamma} = \frac{1}{2g} g_{u^\gamma} = \frac{1}{2}\frac{\partial}{\partial u^\gamma}\log g = \frac{\partial}{\partial u^\gamma}\log\sqrt{g}. \tag{12}$$

From (7), we infer in particular

$$\begin{aligned}&\Gamma_{111} = \tfrac{1}{2}\mathcal{E}_u, && \Gamma_{222} = \tfrac{1}{2}\mathcal{G}_v, \\ &\Gamma_{121} = \mathcal{F}_u - \tfrac{1}{2}\mathcal{E}_v, && \Gamma_{212} = \mathcal{F}_v - \tfrac{1}{2}\mathcal{G}_u, \\ &\Gamma_{112} = \Gamma_{211} = \tfrac{1}{2}\mathcal{E}_v, && \Gamma_{221} = \Gamma_{122} = \tfrac{1}{2}\mathcal{G}_u,\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\Gamma^1_{11} = \frac{1}{2\mathcal{W}^2}\{\mathcal{G}\mathcal{E}_u + \mathcal{F}[\mathcal{E}_v - 2\mathcal{F}_u]\}, && \Gamma^2_{22} = \frac{1}{2\mathcal{W}^2}\{\mathcal{E}\mathcal{G}_v + \mathcal{F}[\mathcal{G}_v - 2\mathcal{F}_v]\}, \\ &\Gamma^2_{11} = \frac{1}{2\mathcal{W}^2}\{\mathcal{E}[2\mathcal{F}_u - \mathcal{E}_v] - \mathcal{F}\mathcal{E}_u\}, && \Gamma^1_{22} = \frac{1}{2\mathcal{W}^2}\{\mathcal{G}[2\mathcal{F}_v - \mathcal{G}_u] - \mathcal{F}\mathcal{G}_v\}, \\ &\Gamma^2_{12} = \frac{1}{2\mathcal{W}^2}\{\mathcal{E}\mathcal{G}_u - \mathcal{F}\mathcal{E}_v\} = \Gamma^2_{21}, && \Gamma^1_{21} = \frac{1}{2\mathcal{W}^2}\{\mathcal{G}\mathcal{E}_v - \mathcal{F}\mathcal{G}_u\} = \Gamma^1_{12}.\end{aligned} \tag{14}$$

In case of a surface X with *orthogonal parameter curves*, that is, with $\mathcal{F} = \langle X_u, X_v\rangle = 0$, we obtain the following simplified formulas:

$$\begin{aligned}&\Gamma_{111} = \tfrac{1}{2}\mathcal{E}_u, && \Gamma_{222} = \tfrac{1}{2}\mathcal{G}_v, \\ &\Gamma_{121} = -\tfrac{1}{2}\mathcal{E}_v, && \Gamma_{212} = -\tfrac{1}{2}\mathcal{G}_u, \\ &\Gamma_{112} = \Gamma_{211} = \tfrac{1}{2}\mathcal{E}_v, && \Gamma_{221} = \Gamma_{122} = \tfrac{1}{2}\mathcal{G}_u\end{aligned} \tag{15}$$

and, because of $\mathcal{W}^2 = \mathcal{E}\mathcal{G}$,

$$\begin{aligned}
&\Gamma^1_{11} = \frac{\mathcal{E}_u}{2\mathcal{E}} = \frac{\partial}{\partial u}\log\sqrt{\mathcal{E}}, && \Gamma^2_{22} = \frac{\mathcal{G}_v}{2\mathcal{G}} = \frac{\partial}{\partial v}\log\sqrt{\mathcal{G}},\\
(16)\qquad &\Gamma^2_{11} = -\frac{\mathcal{E}_v}{2\mathcal{G}}, && \Gamma^1_{22} = -\frac{\mathcal{G}_u}{2\mathcal{E}},\\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{\mathcal{G}_u}{2\mathcal{G}} = \frac{\partial}{\partial u}\log\sqrt{\mathcal{G}}, && \Gamma^1_{21} = \Gamma^1_{12} = \frac{\mathcal{E}_v}{2\mathcal{E}} = \frac{\partial}{\partial v}\log\sqrt{\mathcal{E}}.
\end{aligned}$$

Furthermore, if u, v are *conformal parameters of* X, that is, if

$$\mathcal{E} = \mathcal{G}, \quad \mathcal{F} = 0, \tag{17}$$

or

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0, \tag{17$'$}$$

then the equations for the Christoffel symbols simplify even further. In fact, if

$$\Lambda := \mathcal{E} = \mathcal{G}, \quad \mathcal{F} = 0, \tag{17$''$}$$

then

$$\begin{aligned}
\Gamma_{111} = -\Gamma_{212} = \Gamma_{221} = \Gamma_{122} = \tfrac{1}{2}\Lambda_u,\\
\Gamma_{222} = -\Gamma_{121} = \Gamma_{112} = \Gamma_{211} = \tfrac{1}{2}\Lambda_v
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{\partial}{\partial u}\log\sqrt{\Lambda} = \frac{\Lambda_u}{2\Lambda},\\
\Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{21} = \Gamma^1_{12} = \frac{\partial}{\partial v}\log\sqrt{\Lambda} = \frac{\Lambda_v}{2\Lambda}.
\end{aligned} \tag{18$'$}$$

From the Gauss formula

$$X_{,\beta\gamma} = \Gamma^\delta_{\beta\gamma} X_{,\delta} + b_{\beta\gamma} N$$

we obtain

$$X_{,\beta\gamma\alpha} = \{\Gamma^\tau_{\beta\gamma,\alpha} X_{,\tau} + \Gamma^\delta_{\beta\gamma} X_{,\delta\alpha} + b_{\beta\gamma,\alpha} N + b_{\beta\gamma} N_{,\alpha}\}.$$

Substituting

$$X_{,\delta\alpha} = \Gamma^\tau_{\delta\alpha} X_{,\tau} + b_{\delta\alpha} N \quad \text{and} \quad N_{,\alpha} = -b^\tau_\alpha X_{,\tau},$$

we arrive at

$$X_{,\beta\gamma\alpha} = \{\Gamma^\tau_{\beta\gamma,\alpha} + \Gamma^\delta_{\beta\gamma}\Gamma^\tau_{\delta\alpha} - b_{\beta\gamma} b^\tau_\alpha\} X_{,\tau} + [\Gamma^\delta_{\beta\gamma} b_{\delta\alpha} + b_{\beta\gamma,\alpha}] N.$$

By subtracting the corresponding formula for $X_{,\alpha\gamma\beta}$, we infer that

$$(20) \qquad R^{\tau}_{\alpha\beta\gamma} = g^{\sigma\tau}(b_{\alpha\sigma}b_{\beta\gamma} - b_{\beta\sigma}b_{\alpha\gamma})$$

and

$$(21) \qquad \Gamma^{\delta}_{\beta\gamma}b_{\delta\alpha} - \Gamma^{\delta}_{\alpha\gamma}b_{\delta\beta} + b_{\beta\gamma,\alpha} - b_{\alpha\gamma,\beta} = 0,$$

where the coefficients $R^{\tau}_{\alpha\beta\gamma}$ are defined by

$$(22) \qquad R^{\tau}_{\alpha\beta\gamma} := \Gamma^{\tau}_{\beta\gamma,\alpha} - \Gamma^{\tau}_{\alpha\gamma,\beta} + \Gamma^{\tau}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} - \Gamma^{\tau}_{\beta\delta}\Gamma^{\delta}_{\alpha\gamma}.$$

The equations (20) and (21) are called *Gauss equations* and *Codazzi equations*. Together they are equivalent to the integrability conditions

$$(23) \qquad X_{,\beta\gamma\alpha} = X_{,\alpha\gamma\beta}.$$

Although the Christoffel symbols do not transform as the coefficients of a tensor, the Gauss equations (20) show that the functions $R^{\tau}_{\alpha\beta\gamma}$ behave like tensor coefficients. One calls the mapping R, defined by

$$(24) \qquad R(U,V)W = R^{\delta}_{\alpha\beta\gamma}U^{\alpha}V^{\beta}W^{\gamma}X_{,\delta}$$

for $U = U^{\alpha}X_{,\alpha}, V = V^{\beta}X_{,\beta}, W = W^{\gamma}X_{,\gamma}$, which maps triples of tangential vector fields U,V,W onto tangential vector fields $R(U,V)W$, the *Riemann curvature tensor*. We have

$$(25) \qquad \langle R(U,V)W, Z\rangle = R_{\alpha\beta\gamma\delta}U^{\alpha}V^{\beta}W^{\gamma}Z^{\delta}$$

where U,V,W are given as before, and $Z = Z^{\delta}X_{,\delta}$. Here, the coefficients $R_{\alpha\beta\gamma\delta}$ are defined by

$$(26) \qquad R_{\alpha\beta\gamma\delta} = g_{\delta\tau}R^{\tau}_{\alpha\beta\gamma} = b_{\alpha\delta}b_{\beta\gamma} - b_{\alpha\gamma}b_{\beta\delta}.$$

Remark. The reader should once again be aware that the conventions to define $R^{\tau}_{\alpha\beta\gamma}$ and $R_{\alpha\beta\gamma\delta}$ vary from author to author. We have adopted the convention of Gromoll, Klingenberg, and Meyer [1].

Since $K = b/g$, we infer from (26) that

$$(27) \qquad K = -\frac{R_{1212}}{g}$$

and $R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$, whereas $R_{\alpha\beta\gamma\delta} = 0$ for all other $(\alpha\beta\gamma\delta)$.

We infer from (27) that the Gauss curvature K can solely be computed from the coefficients $g_{\alpha\beta}$ of the first fundamental form although it had been defined by means of I and II. This is the content of Gauss's celebrated *Theorema egregium.* If $\mathcal{F} = 0$, equation (27) takes the form

$$K = -\frac{1}{\sqrt{\mathcal{E}\mathcal{G}}}\left[\frac{\partial}{\partial u}\left\{\frac{1}{\sqrt{\mathcal{E}}}\frac{\partial}{\partial u}\sqrt{\mathcal{G}}\right\} + \frac{\partial}{\partial v}\left\{\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial}{\partial v}\sqrt{\mathcal{E}}\right\}\right]. \tag{28}$$

Let us collect some formulas for the particular case of conformal coordinates u, v.

Lemma. *Suppose that*

$$\mathcal{E} = \mathcal{G} := \Lambda, \quad \mathcal{F} = 0. \tag{29}$$

Then we have

$$\begin{aligned} X_{uu} &= \frac{\Lambda_u}{2\Lambda}X_u - \frac{\Lambda_v}{2\Lambda}X_v + \mathcal{L}N, \\ X_{uv} &= \frac{\Lambda_v}{2\Lambda}X_u + \frac{\Lambda_u}{2\Lambda}X_v + \mathcal{M}N, \\ X_{vv} &= -\frac{\Lambda_u}{2\Lambda}X_u + \frac{\Lambda_v}{2\Lambda}X_v + \mathcal{N}N \end{aligned} \tag{30}$$

and

$$N_u = -\frac{\mathcal{L}}{\Lambda}X_u - \frac{\mathcal{M}}{\Lambda}X_v, \quad N_v = -\frac{\mathcal{M}}{\Lambda}X_u - \frac{\mathcal{N}}{\Lambda}X_v. \tag{31}$$

Moreover,

$$H = \frac{\mathcal{L} + \mathcal{N}}{2\Lambda}, \tag{32}$$

$$K = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{\Lambda^2} = -\frac{1}{\Lambda}\Delta \log\sqrt{\Lambda}, \tag{33}$$

where Δ denotes the Laplace operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Furthermore

$$\mathcal{L}_v - \mathcal{M}_u = \Lambda_v H, \quad \mathcal{M}_v - \mathcal{N}_u = -\Lambda_u H \tag{34}$$

and

$$\left[\tfrac{1}{2}(\mathcal{L} - \mathcal{N}) - i\mathcal{M}\right]_{\bar{w}} = \Lambda H_w \tag{35}$$

where

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

If we introduce the real-valued functions $\alpha(w)$ and $\beta(w)$ by

$$g(w) = \alpha(w) + i\beta(w) := w^2 f(w) \quad \text{where } f(w) := \tfrac{1}{2}(\mathcal{L} - \mathcal{N}) - i\mathcal{M}, \tag{36}$$

and if ρ, θ are polar coordinates defined by $w = u + iv = \rho e^{i\theta}$, then the equations (30) and (31) can be brought into the form

(37)
$$
\begin{aligned}
X_{\rho\rho} &= \frac{\Lambda_\rho}{2\Lambda}X_\rho - \frac{1}{\rho}\frac{\Lambda_\theta}{2\Lambda}\frac{1}{\rho}X_\theta + \left(\frac{\alpha}{\rho^2} + \Lambda H\right)N,\\
\frac{1}{\rho}X_{\rho\theta} &= \frac{1}{\rho}\frac{\Lambda_\theta}{2\Lambda}X_\rho + \left(\frac{1}{\rho} + \frac{\Lambda_\rho}{2\Lambda}\right)\frac{1}{\rho}X_\theta - \frac{\beta}{\rho^2}N,\\
\frac{1}{\rho^2}X_{\theta\theta} &= -\left(\frac{1}{\rho} + \frac{\Lambda_\rho}{2\Lambda}\right)X_\rho + \frac{1}{\rho}\frac{\Lambda_\theta}{2\Lambda}\frac{1}{\rho}X_\theta - \left(\frac{\alpha}{\rho^2} - \Lambda H\right)N
\end{aligned}
$$

and

(38)
$$
\begin{aligned}
N_\rho &= -\frac{1}{\Lambda}\left(\frac{\alpha}{\rho^2} + \Lambda H\right)X_\rho + \frac{\beta}{\rho^2\Lambda}\frac{1}{\rho}X_\theta,\\
\frac{1}{\rho}N_\theta &= \frac{\beta}{\rho^2\Lambda}X_\rho + \frac{1}{\Lambda}\left(\frac{\alpha}{\rho^2} - \Lambda H\right)\frac{1}{\rho}X_\theta.
\end{aligned}
$$

(Here $X_\rho, X_\theta, \dots$ are the partial derivatives with respect to ρ or θ, respectively, of the composite functions $(\rho,\theta) \mapsto X(\rho e^{i\theta})$, etc.)

Proof. Formulas (30) and (31) are the Gauss and Weingarten equations (1) and (2), by virtue of (18′). Equations (32) and (33) immediately follow from the formulas (42) and (43) of Section 1.2, and from the theorema egregium (28). In order to prove the first equation of (34), we consider the equation

$$\mathcal{L} = \langle X_{uu}, N\rangle, \quad \mathcal{M} = \langle X_{uv}, N\rangle$$

whence

$$\mathcal{L}_v - \mathcal{M}_u = \langle X_{uu}, N_v\rangle - \langle X_{uv}, N_u\rangle.$$

By virtue of (29), (30), and (32), we then obtain

$$\mathcal{L}_v - \mathcal{M}_u = \frac{\mathcal{E}_v}{2\mathcal{E}}(\mathcal{L} + \mathcal{N}) = \mathcal{E}_v H.$$

Similarly, the second equation of (34) can be proved. By applying $\frac{\partial}{\partial w}$ to equation (32), we find that

(39)
$$\Lambda H_w = \tfrac{1}{2}(\mathcal{L}_w + \mathcal{N}_w) - H\Lambda_w,$$

and a trivial computation shows that

$$
\begin{aligned}
\mathcal{L}_{\bar w} - \mathcal{N}_{\bar w} &= (\mathcal{L}_w + \mathcal{N}_w) + (-\mathcal{N}_u + i\mathcal{L}_v),\\
-i\mathcal{M}_{\bar w} &= \tfrac{1}{2}(\mathcal{M}_v - i\mathcal{M}_u)
\end{aligned}
$$

whence

$$
\begin{aligned}
\left(\frac{\mathcal{L} - \mathcal{N}}{2} - i\mathcal{M}\right)_{\bar w} &= \frac{\mathcal{L}_{\bar w} - \mathcal{N}_{\bar w}}{2} - i\mathcal{M}_{\bar w}\\
&= \frac{1}{2}(\mathcal{L}_w + \mathcal{N}_w) + \frac{1}{2}(\mathcal{M}_v - \mathcal{N}_u) + \frac{i}{2}(\mathcal{L}_v - \mathcal{M}_u)\\
&= \frac{1}{2}(\mathcal{L}_w + \mathcal{N}_w) - \frac{1}{2}\Lambda_u H + \frac{i}{2}\Lambda_v H\\
&= \frac{1}{2}(\mathcal{L}_w + \mathcal{N}_w) - H\Lambda_w.
\end{aligned}
$$

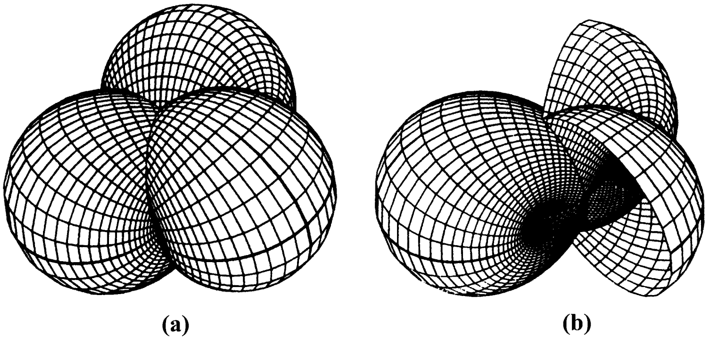


Fig. 1. (a) Wente's surface is a compact surface of constant mean curvature and of genus one. **(b)** In this picture one third of the Wente surface is removed to gain a glimpse into its interior. Courtesy of D. Hoffman, J. Spruck, J. Hoffman, and M. Callahan

Taking (39) into account, we can infer (35).

It is now an easy exercise to derive the equations (37) and (38) from (30) and (31), respectively. □

An immediate consequence of formula (35) is the following observation of H. Hopf:

If $X(u, v)$ is a surface of constant mean curvature H represented by conformal parameters u, v, then

$$f(w) := \frac{\mathcal{L} - \mathcal{N}}{2} - i\mathcal{M}$$

is a holomorphic function of $w = u + iv$.

Finally we will derive *Minding's formula for the geodesic curvature* of a curve on the surface X. To this end, we consider a curve $c(t) = X(\omega(t))$ on X where $\omega : [t_1, t_2] \to \Omega$ is a curve in the parameter domain Ω. Let us begin by first assuming that t is the parameter of arc length s, i.e., $|\dot{c}(t)| = 1$. We consider the orthonormal frame $\{\boldsymbol{t}, \boldsymbol{s}, \mathfrak{N}\}$ consisting of the tangent vector $\boldsymbol{t} = \dot{c}$, the side normal $\boldsymbol{s}$, and the surface normal $\mathfrak{N} = N \circ \omega$. Since $\boldsymbol{s} = \mathfrak{N} \wedge \boldsymbol{t}$ we obtain

$$\kappa_g = \langle \boldsymbol{s}, \dot{\boldsymbol{t}} \rangle = \langle \mathfrak{N} \wedge \boldsymbol{t}, \dot{\boldsymbol{t}} \rangle = [\mathfrak{N}, \boldsymbol{t}, \dot{\boldsymbol{t}}] = [\boldsymbol{t}, \dot{\boldsymbol{t}}, \mathfrak{N}]$$

and therefore

$$\kappa_g = [\dot{c}, \ddot{c}, \mathfrak{N}]. \tag{40}$$

Moreover,

$$\dot{c} = X_{,\alpha}(\omega)\dot{\omega}^\alpha, \quad \ddot{c} = X_{,\beta\gamma}(\omega)\dot{\omega}^\beta\dot{\omega}^\gamma + X_{,\beta}(\omega)\ddot{\omega}^\beta,$$

whence by (1)

$$\ddot{c} = [\ddot{\omega}^\delta + \Gamma^\delta_{\beta\gamma}(\omega)\dot{\omega}^\beta\dot{\omega}^\gamma]X_{,\delta}(\omega) + b_{\beta\gamma}(\omega)\dot{\omega}^\beta\dot{\omega}^\gamma\mathfrak{N}. \tag{41}$$

Since
$$X_{u^1}(\omega) \wedge X_{u^2}(\omega) = \sqrt{g(\omega)}\mathfrak{N},$$
we arrive at
$$\dot{c} \wedge \ddot{c} = \{\sigma^2\dot{\omega}^1 - \sigma^1\dot{\omega}^2\}\sqrt{g(\omega)}\,\mathfrak{N} + b_{\beta\gamma}(\omega)\dot{\omega}^\alpha\dot{\omega}^\beta\dot{\omega}^\gamma X_{,\alpha}(\omega) \wedge \mathfrak{N}$$
where we have set
$$\sigma^\delta := \ddot{\omega}^\delta + \Gamma^\delta_{\beta\gamma}(\omega)\dot{\omega}^\beta\dot{\omega}^\gamma.$$
By virtue of (40), we obtain

$$\kappa_g = \sqrt{g(\omega)}(\sigma^2\dot{\omega}^1 - \sigma^1\dot{\omega}^2) \tag{42}$$

or, equivalently

$$\kappa_g = \sqrt{g(\omega)}[\dot{\omega}^1\ddot{\omega}^2 - \dot{\omega}^2\ddot{\omega}^1 + \Gamma^2_{\beta\gamma}(\omega)\dot{\omega}^1\dot{\omega}^\beta\dot{\omega}^\gamma - \Gamma^1_{\beta\gamma}(\omega)\dot{\omega}^2\dot{\omega}^\beta\dot{\omega}^\gamma]. \tag{43}$$

If $|\dot{c}| \neq 1$, it follows that

$$\kappa_g = \frac{\sqrt{g(\omega)}}{|\dot{c}|^3}[\dot{\omega}^1\ddot{\omega}^2 - \dot{\omega}^2\ddot{\omega}^1 + \Gamma^2_{\beta\gamma}(\omega)\dot{\omega}^1\dot{\omega}^\beta\dot{\omega}^\gamma - \Gamma^1_{\beta\gamma}(\omega)\dot{\omega}^2\dot{\omega}^\beta\dot{\omega}^\gamma] \tag{44}$$

with $|\dot{c}|^2 = g_{\alpha\beta}(\omega)\dot{\omega}^\alpha\dot{\omega}^\beta$.

Let $\omega(t) = (\alpha(t), \beta(t))$, and set

$$\begin{aligned} Q(\alpha, \beta, \dot{\alpha}, \dot{\beta}) := {}& \Gamma^2_{11}(\alpha,\beta)\dot{\alpha}^3 + \{2\Gamma^2_{12}(\alpha,\beta) - \Gamma^1_{11}(\alpha,\beta)\}\dot{\alpha}^2\dot{\beta} \\ & - \{2\Gamma^1_{12}(\alpha,\beta) - \Gamma^2_{22}(\alpha,\beta)\}\dot{\alpha}\dot{\beta}^2 - \Gamma^1_{22}(\alpha,\beta)\dot{\beta}^3. \end{aligned} \tag{45}$$

Then

$$\kappa_g = \frac{\mathcal{W}(\alpha,\beta)[\dot{\alpha}\ddot{\beta} - \dot{\beta}\ddot{\alpha} + Q(\alpha,\beta,\dot{\alpha},\dot{\beta})]}{\{\mathcal{E}(\alpha,\beta)\dot{\alpha}^2 + 2\mathcal{F}(\alpha,\beta)\dot{\alpha}\dot{\beta} + \mathcal{G}(\alpha,\beta)\dot{\beta}^2\}^{3/2}}. \tag{46}$$

In particular, if u, v are conformal coordinates: $\Lambda := \mathcal{E} = \mathcal{G}, \mathcal{F} = 0$, then, according to (18′),

$$\begin{aligned} & \kappa_g\sqrt{\Lambda}\{\dot{\alpha}^2 + \dot{\beta}^2\}^{3/2} \\ & \quad = (\dot{\alpha}\ddot{\beta} - \dot{\beta}\ddot{\alpha}) + (\dot{\alpha}^2 + \dot{\beta}^2)\left[\dot{\beta}\frac{\partial}{\partial u}\log\sqrt{\Lambda} - \dot{\alpha}\frac{\partial}{\partial v}\log\sqrt{\Lambda}\right]. \end{aligned} \tag{47}$$

If $\alpha(t) = R\cos t, \beta(t) = R\sin t, 0 \le t \le 2\pi$, and if $\nu(t) = (\cos t, \sin t)$ points into the exterior of this circular line, then (47) reduces to

$$\kappa_g\sqrt{\Lambda} = \frac{1}{R} + \frac{\partial}{\partial\nu}\log\sqrt{\Lambda}. \tag{48}$$

If $\alpha = R\cos t, \beta = -R\sin t, 0 \le t \le 2\pi$, then we have

$$\kappa_g\sqrt{\Lambda} = -\frac{1}{R} + \frac{\partial}{\partial\nu}\log\sqrt{\Lambda} \tag{47′}$$

where $\nu(t)$ now denotes the interior normal $(-\cos t, \sin t)$.

1.4 Conformal Parameters, Gauss–Bonnet Theorem

Consider two matrix-valued functions

$$P : \Omega \to \mathbb{R}^4, \quad Q : \Omega \to \mathbb{R}^4$$

on a domain Ω of $\mathbb{R}^2$ which are given by

$$P(w) = \begin{bmatrix} p_{11}(w) & p_{12}(w) \\ p_{21}(w) & p_{22}(w) \end{bmatrix}, \quad Q(w) = \begin{bmatrix} q_{11}(w) & q_{12}(w) \\ q_{21}(w) & q_{22}(w) \end{bmatrix}$$

and are assumed to be symmetric:

$$p_{\alpha\beta}(w) = p_{\beta\alpha}(w), \quad q_{\alpha\beta}(w) = q_{\beta\alpha}(w).$$

The functions P and Q are called *conformal to each other* if there exists a function $\mu : \Omega \to \mathbb{R}$ with $\mu(w) > 0$ on Ω such that

$$(1) \qquad P(w) = \mu(w)Q(w) \quad \text{for all } w \in \Omega.$$

Relation (1) is equivalent to

$$(2) \qquad p_{\alpha\beta}(w)\xi^\alpha\xi^\beta = \mu(w)q_{\alpha\beta}(w)\xi^\alpha\xi^\beta$$

for all $w \in \Omega$ and all $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$.

Let now $X : \Omega \to \mathbb{R}^3$, $Y : \Omega \to \mathbb{R}^3$ be regular surfaces of class C^1 which are defined on the same parameter domain Ω, and let

$$g_{\alpha\beta}(w) = \langle X_{u^\alpha}(w), X_{u^\beta}(w)\rangle, \quad \gamma_{\alpha\beta}(w) = \langle Y_{u^\alpha}(w), Y_{u^\beta}(w)\rangle$$

be the coefficients of the first fundamental forms I_X and I_Y of X and Y respectively.

Then the surfaces X and Y are said to be *conformal to each other* if the matrix functions $(g_{\alpha\beta}(w))$ and $(\gamma_{\alpha\beta}(w))$ are conformal to each other, that is, if

$$(3) \qquad g_{\alpha\beta}(w)\xi^\alpha\xi^\beta = \mu(w)\gamma_{\alpha\beta}(w)\xi^\alpha\xi^\beta$$

holds for some $\mu(w) > 0$ and all $w \in \Omega, \xi = (\xi^1, \xi^2) \in \mathbb{R}^2$. Equivalently, we have

$$(4) \qquad \mathrm{I}_X(U) = \mu(w)\ \mathrm{I}_Y(V)$$

for $U = \xi^\alpha X_{,\alpha}(w) \in T_wX$, $V = \xi^\alpha Y_{,\alpha}(w) \in T_wY$ for all $w \in \Omega$, $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$.

If (4) holds, we also call the metric forms I_X and I_Y conformal to each other.

This relation has a simple but important geometric interpretation which explains the notation *conformal*. For this purpose we consider two curves

$\omega(t), \underline{\omega}(t)$ in the parameter domain which intersect at $t = t_0$, i.e., $\omega(t_0) = \underline{\omega}(t_0) := w_0$. They define curves $c := X \circ \omega$, $\underline{c} := X \circ \underline{\omega}$ on X as well as curves $e := Y \circ \omega$, $\underline{e} := Y \circ \underline{\omega}$ on Y, respectively, that intersect at $X(w_0)$ and $Y(w_0)$, in angles φ and ψ given by

$$\cos\varphi = \frac{\langle \dot{c}(t_0), \dot{\underline{c}}(t_0)\rangle}{|\dot{c}(t_0)||\dot{\underline{c}}(t_0)|} = \frac{\mathrm{I}_X(\dot{c}(t_0), \dot{\underline{c}}(t_0))}{\mathrm{I}_X^{1/2}(\dot{c}(t_0)) \cdot \mathrm{I}_X^{1/2}(\dot{\underline{c}}(t_0))}$$

and

$$\cos\psi = \frac{\langle \dot{e}(t_0), \dot{\underline{e}}(t_0)\rangle}{|\dot{e}(t_0)||\dot{\underline{e}}(t_0)|} = \frac{\mathrm{I}_Y(\dot{e}(t_0), \dot{\underline{e}}(t_0))}{\mathrm{I}_Y^{1/2}(\dot{e}(t_0)) \cdot \mathrm{I}_Y^{1/2}(\dot{\underline{e}}(t_0))}.$$

The equation (4) implies that $\cos\varphi = \cos\psi$. In other words, if X and Y are conformal to each other, then, at corresponding points $w \in \Omega$, angles on X and Y are measured in the same way.

As an example, we consider a regular C^3-surface $X : \Omega \to \mathbb{R}^3$ of zero mean curvature and its spherical image $N : \Omega \to \mathbb{R}^3$. From $2H = \kappa_1 + \kappa_2$ and $H = 0$ we infer $\kappa_1 = -\kappa_2$, whence $K \le 0$. Suppose that even $K < 0$. Then X is free of umbilical points. Moreover, we infer from (26) of Section 1.2 that

$$(5) \qquad -K(w)\mathrm{I}_X(V) = \mathrm{III}_X(V) \quad \text{for all } V \in T_wX$$

or

$$-K(w)g_{\alpha\beta}(w)\xi^\alpha\xi^\beta = c_{\alpha\beta}(w)\xi^\alpha\xi^\beta$$

where $g_{\alpha\beta}$ and $c_{\alpha\beta}$ are the coefficients of I_X and III_X, respectively. Let $V = \xi^\alpha X_{,\alpha}(w)$ and $U = \xi^\alpha N_{,\alpha}(w)$. Since

$$c_{\alpha\beta}(w)\xi^\alpha\xi^\beta = \langle S(w)V, S(w)V\rangle = \langle U, U\rangle = I_N(U)$$

where I_N is the first fundamental form of the surface $N : \Omega \to \mathbb{R}^3$, we obtain that

$$(6) \qquad \mathrm{I}_N(U) = -K(w)\mathrm{I}_X(V) \quad \text{for } U = \xi^\alpha N_{,\alpha}(w), V = \xi^\alpha X_{,\alpha}(w),$$

where $w \in \Omega$ and $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$. This, by definition, means that X and N are conformal to each other provided that the mean curvature of X vanishes identically. Thus we have proved: *A zero mean curvature surface X without umbilical points is conformal to its spherical image N.*

Now we turn to another important notion, to the notion of the *conformal type of a regular C^1-surface.*

Two regular C^1-surfaces $X : \Omega \to \mathbb{R}^3$ and $Y : \Omega^* \to \mathbb{R}^3$ are said to be *conformally equivalent* if there exists a C^1-diffeomorphism $\tau : \Omega \to \Omega^*$ such that the surfaces $X : \Omega \to \mathbb{R}^3$ and $Y \circ \tau : \Omega \to \mathbb{R}^3$ are conformal to each other. The mapping τ will be called a *conformal map* of X onto Y.

One checks without difficulty that conformal equivalence is, in fact, an equivalence relation. Thus *a conformal type* is defined as an equivalence class of conformally equivalent surfaces.

Let us, in particular, consider two planar surfaces $X(u,v) = (u,v,0)$, $(u,v) \in \Omega$, and $Y(\xi,\eta) = (\xi,\eta,0), (\xi,\eta) \in \Omega^*$, which are conformally equivalent. Then there exists a C^1-diffeomorphism $\tau : \Omega \to \Omega^*$ given by, say,

$$\xi = \alpha(u,v), \quad \eta = \beta(u,v)$$

such that $X(u,v)$ and $Y(\tau(u,v)) = (\alpha(u,v), \beta(u,v), 0)$ are conformal to each other. Then there is a function $\mu(u,v) > 0$ such that

$$\begin{pmatrix} \alpha_u^2 + \beta_u^2 & \alpha_u\alpha_v + \beta_u\beta_v \\ \alpha_u\alpha_v + \beta_u\beta_v & \alpha_v^2 + \beta_v^2 \end{pmatrix} = \mu(u,v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Hence the two vectors $\omega = (\alpha_u, \beta_u)$ and $\sigma = (\alpha_v, \beta_v)$ of $\mathbb{R}^2$ have the same length $\sqrt{\mu}$ and are orthogonal to each other:

$$|\omega| = |\sigma| \quad \text{and} \quad \langle \omega, \sigma \rangle = 0.$$

From this we infer that either

$$\alpha_u = \beta_v, \quad \alpha_v = -\beta_u \tag{8}$$

or

$$\alpha_u = -\beta_v, \quad \alpha_v = \beta_u \tag{8'}$$

holds. Conversely, both (8) and (8′) imply (7). That is, a diffeomorphism $\tau : \Omega \to \Omega^*$ is a conformal mapping of the planar surface X onto the planar surface Y if it either satisfies the Cauchy-Riemann equations (8), or if it fulfills (8′), that is, if it is either a strictly conformal or an anticonformal map in the usual sense, or in other words, if either $\alpha + i\beta$ or $\alpha - i\beta$ is a biholomorphic map of $w = u + iv$. We mention that a *conformal mapping* $\tau : \Omega \to \Omega^*$ with $\Omega, \Omega^* \subset \mathbb{R}^2$ usually means "*strictly conformal*", i.e. a diffeomorphism τ from Ω to Ω^* with a positive Jacobian J_τ, whereas we shall subsume both strictly conformal mappings ($J_\tau > 0$) and anticonformal mappings ($J_\tau < 0$) under this notion. (However, occasionally we may write "conformal" instead of "strictly conformal" if the meaning is clear from the context.

Next, we consider the planar surface $Y : \Omega \to \mathbb{R}^3$ given by $Y(u,v) = (u,v,0)$, and a regular C^1-surface $X : \Omega \to \mathbb{R}^3$ with coefficients

$$\mathcal{E} = |X_u|^2, \quad \mathcal{F} = \langle X_u, X_v \rangle, \quad \mathcal{G} = |X_v|^2$$

of its metric form I. By definition, X and Y are conformal to each other if there is a function $\mu(w) > 0$ such that

$$\begin{pmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F} & \mathcal{G} \end{pmatrix} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{on } \Omega,$$

or, equivalently, that

$$\mathcal{E} = \mathcal{G}, \quad \mathcal{F} = 0. \tag{9}$$

In other words, the surface $X : \Omega \to \mathbb{R}^3$ is conformal to Y if and only if the parameters u, v are conformal parameters of X.

A celebrated theorem by Lichtenstein states that *each regular surface* $Y : \Omega^* \to \mathbb{R}^3$ *of class* $C^{1,\alpha}$, $0 < \alpha < 1$, *can be mapped conformally onto a planar domain* Ω. This means the following:

There is a diffeomorphism σ from Ω^* onto Ω with inverse $\tau : \Omega \to \Omega^*$, both of class $C^{1,\alpha}$, such that $X := Y \circ \tau : \Omega \to \mathbb{R}^3$ is represented by conformal parameters u, v, i.e., the coefficients $\mathcal{E}, \mathcal{F}, \mathcal{G}$ of the first fundamental form of X satisfy (9).

The proof of this result is usually carried out in two steps. Firstly one shows by employing the theory of elliptic differential equations that sufficiently small pieces of a smooth regular surface can be mapped conformally onto a planar domain. Then, secondly, one infers from the uniformization theorem that the entire surface can be mapped onto a planar domain. We shall, however, not provide the details of this proof since, in this book, we shall mainly be concerned with surfaces of zero mean curvature for which it is fairly easy to verify that they can be transformed to conformal parameters. This will be carried out in the next chapter. In addition, we shall in Section 4.11 provide a variational proof of Lichtenstein's theorem that is related to the solution of Plateau's problem.

Let us now turn to the *Gauss–Bonnet theorem* which, in its simplest form, states the following:

If Ω *is a simply connected bounded open set in* $\mathbb{R}^2$ *with a smooth regular Jordan curve as boundary, and if* $X : \bar{\Omega} \to \mathbb{R}^3$ *is a regular surface of class* C^2, *then*

$$\int_X K\, dA + \int_\Gamma \kappa_g \, ds = 2\pi. \tag{10}$$

Here the first integral is the *total curvature* of the surface X defined by

$$\int_X K\, dA := \int_\Omega K\mathcal{W}\, du\, dv. \tag{11}$$

The second integral, $\int_\Gamma \kappa_g \, ds$, the total geodesic curvature of the boundary curve Γ of X, will be defined as follows: We choose a parametrization $w = \omega(s)$, $0 \le s \le L$, of $\partial\Omega$ which is positively oriented with respect to Ω. Then $c(s) = X(\omega(s))$, $0 \le s \le L$, is a parametrization of the boundary curve Γ of X, and it is assumed that $|\dot{c}(s)| = 1$, that is, s is the parameter of the arc length of Γ. Moreover, $\kappa_g = [\dot{c}, \ddot{c}, N]$ is the geodesic curvature of Γ. We now set $\int_\Gamma \kappa_g \, ds := \int_0^L \kappa_g(s)\, ds$.

Proof of (10). We first assume that X is given in conformal coordinates, i.e., $\mathcal{E} = \mathcal{G} := \Lambda, \mathcal{F} = 0$ on $\bar{\Omega}$, and that $\Omega = \{(u, v) : u^2 + v^2 < 1\}$. Then, $dA = \Lambda\, du\, dv$, and by formula (33) of Section 1.3

$$K = -\frac{1}{\Lambda}\Delta \log \sqrt{\Lambda}.$$

For $\partial\Omega$ we choose the parametrization

$$\omega(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi,$$

which is positively oriented with respect to Ω. Then, by formula (48) of Section 1.3, we have

$$\kappa_g\, ds = \kappa_g \sqrt{\Lambda}\, dt = \left(1 + \frac{\partial}{\partial\nu}\log\sqrt{\Lambda}\right) dt,$$

where ν is the exterior normal of $\partial\Omega$.

By virtue of Gauss's integral theorem, we thus obtain

$$\begin{aligned} -\int_X K\, dA &= \int_\Omega \Delta \log\sqrt{\Lambda}\, du\, dv = \int_0^{2\pi} \frac{\partial}{\partial\nu}\log\sqrt{\Lambda}\, dt \\ &= \int_0^{2\pi} (\kappa_g\sqrt{\Lambda} - 1)\, dt = \int_\Gamma \kappa_g\, ds - 2\pi \end{aligned}$$

which proves (10).

Next we note that $\int_X K\, dA$ remains the same if X is replaced by a surface $X \circ \tau$, where $\tau : \overline{\Omega^*} \to \bar{\Omega}$ is a C^1-diffeomorphism, and similarly $\int_\Gamma \kappa_g\, ds$ is unchanged if we assume the Jacobian of τ to be positive. Hence the left hand side of (10) is an invariant of $X \circ \tau$ with respect to all parameter changes by diffeomorphisms $\tau \in C^1(\overline{\Omega^*}, \mathbb{R}^2)$ with $\bar{\Omega} = \tau(\overline{\Omega^*})$ and $J_\tau > 0$. Moreover, by a slight strengthening of Lichtenstein's theorem, there exists such a parameter transformation τ with the property that $X \circ \tau \colon \overline{\Omega^*} \to \mathbb{R}^3$ is given in conformal parameters. Since each simply connected bounded domain is of the conformal type of the disk, we can assume that Ω^* is the unit disk $\{(u,v)\colon u^2 + v^2 < 1\}$. If we now apply the previous reasoning to $X \circ \tau$, formula (10) will be established in general.

Now we can state an analogous formula for surfaces which are bounded by only piecewise smooth regular curves. To have a clear-cut assumption, we suppose that the simply connected parameter domain $\Omega \subset \mathbb{R}^2$ is bounded by a smooth regular curve $w = \omega(t), a \le t \le b$, which is positively oriented with respect to Ω. Let $a \le t_1 < t_2 < \cdots < t_n \le b$ and $w_1 = \omega(t_1), \ldots, w_n = \omega(t_n)$, and suppose that $X : \bar{\Omega} \to \mathbb{R}^3$ is of class C^0 on $\bar{\Omega}$ and of class C^2 on $\bar{\Omega} \setminus \{w_1, w_2, \ldots, w_n\}$. Finally assume that $c(t) = X(\omega(t))$ is piecewise smooth, that $\dot{c}(t_i \pm 0) \ne 0$, and that $N(w_i) := \lim_{w \to w_i} N(w)$ exists for $1 \le i \le n$.

Then it makes sense to speak of the *interior angles* α_i of X at the vertices $X(w_i)$ corresponding to the points $w_1, \ldots, w_n \in \partial\Omega$, and of its *exterior angles* $\beta_i = \pi - \alpha_i$, where $0 < \alpha_i \le 2\pi$, $-\pi \le \beta_i < \pi$.

By rounding off the corners of X corresponding to $w_1, \ldots, w_n$ from the interior of X, and by carrying out an obvious limit procedure, formula (10) changes to

$$\int_X K\,dA + \int_\Gamma \kappa_g\,ds + \sum_{i=1}^n \beta_i = 2\pi, \tag{12}$$

or, equivalently,

$$\int_X K\,dA + \int_\Gamma \kappa_g\,ds + (n-2)\pi = \sum_{i=1}^n \alpha_i. \tag{12$'$}$$

A second generalization of (10) concerns regular surfaces $X : \bar{\Omega} \to \mathbb{R}^3$ of class C^2 where Ω is an n-fold connected domain in $\mathbb{R}^2$ bounded by n closed, regular, and smooth curves $\omega_1, \dots, \omega_n$. If we introduce the n closed and regular boundary curves $\Gamma_1, \dots, \Gamma_n$ of X given by $c_1 = X \circ \omega_1,\ \dots,\ c_n = X \circ \omega_n$, we obtain the formula

$$\int_X K\,dA + \int_{\Gamma_1} \kappa_g\,ds + \cdots + \int_{\Gamma_n} \kappa_g\,ds + (n-1)2\pi = 2\pi \tag{13}$$

provided that the parameterizations $\omega_1, \dots, \omega_n$ of the boundary curves of $\partial\Omega$ are positively oriented with respect to Ω. □

Sketch of a proof. By $n-1$ suitable lines we can cut the multiply connected domain Ω into a simply connected domain Ω' in such a way that $\partial\Omega'$ possesses $4(n-1)$ corners with exterior angles of value $\pi/2$. Then also the surface $X|_{\Omega'}$ has a boundary with exactly $4(n-1)$ vertices, and all the exterior angles of $X|_{\Omega'}$ at these vertices are right angles. Let us apply (12) to the surface $X|_{\Omega'}$. The integrals $\int \kappa_g\,ds$ over the cuts add to zero because κ_g changes on oppositely oriented edges its sign. Thus we arrive at

$$\int_X K\,dA + \sum_{j=1}^n \int_{\Gamma_j} \kappa_g\,ds + 4(n-1)\frac{\pi}{2} = 2\pi$$

which implies (13). □

Similarly, if M is an orientable manifold of genus g bounded by n smooth, mutually distinct, regular curves, and if $X : M \to \mathbb{R}^3$ is a regular mapping of class C^2, then we infer the *general Gauss–Bonnet formula*

$$\int_X K\,dA + \sum_{j=1}^n \int_{\Gamma_j} \kappa_g\,ds + 4\pi(g-1) + 2\pi n = 0 \tag{14}$$

or, equivalently,

$$\int_X K\,dA + \sum_{j=1}^n \int_{\Gamma_j} \kappa_g\,ds = 2\pi\chi(M) \tag{15}$$

where $\chi(M) = 2(1-g) - n$ is the Euler–Poincaré-number of the manifold M. If we consider an arbitrary triangulation of M with V vertices, E edges and F faces, then

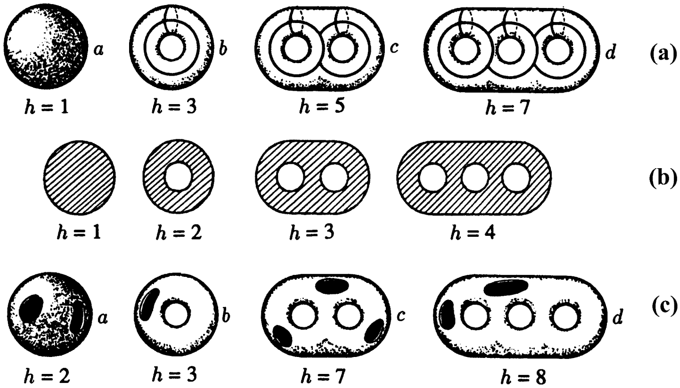


Fig. 1. Orientable surfaces with and without boundaries and of finite connectivity. **(a)** Compact surfaces of genus $g = 0, 1, 2, 3$ without boundary. **(b)** Schlicht domains ($g = 0$) with $1, 2, 3, 4$ boundary curves. **(c)** Compact surfaces with n boundary curves of genus $g = 0, 1, 2, 3$. The number $h = 2g + n$ indicates the order of connectivity of the corresponding surface. Taken from K.H. Naumann and H. Bödeker (in Hilbert and Cohn-Vossen [1])

$$\chi(M) = F - E + V.$$

The proof of (14) or (15) can be performed in a similar way as that of (13). We only have to cut M into a simply connected domain M' which is to be mapped into $\mathbb{R}^2$ whence (12) can be applied. We shall refrain from carrying out the details.

1.5 Covariant Differentiation. The Beltrami Operator

In this section we shall briefly discuss the algebraic formalism connected with the so-called *covariant differentiation*. To simplify notations we restrict ourselves to functions and vector fields of class C^∞ on a regular C^∞-surface $X : \Omega \to \mathbb{R}^3$. By counting the derivatives that are actually needed the reader can easily modify these assumptions. Usually the existence of continuous derivatives up to second or at most third order will suffice.

Denote by $\mathfrak{V}(X)$ the set of tangential vector fields $V : \Omega \to \mathbb{R}^3$ which are of class C^∞. Each $V \in \mathfrak{V}(X)$ can be written in the form

$$V(w) = V^\alpha(w) X_{,\alpha}(w), \quad w \in \Omega,$$

where $V^1, V^2 \in C^\infty(\Omega)$. We can consider $\mathfrak{V}(X)$ as an $\mathfrak{F}(X)$-module over the function space $\mathfrak{F}(X) := C^\infty(\Omega)$, that is, if $f, g \in \mathfrak{F}(X)$ and $V, W \in \mathfrak{V}(X)$, then also $fV + gW \in \mathfrak{V}(X)$ where

$$(fV + gW)(w) = f(w)V(w) + g(w)W(w), \quad w \in \Omega.$$

With each $U = U^\alpha X_{,\alpha} \in \mathfrak{V}(X)$ we can uniquely associate a differential operator $L_U := U^\alpha \partial_\alpha$ defined by

$$(1) \qquad (L_U f)(w) = U^\alpha(w) f_{,\alpha}(w) \quad \text{for } f \in \mathfrak{F}(X)$$

and by

$$(2) \qquad \begin{aligned}(L_U V)(w) &= (U^\alpha \partial_\alpha (V^\beta X_{,\beta}))(w) \\ &= U^\alpha(w) V^\beta(w) X_{,\alpha\beta}(w) + U^\alpha(w) V^\beta_{,\alpha}(w) X_{,\beta}(w)\end{aligned}$$

for $V = V^\beta X_\beta \in \mathfrak{V}(X)$. By the Gauss formula (1) of Section 1.3, we can write

$$(3) \qquad L_U V = [U^\alpha V^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\beta} U^\alpha V^\beta] X_{,\gamma} + b_{\alpha\beta} U^\alpha V^\beta N.$$

Therefore $L_U V$ will in general not be a tangent vector field. If, however, $P = P(w)$ denotes the operator-valued function which associates with every $w \in \Omega$ the orthogonal projection $P(w) : \mathbb{R}^3 \to T_w X$ of $\mathbb{R}^3$ onto the tangent space $T_w X$ of X at w, then we can define a mapping $D : \mathfrak{V}(X) \times \mathfrak{V}(X) \to \mathfrak{V}(X)$ setting $D_U V := P L_U V$ or, more precisely,

$$(4) \qquad (D_U V)(w) = P(w)\{(L_U V)(w)\} \quad \text{for } w \in \Omega.$$

From (3) and (4) we infer that

$$(5) \qquad D_U V = [U^\alpha V^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\beta} U^\alpha V^\beta] X_{,\gamma}$$

holds for $U = U^\alpha X_{,\alpha}, V = V^\beta X_{,\beta} \in \mathfrak{V}(X)$. In particular,

$$(6) \qquad D_{X_{,\alpha}} X_{,\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma}.$$

The mapping D which maps each pair (U, V) of tangential vector fields to another tangential vector field $D_U V$ is called the *covariant differentiation* on the surface X. It satisfies the three rules of a *connection*:

$$(7) \qquad \begin{array}{lll} \text{(i)} & D_U[\alpha V + \beta W] = \alpha D_U V + \beta D_U W & \text{for } \alpha, \beta \in \mathbb{R}, \\ \text{(ii)} & D_{[fU + gV]} W = f D_U W + g D_V W & \text{for } f, g \in \mathfrak{F}(X), \\ \text{(iii)} & D_U[fV] = (L_U f) \cdot V + f \cdot D_U V & \text{for } f \in \mathfrak{F}(X) \end{array}$$

and for $U, V, W \in \mathfrak{V}(X)$. One easily proves that

$$(8) \qquad L_U \langle V, W \rangle = \langle D_U V, W \rangle + \langle V, D_U W \rangle.$$

Moreover, we have the two formulas

$$(9) \qquad D_U V - D_V U - [U, V] = 0$$

and

$$(10) \qquad R(U, V) W = D_U D_V W - D_V D_U W - D_{[U,V]} W$$

where

$$[U, V] := (U^\alpha V^\beta_{,\alpha} - V^\alpha U^\beta_{,\alpha}) X_{,\beta}$$

denotes the *commutator* field of the two vector fields $U = U^\alpha X_{,\alpha}$, $V = V^\alpha X_{,\alpha} \in \mathfrak{V}(X)$ and $R(U, V)W$ is the Riemann curvature tensor.

For each function $f \in \mathfrak{F}(X)$ we can define a differential form ω_f of degree one by setting

$$\omega_f(V) := L_V f = V^\alpha f_{,\alpha}$$

for each $V = V^\alpha X_{,\alpha} \in \mathfrak{V}(X)$. For each f, we can find a uniquely determined vector field $U \in \mathfrak{V}(X)$ such that

$$\omega_f(V) = \langle U, V \rangle \quad \text{for all } V \in \mathfrak{V}(X)$$

holds. Setting $\nabla_X f := U$, we obtain a linear mapping $\nabla_X : \mathfrak{F}(X) \to \mathfrak{V}(X)$ which satisfies

$$L_V f = \langle \nabla_X f, V \rangle \quad \text{for all } V \in \mathfrak{V}(X) \tag{11}$$

and any $f \in \mathfrak{F}(X) = C^\infty(\Omega)$.

Let $f^{,1}$ and $f^{,2}$ be the coordinate functions of $\nabla_X f$ with respect to the base vectors X_1 and X_2, respectively, that is,

$$\nabla_X f = f^{,\alpha} X_{,\alpha}. \tag{12}$$

We claim that

$$f^{,\alpha} = g^{\alpha\beta} f_{,\beta}, \quad \text{where } f_{,\alpha} = \frac{\partial f}{\partial u^\alpha}. \tag{12$'$}$$

In fact, if $U = U^\alpha X_{,\alpha}$ and $V = V^\beta X_{,\beta}$, then

$$\langle U, V \rangle = \langle U^\alpha X_{,\alpha}, V^\beta X_{,\beta} \rangle = U^\alpha V^\beta \langle X_{,\alpha}, X_{,\beta} \rangle = g_{\alpha\beta} U^\alpha V^\beta,$$

and if we choose $U^\alpha = g^{\alpha\gamma} f_{,\gamma}$, we obtain

$$\langle U, V \rangle = g_{\alpha\beta} g^{\alpha\gamma} f_{,\gamma} V^\beta = \delta^\gamma_\beta f_{,\gamma} V^\beta = f_{,\beta} V^\beta = L_V f$$

whence $U = \nabla_X f$.

We call the vector field $\nabla_X f \in \mathfrak{V}(X)$ the *X-gradient of the function* $f \in \mathfrak{F}(X)$. To compute its length, we consider

$$\langle \nabla_X f, \nabla_X f \rangle = g_{\alpha\beta} f^{,\alpha} f^{,\beta} = g_{\alpha\beta} g^{\beta\gamma} f^{,\alpha} f_{,\gamma} = f^{,\alpha} f_{,\alpha}.$$

Thus we obtain

$$|\nabla_X f|^2 = f^{,\alpha} f_{,\alpha} = g^{\alpha\beta} f_{,\alpha} f_{,\beta}. \tag{13}$$

This expression for the square of the X-gradient of f is sometimes called *the first Beltrami differentiator*. In terms of the Gauss symbols $\mathcal{E}, \mathcal{F}, \mathcal{G}$, we have

$$(13') \qquad |\nabla_X f|^2 = \frac{\mathcal{E} f_v^2 - 2\mathcal{F} f_u f_v + \mathcal{G} f_u^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}.$$

Let us now fix some $W \in \mathfrak{V}(X)$, and consider the mapping

$$A : \mathfrak{V}(X) \to \mathfrak{V}(X)$$

which is defined by

$$AV := D_V W \quad \text{for any } V \in \mathfrak{V}(X).$$

This is an $\mathfrak{F}(X)$-linear operation, that is,

$$A(fU + gV) = fA(U) + gA(V)$$

for $f, g \in \mathfrak{F}(X)$ and $U, V \in \mathfrak{V}(X)$.

Let us associate with A the 2×2-matrix

$$\mathcal{A} = (a_\alpha^\gamma), \quad \text{where } a_\alpha^\gamma = W^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\beta} W^\beta$$

if $W = W^\beta X_{,\beta}$.

Clearly $\mathcal{A}$ is the representation of the linear operator A with respect to the base vector fields $X_{,1}$ and $X_{,2}$. In fact, for $V = V^\alpha X_{,\alpha}$, we have according to equation (5) that

$$AV = a_\alpha^\gamma V^\alpha X_{,\gamma}.$$

The trace $a_1^1 + a_2^2$ of $\mathcal{A}$ will be called the X-*divergence of the tangential vector field* $W = W^\beta X_{,\beta}$:

$$(14) \qquad \operatorname{div}_X W := \operatorname{trace} A = a_1^1 + a_2^2 = W^\alpha_{,\alpha} + \Gamma^\alpha_{\alpha\beta} W^\beta.$$

By virtue of formula (12) of Section 1.3, we have

$$\Gamma^\alpha_{\alpha\beta} = \frac{g_{,\beta}}{2g} = \frac{1}{\sqrt{g}} \partial_\beta \sqrt{g}$$

whence

$$\operatorname{div}_X W = W^\alpha_{,\alpha} + \Gamma^\alpha_{\alpha\beta} W^\beta = W^\alpha_{,\alpha} + \frac{1}{\sqrt{g}} (\partial_\beta \sqrt{g}) W^\beta = \frac{1}{\sqrt{g}} \partial_\beta [\sqrt{g} W^\beta],$$

that is,

$$(15) \qquad \operatorname{div}_X W = \frac{1}{\sqrt{g}} [\sqrt{g} W^\alpha]_{,\alpha}.$$

Next we define a linear mapping $\Delta_X : \mathfrak{F}(X) \to \mathfrak{F}(X)$ by

$$(16) \qquad \Delta_X f := \operatorname{div}_X (\nabla_X f).$$

Since

$$\nabla_X f = [g^{\alpha\beta} f_{,\beta}] X_{,\alpha},$$

we infer from (15) that

$$\Delta_X f = \frac{1}{\sqrt{g}} \partial_\alpha [\sqrt{g} g^{\alpha\beta} \partial_\beta f]. \tag{17}$$

We name Δ_X the *Laplace–Beltrami operator* on X. Sometimes $\Delta_X f$ is called *Beltrami's second differentiator*.

Using the Gauss symbols, $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{g}$, we can also write

$$\Delta_X f = \frac{1}{\mathcal{W}} \left[\left(\frac{\mathcal{E} f_v - \mathcal{F} f_u}{\mathcal{W}} \right)_v + \left(\frac{\mathcal{G} f_u - \mathcal{F} f_v}{\mathcal{W}} \right)_u \right]. \tag{18}$$

Another formula for $\Delta_X f$ follows from (14) by virtue of Section 1.3, (9):

$$\begin{aligned} \operatorname{div}_X \nabla_X f &= \operatorname{div}_X \{ f^{,\alpha} X_{,\alpha} \} = f^{,\alpha}_{,\alpha} + \Gamma^\alpha_{\alpha\beta} f^{,\beta} \\ &= \{ g^{\alpha\gamma} f_{,\gamma} \}_{,\alpha} + \Gamma^\alpha_{\alpha\beta} g^{\beta\gamma} f_{,\gamma} \\ &= g^{\alpha\gamma} f_{,\alpha\gamma} + [g^{\alpha\gamma}_{,\alpha} + \Gamma^\alpha_{\alpha\beta} g^{\beta\gamma}] f_{,\gamma} \\ &= g^{\alpha\gamma} f_{,\alpha\gamma} - \Gamma^\gamma_{\alpha\beta} g^{\alpha\beta} f_{,\gamma}. \end{aligned}$$

Thus

$$\Delta_X f = g^{\alpha\beta} \{ f_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} f_\gamma \}. \tag{19}$$

We now want to apply an integration by parts to the integral

$$\int_{X|_B} \langle \nabla_X \varphi, \nabla_X f \rangle \, dA =: J$$

where $B \subset\subset \Omega$ is a domain with a smooth regular boundary ∂B, and $\varphi, f \in \mathfrak{F}(X) = C^\infty(\Omega)$.

Polarization of (13) yields

$$\langle \nabla_X \varphi, \nabla_X f \rangle = g^{\alpha\beta} \varphi_{,\alpha} f_{,\beta}$$

whence

$$\begin{aligned} J &= \int_B g^{\alpha\beta} \varphi_{,\alpha} f_{,\beta} \sqrt{g} \, du \, dv \\ &= \int_B \partial_\alpha (\varphi \sqrt{g} g^{\alpha\beta} f_{,\beta}) \, du \, dv - \int_B \varphi \partial_\alpha (\sqrt{g} g^{\alpha\beta} f_{,\beta}) \, du \, dv \\ &= \int_{\partial B} \varphi \sqrt{g} g^{\alpha\beta} f_{,\beta} \varepsilon_{\alpha\gamma} \, du^\gamma - \int_B \varphi \Delta_X f \sqrt{g} \, du \, dv \end{aligned} \tag{20}$$

where we have introduced $\varepsilon_{\alpha\gamma}$ by

$$\varepsilon_{\alpha\gamma} = \begin{cases} 1 & \text{if } \alpha < \gamma, \\ -1 & \text{if } \alpha > \gamma, \\ 0 & \text{if } \alpha = \gamma. \end{cases} \tag{21}$$

Let us now parametrize the curve $X|_{\partial B}$ by its arc length s such that $X|_{\partial B}$ is positively oriented with respect to $X|_B$; then $\boldsymbol{t} = t^\alpha X_{,\alpha} = \frac{du^\alpha}{ds} X_{,\alpha}$ is the unit tangent vector along $X|_{\partial B}$. Set $\nu_\beta := \sqrt{g}\varepsilon_{\beta\gamma}t^\gamma$, $\nu^\alpha := g^{\alpha\beta}\nu_\beta$ and $\nu := \nu^\alpha X_{,\alpha}$. Then ν is a tangent vector field to X along the boundary curve $X|_{\partial B}$ with the property that $\langle \nu, \boldsymbol{t}\rangle = 0$ and $|\nu| = 1$. Thus ν is the exterior normal to $X|_{\partial B}$ tangent to X (i.e., ν is collinear to the side normal $\boldsymbol{s}$ of $X|_{\partial B}$), and

$$\sqrt{g} g^{\alpha\beta} f_{,\beta}\varepsilon_{\alpha\gamma}t^\gamma = \nu^\beta f_{,\beta} = g_{\alpha\beta} f^{,\alpha}\nu^\beta = \langle \nabla_X f, \nu\rangle.$$

If we introduce the directional derivative

$$\frac{\partial}{\partial \nu} f := \langle \nabla_X f, \nu\rangle$$

of f in direction of the exterior normal ν, we finally infer from (20) that

$$\int_{X|_B} \langle \nabla_X \varphi, \nabla_X f\rangle \, dA = \int_{X|_{\partial B}} \varphi \frac{\partial}{\partial \nu} f \, ds - \int_{X|_B} \varphi \Delta_X f \, dA. \tag{22}$$

Consider now the *generalized Dirichlet integral*

$$E_B(f) := \frac{1}{2}\int_B |\nabla_X f|^2 \sqrt{g}\, du\, dv. \tag{23}$$

Its first variation at f in direction of φ is defined as

$$\delta E_B(f,\varphi) := \frac{d}{d\varepsilon} E_B(f+\varepsilon\varphi)\Big|_{\varepsilon=0} = \int_B \langle \nabla_X \varphi, \nabla_X f\rangle \sqrt{g}\, du\, dv. \tag{24}$$

Hence, the equation

$$\delta E_B(f,\varphi) = 0 \tag{25}$$

holds for all $\varphi \in C_c^\infty(B)$ if and only if

$$\Delta_X f = 0 \quad \text{on } B. \tag{26}$$

That is, the *Laplace–Beltrami equation* (26) *is the Euler equation of the generalized Dirichlet integral* (23).

The $\mathfrak{F}(X)$-linear mapping $H_f : \mathfrak{V}(X) \to \mathfrak{V}(X)$ associated with any $f \in \mathfrak{F}(X)$ and defined by

$$H_f(V) := D_V(\nabla_X f) \quad \text{for } V \in \mathfrak{V}(X) \tag{27}$$

is called the *Hessian tensor of* f, and the bilinear form

(28) $$h_f(U,V) := \langle H_f(U), V\rangle = \langle D_U \nabla_X f, V\rangle,$$

$U, V \in \mathfrak{V}(X)$, is called the *Hessian form of* f.

An analogous computation as before shows that, for $V = V^\alpha X_{,\alpha} \in \mathfrak{V}(X)$, we have

(29) $$H_f(V) = h_\alpha^\beta V^\alpha X_{,\beta}$$

with

(29′) $$h_\alpha^\beta = g^{\beta\gamma}\{f_{,\alpha\gamma} - \Gamma_{\alpha\gamma}^\delta f_{,\delta}\}.$$

It follows easily from these formulas that $h_f(U,V)$ is a symmetric bilinear form.

The trace of the linear mapping H_f is given by $h_\alpha^\alpha = g^{\alpha\gamma}\{f_{,\alpha\gamma} - \Gamma_{\alpha\gamma}^\delta f_{,\delta}\}$, whence by (19)

(30) $$\Delta_X f = \operatorname{trace} H_f.$$

We close our considerations by a brief excursion to geodesics.

Let $\omega : [a,b] \to \Omega$ be a curve in the parameter domain Ω, and $c := X \circ \omega$ be the curve lifted to the surface X.

A vector field $V : [a,b] \to \mathbb{R}^3$ is said to be a *tangential field along* c if $V(t) \in T_{\omega(t)}X$ for all $t \in [a,b]$. Let $\mathfrak{V}_c$ be the class of tangential C^∞-vector fields along c.

For any $V(t) = V^\alpha(t) X_{,\alpha}(\omega(t))$, $t \in [a,b]$, we define the *covariant derivative* $\frac{DV}{dt}(t) \in \mathfrak{V}_c$ by

(31) $$\left(\frac{DV}{dt}\right)(t) = P(\omega(t))\left\{\frac{d}{dt}V(t)\right\}$$

where $P(w)$ is the orthogonal projection of $\mathbb{R}^3$ onto $T_w X$.

We note the particularly important relation

(32) $$\frac{d}{dt}\langle U,V\rangle = \left\langle \frac{DU}{dt}, V\right\rangle + \left\langle U, \frac{DV}{dt}\right\rangle$$

for arbitrary $U, V \in \mathfrak{V}_c$.

Since
$$\frac{d}{dt}V = \dot V^\alpha X_{,\alpha}(\omega) + V^\alpha X_{,\alpha\beta}(\omega)\dot\omega^\beta$$
we infer from the Gauss formulas (1) of Section 1.3 that

(33) $$\frac{DV}{dt} = [\dot V^\gamma + \Gamma_{\alpha\beta}^\gamma(\omega) V^\alpha \dot\omega^\beta] X_{,\gamma}(\omega).$$

If, in particular, $t = s$ (parameter of arc length along c), then $\boldsymbol{t} = \dot c$ and $\ddot c = \dot{\boldsymbol{t}} = \kappa_g \boldsymbol{s} + \kappa_n \mathfrak{N}$ whence

$$\frac{D}{dt}\dot{c} = P(\omega)\ddot{c} = \kappa_g \boldsymbol{s}. \tag{34}$$

Hence, the *geodesic curvature* $\kappa_g(t)$ of $c(t)$, $t = s$, *vanishes identically if and only if*

$$\frac{D}{dt}\dot{c} = 0 \tag{35}$$

or, equivalently, if

$$\frac{d^2\omega^\gamma}{dt^2} + \Gamma^\gamma_{\alpha\beta}(\omega)\frac{d\omega^\alpha}{dt}\frac{d\omega^\beta}{dt} = 0, \quad \gamma = 1, 2. \tag{36}$$

The curves on the surface X satisfying (35) are called *geodesic curves* or simply *geodesics* on X.

A vector field $V = V^\alpha X_{,\alpha}(\omega) \in \mathfrak{V}_c$ is said to be *parallel* (or, more precisely, *autoparallel*) if

$$\frac{DV}{dt} = 0 \tag{37}$$

holds along c, that is, if

$$\dot{V}^\gamma + \Gamma^\gamma_{\alpha\beta}(\omega)V^\alpha\dot{\omega}^\beta = 0 \quad (\gamma = 1, 2). \tag{38}$$

It follows from (32) that

$$\langle U(t), V(t)\rangle \equiv \text{const} \quad \text{for all } t \in [a, b],$$

if U and V are parallel fields in $\mathfrak{V}_c$, and in particular

$$|V(t)| \equiv \text{const} \quad \text{if } \frac{DV}{dt}(t) \equiv 0.$$

Since, by definition, the velocity vector $\dot{c}$ of a geodesic satisfies $\frac{D}{dt}\dot{c} = 0$, we obtain that

$$|\dot{c}(t)| \equiv \text{const},$$

that is, every geodesic $c(t)$ is parametrized proportionally to the arc length. Thus we conversely obtain that *each geodesic has zero geodesic curvature.*

Consider now the *energy functional*

$$E(c) := \frac{1}{2}\int_a^b |\dot{c}|^2\, dt \tag{39}$$

and the *length functional*

$$L(c) := \int_a^b |\dot{c}|\, dt \tag{40}$$

for curves $c = X(\omega) : [a,b] \to \mathbb{R}^3$ on X. The first variations $\delta E(c,V)$ and $\delta L(c,V)$ of E and L at a curve c in direction of $V \in \mathfrak{V}_c$ are given by

$$\delta E(c,V) = \int_a^b \left\langle \dot{c}, \frac{D}{dt} \right\rangle dt \tag{41}$$

and by

$$\delta L(c,V) = \int_a^b \left\langle \dot{c}, \frac{D}{dt} V \right\rangle |\dot{c}|^{-1/2} \, dt. \tag{42}$$

This can be seen by embedding $c(t)$ in a sufficiently differentiable family $\psi(t,\varepsilon), (t,\varepsilon) \in [a,b] \times (-\varepsilon_0, \varepsilon_0)$, of curves $\psi(\cdot,\varepsilon)$ on X such that $\psi(t,0) = c(t)$ and

$$\frac{\partial}{\partial \varepsilon} \psi(t,0) = V(t).$$

Since $\frac{\partial}{\partial \varepsilon} \frac{\partial}{\partial t} \psi(t,\varepsilon)|_{\varepsilon=0} = \dot{V}(t)$, we obtain

$$\begin{aligned}
\delta E(c,V) &= \frac{d}{d\varepsilon} E(\psi(\cdot,\varepsilon)) \Big|_{\varepsilon=0} \\
&= \int_a^b \left\langle \dot{c}, \frac{\partial}{\partial \varepsilon} \frac{\partial}{\partial t} \psi \right\rangle \Big|_{\varepsilon=0} dt \\
&= \int_a^b \langle \dot{c}, \dot{V} \rangle \, dt = \int_a^b \left\langle \dot{c}, \frac{D}{dt} V \right\rangle dt,
\end{aligned}$$

because of $\dot{c} \in \mathfrak{V}_c$. In the same way we can prove (42). For $V \in \mathfrak{V}_c$ with $V(a) = 0$ and $V(b) = 0$ we infer that

$$\delta E(c,V) = -\int_a^b \left\langle \frac{D\dot{c}}{dt}, V \right\rangle dt, \tag{43}$$

and if, in addition, $|\dot{c}(t)| \equiv \text{const} \neq 0$, we obtain also

$$\delta L(c,V) = -\int_a^b |\dot{c}|^{-1/2} \left\langle \frac{D}{dt} \dot{c}, V \right\rangle dt. \tag{44}$$

Therefore, the *equation* (35) *is the Euler equation of the energy functional $E(c)$, and it is also the Euler equation of the length functional $L(c)$ if we restrict ourselves to curves c that are parametrized proportionally to the arc length.*

1.6 Scholia

1 Textbooks

The notes by Klingenberg [1] yield a concise and lucid introduction to differential geometry. A very readable modern text with numerous examples is the book by do Carmo [1]. The famous treatises [1] and [2] by Blaschke with their historical annotations still provide an excellent guide to classical differential geometry. We also refer to the modernized version of Blaschke [2] written by Leichtweiß, cf. Blaschke and Leichtweiß [1].

As a reference to Riemannian geometry, we mention the lecture notes by Gromoll, Klingenberg, and Meyer [1], the monographs by do Carmo [3], Jost [18], Kobayashi and Nomizu [1], Kühnel [2], Spivak [1], Warner [1], Dubrovin, Fomenko, and Novikov [1], and the notes by Chern [1,3] and Hicks [1].

2 Annotations to the History of the Theory of Surfaces

The curvature theory of surfaces began with Euler's investigation [1] from 1760 (printed 1767) concerning the curvature of plane sections of a given surface. Euler's results were supplemented by Meusnier [1], to whom in 1774 Monge had suggested Euler's paper [1] as a starting point for further investigations. The paper Meusnier [1], in which the expression $2H = \kappa_1 + \kappa_2$ appeared for the first time, is the only publication by Meusnier.

The first glimpse to the *inner theory of surfaces* can be found in Euler's publication [2] from 1771 (printed 1772) which is memorable as the first paper which solely operates with the first fundamental form (or the *line element*). It contains the following main result: *All surfaces that are developable into the plane are formed by the tangents of some space curve.* Euler's method of operating with two parameters and with the line element was, according to Speiser, only taken up by Gauss.

In other publications (for instance, [3] and [4]) Euler laid the foundations of a theory of conformal mappings and noticed its connection with the theory of complex analytic functions.

Great merits for the further development have to be attributed to Monge whose stimulating treatise [1] appeared between 1795 and 1807.

The modern development of differential geometry started with the work of Gauss, especially with his prize-winning essay on conformal mappings [1] from 1822 (published in 1825) and his *Disquisitiones generales circa superficies curvas* [3] from 1827 (which appeared in print in 1828) that F. Klein called the *bible of modern differential geometry*.

Already in the spring of 1816, Gauss had suggested to Schumacher as a prize-question for the newly founded "Zeitschrift für Astronomie und verwandte Wissenschaften", to *map two curved surfaces onto each other such that the similarity of smallest parts is preserved.* A letter to Schumacher dated July 5, 1816 documents that the solution of this problem was known to Gauss, and a note of this solution is preserved (cf. vol. 8 of [2], p. 371). Schumacher arranged at the first possible occasion that the Copenhagen Academy of Sciences posed this problem as prize-question for 1821 to the scientific community, namely: "generaliter superficiem datam in alia superficie ita exprimere, ut partes minimae imaginis archetypo fiant similes". When in 1821 no solution came in, the question was renewed for 1822. Having been urged by Schumacher, Gauss sent his contribution to Copenhagen on December 11, 1822, and, in 1823, he obtained the prize of the academy. In 1825, Gauss's paper was published in the last issue of Schumacher's "Astronomische Abhandlungen". The main result of this investigation is that *every sufficiently small piece of a regular, real-analytic surface can be mapped conformally onto a domain in the plane.* Gauss's proof uses the method of characteristics which restricts his reasoning to real-analytic surfaces.

Conformal mappings were, by the way, investigated long before Euler and Gauss by geographers since such mappings naturally arise in the problem of chart-making. Already Hipparch and also Ptolemy knew of the conformality of the stereographic projection of the sphere into the plane. Mercator's chart of the world that was completed in 1569 used another conformal mapping of the sphere, known as Mercator projection. Lambert's conformal projections from 1772 (cf. [1]) as well as Mercator's projection are still used today. The publications of Euler [3] and Lagrange [2] solve special cases of the problem treated by Gauss in 1822. The term *conformal mapping* (konforme Abbildung) was coined by Gauss in 1844 (see vol. 4 of [2], p. 262).

From Gauss's prize-essay, one can draw a direct line to Riemann's thesis from 1851 (see Riemann [1], pp. 3–45, especially pp. 42–43), where the foundations of a *geometrical theory of functions* were laid. As Riemann observed, such a theory provided a global version of Gauss's mapping theorem from 1822.

Although Riemann's reasoning was defective, his thesis stimulated the research of a whole century. A culmination point of the subsequent developments was the proof of the *uniformization theorem* by Poincaré and Koebe (1909), which is the precise form of the global Gauss mapping theorem envisioned by Riemann. It is described in H. Weyl's celebrated monograph: *Die Idee der Riemannschen Fläche* [4].

In 1916, Lichtenstein [3] established the analogue of Gauss's local mapping theorem for surfaces of class $C^{1,\alpha}$ which, together with the uniformization theorem, yields the global theorem stated in Section 1.4. A direct proof of this result by a variational method was proposed by Morrey [8]. The faulty reasoning of Morrey was rectified and completed by Jost [6] and [17]. Another approach, due to Bers and Vekua, is based on the theory of generalized analytic functions (cf., for instance, Vekua [1] and [2]). Simplified proofs were recently given by Sauvigny [13] and Hildebrandt and von der Mosel [6,8].

The expression $2H = \kappa_1 + \kappa_2$ for the *mean curvature* appeared, as we already have noted, first in Meusnier's paper [1] where the equation $H = 0$ for minimal surfaces was derived. Thereafter, Young [1] and Laplace [1] reintroduced this expression in their theories of capillarity from 1805 and 1806, respectively. The prize-essay by Gauss as well as his *Disquisitiones generales* were in part the result of his practical work as director and organizer of the geodesic measurements in the Kingdom of Hannover during the years 1821–1825. The letters of his friends Bessel and Schumacher testify the deep impression the *Disquisitiones* made immediately after their appearance. The wealth of ideas is indeed overwhelming. We find the definition of the *spherical image* that, as Gauss pointed out elsewhere, is derived from the method of the astronomer to describe the locus of a star on the celestial sphere. The Gauss curvature K is defined as the (signed) ratio $\pm\frac{dA_N}{dA_X}$, taking orientation into account, as has been described in Section 1.2. We have introduced Gauss's representation formulas in Section 1.3 and, as the main result, we have presented the *theorema egregium* which opened the way to Riemann's intrinsic geometry of higher dimensional manifolds as it was sketched in Riemann's Habilitation lecture from 1854 (see Riemann [1], pp. 254–269). Dedekind reported in his biographical sketch of Riemann's life how amazed and excited Gauss was by this lecture (Riemann [1], p. 517).

Gauss, moreover, proved that $K = \kappa_1 \cdot \kappa_2$, and introducing the total curvature (curvatura integra) $\int_X K\,dA$, he could compute the sum of the angles in a geodesic triangle Δ on X in terms of the total curvature of X.

From here it was only one more step to the Gauss–Bonnet theorem stated in Section 1.4 that was first formulated and proved by Bonnet [1] in 1848. One only needs a formula for the quantity κ_g, which Bonnet called geodesic curvature, and the formula for integration by parts. Such a formula in terms of the coefficients of the first fundamental form was first published by Minding [1]. It turned out, however, that a similar result was already known to Gauss. It was found in his posthumous papers (see vol. 8 of [2], pp. 386–396). Since Gauss in his *Disquisitiones* announced the publication of further investigations on the curvatura integra (which, however, never appeared), there seems to be no doubt that he, in fact, knew the essence of the Gauss–Bonnet theorem. It should be mentioned that Gauss used the term *side curvature* (Seitenkrümmung) instead of geodesic curvature.

We also note that, with the single exception of Euler [2], all authors before Gauss only considered surfaces given as graphs $z = z(x, y)$ of a function over a planar domain. The point of view mostly taken in our notes is that of Gauss. The more general notion of manifolds (and their immersions) was first envisioned by Riemann [1] and then solidly founded by H. Weyl [4].

The two differential expressions $|\nabla_X f|^2$ and $\Delta_X f$ were introduced by Beltrami [1] in 1864. They nowadays play a fundamental role. The Weingarten equations were first stated in Weingarten's paper [1].

There is a long history regarding the *covariant differentiation* of tangent vectors and of tensors, starting with investigations by Christoffel and Lipschitz. The tensor character of the covariant differentiation was first established by Christoffel in [1], where also the 3-index symbols were defined.

Covariant differentiation was developed into a systematic tool by Ricci and Levi-Civita in 1901. The notion of autoparallel vector fields and of parallel displacement along curves was discovered[1] by Levi-Civita [1] in 1917. This,

[1] H. Weyl ([2], 5. Auflage, p. 325) noted that the theory of parallel displacement is already contained in the kinematic considerations of the *Treatise on Natural Philosophy* by Thomson and Tait (edition 1912), Part I, sect. 135–137.

in turn, led to the definition of an *affine connection* on a general differentiable manifold by H. Weyl [1], using the formulas of (7) in Section 1.5 which nowadays are set at the beginning of the theory.

The equations for the *geodesics*, considered as stationary curves of the length functional, stand at the beginning of the theory of surfaces. The first paper on this subject was published by Euler in 1728, but already in 1698 Johann Bernoulli, Euler's teacher at Basel, had discovered that all shortest lines on a surface have vanishing geodesic curvature. He stated his result in the form that, *at each of its points, the osculating plane of a geodesic curve on a surface X must intersect X perpendicularly.*

3 References to the Sources of Differential Geometry and to the Literature on its History

Euler's contributions to the curvature theory of surfaces can be found in his *Opera Omnia*, Ser. I, vols. 28 and 29 [4].

The classical work of Monge appeared as *Application de l'Analyse à la Géométrie* between 1795 and 1807 [1].

The fundamental results by Gauss, in particular his prize-winning paper from 1822 for the Copenhagen Academy and his *Disquisitiones generales circa superficies curvas*, are collected in his *Werke*, vols. 4 and 8 [2], and vols. 10 and 11 contain essays by Bolza [2], Galle [1], and Stäckel [1] which describe the genesis of Gauss's contributions to differential geometry.

There exist translations of the *Disquisitiones* into German and English. The comments by Dombrowski [2] are particularly interesting.

A rich source of references are the surveys by v. Mangoldt, Lilienthal, Scheffers, Voss, Salkowski, Liebmann, Weitzenböck, and Berwald in the *Encyklopädie der mathematischen Wissenschaften* III.3.

A selected and historically ordered bibliography of the differential-geometric literature which begins with Riemann's Habilitation lecture (1854) and reaches till 1949, is contained in Eisenhart's treatise [3].

Numerous interesting historical annotations can be found in the various editions of Weyl's *Raum, Zeit und Materie* [2], which has also been translated into English.

Of inestimable value are Darboux's *Leçons sur la théorie générale des surfaces* [1], which comprehend a large part of the differential geometric knowledge at the end of the nineteenth century.

We finally mention Klein's lectures [1] on the development of mathematics in the nineteenth century, which provide a comprehensive view of one of the great epochs of mathematics.

Chapter 2

Minimal Surfaces

Since the last century, the name *minimal surfaces* has been applied to surfaces of vanishing mean curvature, because the condition

$$H = 0$$

will necessarily be satisfied by surfaces which minimize area within a given boundary configuration. This was implicitly proved by Lagrange for nonparametric surfaces in 1760, and then by Meusnier in 1776 who used the analytic expression for the mean curvature and determined two minimal surfaces, the catenoid and the helicoid. (The notion of mean curvature was introduced by Young [1] and Laplace [1], but usually it is ascribed to Sophie Germain [1].) In Section 2.1 we shall derive an expression for the first variation of area with respect to general variations of a given surface. From this expression we obtain the equation $H = 0$ as necessary condition for stationary surfaces of the area functional, and we also demonstrate that solutions of the free boundary problem meet their supporting surfaces at a right angle.

In Section 2.2, we particularly investigate nonparametric surfaces, and we state the *minimal surface equation* in divergence and nondivergence form which has to be satisfied by the height function. Finally we prove that, for a nonparametric minimal surface X, the 1-form $N \wedge dX$ is closed. In Section 2.3 it is shown that a nonparametric minimal surface $X(x, y) = (x, y, z(x, y))$ has a real analytic height function $z(x, y)$ and, moreover, that X can be conformally mapped onto some planar domain. This conformal mapping can be constructed explicitly if the domain of definition Ω of the surface X is convex.

Thereafter we prove in Section 2.4 the celebrated Bernstein theorem for nonparametric minimal surfaces and also a quantitative local version of this theorem which was discovered by E. Heinz. Then we show in Section 2.5 that every regular surface $X : \Omega \to \mathbb{R}^3$ satisfies the equation

$$\Delta_X X = 2HN$$

U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 339,
DOI 10.1007/978-3-642-11698-8_2,

and, therefore, minimal surfaces are characterized by the equation

$$\Delta_X X = 0.$$

If X is given by conformal parameters, this relation is equivalent to

$$\Delta X = 0.$$

This observation is used in Section 2.6 to enlarge the class of minimal surfaces. We can now admit surfaces with isolated singularities by defining minimal surfaces as harmonic mappings $X : \Omega \to \mathbb{R}^3$ that are given in conformal parameters.

In Section 2.7 we derive a formula for the mean curvature of surfaces that are defined by implicit equations. This relation is used in the last part of the chapter to demonstrate that a minimal surface provides a minimum of area if it can be embedded into a field of minimal surfaces. Finally, an expression for the second variation of area is given, and we comment on the question when a given minimal surface can be embedded into such a field.

2.1 First Variation of Area. Minimal Surfaces

Let $X : \bar{\Omega} \to \mathbb{R}^3$ be a regular surface of class C^2 with its spherical image $N : \bar{\Omega} \to \mathbb{R}^3$ defined by

$$N = \frac{1}{\mathcal{W}} X_u \wedge X_v, \quad \mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \sqrt{g},$$

and denote by $g_{\alpha\beta}$ and $b_{\alpha\beta}$ (or $\mathcal{E}, \mathcal{F}, \mathcal{G}$ and $\mathcal{L}, \mathcal{M}, \mathcal{N}$, respectively) the coefficients of its first and second fundamental forms. Moreover, H stands for the mean curvature of X. We write $w = (u, v)$, $u^1 = u$, $u^2 = v$, and $X_{,\alpha} = \frac{\partial}{\partial u^\alpha} X$; $\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of the second kind for X introduced in Section 1.3.

We now consider a *variation of* X, that is, a mapping

$$Z : \bar{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3, \quad \varepsilon_0 > 0,$$

of class C^2, with the property that

$$Z(w, 0) = X(w) \quad \text{for all } w \in \bar{\Omega}.$$

This map will be interpreted as a family of surfaces $Z(w, \varepsilon)$, $w \in \bar{\Omega}$, which vary X, and in which X is embedded.

By Taylor expansion, we can write

$$(1) \qquad Z(w, \varepsilon) = X(w) + \varepsilon Y(w) + \varepsilon^2 R(w, \varepsilon)$$

with a continuous remainder term $\varepsilon^2 R(w,\varepsilon)$ of square order, i.e. $R(w,\varepsilon) = O(1)$ as $\varepsilon \to 0$. The vector field

$$Y(w) = \frac{\partial}{\partial \varepsilon} Z(w,\varepsilon)\bigg|_{\varepsilon=0} \in C^1(\bar{\Omega}, \mathbb{R}^3)$$

is called *the first variation of the family of surfaces* $Z(\cdot,\varepsilon)$.

We can write

$$Y(w) = \eta^\beta(w) X_{,\beta}(w) + \lambda(w) N(w) \tag{2}$$

with functions η^1, η^2, λ of class $C^1(\bar{\Omega})$. Then

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon[\eta^\beta_{,\alpha} X_{,\beta} + \eta^\beta X_{,\alpha\beta} + \lambda_{,\alpha} N + \lambda N_{,\alpha}] + \varepsilon^2 R_{,\alpha}.$$

By virtue of the Gauss equations

$$X_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma} + b_{\alpha\beta} N$$

and the Weingarten equations

$$N_{,\alpha} = -b^\beta_\alpha X_{,\beta}, \quad b^\beta_\alpha = b_{\alpha\gamma} g^{\beta\gamma},$$

we obtain that

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon[\xi^\gamma_\alpha X_{,\gamma} + \nu_\alpha N] + \varepsilon^2 R_{,\alpha} \tag{3}$$

where we have set:

$$\begin{aligned} \xi^\gamma_\alpha &= \eta^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\beta}\eta^\beta - b_{\alpha\beta} g^{\beta\gamma}\lambda, \\ \nu_\alpha &= b_{\alpha\beta}\eta^\beta + \lambda_{,\alpha}. \end{aligned} \tag{4}$$

Then, indicating the ε^2-terms by $\cdots$, we find

$$\begin{aligned} |Z_u|^2 &= \mathcal{E} + 2\varepsilon(\xi^1_1 \mathcal{E} + \xi^2_1 \mathcal{F}) + \cdots, \\ |Z_v|^2 &= \mathcal{G} + 2\varepsilon(\xi^1_2 \mathcal{F} + \xi^2_2 \mathcal{G}) + \cdots, \\ \langle Z_u, Z_v\rangle &= \mathcal{F} + \varepsilon[\xi^1_2 \mathcal{E} + (\xi^1_1 + \xi^2_2)\mathcal{F} + \xi^2_1 \mathcal{G}] + \cdots, \end{aligned}$$

whence

$$|Z_u|^2 |Z_v|^2 - \langle Z_u, Z_v\rangle^2 = \mathcal{W}^2[1 + 2\varepsilon(\xi^1_1 + \xi^2_2) + \cdots].$$

We, moreover, have

$$\xi^1_1 + \xi^2_2 = \eta^1_u + \eta^2_v - \lambda b_{\alpha\beta} g^{\alpha\beta} + \Gamma^\alpha_{\alpha\beta}\eta^\beta.$$

Since

$$b_{\alpha\beta} g^{\alpha\beta} = 2H, \quad \Gamma^\alpha_{\alpha\beta} = \frac{1}{2g} g_{,\beta} = \frac{1}{\mathcal{W}} \mathcal{W}_{,\beta}$$

(see formulas (42) of Section 1.2 and (12) of Section 1.3), we infer that

$$\xi_1^1 + \xi_2^2 = \frac{1}{\mathcal{W}}\{(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v\} - 2H\lambda.$$

On account of $\sqrt{1+x} = 1 + x/2 + O(x^2)$ for $|x| \ll 1$, we see that

$$(|Z_u|^2|Z_v|^2 - \langle Z_u, Z_v\rangle^2)^{1/2} = \mathcal{W} + \varepsilon[(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v - 2H\mathcal{W}\lambda] + \cdots .$$

Then we can conclude that the *first variation*

$$\delta A_\Omega(X,Y) := \frac{d}{d\varepsilon}A_\Omega(Z(\cdot,\varepsilon))\Big|_{\varepsilon=0} \tag{5}$$

of the area functional $A_\Omega(X)$ on Ω at X in the direction of a vector field $Y = \eta^\alpha X_\alpha + \lambda N$ is given by

$$\delta A_\Omega(X,Y) = \int_\Omega [(\eta^1\mathcal{W})_u + (\eta^2\mathcal{W})_v - 2H\mathcal{W}\lambda]\, du\, dv. \tag{6}$$

Performing an integration by parts, it follows that

$$\delta A_\Omega(X,Y) = \int_{\partial\Omega} \mathcal{W}(\eta^1\, dv - \eta^2\, du) - 2\int_\Omega \lambda H\mathcal{W}\, du\, dv. \tag{7}$$

This, in particular, implies that

$$\begin{aligned} \delta A_\Omega(X,Y) &= -2\int_X \langle Y,N\rangle H\mathcal{W}\, du\, dv \\ &= -2\int_X \langle Y,N\rangle H\, dA \end{aligned} \tag{8}$$

for all $Y \in C_c^\infty(\Omega,\mathbb{R}^3)$. Since $\lambda = \langle Y,N\rangle$ can be chosen as an arbitrary function of class $C_c^\infty(\Omega)$, the fundamental theorem of the calculus of variations yields:

Theorem 1. *The first variation $\delta A_\Omega(X,Y)$ of A_Ω at X vanishes for all vector fields $Y \in C_c^\infty(\Omega,\mathbb{R}^3)$ if and only if the mean curvature H of X is identically zero.*

In other words, the (regular) stationary points of the area functional—and, in particular, its (regular) minimizers—are exactly the surfaces of zero mean curvature. For this reason, *a regular (i.e. immersed) surface $X : \Omega \to \mathbb{R}^3$ of class C^2 is usually called a minimal surface if its mean curvature function H satisfies*

$$H = 0. \tag{9}$$

We shall later broaden the class of minimal surfaces in order to allow also surfaces with isolated singularities, but then we use conformal parameters u, v.

Let us now formulate a more geometric expression for the first variation of the area. Note that

$$\lambda \mathcal{W} = \langle Y, N\rangle \mathcal{W} = \langle Y, X_u \wedge X_v\rangle = [Y, X_u, X_v] \ (:= \det(Y, X_u, X_v))$$

and, for $Y = \eta^1 X_u + \eta^2 X_v + \lambda N$, we obtain

$$[Y, N, dX] = \eta^1 [X_u, N, X_v\, dv] + \eta^2 [X_v, N, X_u\, du] = \mathcal{W}\ \{\eta^2\, du - \eta^1\, dv\}.$$

Hence, formula (7) implies that

$$-\delta A_\Omega(X, Y) = \int_{\partial\Omega} [Y, N, dX] + 2 \int_\Omega H[Y, X_u, X_v]\, du\, dv. \tag{10}$$

Let $\omega(\mathrm{s})$ be a representation of $\partial\Omega$ in terms of the parameter of arc length s of the boundary $X|_{\partial\Omega}$. Then $c(s) := X(\omega(s))$ is a representation of the boundary of X. Moreover, let $\mathfrak{Y}(s) := Y(\omega(s)), \mathfrak{N}(s) := N(\omega(s))$. Then

$$[Y, N, dX] \circ \omega = \langle \mathfrak{Y}, \mathfrak{N} \wedge \boldsymbol{t}\rangle\, ds = \langle \mathfrak{Y}, \boldsymbol{s}\rangle\, ds$$

where $\boldsymbol{s}$ is the side normal of the boundary curve c of the surface X. Hence we get

$$-\delta A_\Omega(X, Y) = \int_{\partial X} \langle \mathfrak{Y}, \boldsymbol{s}\rangle\, ds + 2 \int_X \langle Y, N\rangle H\, dA. \tag{11}$$

In particular,

$$\delta A_\Omega(X, \lambda N) = -2 \int_X \lambda H\, dA \tag{12}$$

and

$$2H = -\delta A_\Omega(X, N)/A_\Omega(X) \quad \text{if } H = \text{const}. \tag{13}$$

In other words, for surfaces of constant mean curvature H, the expression $-2H$ is just the relative change of the area of the surface with respect to normal variations.

Moreover, we have

$$\delta A_\Omega(X, Y) = -\int_{\partial X} \langle \mathfrak{Y}, \boldsymbol{s}\rangle\, ds \quad \text{if } H = 0, \tag{14}$$

and we obtain the following

Proposition. *If $X : \bar{\Omega} \to \mathbb{R}^3$ is a minimal surface, then the equation*

$$\delta A_\Omega(X, Y) = 0$$

holds for all $Y \in C^1(\bar{\Omega}, \mathbb{R}^3)$ which are orthogonal to the side normal of the boundary ∂X (that is, $\langle \mathfrak{Y}, \boldsymbol{s}\rangle = 0$ on ∂X).

Furthermore, if we assume that

(15) $$\frac{d}{d\varepsilon}A_\Omega(Z(\cdot,\varepsilon))\Big|_{\varepsilon=0} = 0$$

holds for all variations $Z(\cdot,\varepsilon)$ of X whose boundary values lie on some *supporting manifold* $S \subset \mathbb{R}^3$ of dimension two, then it follows that

(16) $\delta A_\Omega(X,Y) = 0$ *holds for all* $Y \in C^1(\bar{\Omega},\mathbb{R}^3)$, *the boundary values of which at* $\partial\Omega$ *are tangential to* S.

From this equation, we firstly infer that X is a minimal surface, and secondly, by once again applying the fundamental theorem of the calculus of variations, we obtain from equation (16) that the side normal of ∂X meets S everywhere at a right angle. This means that X intersects S perpendicularly. Thus we have proved:

Theorem 2. *Suppose that* (15) *holds for all variations* $Z(\cdot,\varepsilon)$ *of* X *with boundary on some supporting surface* S. *Then* X *is a minimal surface which meets* S *orthogonally at its boundary* ∂X.

A minimal surface as in Theorem 2 will be called *a stationary surface to the supporting manifold* S, or *solution of the free boundary problem for* S. The study of such free boundary problems will be emphasized in Section 4.6 and particularly in Vols. 2 and 3. In short, if we consider stationary surfaces in boundary configurations which, in part, consist of fixed curves Γ and, in addition, of free surfaces S (called *support surfaces*), then we deal with minimal surfaces that meet S perpendicularly.

2.2 Nonparametric Minimal Surfaces

We shall now consider surfaces which are given in *nonparametric form*, that is, as graph of a function $z = z(x,y)$ on some domain Ω of $\mathbb{R}^2$. Such a surface can be described by the special parameter representation

$$X(x,y) = (x,y,z(x,y)), \quad (x,y) \in \Omega.$$

(In this case, the parameters are usually denoted by x and y instead of u and v.)

We shall assume that the function $z(x,y)$ is at least of class C^2. Introducing the time-honored abbreviations

(1) $$p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy}$$

we compute that

$$
\mathcal{E} = 1 + p^2, \quad \mathcal{F} = pq, \quad \mathcal{G} = 1 + q^2, \qquad (2)
$$
$$
\mathcal{W}^{\,2} = 1 + p^2 + q^2, \quad N = (\xi, \eta, \zeta),
$$

where

$$
\xi = -p/\sqrt{1 + p^2 + q^2}, \quad \eta = -q/\sqrt{1 + p^2 + q^2}, \qquad (3)
$$
$$
\zeta = 1/\sqrt{1 + p^2 + q^2}.
$$

Moreover,

$$
\mathcal{L} = r/\sqrt{1 + p^2 + q^2}, \quad \mathcal{M} = s/\sqrt{1 + p^2 + q^2}, \qquad (4)
$$
$$
\mathcal{N} = t/\sqrt{1 + p^2 + q^2},
$$

whence finally

$$
H = \frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{2(1 + p^2 + q^2)^{3/2}}, \qquad (5)
$$

$$
K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}. \qquad (6)
$$

Therefore, the equation $H = 0$ is equivalent to the nonlinear second order differential equation

$$
(1 + q^2)r - 2pqs + (1 + p^2)t = 0, \qquad (7)
$$

the so-called *minimal surface equation*. It is necessary and sufficient for a surface $z = z(x, y)$ to be a minimal surface.

For nonparametric surfaces $X(x, y) = (x, y, z(x, y))$ the area functional $A_\Omega(X)$ takes the form

$$
A_\Omega(X) = \int_\Omega \sqrt{1 + p^2 + q^2}\, dx\, dy. \qquad (8)
$$

By Theorem 1 of Section 2.1, a nonparametric minimal surface X, defined by the function $z = z(x, y)$, satisfies $\delta A_\Omega(X, Y) = 0$ for all $Y \in C_c^\infty(\Omega, \mathbb{R}^3)$. In particular for $Y = (0, 0, \zeta)$, $\zeta \in C_c^\infty(\Omega)$, we obtain that

$$
\int_\Omega \left(\frac{p}{\mathcal{W}} \zeta_x + \frac{q}{\mathcal{W}} \zeta_y \right) dx\, dy = 0,
$$

and the fundamental lemma of the calculus of variations yields the Euler equation

$$
\left\{ \frac{p}{\sqrt{1 + p^2 + q^2}} \right\}_x + \left\{ \frac{q}{\sqrt{1 + p^2 + q^2}} \right\}_y = 0 \qquad (9)
$$

for the functional

$$\mathcal{A}(z) := \int_\Omega \sqrt{1+p^2+q^2}\,dx\,dy. \tag{10}$$

Equation (9) can equivalently by written as

$$\operatorname{div} \frac{\nabla z}{\sqrt{1+|\nabla z|^2}} = 0. \tag{11}$$

This relation will be called *the minimal surface equation in divergence form.*

Actually equations (7) and (11) are equivalent. In fact, by means of a straight-forward computation we infer from (5) that

$$\operatorname{div}(\mathcal{W}^{-1}\nabla z) = 2H$$

holds true for any nonparametric surface $z = z(x,y)$. This equation also implies that *any nonparametric surface $X(x,y) = (x,y,z(x,y))$ described by the function $z(x,y)$ is a minimal surface if and only if the* 1*-form*

$$\gamma = -(p/\mathcal{W})\,dy + (q/\mathcal{W})\,dx$$

is a closed differential form on Ω, that is, if and only if

$$\gamma = -dc$$

with some function $c \in C^2(\Omega)$ provided that the domain Ω is simply connected.

There is actually a stronger version of this result which permits a remarkable geometric interpretation. For this purpose, we introduce the differential form

$$N \wedge dX = (\alpha, \beta, \gamma) \tag{12}$$

with the components

$$\alpha = \eta\,dz - \zeta\,dy, \quad \beta = \zeta\,dx - \xi\,dz, \quad \gamma = \xi\,dy - \eta\,dx. \tag{13}$$

Inserting

$$dz = p\,dx + q\,dy, \quad \xi = -p/\mathcal{W}, \quad \eta = -q/\mathcal{W}, \quad \zeta = 1/\mathcal{W},$$

one obtains

$$\begin{aligned} \alpha &= -\frac{pq}{\mathcal{W}}\,dx - \frac{1+q^2}{\mathcal{W}}\,dy, \\ \beta &= \frac{1+p^2}{\mathcal{W}}\,dx + \frac{pq}{\mathcal{W}}\,dy, \\ \gamma &= \frac{q}{\mathcal{W}}\,dx - \frac{p}{\mathcal{W}}\,dy. \end{aligned} \tag{14}$$

Let us introduce the differential expression

$$T := (1 + q^2)r - 2pqs + (1 + p^2)t.$$

Then a straight forward computation shows

$$\begin{aligned}
\left(-\frac{pq}{\mathcal{W}}\right)_y - \left(-\frac{1+q^2}{\mathcal{W}}\right)_x &= -\frac{p}{\mathcal{W}^{\,3}}T,\\
\left(\frac{1+p^2}{\mathcal{W}}\right)_y - \left(\frac{pq}{\mathcal{W}}\right)_x &= -\frac{q}{\mathcal{W}^{\,3}}T,\\
\left(\frac{q}{\mathcal{W}}\right)_y - \left(\frac{-p}{\mathcal{W}}\right)_x &= \frac{1}{\mathcal{W}^{\,3}}T,
\end{aligned}$$

that is,

$$(15) \qquad d\alpha = 2Hp\,dx\,dy, \quad d\beta = 2Hq\,dx\,dy, \quad d\gamma = -2H\,dx\,dy,$$

whence

$$(16) \qquad d(N \wedge dX) = -2H\mathcal{W}N\,dx\,dy$$

or equivalently

$$(16') \qquad d(N \wedge dX) = -2HN\,dA,$$

where dA denotes the area element $\mathcal{W}\,dx\,dy$.

Thus we have proved the following

Theorem 1. *A nonparametric surface $X(x, y) = (x, y, z(x, y))$, described by a function $z = z(x, y)$ of class C^2 on a simply connected domain Ω of $\mathbb{R}^2$, with the Gauss map $N = (\xi, \eta, \zeta)$ is a minimal surface if and only if the vector-valued differential form $N \wedge dX$ is a total differential, i.e. if and only if there is a mapping $X^* \in C^2(\Omega, \mathbb{R}^3)$ such that*

$$(17) \qquad -dX^* = N \wedge dX.$$

If we write

$$(18) \qquad X^* = (a, b, c), \quad N \wedge dX = (\alpha, \beta, \gamma),$$

equation (17) *is equivalent to*

$$(19) \qquad -da = \alpha, \quad -db = \beta, \quad -dc = \gamma.$$

This remarkable theorem will be used to prove that each C^2-solution of the minimal surface equation (7) or (11), respectively, is in fact real analytic, and that it can be mapped conformally onto a planar domain provided that its domain of definition Ω is convex. This will be shown in the next section.

We finally note that nonparametric surfaces, besides being interesting in their own right, serve as useful tools for deriving identities between differential invariants of general surfaces. In fact, locally each regular C^2-surface $X(u, v)$ can, after a suitable rotation of the Cartesian coordinate system in $\mathbb{R}^3$, be written in the nonparametric form stated before. In other words, by a suitable coordinate transformation $w = \varphi(x, y)$ we can pass from $X(w)$ to a strictly equivalent surface $Z(x, y) = X(\varphi(x, y))$ which is of type $(x, y, z(x, y))$ if we have chosen appropriate Cartesian coordinates in $\mathbb{R}^2$. It is evident that for such a representation $Z(x, y)$ many differential expressions have a fairly simple form, and therefore it will be much easier than in the general case to recognize identities. Switching back to the original representation $X(u, v)$, these identities are equally well established provided that the terms involved are known to be invariant with respect to parameter changes.

2.3 Conformal Representation and Analyticity of Nonparametric Minimal Surfaces

Let $X(x, y) = (x, y, z(x, y))$ be a nonparametric minimal surface of class C^2 defined on an open convex set Ω of $\mathbb{R}^2$. We will show that $z(x, y)$ is real analytic and that $X(x, y)$ can be mapped conformally onto some planar domain.

By the Theorem 1 of Section 2.2, there exists a function $a \in C^2(\Omega)$ such that

$$da = \frac{pq}{\mathcal{W}}\,dx + \frac{1 + q^2}{\mathcal{W}}\,dy, \tag{1}$$

where $p = z_x, q = z_y$, and $\mathcal{W} = \sqrt{1 + p^2 + q^2}$.

Then we consider the mapping $\varphi : \Omega \to \mathbb{R}^2$ defined by $\varphi(x, y) = (x, a(x, y))$ which can be expressed by $(x, y) \mapsto (u, v)$ or by the pair of equations

$$u = x, \quad v = a(x, y). \tag{2}$$

Since $a_y = \mathcal{W}^{-1} \cdot (1 + q^2) > 0$ and Ω is convex, the mapping φ is one-to-one, and its Jacobian J_φ satisfies

$$J_\varphi = \frac{\partial(u, v)}{\partial(x, y)} = a_y > 0.$$

Hence φ is a C^2-diffeomorphism which maps Ω onto some domain Ω^* of $\mathbb{R}^2$. Its inverse $\psi : \Omega^* \to \Omega$ of class C^2 is given by

$$x = u, \quad y = f(u, v) \tag{3}$$

with some function $f \in C^2(\Omega^*)$. Since $D\psi(u,v) = [D\varphi(x,y)]^{-1}$, we then obtain

$$\begin{pmatrix} 1 & 0 \\ f_u & f_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_x & a_y \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -a_x/a_y & 1/a_y \end{pmatrix}.$$

On account of (1), we infer that

$$f_u = -\frac{pq}{1+q^2}, \quad f_v = \frac{\mathcal{W}}{1+q^2}, \tag{4}$$

where the arguments u, v and x, y in (4) are related to each other by (3). Next, we transform the function $z(x,y)$ to the new variables u, v and set

$$g(u,v) := z(u, f(u,v)) \tag{5}$$

and

$$Z(u,v) := (u, f(u,v), g(u,v)) = X(\psi(u,v)). \tag{6}$$

Then the differentials dx, dy, $dz = p\,dx + q\,dy$ of the functions $x = u$, $y = f(u,v)$, $z = g(u,v)$ turn out to be

$$\begin{aligned} dx &= du, \\ dy &= df = -\frac{pq}{1+q^2}\,du + \frac{\mathcal{W}}{1+q^2}\,dv, \\ dz &= dg = \frac{p}{1+q^2}\,du + \frac{q\mathcal{W}}{1+q^2}\,dv. \end{aligned} \tag{7}$$

These equations yield the conformality relations

$$|Z_u|^2 = |Z_v|^2 = \frac{1+p^2+q^2}{1+q^2}, \quad \langle Z_u, Z_v \rangle = 0 \tag{8}$$

for the surface $Z = X \circ \psi$ which is strictly equivalent to the nonparametric surface $X(x,y)$.

For the following, we use the two other equations of Section 2.2, (19):

$$-db = \beta, \quad -dc = \gamma,$$

which state that

$$\begin{aligned} db &= -\frac{1+p^2}{\mathcal{W}}\,dx - \frac{pq}{\mathcal{W}}\,dy \\ dc &= -\frac{q}{\mathcal{W}}\,dx + \frac{p}{\mathcal{W}}\,dy. \end{aligned} \tag{9}$$

We introduce a surface

$$Z^*(u,v) = (v, f^*(u,v), g^*(u,v)) \tag{10}$$

for $(u,v) \in \Omega^*$, the components of which are defined by

$$\begin{aligned} f^*(u,v) &:= b(u, f(u,v)),\\ g^*(u,v) &:= c(u, f(u,v)). \end{aligned} \tag{11}$$

It follows from (4) and (9) that

$$\begin{aligned} df^* &= -\frac{\mathcal{W}}{1+q^2}\,du - \frac{pq}{1+q^2}\,dv,\\ dg^* &= -\frac{q\mathcal{W}}{1+q^2}\,du + \frac{p}{1+q^2}\,dv. \end{aligned} \tag{12}$$

Comparing (7) and (12), we see that Z and Z^* satisfy the Cauchy–Riemann equations

$$Z_u = Z_v^*, \quad Z_v = -Z_u^* \tag{13}$$

on Ω^*, which are equivalent to

$$\begin{aligned} f_u &= f_v^*, \quad f_v = -f_u^*,\\ g_u &= g_v^*, \quad g_v = -g_u^*. \end{aligned} \tag{14}$$

Thus $f + if^*$ and $g + ig^*$ are holomorphic functions of the variable $w = u + iv$, and their real and imaginary parts f, g and f^*, g^*, respectively, are harmonic and therefore real analytic functions on Ω^*. It follows from (3) that $\psi : \Omega^* \to \Omega$ is real analytic, and then the same holds for the inverse mapping $\varphi : \Omega \to \Omega^*$. On the other hand, we infer from (5) that

$$z(x,y) = g(\varphi(x,y)) = g(x, a(x,y)), \tag{15}$$

whence $z(x,y)$ is seen to be real analytic on Ω.

Let us collect the results that are so far proved.

Theorem 1. *If $z \in C^2(\Omega)$ is a solution of the minimal surface equation* (7) *or* (11) *of Section* 2.2 *in the domain Ω of $\mathbb{R}^2$, then z is real analytic.*

Remark. Although we have proved this result only for convex domains, the general statement holds as well because we have only to show that z is real analytic on every ball $B_r(c)$ contained in Ω, and this has been proved.

Theorem 2. *Let $X(x,y) = (x, y, z(x,y))$ be a nonparametric minimal surface of class C^2 defined on some convex domain Ω of $\mathbb{R}^2$. Then there exists a real analytic diffeomorphism $\varphi : \Omega \to \Omega^*$ of Ω onto some simply connected domain*

Ω^*, with a real analytic inverse $\psi : \Omega^* \to \Omega$, such that $Z(u,v) = X(\psi(u,v))$ satisfies the conformality conditions

$$|Z_u|^2 = |Z_v|^2, \quad \langle Z_u, Z_v \rangle = 0.$$

The diffeomorphism φ can be chosen as

$$u = x, \quad v = a(x,y)$$

where $a(x,y)$ is a real analytic function which satisfies

$$a_x = \frac{pq}{\mathcal{W}}, \quad a_y = \frac{1+q^2}{\mathcal{W}}$$

with $p = z_x$, $q = z_y$, $\mathcal{W} = \sqrt{1+p^2+q^2}$. Its inverse ψ is described by

$$x = u, \quad y = f(u,v)$$

where f is a solution of

$$v = a(u, f(u,v)).$$

Finally, there is a surface $X^* = (a,b,c)$ on Ω which satisfies

$$dX^* = -N \wedge dX$$

where N denotes the spherical image of X, and the mapping $\Phi : \Omega^* \to \mathbb{C}^3$ defined by

$$\begin{aligned} \Phi(u+iv) &= Z(u,v) + iZ^*(u,v) \\ &:= X(u, f(u,v)) + iX^*(u, f(u,v)) \end{aligned}$$

is a holomorphic function of the complex variable $w = u + iv$.

As we have already noted in Section 1.4, every regular surface of class $C^{1,\alpha}$ can be mapped conformally onto some plane domain, irrespective of its mean curvature and its way of definition. But the previous reasoning shows that, in the case of nonparametric minimal surfaces, it is not necessary to apply Lichtenstein's mapping theorem. For such surfaces $X(x,y) = (x,y,z(x,y))$ defined on a convex domain Ω, the conformal mapping $\psi : \Omega^* \to \Omega$ can be explicitly constructed from the function $z(x,y)$. Moreover, if we introduce the line integral

$$X^*(x,y) := -\int_{(x_0,y_0)}^{(x,y)} N \wedge dX$$

for some $(x_0, y_0) \in \Omega$, we have the additional feature that $\Phi = (X + iX^*) \circ \psi$ is a holomorphic map $\Omega^* \to \mathbb{C}^3$.

Let us conclude this section with a geometric observation made by Riemann and Beltrami. By the transformation (2) we have introduced conformal parameters u, v on a given nonparametric minimal surface $X(x, y)$, in such a way that the coordinate lines $u = \text{const}$ are planar curves that can be generated by intersecting the given surface by the family of parallel planes $x = \text{const}$.

Conversely, if a regular minimal surface X is intersected by a family of parallel planes P none of which is tangent to the given surface, and if each point of X is met by some P, then the intersection lines of these planes with the minimal surface form a family of curves on the surface which locally belong to a net of conformal parameters u, v on the surface.

In fact, picking any sufficiently small piece of X, we can introduce Cartesian coordinates of $\mathbb{R}^3$ in such a way that the planes P are given as coordinate planes $x = \text{const}$, and that this piece can be written as a nonparametric surface $(x, y, z(x, y))$ over some domain Ω contained in the plane $z = 0$. Then the assertion follows from the previous result.

2.4 Bernstein's Theorem

In this section we want to prove Bernstein's celebrated theorem that every solution of the minimal surface equation defined on the whole plane must be an affine linear function.

To this end we consider an arbitrary nonparametric minimal surface $X(x^1, x^2) = (x^1, x^2, z(x^1, x^2))$ defined on a convex domain Ω of $\mathbb{R}^2$. Its height function $z(x^1, x^2)$ which is supposed to be of class C^2 on Ω will then automatically be real analytic. The coefficients of the first fundamental form of X are given by $g_{\alpha\beta} = \delta_{\alpha\beta} + z_{,\alpha} \cdot z_{,\beta}$. Let $\mathcal{W}^2 = g = \det(g_{\alpha\beta})$, and set

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta}/\mathcal{W}. \tag{1}$$

We have $\det(\bar{g}_{\alpha\beta}) = 1$ and

$$(\bar{g}^{\alpha\beta}) := (\bar{g}_{\alpha\beta})^{-1} = \begin{pmatrix} \bar{g}_{22} & -\bar{g}_{12} \\ -\bar{g}_{21} & \bar{g}_{11} \end{pmatrix}.$$

Since $z(x^1, x^2)$ is a solution of the minimal surface equation, there exist real analytic functions $\tau^\alpha(x^1, x^2), \alpha = 1, 2$, on Ω such that

$$d\tau^\alpha = \bar{g}_{\alpha\beta} dx^\beta, \quad \alpha = 1, 2. \tag{2}$$

(This follows from the equations (14) and (19) of Section 2.2, setting $\tau^1 = -b$ and $\tau^2 = a$.) We use these functions to define a real analytic mapping $\psi : \Omega \to \mathbb{R}^2$ by setting $\sigma = \psi(x) := x + \tau(x)$ or, in components,

$$\begin{aligned} \sigma^1 &= x^1 + \tau^1(x^1, x^2), \\ \sigma^2 &= x^2 + \tau^2(x^1, x^2). \end{aligned} \tag{3}$$

Since $B = D\tau = (\tau^\alpha_{,\beta}) = (\bar{g}_{\alpha\beta})$, the matrix B is symmetric and positive definite and we infer that, for arbitrary $x = (x^1, x^2)$ and $y = (y^1, y^2) \in \Omega$,

$$\langle x - y, \tau(x) - \tau(y)\rangle \geq 0.$$

Then it follows that

$$\begin{aligned} |\psi(x) - \psi(y)|^2 &= |x - y|^2 + |\tau(x) - \tau(y)|^2 + 2\langle x - y, \tau(x) - \tau(y)\rangle \\ &\geq |x - y|^2 \end{aligned}$$

or

$$|\psi(x) - \psi(y)| \geq |x - y|. \tag{4}$$

Therefore ψ maps Ω in a 1–1 way onto $\Omega^* := \psi(\Omega)$. Moreover,

$$\begin{aligned} \rho &:= \det\left(\frac{\partial\psi^\alpha}{\partial x^\beta}\right) = 2 + \bar{g}_{11} + \bar{g}_{22} \\ &= 2 + \mathcal{W} + 1/\mathcal{W} \geq 2, \end{aligned} \tag{5}$$

and thus $\psi : \Omega \to \Omega^*$ is a diffeomorphism. Now we define a second mapping $h(\sigma) = (h^1(\sigma), h^2(\sigma))$ for $\sigma \in \Omega^*$ by

$$\begin{aligned} h^1(\sigma) &= x^1 - \tau^1(x) \\ h^2(\sigma) &= -x^2 + \tau^2(x) \end{aligned} \quad \text{where } \sigma = \psi(x). \tag{6}$$

From the chain rule and from

$$\begin{aligned} \left(\frac{\partial\psi^\alpha}{\partial x^\beta}\right)^{-1} &= \begin{pmatrix} 1 + \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & 1 + \bar{g}_{22} \end{pmatrix}^{-1} \\ &= \frac{1}{2 + \mathcal{W} + 1/\mathcal{W}} \begin{pmatrix} 1 + \bar{g}_{22} & -\bar{g}_{12} \\ -\bar{g}_{21} & 1 + \bar{g}_{11} \end{pmatrix} \end{aligned} \tag{6'}$$

it follows that the derivative $Dh(\sigma)$ of $h(\sigma)$ is given by

$$\left(\frac{\partial h^\alpha}{\partial\sigma^\beta}\right) = \frac{1}{2 + \mathcal{W} + 1/\mathcal{W}} \begin{pmatrix} \bar{g}_{22} - \bar{g}_{11} & -2\bar{g}_{12} \\ 2\bar{g}_{21} & \bar{g}_{22} - \bar{g}_{11} \end{pmatrix} \circ \psi^{-1} \tag{7}$$

or

$$\left(\frac{\partial h^\alpha}{\partial\sigma^\beta}\right) = \frac{1}{(\mathcal{W} + 1)^2} \begin{pmatrix} g_{22} - g_{11} & -2g_{12} \\ 2g_{21} & g_{22} - g_{11} \end{pmatrix} \circ \psi^{-1}. \tag{8}$$

This shows that

$$H(\sigma) := h^1(\sigma) + ih^2(\sigma) \tag{9}$$

is a holomorphic function of $\sigma = \sigma^1 + i\sigma^2$ in Ω^* with the complex derivative

$$H'(\sigma) = \frac{q^2 - p^2 + 2ipq}{(\mathcal{W}+1)^2} = \left(\frac{ip+q}{1+\mathcal{W}}\right)^2 \tag{10}$$

where in the expressions $p = z_{,1}$, $q = z_{,2}$, and $\mathcal{W} = \sqrt{1+p^2+q^2}$ on the right-hand side one has to replace x by $\psi^{-1}(\sigma)$. We finally note that

$$|H'(\sigma)| = \frac{p^2+q^2}{(1+\mathcal{W})^{\ 2}} < \left(\frac{\mathcal{W}}{1+\mathcal{W}}\right)^2 < 1. \tag{11}$$

The image $\Omega^* = \psi(\Omega)$ of the convex set Ω clearly is a simply connected domain. If Ω is the whole plane $\mathbb{R}^2 \mathrel{\hat{=}} \mathbb{C}$, then one can infer from (4) that also $\Omega^* = \mathbb{C}$. Then, by Liouville's theorem and by (11), the entire function $H'(\sigma)$ must be constant. Thus, for $\mu := p/(1+\mathcal{W})$, $\nu := q/(1+\mathcal{W}\ )$, we infer that

$$\mu^2 - \nu^2 = c_1, \quad 2\mu\nu = c_2$$

for appropriate constants c_1 and c_2, whence

$$\mu^2 + \nu^2 = \sqrt{c_1^2 + c_2^2}.$$

This shows that the continuous functions μ and ν must be constant, and that there exists a constant $c \geq 0$ such that

$$p^2 + q^2 = c(1 + \sqrt{1+p^2+q^2})^2$$

which implies $p^2 + q^2 = \text{const}$, and therefore

$$p = \alpha_1 \quad \text{and} \quad q = \alpha_2$$

for some numbers α_1 and α_2, that is

$$z(x^1, x^2) = \alpha_0 + \alpha_1 x^1 + \alpha_2 x^2. \tag{12}$$

Thus a nonparametric minimal surface $X(x^1, x^2)$ which is defined on all of $\mathbb{R}^2$ has to be a plane. But this is the assertion of *Bernstein's theorem* from 1916 which we will state as

Theorem 1. *Every C^2-solution of the minimal surface equation on $\mathbb{R}^2$ has to be an affine linear function.*

In order to exploit the previous formulas more thoroughly, we introduce the function

$$F(\sigma) := \frac{p}{1+\mathcal{W}} - i\frac{q}{1+\mathcal{W}} \tag{13}$$

of $\sigma = \sigma^1 + i\sigma^2$. Here and in the sequel, x has to be replaced by $\psi^{-1}(\sigma)$ so that, as in (10), (11), and (13), the right-hand sides are to be understood as functions of σ. (We omit to write the composition by ψ^{-1}, because the formulas would then become rather cumbersome. For instance formula (13) should correctly have been written as

$$F := \left(\frac{p - iq}{1 + \mathcal{W}}\right) \circ \psi^{-1}.$$

We think that the reader will have no difficulties with our sloppy but more suggestive notation.)

Comparing (10) with (13), we see that

$$H' = (iF)^2. \tag{14}$$

Since H is holomorphic on Ω^*, we infer that also F is a holomorphic function. Furthermore,

$$|H'| = |F|^2 = \frac{p^2 + q^2}{(1 + \mathcal{W}\ )^2} = \frac{\mathcal{W}^{\ 2} - 1}{(1 + \mathcal{W}\ )^2} = \frac{\mathcal{W} - 1}{\mathcal{W} + 1} \tag{15}$$

whence

$$1 + |F|^2 = \frac{2\mathcal{W}}{\mathcal{W} + 1}$$

and

$$\Lambda := \left(\frac{\mathcal{W}}{\mathcal{W} + 1}\right)^2 = \frac{1}{4}[1 + |F|^2]^2. \tag{16}$$

Let $\gamma_{\mu\nu}(\sigma)$ be the coefficients of the first fundamental form of $Z := X \circ \psi^{-1}$. By the chain rule, we have

$$\gamma_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \sigma^\mu} \frac{\partial x^\beta}{\partial \sigma^\nu} = \mathcal{W} \bar{g}_{\alpha\beta} \frac{\partial x^\alpha}{\partial \sigma^\mu} \frac{\partial x^\beta}{\partial \sigma^\nu}.$$

By (5) and (6′), we obtain

$$\left(\frac{\partial x^\alpha}{\partial \sigma^\mu}\right) = \frac{1}{\rho}(\delta^{\alpha\mu} + \bar{g}^{\alpha\mu})$$

and therefore

$$\begin{aligned} \gamma_{\mu\nu} &= \frac{\mathcal{W}}{\rho^2}(\bar{g}_{\mu\beta} + \delta_{\mu\beta})(\delta^{\beta\nu} + \bar{g}^{\beta\nu}) \\ &= \frac{\mathcal{W}}{\rho^2}(\bar{g}_{\mu\nu} + \delta_{\mu\nu} + \delta^{\mu\nu} + \bar{g}^{\mu\nu}). \end{aligned}$$

On account of (5), we arrive at

$$(\gamma_{\mu\nu}) = \frac{\mathcal{W}}{\rho^2} \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} = \frac{\mathcal{W}}{\rho} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho = 2 + \mathcal{W} + 1/\mathcal{W} = \frac{(\mathcal{W}+1)^2}{\mathcal{W}}, \quad \text{or} \quad \frac{\mathcal{W}}{\rho} = \Lambda,$$

whence

$$\gamma_{\mu\nu} = \Lambda \delta_{\mu\nu}. \tag{17}$$

Hence $Z = X \circ \psi^{-1}$ is represented by conformal parameters. By virtue of the theorema egregium (cf. Section 1.3, (32)), the Gauss curvature $\mathcal{K}(\sigma)$ of $Z(\sigma)$ is given by

$$\mathcal{K} = -\frac{1}{2\Lambda}\Delta \log \Lambda$$

or, equivalently,

$$\mathcal{K} = -\frac{\Lambda \Delta \Lambda - |\nabla \Lambda|^2}{2\Lambda^3}. \tag{18}$$

To simplify the computations, we set $\alpha = \operatorname{Re} F, \beta = \operatorname{Im} F$. Then it follows that

$$\begin{aligned}
&F = \alpha + i\beta, \quad |F|^2 = \alpha^2 + \beta^2, \\
&\Lambda = \tfrac{1}{4}\{1 + \alpha^2 + \beta^2\}^2, \\
&\alpha_{\sigma^1} = \beta_{\sigma^2}, \quad \alpha_{\sigma^2} = -\beta_{\sigma^1}, \quad \Delta\alpha = 0, \quad \Delta\beta = 0.
\end{aligned}$$

From these formulas, we derive

$$\begin{aligned}
\Lambda_{\sigma^1}^2 + \Lambda_{\sigma^2}^2 &= \{1 + \alpha^2 + \beta^2\}^2[(\alpha\alpha_{\sigma^1} + \beta\beta_{\sigma^1})^2 + (\alpha\alpha_{\sigma^2} + \beta\beta_{\sigma^2})^2] \\
&= \{1 + \alpha^2 + \beta^2\}^2[\alpha^2(\alpha_{\sigma^1}^2 + \alpha_{\sigma^2}^2) + \beta^2(\beta_{\sigma^1}^2 + \beta_{\sigma^2}^2)] \\
&= 4\Lambda|F|^2|F'|^2
\end{aligned}$$

and

$$\begin{aligned}
\Delta\Lambda &= 2\sum_{\nu=1}^{2}(\alpha\alpha_{\sigma^\nu} + \beta\beta_{\sigma^\nu})^2 + \{1 + \alpha^2 + \beta^2\}(|\nabla\alpha|^2 + |\nabla\beta|^2) \\
&= 2|F|^2|F'|^2 + 2\{1 + |F|^2\}|F'|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
|\nabla\Lambda|^2 &= 4\Lambda|F|^2|F'|^2, \\
\Delta\Lambda &= 2(1 + 2|F|^2)|F'|^2.
\end{aligned} \tag{19}$$

By inserting these relations in (18), we arrive at the important equation

$$\mathcal{K} = -|F'|^2/\Lambda^2 \tag{20}$$

which, on account of (16), can also be written as

$$\mathcal{K} = -|F'|^2 \left(1 + \frac{1}{\mathcal{W}}\right)^4. \tag{20'}$$

Fix now some disk $B_r(x_0) \subset\subset \Omega$, and set $\sigma_0 := \psi(x_0)$. It follows from (4) that $B_r(\sigma_0) \subset\subset \Omega^*$.

Next we set $c := F(\sigma_0)$. By (11) or (15), we have $|c| < 1$. Thus

$$R(\sigma) := \frac{\sigma - c}{1 - \bar{c}\sigma}$$

defines a conformal mapping of the unit disk $B = B_1(0)$ onto itself which satisfies

$$R(c) = 0 \quad \text{and} \quad R'(c) = \frac{1}{1 - |c|^2} > 1.$$

Secondly we consider the linear mapping

$$L(\sigma) := \sigma_0 + r\sigma$$

of B onto $B_r(\sigma_0)$ that fulfills

$$L(0) = \sigma_0 \quad \text{and} \quad L'(0) = r.$$

Then the composition

$$M := R \circ F \circ L,$$

which can also be described by

$$M(\sigma) = \frac{F(\sigma_0 + r\sigma) - c}{1 - \bar{c}F(\sigma_0 + r\sigma)},$$

is a holomorphic mapping of B into itself since $|F| < 1$, and $M(0) = 0$. On account of Schwarz's lemma it follows that $|M'(0)| \leq 1$.

Since $M'(0) = R'(c)F'(\sigma_0)r$ and $R'(c) > 1$, we obtain

$$|F'(\sigma_0)| \leq 1/r,$$

and we infer from (20′) that

$$|\mathcal{K}(\sigma_0)| \leq \frac{1}{r^2}\left(1 + \frac{1}{\mathcal{W}(x_0)}\right)^4 \leq \frac{1}{r^2}(1+1)^4 = \frac{16}{r^2}.$$

The Gauss curvatures K and $\mathcal{K}$ of X and Z, respectively, are related to each other by

$$\mathcal{K} = K \circ \psi^{-1}.$$

Thus we have proved

$$|K(x_0)| \leq \frac{16}{r^2}, \tag{21}$$

and we can formulate the following assertion:

Theorem 2. *If a disk of center x_0 and radius r is contained in the domain of definition of a nonparametric minimal surface $X(x^1, x^2) = (x^1, x^2, z(x^1, x^2))$, then its Gauss curvature in x_0 can be estimated by*

$$|K(x_0)| \le \frac{16}{r^2}. \tag{22}$$

This result, which is due to E. Heinz [1], can be considered as a quantitative and local version of Bernstein's theorem that follows from Theorem 2 if we let $r \to \infty$.

2.5 Two Characterizations of Minimal Surfaces

We shall prove two results that more or less were already established in Section 2.2. Yet the formulas to be developed here will shed light on the problem from a different angle.

Theorem 1. *If $X : \Omega \to \mathbb{R}^3$ is a regular surface of class C^2 with mean curvature H and with the spherical map $N : \Omega \to \mathbb{R}^3$, then*

$$\Delta_X X = 2HN, \tag{1}$$

where Δ_X denotes the Laplace–Beltrami operator on the surface X.

This implies the following characterization of minimal surfaces:

Corollary 1. *A regular C^2-surface X is a minimal surface if and only if*

$$\Delta_X X = 0 \tag{2}$$

holds.

Suppose now that the parameters u, v of $X(u, v)$ are conformal. Then $\mathcal{W} = \mathcal{E} = \mathcal{G}$, and Section 1.5, (17) implies that

$$\Delta_X = \frac{1}{\mathcal{W}} \Delta \tag{3}$$

where Δ denotes the ordinary Laplace operator $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$. Moreover, we have

$$\mathcal{W} N = X_u \wedge X_v$$

and therefore

Corollary 2. *If $X(u, v)$ is a regular C^2-surface represented by conformal parameters, then*

$$\Delta X = 2H X_u \wedge X_v. \tag{4}$$

In particular, X is a minimal surface if and only if

$$\Delta X = 0 \tag{5}$$

holds.

Proof of Theorem 1. It obviously suffices to establish (1) in a sufficiently small neighborhood Ω' of every point of Ω. Moreover, because of the invariant character of the expressions on both sides of (1), we only have to verify the assertion for some surface that is strictly equivalent to $X|_\Omega$. Since every regular C^2-surface is locally equivalent (in a strict way) to some nonparametric surface we have convinced ourselves that it suffices to prove (1) for an arbitrary nonparametric surface

$$X(x^1, x^2) = (x^1, x^2, z(x^1, x^2)), \quad (x^1, x^2) \in \Omega.$$

Set as usual,

$$p = z_{,1}, \quad q = z_{,2}, \quad \mathcal{W} = \sqrt{1 + p^2 + q^2}.$$

Then the Gauss equations

$$X_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma} + b_{\alpha\beta} N$$

of Section 1.3 take the form

$$\begin{pmatrix} 0 \\ 0 \\ z_{,\alpha\beta} \end{pmatrix} = \Gamma^1_{\alpha\beta} \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} + \Gamma^2_{\alpha\beta} \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} + b_{\alpha\beta} \begin{pmatrix} -p/\mathcal{W} \\ -q/\mathcal{W} \\ 1/\mathcal{W} \end{pmatrix}$$

whence

$$(6) \qquad \begin{aligned} &\Gamma^1_{\alpha\beta} = b_{\alpha\beta} \frac{p}{\mathcal{W}}, \quad \Gamma^2_{\alpha\beta} = b_{\alpha\beta} \frac{q}{\mathcal{W}}, \\ &z_{,\alpha\beta} = \Gamma^1_{\alpha\beta} p + \Gamma^2_{\alpha\beta} q + \frac{b_{\alpha\beta}}{\mathcal{W}}, \end{aligned}$$

and this implies

$$(7) \qquad z_{,\alpha\beta} = b_{\alpha\beta} \mathcal{W}.$$

Since

$$2H = b_{\alpha\beta} g^{\alpha\beta}$$

(see (42) of Section 1.2), it follows that

$$(8) \qquad g^{\alpha\beta} \Gamma^1_{\alpha\beta} = \frac{2H}{\mathcal{W}} p, \quad g^{\alpha\beta} \Gamma^2_{\alpha\beta} = \frac{2H}{\mathcal{W}} q, \quad g^{\alpha\beta} z_{,\alpha\beta} = 2H\mathcal{W}.$$

On account of Section 1.5, (19), we have

$$\Delta_X f = g^{\alpha\beta} [f_{,\alpha\beta} - \Gamma^\gamma_{\alpha\beta} f_{,\gamma}]$$

for an arbitrary function $f \in C^2(\Omega)$. Then, by virtue of (8),

$$(9) \qquad \Delta_X f = g^{\alpha\beta} f_{,\alpha\beta} - \frac{2H}{\mathcal{W}} z_{,\gamma} f_{,\gamma}.$$

Specializing this formula to the functions $f(x^1,x^2) = x^1$, x^2, and $z(x^1,x^2)$ respectively, we obtain

$$\Delta_X x^1 = 2H\left(-\frac{p}{\mathcal{W}}\right), \quad \Delta_X x^2 = 2H\left(-\frac{q}{\mathcal{W}}\right), \quad \Delta_X z = 2H\left(\frac{1}{\mathcal{W}}\right), \tag{10}$$

and this is equivalent to

$$\Delta_X X = 2HN.$$

This completes the proof of Theorem 1. □

The following result relates the Beltrami operators Δ_X and Δ_N of a minimal surface X and its Gauss map N to each other.

Proposition. *If N is the Gauss map of a minimal surface X, then*

$$\Delta_X = |K|\Delta_N. \tag{11}$$

Proof. Since $\mathrm{I}_N = \mathrm{III}_X$ and (Section 1.2, (26))

$$K\mathrm{I}_X - 2H\mathrm{II}_X + \mathrm{III}_X = 0,$$

it follows from $H = 0$ that $K \le 0$ and

$$\mathrm{I}_N = -K\mathrm{I}_X = |K|\mathrm{I}_X. \tag{12}$$

Hence, if X is represented conformally, then the same holds for N, and we infer from (12) and from relation (17) of Section 1.5 that

$$\Delta_X = \frac{1}{\mathcal{W}}\,\Delta, \quad \Delta_N = \frac{1}{|K|\mathcal{W}}\,\Delta$$

whence

$$\Delta_X = |K|\Delta_N.$$

Since both sides are invariant expressions with respect to parameter changes, we conclude on account of Section 2.3, Theorem 2, the general validity of (11). □

Now we turn to the second characterization of minimal surface which follows from

Theorem 2. *Let $X(u,v)$ be a regular surface of class C^2 defined on some domain Ω of $\mathbb{R}^2$, and let $N : \Omega \to \mathbb{R}^3$ be its Gauss map. Then*

$$N_v \wedge X_u - N_u \wedge X_v = 2H\mathcal{W}N \tag{13}$$

and

$$(N \wedge X_u)_v - (N \wedge X_v)_u = 2H\mathcal{W}N. \tag{14}$$

Proof. Set $u^1 = u$ and $u^2 = v$. Then the Weingarten equations

$$N_{,\alpha} = -b_\alpha^\beta X_{,\beta} = -g^{\gamma\beta} b_{\alpha\gamma} X_{,\beta}$$

imply

$$\begin{aligned} N_{,2} \wedge X_{,1} &= -g^{\gamma 2} b_{2\gamma} X_{,2} \wedge X_{,1}, \\ N_{,1} \wedge X_{,2} &= -g^{\gamma 1} b_{1\gamma} X_{,1} \wedge X_{,2} \end{aligned}$$

whence

$$N_{,2} \wedge X_{,1} - N_{,1} \wedge X_{,2} = \{g^{\gamma 1} b_{1\gamma} + g^{\gamma 2} b_{2\gamma}\} X_{,1} \wedge X_{,2}.$$

Since

$$g^{\gamma\alpha} b_{\alpha\gamma} = 2H, \quad X_u \wedge X_v = \mathcal{W} N,$$

the relation (13) is established, and (14) is a direct consequence of (13). □

Because of

$$N \wedge \, dX = N \wedge X_u \, du + N \wedge X_v \, dv$$

equation (14) is equivalent to

$$d(N \wedge dX) = -2H\mathcal{W}N \, du \, dv = -2HN \, dA. \tag{15}$$

This implies

Corollary 3. *A regular C^2-surface $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if the differential form $N \wedge dX$ is closed, that is,*

$$d(N \wedge dX) = 0. \tag{16}$$

If Ω is a simply connected domain, condition (16) is equivalent to the statement that

$$\Psi(u, v) := \int_{(u_0, v_0)}^{(u,v)} N \wedge dX$$

is a path-independent line integral.

Remark. Since formula (16) is invariant with respect to parameter changes and has only to be proved locally, it follows as well from Section 2.2, (17).

2.6 Parametric Surfaces in Conformal Parameters. Conformal Representation of Minimal Surfaces. General Definition of Minimal Surfaces

Now we will provide another proof of the result stated in Corollary 2 of Section 2.5 which is particularly simple because it uses only a minimum of differential geometric formulas.

Theorem 1. *Let $X(u,v)$ be a regular surface of class $C^2(\Omega,\mathbb{R}^3)$ given by conformal parameters u and v, that is,*

$$(1) \qquad |X_u|^2 = |X_v|^2 \quad \textit{and} \quad \langle X_u, X_v\rangle = 0.$$

Then necessary and sufficient for a real-valued function $H(u,v)$ to represent the mean curvature of the surface X is that Rellich's equation

$$(2) \qquad \Delta X = 2H X_u \wedge X_v$$

holds in Ω.

In particular, $X : \Omega \to \mathbb{R}^3$ is a minimal surface if and only if

$$(3) \qquad \Delta X = 0.$$

Proof. The equation (1) can be written as

$$\Lambda := \mathcal{E} = \mathcal{G} = \mathcal{W}, \quad \mathcal{F} = 0 \quad \text{in } \Omega.$$

According to Section 1.3, (31), the mean curvature is simply

$$H = \frac{1}{2\Lambda}(\mathcal{L} + \mathcal{N}).$$

Recalling that $\mathcal{L} = \langle X_{uu}, N\rangle$, $\mathcal{N} = \langle X_{vv}, N\rangle$, we obtain

$$(4) \qquad \langle \Delta X, N\rangle = \langle X_{uu} + X_{vv}, N\rangle = 2\Lambda H.$$

On the other hand, differentiating (1) with respect to u and v yields

$$\begin{aligned} \langle X_u, X_{uu}\rangle &= \langle X_v, X_{vu}\rangle, \quad \langle X_{uu}, X_v\rangle + \langle X_u, X_{uv}\rangle = 0,\\ \langle X_v, X_{vv}\rangle &= \langle X_u, X_{uv}\rangle, \quad \langle X_{vv}, X_u\rangle + \langle X_v, X_{uv}\rangle = 0 \end{aligned}$$

and therefore

$$(5) \qquad \langle \Delta X, X_u\rangle = 0 \quad \text{and} \quad \langle \Delta X, X_v\rangle = 0.$$

In other words, ΔX is proportional to N.

Since $|N| = 1$, it follows from (4) that

$$\Delta X = 2\Lambda H N,$$

and, by virtue of $\Lambda N = \mathcal{W} N = X_u \wedge X_v$, we arrive at (2), and the theorem is proved. □

The previous theorem provides another approach to the general formula

$$(6) \qquad \Delta_X X = 2HN$$

proved in Theorem 1 of Section 2.5. One only has to show that an arbitrary regular surface X of class C^2 is locally strictly equivalent to a surface represented by conformal parameters. Then (6) follows from (2) by an invariance reasoning.

The possibility to introduce conformal parameters on an arbitrary regular C^2-surface is expressed in Lichtenstein's theorem, stated in Section 1.4. For minimal surfaces ($H = 0$) we have—independently of the general Lichtenstein theorem—given two different proofs that it is possible to introduce conformal parameters in the small (cf. Sections 2.3 and 2.4). Thus the equation

$$\Delta_X X = 0, \tag{7}$$

which characterizes minimal surfaces, is independently verified.

In order to transform a regular minimal surface globally to conformal parameters, one can combine Theorem 2 of Section 2.3 with the uniformization theorem proved by Koebe and Poincaré. We cannot give the proof of this celebrated theorem. Instead, in Section 4.11 we shall present a *variational proof* of Lichtenstein's theorem which is based on the solution of a Plateau-type problem. Here we merely state the global version of Theorem 2 in Section 2.3:

Theorem 2. *Every regular surface $X : \Omega \to \mathbb{R}^3$ of class C^2, whether minimal or not, is strictly equivalent to a surface represented by conformal parameters.*

Still it should be noted that the following discussion will not rest on unfortified ground since existence proofs for minimal surfaces that will be given later yield the existence of minimal surfaces represented by conformal parameters.

While the equations (6) and (7) only make sense for regular surfaces, the equations (2) and (3) can also be formulated for surfaces with $\mathcal{W} = 0$. This enables us to give a definition of minimal surfaces that includes surfaces with isolated singularities, called branch points, that will be studied in the next chapter.

Definition 1. *A nonconstant surface $X : \Omega \to \mathbb{R}^3$ of class C^2 is said to be a* **minimal surface** *if it satisfies the conformality relations*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{1}$$

as well as the equation

$$\Delta X = 0 \tag{3}$$

on Ω.

If equation (3) is replaced by (2), we speak of a surface with the mean curvature function $H(u, v)$.

By (1) and (3) we can also define minimal surfaces $X : \Omega \to \mathbb{R}^n$ in $\mathbb{R}^n$, $n \geq 2$, and most of the results in these notes will be true independently of the dimension n. For convenience we shall ordinarily restrict ourselves to the case $n = 3$, and we shall only occasionally touch minimal surfaces $X : \Omega \to M$ in an n-dimensional Riemannian manifold M. If, in local coordinates $x^1, \dots, x^n$ on M, the line element ds^2 of M has the form

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k \tag{8}$$

(summation with respect to Latin indices $i, k, \dots$ from 1 to n), then a surface $X \in C^2(\Omega, M)$ is said to be a minimal surface in M, if its local components $x^1(u,v), \dots, x^n(u,v)$ satisfy

$$g_{ik}(X)x_u^i x_u^k = g_{ik}(X)x_v^i x_v^k, \quad g_{ik}(X)x_u^i x_v^k = 0 \tag{9}$$

and

$$\Delta x^l + \Gamma_{ik}^l(X)(x_u^i x_u^k + x_v^i x_v^k) = 0 \tag{10}$$

where Γ_{ik}^l are the Christoffel symbols of second kind with respect to ds^2.

Although there will be no systematic treatment of (9) and (10) in our notes, these equations will turn up when we replace Cartesian coordinates by general curvilinear coordinates in $\mathbb{R}^3$ which will be essential for the investigation of the boundary behavior of minimal surfaces.

2.7 A Formula for the Mean Curvature

Let us consider a family $\{\mathfrak{F}_c\}_{c \in J}$ of regular C^2-surfaces $\mathfrak{F}_c$ which are embedded in $\mathbb{R}^3$, implying that none of these surfaces has selfcuttings or selftangencies. We also assume that the family depends in a C^2-way on the parameter c.

A set $\mathcal{S}$ of $\mathbb{R}^3$ is said to be *simply covered by the surfaces of the family* $\{\mathfrak{F}_c\}$ if each point $x = (x^1, x^2, x^3)$ of $\mathcal{S}$ is contained in exactly one of the surfaces.

Consider now a domain G in $\mathbb{R}^3$ whose closure $\bar{G}$ is simply covered by a family of C^2-surfaces $\mathfrak{F}_c$ in the sense that there is a function $S \in C^2(\bar{G})$ with $\nabla S(x) \neq 0$ for all $x \in \bar{G}$, such that the *leaves* $\mathfrak{F}_c$ of the *foliation* $\{\mathfrak{F}_c\}$ can be described as its level surfaces

$$\mathfrak{F}_c = \{x \in \bar{G} : S(x) = c\}. \tag{1}$$

Then

$$Q(x) := |\nabla S(x)|^{-1} \cdot \nabla S(x) \tag{2}$$

defines a field $Q \in C^1(\bar{G}, \mathbb{R}^3)$ of unit vectors that is orthogonal to all surfaces $\mathfrak{F}_c$; it is called the *normal field* of the foliation $\{\mathfrak{F}_c\}$.

Theorem 1. *If G is a domain in $\mathbb{R}^3$, and if S is a function of class $C^2(\bar{G})$ such that $\nabla S(x) \neq 0$ on $\bar{G}$, then the mean curvature $H(x)$ of the level surface*

$$\mathfrak{F}_c = \{x \in \bar{G} : S(x) = c\}$$

passing through $x \in \bar{G}$ is given by the equation

$$\operatorname{div} Q(x) = -2H(x) \tag{3}$$

where $Q(x) = |\nabla S(x)|^{-1} \cdot \nabla S(x)$ denotes the normal field of the foliation $\{\mathfrak{F}_c\}$.

Proof. Pick some point $x_0 \in G$, some $r > 0$ with $B_r(x_0) \subset\subset G$, and let x_0 be contained in $\mathfrak{F}_{c_0}$. For $\mathfrak{F} := \bar{B}_r(x_0) \cap \mathfrak{F}_{c_0}$ we then choose a regular C^2-parametrization $X(w), w \in \bar{\Omega}$ such that its surface normal $N(w) = N_X(w)$ satisfies

$$N(w) = Q(X(w)) \quad \text{for all } w \in \bar{\Omega}.$$

We can also achieve that $x_0 = X(w_0)$ for some $w_0 \in \Omega$.

For some sufficiently small $\varepsilon_0 > 0$ we define the *normal variation*

$$Z(w, \varepsilon) = X(w) + \varepsilon N(w), \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0],$$

of the surface $\mathfrak{F}$ represented by $X(w)$. Let $\mathfrak{S}_\varepsilon$ be the surface with the parameter representation $Z(\cdot, \varepsilon)$, and denote by $\mathfrak{C}_\varepsilon$ the *collar*

$$\{X(w) + \lambda N(w) : w \in \partial\Omega, 0 \leq \lambda \leq \varepsilon\}.$$

The two caps $\mathfrak{F}$ and $\mathfrak{S}_\varepsilon$ together with the collar $\mathfrak{C}_\varepsilon$ bound a domain U_ε in $\mathbb{R}^3$ over which we will integrate $\operatorname{div} Q$. Performing an integration by parts, we obtain

$$\int_{U_\varepsilon} \operatorname{div} Q \, dX = \int_{\partial U_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle \, dA$$

where $\bar{N}_\varepsilon$ denotes the exterior normal of ∂U_ε. Note that

$$\bar{N}_\varepsilon = -N = -Q \quad \text{on } \mathfrak{F}.$$

By virtue of Taylor's theorem, we infer that

$$\langle Q, \bar{N}_\varepsilon \rangle = O(\varepsilon) \quad \text{on } \mathfrak{C}_\varepsilon$$

whence

$$\int_{\mathfrak{C}_\varepsilon} \langle Q, \bar{N}_\varepsilon \rangle dA = O(\varepsilon^2).$$

If we apply formula (13) of Section 2.5, we obtain for $Z(w, \varepsilon) = X(w) + \varepsilon N(w)$ the relations

$$\begin{aligned} Z_u \wedge Z_v &= X_u \wedge X_v + \varepsilon\{X_u \wedge N_v + N_u \wedge X_v\} + \varepsilon^2\{N_u \wedge N_v\} \\ &= \mathcal{W}N - \varepsilon 2H\mathcal{W}N + \varepsilon^2 N_u \wedge N_v, \end{aligned}$$

and, by $N = Q \circ X$, it follows that

$$\begin{aligned}
\int_{\mathfrak{S}_\varepsilon} \langle Q, \bar N_\varepsilon\rangle dA &= \int_\Omega \langle Q(X+\varepsilon N), Z_u \wedge Z_v\rangle \, du\, dv \\
&= \int_\Omega \langle Q(X+\varepsilon N), \mathcal{W}N - \varepsilon 2H\mathcal{W}N + \varepsilon^2 N_u \wedge N_v\rangle \, du\, dv \\
&= \int_{\mathfrak{F}} dA - \varepsilon \int_{\mathfrak{F}} 2H\, dA \\
&\quad + \int_\Omega \langle Q(X+\varepsilon N) - Q(X), \mathcal{W}N - \varepsilon 2H\mathcal{W}N\rangle \, du\, dv + O(\varepsilon^2).
\end{aligned}$$

The relations $|Q(x)| = 1$ and $N = Q \circ X$ imply that

$$\langle Q(X(w) + \varepsilon N(w)) - Q(X(w)), N(w)\rangle = O(\varepsilon^2).$$

Thus we obtain from the previous computation that

$$\int_{\mathfrak{S}_\varepsilon} \langle Q, \bar N_\varepsilon\rangle \, dA = \int_{\mathfrak{F}} dA - \varepsilon \int_{\mathfrak{F}} 2H\, dA + O(\varepsilon^2).$$

Since

$$\int_{U_\varepsilon} \operatorname{div} Q \, dX = \int_{\mathfrak{S}_\varepsilon} \langle Q, \bar N_\varepsilon\rangle \, dA - \int_{\mathfrak{F}} dA + \int_{\mathfrak{C}_\varepsilon} \langle Q, \bar N_\varepsilon\rangle \, dA,$$

it follows that

$$\frac{1}{\varepsilon} \int_{U_\varepsilon} \operatorname{div} Q\, dX = -\int_{\mathfrak{F}} 2H\, dA + O(\varepsilon).$$

As $\varepsilon \to +0$, we arrive at the equation

$$\int_{\mathfrak{F}} \operatorname{div} Q\, dA = -2 \int_{\mathfrak{F}} H\, dA.$$

Here $\mathfrak{F}$ stands for $\mathfrak{F}_{c_0} \cap \bar B_r(x_0)$. Dividing both sides by $\int_{\mathfrak{F}} dA$, and letting r tend to zero, we arrive at

$$\operatorname{div} Q(x_0) = -2H(x_0).$$

Since x_0 was chosen as an arbitrary point of G, and since both sides of this equation are continuous functions on $\bar G$, we finally obtain

$$\operatorname{div} Q(x) = -2H(x)$$

for all $x \in \bar G$ which proves that theorem. □

Remark 1. With $D_i = \frac{\partial}{\partial x^i}$ and $Q = (Q_1, Q_2, Q_3)$ we can write

$$\operatorname{div} Q = D_i Q_i = D_i \left\{ \frac{S_{x^i}}{\sqrt{S_{x^k} S_{x^k}}} \right\} = \frac{S_{x^i x^i}}{\sqrt{S_{x^k} S_{x^k}}} - S_{x^i x^k} \frac{S_{x^i} S_{x^k}}{\{S_{x^l} S_{x^l}\}^{3/2}}.$$

If we introduce the Hessian

$$h_S(\xi, \eta) = S_{x^i x^k} \xi^i \xi^k$$

and the Laplacian

$$\Delta S = S_{x^i x^i} = (D_1^2 + D_2^2 + D_3^2) S$$

we can write

$$\operatorname{div} Q = \frac{1}{|\nabla S|} \left\{ \Delta S - \frac{1}{|\nabla S|^2} h_S(\nabla S, \nabla S) \right\}.$$

Thus (3) can be written as

$$H = \frac{1}{2|\nabla S|} \left\{ \frac{1}{|\nabla S|^2}\ h_S(\nabla S, \nabla S) - \Delta S \right\}. \tag{4}$$

This and related formulas for curvature quantities can also be derived by the technique of covariant differentiation applied to manifolds which are implicitly defined. This has in detail been carried out by P. Dombrowski [1].

Remark 2. Consider the nonparametric surface which is given as graph of a function $\psi(x, y), (x, y) \in \bar{\Omega} \subset \mathbb{R}^2$. We can embed $z = \psi(x, y)$ into the family of surfaces

$$z = \psi(x, y) + c$$

which simply cover $\bar{G} := \bar{\Omega} \times \mathbb{R}$. They are the level surfaces

$$S(x, y, z) = c$$

of the function $S(x, y, z) := z - \psi(x, y)$, for which we obtain

$$Q(x, y, z) = \frac{1}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \cdot (-\psi_x, -\psi_y, 1)$$

whence

$$\operatorname{div} Q = -\left\{ \frac{\psi_x}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \right\}_x - \left\{ \frac{\psi_y}{\sqrt{1 + \psi_x^2 + \psi_y^2}} \right\}_y .$$

Thus in this particular case equation (3) takes the form

$$\operatorname{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} = 2H \tag{5}$$

which is equivalent to formula (5) of Section 2.2. If $H = 0$, we obtain the minimal surface equation in divergence form (see Section 2.2, (11)):

$$\operatorname{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} = 0. \tag{6}$$

From the theorem proved above we obtain the following

Corollary 1. *If $\bar{G}$ is simply covered by a foliation of minimal surfaces $\mathfrak{F}_c$ which are the level surfaces of a function $S \in C^2(\bar{G})$ with $\nabla S(x) \neq 0$ on $\bar{G}$, then the normal field $Q = |\nabla S|^{-1} \cdot \nabla S$ of this foliation is divergence-free, that is, the equation*

$$\operatorname{div} Q = 0 \tag{7}$$

holds on $\bar{G}$.

2.8 Absolute and Relative Minima of Area

We begin with a result of the Weierstrass field theory for minimal surfaces which, in a somewhat different form, was developed by H.A. Schwarz.

Lemma 1. *Suppose that G is a domain in $\mathbb{R}^3$ and that $Q \in C^1(\bar{G}, \mathbb{R}^3)$ is a vector field on $\mathbb{R}^3$ with the properties that*

$$|Q(x)| = 1 \quad \text{and} \quad \operatorname{div} Q(x) = 0 \quad \text{in } G. \tag{1}$$

Moreover, let $\mathfrak{F}$ be a regular C^1-surface embedded in G whose surface normal $N_{\mathfrak{F}}$ coincides on $\mathfrak{F}$ with the vector field Q. Then, for every regular C^1-surface $\mathfrak{S}$ that is contained in G and has the same boundary as $\mathfrak{F}$, we have

$$\int_{\mathfrak{F}} dA \leq \int_{\mathfrak{S}} dA. \tag{2}$$

Proof. Let us first assume that the surfaces $\mathfrak{F}$ and $\mathfrak{S}$ bound a domain U whose exterior surface normal on $\mathfrak{F}$ points in the opposite direction of $Q|_{\mathfrak{F}} = N_{\mathfrak{F}}$. Then we infer from Gauss's theorem that

$$\begin{aligned} \int_U \operatorname{div} Q \, dX &= \int_{\partial U} \langle Q, N_{\partial U} \rangle \, dA \\ &= \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}} \rangle \, dA - \int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}} \rangle \, dA. \end{aligned} \tag{3}$$

Because of (1), the left hand side is vanishing, and therefore

$$\int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}} \rangle \, dA = \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}} \rangle \, dA. \tag{4}$$

On account of

$$\langle Q, N_{\mathfrak{F}} \rangle = 1$$

and of

$$\langle Q, N_{\mathfrak{S}} \rangle \leq |Q||N_{\mathfrak{S}}| = 1,$$

we obtain

$$\int_{\mathfrak{F}} dA \leq \int_{\mathfrak{S}} dA. \tag{5}$$

If $\mathfrak{S}$ is a general surface as stated in the theorem, the same result holds. This can be proved in essentially the same way by applying the calculus of differential forms and the general Stokes theorem for 1-forms (see, for instance, F. Warner [1]). □

Remark. It is easy to see that the equality sign in (5) holds if and only if $\mathfrak{F}$ and $\mathfrak{S}$ are strictly equivalent.

Lemma 2. *Let the assumptions of Lemma 1 be satisfied, with the following alteration: The boundaries $\partial\mathfrak{F}$ and $\partial\mathfrak{S}$ of $\mathfrak{F}$ and $\mathfrak{S}$ are not necessarily the same but lie on a surface T which is tangent to the vector field Q (that is, $Q(x)$ is a tangent vector to T at every point $x \in T$), and are supposed to be homologous to each other:*

$$\partial\mathfrak{F} \sim \partial\mathfrak{S} \quad \text{on } T.$$

Then the inequality (5) is still satisfied.

Proof. Let us choose a surface $\mathfrak{C} \subset T$ such that $\partial\mathfrak{C} = \partial\mathfrak{S} \setminus \partial\mathfrak{F}$. Applying Gauss's theorem, we obtain

$$\int_U \operatorname{div} Q \, dX = \int_{\mathfrak{S}} \langle Q, N_{\mathfrak{S}} \rangle \, dA - \int_{\mathfrak{F}} \langle Q, N_{\mathfrak{F}} \rangle \, dA + \int_{\mathfrak{C}} \langle Q, N_{\mathfrak{C}} \rangle \, dA.$$

Since

$$\int_{\mathfrak{C}} \langle Q, N_{\mathfrak{C}} \rangle \, dA = 0$$

we arrive once again at (4), from where the proof proceeds as before. □

By combining Lemma 1 or Lemma 2 with the corollary stated in Section 2.7, we obtain the following

Theorem 1. *A C^2-family of regular, embedded C^2-surfaces $\mathfrak{F}_c$ which cover a domain G simply is a family of minimal surfaces if and only if its normal field is divergence-free. Such a foliation by minimal surfaces is area minimizing in the following sense:*

(i) *Let $\mathfrak{F}$ be a piece of some of the minimal leaves $\mathfrak{F}_c$, with $\mathfrak{F} \subset\subset G$. Then we have*

$$\int_{\mathfrak{F}} dA \leq \int_{\mathfrak{S}} dA \tag{6}$$

for each regular C^1-surface $\mathfrak{S}$ contained in G with $\partial\mathfrak{S} = \partial\mathfrak{F}$.

(ii) *Let T be a surface in G which, in all of its points, is tangent to the normal field of the minimal foliation, and suppose that T cuts out of each leaf $\mathfrak{F}_c$ some piece $\mathfrak{F}_c^*$ whose boundary $\partial\mathfrak{F}_c^*$ lies on T. Then we have*:

$$\int_{\mathfrak{F}^*_{c_1}} dA = \int_{\mathfrak{F}^*_{c_2}} dA \tag{7}$$

for all admissible parameter values c_1 and c_2, and secondly,

$$\int_{\mathfrak{F}_c} dA \le \int_{\mathfrak{S}} dA \tag{8}$$

for all regular C^1-surfaces $\mathfrak{S}$ contained in G whose boundaries $\partial\mathfrak{S}$ are homologous to $\partial\mathfrak{F}_c$ on T.

The identity (7) is the minimal surface version of A. Kneser's *transversality theorem.*

The integral $\int_{\mathfrak{F}} \langle Q, N\rangle\, dA$ appearing in the previous reasoning, is the so-called *Hilbert's independent integral* associated with the area functional $\int_{\mathfrak{F}} dA$. If we express $\mathfrak{F}$ by its representation $X(u,v)$, $(u,v) \in \bar{\Omega}$, Hilbert's independent integral takes the form

$$\int_{\Omega} \langle Q(X), X_u \wedge X_v\rangle\, du\, dv. \tag{9}$$

The aforestated results can be summarized as follows:

A regular embedded minimal surface $\mathfrak{F}$ yields a relative minimum of area among all surfaces having the same boundary as $\mathfrak{F}$, if it can be embedded in a foliation (*or field*) *of minimal surfaces in the sense described before. In fact, $\mathfrak{F}$ is an absolute minimum of area among all surfaces with the same boundary which lie in the domain covered by the field.*

Not every minimal surface will have minimal area among all surfaces having the same boundary. It is, in fact, not difficult to find examples of non-minimizing surfaces of vanishing mean curvature. Yet the result just proved shows that a minimal surface yields a relative minimum of area if it can be embedded into a field of minimal surfaces. Thus we ask the question:

When can a minimal surface be embedded in a field of minimal surfaces?

An answer to this question was given by H.A. Schwarz. He proved that *each interior piece of a given regular embedded minimal surface X can be embedded in a field of minimal surfaces if the first eigenvalue of the second variation of the area functional at X is positive.*

Presently we will not prove this result, but refer to Chapter 5 of this volume and also to Volume 1 of Schwarz's collected papers [2] as well as to Chapter I, Section 6, pp. 86–110 of Nitsche's lectures [28] where several examples and further applications are discussed.

However, we shall at least derive an expression for the *second variation of area* $\delta^2 A(X, Y)$ of a regular C^2-surface $X : \bar{\Omega} \to \mathbb{R}^3$ with respect to *normal variations* $Y = \varphi N$.

Here Ω is assumed to be a bounded domain in $\mathbb{R}^2$, and φ is supposed to be of class $C^1(\bar{\Omega})$. Let

$$Z := X + \varepsilon Y, \quad Y = \varphi N. \tag{10}$$

Then

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon\varphi N_{,\alpha} + \varepsilon\varphi_{,\alpha} N,$$

whence

$$\zeta_{\alpha\beta} := \langle Z_{,\alpha}, Z_{,\beta}\rangle = \langle X_{,\alpha}, X_{,\beta}\rangle + 2\varphi\varepsilon\langle X_{,\alpha}, N_{,\beta}\rangle + \varphi^2\varepsilon^2\langle N_{,\alpha} N_{,\beta}\rangle + \varphi_{,\alpha}\varphi_{,\beta}\varepsilon^2$$

and therefore

$$\zeta_{\alpha\beta} = g_{\alpha\beta} - \varepsilon 2\varphi b_{\alpha\beta} + \varepsilon^2\{\varphi^2 c_{\alpha\beta} + \varphi_{,\alpha}\varphi_{,\beta}\}. \tag{11}$$

Then

$$\begin{aligned} \det(\zeta_{\alpha\beta}) &= \zeta_{11}\zeta_{22} - \zeta_{12}\zeta_{21} \\ &= g[1 - \varepsilon 2\varphi g^{\alpha\beta} b_{\alpha\beta} + \varepsilon^2\{g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + \varphi^2 g^{\alpha\beta} c_{\alpha\beta} + 4\varphi^2 b/g\}] + O(\varepsilon^3), \end{aligned}$$

where $g = \det(g_{\alpha\beta})$ and $b = \det(b_{\alpha\beta})$.

From

$$K\mathrm{I} - 2H\mathrm{II} + \mathrm{III} = 0$$

we infer the analogous relation for the corresponding bilinear forms whence

$$Kg_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} = 0 \tag{12}$$

or

$$c_{\alpha\beta} = 2Hb_{\alpha\beta} - Kg_{\alpha\beta}. \tag{13}$$

Because of

$$g^{\alpha\beta} b_{\alpha\beta} = 2H, \quad g^{\alpha\beta} g_{\alpha\beta} = 2, \quad b = Kg$$

we infer that

$$\det(\zeta_{\alpha\beta}) = g[1 - \varepsilon 4\varphi H + \varepsilon^2\{|\nabla_X\varphi|^2 + \varphi^2(4H^2 + 2K)\}] + O(\varepsilon^3). \tag{14}$$

Moreover,

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \quad \text{for } |x| \ll 1,$$

and therefore

$$\sqrt{1 + \varepsilon\alpha + \varepsilon^2\beta} = 1 + \frac{\alpha}{2}\varepsilon + \left(\frac{\beta}{2} - \frac{\alpha^2}{8}\right)\varepsilon^2 + O(\varepsilon^3)$$

for $|\varepsilon| \ll 1$. Thus we see that

$$\sqrt{\det(\zeta_{\alpha\beta})} = \sqrt{g}[1 - \varepsilon 2\varphi H + \varepsilon^2\{\tfrac{1}{2}|\nabla_X \varphi|^2 + K\varphi^2 + 2H^2\varphi^2 - 2H^2\varphi^2\}] + O(\varepsilon^3)$$

or

$$\sqrt{\det(\zeta_{\alpha\beta})} = \sqrt{g}[1 - \varepsilon 2\varphi H + \varepsilon^2\{\tfrac{1}{2}|\nabla_X \varphi|^2 + K\varphi^2\}] + O(\varepsilon^3)\}. \tag{15}$$

From this expansion, we derive for the second variation

$$\delta^2 A_\Omega(X, Y) := \frac{d^2}{d\varepsilon^2} A(X + \varepsilon Y)\bigg|_{\varepsilon=0} \tag{16}$$

of X in the normal direction $Y = \varphi N$ the formula

$$\delta^2 A_\Omega(X, Y) = \int_\Omega \{|\nabla_X \varphi|^2 + 2K\varphi^2\}\, dA \tag{17}$$

which can be considered as a quadratic form on the Sobolev space $H_2^1(\Omega)$.

We restrict

$$J(\varphi) := \delta^2 A_\Omega(X, \varphi N) \tag{18}$$

to the Sobolev space $\mathring{H}_2^1(\Omega)$ of functions $\varphi \in H_2^1(\Omega)$ with (generalized) boundary values zero on $\partial\Omega$.

Consider the isoperimetric problem

$$J(\varphi) \to \min \quad \text{for } \varphi \in \mathring{H}_2^1(\Omega) \quad \text{with } \int_\Omega \varphi^2 dA = 1. \tag{19}$$

Its solution satisfies

$$\begin{aligned} -\Delta_X \varphi + 2K\varphi &= \mu\varphi \quad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{20}$$

where μ is the smallest real number, for which a nontrivial solution φ of these two equations exists; in other words, $\mu = J(\varphi)$ is the smallest eigenvalue of the operator $-\Delta_X + 2K$ on Ω with respect to zero boundary values.

In the sequel we shall often write $\delta^2 A(X, \varphi)$ instead of $\delta^2 A(X, \varphi N)$.

2.9 Scholia

1 References to the Literature on Nonparametric Minimal Surfaces

The modern theory of the nonparametric minimal surface equation and of related equations begins with the celebrated papers of S. Bernstein [1–4] and with the work of Korn [1,2] and Müntz [1]. The central problem of interest concerning nonparametric minimal surfaces was at that time the solution of Plateau's problem. A new attack on this problem was started by Müntz [2] in 1925 which, however, proved to be faulty (see Radó [11], and also Müntz [3]). The final solution of Plateau's problem in the context of nonparametric minimal surfaces in two dimensions was achieved by Haar in his pioneering paper [3]. Important supplements were given by Radó; see Haar [5], Radó [2,8,15]. In his survey [21], Radó gave a lucid presentation of the development until 1933.

After 1945, many new and surprising results on two-dimensional nonparametric minimal surfaces were found. In particular we mention the work of Bers, Finn, Heinz, E. Hopf, Jörgens and J.C.C. Nitsche. A beautiful and very complete presentation of the whole theory of two-dimensional nonparametric minimal surfaces can be found in Nitsche's treatise [28]; for an updated version see [37]. Certain aspects of the theory based on the work of Sauvigny are presented in Chapters 5 and 7 of this volume.

Even more astounding is the development of the theory of n-dimensional nonparametric minimal surfaces which is to a large extent described in the monographs of Gilbarg and Trudinger [1], Giusti [4], and Massari and Miranda [1]. Finn's treatise [11] leads the reader into the fascinating field of free boundary problems connected with the phenomenon of capillarity.

The theory of nonparametric minimal surfaces of codimension $m > 1$ was initiated by Osserman [11]. Here many new problems arise as was shown by Lawson and Osserman [1]. Osserman proved:

Let $\mathcal{M}$ be an n-dimensional submanifold in $\mathbb{R}^{n+p}$ which is the graph of a function $f \in C^2(\Omega, \mathbb{R}^p)$, $\Omega \subset \mathbb{R}^n$. Let $\gamma_{\alpha\beta}(x) := \delta_{\alpha\beta} + f^i_{x^\alpha}(x) f^i_{x^\beta}(x)$ be the metric tensor of $\mathcal{M}, \gamma := \det(\gamma_{\alpha\beta})$ and $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. Then $\mathcal{M}$ is a minimal submanifold of $\mathbb{R}^{n+p}$ if and only if

$$\frac{1}{\sqrt{\gamma}} D_\beta \{\sqrt{\gamma}\gamma^{\alpha\beta} D_\alpha f^i\} = 0, \quad 1 \le i \le p, \tag{1}$$

that is, if and only if the coordinate functions f^i of the mapping f are harmonic with respect to the metric of $\mathcal{M}$. Equivalently we can write

$$\Delta_{\mathcal{M}} f = 0. \tag{2}$$

The equations (1) imply that

$$D_\alpha \{\sqrt{\gamma}\gamma^{\alpha\beta}\} = 0, \quad 1 \le \beta \le n. \tag{3}$$

Therefore the equations (1) are equivalent to the system

$$\gamma^{\alpha\beta} D_\alpha D_\beta f^i = 0, \quad 1 \le i \le p. \tag{4}$$

Morrey [4] proved that any weak solution $f \in C^1(\Omega, \mathbb{R}^p)$, $\Omega \subset \mathbb{R}^n$, of (4) is real analytic. On the other hand, Lawson and Osserman [1] found for $n = 4$, $p = 3$ an example of a Lipschitz continuous weak solution of (1) which is not of class C^1. Furthermore, if Ω is the unit ball in $\mathbb{R}^4$, they discovered a quadratic polynomial $\varphi : \partial\Omega \to \mathbb{R}^3$ which cannot be extended to a mapping

$$f \in C^0(\bar{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$$

solving (1) in Ω. Harvey and Lawson [4] later proved that the singular solution of (1) found by Lawson and Osserman is, in fact, area-minimizing with respect to its boundary values.

Moreover, Lawson and Osserman [1] pointed out that, differently from the case of codimension $p = 1$, the solutions of (1) are no longer uniquely determined by their boundary values. Even if $n = 2$ and Ω is the unit disk, there is a real analytic map $\varphi : \partial\Omega \to \mathbb{R}^2$ to which there correspond three distinct solutions u of (1) in $\bar{\Omega}$ satisfying $u|_{\partial\Omega} = \varphi$.

2 Bernstein's Theorem

Bernstein's theorem is one of the most fascinating results in the theory of nonlinear elliptic differential equations. First published in 1916, it has attracted time and again the attention of analysis since the German translation of Bernstein's paper [4] appeared in 1927. Much later, a gap was discovered in Bernstein's original proof which succeedingly was closed by E.J. Mickle [1] and E. Hopf [3].

A discussion of various ramifications and generalizations of Bernstein's theorem can be found in Osserman [5], Nitsche [28], Giusti [4], Gilbarg and Trudinger [1], Hildebrandt [14,17]. The results presented in Sections 2.2–2.5 are essentially taken from the work of Radó, Nitsche and Heinz.

We mention that for nonparametric n-dimensional minimal surfaces of codimension one Bernstein's theorem holds true if $n \leq 7$, whereas Bombieri, de Giorgi, and Giusti [1] derived from the Simons cone $C = \{x \in \mathbb{R}^8 : x = (y, z),\ y, z \in \mathbb{R}^4 \text{ and } |y|^2 = |z|^2\}$ an example which shows that Bernstein's theorem becomes false if $n \geq 8$. A slight error in their reasoning was pointed out by Luckhaus who also saw how it can be removed (cf. Dierkes [5]).

Another major achievement was the paper of Schoen, Simon, and Yau [1] who proved a generalization of Heinz's estimate (22) stated in Theorem 2 of Section 2.4 to all dimensions $n \leq 5$, thereby obtaining another proof for Bernstein's theorem in dimensions $n \leq 5$. Improvements of this work were made by Simon [1,4]. We present some of these results in Vol. 3, Chapter 3.

A Bernstein theorem in arbitrary dimension and codimension was proved by Hildebrandt, Jost, and Widman [1]:

If $f : \mathbb{R}^n \to \mathbb{R}^p$ is an entire solution of the minimal surface system (1) (*i.e., a solution on all of* $\mathbb{R}^n$) *such that* $\sqrt{\gamma(x)} \leq \beta_0$ *on* $\mathbb{R}^n$ *where* β_0 *is a number* *satisfying* $\beta_0 < \cos^{-m}(\pi/2\sqrt{2m})$ *and* $m := \min\{n, p\} \geq 2$, *then* f *is linear, and therefore its graph represents an affine n-plane in* $\mathbb{R}^{n+p}$.

In this theorem $\gamma(x)$ denotes the function $\det(\gamma_{\alpha\beta}(x))$ where $\gamma_{\alpha\beta}(x) = \delta_{\alpha\beta} + f^i_{x^\alpha}(x) f^i_{x^\beta}(x)$. Note that a better result holds true if $m = 1$. A related result was proved by Fischer-Colbrie [1]. In Vol. 3 we present a fairly comprehensive presentation of Bernstein-type theorems.

3 Stable Minimal Surfaces

It is a rather difficult problem to decide whether a given specific minimal surface spanned by a closed curve Γ is actually area minimizing, that is, whether it is an absolute or at least relative minimizer of the area functional among all surfaces of the same topological type bounded by Γ. Suppose that the minimal surface $X : \Omega \to \mathbb{R}^3$ is defined on a bounded domain Ω of $\mathbb{R}^2$. Then it is easy to see that the condition

$$\delta^2 A_\Omega(X, \varphi) \geq 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^3) \tag{5}$$

is necessary for any relative minimizer X within Γ. Let $\lambda_1(\Omega)$ be the smallest eigenvalue of the *second-variation operator* $-\Delta_X + 2K$ on Ω with respect to zero boundary values. Then, by a classical result of the calculus of variations, X is a relative minimizer of area with respect to the C^1-topology if X is a regular minimal surface of class $C^2(\bar{\Omega}, \mathbb{R}^3)$ satisfying

$$\lambda_1(\Omega) > 0. \tag{6}$$

A minimal surface $X : \Omega \to \mathbb{R}^3$ defined on a parameter domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary is said to be *strictly stable* if it is of class $C^2(\bar{\Omega}, \mathbb{R}^3)$, regular (i.e. free of branch points) on $\bar{\Omega}$ and satisfies $\lambda_1(\Omega) > 0$. If $\lambda_1(\Omega) \geq 0$, the surface X is called *stable*.

In certain situations one can show that a stable minimal surface can be embedded in a field, that is, it can be viewed as a leaf of a suitable foliation in $\mathbb{R}^3$ whose leaves are all minimal surfaces. Then we obtain that such a stable surface actually is a relative minimizer of area with respect to the C^0-topology. Such a field construction plays an essential role in the proof of Nitsche's uniqueness theorem (see Section 4.9, and, for details, Sections 5.6 and 5.7).

Barbosa and do Carmo [1] proved that any immersed minimal surface $X : \Omega \to \mathbb{R}^3$ is strictly stable if the image $N(\Omega)$ of Ω under the Gauss map $N : \Omega \to S^2$ corresponding to X has area less than 2π. Later these authors showed in [4] that the assumption $\int K \, dA < \frac{4}{3}\pi$ implies strict stability of any immersed minimal surface $X : \Omega \to \mathbb{R}^n$, for an arbitrary $n \geq 3$, if Ω is simply connected.

Stable minimal surfaces are an important subclass in the set of all minimal surfaces. Roughly speaking, we can view strictly stable minimal surfaces as those surfaces of mean curvature zero that can experimentally be realized by soap films. In some respect they behave like nonparametric minimal surfaces. For instance, R. Schoen [2] proved an analogue of Heinz's estimate (22) stated in Section 2.4 for stable surfaces which, in turn, implies Bernstein's theorem for such surfaces. Moreover, Schoen's estimate also yields an earlier result of do Carmo and Peng [1] and of Fischer-Colbrie and R. Schoen [1], namely that a complete stable minimal surface in $\mathbb{R}^3$ has to be a plane.

For a fairly detailed discussion of the second variation of area and of stable minimal surfaces we refer to Chapter 5 as well as to Nitsche [28], pp. 86–109, and for an updated version to Nitsche [37], pp. 90–116. There the reader will also find a survey of the fundamental contributions of H.A. Schwarz to this problem which are mainly contained in his Festschrift for the 70th birthday of Weierstrass (cf. Schwarz [2], vol. 1, pp. 223–269).

4 Foliations by Minimal Surfaces

In Section 2.8 as well as in Subsection 3 of these Scholia we saw that any leaf of a foliation by minimal surfaces is area minimizing. This is the basic content of Weierstrass's approach to the calculus of variations. Its main ingredients are the Weierstrass field construction (that is, the embedding of a given minimal surface into a field consisting of a foliation with minimal leaves) and Hilbert's independent integral. The method presented in Section 2.8 furnishes a simplification of the original form of the independent integral stated by Schwarz. This simplified version is based on the calculus of differential forms and provides a flexible and important tool in differential geometry which is very easy to handle. For applications and further results we refer to the basic work of Harvey and Lawson [3,4] and of Lawlor and Morgan [1].

Other contributions on foliations by minimal submanifolds of a given Riemannian manifold are due to Haefliger [1], Rummler [1], and Sullivan [1].

In the Sections 5.6 and 5.7 we discuss field constructions for immersed minimal surfaces that are not embedded. They are the geometric basis for Tomi's finiteness theorem and Nitsche's uniqueness theorem.

Chapter 3

Representation Formulas and Examples of Minimal Surfaces

In this chapter we present the elements of the classical theory of minimal surfaces developed during the nineteenth century. We begin by representing minimal surfaces as real parts of holomorphic curves in $\mathbb{C}^3$ which are isotropic. This leads to useful and handy formulas for the line element, the Gauss map, the second fundamental form and the Gauss curvature of minimal surfaces. Moreover we obtain a complete description of all interior singular points of two-dimensional minimal surfaces as branch points of $\mathbb{C}^3$-valued power series, and we derive a normal form of a minimal surface in the vicinity of a branch point. Close to a branch point of order m, a minimal surface behaves, roughly speaking, like an m-fold cover of a disk, a property which is also reflected in the form of lower bounds for its area. Other by-products of the representation of minimal surfaces as real parts of isotropic curves in $\mathbb{C}^3$ are results on *adjoint* and *associated minimal surfaces* that were discovered by Bonnet.

In Section 3.3 we turn to the representation formula of Enneper and Weierstrass which expresses a given minimal surface in terms of integrals involving a holomorphic function μ and a meromorphic function ν. Conversely, any pair of such functions μ, ν can be used to define minimal surfaces provided that $\mu\nu^2$ is holomorphic. In the older literature this representation was mostly used for a local discussion of minimal surfaces. Following the example of Osserman (see [10] and [24]), the representation formula has become very important for the treatment of global questions for minimal surfaces. As an example of this development we describe in Section 3.7 the results concerning the omissions of the Gauss map of a complete regular minimal surface. These results are the appropriate generalization of Picard's theorem in function theory to differential geometry and culminate in the remarkable theorem of Fujimoto that the Gauss map of a nonplanar complete and regular minimal surface cannot miss more than four points on the Riemann sphere. Important steps to the final version of this result which can also be viewed as a generalization of Bernstein's theorem were taken by Osserman and Xavier. The proof given in

U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 339,
DOI 10.1007/978-3-642-11698-8_3,

Section 3.7 is very close to Osserman's original approach and is due to Mo and Osserman [1].

Moreover, most of the sophisticated examples of minimal surfaces and, in particular, of families of complete embedded minimal surfaces and also of periodic surfaces of zero mean curvature are best described via the Enneper-Weierstrass formula. We shall not attempt to present a complete picture of this part of the theory which in recent years has gathered new momentum, but we shall content ourselves with a few examples mentioned in Section 3.6 and with a very short survey given in the Scholia Section 3.8. Instead of a careful discussion we include various figures depicting old and new examples of these fascinating species.

A few of the known classical minimal surfaces are briefly described in Section 3.5, and these surfaces are illustrated by numerous figures so that the reader has sufficient visual examples for the investigations carried out in the following chapters. We do not aim at completeness but we refer the reader to Nitsche's encyclopaedic treatise [28] as well as to the literature cited in Subsection 1 of the Scholia, Section 3.8. A brief survey of some of the newer examples can be found in Subsections 4 and 5 of the Scholia. For a detailed presentation of recent results on complete minimal surfaces we in particular refer to work of H. Karcher [1–5], to the encyclopaedia article by Karcher and Hoffmann in EMS, and to the collection of papers in GTMS.

The Enneper–Weierstrass representation formula of a minimal surface $X : \Omega \to \mathbb{R}^3$ is still somewhat arbitrary since the composition $Y = X \circ \tau$ of X with a conformal mapping $\tau : \Omega^* \to \Omega$ describes the same geometric object as X. Thus one can use a suitable map τ to eliminate one of the two functions μ, ν in the Weierstrass formula; consequently every minimal surface viewed as a geometric object, i.e., as an equivalence class of conformally equal surfaces, corresponds to *one* holomorphic function $\mathfrak{F}(\omega)$. Weierstrass derived a representation of this kind where $\mathfrak{F}$ is defined on the stereographic projection of the spherical image of the considered minimal surface. The Gauss curvature and the second fundamental form of a minimal surface can be expressed in a very simple way in terms of the functions μ, ν, or $\mathfrak{F}$.

Finally in Section 3.4 we discuss several contributions by H.A. Schwarz to the theory of minimal surfaces, in particular his solution of Björling's problem. This is just the Cauchy problem for minimal surfaces and an arbitrarily prescribed real analytic initial strip, and it is known to possess a unique solution due to the theorem of Cauchy–Kovalevskaya. Schwarz found a beautiful integral representation of this solution which can be used to construct interesting minimal surfaces, such as surfaces containing given curves as geodesics or as lines of curvature. As an interesting application of Schwarz's solution we treat his reflection principles for minimal surfaces.

3.1 The Adjoint Surface. Minimal Surfaces as Isotropic Curves in $\mathbb{C}^3$. Associate Minimal Surfaces

Let us begin by recalling the general definition of a minimal surface, given in Section 2.6.

A nonconstant surface $X : \Omega \to \mathbb{R}^3$ of class C^2 is said to be a minimal surface if it satisfies the equations

$$\Delta X = 0 \tag{1}$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{2}$$

on Ω.

If a minimal surface

$$X(u,v) = (x(u,v), y(u,v), z(u,v))$$

is defined on a simply connected domain Ω of $\mathbb{R}^2 \mathrel{\hat{=}} \mathbb{C}$, then we define an *adjoint surface*

$$X^*(u,v) = (x^*(u,v), y^*(u,v), z^*(u,v))$$

to $X(u,v)$ on Ω as solution of the *Cauchy–Riemann equations*

$$X_u = X_v^*, \quad X_v = -X_u^* \tag{3}$$

in Ω.

Clearly, all adjoint surfaces to some given minimal surface X differ only by a constant vector; thus we may speak of *the adjoint surface* $X^*(u,v)$ of some minimal surface $X(u,v)$ which is defined on a simply connected domain Ω of $\mathbb{R}^2$.

The equations (1)–(3) immediately imply

$$\Delta X^* = 0, \quad |X_u^*|^2 = |X_v^*|^2, \quad \langle X_u^*, X_v^* \rangle = 0,$$

that is, the adjoint surface X^* to some minimal surface X is a minimal surface.

Consider an arbitrary harmonic mapping $X : \Omega \to \mathbb{R}^3$ of a simply connected domain Ω in $\mathbb{R}^2 \mathrel{\hat{=}} \mathbb{C}$, and let X^* be the adjoint harmonic mapping to X, defined as a solution of (3). Then

$$f(w) := X(u,v) + iX^*(u,v), \quad w = u + iv \in \Omega \tag{4}$$

is a holomorphic mapping of Ω into $\mathbb{C}^3$ with components

$$\begin{aligned} \varphi(w) &= x(u,v) + ix^*(u,v), \\ \psi(w) &= y(u,v) + iy^*(u,v), \\ \chi(w) &= z(u,v) + iz^*(u,v), \end{aligned} \tag{5}$$

which can be considered as a *holomorphic curve* in $\mathbb{C}^3$. Its complex derivative $f' = \frac{df}{dw}$ is given by

$$f' = X_u + iX_u^* = X_u - iX_v, \tag{6}$$

whence it follows that

$$\langle f', f' \rangle = |X_u|^2 - |X_v|^2 - 2i\langle X_u, X_v \rangle. \tag{7}$$

Consequently, the conformality relations (2) are satisfied if and only if the *isotropy relation*

$$\langle f', f' \rangle = 0 \tag{8}$$

is fulfilled.

A holomorphic curve satisfying relation (8) is said to be an *isotropic curve.*

Using this notation, we obtain the following result:

Proposition 1. *If $X : \Omega \to \mathbb{R}^3$ is a minimal surface on a simply connected parameter domain Ω in $\mathbb{R}^2$, then the holomorphic curve $f : \Omega \to \mathbb{C}^3$, defined by* (3) *and* (4)*, is a nonconstant isotropic curve. Conversely, if $f : \Omega \to \mathbb{C}^3$ is a nonconstant isotropic curve in $\mathbb{C}^3$, then*

$$X(u, v) := \operatorname{Re} f(w), \quad X^*(u, v) := \operatorname{Im} f(w) \tag{9}$$

defines two minimal surfaces $X : \Omega \to \mathbb{R}^3$ and $X^* : \Omega \to \mathbb{R}^3$ on Ω, whether or not Ω is simply connected.

We say that $X^*(u, v)$, $w \in \Omega$, is an *adjoint surface* to some minimal surface $X(u, v)$, $w \in \Omega$, if there is an isotropic curve $f : \Omega \to \mathbb{C}^3$ such that (9) is satisfied.

If X^* is adjoint to X, then $-X$ is adjoint to X^*, i.e.,

$$X^{**} = -X. \tag{10}$$

The isotropy condition (8) for a curve $f(w) = (\varphi(w), \psi(w), \chi(w))$ means that the derivatives of the three holomorphic functions φ, ψ, χ are coupled by the relation

$$\varphi'^2 + \psi'^2 + \chi'^2 = 0. \tag{11}$$

Let us introduce the two Wirtinger operators

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{12}$$

Then the equations (1) and (2) can equivalently be written as

$$X_{w\bar{w}} = 0 \tag{13}$$

and

$$(14) \qquad \langle X_w, X_w \rangle = 0,$$

respectively, and we also have

$$(14') \qquad f' = 2X_w.$$

Suppose now that $X : \Omega \to \mathbb{R}^3$ is a minimal surface on some domain Ω. Then we have

$$\mathcal{W} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \mathcal{E} = \mathcal{G} = \tfrac{1}{2}(\mathcal{E} + \mathcal{G}).$$

By restricting ourselves to simply connected subdomains Ω' of Ω, we can assume that there is an isotropic curve f such that $X = \operatorname{Re} f$, $f = (\varphi, \psi, \chi)$. Since $|f'|^2 = |\nabla X|^2 = 4|X_w|^2$, we obtain

$$(15) \qquad \mathcal{W} = |X_u|^2 = \tfrac{1}{2}|\nabla X|^2 = \tfrac{1}{2}|f'|^2 = 2|X_w|^2.$$

Thus the zeros of $\mathcal{W}$ are the common zeros of the three holomorphic functions φ', ψ', χ' and must, therefore, be isolated in Ω, except if $X(w) \equiv \text{const}$, which is excluded.

Proposition 2. *The singular points w of a minimal surface $X : \Omega \to \mathbb{R}^3$ on a domain Ω are isolated. They are exactly the zeros of the function $|X_u|$ in Ω.*

As we shall see, the behavior of a minimal surface in the neighborhood of one of its singular points resembles the behavior of a holomorphic function $\varphi(w)$ in the neighborhood of a zero of its derivative $\varphi'(w)$. Therefore the singular points of minimal surfaces are called *branch points*. We shall look at them more closely in the next section.

The following statements are an immediate consequence of the equations (1)–(3).

Proposition 3. *Let $X^* : \Omega \to \mathbb{R}^3$ be an adjoint surface to the minimal surface $X : \Omega \to \mathbb{R}^3$.*

(i) *We have $X^*(w) \not\equiv \text{const}$.*

(ii) *Some point $w_0 \in \Omega$ is a branch point of X if and only if it is a branch point of X^*.*

(iii) *Denote by $N(w)$ and $N^*(w)$ the Gauss maps of $X(w)$ and $X^*(w)$ respectively, which are defined on the set Ω' of regular points of X in Ω. Then we have*

$$(16) \qquad N(w) \equiv N^*(w) \quad \text{on } \Omega'.$$

Moreover, the tangent spaces of X and X^* coincide:

$$T_w X = T_w X^* \quad \text{for all } w \in \Omega',$$

and also the first fundamental forms of X and X^* agree:

$$I_X(V,W) = I_{X^*}(V,W) \quad \text{for all } V,W \in T_wX,\ w \in \Omega',$$

i.e., the surfaces X and X^* are isometric to each other. Therefore the Gauss curvatures K and K^* of X and X^* are the same:

$$K(w) = K^*(w) \quad \text{for all } w \in \Omega'.$$

The Weingarten maps S and S^* of X and X^* respectively differ by a rotation of 90 degrees on all tangent spaces T_wX, with $w \in \Omega'$.

Later on, we shall exhibit other relations between X, X^*, and their Gauss map N. Presently, we want to formulate a consequence of the Propositions 1 and 2.

Proposition 4. *Let Ω be a simply connected domain in $\mathbb{C}$, $X_0 \in \mathbb{R}^3$, $w_0 \in \Omega$, and suppose that $\Phi(w) = (\Phi_1(w), \Phi_2(w), \Phi_3(w)) \not\equiv 0$ is a holomorphic mapping of Ω into $\mathbb{C}^3$ which satisfies*

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0 \tag{17}$$

on Ω. Then the formula

$$X(w) = X_0 + \operatorname{Re} \int_{w_0}^{w} \Phi(\omega)\, d\omega, \quad w \in \Omega, \tag{18}$$

defines a minimal surface $X : \Omega \to \mathbb{R}^3$, and, for every $X_0^* \in \mathbb{R}^3$, the formula

$$X^*(w) = X_0^* + \operatorname{Im} \int_{w_0}^{w} \Phi(\omega)\, d\omega, \quad w \in \Omega, \tag{19}$$

yields an adjoint surface to X. The branch points of X are exactly the zeros of Φ.

Conversely, if $X : \Omega \to \mathbb{R}^3$ is a minimal surface defined on a simply connected domain Ω, then there is a holomorphic mapping $\Phi : \Omega \to \mathbb{C}^3$ satisfying (17) such that

$$X(w) = X(w_0) + \operatorname{Re} \int_{w_0}^{w} \Phi(\omega)\, d\omega$$

holds for arbitrary $w, w_0 \in \Omega$.

Remark. If Ω is not simply connected, then the integral (18) still defines a minimal surface on Ω provided that the differential form $\Phi\, d\omega$ only has purely imaginary periods, i.e., that $\int_\gamma \Phi(\omega)\, d\omega$ is a purely imaginary number for every closed path γ contained in Ω.

Formula (18) yields

$$X_w = \tfrac{1}{2}\Phi.$$

More generally, if $X : \Omega \to \mathbb{R}^3$ is a minimal surface given in the form

$$X(w) = \operatorname{Re} f(w),$$

where $f : \Omega \to \mathbb{C}^3$ denotes an isotropic curve with the derivative

$$f' = \Phi = (\Phi_1, \Phi_2, \Phi_3),$$

then we infer from

$$f' = X_u - iX_v$$

that

$$X_u = \operatorname{Re} \Phi, \quad X_v = -\operatorname{Im} \Phi.$$

Consequently, we obtain

$$X_u \wedge X_v = \operatorname{Im}(\Phi_2\bar{\Phi}_3, \Phi_3\bar{\Phi}_1, \Phi_1\bar{\Phi}_2). \tag{20}$$

The line element $ds = |dX|$ takes the form

$$ds^2 = \lambda\{du^2 + dv^2\}$$

where

$$\Lambda := |X_u|^2 = \tfrac{1}{2}|\nabla X|^2 = \tfrac{1}{2}|f'|^2 = \tfrac{1}{2}|\Phi|^2 = \mathcal{W}. \tag{21}$$

Thus the spherical image $N : \Omega' \to S^2$, $N = \Lambda^{-1} X_u \wedge X_v$, $\Omega' := \{w \in \Omega : \Lambda(w) \neq 0\}$, is given by

$$N = 2|\Phi|^{-2} \operatorname{Im}(\Phi_2\bar{\Phi}_3, \Phi_3\bar{\Phi}_1, \Phi_1\bar{\Phi}_2). \tag{22}$$

Moreover, the equation $f' = X_u - iX_v$ implies

$$f'' = X_{uu} - iX_{uv} = -X_{vv} - iX_{uv}$$

whence

$$\langle f'', N\rangle = \mathcal{L} - i\mathcal{M} = -\mathcal{N} - i\mathcal{M}$$

on Ω'. Therefore we obtain the well-known relation

$$\mathcal{L} = -\mathcal{N}, \tag{23}$$

expressing the fact that X has zero mean curvature, and also

$$|\langle f'', N\rangle|^2 = \mathcal{L}^2 + \mathcal{M}^2. \tag{24}$$

By the observation of H. Hopf (cf. Section 1.3), the function $\frac{1}{2}(\mathcal{L} - \mathcal{N}) - i\mathcal{M}$ is holomorphic on Ω'. Thus we obtain that *the function*

$$l(w) := \mathcal{L}(w) - i\mathcal{M}(w) = \langle f''(w), N(w)\rangle \tag{25}$$

is holomorphic on Ω'.

We infer from (23) and (24) that the Gauss curvature K of X on Ω' is given by

$$K = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{\mathcal{W}^2} = -\frac{\mathcal{L}^2 + \mathcal{M}^2}{\mathcal{W}^2} = -\frac{|l|^2}{\Lambda^2} \tag{26}$$

or

$$K = -4|\Phi|^{-4}|\langle \Phi', N\rangle|^2, \quad \Phi = f'. \tag{26'}$$

We conclude that $K(w) \le 0$ *on* Ω', *and that* $K(w) = 0$ *if and only if* $l(w) = 0$ *holds.*

Note that $K \le 0$ also follows from $H = 0$ because of $2H = \kappa_1 + \kappa_2$ and $K = \kappa_1 \kappa_2$.

Umbilical points w of a surface $X(w)$ are regular points where both principal curvatures κ_1 and κ_2 are equal. Since $H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$ and $K = \kappa_1\kappa_2 \le 0$, the umbilical points $w \in \Omega'$ of a minimal surface $X : \Omega \to \mathbb{R}^3$ are characterized by the condition

$$K(w) = 0,$$

or equivalently, by

$$\mathcal{L}(w) = 0, \quad \mathcal{M}(w) = 0, \quad \mathcal{N}(w) = 0.$$

Since umbilical points of X are precisely the zeros of the holomorphic function $l : \Omega' \to \mathbb{C}$, they must either be isolated, or else $\mathcal{L}(w) \equiv 0$, $\mathcal{M}(w) \equiv 0$, and $\mathcal{N}(w) \equiv 0$ on Ω' which implies that $X(w)$, $w \in \Omega$, is a planar surface, taking the Weingarten equations (48) of Section 1.2 into account.

In the next section, we shall prove that $N(w)$ approaches a limit N_0 as w tends to some branch point $w_0 \in \Omega$. This implies that $l(w) = \mathcal{L}(w) - i\mathcal{M}(w)$ is actually holomorphic on Ω, since isolated singularities of holomorphic functions are removable if they are continuity points.

By means of the function $l(w), w \in \Omega$, it is easy to characterize the asymptotic lines and the curvature lines of a nonconstant minimal surface $X(w)$, $w \in \Omega$.

Let $\omega(t) = (\alpha(t), \beta(t))$, $t \in \mathcal{I}$, be a C^1-curve in Ω, i.e., $\omega(\mathcal{I}) \subset \Omega$. On account of Section 1.2, (48), this curve is an *asymptotic line* of X if and only if

$$\mathcal{L}\dot{\alpha}^2 + 2\mathcal{M}\dot{\alpha}\dot{\beta} + \mathcal{N}\dot{\beta}^2 = 0,$$

and, by 1.2, (53), it is a *line of curvature* if and only if

$$(\mathcal{E}\mathcal{M} - \mathcal{F}\mathcal{L})\dot{\alpha}^2 + (\mathcal{E}\mathcal{N} - \mathcal{G}\mathcal{L})\dot{\alpha}\dot{\beta} + (\mathcal{F}\mathcal{N} - \mathcal{G}\mathcal{M})\dot{\beta}^2 = 0.$$

Here $\mathcal{E}, \dots, \mathcal{L}, \dots$ have to be understood as $\mathcal{E}(\omega), \dots, \mathcal{L}(\omega), \dots$. Since $\mathcal{E} = \mathcal{G}$, $\mathcal{F} = 0$, $\mathcal{E} = -\mathcal{N}$, we obtain:

The asymptotic lines are described by

$$\mathcal{L}(\dot{\alpha}^2 - \dot{\beta}^2) + 2\mathcal{M}\dot{\alpha}\dot{\beta} = 0, \tag{27}$$

and the lines of curvature are characterized by

$$\mathcal{M}(\dot{\alpha}^2 - \dot{\beta}^2) - 2\mathcal{L}\dot{\alpha}\dot{\beta} = 0. \tag{28}$$

Let us introduce the complex valued quadratic form $\Xi(\dot{\omega})$, depending on $\dot{\omega} = (\dot{\alpha}, \dot{\beta})$, by

$$\Xi(\dot{\omega}) := l(\omega)(\dot{\alpha}^2 + i\dot{\beta}^2), \quad l = \mathcal{L} - i\mathcal{M}. \tag{29}$$

Then the asymptotic lines and the curvature lines are given by

$$\operatorname{Re}\Xi(\dot{\omega}) = 0 \quad \text{and} \quad \operatorname{Im}\Xi(\dot{\omega}) = 0 \tag{30}$$

respectively, or, in other words, by

$$\operatorname{Re} l(w)(dw)^2 = 0 \quad \text{and} \quad \operatorname{Im} l(w)(dw)^2 = 0, \tag{30$'$}$$

using the holomorphic quadratic differential $l(w)(dw)^2$.

Collecting these results, we obtain:

Proposition 5. *Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface given by $X = \operatorname{Re} f$, where $f : \Omega \to \mathbb{C}^3$ is an isotropic curve with $f' = \Phi = (\Phi_1, \Phi_2, \Phi_3)$. Then its spherical image $N(w)$, $w \in \Omega'$, on the set of regular points $\Omega' := \{w \in \Omega\colon \Lambda(w) \neq 0\}$, $\Lambda := |X_u|$, is given by (22), and its Gauss curvature K on Ω' can be computed from (26) or (26$'$). On Ω', the curvature $K(w)$ is strictly negative, except for umbilical points, where $K(w)$ is vanishing. The umbilical points of X are exactly the zeros of the holomorphic function $l(w) = \mathcal{L}(w) - i\mathcal{M}(w)$, $w \in \Omega$. If X is a nonplanar surface, then its umbilical points are isolated. Moreover, the asymptotic lines of X are described by*

$$\operatorname{Re} l(w)(dw)^2 = 0,$$

and the curvature lines by

$$\operatorname{Im} l(w)(dw)^2 = 0.$$

Now we want to define the family of *associate minimal surfaces* to a given minimal surface $X : \Omega \to \mathbb{R}^3$ which is given as the real part of some isotropic curve $f : \Omega \to \mathbb{C}^3$. That is,

$$f(w) = X(w) + iX^*(w), \quad w = u + iv \in \Omega,$$

where

$$\langle f'(w), f'(w)\rangle \equiv 0 \quad \text{on } \Omega.$$

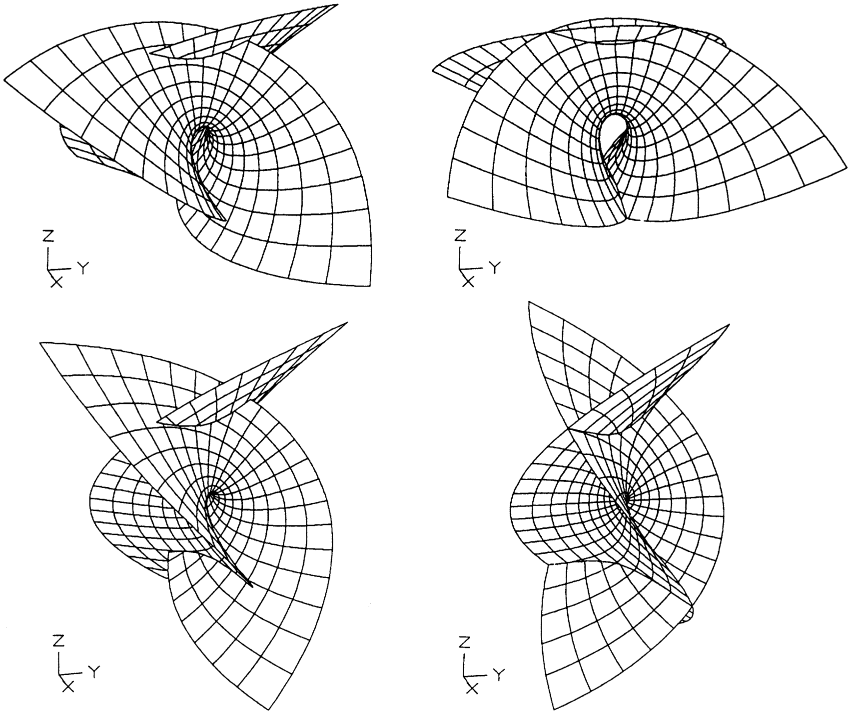


Fig. 1. The bending process leading to the associate surfaces of Enneper's surface corresponding to the square $[-2,2]^2$, counter-clockwise from top right: $\theta = 0$, $\pi/6$, $\pi/3$, and $\pi/2$

Then, for every $\theta \in \mathbb{R}$, also

$$g(w,\theta) := e^{-i\theta} f(w), \quad w \in \Omega \tag{31}$$

describes an isotropic curve, and

$$Z(w,\theta) := \operatorname{Re}\{e^{-i\theta} f(w)\} = X(w)\cos\theta + X^*(w)\sin\theta \tag{32}$$

defines a one-parameter family of minimal surfaces with the property that

$$Z(w,0) = X(w), \quad Z\left(w, \frac{\pi}{2}\right) = X^*(w). \tag{33}$$

The surfaces $Z(w,\theta)$, $w \in \Omega$, are called *associate minimal surfaces* to the surface $X(w)$, $w \in \Omega$. Relation (3) yields

$$\begin{aligned} Z_u &= X_u \cos\theta - X_v \sin\theta, \\ Z_v &= X_v \cos\theta + X_u \sin\theta, \end{aligned}$$

and therefore, by virtue of (2),

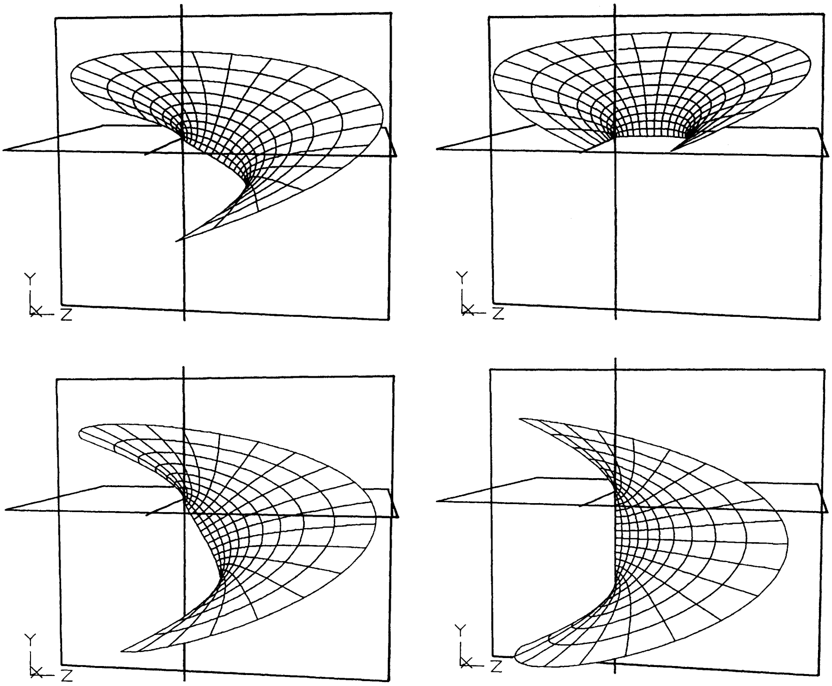


Fig. 2. The associates of Catalan's surface, counter-clockwise from top right ($\theta = 0, \pi/6, \pi/3, \pi/2$). The image of the curve $v = 0$ on Catalan's contained in the plane $y = 0$ is a geodesic. Its Gauss image on S^2 is an arc of a great circle

$$|Z_u|^2 = |Z_v|^2 = |X_u|^2 = |X_v|^2, \quad \langle Z_u, Z_v \rangle = 0.$$

As before, we denote by $\Omega' = \{w \in \Omega \colon \Lambda(w) \neq 0\}$, $\Lambda := |X_u|$, the domain of regular points of X in Ω. Then Ω' is also the domain of regular points for each of the associate surfaces $Z(\cdot, \theta)$, and also the tangent spaces $T_w X$ and $T_w Z(\cdot, \theta)$ of X and $Z(\cdot, \theta)$ coincide for all $w \in \Omega'$ and every $\theta \in \mathbb{R}$. Therefore the Gauss map $N : \Omega' \to S^2$ of X agrees with the spherical image of each of its associate surfaces. Moreover, we have

$$\langle dZ(\cdot, \theta), dZ(\cdot, \theta) \rangle = \langle dX, dX \rangle \tag{34}$$

for all $\theta \in \mathbb{R}$, that is, all associate minimal surfaces have the same first fundamental form and, therefore, all associate surfaces are isometric to each other.

Consider now, for every $\theta \in \mathbb{R}$, the holomorphic function

$$l(\theta) = \mathcal{L}(\theta) - i\mathcal{M}(\theta) := \langle g''(\cdot, \theta), N \rangle$$

which characterizes the asymptotic lines, the curvature lines, and the umbilical points of the associate minimal surface $Z(\cdot, \theta)$. Because of $g''(w, \theta) = e^{-i\theta} f''(w)$, we obtain

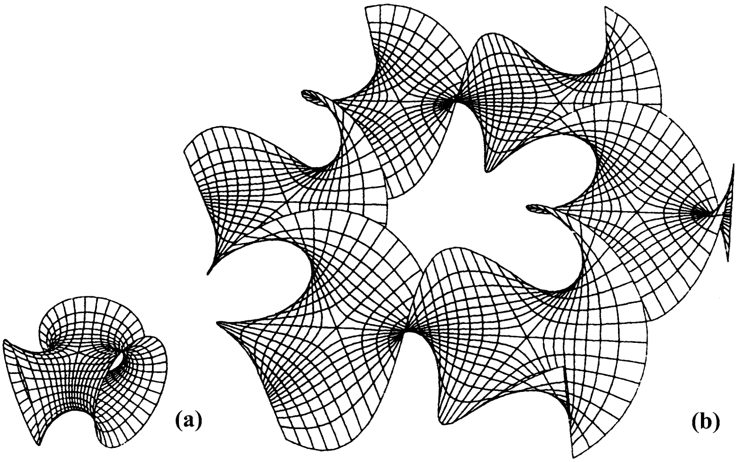


Fig. 3. (a) The Jorge–Meeks catenoid. With courtesy of J. Hahn and K. Polthier. **(b)** An associate minimal surface to the Jorge–Meeks catenoid. Courtesy of K. Polthier and M. Wohlgemuth

$$l(\theta) = e^{-i\theta} l(0) = [\mathcal{L}\cos\theta - \mathcal{M}\sin\theta] - i[\mathcal{L}\sin\theta + \mathcal{M}\cos\theta] \tag{35}$$

where $l(0) = l = \mathcal{L} - i\mathcal{M}$ is the characteristic function for $X = Z(\cdot, 0)$. It follows that $l(\frac{\pi}{2}) = -\mathcal{M} - i\mathcal{L}$. Set

$$\begin{aligned} \xi &:= \mathcal{L}(\dot\alpha^2 - \dot\beta^2) + 2\mathcal{M}\dot\alpha\dot\beta, \\ \eta &:= -\mathcal{M}(\dot\alpha^2 - \dot\beta^2) + 2\mathcal{L}\dot\alpha\dot\beta. \end{aligned}$$

Then we obtain

$$\begin{aligned} l(0)(\dot\alpha + i\dot\beta)^2 &= \xi + i\eta, \\ l\left(\frac{\pi}{2}\right)(\dot\alpha + i\dot\beta)^2 &= \eta - i\xi. \end{aligned} \tag{36}$$

Since $X = Z(\cdot, 0)$ and $X^* = Z(\cdot, \frac{\pi}{2})$, we infer from Proposition 5 that the asymptotic lines of X are the curvature lines of X^*, and conversely, the curvature lines of X are the asymptotic lines of X^*. Thus we have found:

Proposition 6. *All associate surfaces $Z(\cdot, \theta)$ are in isometric correspondence to each other. Each associate surface can be obtained from the original surface X by a bending procedure which, at every stage, passes through a minimal surface. For each $w \in \Omega$, all tangent spaces $T_w Z(\cdot, \theta)$ coincide as θ varies in $\mathbb{R}$. Finally if θ and θ' differ by $\frac{\pi}{2}$, then the asymptotic lines of $Z(\cdot, \theta)$ are the curvature lines of $Z(\cdot, \theta')$, and the curvature lines of $Z(\cdot, \theta)$ are the asymptotic lines of $Z(\cdot, \theta')$.*

One calls the bending procedure $X \mapsto Z(\cdot, \theta)$ *Bonnet's transformation.* For w fixed, the points $Z(w, \theta)$ describe an ellipse as θ varies between 0 and 2π.

To the first assertion of Proposition 6 one also can state a converse due to H.A. Schwarz [1], vol. I, p. 175.

Proposition 7. *Let $X : \Omega \to \mathbb{R}^3$ and $\hat{X} : \hat{\Omega} \to \mathbb{R}^3$ be two minimal surfaces defined on simply connected domains Ω and $\hat{\Omega}$ respectively. Suppose also that X and $\hat{X}$ are isometric to each other, and that $Z(\cdot,\theta)$, $\theta \in \mathbb{R}$, is a family of associate minimal surfaces to X. Then $\hat{X}$ is congruent to one of the surfaces $Z(\cdot,\theta)$. More precisely, there are a conformal mapping τ of Ω onto $\hat{\Omega}$, a motion T of $\mathbb{R}^3$, possibly followed by a reflection, and some $\theta_0 \in \mathbb{R}$ such that*

$$T \circ \hat{X} \circ \tau = Z(\cdot, \theta_0).$$

For a proof of this Proposition we refer to Nitsche [28], § 177, pp. 164–165, and to Calabi [1].

Let us return to the representation (18) in Proposition 4 which, in principle, yields all simply connected minimal surfaces. However, we have to satisfy the isotropy relation (17) which prevents us from inserting arbitrary holomorphic functions Φ_1, Φ_2, Φ_3. We can overcome this difficulty in the following way:

Let Ω be a sufficiently small neighborhood of w_0 and suppose that $\Phi_1(w) \neq 0$ on Ω. Then we can assume that the holomorphic function

$$\sigma(w) := \int_{w_0}^{w} \Phi_1(\underline{w})\, d\underline{w}$$

yields an invertible mapping of Ω onto $\Omega^* := \sigma(\Omega)$. Let $w = \tau(\zeta), \zeta \in \Omega^*$, be the inverse of $\zeta = \sigma(w), w \in \Omega$, and set

$$h(\zeta) := \int_0^{\zeta} \frac{\Phi_2 \circ \tau}{\Phi_1 \circ \tau}(\underline{\zeta})\, d\underline{\zeta}.$$

Then we obtain

$$\zeta = \int_{w_0}^{w} \Phi_1(\underline{w})\, d\underline{w}, \quad h(\zeta) = \int_{w_0}^{w} \Phi_2(\underline{w})\, d\underline{w},$$

and from

$$\Phi_3^2 = -\{\Phi_1^2 + \Phi_2^2\}$$

we infer that

$$\Phi_3(w)\, dw = i\sqrt{\Phi_1(w)^2 + \Phi_2(w)^2}\, dw = i\sqrt{1 + h'(\zeta)^2}\, d\zeta$$

if $\Phi_3(w) \neq 0$ on Ω. *Hence we see that $X|_\Omega$ is equivalent to the representation $Y := X \circ \tau$, which can be written as*

$$Y(\zeta) = X_0 + \mathrm{Re}\left(\zeta, h(\zeta), i\int_0^{\zeta} \sqrt{1 + h'(\underline{\zeta})^2}\, d\underline{\zeta}\right) \tag{37}$$

for $\zeta \in \Omega^*$.

Conversely, if $h(\zeta)$ *is holomorphic on* Ω^* and $1 + h'(\zeta)^2 \neq 0$ *for* $\zeta \in \Omega^*$, *then* (37) *defines a minimal surface* $Y(\zeta)$, $\zeta \in \Omega^*$, provided that Ω^* *is a simply connected domain in* $\mathbb{C}$.

This is the classical *representation formula of Monge*, stating that *every minimal surface is locally equivalent to some holomorphic function and, conversely, that essentially every holomorphic function h generates a minimal surface.* In Section 3.3 we shall derive global representation formulas for minimal surfaces.

3.2 Behavior of Minimal Surfaces Near Branch Points

Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface on a domain Ω in $\mathbb{R}^2 \hat{=} \mathbb{C}$. For some $w_0 \in \Omega$, we choose a disk $B_R(w_0) \subset\subset \Omega$. Then, by virtue of Section 3.1, Proposition 1, there is an isotropic curve $f : B_R(w_0) \to \mathbb{C}^3$ such that

$$X(w) = \operatorname{Re} f(w) \tag{1}$$

holds for all $w \in B_R(w_0)$. As we have seen in Section 3.1, the point w_0 is a branch point of X if and only if

$$f'(w_0) = 0. \tag{2}$$

We now want to derive an asymptotic expansion for $X(w)$ in the neighborhood $B_R(w_0)$ of w_0, using the formula

$$f(w) = X(w) + iX^*(w), \quad w \in B_R(w_0). \tag{3}$$

Suppose that $f(w) \not\equiv \text{const}$, and that $f'(w_0) = 0$. Then there is an integer $m \geq 1$ such that

$$f^{(k)}(w_0) = 0 \quad \text{for } 1 \leq k \leq m, \quad f^{(m+1)}(w_0) \neq 0. \tag{4}$$

Thus we obtain the Taylor expansion

$$f(w) = f(w_0) + \frac{1}{(m+1)!} f^{(m+1)}(w_0)(w - w_0)^{m+1} + \cdots \tag{5}$$

on $B_R(w_0)$, and therefore also

$$f'(w) = \frac{1}{m!} f^{(m+1)}(w_0)(w - w_0)^m + \cdots .$$

Set $X_0 := X(w_0)$ and

$$A = \frac{1}{2}(\alpha - i\beta) := \frac{1}{2m!} f^{(m+1)}(w_0), \quad B := \frac{2}{m+1} A.$$

Then we conclude from

$$2X_w(w) = X_u(w) - iX_v(w) = f'(w)$$

that

$$X_w(w) = A(w - w_0)^m + O(|w - w_0|^{m+1}) \quad \text{as } w \to w_0,$$

and

$$X(w) = X_0 + \operatorname{Re}\{B(w - w_0)^{m+1} + O(|w - w_0|^{m+2})\}.$$

The conformality relation

$$\langle X_w, X_w \rangle = 0$$

implies that

$$\langle A, A \rangle = 0$$

holds, whence

$$|\alpha|^2 = |\beta|^2, \quad \langle \alpha, \beta \rangle = 0,$$

and $A \neq 0$ yields $|\alpha| = |\beta| > 0$. Moreover,

$$\begin{aligned}
X_u(w) &= \operatorname{Re} f'(w) = \alpha \operatorname{Re}(w - w_0)^m + \beta \operatorname{Im}(w - w_0)^m + \cdots, \\
X_v(w) &= -\operatorname{Im} f'(w) = -\alpha \operatorname{Im}(w - w_0)^m + \beta \operatorname{Re}(w - w_0)^m + \cdots,
\end{aligned}$$

where the remainder terms are of order $O(|w - w_0|^{m+1})$. Hence we conclude that

$$X_u(w) \wedge X_v(w) = (\alpha \wedge \beta)|w - w_0|^{2m} + O(|w - w_0|^{2m+1}) \quad \text{as } w \to w_0.$$

This implies that $N(w)$ tends to a limit vector N_0 as $w \to w_0$:

$$\lim_{w \to w_0} N(w) = N_0 = \frac{\alpha \wedge \beta}{|\alpha \wedge \beta|}.$$

Consequently, the Gauss map $N(w)$ of a minimal surface $X(w)$, $w \in \Omega$, is well-defined on all of Ω as a continuous mapping into S^2. In fact, $N : \Omega \to S^2$ is a harmonic mapping of Ω into the unit sphere S^2 (cf. Section 5.1) which satisfies

$$\Delta N + N|\nabla N|^2 = 0 \quad \text{in } \Omega.$$

Therefore N is real analytic (this also follows from the discussion in Section 3.2). From formula (6) in Section 1.4 we then infer

$$|N_u|^2 = |N_v|^2, \quad \langle N_u, N_v \rangle = 0$$

and

$$|\nabla N|^2 = -K|\nabla X|^2,$$

whence also

$$2|N_u \wedge N_v| = |\nabla N|^2,$$

and formula (44) in Section 1.2 yields

$$N_u \wedge N_v = K X_u \wedge X_v.$$

Since $K \leq 0$, one concludes

$$\varDelta N = 2 N_u \wedge N_v,$$

i.e. N is a surface of constant mean curvature one (cf. Chapter 5, and also Vol. 3, Section 2.3).

We now want to put X into some *normal form* which will explain the term *branch point.* Set

$$a := \frac{|\alpha|}{m+1} = \frac{|\beta|}{m+1}$$

and

$$e_1 := \frac{\alpha}{|\alpha|}, \quad e_2 := \frac{\beta}{|\beta|}, \quad e_3 := e_1 \wedge e_2 = N_0.$$

Then we can rewrite the formula

$$X(w) = X_0 + \operatorname{Re}\left\{ \frac{\alpha - i\beta}{m+1}(w - w_0)^{m+1} + O(|w - w_0|^{m+2}) \right\}$$

as

$$X(w) = X_0 + a e_1 \operatorname{Re}(w - w_0)^{m+1} + a e_2 \operatorname{Im}(w - w_0)^{m+1} + O(|w - w_0|^{m+2}).$$

If we rotate the axes of the given coordinate system in $\mathbb{R}^3$ such that e_1, e_2, and e_3 point in the directions of the new positive x, y, and z-axes respectively, we obtain

$$\begin{aligned} x(w) + i y(w) &= (x_0 + i y_0) + a(w - w_0)^{m+1} + O(|w - w_0|^{m+2}), \\ z(w) &= z_0 + O(|w - w_0|^{m+2}). \end{aligned} \tag{6}$$

This *normal form of a minimal surface* $X(w) = (x(w), y(w), z(w))$ shows that a minimal surface X behaves in a neighborhood of one of its branch points w_0 like a branch point of m-th order of a Riemann surface. Thus we shall denote the integer m, defined by (4), as the *order of the branch point* w_0 *of the minimal surface* X. *If we define* $m = 0$ *for regular points, we may consider regular points as branch points of order zero.*

Remark 1. In Vol. 2, Section 6, we denote the *order* of a branch point w_0 by n, while m is used for the *index* of w_0.

Let us collect some of the previous results in the following

Proposition 1. *If* $w_0 \in \Omega$ *is a branch point of a minimal surface* $X : \Omega \to \mathbb{R}^3$*, then there is a vector* $A \in \mathbb{C}^3$*,* $A \neq 0$*, and an integer* $m \geq 1$*, the so-called order of the branch point* w_0*, such that the following holds*:

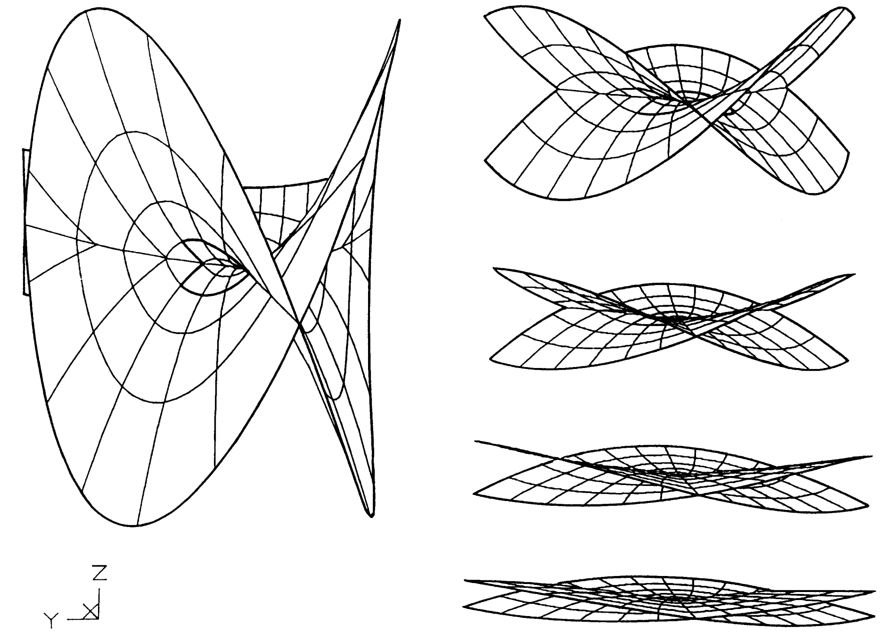


Fig. 1. $w = 0$ is a branch point of order one of Catalan's surface. The parts of the surface corresponding to the shrinking neighborhoods $[-2^n/10, 2^n/10]^2$ for $n = 5, 4, 3, 2, 1$ illustrate the convergence of the tangent planes in the vicinity of a branch point, a general property of all two-dimensional minimal surfaces. Note that the second picture shows an enlarged detail of the first one, the third one an enlarged detail of the second one, etc.

$$X_w(w) = A(w - w_0)^m + O(|w - w_0|^{m+1}) \quad \textit{as } w \to w_0, \tag{7}$$

and N is a surface of constant mean curvature one;

$$X(w) = X_0 + \mathrm{Re}[B(w - w_0)^{m+1}] + O(|w - w_0|^{m+2}), \tag{8}$$

where $B = \frac{2}{m+1}A$, and $A = \frac{1}{2}(\alpha - i\beta)$ is an isotropic vector in $\mathbb{C}^3 \setminus \{0\}$:

$$\langle A, A\rangle = 0 \tag{9}$$

or

$$|\alpha|^2 = |\beta|^2 > 0, \quad \langle \alpha, \beta\rangle = 0, \quad \alpha, \beta \in \mathbb{R}^3. \tag{9'}$$

The normal $N(w)$ tends to the limit

$$N_0 = \frac{\alpha \wedge \beta}{|\alpha \wedge \beta|}, \tag{10}$$

and the tangent plane of X at w converges to a limiting position as $w \to w_0$.

Consequently, the function $l(w) = \mathcal{L}(w) - i\mathcal{M}(w)$ is holomorphic on Ω, and the spherical image map $N(w)$ is a continuous map from Ω into S^2.

Next we want to derive a *lower bound for the area of minimal surfaces.* Suppose that $B_R(P)$ is a ball in $\mathbb{R}^3$, the center P of which lies on the trace of some minimal surface $X : \Omega \to \mathbb{R}^3$ extending beyond $B_R(P)$, i.e., there are no boundary points of $X(\Omega)$ within $B_R(P)$. Let $w_0 \in \Omega$ be a branch point of X of order m, and suppose that $P = X(w_0)$. Then the normal form (6) suggests that the area of $X(\Omega) \cap B_R(P)$ is at least as large as the area of $m+1$ plane equatorial disks of $B_R(P)$, provided that the radius R is sufficiently small. In fact, we can prove:

Proposition 2. *Suppose that $X : \Omega \to \mathbb{R}^3$ is a minimal surface defined on a bounded simply connected domain Ω. Moreover, let $w_0 \in \Omega$ be a branch point of order $m \geq 0$, $X_0 = X(w_0)$, and let $R > 0$ be some number such that*

$$\liminf_{k\to\infty} |X(w_k)| \geq R \tag{11}$$

holds for every sequence $\{w_k\}$ of points $w_k \in \Omega$ with dist $(w_k, \partial\Omega) \to 0$ as $k \to \infty$. Then the area $A(X)$ of the surface X satisfies

$$A(X) \geq (m+1)\pi(R^2 - |X_0|^2). \tag{12}$$

Equality holds if and only if the image of X lies in a plane through the point X_0 which is perpendicular to the line from 0 to X_0.

Proof. Since Ω can be mapped conformally onto the unit disk such that w_0 is transformed into the origin, we may assume that $w_0 = 0, X_0 = X(0)$, and $\Omega = \{w\colon |w| < 1\}$.

Then we can find an isotropic curve $f : \Omega \to \mathbb{C}^3$ satisfying $f(0) = X_0 = X(0)$ and

$$f = X + iX^*,$$

where $X^* : \Omega \to \mathbb{R}^3$ is an adjoint surface to X with $X^*(0) = 0$. We can represent $f(w)$ by the Taylor series

$$f(w) = X_0 + \sum_{k=m+1}^{\infty} A_k w^k, \quad A_k \in \mathbb{C}^3,$$

which is convergent for $|w| < 1$. Applying Cauchy's integral formula to the holomorphic function $F(w) := \langle f(w), f(w)\rangle, |w| < 1$, it follows that

$$\int_0^{2\pi} F(re^{i\theta})\, d\theta = 2\pi F(0),$$

and therefore

$$\int_0^{2\pi} |X(re^{i\theta})|^2\, d\theta - \int_0^{2\pi} |X^*(re^{i\theta})|^2\, d\theta = 2\pi|X_0|^2,$$

for every $r \in (0,1)$.

On the other hand, we obtain

$$\begin{aligned}\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta &= \int_0^{2\pi} |X(re^{i\theta})|^2\,d\theta + \int_0^{2\pi} |X^*(re^{i\theta})|^2\,d\theta\\ &= 2\pi\left\{|X_0|^2 + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}\right\}.\end{aligned}$$

Combining the two identities, we arrive at

$$\int_0^{2\pi} |X(re^{i\theta})|^2\,d\theta = 2\pi|X_0|^2 + \pi \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}.$$

Setting

$$\mu(r) := \min_{|w|=r} |X(w)|^2,$$

we deduce the estimate

$$\mu(r) \le |X_0|^2 + \frac{1}{2}\sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}.$$

Moreover, the area $A(r)$ of the image of $\{w\colon |w| < r\}$ under the mapping X is given by

$$\begin{aligned}A(r) &= \frac{1}{2}\int_{|w|<r} |\nabla X|^2\,du\,dv = \frac{1}{2}\int_0^r \int_0^{2\pi} |f'(te^{i\theta})|^2 t\,dt\,d\theta\\ &= \frac{\pi}{2}\sum_{k=m+1}^{\infty} k|A_k|^2 r^{2k}.\end{aligned}$$

Thus we infer that

$$\frac{\pi}{2}\sum_{k=m+2}^{\infty} [k-(m+1)]|A_k|^2 r^{2k} - (m+1)\pi|X_0|^2 \le A(r) - (m+1)\pi\mu(r). \tag{13}$$

By assumption (11), we have $\liminf_{r\to 1} \mu(r) \ge R^2$, whence

$$\frac{\pi}{2}\sum_{k=m+2}^{\infty} [k-(m+1)]|A_k|^2 + (m+1)\pi(R^2 - |X_0|^2) \le \lim_{r\to 1} A(r) = A(X), \tag{14}$$

and inequality (12) is proved.

Suppose now that equality holds in (12). Then we infer form (14) that $A_k = 0$ for $k \ge m+2$, whence

$$f(w) = X_0 + A_{m+1}w^{m+1}.$$

Let $A_{m+1} = a+ib$, $a, b \in \mathbb{R}^3$. Since f is isotropic, we obtain $\langle A_{m+1}, A_{m+1}\rangle = 0$, or $|a| = |b|, \langle a, b\rangle = 0$. Therefore, the vectors $e_1 := \frac{a}{|a|}$ and $e_2 := -\frac{b}{|b|}$ are orthonormal, and we have

$$X(w) = X_0 + |a|r^{m+1}\{e_1 \cos(m+1)\theta + e_2 \sin(m+1)\theta\}. \tag{15}$$

This yields

$$A(X) = (m+1)|a|^2\pi.$$

On the other hand, we have assumed that

$$A(X) = (m+1)\pi(R^2 - |X_0|^2)$$

holds. Then we conclude that

$$|a|^2 = R^2 - |X_0|^2. \tag{16}$$

Set $e_3 := e_1 \wedge e_2$. Then e_1, e_2, e_3 form an orthonormal frame in $\mathbb{R}^3$, and we can write

$$X_0 = \sum_{k=1}^{3} c_k e_k.$$

In conjunction with (15), it follows that

$$\begin{aligned} |X(e^{i\theta})|^2 &= (c_1 + |a|\cos(m+1)\theta)^2 + (c_2 + |a|\sin(m+1)\theta)^2 + c_3^2 \\ &= |X_0|^2 + |a|^2 + 2|a|\{c_1 \cos(m+1)\theta + c_2 \sin(m+1)\theta\}, \end{aligned}$$

and, on account of (16), we conclude that

$$|X(e^{i\theta})|^2 = R^2 + 2|a|\{c_1 \cos(m+1)\theta + c_2 \sin(m+1)\theta\}.$$

Therefore, unless $c_1 = c_2 = 0$, we can find an angle θ such that $|X(e^{i\theta})| < R$, which contradicts (11). Hence we see that

$$X_0 = c_3(e_1 \wedge e_2),$$

and formula (15) shows that $X(w)$ lies in an affine plane, perpendicular to the vector X_0, which contains the point with the position vector X_0.

Introducing suitable Cartesian coordinates x, y, z in $\mathbb{R}^3$, we obtain the normal form

$$\begin{aligned} x + iy &= \sqrt{R^2 - |X_0|^2}\, w^{m+1}, \quad |w| < 1, \\ z &= 0 \end{aligned}$$

for the minimal surface $X(w) = (x(w), y(w), z(w)), |w| < 1$, in the case that equality holds in (12).

This completes the proof of Proposition 2. $\square$

3.3 Representation Formulas for Minimal Surfaces

In Proposition 4 of Section 3.1 we have stated that for every holomorphic map

$$\Phi(w) = (\Phi_1(w), \Phi_2(w), \Phi_3(w)), \quad w \in \Omega,$$

of a simply connected domain Ω in $\mathbb{C}$ with $\Phi(w) \not\equiv 0$ and

$$\langle \Phi, \Phi \rangle = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0, \tag{1}$$

the formula

$$X(w) = X_0 + \operatorname{Re} \int_{w_0}^{w} \Phi(\zeta)\, d\zeta, \quad w \in \Omega, \tag{2}$$

with $w_0 \in \Omega$ and $X_0 \in \mathbb{R}^3$, defines a minimal surface $X : \Omega \to \mathbb{R}^3$, and every such surface can be obtained in this way. At the end of Section 3.1 we have derived local solutions Φ of the isotropy equation (1). In this section we shall first determine all (global) holomorphic mappings $\Phi : \Omega \to \mathbb{C}^3$ satisfying (1). This in turn will lead us to the celebrated Enneper–Weierstrass representation formulas of minimal surfaces which, in particular, can be used to establish explicit expressions for the normal image, the Gauss curvature, and for the asymptotic and curvature lines of minimal surfaces.

Lemma 1. *If $\mu(w)$ is a holomorphic function and $\nu(w)$ is a meromorphic function in a domain Ω in $\mathbb{C}$ such that $\mu(w) \not\equiv 0$ and that μ has a zero of order at least $2n$ where ν has a pole of order n, then the functions*

$$\Phi_1 = \frac{1}{2}\mu(1 - \nu^2), \quad \Phi_2 = \frac{i}{2}\mu(1 + \nu^2), \quad \Phi_3 = \mu\nu \tag{3}$$

are holomorphic in Ω, and the triple $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ satisfies (1) and $\Phi(w) \not\equiv 0$. Conversely, every triple $\Phi = (\Phi_1, \Phi_2, \Phi_3) \not\equiv 0$ of holomorphic functions on Ω satisfying (1) can be written in the form (3) if and only if $\Phi_1 - i\Phi_2 \not\equiv 0$.

Proof. The first part of the lemma follows by a straight-forward computation. In order to prove the converse, we note that the assumption $\Phi_1 - i\Phi_2 \not\equiv 0$ certainly is necessary for (3) to hold. In fact, (1) is equivalent to

$$(\Phi_1 - i\Phi_2)(\Phi_1 + i\Phi_2) + \Phi_3^2 = 0. \tag{4}$$

Hence $\Phi_1 - i\Phi_2 = 0$ yields $\Phi_3 = 0$, and $\Phi_3 = \mu\nu$ would imply $\mu = 0$ or $\nu = 0$. Since $\mu = 0$ would give $\Phi = 0$, we would have $\nu = 0$ and therefore $\Phi_1 = \mu/2, \Phi_2 = i\mu/2$; thus $\Phi_1 + i\Phi_2 = 0$. Consequently $\Phi_1 = \Phi_2 = \Phi_3 = 0$, which contradicts $\Phi \not\equiv 0$.

Suppose now that $\Phi_1 - i\Phi_2 \not\equiv 0$. Then the formulas

$$\mu := \Phi_1 - i\Phi_2, \quad \nu := \frac{\Phi_3}{\Phi_1 - i\Phi_2} \tag{5}$$

define a holomorphic function μ and a meromorphic function ν in Ω, satisfying $\mu\nu = \Phi_3$. Moreover, (4) implies

$$\Phi_1 + i\Phi_2 = -\frac{\Phi_3^2}{\Phi_1 - i\Phi_2} = -\mu\nu^2 \tag{6}$$

which, together with

$$\Phi_1 - i\Phi_2 = \mu,$$

yields

$$\Phi_1 = \frac{\mu}{2}(1-\nu^2), \quad \Phi_2 = i\frac{\mu}{2}(1+\nu^2).$$

Finally, the relation $\mu\nu^2 = -(\Phi_1 + i\Phi_2)$ shows that the function $\mu\nu^2$ is holomorphic. Therefore if $w_0 \in \Omega$ is a pole of order n of ν, then w_0 is a zero of order at least $2n$ for μ. □

In conjunction with (2), this lemma yields the following result:

Theorem 1 (Enneper–Weierstrass representation formula). *For every nonplanar minimal surface*

$$X(w) = (x(w), y(w), z(w)), \quad w \in \Omega,$$

defined on a simply connected domain Ω in $\mathbb{C}$, there are a holomorphic function μ and a meromorphic function ν in Ω with $\mu \not\equiv 0, \nu \not\equiv 0$ such that $\mu\nu^2$ is holomorphic in Ω, and that

$$\begin{aligned} x(w) &= x_0 + \operatorname{Re} \int_{w_0}^{w} \frac{1}{2}\mu(1-\nu^2)\, d\zeta, \\ y(w) &= y_0 + \operatorname{Re} \int_{w_0}^{w} \frac{i}{2}\mu(1+\nu^2)\, d\zeta, \\ z(w) &= z_0 + \operatorname{Re} \int_{w_0}^{w} \mu\nu\, d\zeta \end{aligned} \tag{7}$$

holds for $w, w_0 \in \Omega$ and $X_0 = (x_0, y_0, z_0) = X(w_0)$.

Conversely, two functions μ and ν as above define by means of (7) a minimal surface $X : \Omega \to \mathbb{R}^3$ provided that Ω is simply connected.

Remark. A point $w \in \Omega$ is a branch point of a minimal surface $X : \Omega \to \mathbb{R}^3$ represented by (1) and (2) if and only if $\Phi_1(w) = \Phi_2(w) = \Phi_3(w) = 0$. Thus, $w \in \Omega$ is a branch point of a minimal surface $X : \Omega \to \mathbb{R}^3$ represented by (7) if and only if both μ and $\mu\nu^2$ are vanishing at w. The set of regular points $\Omega' := \{w \in \Omega\colon \Lambda(w) \neq 0\}$ is therefore given by

$$\Omega' = \{w \in \Omega\colon |\mu(w)|(1 + |\nu(w)|^2) \neq 0\}.$$

The function ν has an important geometric meaning. It will turn out that ν is just the stereographic projection of the spherical image N of X onto the x, y-plane.

Before we prove this, we want to derive explicit expressions for the spherical image N and for the Gauss curvature of a minimal surface $X : \Omega \to \mathbb{R}^3$ given by

$$X(w) = \operatorname{Re} f(w), \tag{8}$$

with an isotropic curve $f : \Omega \to \mathbb{R}^3$ satisfying

$$\begin{aligned} f' &= X_u - iX_v = \Phi = (\Phi_1, \Phi_2, \Phi_3) \\ &= \left(\frac{1}{2}\mu(1-\nu^2), \frac{i}{2}\mu(1+\nu^2), \mu\nu\right), \end{aligned} \tag{9}$$

where μ and ν satisfy the assumptions stated in Theorem 1. Then the function

$$\Lambda := |X_u|^2 = \tfrac{1}{2}|\nabla X|^2 = \tfrac{1}{2}|f'|^2 = \tfrac{1}{2}|\Phi|^2$$

can be written as

$$\Lambda = \tfrac{1}{4}|\mu|^2(1+|\nu|^2)^2, \tag{10}$$

and the line element $ds = |dX|$ takes the form

$$ds^2 = \Lambda\{du^2 + dv^2\}.$$

By virtue of Section 3.1 (20), it follows that

$$X_u \wedge X_v = \tfrac{1}{4}|\mu|^2\{1+|\nu|^2\}(2\operatorname{Re}\nu, 2\operatorname{Im}\nu, |\nu|^2 - 1).$$

Taking (10) into account, we obtain the representation

$$N = \frac{1}{1+|\nu|^2}(2\operatorname{Re}\nu, 2\operatorname{Im}\nu, |\nu|^2 - 1) \tag{11}$$

for the spherical image $N : \Omega \to S^2$ of X.

In order to compute K, we first note that

$$f' = \mu\left(\frac{1}{2}(1-\nu^2), \frac{i}{2}(1+\nu^2), \nu\right)$$

implies

$$f'' = \frac{\mu'}{\mu} f' + \mu\nu' g, \quad g := (-\nu, i\nu, 1),$$

on $\{w \in \Omega \colon \mu(w) \neq 0\}$. From (11) we infer that

$$\langle N, g\rangle = -1.$$

Since the zeros of μ are isolated and $\langle N, f' \rangle = 0$, we obtain

$$\langle N, f'' \rangle = -\mu\nu',$$

and on account of (25) of Section 3.1, we arrive at

$$l = \mathcal{L} - i\mathcal{M} = -\mu\nu'. \tag{12}$$

The branch points of $X(w)$ are removable singularities of $l(w) = -\mu(w)\nu'(w)$ since $l(w)$ remains bounded if w approaches such a point. Hence $l(w)$ is holomorphic in Ω.

Recall that $\mu\nu^2$ is holomorphic in Ω, and that $w \in \Omega$ is a branch point of X if and only if both $\mu(w) = 0$ and $\mu(w)\nu^2(w) = 0$ are satisfied. If $\Lambda(w) \neq 0$, then $l(w) = 0$ if either $\nu'(w) = 0$, or w is a pole for ν and a zero of at least third order for μ.

Moreover, the formulas (10) and (12) together with (26) of Section 3.1 yield

$$K = -\left\{ \frac{4|\nu'|}{|\mu|(1 + |\nu|^2)^2} \right\}^2 \tag{13}$$

for the Gauss curvature of X on $\Omega' = \{w \in \Omega\colon \Lambda(w) \neq 0\}$. Then we infer for any $w \in \Omega'$ which is not a pole of ν that $K(w) \neq 0$ *holds if and only if* $\nu'(w) \neq 0$ *is satisfied.*

Moreover, Proposition 5 of Section 3.1 yields:

A curve $\gamma(t) = \alpha(t) + i\beta(t)$ contained in Ω (with $\alpha(t), \beta(t) \in \mathbb{R}$) describes an asymptotic line of the minimal surface X if and only if

$$\mathrm{Re}\{\mu(\gamma)\nu'(\gamma)\dot{\gamma}^2\} = 0, \tag{14}$$

and the lines of curvature are characterized by

$$\mathrm{Im}\{\mu(\gamma)\nu'(\gamma)\dot{\gamma}^2\} = 0. \tag{15}$$

Now we want to give *a geometric interpretation of the function* $\nu(w)$ that enters into the representation formula (7).

Let us identify the complex plane $\mathbb{C} = \{x + iy\colon x, y \in \mathbb{R}\}$ with the x, y-plane $\{(x, y, z)\colon z = 0\}$ in $\mathbb{R}^3$, and let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the compactification of $\mathbb{C}$ with the point at infinity. As usual, we introduce the Riemann sphere $S^2 = \{(x, y, z)\colon x^2 + y^2 + z^2 = 1\}$, and denote by $P = (0, 0, 1)$ its north pole. Then the *stereographic projection*

$$\sigma : S^2 \to \bar{\mathbb{C}}$$

is the 1–1-mapping of S^2 onto the compactified complex plane $\bar{\mathbb{C}}$ which associates each point $Q \in S^2, Q \neq P$, with the intersection point ω of the x, y-plane

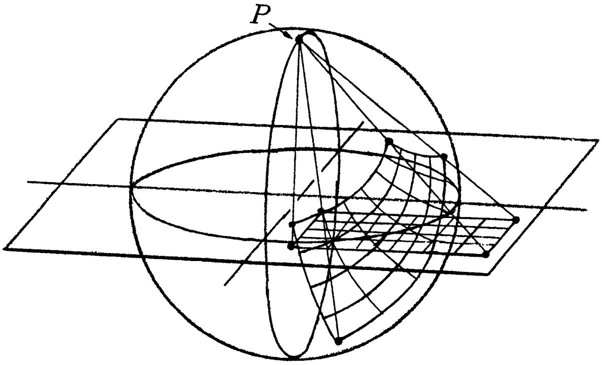


Fig. 1. The stereographic projection.

with the straight line through P and Q, whereas P is mapped to ∞ (cf. Fig. 1). Let $\rho : \bar{\mathbb{C}} \to S^2$ be the inverse of σ. Then the image $\rho(\omega) := (a, b, c) = Q$ of some point $\omega = \xi + i\eta \in \mathbb{C}$ is given by

$$a = \frac{2\xi}{1 + \xi^2 + \eta^2}, \quad b = \frac{2\eta}{1 + \xi^2 + \eta^2}, \quad c = \frac{\xi^2 + \eta^2 - 1}{1 + \xi^2 + \eta^2} \tag{16}$$

and $\rho(\infty) = P$.

The formula (16) can be written as

$$\rho(\omega) = \frac{1}{1 + |\omega|^2}(2\operatorname{Re}\omega, 2\operatorname{Im}\omega, |\omega|^2 - 1), \tag{17}$$

and we see that $\rho(\omega) \to (0, 0, 1) = P$ as $|\omega| \to \infty$. A straightforward computation yields for $\omega = \sigma(Q)$ the formula

$$\omega = \frac{a + ib}{1 - c}. \tag{18}$$

If we now compare the formula (11) for the spherical image $N(w)$ of a minimal surface $X(w), w \in \Omega$, given by (8) and (9), with the expression (17) for $\rho = \sigma^{-1}$, then we see that

$$N(w) = \rho(\nu(w)), \tag{19}$$

whence

$$\nu(w) = \sigma(N(w)). \tag{20}$$

Thus *the meromorphic function* $\nu : \Omega \to \bar{\mathbb{C}}$ *is nothing but the stereographic projection of the normal image* N *of the given minimal surface* X:

$$\nu = \sigma \circ N, \quad N = \rho \circ \nu. \tag{21}$$

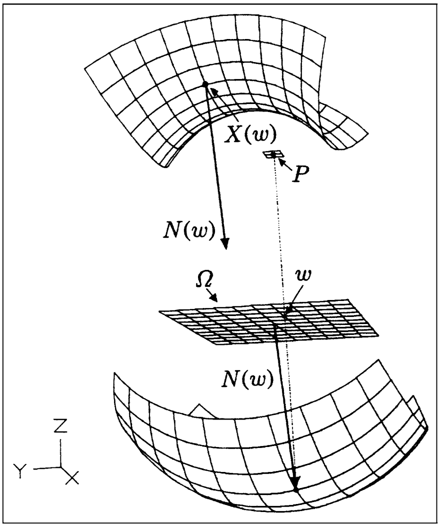


Fig. 2. The part of Enneper's surface corresponding to the rectangle $\Omega = [-1/2, 1/2]^2$ floats at the top of the picture; w is mapped to $X(w)$. The unit normal vector $N(w)$ of Enneper's surface at the point $X(w)$ is shown twice, one copy has its foot on the surface, the other one at the origin of space. The Gauss image, i.e. the set of all unit normals of this part of Enneper's surface is displayed at the bottom. For minimal surfaces the Gauss map corresponds to the meromorphic function $\nu(w)$ appearing in Weierstrass's representation formula (7) via the inverse of the stereographic projection from Ω to S^2. The latter is indicated by the dotted line starting at the north pole P, and for Enneper's surface $\nu(w) = w$

In particular, $w \in \Omega$ is a pole of ν if and only if the point $N(w) \in S^2$ is the north pole P.

Furthermore, the mapping

$$\omega = \nu(w), \quad w \in \Omega, \tag{22}$$

is a biholomorphic mapping of Ω onto $\Omega^* := \nu(\Omega)$ if the following two conditions are satisfied:

$$\begin{aligned} &\text{(i)} \quad N(w) \neq P \quad \text{for all } w \in \Omega; \\ &\text{(ii)} \quad \text{the mapping } N : \Omega \to S^2 \text{ is injective.} \end{aligned} \tag{23}$$

If $\nu : \Omega \to \Omega^*$ is biholomorphic, then we have $\nu'(w) \neq 0$ for all $w \in \Omega$, and this implies $K(w) < 0$ on Ω, i.e., X has no umbilical points.

Suppose now that $\nu : \Omega \to \Omega^*$ is a biholomorphic mapping, and let

$$w = \tau(\omega), \quad \omega \in \Omega^*, \tag{24}$$

be its inverse. Then the reparametrization $Y = X \circ \tau$ of the minimal surface X is again a minimal surface, and

$$Y(\omega) = X(\tau(\omega)), \quad \omega \in \Omega^*. \tag{25}$$

Let us introduce the function

$$\mathfrak{F}(\omega) := \frac{1}{2}\frac{\mu(\tau(\omega))}{\nu'(\tau(\omega))} = \frac{1}{2}\tau'(\omega)\mu(\tau(\omega)) \tag{26}$$

which is holomorphic in Ω^*. Then we infer from (7) the following *representation formula of Weierstrass*:

$$Y(\omega) = X_0 + \operatorname{Re}\begin{bmatrix} \int_{\omega_0}^{\omega}(1-\underline{\omega}^2)\mathfrak{F}(\underline{\omega})d\underline{\omega} \\ \int_{\omega_0}^{\omega} i(1+\underline{\omega}^2)\,\mathfrak{F}(\underline{\omega})d\underline{\omega} \\ \int_{\omega_0}^{\omega} 2\underline{\omega}\mathfrak{F}(\underline{\omega})\,d\underline{\omega} \end{bmatrix} \tag{27}$$

where $\omega, \omega_0 \in \Omega^*$, and $X_0 = X(w_0) = Y(\omega_0)$, $\omega_0 = \nu(w_0)$.

Instead of two (essentially) arbitrary functions μ and ν as in (7), the expression (27) only involves an arbitrary function $\mathfrak{F}(\omega)$. Conversely, for every holomorphic function $\mathfrak{F}(\omega) \not\equiv 0$ on a simply connected domain Ω^* in $\mathbb{C}$, the formula (27) defines a minimal surface $Y : \Omega^* \to \mathbb{R}^3$. In other words, *to each holomorphic function $\mathfrak{F} \neq 0$ corresponds some minimal surface, and vice versa.* Thus we have also recovered the result of Monge from the end of Section 3.1.

From (27), we can derive an *integral-free representation* formula by introducing a function $F(\omega)$ such that $F^{(3)}(\omega) = \mathfrak{F}(\omega)$, and performing some partial integrations. Let $Y = (Y^1, Y^2, Y^3)$. Then, for suitable constants c_1, c_2, c_3, we obtain

$$\begin{aligned} Y^1(\omega) &= \operatorname{Re}\{(1-\omega^2)F''(\omega) + 2\omega F'(\omega) - 2F(\omega)\} + c_1, \\ Y^2(\omega) &= \operatorname{Re}\{i(1+\omega^2)F''(\omega) - 2i\omega F'(\omega) + 2iF(\omega)\} + c_2, \\ Y^3(\omega) &= \operatorname{Re}\{2\omega F''(\omega) - 2F'(\omega)\} + c_3. \end{aligned} \tag{28}$$

Weierstrass ([1], pp. 48–50) has used this representation to prove the following theorem:

If $F(\omega)$ is an algebraic function of ω, then (28) defines an algebraic minimal surface, and conversely, every algebraic minimal surface possesses a parameter representation $Y(\omega)$ of type (28) with an algebraic function $F(\omega)$.

Let us now put together the main results for the representation formula (27). We first note that (27) goes over into (7) if we replace ω and ω_0 and w and $w_0, Y(\omega)$ and $X(w)$, and set $\mu(w) := 2\mathfrak{F}(w)$ and $\nu(w) := w$. Then we arrive at the following result:

Theorem 2. *Let $\mathfrak{F}(w)$ be a holomorphic function in a simply connected domain Ω of $\mathbb{C}, \mathfrak{F}(w) \not\equiv 0$, and set*

$$\Phi(w) = ((1-w^2)\mathfrak{F}(w), i(1+w^2)\mathfrak{F}(w), 2w\mathfrak{F}(w)). \tag{29}$$

Then

$$X(w) = X_0 + \operatorname{Re} \int_{w_0}^{w} \Phi(\underline{w})\, d\underline{w}, \quad w \in \Omega, \tag{30}$$

defines a minimal surface $X : \Omega \to \mathbb{R}^3$ *with the surface normal*

$$N(w) = \frac{1}{1 + u^2 + v^2}(2u, 2v, u^2 + v^2 - 1), \quad w = u + iv. \tag{31}$$

If σ *denotes the stereographic projection from the north pole* $P = (0, 0, 1)$ *of* $S^2 = \{(x, y, z) \colon x^2 + y^2 + z^2 = 1\}$ *onto the* x, y*-plane, then we have*

$$\sigma(N(w)) = w.$$

The line element $ds = |dX|$ *on the surface* X *is given by*

$$ds^2 = \Lambda(w)\{du^2 + dv^2\} \tag{32}$$

where

$$\Lambda(w) = |\mathfrak{F}(w)|^2(1 + u^2 + v^2)^2, \quad w = u + iv. \tag{32'}$$

Thus the set $\Omega' := \{w \in \Omega \colon \Lambda(w) \neq 0\}$ *of regular points of the minimal surface* X *is described by*

$$\Omega' = \{w \in \Omega \colon \mathfrak{F}(w) \neq 0), \tag{33}$$

and its Gauss curvature $K(w)$ *on* Ω' *is given by*

$$K(w) = -\frac{4}{|\mathfrak{F}(w)|^2(1 + u^2 + v^2)^4}. \tag{34}$$

The coefficients $\mathcal{L}, \mathcal{M}, \mathcal{N}$ *of the second fundamental form satisfy* $\mathcal{L} + \mathcal{N} = 0$ *and can be obtained from the holomorphic function*

$$l(w) = \mathcal{L}(w) - i\mathcal{M}(w) = -2\mathfrak{F}(w), \quad w \in \Omega. \tag{35}$$

The branch points $w \in \Omega$ *of* X *are the zeros of the holomorphic function* $\mathfrak{F}(w), w \in \Omega$*. Thus* X *has no umbilical points since an umbilical point is a zero of* l *on the set of regular points* Ω'*. The directions* (du, dv) *of the asymptotic lines are characterized by the equation*

$$\operatorname{Re} \mathfrak{F}(w)(dw)^2 = 0, \tag{36}$$

and the lines of curvature are described by

$$\operatorname{Im} \mathfrak{F}(w)(dw)^2 = 0. \tag{37}$$

Finally, the associate minimal surfaces $Z(w,\theta)$ *to* $X(w)$ *are given by*

$$Z(w,\theta) = X_0 + \operatorname{Re}\int_{w_0}^{w} e^{-i\theta}\Phi(\underline{w})\,d\underline{w}; \tag{38}$$

their Weierstrass function $\tilde{\mathfrak{F}}(w,\theta)$ *is simply*

$$\tilde{\mathfrak{F}}(w,\theta) = e^{-i\theta}\mathfrak{F}(w). \tag{39}$$

Conversely, if $f : \tilde{\Omega} \to \mathbb{C}^3$ *is an isotropic map on a simply connected domain* $\tilde{\Omega} \subset \mathbb{C}$, *then the minimal surface*

$$\tilde{X}(w) = X_0 + \operatorname{Re}\int_{w_0}^{w} f'(\underline{w})\,d\underline{w}, \quad w \in \tilde{\Omega},$$

has an equivalent representation $X : \Omega \to \mathbb{R}^3$ *on* $\Omega := \sigma(\tilde{N}(\tilde{\Omega}))$, *given by* (29) *and* (30), *provided that its normal* $\tilde{N}(w)$ *satisfies condition* (23):

(i) $\tilde{N}(w) \neq$ *north pole of* S^2 *for all* $w \in \tilde{\Omega}$;

(ii) *the mapping* $\tilde{N} : \tilde{\Omega} \to S^2$ *is injective.*

Remark. In our computation of the Gauss curvature K we have used the theorema egregium. Yet for minimal surfaces we can obtain K in a much simpler way, basically by going back to the definition of K. First we note that the spherical image

$$N(w) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1), \quad w = u + iv,$$

is given by conformal parameters u, v; in fact, a straightforward computation yields

$$\begin{aligned} |N_u(w)|^2 = |N_v(w)|^2 &= \frac{4}{(1+u^2+v^2)^2}, \\ \langle N_u(w), N_v(w)\rangle &= 0. \end{aligned} \tag{40}$$

(Note that N is just the inverse $\rho = \sigma^{-1}$ of the stereographic projection $\sigma : S^2 \to \bar{\mathbb{C}}$ restricted to Ω. The equations (40) express the fact that ρ and therefore also σ are conformal mappings.)

From (40) we obtain for the third fundamental form of X the expression

$$\mathrm{III}(du, dv) = \frac{4}{(1+u^2+v^2)^2}\{du^2 + dv^2\} \tag{41}$$

whereas (32) implies

$$\mathrm{I}(du, dv) = \Lambda(w)\{du^2 + dv^2\}.$$

On the other hand, it follows from Section 1.2, (26) that

$$\mathrm{III}(du, dv) = -KI(du, dv)$$

whence

$$K(w) = -\frac{4}{(1+u^2+v^2)^2 \Lambda(w)}. \tag{42}$$

On account of (32′), this relation is equal to (34).

Another possibility to compute K directly is to employ formula (44) (or (45)) of Section 1.2.

Goursat has found a procedure to generate from a given minimal surface $X : \Omega \to \mathbb{R}^3$ and its adjoint $X^* : \Omega \to \mathbb{R}^3$ a one-parameter family of minimal surfaces $Y(w, \kappa)$, $w \in \Omega$, where the parameter κ varies in $\mathbb{R}$, $\kappa \neq 0$. The *Goursat transformation* resembles Bonnet's transformation described in Section 3.1 but is less restrictive. It is defined by

$$Y(w, \kappa) = X_0 + \mathrm{Re} \int_{w_0}^{w} \Psi(\underline{w}, \kappa)\, d\underline{w} \tag{43}$$

where

$$\Psi(w, \kappa) = \left(\left(\frac{1}{\kappa} - \kappa w^2 \right) \mathfrak{F}(w), i \left(\frac{1}{\kappa} + \kappa w^2 \right) \mathfrak{F}(w), 2w\mathfrak{F}(w) \right), \tag{44}$$

and

$$\begin{aligned} X(w) + iX^*(w) &= X_0 + \int_{w_0}^{w} \Phi(\underline{w})\, d\underline{w}, \\ \Phi(w) &= ((1-w^2)\mathfrak{F}(w), i(1+w^2)\mathfrak{F}(w), 2w\mathfrak{F}(w)). \end{aligned}$$

If $X_0 = 0, Y = (\xi, \eta, \zeta), X = (x, y, z), X^* = (x^*, y^*, z^*)$, we can write

$$\begin{aligned} \xi &= \frac{1+\kappa^2}{2\kappa} x + \frac{1-\kappa^2}{2\kappa} y^*, \\ \eta &= \frac{1+\kappa^2}{2\kappa} y + \frac{\kappa^2-1}{2\kappa} x^*, \\ \zeta &= z. \end{aligned} \tag{45}$$

For fixed w and varying κ, the points $Y(w, \kappa)$ describe a branch of a parabola.

Goursat's transformation maps asymptotic lines into asymptotic lines and lines of curvature into lines of curvature. For further details, we refer to Goursat [1,2] (first and second mémoire).

Now we shall prove another representation formula, due to Weierstrass, which is often found in the literature:

Theorem 3 (Weierstrass representation formula). *For every regular minimal surface $X : \Omega \to \mathbb{R}^3$ on a simply connected domain Ω, there exist two holomorphic functions G and H without common zeros such that*

$$
\begin{aligned}
x(w) &= x_0 + \mathrm{Re}\int_{w_0}^{w} (G^2 - H^2)\, d\zeta, \\
y(w) &= y_0 + \mathrm{Re}\int_{w_0}^{w} i(G^2 + H^2)\, d\zeta, \qquad (46)\\
z(w) &= z_0 + \mathrm{Re}\int_{w_0}^{w} 2GH\, d\zeta
\end{aligned}
$$

holds for $w, w_0 \in \Omega$ and $X_0 = X(w_0)$. Conversely, if G and H are two holomorphic functions on a simply connected domain Ω such that $|G(w)|^2 + |H(w)|^2 \not\equiv 0$, then (46) defines a nonconstant minimal surface which is regular if and only if G and H have no zeros in common.

Proof. The second part follows by a straightforward computation. In order to verify the first part, we consider an arbitrary minimal surface $X : \Omega \to \mathbb{R}^3$ given by

$$X(w) = X_0 + \mathrm{Re}\int_{w_0}^{w} \Phi(\zeta)\, d\zeta, \quad w \in \Omega,$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3) : \Omega \to \mathbb{C}^3$ is a holomorphic mapping satisfying

$$|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 > 0 \qquad (47)$$

and

$$(\Phi_1 - i\Phi_2)(\Phi_1 + i\Phi_2) = -\Phi_3^2. \qquad (48)$$

The last equation, which is equivalent to (1), implies that every zero of $\Phi_1 - i\Phi_2$ or of $\Phi_1 + i\Phi_2$ is also a zero of Φ_3. Then we infer that, because of (47), the two functions $\Phi_1 - i\Phi_2$ and $\Phi_1 + i\Phi_2$ cannot have common zeros. Since every zero of Φ_3^2 is of even order, it follows that the zeros of both $\Phi_1 - i\Phi_2$ and $\Phi_1 + i\Phi_2$ are of even order. Then the functions

$$G := \sqrt{\tfrac{1}{2}(\Phi_1 - i\Phi_2)}, \quad H := \sqrt{-\tfrac{1}{2}(\Phi_1 + i\Phi_2)}$$

are single-valued holomorphic functions which, for suitably chosen square roots, satisfy

$$2GH = \Phi_3,$$

and clearly

$$G^2 - H^2 = \Phi_1, \quad i(G^2 + H^2) = \Phi_2.$$

Moreover, the functions G and H have no common zeros. $\square$

Remark. 1. If we omit the assumption (47), then not every minimal surface $X(w) = X_0 + \mathrm{Re} \int_{w_0}^{w} \Phi(\zeta)\, d\zeta$ can be written in the form (46). For instance, let

$$\Phi_1(w) = 3w, \quad \Phi_2(w) = 5iw, \quad \Phi_3(w) = 4w,$$

where Ω is a small disk centered at $w = 0$. If there were functions G and H such that

$$3w = G(w)^2 - H(w)^2, \quad 5iw = i\{G(w)^2 + H(w)^2\}, \quad 4w = 2G(w)H(w),$$

it would follow that $G^2(w) = 4w$. However, there is no (single-valued) holomorphic solution $G(w)$ of this equation in Ω.

2. Weierstrass has derived the representation (30) with Φ given by (29) from (46), by introducing a new variable

$$\omega = \frac{H(w)}{G(w)} = \frac{\Phi_1(w) + i\Phi_2(w)}{-\Phi_3(w)}$$

(arranging everything in such a way that the mapping $w \mapsto \omega$ is biholomorphic). Then

$$(\Phi_1 - i\Phi_2)(\Phi_1 + i\Phi_2) = -\Phi_3^2$$

implies that

$$\frac{1}{\omega} = \frac{\Phi_1(w) - i\Phi_2(w)}{\Phi_3(w)},$$

and it follows that

$$G^2(w)\frac{dw}{d\omega} = \frac{1}{2}(\Phi_1(w) - i\Phi_2(w))\frac{dw}{d\omega} = \mathfrak{F}(\omega).$$

Then one can pass from (46) to the desired equations.

As a remarkable application of the Enneper–Weierstrass representation formula we present the following[1]

Theorem of R. Krust. *If an embedded minimal surface $X : B \to \mathbb{R}^3$, $B = \{w \in \mathbb{C}\colon |w| < 1\}$, can be written as a graph over a convex domain in a plane, then the corresponding adjoint surface $X^* : B \to \mathbb{R}^3$ is a graph as well.*

First we write the representation formula (7) in a different way. Let us introduce the two meromorphic functions g and h by

$$g := \nu, \quad h' := \mu\nu.$$

Then we have

$$dh = \mu\nu\, d\zeta,$$

[1] Oral communication of R. Krust to H. Karcher. Our proof is borrowed from Karcher's note [3].

and we can write (7) in the form

$$X(w) = X(w_0) + \operatorname{Re} \int_{w_0}^{w} \psi'(\zeta)\, d\zeta, \tag{49}$$

where $d\psi(\zeta) = \psi'(\zeta)\, d\zeta$ is given by

$$d\psi = \left[\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right] dh. \tag{50}$$

Note that the 1-forms $d\psi$ and dh are single-valued on B. The Gauss map $N : B \to S^2$ associated with X is given by

$$N = \frac{1}{1 + |g|^2}(2 \operatorname{Re} g, 2 \operatorname{Im} g, |g|^2 - 1). \tag{51}$$

Proof of Krust's Theorem. We can assume that X is nonplanar, that it can be represented as a graph above the x, y-plane, and that $N(w)$ always points into the lower hemisphere of S^2. Then we infer from (51) that the function g appearing in the Weierstrass representation (49), (50) of X satisfies

$$|g(w)| < 1 \quad \text{for all } w \in B. \tag{52}$$

Moreover, we can also suppose that $w_0 = 0$ and $X(w_0) = 0$. Introducing the functions $\sigma(w)$ and $\tau(w)$, $w \in B$, by

$$\sigma(w) := -\int_0^w \frac{g}{2}\, dh, \quad \tau(w) := \int_0^w \frac{1}{2g}\, dh, \tag{53}$$

we can write the first two coordinate functions $x(w)$ and $y(w)$ of $X(w)$ as

$$x(w) = \operatorname{Re}[\sigma(w) + \tau(w)], \quad y(w) = \operatorname{Re} i[\tau(w) - \sigma(w)]. \tag{54}$$

Then the orthogonal projections

$$\pi(w) := x(w) + iy(w), \quad \pi^*(w) := x^*(w) + iy^*(w) \tag{55}$$

of $X(w)$ and of its adjoint $X^*(w) = (x^*(w), y^*(w), z^*(w))$ onto the x, y-plane can be written as

$$\pi = \bar{\tau} + \sigma, \quad \pi^* = i(\bar{\tau} - \sigma). \tag{56}$$

Pick any two points w_1 and w_2 in B, $w_1 \neq w_2$ and set $p_1 := \pi(w_1)$, $p_2 := \pi(w_2)$. Since $D := \pi(B)$ is a convex domain in the x, y-plane, we can connect p_1 and p_2 within D by a line segment $\mathfrak{L} : [0, 1] \to D$ such that $\mathfrak{L}(0) = p_1$ and $\mathfrak{L}(1) = p_2$. Then there is a piecewise smooth curve $\gamma : [0, 1] \to B$ such that $\mathfrak{L} = \pi \circ \gamma$. We can assume that $|\dot{\mathfrak{L}}(t)| = |p_2 - p_1|$ for all $t \in [0, 1]$ whence

$$p_2 - p_1 = \mathfrak{L}\,(1) - \mathfrak{L}(0) = \dot{\mathfrak{L}}(t) \quad \text{for all } t \in [0, 1]$$

and therefore

$$p_2 - p_1 = \left[\overline{\frac{1}{2g(w)} h'(w)\dot{\gamma}(t)} - \frac{g(w)}{2} h'(w)\dot{\gamma}(t) \right] \Big|_{w=\gamma(t)} . \tag{57}$$

Consider the scalar product S of the two vectors $p_2 - p_1$ and $i[\pi^*(w_2) - \pi^*(w_1)]$ of $\mathbb{R}^2$:

$$\begin{aligned} S &:= \langle p_2 - p_1, i[\pi^*(w_2) - \pi^*(w_1)] \rangle \\ &= \left\langle p_2 - p_1, -\int_\gamma \left(\frac{g}{2} dh + \overline{\frac{1}{2g} dh} \right) \right\rangle . \end{aligned} \tag{58}$$

Since for any two vectors $w_1, w_2 \in \mathbb{R}^2 \hat{=} \mathbb{C}$ we have $\langle w_1, w_2 \rangle = \mathrm{Re}(w_1 \bar{w}_2)$, it follows from (57) that

$$\begin{aligned} S &= \int_0^1 \mathrm{Re} \left\{ \left[\left(\overline{\frac{g}{2} h'} + \frac{1}{2g} h' \right) \circ \gamma \right] \overline{\dot{\gamma}(t)} \left[\left(\frac{g}{2} h' - \overline{\frac{1}{2g} h'} \right) \circ \gamma \right] \dot{\gamma}(t) \right\} dt \\ &= \int_0^1 \frac{1}{4} |\dot{\gamma}(t)|^2 \left[|g(\gamma(t))|^2 - \frac{1}{|g(\gamma(t))|^2} \right] |h'((\gamma(t))|^2 \, dt . \end{aligned}$$

Then we infer from (52) that $S < 0$. Therefore we obtain from (58) that $\pi^*(w_2) - \pi^*(w_1) \neq 0$ for any pair of distinct points $w_1, w_2 \in B$, and we conclude that the adjoint surface X^* is a graph.

Remark. Similarly one proves that all associate surfaces $X : B \to \mathbb{R}^3$ of a minimal embedding $X : B \to \mathbb{R}^3$ are graphs if $X(B)$ can be written as a graph over a convex domain in a plane.

3.4 Björling's Problem. Straight Lines and Planar Lines of Curvature on Minimal Surfaces. Schwarzian Chains

Given a real analytic strip S in $\mathbb{R}^3$, Björling's problem is to find a minimal surface X containing this strip in its interior. This is but a special case of the general theorem by Cauchy–Kovalevskaya whence we will expect to find a uniquely determined solution. Following an idea by H.A. Schwarz, this solution can be given by an explicit formula in terms of the initial data, i.e., in terms of the prescribed strip S. Schwarz's solution of the Björling problem yields a beautiful method for generating minimal surfaces with interesting geometric properties.

Let us now describe the problem in detail. We consider a real-analytic strip

$$S = \{(c(t), n(t)) \colon t \in I\}$$

consisting of a real-analytic curve $c : I \to \mathbb{R}^3$ with $\dot{c}(t) \neq 0$ (or at least $\dot{c}(t) = 0$ only in isolated points $t \in I$), and of a real-analytic vector field $n : I \to \mathbb{R}^3$ along c, with $|n(t)| \equiv 1$ and $\langle \dot{c}(t), n(t) \rangle \equiv 0$.

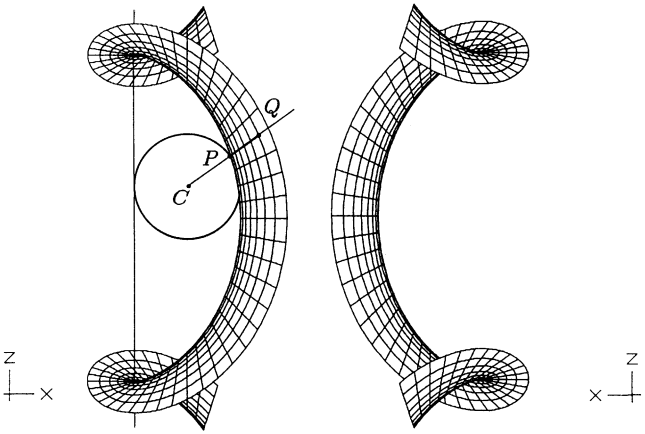


Fig. 1. A cycloid is the curve generated by a point P on the periphery of a *circle with center* C rolling along a *straight line*. Catalan's surface, whose part corresponding to $-3\pi/5 \le u \le 13\pi/5, -2\pi/5 \le v \le 0$ has been drawn here, solves Björling's problem to find a minimal surface passing through the cycloid in such a way that the surface normal coincides with the cycloid's principal normal vector. The two parallel projections onto the x, z-plane show that the curves $u =$ constant (e.g. the curve passing through the points P and Q) are planar and perpendicular to the x, z-plane. Each of them is, in fact, a parabola having its apex on the cycloid

We assume that I is an open interval in $\mathbb{R}$.

Björling's problem consists in finding a minimal surface $X : \Omega \to \mathbb{R}^3$ with $I \subset \Omega$ such that the following conditions are satisfied:

$$\begin{aligned} &\text{(i)} \quad X(u,0) = c(u) \quad \text{for } u \in I,\\ &\text{(ii)} \quad N(u,0) = n(u) \quad \text{for } u \in I, \end{aligned}$$

N being the normal of $X, N : \Omega \to \mathbb{R}^3$.

Theorem 1. *For any prescribed real-analytic strip* $S = \{(c(t), n(t)) \colon t \in I\}$, *the corresponding Björling problem has exactly one solution* $X(u,v)$, *given by*

$$X(u,v) = \mathrm{Re}\left\{ c(w) - i \int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w}) \right\}, \tag{1}$$

$w = u + iv \in \Omega$, $u_0 \in I$, *where* Ω *is a simply connected domain with* $I \subset \Omega$ *in which the power-series expansions of both* c *and* n *are converging.*

Remark. 1. The uniqueness is to be understood in the following sense: If $\tilde{X}(u,v)$, $w = u + iv \in \tilde{\Omega}$, is another solution, then $X(u,v) = \tilde{X}(u,v)$ for $u + iv \in \Omega \cap \tilde{\Omega}$.

2. Formula (1) means the following: One determines holomorphic extensions $c(u + iv)$ and $n(u + iv)$ of the real-analytic functions $c(t)$ and $n(t)$,

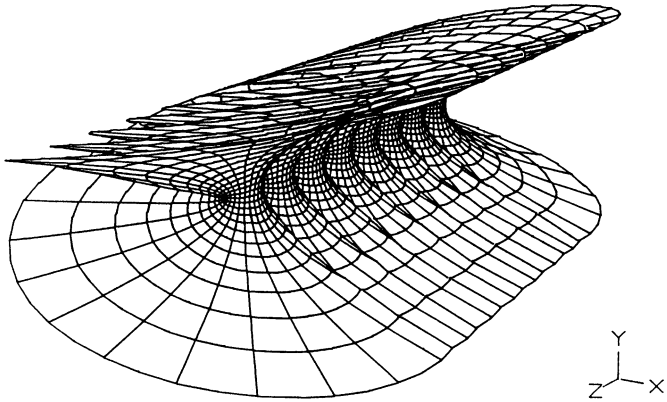


Fig. 2. A large piece of Catalan's surface generated by the cycloid via Björling's problem

$t \in I$, to a suitable simply-connected domain Ω with $I \subset \Omega$, and then one determines the line integral

$$\int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w}) = \int_{u_0}^{w} n(\underline{w}) \wedge c'(\underline{w})\, d\underline{w}$$

where $c'(w)$ is the complex derivative of the holomorphic function $c(w)$.

Proof of Theorem 1. Suppose that $X(u,v)$ is a solution of Björling's problem, defined in the simply connected domain Ω, and let $X^* : \Omega \to \mathbb{R}^3$ be its adjoint surface with $X^*(u_0, 0) = 0$, $u_0 \in I$. Then

$$f(w) = X(u,v) + iX^*(u,v), \quad w = u + iv \in \Omega,$$

is an isotropic curve with $X = \operatorname{Re} f$ and

$$f' = X_u + iX_u^* = X_u - iX_v.$$

Since $X_v = N \wedge X_u$, it follows that

$$f' = X_u - iN \wedge X_u$$

whence

$$f'(u) = \dot{c}(u) - in(u) \wedge \dot{c}(u)$$

and therefore

$$f(u) = c(u) - i \int_{u_0}^{u} n(t) \wedge dc(t) \quad \text{for all } u \in I.$$

This implies

$$f(w) = c(w) - i \int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w}), \quad w \in \Omega, \tag{2}$$

since both sides are holomorphic functions of w. Hence any possible solution X must be of the form (1), which yields the uniqueness.

Now we shall prove that (1), in fact, yields the solution to Björling's problem. To this end, we consider the holomorphic curve $f : \Omega \to \mathbb{C}^3$ defined by (2). For $w \in I$, we have

$$\operatorname{Re} f'(w) = \dot{c}(w), \quad \operatorname{Im} f'(w) = -n(w) \wedge \dot{c}(w).$$

Since the real vectors $\dot{c}(w)$ and $\dot{c}(w) \wedge n(w)$ are orthogonal to each other and have the same length, we infer that

$$\langle f'(w), f'(w) \rangle = 0 \quad \text{for all } w \in I,$$

and therefore also

$$\langle f'(w), f'(w) \rangle = 0 \quad \text{for all } w \in \Omega.$$

Hence $X(u, v) = \operatorname{Re} f(w), w = u + iv \in \Omega$, is a minimal surface. Since $c(w), n(w)$, and $c'(w)$ are real for $w \in I$, we infer that

$$X(u, 0) = \operatorname{Re} f(u) = c(u) \quad \text{for } u \in I, \tag{3}$$

and

$$X_u(u, 0) - iX_v(u, 0) = f'(u) = \dot{c}(u) - in(u) \wedge \dot{c}(u), \quad u \in I,$$

whence

$$X_u(u, 0) = \dot{c}(u), \quad X_v(u, 0) = n(u) \wedge \dot{c}(u). \tag{4}$$

Moreover, we have

$$X_v(u, 0) = N(u, 0) \wedge X_u(u, 0).$$

Because of

$$\langle X_u(u, 0), X_v(u, 0) \rangle = 0$$

and of

$$\langle n(u), \dot{c}(u) \rangle = 0, \quad |N(u, 0)| = |n(u)| = 1,$$

we infer that

$$N(u, 0) = n(u). \qquad \square$$

Corollary 1. *Let $X(u, v)$ be the solution of Björling's problem, given by* (1). *Then we have*

$$X(u, -v) = \operatorname{Re}\{c(w) + i \int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w})\}, \quad w = u + iv. \tag{5}$$

Proof. The surface $\tilde{X}(u,v) := X(u,-v)$ is again a minimal surface with the normal $\tilde{N}(u,v) = -N(u,-v)$. Hence $\tilde{X}$ solves Björling's problem for the strip

$$\tilde{S} = \{(c(t), -n(t))\colon t \in I\}$$

and is, therefore, given by

$$\tilde{X}(u,v) = \mathrm{Re}\left\{c(w) + i\int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w})\right\}. \qquad \square$$

The formulae (1) and (5) imply the following two symmetry principles discovered by H.A. Schwarz:

Theorem 2. (i) *Every straight line contained in a minimal surface is an axis of symmetry of the surface.*

(ii) *If a minimal surface intersects some plane E perpendicularly, then E is a plane of symmetry of the surface.*

In fact, this theorem is an immediate consequence of the following

Lemma 1. *Let $X(u,v) = (x(u,v), y(u,v), z(u,v))$, $w = u + iv \in \Omega$, be a minimal surface whose domain of definition Ω contains some interval I that lies on the real axis.*

(i) *If, for all $u \in I$, the points $X(u,0)$ are contained in the x-axis, then we have for $w = u + iv \in \Omega$ with $u \in I$ and $\overline{w} = u - iv \in \Omega$ that*

$$\begin{aligned} x(u,-v) &= x(u,v), \\ y(u,-v) &= -y(u,v), \\ z(u,-v) &= -z(u,v). \end{aligned} \tag{6}$$

(ii) *If the curve $\Sigma = \{X(u,0)\colon u \in I\}$ is contained in the x,y-plane E, and if the surface X intersects orthogonally at Σ, then it follows*

$$\begin{aligned} x(u,-v) &= x(u,v), \\ y(u,-v) &= y(u,v), \\ z(u,-v) &= -z(u,v) \end{aligned} \tag{7}$$

for $u \in I$ and $w, \overline{w} \in \Omega$.

Proof. (i) Set $c(u) := X(u,0)$ and $n(u) := N(u,0)$. By assumption, we have

$$c(u) = (c^1(u), 0, 0), n(u) = (0, n^2(u), n^3(u)),$$

and therefore

$$n(u) \wedge \dot{c}(u) = (0, \dot{c}^1(u)n^3(u), -\dot{c}^1(u)n^2(u)).$$

On account of (1) and (5), we then arrive at the formulae (6).

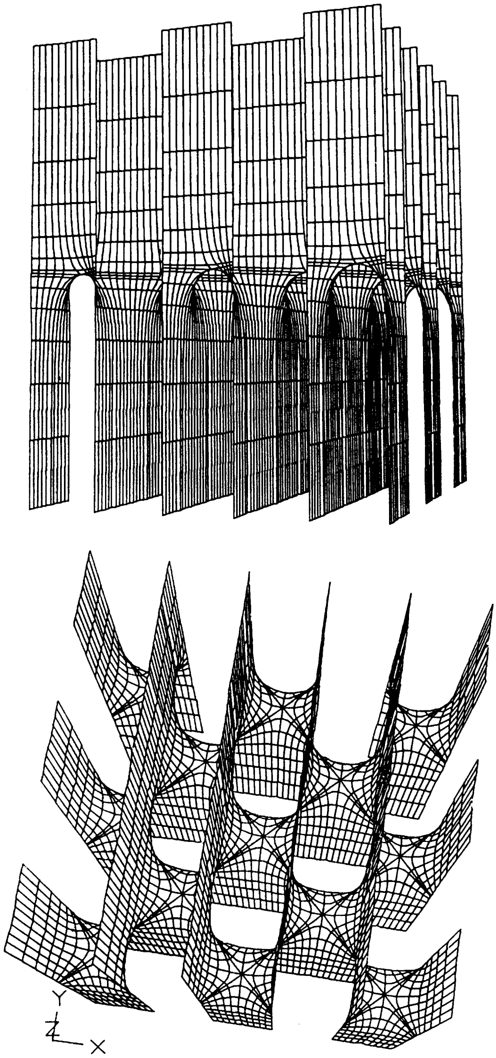


Fig. 3. Lines of symmetry of Scherk's surface demonstrate Schwarz's first reflection principle

(ii) If X intersects $E = \{z = 0\}$ at $c(u) := X(u,0)$ orthogonally, and if $n(u) := N(u,0)$, it follows that

$$c(u) = (c^1(u), c^2(u), 0), \quad n(u) = (n^1(u), n^2(u), 0),$$

whence

$$n(u) \wedge \dot{c}(u) = (0, 0, n^1(u)\dot{c}^2(u) - n^2(u)\dot{c}^1(u)).$$

In conjunction with (1) and (5), we then obtain the identities (7). $\square$

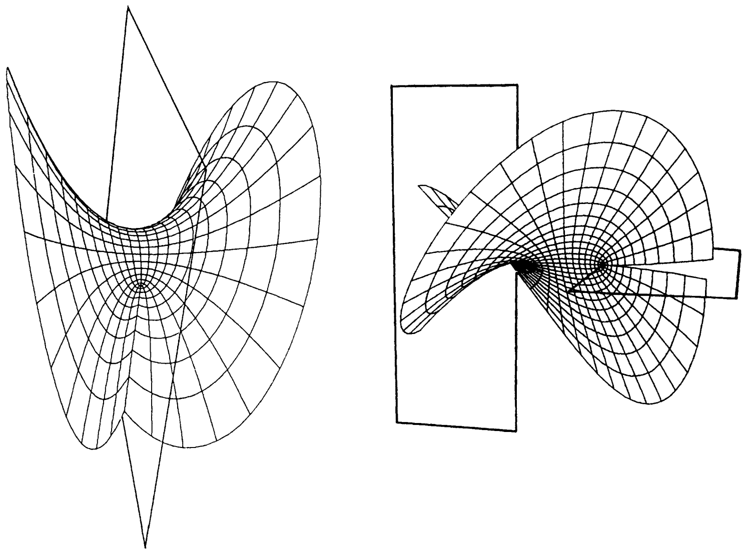


Fig. 4. Planes of symmetry in Catalan's and Henneberg's surfaces

Lemma 2. *Let $X(w)$, $w \in \Omega$, be a regular surface of class $C^3(\Omega, \mathbb{R}^3)$, and let $c(t) = X(\omega(t))$, $t \in I$, be a regular curve on X defined by some C^3-curve $\omega : I \to \Omega$. Then the following holds:*

(i) *The curve c is both a geodesic and an asymptotic line if and only if it is a straight line.*

(ii) *Let c be a geodesic on X. Then c is also a line of curvature if and only if it is a plane curve.*

(iii) *Suppose that c is contained in a plane E. Then c is a line of curvature on X if and only if X intersects E along c at a constant angle φ (if $\varphi = \frac{\pi}{2}$, then c is a geodesic).*

Proof. We may assume that t coincides with the parameter of arc length s. Let $\{\boldsymbol{t}(s), \boldsymbol{s}(s), \mathfrak{N}(s)\}$ be the moving frame along $c(s)$, consisting of the tangent vector $\boldsymbol{t}(s) = \dot{c}(s)$, the side normal $\boldsymbol{s}(s)$, and the surface normal $\mathfrak{N}(s) = N(\omega(s))$. Secondly, we consider the frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$, where $\boldsymbol{n}(s)$ is the principal normal, and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$ stands for the binormal vector of $c(s)$. Let us recall the formula (14) of Section 1.2:

$$\dot{\boldsymbol{t}} = \kappa_g \boldsymbol{s} + \kappa_n \mathfrak{N}, \tag{8}$$

where $\kappa_n = \kappa \cos\theta$ is the normal curvature of $c, \kappa_g = \pm\kappa \sin\theta$ the geodesic curvature of c, $\cos\theta = \langle \boldsymbol{n}, \mathfrak{N} \rangle$, and

$$\dot{\boldsymbol{t}} = \kappa \boldsymbol{n},$$

where κ denotes the curvature of c.

(i) Suppose now that $\kappa_n(s) \equiv 0$ and $\kappa_g(s) \equiv 0$. Then the relation (8) implies $\boldsymbol{t}\,(s) \equiv$ const, whence $c(s)$ must be a straight line. Conversely, if $c(s)$ is a straight line, then $\dot{\boldsymbol{t}}(s) \equiv 0$, and therefore $\kappa_n(s) \equiv 0$ as well as $\kappa_g(s) \equiv 0$. Thus the first assertion is proved.

(ii) Suppose that $c(s)$ is a geodesic line, i.e., $\kappa_g(s) \equiv 0$, or $\boldsymbol{n}(s) \equiv \pm\mathfrak{N}(s)$. We may assume that $\boldsymbol{n}(s) = \mathfrak{N}(s)$. Then the identity

$$\dot{\boldsymbol{b}} = \dot{\boldsymbol{t}} \wedge \boldsymbol{n} + \boldsymbol{t} \wedge \dot{\boldsymbol{n}} = \kappa\boldsymbol{n} \wedge \boldsymbol{n} + \boldsymbol{t} \wedge \dot{\boldsymbol{n}} = \boldsymbol{t} \wedge \dot{\boldsymbol{n}}$$

yields

$$\dot{\boldsymbol{b}} = \boldsymbol{t} \wedge \dot{\mathfrak{N}}.$$

Since $\dot{\mathfrak{N}}(s) \in T_{\omega(s)}X$, we can write

$$\dot{\mathfrak{N}} = \gamma_1\boldsymbol{t} + \gamma_2\boldsymbol{s}, \tag{9}$$

whence

$$\dot{\boldsymbol{b}} = \gamma_2\mathfrak{N}.$$

It follows that $\dot{\boldsymbol{b}}(s) \equiv 0$ if and only if $\gamma_2(s) \equiv 0$, that is, if and only if we have $\dot{\mathfrak{N}}(s) \equiv \gamma_1(s)\boldsymbol{t}(s)$. Thus we conclude that $c(s)$ is planar if and only if c is a line of curvature.

(iii) Introduce $\varphi(s)$ as the angle between the tangent plane of the surface at $w = \omega(s)$ and the osculating plane of the curve c for the parameter value s, i.e.,

$$\cos\varphi = \langle \mathfrak{N}, \boldsymbol{b}\rangle.$$

Then we obtain

$$\frac{d}{ds}\cos\varphi = \langle \dot{\mathfrak{N}}, \boldsymbol{b}\rangle + \langle \mathfrak{N}, \dot{\boldsymbol{b}}\rangle. \tag{10}$$

If c is a planar curve, we have $\dot{\boldsymbol{b}} = 0$, and it satisfies

$$-\dot{\mathfrak{N}} = k\boldsymbol{t}, \quad k = \kappa_1 \text{ or } \kappa_2,$$

if it is a line of curvature. Hence a planar line of curvature fulfills

$$\frac{d}{ds}\cos\varphi = -k\langle \boldsymbol{t}, \boldsymbol{b}\rangle = 0$$

or $\varphi(s) \equiv$ const.

Conversely, suppose that c is a plane curve such that $\varphi(s) \equiv$ const. Then we have $\dot{\boldsymbol{b}}(s) \equiv 0$, and (10) implies

$$\langle \dot{\mathfrak{N}}(s), \boldsymbol{b}(s)\rangle \equiv 0. \tag{11}$$

Moreover, we can use formula (9) from part (ii):

$$\dot{\mathfrak{N}}(s) \equiv \gamma_1(s)\boldsymbol{t}(s) + \gamma_2(s)\boldsymbol{s}(s), \tag{12}$$

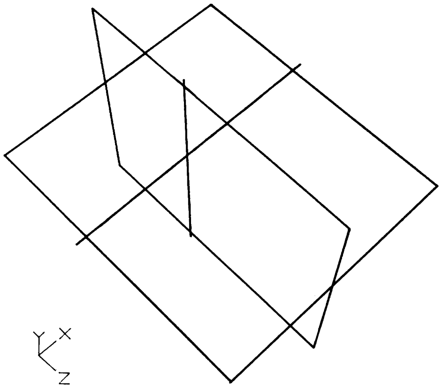


Fig. 5. The affine spaces shown here are the setting in which a part of Catalan's surface is deformed into its adjoint surface. It will be illustrated that if a minimal surface is perpendicular to a plane along a part of the boundary of its domain of definition, then its adjoint minimal surface maps that part of the boundary onto a straight line perpendicular to that plane, and vice versa (Section 3.4, Proposition 1)

and (11) yields

$$\dot{\mathfrak{N}}(s) \equiv \gamma_1(s)\boldsymbol{t}(s) + \gamma_3(s)\boldsymbol{n}(s) \tag{13}$$

for an appropriate function $\gamma_3(s)$. Thus, for every admissible value of the parameter s, at least one of the two relations

$$\gamma_2(s) = \gamma_3(s) = 0 \quad \text{or} \quad \boldsymbol{s}(s) = \boldsymbol{n}(s).$$

must be satisfied. Suppose that, for some admissible value s_0, the equation $\boldsymbol{s}(s_0) = \boldsymbol{n}(s_0)$ holds. Then we have $\boldsymbol{b}(s_0) = \pm\mathfrak{N}(s_0)$, i.e. $\varphi(s_0) = 0$. On the other hand, since $\varphi(s) \equiv$ const, we find $\varphi(s) \equiv 0$, i.e., $\boldsymbol{b}(s) \equiv \mathfrak{N}(s)$ or $\boldsymbol{b}(s) \equiv -\mathfrak{N}(s)$. Since $\dot{\boldsymbol{b}}(s) \equiv 0$, we infer that $\dot{\mathfrak{N}}(s) \equiv 0$ and therefore (12) and (13) yield $\gamma_2(s) \equiv 0$ and $\gamma_3(s) \equiv 0$. Hence in all cases our assumptions imply $\gamma_2(s) \equiv 0$ and $\gamma_3(s) \equiv 0$, whence $\dot{\mathfrak{N}}(s) \equiv \gamma_1(s)\boldsymbol{t}(s)$, which means that $c(s)$ is a line of curvature. □

Supplement. *It is easy to see that the assertions of Lemma 2 remain valid if X is assumed to be a minimal surface with branch points and if $c(t)$ is supposed to be regular except for isolated points $t \in I$.*

Now we can construct minimal surfaces with interesting special properties by combining Schwarz's formula (1) and Lemma 2. Before doing this, we want to state a few observations, following from Lemma 2, which are pertinent to the so-called *Schwarzian chain problem*.

Proposition 1. *Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface with the normal mapping $N : \Omega \to S^2$, and assume that $X^* : \Omega \to \mathbb{R}^3$ is an adjoint minimal*

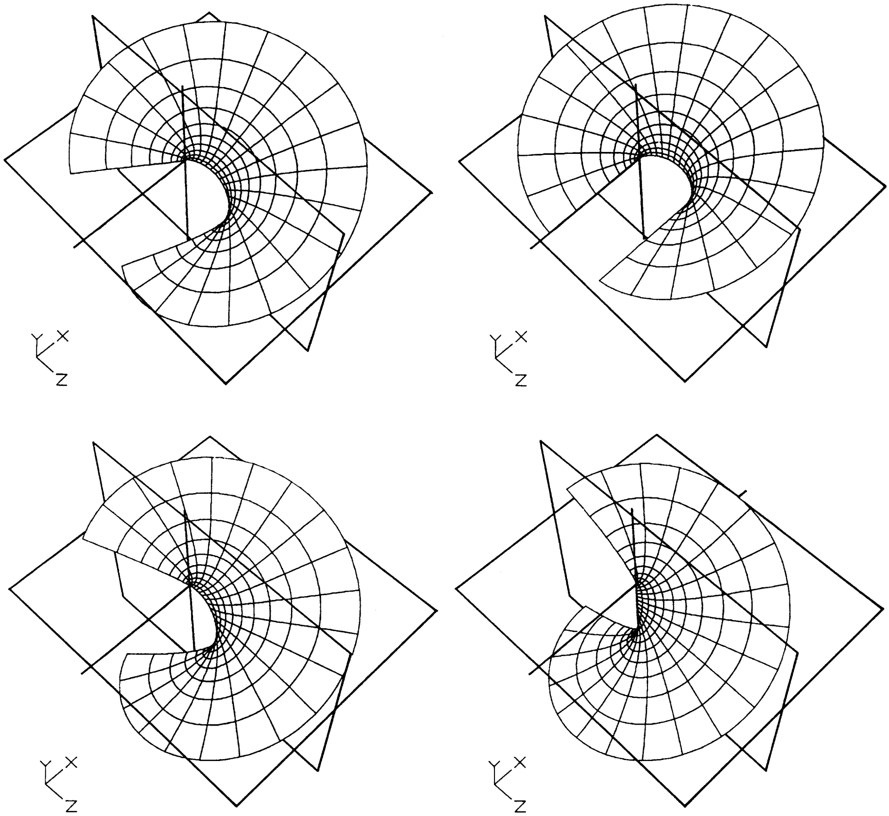


Fig. 6. The bending process for the fundamental part $0 \le u \le 2\pi$, $0 \le v$ $(\le \pi) < \infty$ of Catalan's surface into its adjoint surface through the family of its associated surfaces (counter-clockwise from top right: $\theta = 0, \pi/6, \pi/3$, and $\pi/2$). Catalan's surface is perpendicular to $y = 0$ along $v = 0$ and maps $u = 0$ and $u = 2\pi$ onto straight lines orthogonal to $x = 0$. Proposition 1 describes the resulting properties of the adjoint and the Gauss map

surface of X (hence, X^* has the same normal mapping as X). Choose some C^3-curve $\omega : I \to \Omega$ with $\dot{\omega}(t) \neq 0$ except for isolated points t in the interval I, and consider the curves $c := X \circ \omega$ and $c^* := X^* \circ \omega$. Both have the same spherical image $\gamma := N \circ \omega$, and the following holds:

(i) *If c is a straight arc, i.e. $c(I)$ is contained in some straight line L, then $\gamma(I)$ is contained in the great circle C of S^2 that lies in the plane E_0 through the origin which is perpendicular to L. Moreover, c is both a geodesic and an asymptotic line of X, and c^* is a planar geodesic of X^*. The curve c^* lies in some plane E parallel to E_0, and X^* intersects E orthogonally along c^*.*

(ii) *If c is a planar geodesic on X, i.e. the orthogonal intersection of X with some plane E, then $\gamma(I)$ lies in the great circle $C = E_0 \cap S^2$ where E_0 is the plane parallel to E with $0 \in E_0$, and c^* is a straight arc (and hence a*

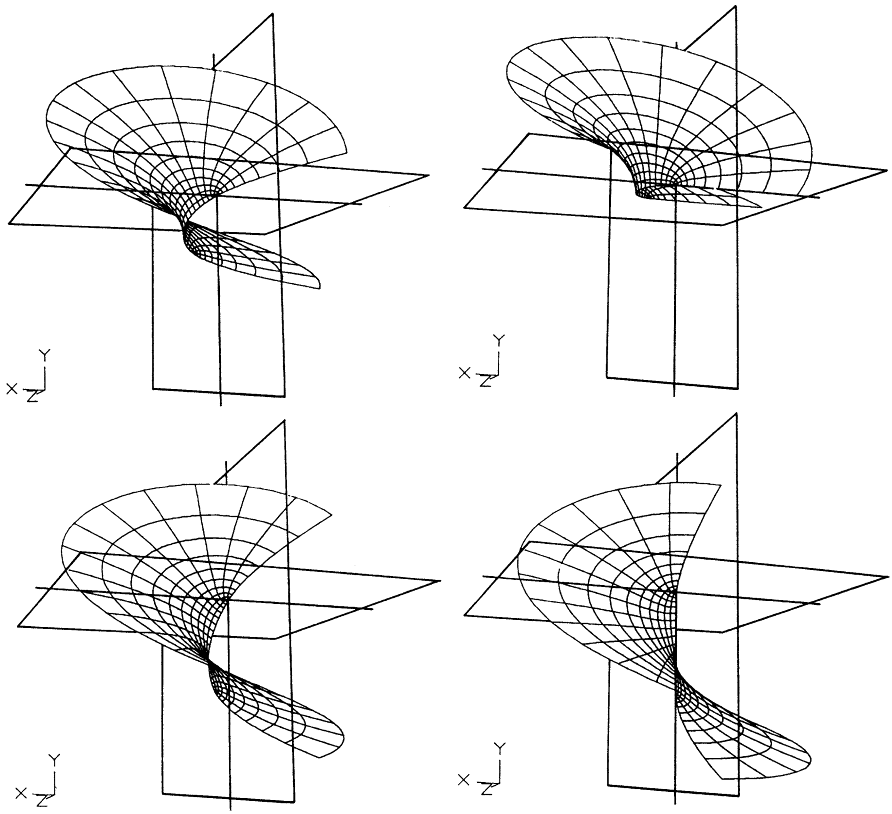


Fig. 7. Another view of the bending of Catalan's surface. The curve $v = 0$ is a geodesic, and the lines $u = 0$ and $u = 2\pi$ are asymptotic lines of the surface. The Gauss images of these lines lie on great circles of the sphere S^2

geodesic asymptotic line) on X^*. Moreover, $c^*(I)$ is contained in some straight line L perpendicular to E.

The *proof* of these very useful facts is either obvious or a direct consequence of Lemma 2 and of Proposition 6 in Section 3.1. In particular, we emphasize the following observation:

Straight arcs and planar geodesics on a minimal surface X are mapped by the normal N of X into great circles on the Riemann sphere S^2.

Similarly, one sees:

Planar lines of curvature on X are mapped by N into circles on S^2.

By virtue of Theorem 2, we also obtain:

Straight arcs and planar geodesics on a minimal surface X are lines of rotational symmetry or of mirror symmetry respectively.

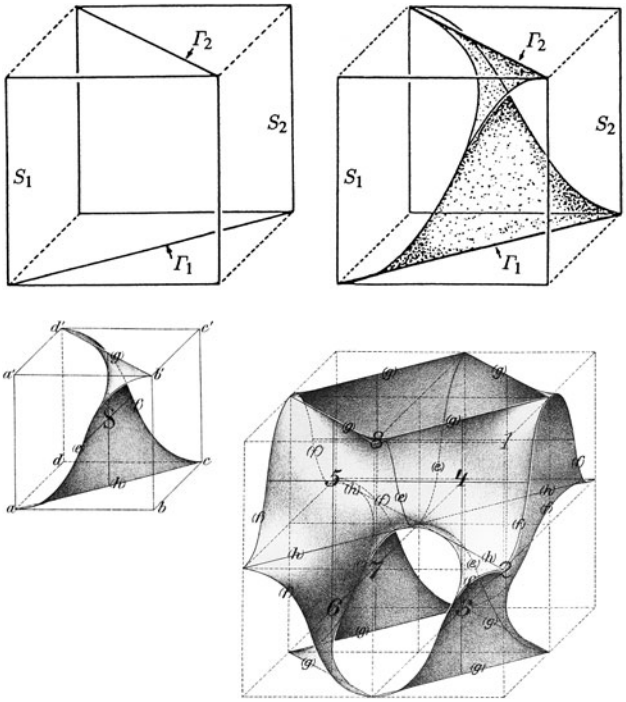


Fig. 8. A Schwarzian chain problem consisting of two plane faces of a cube and two straight lines. Its solution can be used to construct periodic minimal surfaces. Lithograph by H.A. Schwarz

Consider now a minimal surface $X : \Omega_0 \to \mathbb{R}^3$ without branch points, and a simply connected subdomain Ω of Ω_0 with $\bar{\Omega} \subset \Omega_0$. Suppose also that the normal N of X yields an injective mapping of Ω_0 into S^2, and that the boundary of $X(\bar{\Omega})$ consists of finitely many straight arcs and planar geodesics (i.e., of orthogonal intersections of X with planes). In other words, the minimal surface $X : \bar{\Omega} \to \mathbb{R}^3$ is spanned into a frame $\{L_1, \ldots, L_j, E_1, \ldots, E_k\}$ consisting of finitely many straight lines $L_1, \ldots, L_j$ and planes $E_1, \ldots, E_k$. Such a frame is usually called a *Schwarzian chain* $\mathfrak{C}$. The boundary $X : \partial\Omega \to \mathbb{R}^3$ of the minimal surface $X : \bar{\Omega} \to \mathbb{R}^3$ by assumption lies on a Schwarzian chain, and along its boundary, X is perpendicular to all planar parts of the chain $\mathfrak{C}$. *We say that* $X : \bar{\Omega} \to \mathbb{R}^3$ *is a minimal surface solving the Schwarzian chain problem for the chain* $\mathfrak{C}$.

By Proposition 1 the boundary $N : \partial\Omega \to S^2$ of the spherical image $N : \bar{\Omega} \to S^2$ of a solution $X : \bar{\Omega} \to \mathbb{R}^3$ of a Schwarzian chain problem consists of circular arcs, all of which belong to great circles on S^2. Moreover, the stereographic projection $\sigma : S^2 \to \bar{\mathbb{C}}$ maps circles on S^2 onto circles (or straight lines) in $\bar{\mathbb{C}}$. As described in Section 3.3, we can introduce new coordinates ω by some holomorphic mapping $w = \tau(\omega)$, $\omega \in \Omega^*$, such that

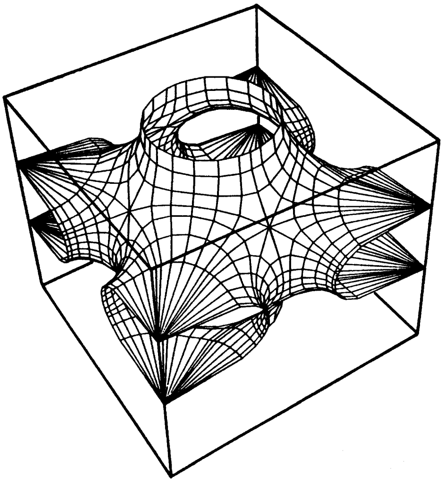


Fig. 9. A fundamental cell of A. Schoen's $S' - S''$ cell in a cuboid whose top and bottom faces are squares. The Schwarzian chain $\langle S_1, S_2, \dots, S_6 \rangle$ consists of the six faces of the cuboid. It is spanned by a minimal surface which is clearly not of the type of the disk; it consists of sixteen congruent pieces. (Varying the surface normal in the branch points of the Gauss map up and down leads to a one-parameter family of minimal surfaces in cuboids of different height.) Courtesy of K. Polthier

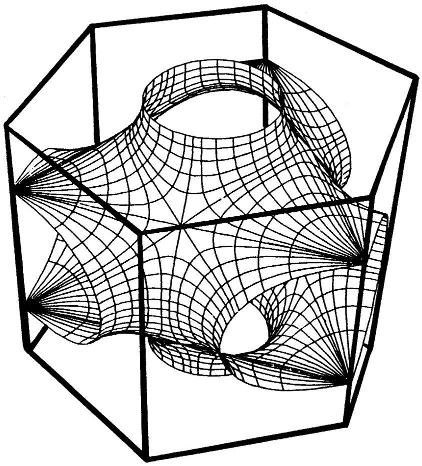


Fig. 10. A Schwarzian chain consisting of the faces of a hexagonal prism. Courtesy of K. Polthier

the equivalent representation

$$Y(\omega) := X(\tau(\omega)), \quad \omega \in \Omega^*,$$

is defined on a domain Ω^* bounded by circular arcs, if we assume that $N(w) \neq$ north pole for $w \in \Omega_0$. Moreover, there is a holomorphic function $\mathfrak{F}(\omega)$, $\omega \in$

Ω^*, with $\mathfrak{F}(\omega) \neq 0$, such that Y is given by

$$
\begin{aligned}
&Y(\omega) = X_0 + \mathrm{Re} \int_{\omega_0}^{\omega} \Phi(\underline{\omega})\, d\underline{\omega}, \quad X_0 \in \mathbb{R}^3, \\
&\Phi(\omega) = ((1-\omega^2)\mathfrak{F}(\omega), i(1+\omega^2)\mathfrak{F}(\omega), 2\omega\mathfrak{F}(\omega)).
\end{aligned} \tag{14}
$$

The functions $\nu(w) = \sigma(N(w))$ and $l(w) = \mathcal{L}(w) - i\mathcal{M}(w)$ are holomorphic on Ω_0, ν yields the inverse of τ, and we have $l(w) \neq 0$ for $w \in \Omega_0$.

Fix some $w_0 \in \Omega$ and set

$$
p(w) := \int_{w_0}^{w} \sqrt{l(\underline{w})}\, d\underline{w}, \quad w \in \bar{\Omega}. \tag{15}
$$

This defines a holomorphic function $p(w)$, $w \in \bar{\Omega}$. Since $p'(w) = \sqrt{l(w)} \neq 0$, we obtain by

$$
\zeta = p(w), \quad w \in \bar{\Omega},
$$

a conformal mapping of Ω onto some domain Ω^{**} in the ζ-plane. Note that

$$
d\zeta = p'(w)\, dw = \sqrt{l(w)(dw)^2}. \tag{16}
$$

Moreover, we know that the asymptotic lines on X are given by $\mathrm{Re}\, l(w)(dw)^2 = 0$, and the relation $\mathrm{Im}\, l(w)(dw)^2 = 0$ yields the lines of curvature (cf. Section 3.1, Proposition 5). Thus the ζ-images of the asymptotic lines $w = w(t)$ lie on straight lines which intersect the real axis at an angle of 45° or of 135°, whereas the lines of curvature $w = w(t)$ are mapped by $\zeta = p(w)$ into straight lines in the ζ-plane which are parallel either to the real axis or to the imaginary axis.

For the solution $X : \bar{\Omega} \to \mathbb{R}^3$ of the Schwarzian chain problem, the boundary $\partial\Omega$ consists of arcs corresponding to asymptotic lines and to lines of curvature. Hence the conformal mapping $\zeta = p(w)$ defined by (15) maps Ω onto some polygonal domain Ω^{**} in the ζ-plane. If we compose the two conformal mappings

$$
\tau : \Omega^* \to \Omega, \quad p : \Omega \to \Omega^{**},
$$

then $q = p \circ \tau : \Omega^* \to \Omega^{**}$ yields a conformal mapping of Ω^* onto Ω^{**}. By the arguments given in Section 3.3 (cf. in particular the formulae (12), (26) and (35)), we obtain from (15) the relation

$$
\begin{aligned}
q(\omega) = p(\tau(\omega)) &= \int_{\omega_0}^{\omega} \sqrt{l(\tau(\underline{\omega}))}\tau'(\omega)\, d\underline{\omega} \\
&= \int_{\omega_0}^{\omega} \sqrt{-\mu(\tau(\underline{\omega}))\nu'(\tau(\underline{\omega}))\tau'(\underline{\omega})^2}\, d\underline{\omega} \\
&= \int_{\omega_0}^{\omega} \sqrt{-2\mathfrak{F}(\underline{\omega})}\, d\underline{\omega},
\end{aligned}
$$

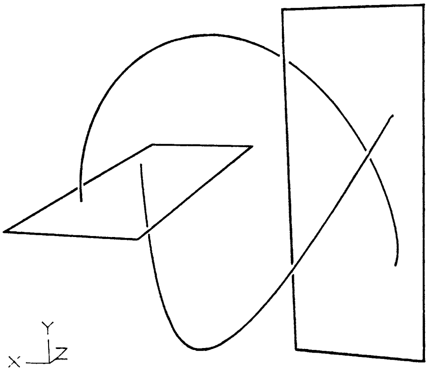


Fig. 11. A (generalized) Schwarzian chain made up of two analytic curves connecting two plane rectangles. The Schwarzian chain problem is to find a minimal surface spanning this configuration

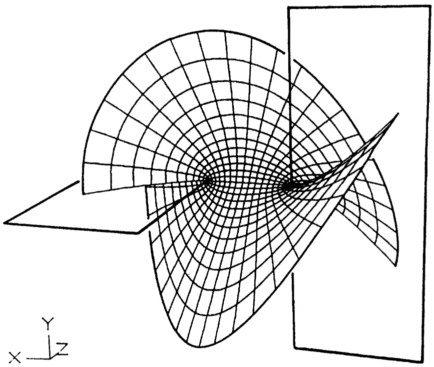


Fig. 12. This particular Schwarzian chain problem is solved by the part of Henneberg's surface corresponding to the rectangle $-0.3\pi \le u \le 0.3\pi$, $0 \le v \le \pi/4$. The surface maps the sides parallel to the v-axis onto the two analytic boundary curves of the chain whereas the two others correspond to Neil's parabolas along which the minimal surface is perpendicular to the two rectangles

that is,

$$q(\omega) = \int_{\omega_0}^{\omega} \sqrt{-2\mathfrak{F}(\underline{\omega})}\, d\underline{\omega}. \tag{17}$$

Hence the Weierstrass function $\mathfrak{F}(\omega)$ used in the representation (14) can be computed from the conformal mapping $q : \Omega^* \to \Omega^{**}$ by the formula

$$\mathfrak{F}(\omega) = -\frac{1}{2}\left(\frac{dq(\omega)}{d\omega}\right)^2. \tag{18}$$

By our assumptions, the mapping $\tau : \Omega^* \to \Omega$ is 1–1. If we also assume that $p : \Omega \to \Omega^{**}$ is 1–1, then $q = p \circ \tau$ provides a biholomorphic mapping of Ω^*

onto Ω^{**} whose extension to $\bar{\Omega}^*$ maps the vertices of the circular polygonal domain Ω^* into the vertices of the polygonal domain Ω^{**}. (Note that the vertices of Ω^* are well defined by the chain $\mathfrak{C}$ since we have assumed that the normal mapping is defined on $\bar{\Omega}^*$.)

This reasoning, using the mappings $\tau : \Omega^* \to \Omega$ and $p : \Omega \to \Omega^{**}$ together with symmetry arguments, yields a handy method to solve the Schwarzian chain problem in many interesting cases by explicit formulas. It can also be used to construct many specimen of complete and, in particular, of periodic minimal surfaces. For details, we refer to Karcher [1–3].

During the 19th century, function theoretic methods were the only known tools for proving existence of minimal surfaces spanning a given boundary configuration. These methods, however, limited the study of existence questions to frames consisting only of straight lines and planar parts. In the following chapters we shall develop another approach that is suitable for tackling more general boundary problems for minimal surfaces. Yet this approach will only yield the existence of minimal surfaces within a prescribed boundary configuration and does not give explicit formulas for solutions of a given boundary problem. One has to use numerical methods to obtain further information on the geometric shape of solutions. The classical methods of function theory, on the other hand, have the appeal that they furnish explicit representation formulas from which, in principle, one can read off all desired geometric properties of solutions. Surveys of and references to the classical results can be found in Riemann [1], Schwarz [2], Weierstrass [1–5], Enneper [1], Darboux [1], von Lilienthal [1], Blaschke [1], and Nitsche [28,37].

Now we are going to construct minimal surfaces with interesting special properties by combining Lemma 2 with Schwarz's solution (1) of the Björling problem.

Firstly, Lemma 2(i) yields:

Proposition 2. *Let $S = \{(c(t), n(t)) \colon t \in I\}$ be a real analytic strip whose supporting curve $c(t)$, $t \in I$, is a straight line. Then*

$$X(u,v) = \mathrm{Re}\left\{ c(w) - i \int_{u_0}^{w} n(\underline{w}) \wedge dc(\underline{w}) \right\}, \quad w = u + iv,$$

$u_0 \in I$, defines a minimal surface with $c(u) = X(u,0)$ as geodesic. Moreover, c is also an asymptotic line of X, and the surface normal $N(u,v)$ of X coincides on I with n, i.e., $N(u,0) = n(u)$.

Consider now a real analytic strip $S = \{(c(t), n(t)) \colon t \in I\}$ whose supporting curve c is contained in a plane E with a normal vector e which satisfies

$$\langle n(t), e \rangle \equiv \cos\varphi, \quad t \in I,$$

for some constant angle φ. Then (1) defines a minimal surface X for which c is a line of curvature contained in the plane E which intersects X at c under the constant angle φ.

Conversely, if $c(t)$, $t \in I$, is a real analytic regular curve contained in a plane E, and if $\varphi(t)$ is a real analytic function, then

$$(19) \qquad \begin{aligned} S &= \{(c(t), n(t)) \colon t \in I\} \quad \text{with} \\ n(t) &:= e \cos\varphi(t) + \dot{c}(t) \wedge e \frac{1}{|\dot{c}(t)|} \sin\varphi(t) \end{aligned}$$

is a real analytic strip such that $n(t)$ intersects E at its point of support $c(t)$ under the angle $\varphi(t)$. By inserting (19) into (1), we obtain:

Proposition 3. *Let $c(t)$, $t \in I$, be some real analytic regular curve contained in a plane E with a normal vector e, and let φ be some constant angle. Then, for $w = u + iv$ and $u_0 \in I$,*

$$(20) \qquad \begin{aligned} X(w) = \mathrm{Re}\Big\{ & c(w) - ie \wedge [c(w) - c(u_0)] \cos\varphi \\ & - i \sin\varphi \int_{u_0}^{w} \langle c'(\underline{w}), c'(\underline{w})\rangle^{1/2} \, d\underline{w} \, e \Big\} \end{aligned}$$

defines a minimal surface containing $c(u) = X(u,0)$ as a planar line of curvature. Moreover, X intersects E along c at a constant angle φ. Finally, if $\varphi = \frac{\pi}{2}$, then c furnishes a planar geodesic on the surface X given by (20).

By choosing E as the x,z-plane, we in particular obtain:

Proposition 4. *If $c(t) = (\xi(t), 0, \zeta(t))$, $t \in \mathbb{R}$, is a real analytic regular curve contained in the x,z-plane E, then*

$$(21) \qquad X(u,v) = \left(\mathrm{Re}\, \xi(w), \mathrm{Im} \int_0^w \{\xi'(\underline{w})^2 + \zeta'(\underline{w})^2\}^{1/2} \, d\underline{w}, \mathrm{Re}\, \zeta(w) \right)$$

defines a minimal surface X that intersects E perpendicularly along c. Moreover, the curve c is a planar line of curvature on X; in fact, c is a planar geodesic.

If c is a smooth regular curve with nonvanishing curvature, then the principal normal $\boldsymbol{n}$ and the binormal $\boldsymbol{b}$ of c are well defined. If c is a geodesic or an asymptotic line on a regular surface X, then the surface normal N of X can be identified along c with $\boldsymbol{n}$ or with $\pm\boldsymbol{b}$, respectively. Thus we infer from Theorem 1:

Proposition 5. *Let c be a regular real analytic curve of nonvanishing curvature. Then there exists a minimal surface X containing c as geodesic; X is the only such surface if we assume that the surface normal of X along c coincides with the principal normal $\mathcal{N}$ of c. Secondly, there is a minimal surface Y containing c as an asymptotic line, and there is no other such surface if we require that the surface normal of Y along c agrees with the binormal $\boldsymbol{b}$ of c.*

3.5 Examples of Minimal Surfaces

In this section we shall briefly discuss some of the classical minimal surfaces found in the nineteenth century, as well as some new examples. Detailed accounts and further information can be found in the treatises of Darboux [1] and Nitsche [28,37], in the lecture notes of Barbosa and Colares [1], and in the papers and reports of Hoffman [1–4], Hoffman and Meeks [1–10], Karcher [1–5], Hoffman and Karcher [1] and Karcher and Polthier [1].

3.5.1 Catenoid and Helicoid

Figure 1 shows a part of the *catenoid*. This minimal surface owes its name to the fact that it can be obtained by rotating a certain *catenary* (or *chain line*) about some axis. If we choose the z-axis as axis of rotation, all catenoids are generated by rotating the catenaries

$$x = \alpha \cosh\left(\frac{z - z_0}{\alpha}\right), \quad z \in \mathbb{R}, \tag{1}$$

where z_0 and α are arbitrary constants, $\alpha \neq 0$. It is one of the classical results of the calculus of variations that every nonplanar rotationally symmetric minimal surface is congruent to a piece of a catenoid. We leave the simple proof of this fact as an exercise to the reader.

Clearly, every catenoid is a doubly connected minimal surface which can be parametrized[2] by

$$\begin{aligned} x(u,v) &= \alpha \cosh u \cos v, \\ y(u,v) &= -\alpha \cosh u \sin v, \\ z(u,v) &= \alpha u \end{aligned} \tag{2}$$

with $-\infty < u < \infty$, $0 \le v < 2\pi$, if we choose $z_0 = 0$.

Note that the representation (2) is defined for all $w = u + iv \in \mathbb{C}$. Hence the mapping $X : \mathbb{C} \to \mathbb{R}^3$, $X(w) := (x(w), y(w), z(w))$, represents the *universal covering surface* of the catenoid generated by the meridian (1). The mapping $X : \mathbb{C} \to \mathbb{R}^3$ is harmonic and conformal. In fact, by means of the formulas

$$\begin{aligned} \cosh(u+iv) &= \cosh u \cos v + i \sinh u \sin v, \\ \sinh(u+iv) &= \sinh u \cos v + i \cosh u \sin v \end{aligned} \tag{3}$$

we infer that

$$X(w) = \mathrm{Re}\, f(w) \tag{4}$$

where $f : \mathbb{C} \to \mathbb{C}^3$ denotes the isotropic curve given by

[2] We could as well define $y(u,v) = \alpha \cosh u \sin v$; this would amount to a change of the sense of rotation.

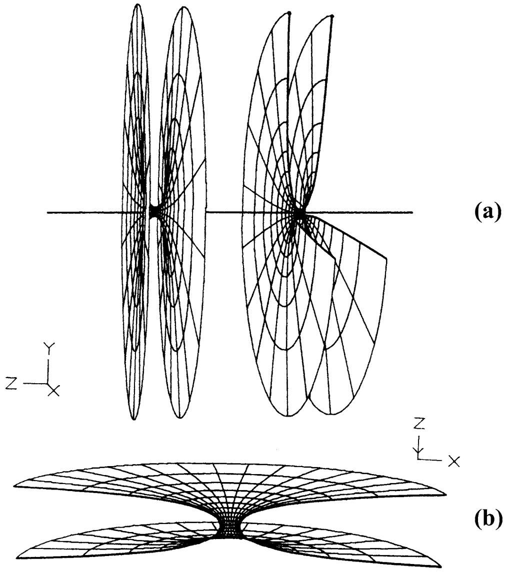


Fig. 1. (a) Three quarters and all of the subset $|z| \leq 1.4\pi$ of the catenoid. The *line* along which the model is cut open is a catenary, the curve described by a hanging chain. The parts contained in *large, origin-centered balls* indicate a global view of the catenoid. They look like two parallel plane disks connected by a thin funnel. **(b)** The part $|u| \leq 1.2\pi$, $0 \leq v \leq \pi$, of the catenoid

$$f(w) = (\alpha \cosh w, \alpha i \sinh w, \alpha w). \tag{5}$$

In order to find the Weierstrass function $\mathfrak{F}(\omega)$ and the representation $Y(\omega)$ of the catenoid given by Section 3.3, (27), or precisely by

$$\begin{aligned} x &= \alpha + \operatorname{Re} \int_1^\omega (1 - \omega^2)\mathfrak{F}(\omega)\, d\omega, \\ y &= \operatorname{Re} \int_1^\omega i(1 + \omega^2)\mathfrak{F}(\omega)\, d\omega, \\ z &= \operatorname{Re} \int_1^\omega 2\omega\mathfrak{F}(\omega)\, d\omega, \end{aligned} \tag{6}$$

we introduce the new variable $\omega = e^{-w}$ instead of $w = u + iv$. Set $r = |\omega|$ and $\theta = \arg \omega$, i.e. $\omega = re^{i\theta}$. Then $\log \omega = \log r + i\theta = -u - iv$, and we can write (2) in the new form

$$\begin{aligned} x &= \frac{\alpha}{2}\left(\frac{1}{r} + r\right)\cos\theta = \operatorname{Re} \frac{\alpha}{2}\left(\frac{1}{\omega} + \omega\right), \\ y &= \frac{\alpha}{2}\left(\frac{1}{r} + r\right)\sin\theta = \operatorname{Re} \frac{i\alpha}{2}\left(\frac{1}{\omega} - \omega\right), \\ z &= -\alpha \log r = -\operatorname{Re} \alpha \log \omega. \end{aligned} \tag{7}$$

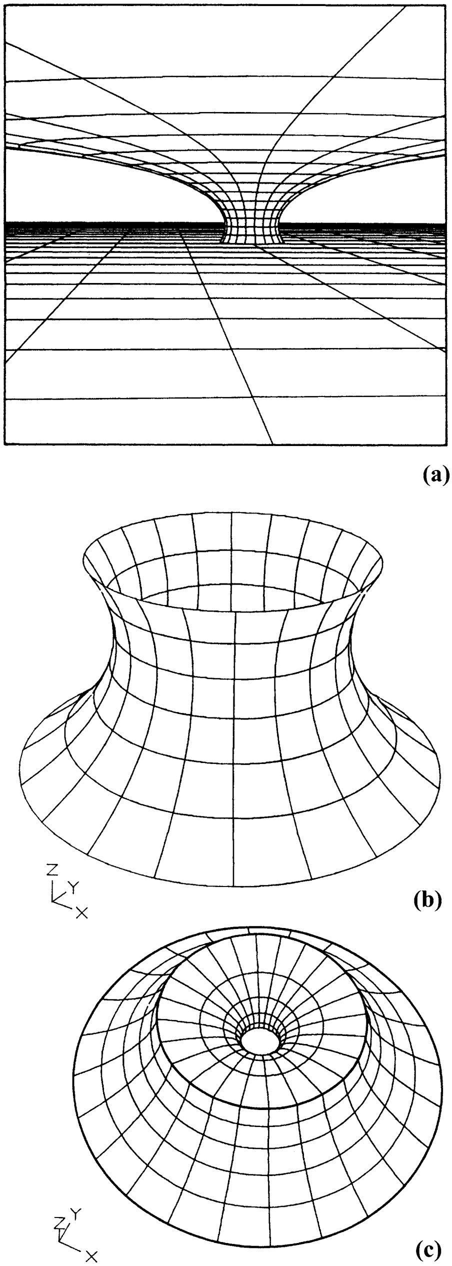


Fig. 2. (a) A view of the part $u \geq -\pi/5$ of the catenoid sitting on the plane $z = -\pi/5$. **(b)** The subset $-\pi/5 \leq z \leq \pi/10$ shows the behaviour of the catenoid close to its plane of symmetry $z = 0$. **(c)** Two catenoids sitting in the same boundary configuration

By a straight-forward computation we infer that these equations are identical with (6) if we choose

$$\mathfrak{F}(\omega) = -\frac{\alpha}{2\omega^2}, \quad \omega \in \mathbb{C} \setminus \{0\}. \tag{8}$$

The geometrical meaning of the parameter ω implies that *the normal map $N(\omega)$ of the representation $Y(\omega)$ of the catenary given by* (6) *or* (7)*, respectively, omits exactly two points on the Riemann sphere S^2*, the north pole $\rho(\infty)$ and the south pole $\rho(0)$. Cut the ω-plane along the positive part of the real axis and denote the resulting set $\{\omega = \xi + i\eta \colon |\omega| > 0,\ 0 < \arg\omega < 2\pi\}$ by $\mathbb{C}'$. Then N maps $\mathbb{C}'$ one-to-one onto S^2 minus a meridian connecting $\rho(0)$ and $\rho(\infty)$, and we infer that the area of the spherical image N is given by

$$\int dA_N = \int_{\mathbb{C}'} |N_\xi \wedge N_\eta|\, d\xi\, d\eta = 4\pi.$$

Since

$$dA_N = -K\, dA_Y \tag{9}$$

(cf. Section 1.2, (44)), we infer that the total curvature of the catenoid has the value -4π:

$$\int_Y K\, dA = -4\pi. \tag{10}$$

From (5), we read off that the adjoint surface

$$X^*(w) := \operatorname{Im} f(w) \tag{11}$$

of the catenoid (2) is given by

$$\begin{aligned} x^*(u,v) &= \alpha \sinh u \sin v, \\ y^*(u,v) &= \alpha \sinh u \cos v, \\ z^*(u,v) &= \alpha v \end{aligned} \tag{12}$$

or

$$X^* = \alpha Y(v) + \sinh u Z(v)$$

with

$$Y(v) = (0,0,v), \quad Z(v) = (\sin v, \cos v, 0).$$

Thus, for every $v \in \mathbb{R}$, the curve $X^*(\cdot, v)$ is a straight line which meets the z-axis perpendicularly. If we fix $u \neq 0$, then $X^*(u,\cdot)$ describes a *helix* of pitch $2\pi|\alpha|$. This helix is left-handed for $\alpha > 0$ and right-handed for $\alpha < 0$. We see that X^* is generated by a screw motion of some straight line $\mathfrak{L}$ meeting the z-axis perpendicularly, whence X^* is called *helicoid* or *screw surface*. *Thus*

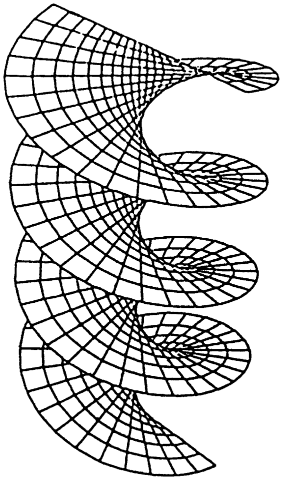


Fig. 3. A part of the helicoid, a ruled minimal surface

the helicoid X^*, the adjoint of the catenoid X, is a ruled surface with the z-axis as its directrix.

We claim that *the point set represented by some ruled surface*

$$X(u, v) = a(v) + ub(v)$$

with $a(v), b(v) \in \mathbb{R}^3$, which is regular, skew (i.e. $[a', b, b'] \neq 0$) and of zero mean curvature, must be congruent to a piece of the helicoid (E. Catalan [1]).

For the proof of this fact we can assume that $|b| = 1$ and $|b'| = 1$ whence

$$\langle b, b' \rangle = 0, \quad \langle b', b'' \rangle = 0.$$

Moreover, we can also assume that

$$\langle a'(v), b(v) \rangle = 0, \quad \langle a'(0), b'(0) \rangle = 0$$

as we can pass from $a(v)$ to a new directrix $\bar{a}(v)$ given by

$$\begin{aligned} \bar{a}(v) &= a(v) - \lambda(v) b(v), \\ \lambda(v) &= \langle a'(0), b'(0) \rangle + \int_0^v \langle a'(t), b(t) \rangle \, dt. \end{aligned}$$

This yields

$$\mathcal{F} = 0, \quad \mathcal{L} = 0,$$

and the equation $H = 0$ is equivalent to $\mathcal{N} = 0$ whence $\langle N, X_{vv} \rangle = 0$, and therefore

$$\det(X_u, X_v, X_{vv}) = 0;$$

see Section 1.2, (31) and (43). Collecting the powers of u, we obtain the three relations

$$\det(b, a', a'') = 0, \quad \det(b, b', b'') = 0,$$
$$\det(b, a', b'') + \det(b, b', a'') = 0.$$

Since b' is perpendicular to b and b'', the second relation yields $b'' = \langle b'', b\rangle b$. Hence $(b \wedge b')' = 0$, and we infer from $|b| = 1$ that the curve $b(v)$ describes a unit circle in a fixed plane E. Now, from the third relation, we obtain $\det(b, b', a'') = 0$, and therefore $a'' \in E$ as well as

$$a'' = \langle a'', b\rangle b + \langle a'', b'\rangle b'.$$

Inserting this expression for a'' into the equation $\det(b, a', a'') = 0$, we infer

$$\langle a'', b'\rangle \det(b, a', b') = 0.$$

The determinant does not vanish for v close to zero (since its columns are mutually orthogonal at $v = 0$ and $X(u, v)$ is a regular surface), and therefore $\langle a'', b'\rangle = 0$. Hence $\langle a', b'\rangle' = 0$ and $\langle a'(0), b'(0)\rangle = 0$ implies $\langle a', b'\rangle = 0$. Together with $\langle a', b\rangle = 0$ we obtain that a' is perpendicular to $\operatorname{span}\{b, b'\} = E$. Since E does not depend on v, we conclude that also a'' is orthogonal to E. On the other hand we know that $a'' \in E$. Thus we obtain $a'' = 0$, i.e., the directrix $a(v)$ is a straight line, and we have proved that $X(u, v)$ is a piece of a helicoid since $\langle a', b\rangle = 0$.

There are various other proofs of this characterization of the helicoid. We particularly mention the elegant approach of H.A. Schwarz by means of the solution of a suitable Björling problem (see Schwarz [2], vol. I, pp. 181–182).

The coordinates of the *associate surfaces*

$$Z(w, \theta) = \operatorname{Re}\{e^{-i\theta} f(w)\}, \quad \theta \in \mathbb{R}, \tag{13}$$

to the catenoid $X(w)$ as well as to the helicoid $X^*(w)$ are given by

$$\begin{aligned} x &= \alpha \cosh u \cos v \cos\theta + \alpha \sinh u \sin v \sin\theta, \\ y &= -\alpha \cosh u \sin v \cos\theta + \alpha \sinh u \cos v \sin\theta, \\ z &= \alpha u \cos\theta + \alpha u \sin\theta. \end{aligned} \tag{14}$$

The bending process of deforming the catenoid X into the helicoid X^* via the associate surfaces $Z(w, \theta)$, $0 \le \theta \le \frac{\pi}{2}$, is depicted in Fig. 4.

3.5.2 Scherk's Second Surface: The General Minimal Surface of Helicoidal Type

Consider the minimal surface $Y(\omega)$ defined by

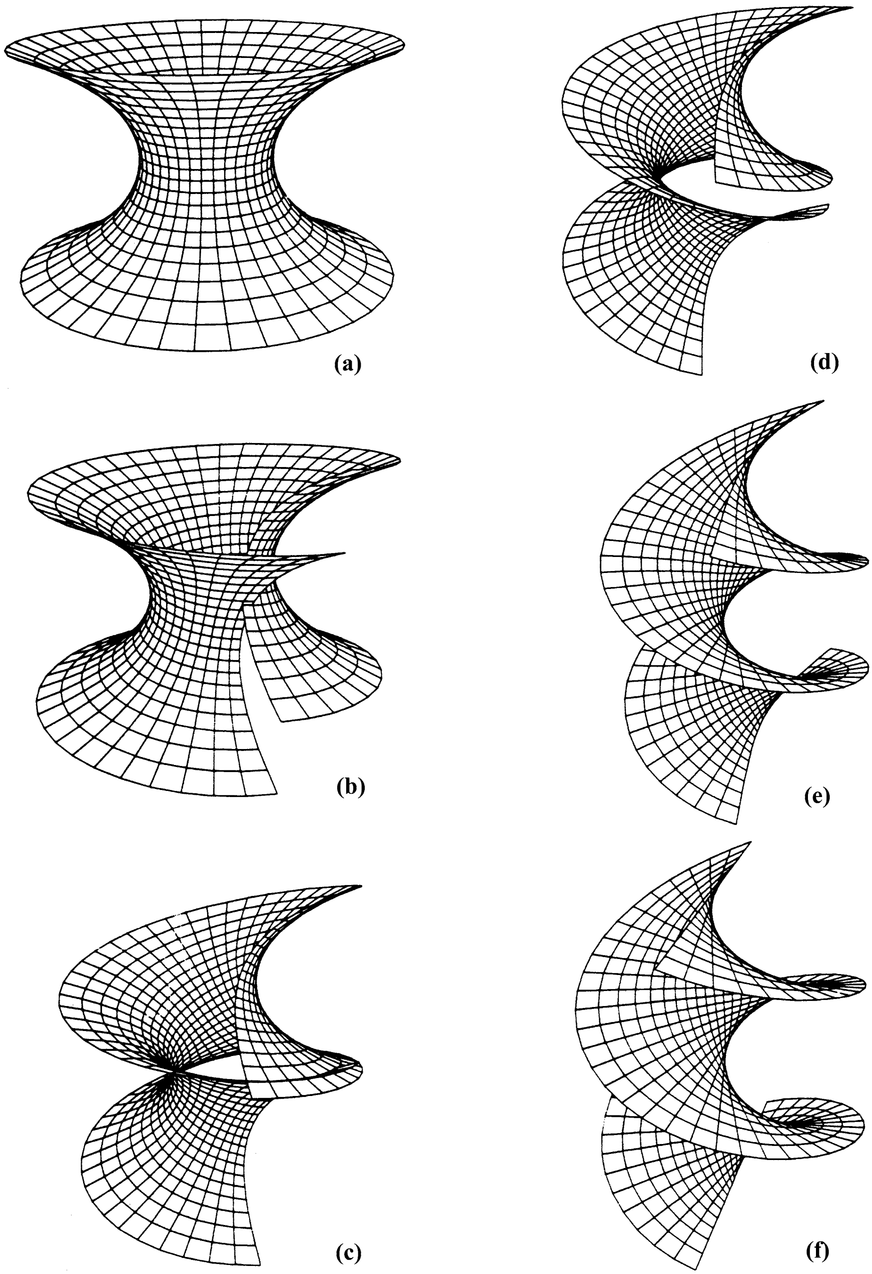


Fig. 4. The catenoid, a minimal surface of rotation, can be bent through its family of associate minimal surfaces into the helicoid, its adjoint surface, which is a ruled surface

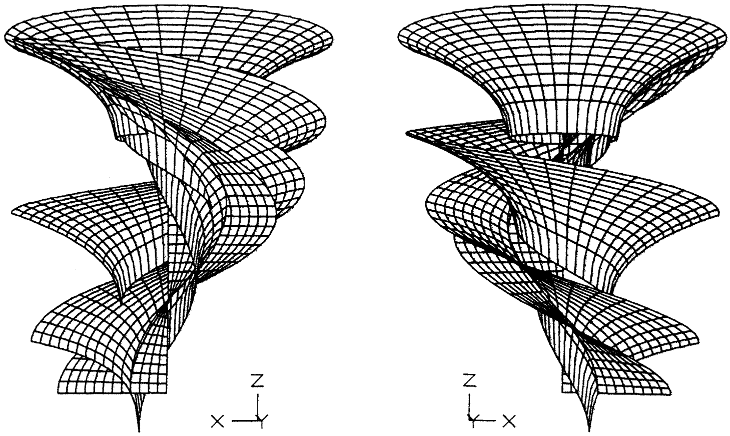


Fig. 5. Scherk's second surface is the family of associate surfaces of the catenoid viewed in a different way. Every member of this family is generated by a screw motion of a planar curve. The illustration shows the parts next to the z-axis of the associate surfaces with parameter values $\theta = k\pi/6$ for $k = 0, 1, 2, 3$

$$\begin{aligned} x &= \alpha + \operatorname{Re} \int_1^\omega (1 - \omega^2)\mathfrak{F}(\omega)\, d\omega, \\ y &= \operatorname{Re} \int_1^\omega i(1 + \omega^2)\mathfrak{F}(\omega)\, d\omega, \\ z &= \gamma + \operatorname{Re} \int_1^\omega 2\omega\mathfrak{F}(\omega)\, d\omega \end{aligned} \tag{15}$$

with the Weierstrass function

$$\mathfrak{F}(\omega) = \frac{-(\alpha - i\beta)}{2\omega^2}, \quad \alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 \neq 0. \tag{16}$$

For $\alpha = 0$ or $\beta = 0$, we obtain a helicoid or a catenoid, respectively. If we switch by $\omega = e^{-w}$, $w = u + iv$, from ω to the new variable w, then (15) is transformed into

$$\begin{aligned} x &= \alpha \cosh u \cos v + \beta \sinh u \sin v, \\ y &= -\alpha \cosh u \sin v + \beta \sinh u \cos v, \\ z &= \alpha u + \beta v + \gamma. \end{aligned} \tag{17}$$

This is a parameter representation of a family of minimal surfaces. For a fixed choice of α, β, γ, we want to denote a surface (17) as *Scherk's second surface.* This family comprises the catenoid ($\beta = 0$) and the helicoid ($\alpha = 0$). In fact, we can write $X(w)$ in the following way, using the formulae

$$X^{\text{cat}}(w) = (\cosh u \cos v, -\cosh u \sin v, u)$$

for the catenoid and

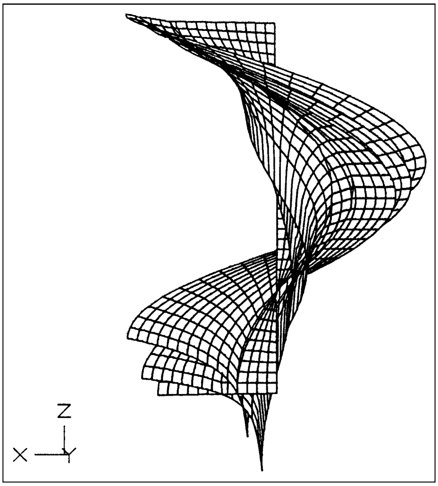


Fig. 6. Another view of parts of three members of the family of minimal surfaces called Scherk's second surface ($\theta = 60, 75, 90$ degrees)

$$X^{\mathrm{hel}}(w) = (\sinh u \sin v, \sinh u \cos v, v)$$

for the helicoid and choosing $\gamma = 0$:

$$X(w) = \alpha X^{\mathrm{cat}}(w) + \beta X^{\mathrm{hel}}(w). \tag{17'}$$

As we can write

$$\alpha = c\cos\theta, \quad \beta = c\sin\theta \quad \text{with } c = \sqrt{\alpha^2+\beta^2},$$

it follows that

$$X(w) = c[\cos\theta X^{\mathrm{cat}}(w) + \sin\theta X^{\mathrm{hel}}(w)].$$

In other words, Scherk's second surface is nothing but an associate surface of the catenoid.

We want to show that (17) *provides a minimal surface of helicoid type generated by a screw motion of some planar curve $z = h(\rho)$ about the z-axis.* (One can easily prove that there exists no other nonplanar minimal surface of helicoidal type; cf. Nitsche [28], pp. 62–63). To this end, we introduce cylindrical coordinates ρ, φ, z instead of the Cartesian coordinates x, y, z by

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z. \tag{18}$$

From the first two formulas of (17), we infer that

$$\begin{aligned}\tan(\varphi + v) &= \frac{\sin\varphi\cos v + \cos\varphi\sin v}{\cos\varphi\cos v - \sin\varphi\sin v}\\ &= \frac{y\cos v + x\sin v}{x\cos v - y\sin v} = \frac{\beta}{\alpha}\tanh u\end{aligned}$$

whence

$$v = -\varphi + \arctan\left(\frac{\beta}{\alpha}\tanh u\right). \tag{19}$$

Moreover, the formulas

$$\rho^2 - \alpha^2 = (\alpha^2+\beta^2)\sinh^2 u, \quad \rho^2+\beta^2 = (\alpha^2+\beta^2)\cosh^2 u$$

yield

$$\frac{\rho^2-\alpha^2}{\rho^2+\beta^2} = \tanh^2 u < 1$$

and

$$\tanh u = \pm\sqrt{\frac{\rho^2-\alpha^2}{\rho^2+\beta^2}},$$

where the plus sign holds for $u \geq 0$, and the minus sign is to be taken if $u \leq 0$. Thus

$$u = \tanh^{-1}\left(\pm\sqrt{\frac{\rho^2-\alpha^2}{\rho^2+\beta^2}}\right),$$

and the identity

$$\tanh^{-1}\xi = \frac{1}{2}\log\frac{1+\xi}{1-\xi} \quad \text{for } |\xi|<1$$

implies

$$u = \frac{1}{2}\log\frac{\sqrt{\rho^2+\beta^2}\pm\sqrt{\rho^2-\alpha^2}}{\sqrt{\rho^2+\beta^2}\mp\sqrt{\rho^2-\alpha^2}}.$$

A brief computation yields

$$u = -\log\sqrt{\alpha^2+\beta^2} + \log(\sqrt{\rho^2+\beta^2}\pm\sqrt{\rho^2-\alpha^2}). \tag{20}$$

Combining the relations (17)–(20), we arrive at

$$\begin{aligned} x &= \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = -\beta\varphi + h(\rho), \\ h(\rho) &:= \alpha\log(\sqrt{\rho^2+\beta^2}\pm\sqrt{\rho^2-\alpha^2}) \\ &\qquad + \beta\arctan\left(\pm\frac{\beta}{\alpha}\sqrt{\frac{\rho^2-\alpha^2}{\rho^2+\beta^2}}\right) + \gamma - \alpha\log\sqrt{\alpha^2+\beta^2}. \end{aligned} \tag{21}$$

This representation shows that Scherk's second surface is a helicoidal surface generated by a screw motion of a planar curve $z = h(\rho)$ about the z-axis.

3.5.3 The Enneper Surface

The minimal surface $X(w), w \in \mathbb{C}$, given by the Weierstrass representation (29), (30) of Section 3.3 with

$$\mathfrak{F}(w) \equiv 1$$

is the *Enneper surface*:

$$X(w) = \mathrm{Re}\left(\int_0^w (1-w^2)\mathfrak{F}(w)\, dw, \int_0^w i(1+w^2)\mathfrak{F}(w)\, dw, \int_0^w 2w\mathfrak{F}(w)\, dw \right),$$

that is,

$$X(w) = \mathrm{Re}\left(w - \frac{w^3}{3}, iw + \frac{iw^3}{3}, w^2 \right). \tag{22}$$

Thus the components of the Enneper surface are given by

$$\begin{aligned} x &= u - \tfrac{1}{3}u^3 + uv^2, \\ y &= -v - u^2 v + \tfrac{1}{3}v^3, \\ z &= u^2 - v^2 \end{aligned} \tag{23}$$

for $w = u + iv \in \mathbb{C}$.

The Gauss curvature $K(w)$ of $X(w)$ has the form

$$K(w) = -\frac{4}{(1+|w|^2)^4}, \tag{24}$$

and

$$N(w) = \frac{1}{1+|w|^2}(2\,\mathrm{Re}\, w, 2\,\mathrm{Im}\, w, |w|^2 - 1) \tag{25}$$

is *its spherical image*, which *omits exactly one point on the Riemann surface*, the north pole $\rho(\infty)$. Moreover, the mapping $N : \mathbb{C} \to S^2 \setminus \{\rho(\infty)\}$ is one-to-one whence $\int dA_N = 4\pi$, and by

$$dA_N = -K\, dA_X$$

we obtain

$$\int_X K\, dA = -4\pi \tag{26}$$

for the total curvature of the Enneper surface. This formula can also be verified by a direct computation using (24) as well as

$$|dX(w)|^2 = |\mathfrak{F}(w)|^2(1+|w|^2)^2|dw|^2.$$

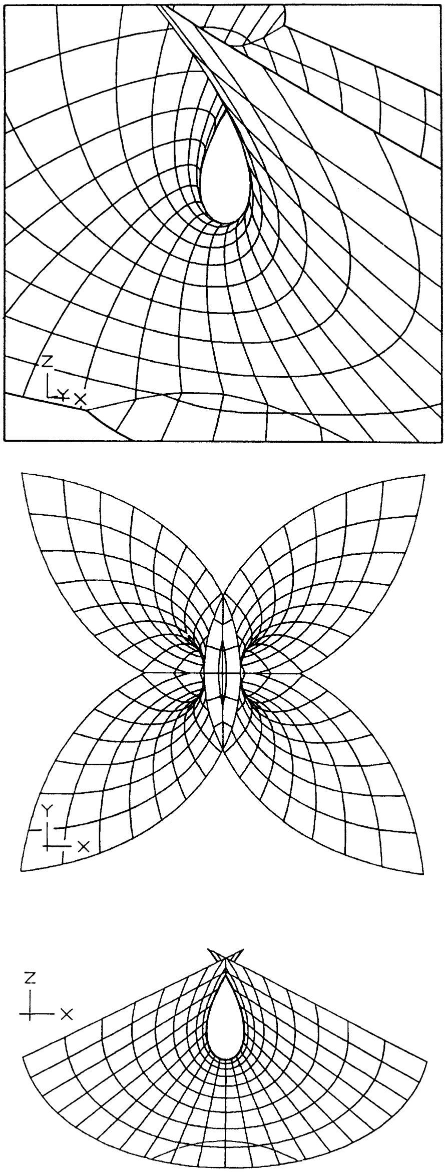


Fig. 7. These views of the subset of Enneper's surface corresponding to $|u| \leq 2$, $|v| \leq 2$ reveal the behavior of the surface close to the origin. The planes and lines of symmetry of the surface can be seen in the two projections onto the coordinate planes

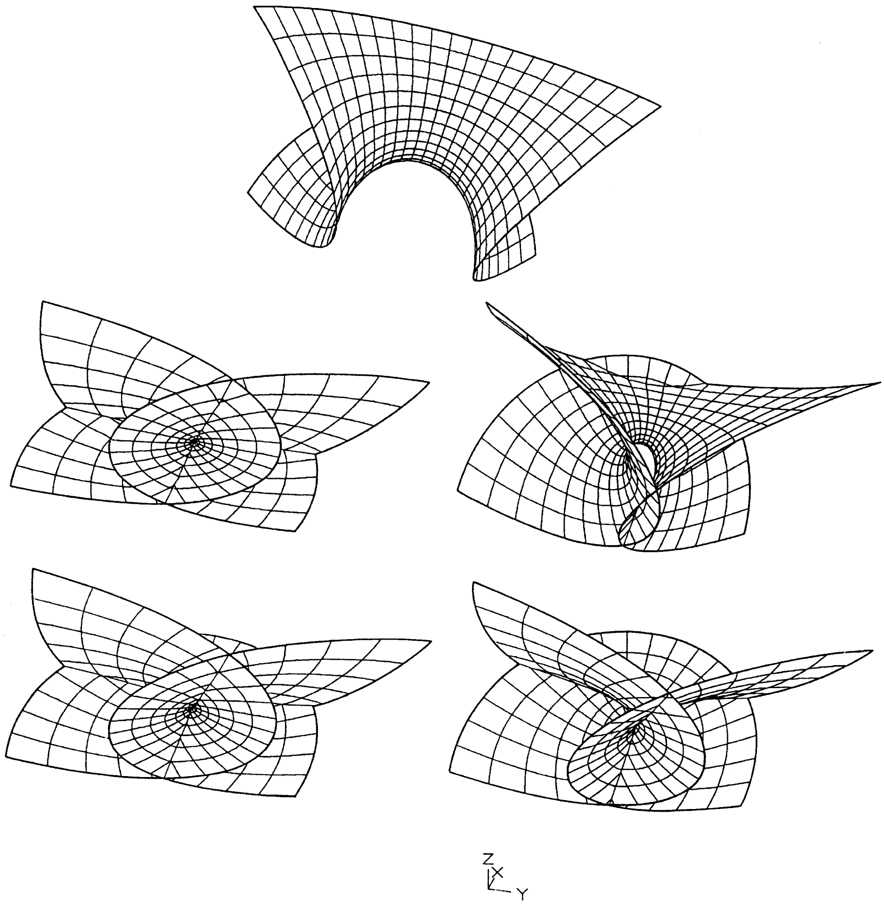


Fig. 8. Large parts of Enneper's surface: the parts shown correspond to the squares $[-R, R]^2, R = 1, 2, 4, 8, 16$, of the parameter plane (clockwise from the top). The shapes of the rescaled figures converge in view of the convergence of $X(Rw)/R^3$

The trace of the Enneper surface X is congruent to the traces of its associate surfaces

$$Z(w, \theta) = \mathrm{Re}\left\{ e^{-i\theta}\left(w - \frac{w^3}{3}\right), ie^{-i\theta}\left(w + \frac{w^3}{3}\right), e^{-i\theta}w^2 \right\}.$$

This can be seen as follows: First we introduce new Cartesian coordinates ξ, η, z instead of x, y, z by a rotation about the z-axis with the angle $-\frac{\theta}{2}$:

$$\xi + i\eta = e^{-i\theta/2}(x + iy).$$

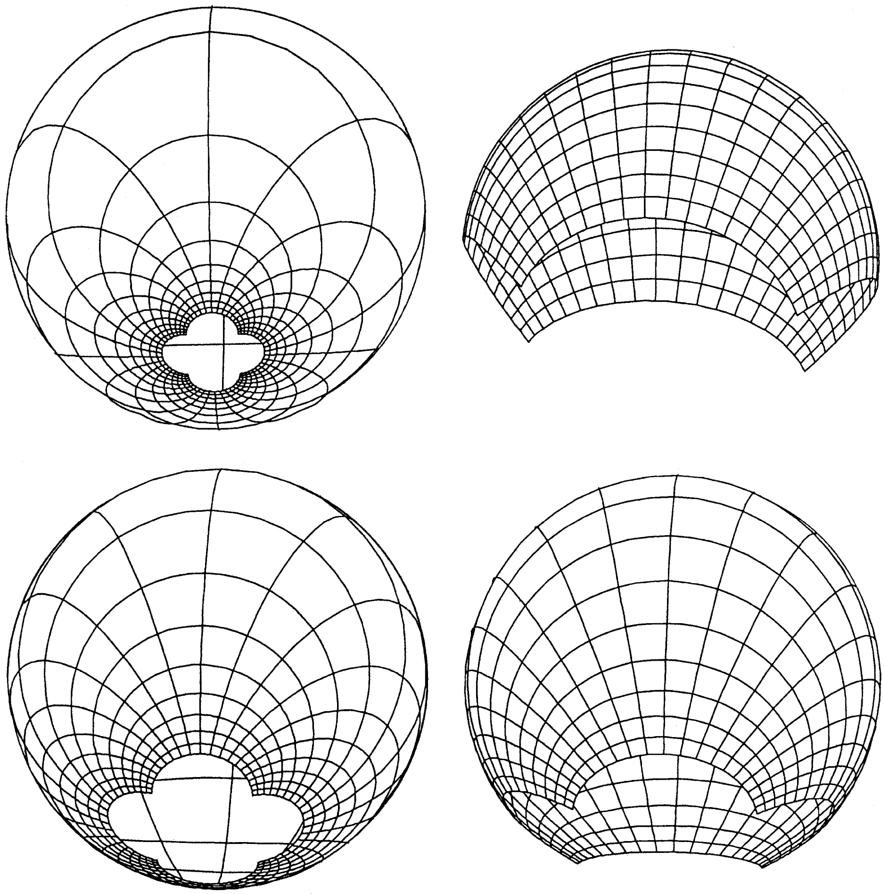


Fig. 9. The (negative of the) Gauss map of the first four parts of Enneper's surface shown before

Then the new coordinates $\xi(w)$, $\eta(w)$ of the associate surface will be obtained from

$$\begin{aligned}\xi + i\eta &= e^{-i\theta/2}[\operatorname{Re}(e^{-i\theta}w) + i\operatorname{Re}(ie^{-i\theta}w)] \\ &\quad + e^{-i\theta/2}\left[\operatorname{Re}\left(-\frac{1}{3}\, e^{-i\theta}w^3\right) + i\operatorname{Re}\left(ie^{-i\theta}\frac{w^3}{3}\right)\right].\end{aligned}$$

Let us now introduce the new independent variable $\zeta = e^{-i\theta/2}w$. Using the identities

$$\operatorname{Re} c + i\operatorname{Re} ic = \bar{c}, \quad \operatorname{Re} c + i\operatorname{Re}(-ic) = c$$

for $c \in \mathbb{C}$, it follows that

$$\xi + i\eta = \operatorname{Re}(\zeta - \tfrac{1}{3}\zeta^3) + i\operatorname{Re} i(\zeta + \tfrac{1}{3}\zeta^3).$$

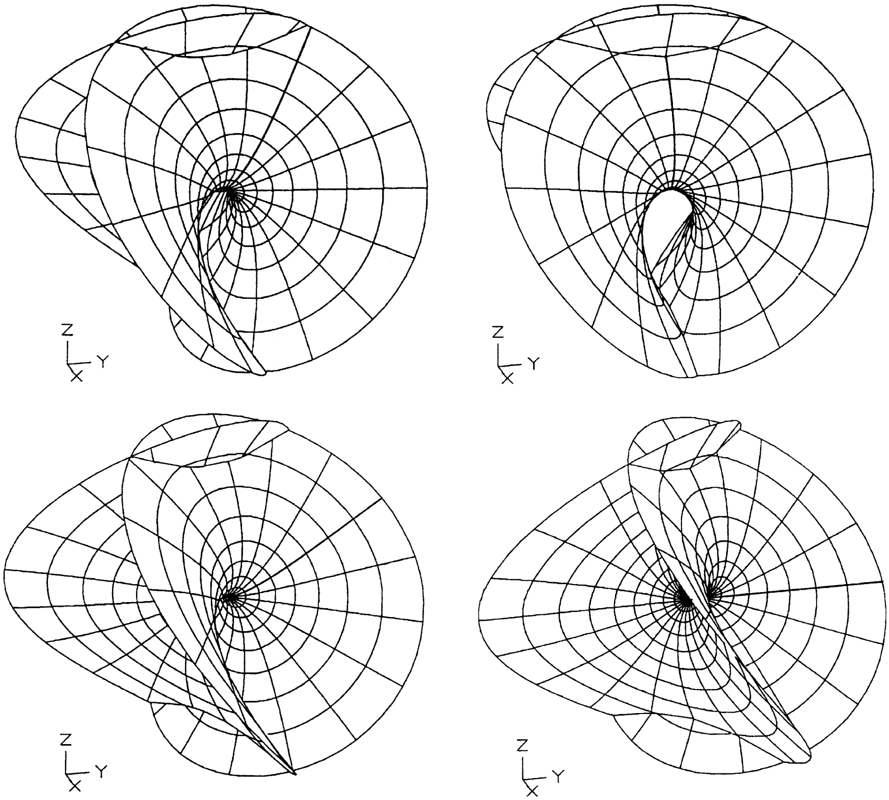


Fig. 10. For every parameter θ the subset of the associate surface corresponding to a disk B_R centered at $w = 0$ is obtained by rotating the same subset of Enneper's surface around the z-axis by an angle $\theta/2$ (counter-clockwise from top right $\theta = 0, \pi/6, \pi/3$, and $\pi/2$)

Thus we arrive at

$$\begin{aligned} \xi &= \operatorname{Re}(\zeta - \tfrac{1}{3}\zeta^3), \\ \eta &= \operatorname{Re} i(\zeta + \tfrac{1}{3}\zeta^3), \\ z &= \operatorname{Re}\zeta^2. \end{aligned}$$

Comparing these expressions with (22), we see that Enneper's surface and its associates are the same geometric objects.

3.5.4 Bour Surfaces

Bour's surfaces are given by

$$X(w) = X_0 + \operatorname{Re}\left(\int_1^w (1-w^2)\mathfrak{F}(w)\,dw, \int_1^w i(1+w^2)\mathfrak{F}(w)\,dw, \int_1^w 2w\mathfrak{F}(w)\,dw\right)$$

with the Weierstrass function

$$\mathfrak{F}(w) = cw^{m-2}, \quad w \in \mathbb{C} \text{ (or } \mathbb{C} \setminus \{0\}), \tag{27}$$

where $m \in \mathbb{R}$ and $c \in \mathbb{C}$, $c \neq 0$. This class of minimal surfaces clearly contains the previously considered examples where we had $m = 0$ or $m = 2$. It was proved by Bour that the surfaces with (27) are exactly those minimal surfaces which are developable onto some surface of revolution; cf. Schwarz [2], pp. 184–185, and Darboux [1], vol. 1, in particular pp. 392–395. Further references can be found in Nitsche [28], p. 57.

3.5.5 Thomsen Surfaces

Surfaces which are both minimal surfaces as well as affine minimal surfaces in the sense of Blaschke [1] have been discussed by Thomsen. A comprehensive discussion and a new derivation of all such surfaces can be found in Barthel, Volkmer, and Haubitz [1]. It turns out that, besides the Enneper surfaces, all other surfaces of this type belong to one of two families. The first family is given by

$$\begin{aligned} X(w) \;=\; & X_0 + \operatorname{Re} \alpha^{-2}(\alpha\beta w + \sqrt{1+\beta^2}\sinh\ \alpha w, \\ & - i\alpha\sqrt{1+\beta^2} w - i\beta \sinh \alpha w, -i\cosh \alpha w) \end{aligned} \tag{28}$$

or

$$\begin{aligned} x &= x_0 + \alpha^{-2}\{\alpha\beta u + \sqrt{1+\beta^2}\sinh \alpha u \cos \alpha v\}, \\ y &= y_0 + \alpha^{-2}\{\alpha\sqrt{1+\beta^2} v + \beta \cosh \alpha u \sin \alpha v\}, \\ z &= z_0 + \alpha^{-2}\{\sinh \alpha u \sin \alpha v\}, \end{aligned} \tag{29}$$

and the second family is obtained from the first by interchanging x and y as well as u and v; here we have assumed $\alpha > 0$.

For $\beta = 0$, the first family yields the left-handed helicoid, the second family the right-handed helicoid. One passes from one family to the other via the Enneper surface or some plane, respectively. Four views of a Thomsen surface are depicted in Fig. 11.

3.5.6 Scherk's First Surface

The nonparametric surface $z = \psi(x, y)$, defined by

$$e^z = \frac{\cos y}{\cos x} \tag{30}$$

or equivalently, by

$$z = \log \frac{\cos y}{\cos x} \tag{31}$$

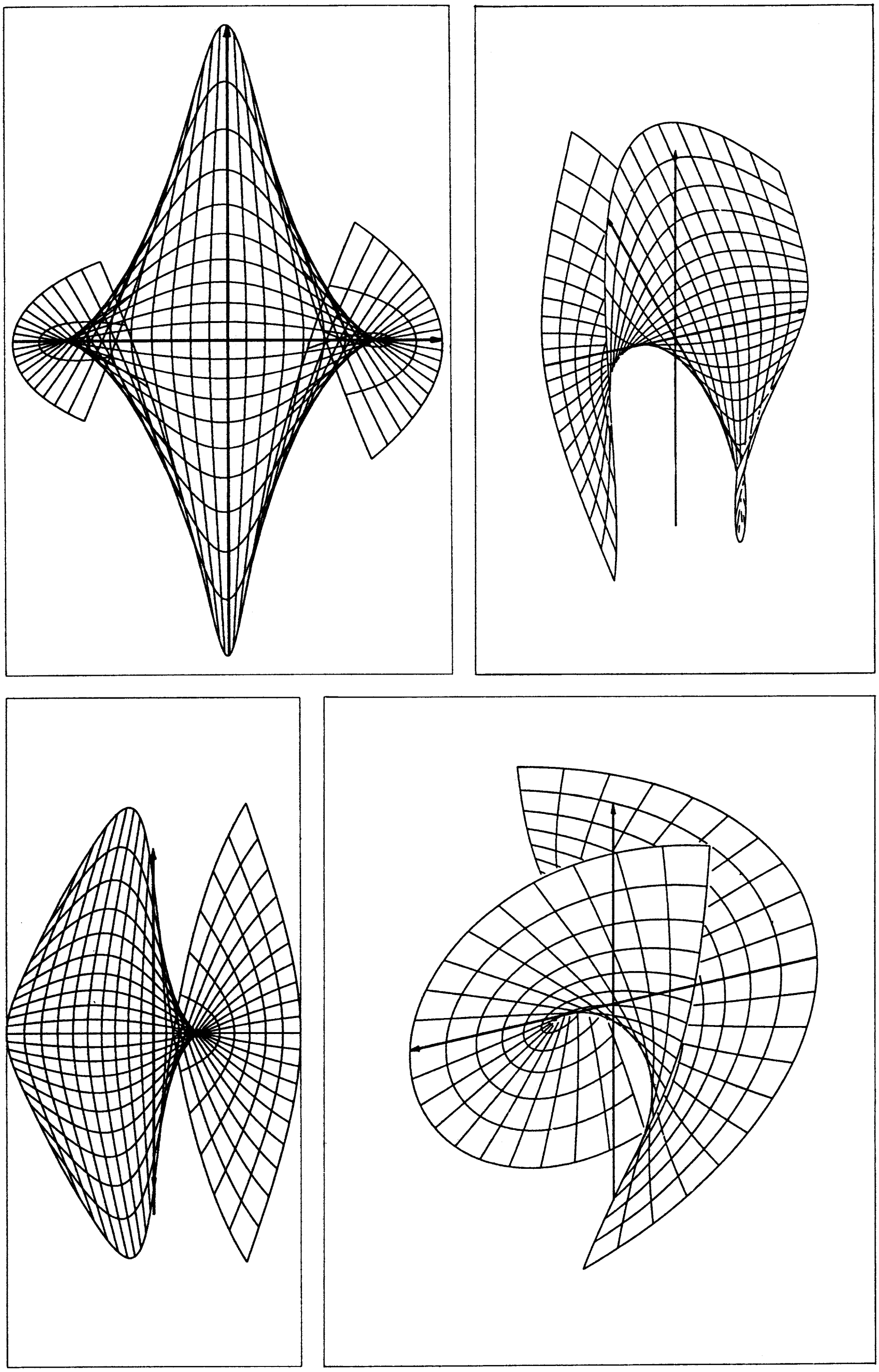


Fig. 11. Four different views of a piece of a Thomsen surface. Courtesy of I. Haubitz

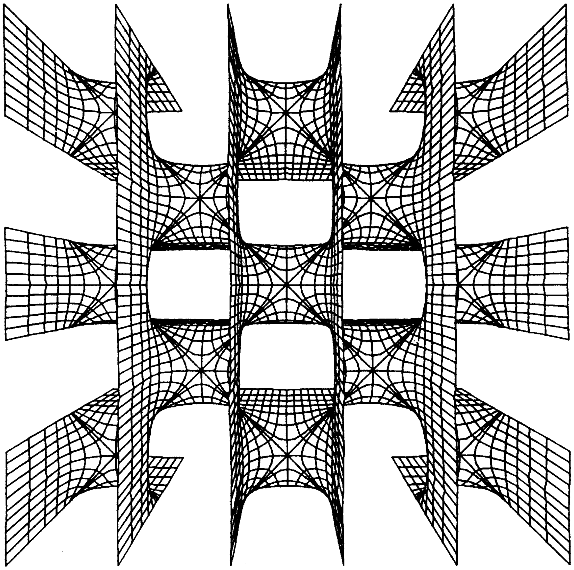


Fig. 12. The part $|z| < 10$, $|x|, |y| < 5\pi/2$, of Scherk's first surface seen from $z = +\infty$

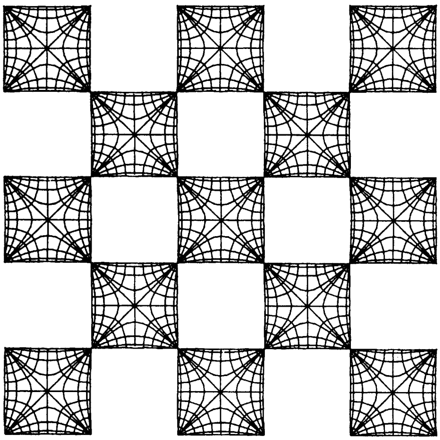


Fig. 13. Scherk's first surface is a non-parametric minimal surface defined on the set $0 < \cos(y)/\cos(x) < +\infty$, which is made up of the black squares of the infinite checker board shown in the figure

on the black squares

$$\Omega_{k,l} := \left\{ (x,y) \colon |x - \pi k| < \frac{\pi}{2}, |y - \pi l| < \frac{\pi}{2} \right\},$$

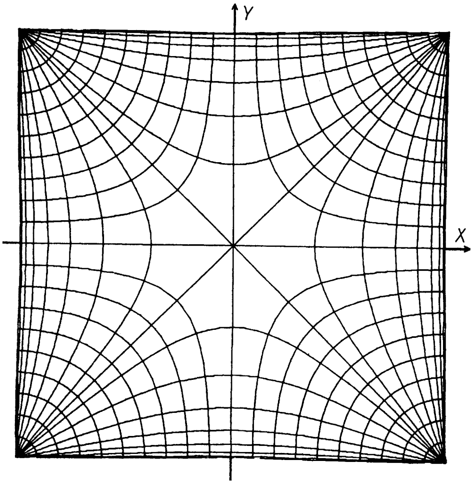


Fig. 14. A closer view of one of the black squares shows the level lines of the surface emanating from the corners. They satisfy $\cos(y)/\cos(x) = \text{constant}$, and the gradient lines perpendicular to them solve the equation $\sin(x)\sin(y) = \text{constant}$

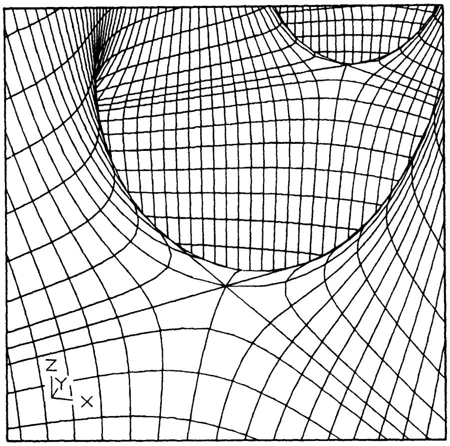


Fig. 15. A view of Scherk's first surface in the vicinity of the plane $z = 0$. The level curves $z = \text{constant}$ include the straight lines $x = \pm y$ as axes of symmetry

$k, l \in \mathbb{Z}$, $k + l = \text{even}$, of the infinite checkerboard shown in Fig. 7, satisfies the nonparametric minimal surface equation

$$(1 + \psi_y^2)\psi_{xx} - 2\psi_x\psi_y\psi_{xy} + (1 + \psi_x^2)\psi_{yy} = 0.$$

This surface is *Scherk's doubly periodic surface* which we want to call *Scherk's first minimal surface*; clearly it is periodic both in the x- and in the y-direction. The graph is repeated on each black square $\Omega_{k,l}$.

The parameter lines shown in our illustrations of Scherk's surface all have the form

$$x = x(t), \quad y = y(t), \quad z = \psi(x(t), y(t))$$

with t varying in the interval $[0, 1]$, and $\psi(x, y) = \log \frac{\cos y}{\cos x}$. Any of the projected curves $(x(t), y(t))$ is either a level line or a gradient line of ψ, that is, we either have

$$\psi(x(t), y(t)) = \text{const},$$

or else

$$\begin{aligned} \frac{dx}{dt} &= \psi_x(x, y) = \tan x, \\ \frac{dy}{dt} &= \psi_y(x, y) = -\tan y. \end{aligned}$$

The gradient lines have the interesting property that they are just the solutions to the equation

$$\sin x \sin y = \text{const}.$$

Let us show that Scherk's surface has the Weierstrass representation

$$\begin{aligned} x &= -\pi + \operatorname{Re} \int_0^w (1 - w^2)\mathfrak{F}(w)\, dw \\ y &= \pi + \operatorname{Re} \int_0^w i(1 + w^2)\mathfrak{F}(w)\, dw, \\ z &= 0 + \operatorname{Re} \int_0^w 2w\mathfrak{F}(w)\, dw \end{aligned}$$

with

$$\text{(32)} \qquad \mathfrak{F}(w) = \frac{2}{1 - w^4} = \frac{2}{(1 + w)(1 - w)(w + i)(w - i)}$$

on the parameter domain $\mathbb{C} \setminus \{\pm 1, \pm i\}$. This will show that the spherical image $N(w)$ of the Scherk surface $X(w)$ omits exactly four points on S^2, namely the points ± 1 and $\pm\, i$ on the equator. Since

$$\begin{aligned} (1 - w^2)\mathfrak{F}(w) &= \frac{2}{1 + w^2} = \frac{i}{w + i} - \frac{i}{w - i}, \\ i(1 + w^2)\mathfrak{F}(w) &= \frac{2i}{1 - w^2} = \frac{i}{w + 1} - \frac{i}{w - 1}, \\ 2w\mathfrak{F}(w) &= \frac{4w}{1 - w^4} = \frac{2w}{w^2 + 1} - \frac{2w}{w^2 - 1}, \end{aligned}$$

we infer that

$$\text{(33)} \qquad X(w) = \operatorname{Re}\left(i \log \frac{w + i}{w - i}, i \log \frac{w + 1}{w - 1}, \log \frac{w^2 + 1}{w^2 - 1} \right)$$

(using the branch with $\log 1 = 0$), and therefore

$$X(w) = \left(-\arg\frac{w+i}{w-i}, -\arg\frac{w+1}{w-1}, \log\left|\frac{w^2+1}{w^2-1}\right|\right). \tag{34}$$

Let us first restrict our considerations to the set $\{w\colon |w| \le 1, w \ne \pm 1, \pm i\}$. From

$$\frac{w+i}{w-i} = \frac{|w|^2-1}{|w-i|^2} + i\frac{w+\bar{w}}{|w-i|^2}, \quad \frac{w+1}{w-1} = \frac{|w|^2-1}{|w-1|^2} + \frac{\bar{w}-w}{|w-1|^2}$$

we infer that

$$\operatorname{Re}\frac{w+i}{w-i} = \frac{|w|^2-1}{|w-i|^2} \le 0, \quad \operatorname{Re}\frac{w+1}{w-1} = \frac{|w|^2-1}{|w-1|^2} \le 0,$$

whence

$$\frac{\pi}{2} \le \arg\frac{w+i}{w-i}, \arg\frac{w+1}{w-1} \le \frac{3\pi}{2}$$

and therefore

$$-\frac{3\pi}{2} \le x, y \le -\frac{\pi}{2}.$$

We conclude that the mapping $(x(w), y(w))$, formed by the first two components of (34), maps the disk $\{w\colon |w| < 1\}$ one-to-one onto the square $\Omega_{-1,-1}$.

It follows that

$$\cos x = \frac{|w|^2-1}{|w-i|^2}\frac{|w-i|}{|w+i|} = \frac{|w|^2-1}{|w^2+1|},$$
$$\cos y = \frac{|w|^2-1}{|w-1|^2}\frac{|w-1|}{|w+1|} = \frac{|w|^2-1}{|w^2-1|},$$

and therefore

$$\frac{\cos y(w)}{\cos x(w)} = \left|\frac{w^2+1}{w^2-1}\right| = e^{z(w)}.$$

This proves that the representation $X(w)$, $|w| < 1$, defined by (34), parametrizes Scherk's surface (30). Moreover, the mapping $X(w) = (x(w), y(w), z(w))$ has the following properties:

(i) Let $|w| = 1$, $w \ne \pm 1, \pm i$. Setting $w = e^{i\varphi}$, we obtain

$$\left|\frac{w^2+1}{w^2-1}\right| = |\cot\varphi|$$

and therefore

$$z(w) = \log|\cot\varphi|.$$

Furthermore, we have $x(e^{i\varphi}) = -\frac{\pi}{2}$, $y(e^{i\varphi}) = -\frac{3\pi}{2}$ for all $\varphi \in (0, \frac{\pi}{2})$. Hence $X(w)$ maps the open arc $\{e^{i\varphi}\colon 0 < \varphi < \frac{\pi}{2}\}$ of the unit circle $\{|w| = 1\}$ onto the straight line through $(-\frac{\pi}{2}, -\frac{3\pi}{2}, 0)$ which is parallel to the z-axis.

More generally, if $C_1, \dots, C_4$ denote the four open quartercircles on $\{|w| = 1\}$ between the points $1, i, -1, -i$ and if $L_1, \dots, L_4$ are the parallels to the z-axis through the vertices $P_1, \dots, P_4$ of the square $\Omega_{-1,-1}$, then X provides a 1–1-mapping of C_j onto L_j (cf. Figs. 17, 18).

(ii) The rays $w = re^{i\theta}$, $r \ge 0$, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ satisfy

$$\cos x(w) = \frac{r^2 - 1}{|\pm ir^2 + 1|} = \frac{r^2 - 1}{|\pm ir^2 - 1|} = \cos y(w),$$

whence $z(w) = 0$. Therefore X maps these rays onto straight halflines in the plane $\{z = 0\}$ emanating from the center $(-\pi, -\pi)$ of $\Omega_{-1,-1}$ and passing through $P_1, \dots, P_4$.

(iii) Similarly, the rays $w = re^{i\theta}$, $r \ge 0$, $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are mapped by $(x(w), y(w))$ onto the straight halflines emanating from $(-\pi, -\pi)$ which are parallel to the x-axis or to the y-axis respectively. (In this case, however, the curve $X(w)$ is no longer a straight line since $z(w)$ is nonlinear.)

Applying Schwarz's reflection principle for holomorphic functions and his symmetry principle for minimal surfaces (Section 3.4, Theorem 2(i)), we infer that a reflection of $\{w\colon |w| \le 1,\ w \ne \pm 1, \pm i\}$ at one of the circular arcs $C_1, \dots, C_4$ corresponds to a reflection of the surface $X(w)$ at one of the straight lines $L_1, \dots, L_4$. More precisely, each of the four quarterdisks $B_1, \dots, B_4$ excised from $\{w\colon |w| < 1\}$ by the u- and v-axes corresponds to one of the four congruent subsquares $Q_1, \dots, Q_4$ of $\Omega_{-1,-1}$ having $(-\pi, -\pi)$ as one of their corner points (cf. Fig. 17), and the representation X maps the mirror image B_j^* of B_j onto the part of Scherk's surface obtained from the graph over the square Q_j by reflection in the straight line L_j.

This way it becomes clear which part of Scherk's surface (30) is parametrized by the representation $X : \mathbb{C} \setminus \{\pm 1, \pm i\} \to \mathbb{R}^3$. If we lift X from the 4-punctured plane to the corresponding universal covering surface, we obtain a parametrization of the full Scherk surface in $\mathbb{R}^3$ sitting as a graph over the black squares of the infinite checkerboard, except for the straight lines parallel to the z-axis through the vertices of the black squares. These lines are also contained in the complete Scherk surface. In addition to these lines of symmetry, we have two further families of parallel lines of symmetry which sit in the plane $\{z = 0\}$ and cross each other at an angle of 90 degrees. As we know, these straight lines are asymptotic lines of the Scherk surface given by $\arg w = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ in the representation X. This can also be seen by investigating the quadratic differential $\mathfrak{F}(w)(dw)^2$. Looking at the rays $\{w = re^{i\varphi},\ r \ge 0,\ \varphi = \text{fixed}\}$, we obtain $(dw)^2 = \frac{w^2}{r^2}\, dr^2$, and therefore

$$\mathfrak{F}(w)(dw)^2 = \frac{2w^2\, dr^2}{r^2(1 - w^4)} = \frac{-2\, dr^2}{r^2(w + \frac{1}{w})(w - \frac{1}{w})}.$$

Setting $w = e^{\omega}$, $\operatorname{Re} \omega = \log r, \operatorname{Im} \omega = \varphi$, it follows that

$$\mathfrak{F}(w)(dw)^2 = \frac{-(\frac{1}{2})\, dr^2}{r^2 \sinh \omega \cosh \omega} = \frac{-dr^2}{r^2 \sinh 2\omega}$$

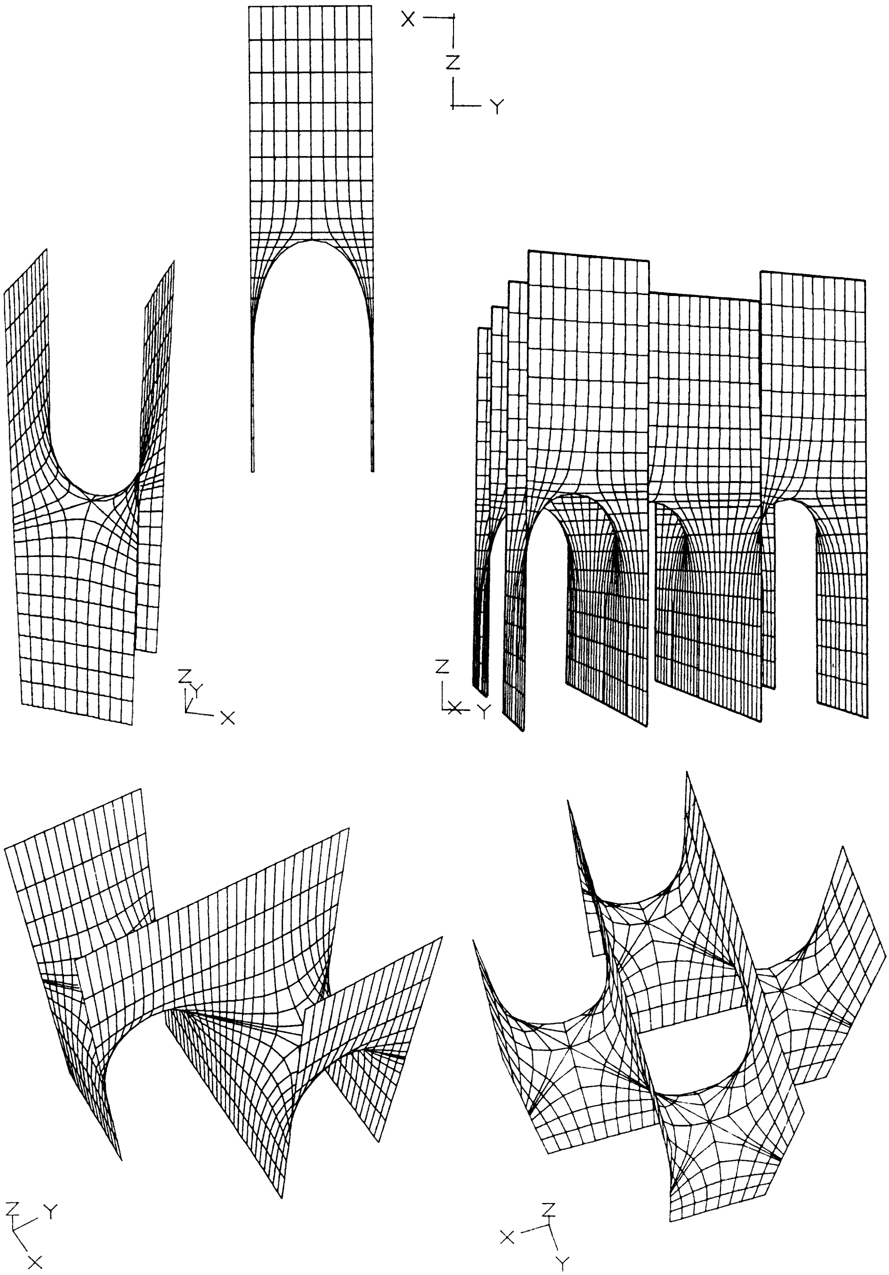


Fig. 16. Scherk's first surface is made up of infinitely many copies of its subset contained in the slab $-\pi/2 < x, y < \pi/2$ of which $|z| \leq 6$ is shown here. Each of the four straight edges of the slab parallel to the z-axis forms a part of the boundary of this fundamental saddle-shaped piece of the surface, and through repeated reflections in these edges Scherk's surface can be built (counter-clockwise from top left)

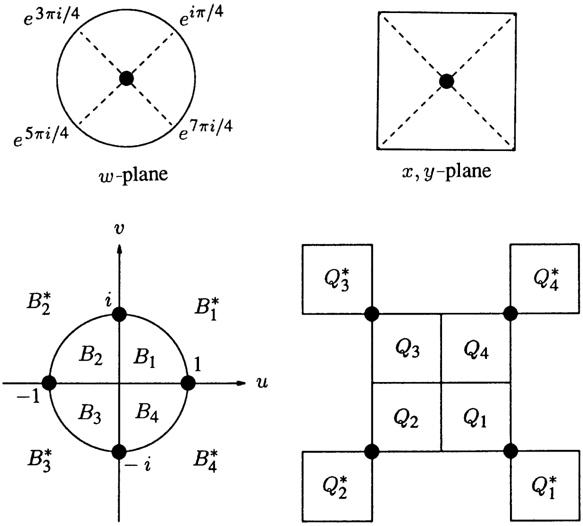


Fig. 17. Construction of Scherk's surface

and

$$\sinh 2\omega = \sinh(2\log r)\cos 2\varphi + i\cosh(2\log r)\sin 2\varphi.$$

Recall now that $\{w = re^{i\varphi} \colon rg \geq 0\}$ is an asymptotic line if $\mathfrak{F}(w)(dw)^2 \in i\mathbb{R}$, and that it is a line of curvature if $\mathfrak{F}(w)(dw)^2 \in \mathbb{R}$. Thus the formula $\varphi = (2k+1)\pi/4$, $k \in \mathbb{Z}$, yields asymptotic lines, and $\varphi = k\pi/2$, $k \in \mathbb{Z}$, provides lines of curvature. As we had already proved, the curves $X(re^{i\varphi}), \varphi = k\pi/2$, are planar curves contained in planes $x = \text{const}$ or $y = \text{const}$ respectively, which turn out to be planes of symmetry for Scherk's surface. This can either be verified by a direct computation or by applying formula (31) of Section 3.3.

If we restrict $X(w)$ to the quarter disk

$$\left\{ w = re^{i\varphi} \colon 0 \leq r \leq 1,\ 0 \leq \varphi \leq \frac{\pi}{2},\ w \neq 1, i \right\},$$

we obtain a minimal surface within the Schwarzian chain formed by the straight line $L = \{x = -\frac{\pi}{2}, y = -\frac{3\pi}{2}\}$ and by the planes $E_1 = \{y = -\pi\}$ and $E_2 = \{x = -\pi\}$. Moreover, X meets the two planes perpendicularly in planar lines of curvature which are plane geodesics of X. In other words, this part of X solves the Schwarzian chain problem for the chain $\{L, E_1, E_2\}$. Then the adjoint surface X^* solves the chain problem for a chain $\{E, L_1, L_2\}$ consisting of a plane E and two straight lines L_1 and L_2 (cf. Fig. 19).

We infer that both X and X^* can be built, by reflection, from elementary pieces which are solutions of Schwarzian chain problems. This situation is typ-

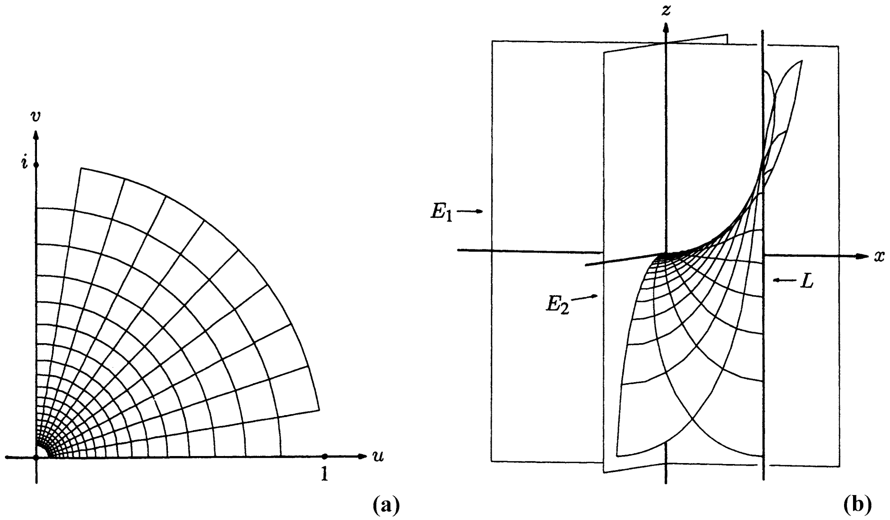


Fig. 18. A conformal representation of Scherk's surface. The part corresponding to a quarter of the unit disk **(a)** solves a Schwarzian chain problem for two perpendicular planes E_1, E_2 and a straight line L parallel to them **(b)**

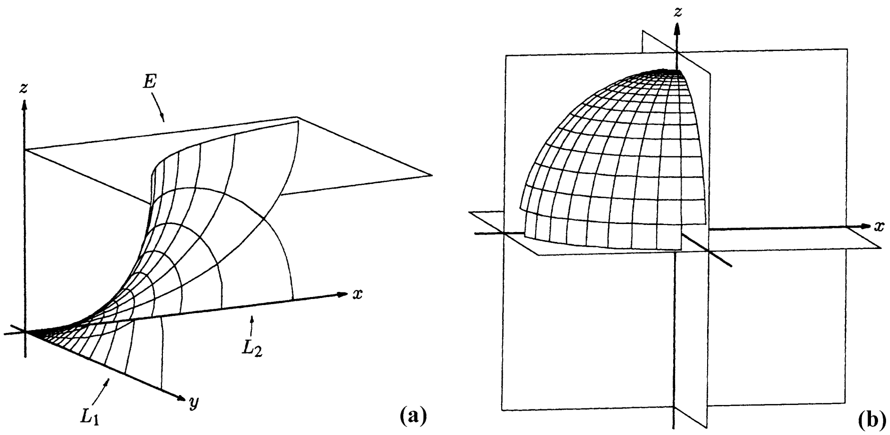


Fig. 19. **(a)** The corresponding part of the adjoint surface of Scherk's surface solves a Schwarzian chain problem for two straight lines L_1, L_2, and a plane E perpendicular to E_1, E_2, and L respectively; cf. Fig. 18. **(b)** The common (negative of the) Gauss map of these surfaces

ical of all cases where we have sufficiently many planes and lines of symmetry. In our present case, the two elementary pieces are mapped by their spherical image N bijectively onto some spherical triangle bounded by great-circular arcs (cf. Fig. 19).

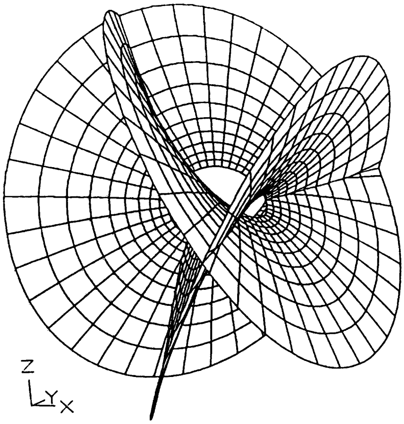


Fig. 20. Part of Henneberg's surface

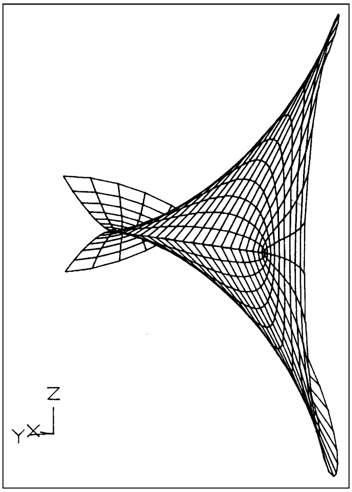


Fig. 21. Henneberg's surface maps the whole v-axis onto a straight line segment of length 2 on the x-axis. (Here we have depicted the part of the surface corresponding to $0 \le u \le \pi/5$, $0 \le v \le \pi$.) The end points of these straight line segments are the two branch points on the surface; the limiting tangent plane in one of them is the x, y-plane, in the other one it is the x, z-plane

3.5.7 The Henneberg Surface

Many interesting minimal surfaces are obtained by solving Björling's problem for a given real analytic strip

$$\Sigma = \{(c(t), n(t)) \colon t \in I\}$$

where c is a given regular, real analytic curve and n its principal normal. If we in addition assume that c is contained in a plane E, then the solution X

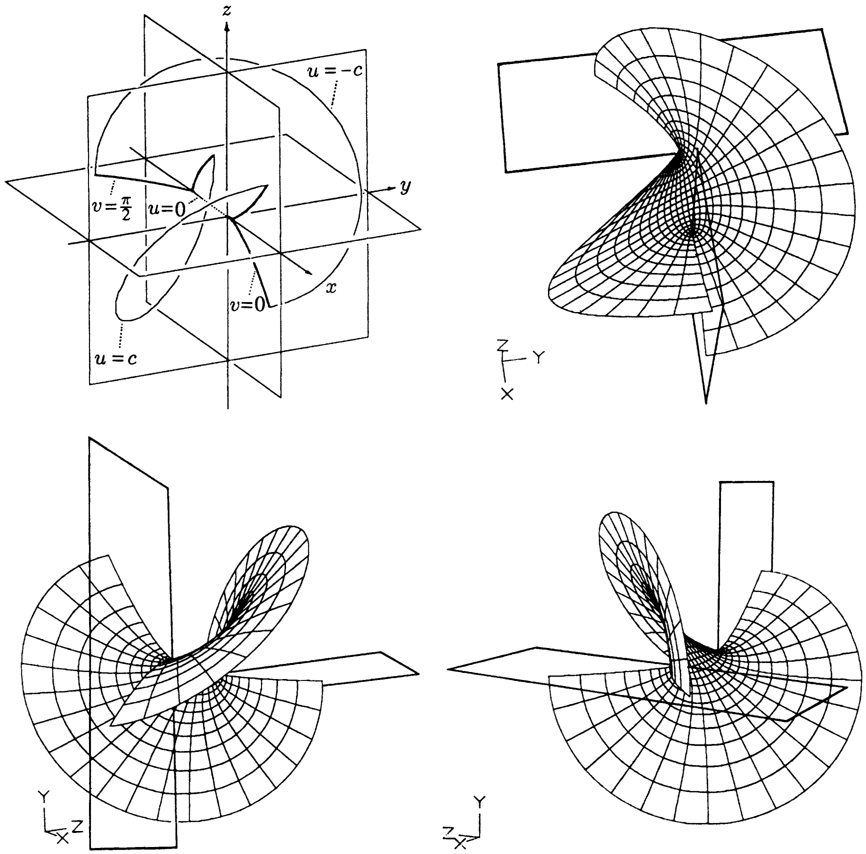


Fig. 22. The curves $v = 0$ and $v = \pi/2$ on Henneberg's surface are Neil parabolas in the x, z-plane and the y, z-plane respectively. For instance the curve $v = 0$ satisfies $2x^3 = 9y^2$, $z = 0$. Along these curves, the surface is perpendicular to the said planes as is shown in our views of Henneberg's surface depicting the parts $|u| \le 3\pi/10$, $0 \le v \le \pi/2$

of Björling's problem for Σ is a minimal surface meeting E perpendicularly at c, and c is a planar geodesic of X as well as a line of curvature.

Let c be given by

$$\begin{aligned} c(t) &= (x(t), 0, z(t)) \\ &= (\cosh(2t) - 1, 0, -\sinh t + \tfrac{1}{3}\sinh(3t)). \end{aligned} \tag{35}$$

From the identities

$$\cosh 2t = 1 + 2\sinh^2 t, \quad \tfrac{1}{3}\sinh(3t) - \sinh t = \tfrac{4}{3}\sinh^3 t$$

we infer that $c(t)$ is a parametrization of *Neil*'s parabola

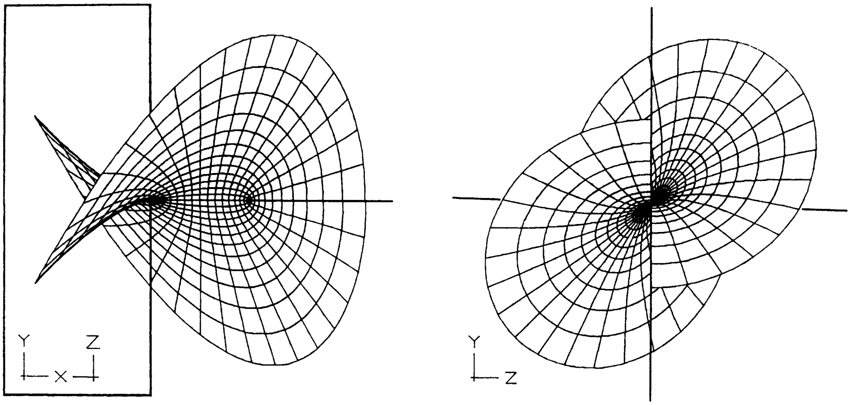


Fig. 23. Parallel projections of the part of Henneberg's surface corresponding to parameter values $|u| \leq 3\pi/10$, $0 \leq v \leq \pi/2$. In particular, one can see that along the two Neil parabolas the surface meets the planes $y = 0$ and $z = 0$ vertically

$$2x^3 = 9z^2 \tag{36}$$

in the plane $\{y = 0\}$. By carrying out Schwarz's construction (cf. formula (1) of Section 3.4), we obtain as solution $X(u, v) = (x(u, v), y(u, v), z(u, v))$ of Björling's problem the *Henneberg surface*

$$\begin{aligned} x &= -1 + \cosh 2u \cos 2v, \\ y &= \sinh u \sin v + \tfrac{1}{3} \sinh 3u \sin 3v, \\ z &= -\sinh u \cos v + \tfrac{1}{3} \sinh 3u \cos 3v. \end{aligned} \tag{37}$$

An isotropic curve $f : \mathbb{C} \to \mathbb{C}^3$ with

$$X(u, v) = \operatorname{Re} f(w), \quad w = u + iv,$$

is given by

$$f(w) = \left(-1 + \cosh 2w, -i \cosh w - \frac{i}{3} \cosh 3w, -\sinh w + \frac{1}{3} \sinh 3w \right). \tag{38}$$

Hence the adjoint surface X^* to X has the form

$$\begin{aligned} x^* &= \sinh 2u \sin 2v, \\ y^* &= -\cosh u \cos v - \tfrac{1}{3} \cosh 3u \cos 3v, \\ z^* &= -\cosh u \sin v + \tfrac{1}{3} \cosh 3u \sin 3v. \end{aligned} \tag{39}$$

The curve $X^*(0, v) = (0, -\frac{4}{3} \cos^3 v, -\frac{4}{3} \sin^3 v)$ lies in the plane $\{x^* = 0\}$ and satisfies

$$y^{*2/3} + z^{*2/3} = (\tfrac{4}{3})^{2/3}, \tag{40}$$

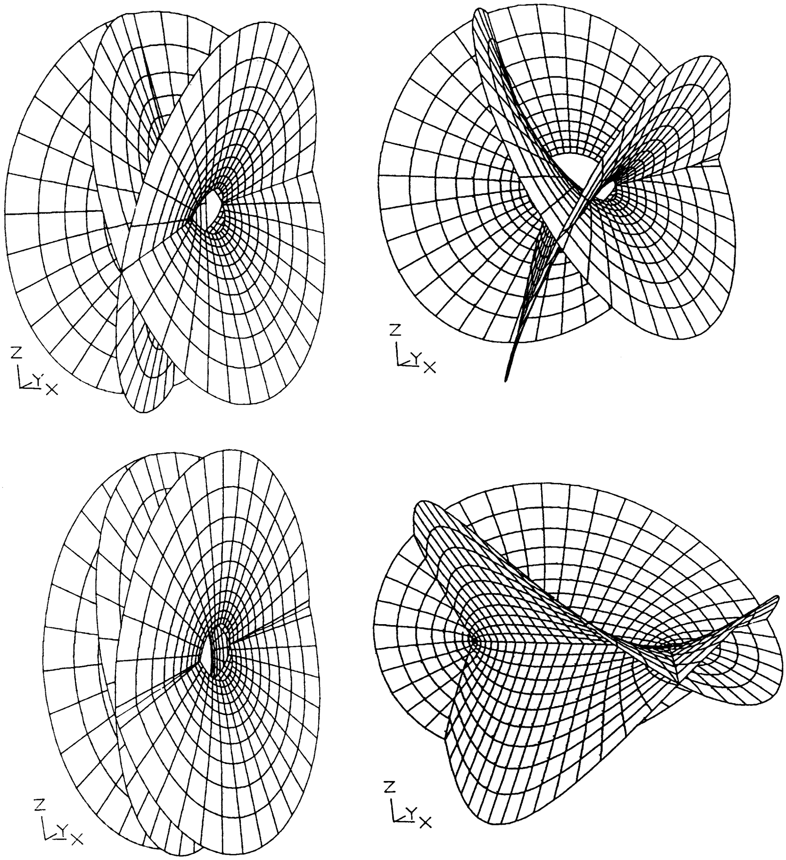


Fig. 24. The parts of Henneberg's surface corresponding to the parameter sets $k\pi/5 \leq |u| \leq (k+1)\pi/5$ for $k = 0, 1, 2, 3$ (counter-clockwise from bottom right) reveal its large scale behavior. Every part of the surface shown in one drawing fits into the hole at the center of the following illustration. In view of the equation $X(-u, v+\pi) = X(u, v)$ each such subset of the surface has two layers glued together and therefore appears to consist of one piece only

that is, the adjoint surface X^* contains an asteroid. This asteroid is a planar geodesic of X^* since $X(0, v) = (-1 + \cos 2v, 0, 0)$ is a straight line and, therefore, a geodesic asymptotic line of X; cf. Section 3.4, Proposition 1. Thus X^* meets the plane $\{x^* = 0\}$ perpendicularly at an asteroid as trace. The straight line $X^*(u, 0) = (0, -\frac{4}{3}\cosh^3\, u, 0) = y$-axis is a line of symmetry for X^*.

Remark. Note that in our figures the coordinate function $y^*(u, v)$ in (39) is replaced by $y^* - \frac{4}{3}$. In this way, the origin remains invariant if we bend X into X^* via the associate surfaces to X.

We furthermore note that both $X(u, v)$ and $X^*(u, v)$ are periodic in v with the period 2π.

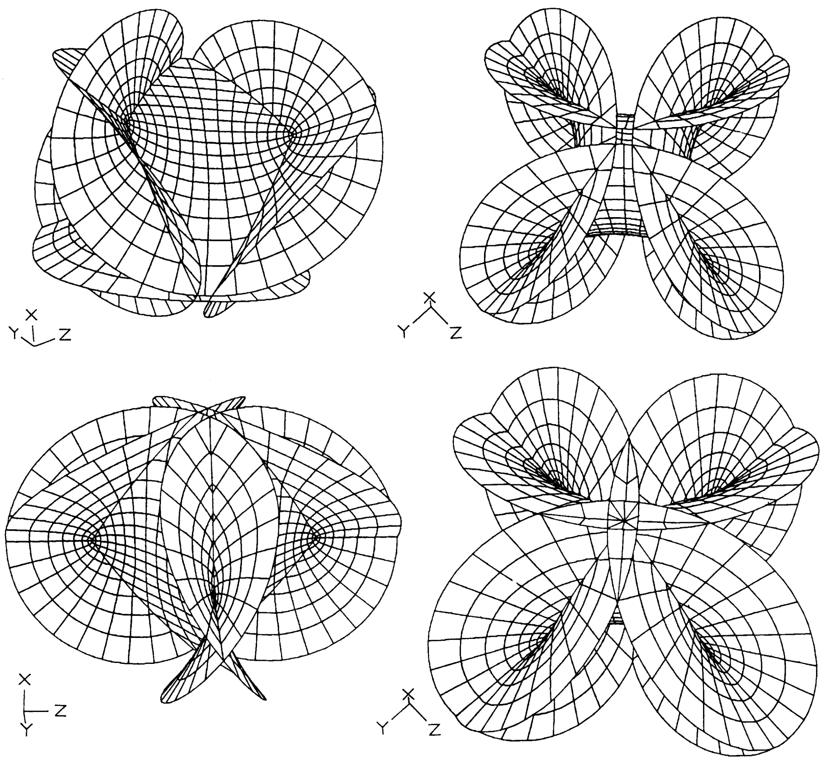


Fig. 25. Some views of parts of the adjoint of Henneberg's surface corresponding to $|u| \leq \pi/5$ and $|u| \leq 9\pi/40$. The adjoint surface encloses a central cavity whose boundary is homeomorphic to the unit sphere and consists of pieces of minimal surfaces. The curve $u = 0$ on the adjoint surface is an asteroid in the y, z-plane connecting the four branch points of the adjoint surface. Along this curve it is orthogonal to the y, z-plane

With the periodicity strip $\{0 \leq v < 2\pi\}$, Henneberg's surface contains four of Neil's parabolas as planar geodesics:

$$
\begin{aligned}
X(u, 0) &= \left(-1 + \cosh 2u, 0, -\sinh u + \frac{1}{3}\sinh 3u\right), \\
X\left(u, \frac{\pi}{2}\right) &= \left(-1 - \cosh 2u, \sinh u - \frac{1}{3}\sinh 3u, 0\right), \\
X(u, \pi) &= \left(-1 + \cosh 2u, 0, \sinh u - \frac{1}{3}\sinh 3u\right), \\
X\left(u, \frac{3\pi}{2}\right) &= \left(-1 - \cosh 2u, -\sinh u + \frac{1}{3}\sinh 3u, 0\right).
\end{aligned}
\tag{41}
$$

However, only two of these four parabolas are geometrically different. Each of these Neil parabolas is periodically repeated on the surface $X(u, v)$. Henneberg's surface intersects the planes $\{y = 0\}$ and $\{z = 0\}$, respectively, at these Neil parabolas orthogonally.

We also observe that the branch points $w = u + iv$ of X and X^* are given by

$$u = 0, \quad v = \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

Moreover, the point set in $\mathbb{R}^3$ represented by $X(u, v)$ is nonorientable. In fact, we easily infer from (39) that

$$\begin{aligned}
&X(u, v) = X(-u, v + \pi), \quad X_u(u, v) = -X_u(-u, v + \pi),\\
&X_v(u, v) = X_v(-u, v + \pi)
\end{aligned}$$

holds for all $w \in \mathbb{C}$. Let $\omega(t)$, $0 \le t \le 1$, be a smooth path in $\mathbb{C}$, avoiding the branch points $w = \frac{1}{2}ik\pi$, joining some point (u, v) with $(-u, v + \pi)$, say $\omega(t) = (2t - 1, \pi(t - \frac{1}{4}))$, $0 \le t \le 1$. Then $\xi(t) := X(\omega(t))$, $0 \le t \le 1$, describes a closed regular loop on Henneberg's surface, but $N(\omega(0)) = -N(\omega(1))$. Thus, if we move around the loop $\xi(t)$ and return to the initial point, the surface normal $N(\omega(0))$ has changed to its opposite. If we slightly thicken the path $\omega(t)$, its image on X will be a Möbius strip (cf. Figs. 27–29). In other words, *Henneberg's surface is a one-sided minimal surface*.

Let us finally mention that the Weierstrass function $\mathfrak{F}(\omega)$ of Henneberg's surface is given by

$$\mathfrak{F}(\omega) = -\frac{i}{2}\left(1 - \frac{1}{\omega^4}\right) \tag{42}$$

if we change the coordinates in $\mathbb{R}^3$ by an orthogonal transformation in such a way that x, y, z become $-z, -y, x$, respectively.

3.5.8 Catalan's Surface

Solving Björling's problem for the strip consisting of the cycloid

$$c(t) = (1 - \cos t, 0, t - \sin t), \quad t \in \mathbb{R} \tag{43}$$

and its principal normal, we obtain *Catalan's surface*

$$X(u, v) = (x(u, v), y(u, v), z(u, v)),$$

given by

$$\begin{aligned}
x &= 1 - \cos u \cosh v,\\
y &= 4 \sin \frac{u}{2} \sinh \frac{v}{2},\\
z &= u - \sin u \cosh v.
\end{aligned} \tag{44}$$

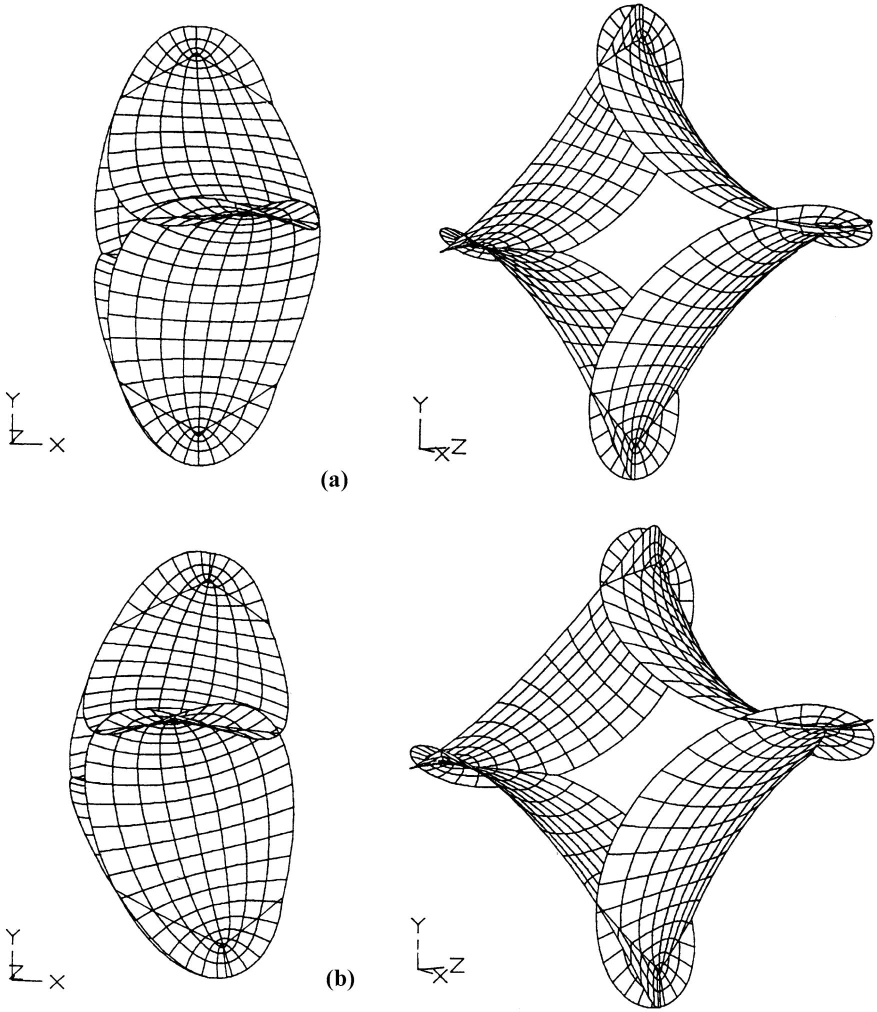


Fig. 26. The bending process for Henneberg's surface into its adjoint surface is so intricate that it is shown here from two different points of view in a long sequence of illustrations. We have arranged for $X^\theta(0) = \text{const}$ for all times θ. The parts of the surfaces depicted here correspond to $|u| \le \pi/10$; the parameter values θ of the associated surfaces are 90, 75, 60, 45, 30, 20, 10, 0 degrees respectively. The bending process starts with a part of the adjoint surface which has a quadruple symmetry and passes through an asteroid in the y, z-plane connecting the four branch points of the surface, the images of $u = 0, v = 0, \pi/2, \pi, 3\pi/2$. The two boundary curves of this part of Henneberg's adjoint surface alternate between the halfspaces $x > 0$ and $x < 0$. In the bending process from the adjoint surface to Henneberg's surface the branch point opposite the origin moves up to the origin of Henneberg's surface, another branch point. The other two branch points move up to the x, z-plane and simultaneously approach each other until they finally meet on the x-axis. In this process the surface is folded together so that one ends up with the double layer of Henneberg's surface for which half of the surface and two of the four branch points seem to have disappeared

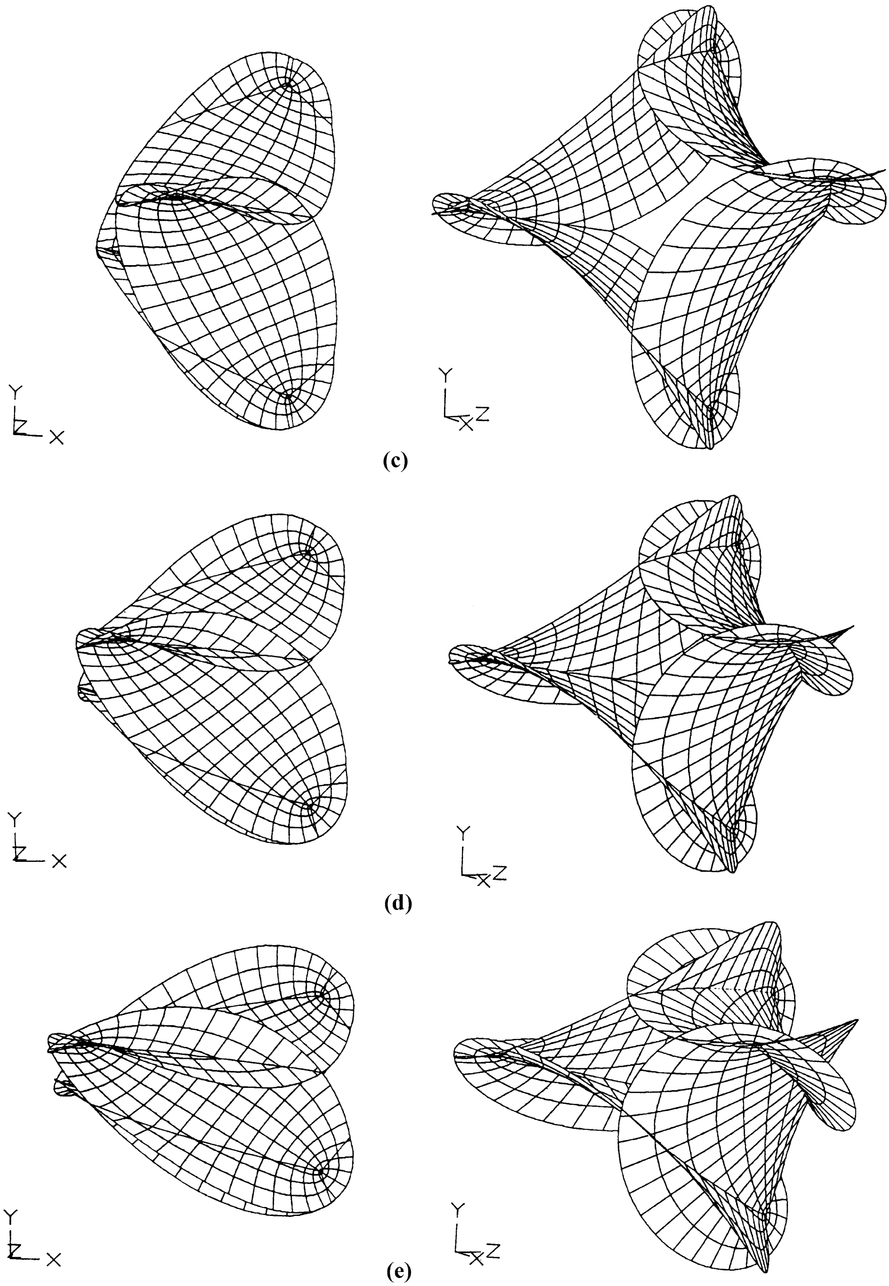


Fig. 26. c–e.

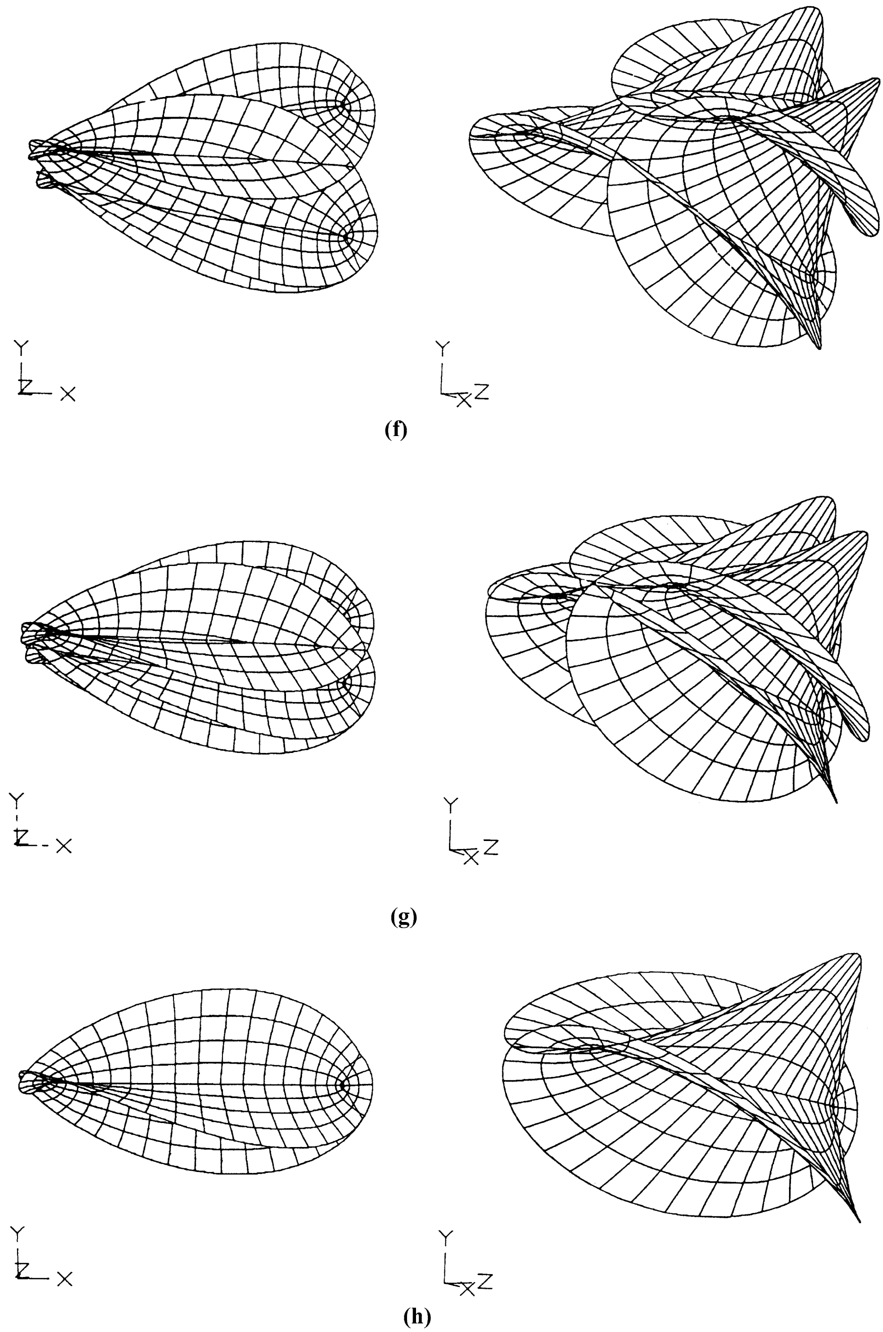


Fig. 26. f–h.

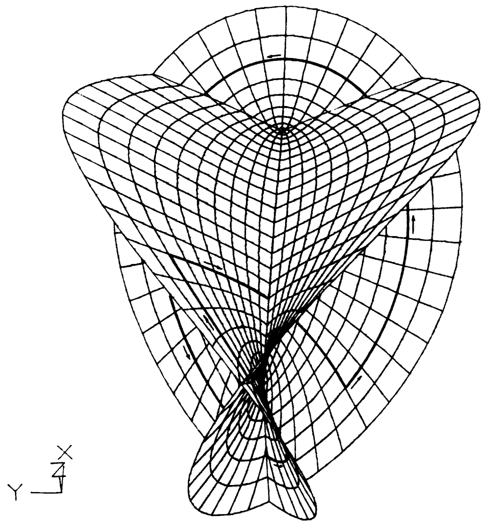


Fig. 27. Henneberg's surface is non-orientable. After a walk on the surface along the emphasized circuit you will find yourself upside down. This results from the equations $X(-u, v+\pi) = X(u,v)$ and $N(-u, v+\pi) = -N(u,v)$ valid on Henneberg's surface

Catalan's surface $X(u,v)$ contains the cycloid $c(u) = X(u,0)$ as a planar geodesic, and we infer from $X(0,v) = (1-\cosh v, 0, 0)$ that the x-axis is both an asymptotic line and a line of symmetry for X.

The branch points of X lie on the u-axis and are given by $(u,v) = (2\pi k, 0)$, $k \in \mathbb{Z}$. Their image points $X(u,v)$ are the cusps of the cycloid $c(u) = X(u,0)$.

Catalan's surface is periodic in the z-direction: The translation in the parameter plane mapping $u+iv$ onto $u+4\pi+iv$ corresponds to a 4π-shift of the surface along the z-axis.

Catalan's surface also has a number of other symmetries; for example, complex conjugation in the parameter plane (i.e., the map $u+iv$ to $u-iv$) corresponds to a reflection of Catalan's surface across the x,z-plane. Moreover all planes $z = (2k+1)\pi$, $k \in \mathbb{Z}$, are planes of symmetry of Catalan's surface.

Reflection in the parameter plane across the v-axis (i.e., the map $u+iv$ to $-u+iv$) corresponds to a reflection of the surface across the x-axis. More generally, all lines $y=0$, $z=2\pi k$, $k \in \mathbb{Z}$, are lines of symmetry of Catalan's surface.

These properties imply that Catalan's surface is made up of denumerably many copies of the fundamental piece corresponding to

$$0 \le u \le 2\pi, \quad 0 \le v.$$

The part $v=0$ of the boundary of this fundamental piece lies on the cycloid and is perpendicular to the x,z-plane as the following equation shows:

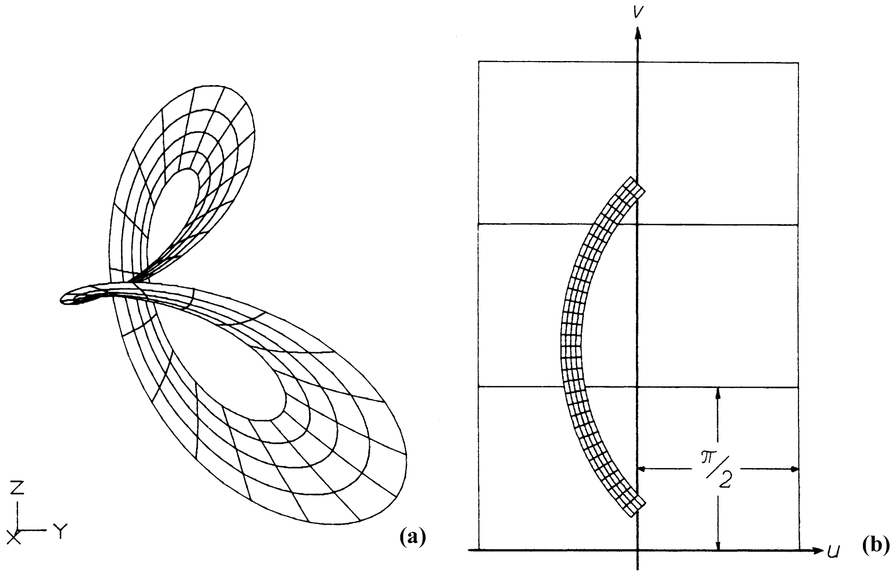


Fig. 28. Henneberg's surface contains a minimal Möbius band with a C^1-smooth boundary curve **(a)**. It corresponds to the quarter of the annulus in the parameter plane shown in **(b)**

$$X_v(u,0) = (0, 2\sin(u/2), 0) \quad \text{for all } u \in \mathbb{R}.$$

The other two boundaries of this fundamental piece, $u = 0$ and $u = 2\pi$ lie on the x-axis and the straight line $y = 0$, $z = 2\pi$ parallel to it respectively. Repeated reflections across the straight lines on the boundary and across the x, z-plane will then build up the complete surface as shown in our illustrations.

Consider now the rolling wheel in the plane $\{y = 0\}$ which is generating the cycloid (43). If we introduce the complex coordinates $\xi = x + iz$ in the x, z-plane, the center of the wheel is described by $\xi = 1 + iu$, and the cycloid is given by $\xi = 1 + iu - e^{iu}$ where u denotes the rotation angle of the rolling wheel which generates the cycloid. Let $R := \{(1 + iu) - (\rho + 1)e^{iu} \colon \rho > 0\}$ be the ray on the straight line through the centerpoint $1 + iu$ and the point $c(u) := 1 + iu - e^{iu}$ on the cycloid, emanating at $c(u)$ and pointing in direction of $-e^{iu}$.

For fixed $u \in \mathbb{R}$, the projection of $X(u, v)$ onto the plane $\{y = 0\}$ is given by

$$\xi = 1 + iu - e^{iu} \cosh v.$$

Hence the curve $X(u, v), v \in \mathbb{R}$, lies in the plane E that is perpendicular to the x, z-plane and contains the ray R. Using Cartesian coordinates ρ and y in E, we can describe $X(u, \cdot)$ by the formulas

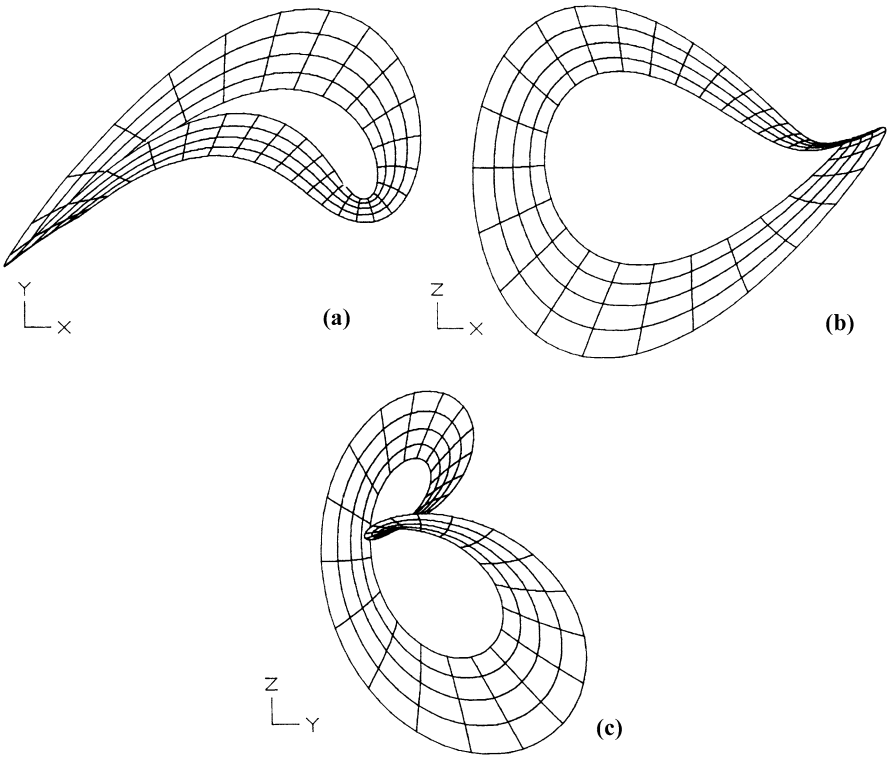


Fig. 29. The projections onto the three coordinate planes convey the shape of this Möbius band. Look at the x, y-projection **(a)** of the Möbius band, then turn it around the x-axis to obtain the x, z-projection **(b)**. Finally rotate it around the z-axis to end up with the y, z-projection **(c)**

$$\rho = \cosh v - 1 = 2\sinh^2 \frac{v}{2}, \qquad (45)$$
$$y = 4 \sin \frac{u}{2} \sinh \frac{v}{2}$$

with $v \in \mathbb{R}$. Hence $X(u, \cdot)$ yields a parametrization of the parabola

$$y^2 = a\rho \qquad (46)$$

with $a := 8 \sin^2 \frac{u}{2}$ in the plane E. Thus Catalan's surface X is swept out by a one-parameter family of parabolas $\mathcal{P}(u), u \in \mathbb{R}$. The vertex of $\mathcal{P}(u)$ moves on the cycloid $c(u)$, and the plane $E(u)$ of $\mathcal{P}(u)$ intersects the x, z-plane perpendicularly and contains the straight line through $c(u)$ and the center $\xi = 1 + iu$ of the rolling wheel.

From (3) and (44) we infer that

$$X(u, v) = \operatorname{Re} f(w), \quad w = u + iv,$$

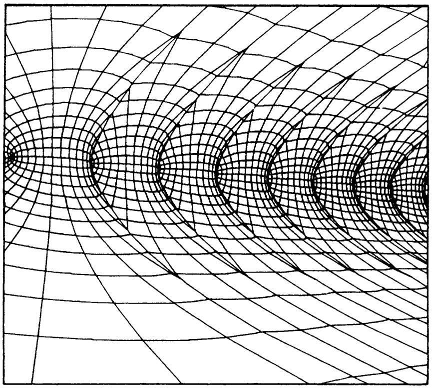


Fig. 30. Catalan's surface as seen from the halfplane $y = 0$, $x > 0$. All points of Catalan's surface remain outside the parabolic cylinder $8(x-2) > y^2$, but the curves $v = (2k+1)\pi$ on Catalan's surface lie on its boundary

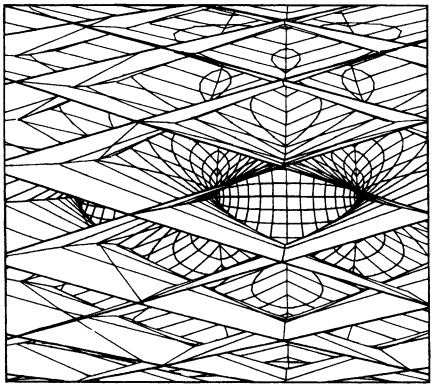


Fig. 31. The view of Catalan's surface from the opposite halfplane $y = 0$, $x < 0$ is quite different. The surface partitions the halfspace $x < 0$ into boxes of rhomboid cross sections

where $f : \mathbb{C} \to \mathbb{C}^3$ is an isotropic curve given by

$$f(w) = \left(1 - \cosh(iw), 4i\cosh\left(\frac{iw}{2}\right), w + i\sinh(iw)\right). \tag{47}$$

This implies that the adjoint surface $X^*(u,v)$ of Catalan's surface has the representation

$$\begin{aligned} x^* &= \sin u \sinh v, \\ y^* &= 4\cos\frac{u}{2}\cosh\frac{v}{2}, \\ z^* &= v - \cos u \sinh v \end{aligned} \tag{48}$$

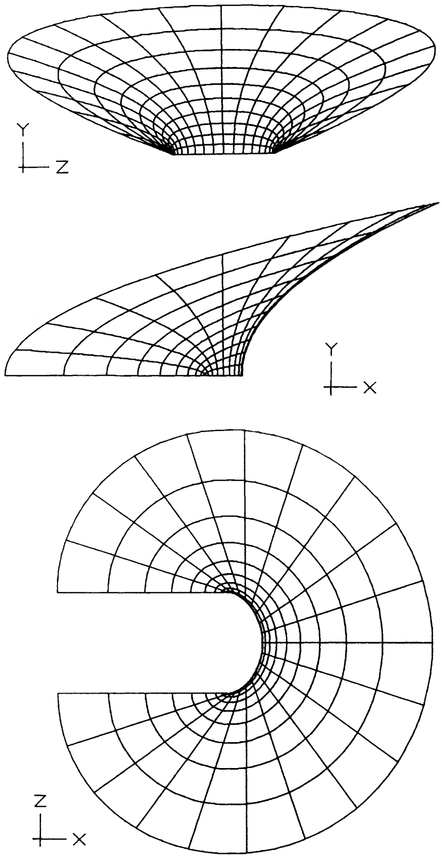


Fig. 32. Catalan's surface is made up of infinitely many copies congruent to its fundamental subset defined by $0 \le u \le 2\pi$, $0 \le v$ and shown here (for $v \le \pi$). Every curve $u =$ constant defines a parabola on the surface having its apex on the cycloid $v = 0$ along which the surface is perpendicular to the x, z-plane. The parabolas $u = 0$ and $u = 2\pi$ degenerate into straight lines, and $z = \pi$ is another plane of symmetry of the surface

with the y-axis as line of symmetry and the y, z-plane as plane of symmetry. The adjoint surface X^* intersects the plane $\{x = 0\}$ perpendicularly along the curve

$$X^*(0, v) = \left(0, 4\cosh\frac{v}{2}, v - \sinh v\right).$$

Points (x, y, z) on Catalan's surface satisfy the following inequality

$$8(x - 2) \le y^2,$$

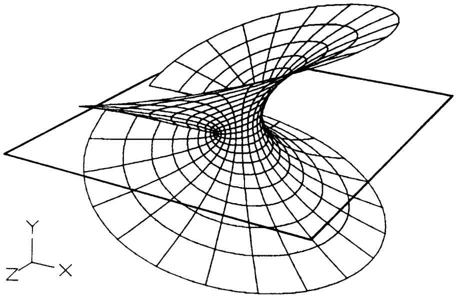


Fig. 33. Reflecting the fundamental piece defined by $0 \le u \le 2\pi$, $0 \le v$ in the x, z-plane yields the part $0 \le u \le 2\pi$ of Catalan's surface. According to the reflection principle every minimal surface which is perpendicular to a plane along a part of its boundary can be extended by reflection as a minimal surface (Section 4.8)

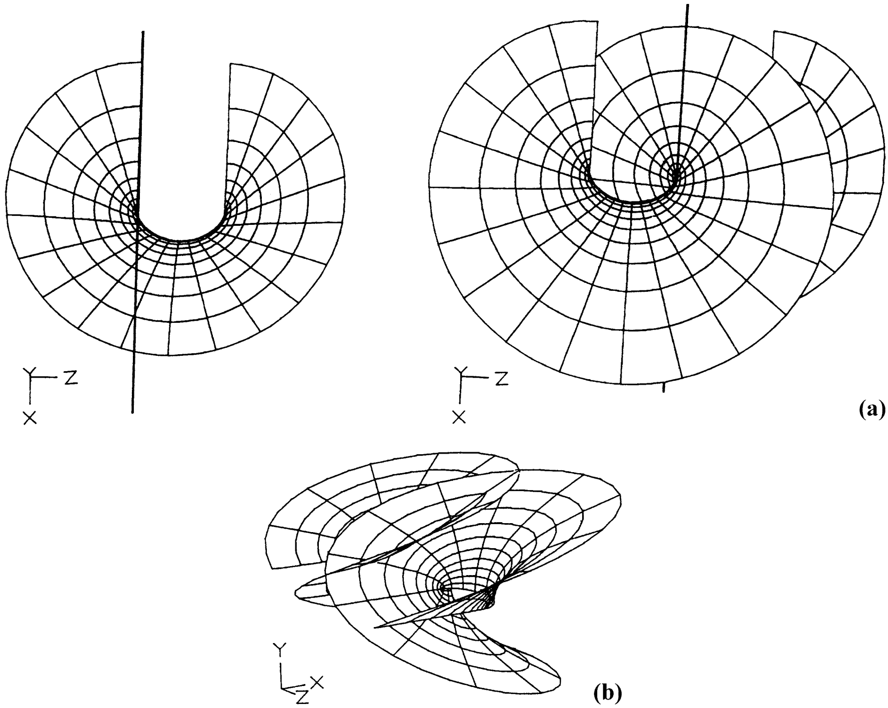


Fig. 34. **(a)** The part of Catalan's surface obtained by reflecting the fundamental piece $0 \le u \le 2\pi$, $v \ge 0$ in the x-axis. **(b)** Repetition of this reflection

i.e., the surface avoids the parabolic cylinder defined by this inequality. This is illustrated in Figs. 30 and 32; note also that the curves $u = (2k+1)\pi$, $k \in \mathbb{Z}$, lie on the boundary of the cylinder.

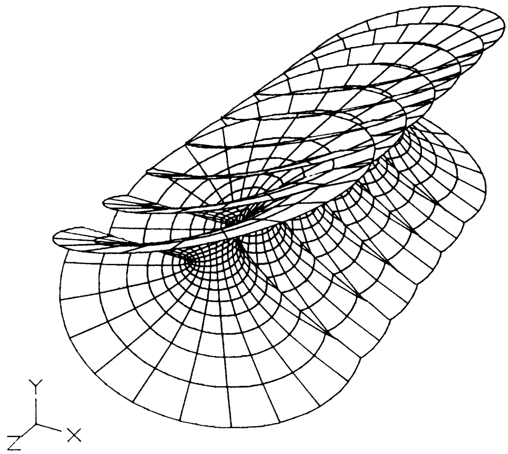


Fig. 35. Starting from the fundamental piece, the complete Catalan surface can be built by repeated reflections across straight lines and the z, x-plane

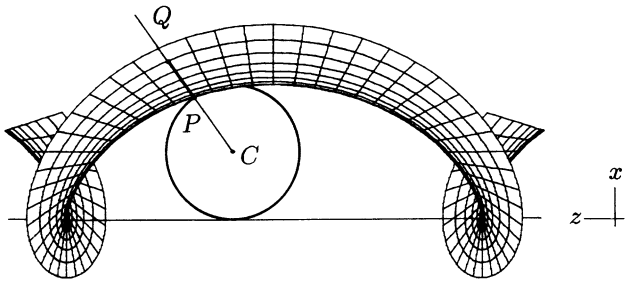


Fig. 36. Construction of Catalan's surface via a Björling problem corresponding to cycloid

The estimate can be obtained by using the following formulas for the trigonometric and hyperbolic functions:

$$\begin{aligned} y^2 &= 16\sin^2(u/2)\sinh^2(v/2) = 4(1-\cos(u))(\cosh(v)-1) \\ &= 4(-1-\cos(u)\cosh(v)+\cosh(v)+\cos(u)), \end{aligned}$$

$$x - 2 = -1 - \cos(u)\cosh(v),$$

$$\begin{aligned} \cosh(v)\cos(u) &\geq -\cos(u)\cosh(v) + \cos(u) \\ &\geq -\cos(u)\cosh(v) - 1 = x - 2, \end{aligned}$$

which clearly imply $8(x-2) \leq y^2$.

Finally we note that, except for a suitable orthogonal transformation of the Cartesian coordinates in $\mathbb{R}^3$, the Weierstrass function of Catalan's surface is of the form

$$\mathfrak{F}(\omega) = i\left(\frac{1}{\omega} - \frac{1}{\omega^3}\right).$$

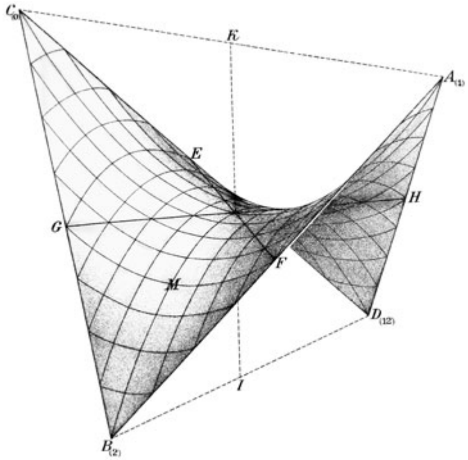


Fig. 37. Schwarz's surface. Lithograph by H.A. Schwarz

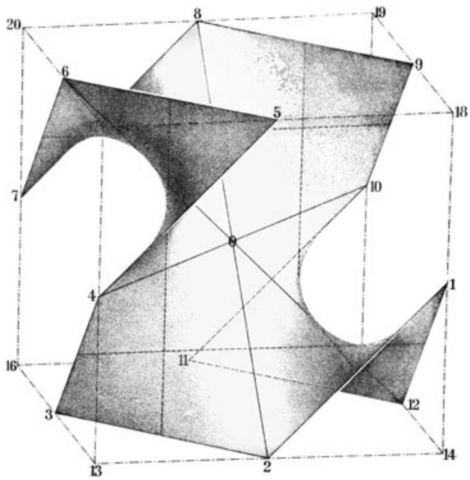


Fig. 38. Extension of Schwarz's surface by reflection. Lithograph by H.A. Schwarz

Remark. In our figures, we have instead of (48) used a translated surface, given by

$$y^* = 4\cos\frac{u}{2}\cosh\frac{v}{2} - 4. \tag{48$'$}$$

Then the origin is kept fixed if one deforms X into X^*.

3.5.9 Schwarz's Surface

This celebrated surface is a disk-type minimal surface $X : B \to \mathbb{R}^3$ which is bounded by a (nonplanar) quadrilateral Γ, see Fig. 37. By the general

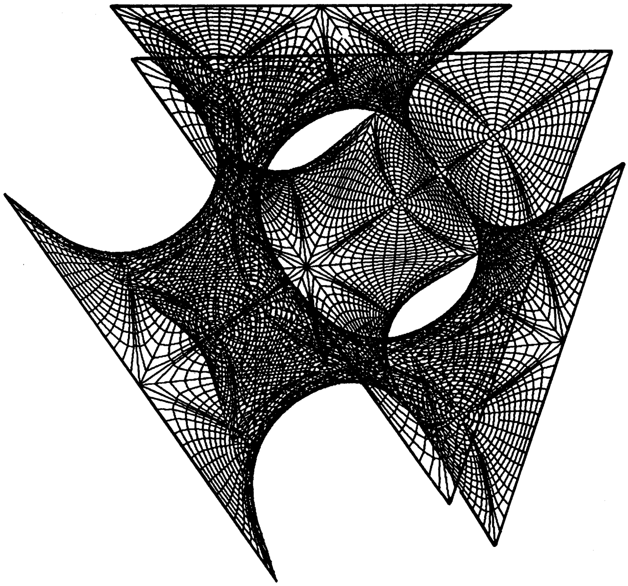


Fig. 39. A part of Schwarz's periodic surface. Courtesy of O. Wohlrab

theory to be developed in the following, there is exactly one such minimal surface which, by the reflection principle, can be continued without limit as a minimal surface if we reflect it at its boundary edges. If the edges are equally long and if the angles at the vertices are $\pi/3$, then we obtain an embedded triply-periodic minimal surface. Its adjoint surface is also triply periodic and embedded. It can be obtained by spanning a symmetric quadrilateral with two angles of $\pi/2$ and two angles of $\pi/3$. Of course, H.A. Schwarz found these two surfaces explicitly (by means of hyperelliptic integrals using the Weierstrass representation formula Section 3.3 (27) with the Weierstrass function

$$\mathfrak{F}(\omega) = \frac{\kappa}{\sqrt{1 - 14\omega^4 + \omega^8}}$$

where κ is a suitable positive constant). As this representation was carefully described by Schwarz himself (see [2], vol. 1) as well as by Bianchi [1] and Nitsche [28,37], we refer the reader to these sources for the study of the classical approach.

3.6 Complete Minimal Surfaces

In this section we want to consider *global minimal* surfaces $\mathcal{X} : M \to \mathbb{R}^3$ in $\mathbb{R}^3$ defined on Riemann surfaces M without boundary.

Let us assume that M is a two-dimensional manifold without boundary which is endowed with a *complex* (or: *conformal*) *structure* c. Such a structure

c is an atlas of charts $\varphi : G \to \mathbb{R}^2$ with the property that the transition map $\varphi \circ \tilde{\varphi}^{-1}$ between any two charts $\varphi : G \to \mathbb{R}^2$ and $\tilde{\varphi} : \tilde{G} \to \mathbb{R}^2$ is a biholomorphic mapping of $\tilde{\varphi}(G \cap \tilde{G})$ onto $\varphi(G \cap \tilde{G})$. A pair (M, c) consisting of a two-manifold M and of a complex structure c is called a *Riemann surface.*

A mapping $\mathcal{X} : M \to \mathbb{R}^3$ is *harmonic* if, for any chart $\varphi : G \to \mathbb{R}^2$, the mapping $X := \mathcal{X} \circ \varphi^{-1}$ is harmonic. Since the composition $X \circ \chi$ of a harmonic mapping X with a conformal (i.e., biholomorphic) mapping χ is also harmonic, this definition of harmonicity of $\mathcal{X}$ is compatible with the complex structure c.

Secondly, we call a nonconstant mapping $\mathcal{X} : M \to \mathbb{R}^3$ a *minimal surface* with the parameter domain M if, for any chart $\varphi : G \to \mathbb{R}^2$, the mapping $X := \mathcal{X} \circ \varphi^{-1}$ is a minimal surface in the sense of Section 2.6. That is, for any chart $\{G, \varphi\}$ of the structure c, the map $X(w) = X(u, v)$ defined by $X := \mathcal{X} \circ \varphi^{-1}$ satisfies

$$\Delta X = 0 \tag{1}$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{2}$$

Again this definition of a minimal surface is compatible with the conformal structure c of M. This can be seen as follows. The map $\Phi(w) = X_u(u, v) - iX_v(u, v), w = u + iv$, is holomorphic if and only if X is harmonic. Moreover, if $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ is holomorphic, then also

$$\langle \Phi, \Phi \rangle = \Phi_1^2 + \Phi_2^2 + \Phi_3^2$$

is holomorphic, i.e. $\langle X_w, X_w \rangle \, dw^2$ is a holomorphic quadratic differential. Thus, for any harmonic X, the equations (2) are equivalent to the fact that the holomorphic quadratic differential $\langle X_w, X_w \rangle \, dw^2$ vanishes, and we see that the equations (1) and (2) are preserved with respect to biholomorphic changes of the variables $w = u + iv$. Hence the definition of minimality is compatible with the structure c.

A minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ defined on a Riemann surface M as parameter domain will be called a *global minimal surface*.

A global minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ is said to be regular if, for any chart $\{G, \varphi\}$ of M, the surface $X = \mathcal{X} \circ \varphi^{-1}$ is regular. Moreover, $p_0 \in M$ is said to be a *branch point* of $\mathcal{X}$ if, for some chart $\{G, \varphi\}$ satisfying $p_0 \in G$, the point $w_0 = \varphi(p_0)$ is a branch point of $X = \mathcal{X} \circ \varphi^{-1}$. It can easily be seen that this definition of a branch point holds for any chart $\{G, \varphi\}$ with $p_0 \in G$ if it holds for a single one, and the order of the branch point is independent of the chart.

The Gauss map $\mathcal{N} : M \to S^2$ of a global minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ is defined by means of the charts $\{G, \varphi\}$ of the conformal structure c of M by

$$\mathcal{N}(\omega) := N(\varphi(\omega))$$

where

$$N = |X_u \wedge X_v|^{-1} X_u \wedge X_v$$

is the surface normal of $X = \mathcal{X} \circ \varphi^{-1}$. This definition of $\mathcal{N}$ holds in the classical sense if $\mathcal{X}$ is free of branch points. Otherwise, if p_0 is a branch point of $\mathcal{X}$ and $w_0 = \varphi(p_0)$, then $N(w_0)$ is defined by $N(w_0) = \lim_{w \to w_0} N(w)$, and correspondingly,

$$\mathcal{N}(p_0) = \lim_{\omega \to p_0} \mathcal{N}(\omega).$$

This definition of $\mathcal{N}$ is compatible with the structure c of M since the transition maps $\varphi \circ \tilde{\varphi}^{-1}$ between charts are biholomorphic and therefore orientation preserving.

Remark. If one admits parameter domains (M, c) with a structure c where the transition maps $\psi := \varphi \circ \tilde{\varphi}^{-1}$ are not necessarily holomorphic but either holomorphic or antiholomorphic (i.e., either ψ or $\bar{\psi}$ is holomorphic), then we include also *nonorientable parameter domains* such as the Klein bottle into the class of admissible parameter domains of minimal surfaces. For instance, the minimal surface $X : \mathbb{C} \to \mathbb{R}^3$ defined by

$$\begin{aligned} X(w) &:= \mathrm{Re}\left[\frac{i}{p(w)}(w^5 - w), -i(w^5 + w), \frac{2}{3}(w^6 + 1) \right] + \left(0, 0, \frac{1}{2}\right), \\ p(w) &:= w^6 + \sqrt{5}\, w^3 - 1, \quad w \in \mathbb{C}, \end{aligned}$$

is a minimal surface of the topological type of the projective plane (see Pinkall [1]). Its inversion in S^2, given by $Z(w) := |X(w)|^{-2} X(w)$, is a Willmore surface, i.e., a critical point of the functional $\int H^2 \, dA$ (see Fig. 1).

Again it makes sense to define minimal surfaces $\mathcal{X} : M \to \mathbb{R}^3$ by means of equations (1) and (2) which are to be satisfied by $X = \mathcal{X} \circ \varphi^{-1}$ for any chart $\{U, \varphi\}$ of the structure c. In this way we are led to nonorientable minimal surfaces such as the Henneberg surface. However, we can always pass from M to the orientable double-cover $\tilde{M}$ of M, and $\mathcal{X}$ can be lifted as a minimal surface from M to $\tilde{M}$. Thus nothing is lost if we assume in the sequel that M is orientable.

From now on we want to restrict our attention to regular and orientable global minimal surfaces $\mathcal{X} : M \to \mathbb{R}^3$. On the parameter domain M of such a manifold we can introduce a Riemannian metric $\langle\langle \xi, \eta \rangle\rangle$ as pull-back of the Euclidean metric of $\mathbb{R}^3$ to M via the mapping $\mathcal{X}$. Introducing local coordinates $w = u^1 + iu^2 = \varphi(\omega)$ by means of a chart $\{G, \varphi\}$, the *induced metric* $\langle\langle \xi, \eta \rangle\rangle$ is given by

$$\langle\langle \xi, \eta \rangle\rangle = \langle \xi^\alpha X_{u^\alpha}, \eta^\beta X_{u^\beta} \rangle \tag{3}$$

for $\xi = (\xi^1, \xi^2), \eta = (\eta^1, \eta^2)$, where $X = \mathcal{X} \circ \varphi^{-1}$. In other words, we have

$$\langle\langle \xi, \eta \rangle\rangle = g_{\alpha\beta}(w) \xi^\alpha \eta^\beta \tag{4}$$

where $g_{\alpha\beta}(w) = \langle X_{u^\alpha}(w), X_{u^\beta}(w) \rangle$.

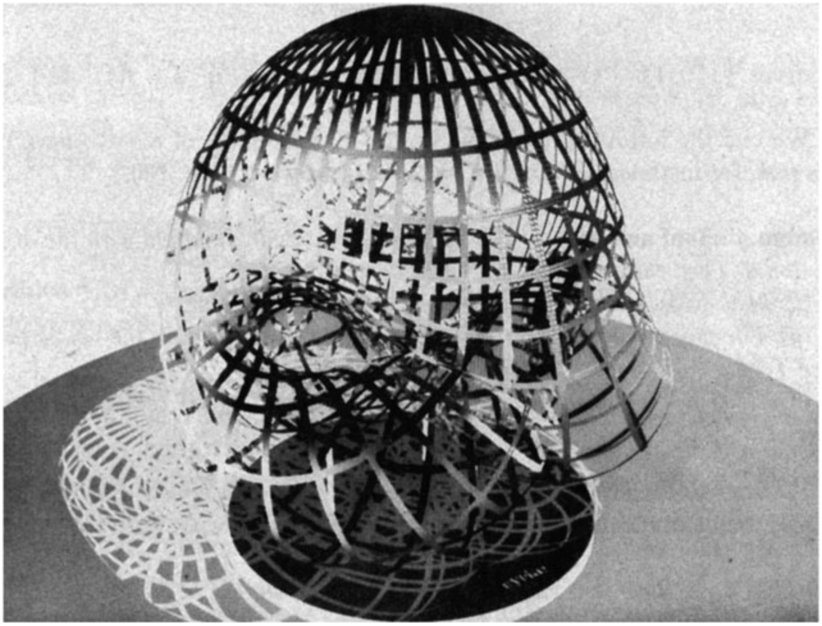


Fig. 1. A photograph of a model of the Wilmore surface $Z : \mathbb{C} \to \mathbb{R}^3$ which is exhibited at the entrance to the library of the Mathematics Research Institute Oberwolfach (Black Forest). Since $Z(\mathbb{C})$ is topologically a projective plane, the surface Z is a realization of a Boy surface (see Hilbert and Cohn-Vossen [1], pp. 276–283). Courtesy of Archive of Mathematisches Forschungsinstitut Oberwolfach

Definition 1. *A regular global minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ is said to be complete if its parameter domain M endowed with the induced Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ of $\mathbb{R}^3$ via $\mathfrak{X}$ is a complete Riemannian manifold.*

We recall that a Riemannian manifold M with a metric $\langle\langle\cdot,\cdot\rangle\rangle$ is said to be complete if it is a complete metric space with respect to its distance function $d(p, q)$. Here the distance $d(p, q)$ of any two points p, q of M is defined as infimum of the lengths

$$l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt$$

of curves $\gamma : [0, 1] \to M$ connecting p, q, i.e., $p = \gamma(0)$, $q = \gamma(1)$, and $\|\dot{\gamma}\| = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle^{1/2}$.

We cite the following criterion for the completeness of Riemannian manifolds (see, for instance, Gromoll, Klingenberg, and Meyer [1], p. 166):

Theorem of Hopf and Rinow. *Let M be a Riemannian manifold with the distance function d. Then the following statements are equivalent*:

(i) *M is complete, i.e. (M, d) is a complete metric space.*

(ii) *For any $p \in M$, the exponential map $\exp_p$ is defined on the whole tangent space T_pM.*

(iii) *If G is a bounded subset of the metric space (M, d), then its closure $\bar{G}$ is compact.*

In order to formulate another condition for completeness that will be particularly useful for the discussion of global minimal surfaces, we need the following

Definition 2. *A* **divergent path** *on a Riemannian manifold M is a continuous curve $\gamma : [0, 1] \to M$ such that, for any compact subset K of M, there is a number $t_0(K)$ such that $\gamma(t)$ is contained in the complement $M \setminus K$ for all $t > t_0(K)$.*

In other words: A divergent path on M is a ray that ultimately leaves every compact subset of M.

Proposition 1. *A Riemannian manifold M is complete if and only if every divergent C^1-path $\gamma : [0, 1) \to M$ has infinite length.*

Proof. (i) If M is complete and $\gamma : [0, 1) \to M$ is an arbitrary C^1-path of finite length, then $\gamma([0, 1))$ is bounded. Consequently, the closure of $\gamma([0, 1))$ is compact by the Hopf–Rinow theorem, and therefore γ is not divergent.

(ii) Conversely, if M is not complete, then we can find a geodesic $\gamma : [0, 1) \to M$ having $[0, 1)$ as its maximal domain of definition (to the right). The curve γ is divergent since otherwise $\lim_{t\to 1-0} \gamma(t)$ would exist and $\gamma(t)$ could be extended beyond $t = 1$. Since γ is a geodesic, its speed $\|\dot{\gamma}(t)\|$ is constant for all $t \in [0, 1)$ and therefore the length $l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt$ of γ is finite. □

Let us now consider a global minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ which is not necessarily regular. Then $\mathcal{X}$ may have isolated singularities on M, branch points, and its parameter domain M can be viewed as a generalized Riemannian 2-manifold with isolated singular points whose metric tensor $(g_{\alpha\beta}(w))$ is defined as before by $g_{\alpha\beta}(w) = \langle X_{u^\alpha}(w), X_{u^\beta}(w)\rangle$, $X = \mathcal{X} \circ \varphi^{-1}$, for any chart $\{G, \varphi\}$ of the complex structure c of M. The only difference is now that $(g_{\alpha\beta}(w))$ will vanish at points $w = w_0$ corresponding to branch points of $\mathcal{X}$. Thus the notion of the length of a curve in M retains its meaning, and the same holds for the notions distance function, closed set, compact set in M, as well as for the notion divergent path on M. This leads us to

Definition 3. *A* **divergent path** *on a global minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ is a continuous curve $\Gamma : [0, 1) \to \mathbb{R}^3$ of the form $\Gamma = \mathcal{X} \circ \gamma$ where $\gamma : [0, 1) \to M$ is a divergent path on the generalized Riemannian manifold M endowed with the metric of $\mathbb{R}^3$ via the mapping $\mathcal{X}$.*

Furthermore, Proposition 1 suggests the following

Definition 4. *A global minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ is called* **complete** *if the length of every divergent C^1-path Γ on $\mathcal{X}$ is infinite.*

Note that a regular minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ is complete in the sense of Definition 4 if it is complete in the sense of Definition 1. Thus Definition 4 can be viewed as a legitimate extension of our preceding definition of a complete global minimal surface. *In the sequel we shall drop the epithet global if we speak of a minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ with a Riemann surface M as a parameter domain.*

If one wants to consider minimal surfaces in the large, one has to deal with surfaces $\tilde{\mathcal{X}} : \tilde{M} \to \mathbb{R}^3$ which are defined on Riemann surfaces $\tilde{M}$. However, in certain situations the investigation can be simplified by passing from $\tilde{M}$ to its universal covering M which is a simply connected manifold of the same dimension as $\tilde{M}$. Any minimal surface $\tilde{\mathcal{X}} : \tilde{M} \to \mathbb{R}^3$ can be lifted from $\tilde{M}$ to M as a minimal surface $\mathcal{X} : M \to \mathbb{R}^3$, and we shall see that $\mathcal{X}$ is complete if and only if $\tilde{\mathcal{X}}$ is complete.

Recall that the universal covering of $\tilde{M}$ is, precisely speaking, a mapping $\pi : M \to \tilde{M}$ of a simply connected two-dimensional manifold M with the property that every point p of $\tilde{M}$ has a neighborhood U such that $\pi^{-1}(U)$ is the disjoint union of open sets S_i in M, called the sheets of the covering above U, each of which is mapped homeomorphically by π onto U.[3]

If $\tilde{M}$ is a Riemann surface with the conformal structure $\tilde{c}$, then π^{-1} induces a conformal structure c on M such that $\pi : (M, c) \to (\tilde{M}, \tilde{c})$ becomes a holomorphic mapping of the Riemann surface (M, c) onto the Riemann surface $(\tilde{M}, \tilde{c})$. Consequently, if $\tilde{\mathcal{X}} : \tilde{M} \to \mathbb{R}^3$ is a minimal surface with $\tilde{M}$ as parameter domain, and if $\pi : M \to \tilde{M}$ is the universal covering of $\tilde{M}$, then $\mathcal{X} := \tilde{\mathcal{X}} \circ \pi$ defines a mapping $\mathcal{X} : M \to \mathbb{R}^3$ which is again a minimal surface. We call this map the *universal covering of the minimal surface* $\tilde{\mathcal{X}}$. Note that $\mathcal{X}$ is regular if and only if $\tilde{\mathcal{X}}$ is regular, and the images of the Gauss maps $\mathcal{N}$ and $\tilde{\mathcal{N}}$ of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ coincide.

Proposition 2. *A minimal surface $\tilde{\mathcal{X}} : \tilde{M} \to \mathbb{R}^3$ is complete if and only if its universal covering $\mathcal{X} : M \to \mathbb{R}^3$ is complete.*

Proof. If $\tilde{\mathcal{X}}$ is regular, the result is an immediate consequence of statement (ii) of the Hopf–Rinow theorem since the projection $\pi : M \to \tilde{M}$ is a local isometry.

To prove the result in general, we have to use Definition 4.

Suppose first that $\mathcal{X}$ is complete. We consider an arbitrary divergent path $\tilde{\varGamma}$ on $\tilde{\mathcal{X}}$. Lifting $\tilde{\varGamma}$ to the covering surface $\mathcal{X}$, we obtain a divergent path $\varGamma$ on $\mathcal{X}$ which must have infinite length as $\mathcal{X}$ is complete. Since $\pi : M \to \tilde{M}$ is a local isometry, it follows that $\tilde{\varGamma}$ has infinite length, and we conclude that $\tilde{\mathcal{X}}$ is complete.

Conversely, let now $\tilde{\mathcal{X}}$ be complete. Consider an arbitrary divergent path $\varGamma$ on $\mathcal{X}$ given by $\varGamma = \mathcal{X} \circ \gamma, \gamma : [0, 1) \to M$. We have to show that the length of $\varGamma$ is infinite. We look at the paths $\tilde{\gamma} := \pi \circ \gamma$ on $\tilde{M}$ and $\tilde{\varGamma} := \tilde{\mathcal{X}} \circ \tilde{\gamma} = \mathcal{X} \circ \gamma = \varGamma$

[3] Concerning the universal covering we refer the reader to Weyl [4], Springer [1], Greenberg [1].

on $\tilde{\mathfrak{X}}$, respectively. If $\tilde{\gamma}$ is divergent, then the completeness of $\tilde{\mathfrak{X}}$ implies that $\tilde{\gamma}$ has infinite length whence also γ has infinite length since π is locally an isometry.

On the other hand, if $\tilde{\gamma}$ is not divergent, then there is a compact subset K of $\tilde{M}$ and a sequence of parameter values t_n in $[0, 1)$ converging to 1 such that $\tilde{\gamma}(t_n)$ belongs to K for all n. Passing to a subsequence we may assume that the points $\tilde{\gamma}(t_n)$ converge to a point $p_* \in \tilde{M}$. Then we choose a chart $\varphi : G \to \mathbb{R}^2$ around p_* such that $\varphi(p_*) = 0$, and that $\pi^{-1}(G)$ is the disjoint union of open sheets S_i. Since the branch points are isolated, there is an $\varepsilon > 0$ such that $\Omega_\varepsilon := B_\varepsilon(0) \setminus \bar{B}_{\varepsilon/2}(0)$ is contained in $\varphi(G)$ and that the metric of M is positive definite on $\varphi^{-1}(\bar{\Omega}_\varepsilon)$. Since the points $\tilde{\gamma}(t_n)$ converge to p_*, almost all of them belong to the compact set $\varphi^{-1}(\bar{B}_{\varepsilon/2}(0))$. Therefore and since γ is divergent, the points $\gamma(t_n)$ are distributed over infinitely many sheets S_i. From this fact we infer that the path $\varphi \circ \tilde{\gamma}$ has to cross Ω_ε an infinite number of times, implying that the length of $\tilde{\gamma}$ is infinite. Therefore also the length of γ is infinite.

Thus $\mathfrak{X}$ is shown to be complete if $\tilde{\mathfrak{X}}$ is complete. □

Let us note a simple but basic result on parameter domains M of global minimal surfaces $\mathfrak{X} : M \to \mathbb{R}^3$ satisfying $\partial M = \emptyset$.

Proposition 3. *The parameter domain M of a global minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ cannot be compact, i.e. there are no compact minimal surfaces.*

Proof. If M were compact, each of the components $\mathfrak{X}^j(p)$ of $\mathfrak{X}(p)$ would assume its maximum in some point $p_j \in M$, and since the functions $\mathfrak{X}^j(p)$ are harmonic on M, the maximum principle would imply that $\mathfrak{X}^j(p) \equiv \text{const}$ on M for $j = 1, 2, 3$. Since $\mathfrak{X}(p)$ is supposed to be nonconstant, this is a contradiction. □

By the uniformization theorem, a simply connected Riemann surface is either of the conformal type of the sphere S^2, or of the complex plane $\mathbb{C}$, or of the unit disk $B = \{w : |w| < 1\}$. Because of Proposition 3 the first case is excluded, and we obtain

Proposition 4. *If the parameter domain M of a global minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ is simply connected, then M is conformally equivalent to the complex plane or to the unit disk.*

A minimal surface $\mathfrak{X} : M \to \mathbb{R}^3$ is said to be of *parabolic type* if $M \sim \mathbb{C}$, and of *hyperbolic type* if $M \sim B$. If M is not simply connected, we may pass to the universal covering $\hat{\mathfrak{X}} : \hat{M} \to \mathbb{R}^3$ whose parameter domain $\hat{M}$ is simply connected, and we call $\mathfrak{X}$ to be of parabolic or hyperbolic type if its universal covering $\hat{\mathfrak{X}}$ is of parabolic or hyperbolic type respectively.

3.7 Omissions of the Gauss Map of Complete Minimal Surfaces

A minimal surface which is a graph over $\mathbb{R}^2$ is a complete minimal surface whose Gauss map omits a whole hemisphere of S^2, and Bernstein's theorem states that such a surface must necessarily be a plane. More generally one may ask how large the set of omissions of the Gauss map for an arbitrary nonplanar and complete minimal surface in $\mathbb{R}^3$ can be. In order to get a feeling for what can be true we first consider some special cases and a few examples before we state the main result of this section.

Again we shall throughout consider global minimal surfaces $\mathcal{X} : M \to \mathbb{R}^3$ whose parameter domains M are Riemann surfaces without boundary, i.e.

$$\partial M = \emptyset. \tag{1}$$

A first information is provided by the following result.

Proposition 1. *The Gauss map of a minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ of parabolic type misses at most two points unless $\mathcal{X}(M)$ is contained in a plane.*

Proof. If $\mathcal{X}$ is of parabolic type, then the corresponding universal covering $\hat{\mathcal{X}} : \hat{M} \to \mathbb{R}^3$ is defined on a parameter domain $\hat{M}$ that is conformally equivalent to the complex plane $\mathbb{C}$. Since the spherical images of $\mathcal{X}$ and $\hat{\mathcal{X}}$ are the same, it suffices to prove the following result:

Lemma 1. *The Gauss map of a minimal surface $X : \mathbb{C} \to \mathbb{R}^3$ misses at most two points if $X(\mathbb{C})$ is not contained in a plane.*

Proof. We represent X by a Weierstrass representation formula

$$X(w) = X(0) + \mathrm{Re}\left(\int_0^w \frac{1}{2}\mu(1-\nu^2)\,d\zeta, \int_0^w \frac{i}{2}\mu(1+\nu^2)\,d\zeta, \int_0^w \mu\nu\,d\zeta\right) \tag{2}$$

where $\mu(\zeta)$ is holomorphic, $\nu(\zeta)$ is meromorphic, $\mu(\zeta) \not\equiv 0, \nu(\zeta) \not\equiv 0$, and $\mu\nu^2$ is holomorphic on $\mathbb{C}$. As we have seen in Section 3.3, the meromorphic mapping ν is just the Gauss map N of X followed by the stereographic projection $\sigma : S^2 \to \bar{\mathbb{C}}$ of the Riemann sphere into the complex plane, i.e., $\nu = \sigma \circ N$. As Picard's theorem implies that ν misses at most two values of $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the assertion of the lemma follows from the representation $N = \sigma^{-1} \circ \nu$. □

Now we shall use formula (2) to construct some examples. Let Ω be the complex plane $\mathbb{C}$ or the unit disk B, and suppose that μ and $\mu\nu^2$ are holomorphic and nowhere vanishing on Ω. Then formula (2) defines a regular minimal surface $X : \Omega \to \mathbb{R}^3$ which has the line element

$$ds = \lambda|dw|, \quad \lambda = \tfrac{1}{2}|\mu|(1+|\nu|^2) \tag{3}$$

(see Section 3.3, (10)). Hence we can compare the line element ds on Ω with the ordinary Euclidean line element $|dw|$. Moreover, compact sets in (Ω, ds) correspond to compact sets in the domain Ω equipped with the Euclidean metric $|dw|$, and divergent paths in (Ω, ds) correspond to divergent paths in $(\Omega, |dw|)$, and vice versa. Recall that by definition the surface $\mathfrak{X}$ (or, equivalently, the manifold (Ω, ds)) is complete if every divergent path $\gamma : [0,1) \to \Omega$ has infinite length, that is, if

$$\int_\gamma \lambda |dw| = \frac{1}{2} \int_\gamma |\mu|(1+|\nu|^2)|dw| = \infty. \tag{4}$$

Then we obtain the following

Examples.

$\boxed{1}$ If $\mu(w) = w^2$ and $\nu(w) = p(w)/w$ where $p(w)$ is a polynomial of degree not less than two satisfying $p(0) \neq 0$, then μ and $\mu\nu^2$ are holomorphic, and ν maps $\mathbb{C}$ onto $\mathbb{C}$. Moreover, there is a number $\delta > 0$ such that $|\lambda(z)| \geq \delta$ for all $z \in \mathbb{C}$ whence

$$\int_\gamma \lambda |dw| \geq \delta \int_\gamma |dw|$$

for any path $\gamma : [0,1) \to \mathbb{C}$. By the preceding observations we infer that formula (2) defines a complete regular minimal surface $X : \mathbb{C} \to \mathbb{R}^3$ the Gauss map of which omits no points of S^2.

$\boxed{2}$ If we choose $\mu(w) = c$ and $\nu(w) = p(w)$ for some constant $c \neq 0$ and some polynomial $p(w)$ of degree at least one, then ν maps $\mathbb{C}$ onto $\mathbb{C}$, and a similar reasoning as in $\boxed{1}$ shows that (2) defines a complete regular minimal surface $X : \mathbb{C} \to \mathbb{R}^3$ whose Gauss map omits exactly one point, the north pole of S^2. In particular, if we choose $\mu(w) = \frac{1}{2}$ and $\nu(w) = w$, formula (2) yields Enneper's surface.

$\boxed{3}$ If we take $\mu(w) = 1, \nu(w) = e^w$, and $\Omega = \mathbb{C}$, then $\nu(w)$ omits exactly the value zero, and we infer that (2) defines a complete regular minimal surface $X : \mathbb{C} \to \mathbb{R}^3$ whose Gauss map omits exactly two points of S^2, the north pole and the south pole. The same holds true for the catenoid (after a suitable rotation).

$\boxed{4}$ Now we want to construct minimal surfaces $X : \Omega \to \mathbb{R}^3$ whose Gauss map omits a finite number of points. In fact, we want to prescribe a finite set $E = \{a_1, a_2, \ldots, a_{n+1}\}$ on S^2 which is to be omitted by the Gauss map of X. Without loss of generality we can assume that a_{n+1} is the north pole of S^2 as a_{n+1} can be moved into this position by a suitable rotation of $\mathbb{R}^3$. Let $w_1, w_2, \ldots, w_n, \infty$ be the images of $a_1, a_2, \ldots, a_n, a_{n+1}$ under the stereographic projection σ of S^2 onto $\mathbb{C}$. Then we choose

$$\Omega := \mathbb{C} \setminus \{w_1, w_2, \dots, w_n\}, \quad \mu(w) := \prod_{k=1}^{n} (w - w_k)^{-1}, \quad \nu(w) := w.$$

Since Ω is not simply connected, the surface $X : \Omega \to \mathbb{R}^3$ defined by (1) is multiple-valued as its values depend on the paths of integration. However, the universal covering $\hat{X} : \hat{\Omega} \to \mathbb{R}^3$ of X will be single-valued and the Gauss maps of X and $\hat{X}$ omit the same set of points E. Moreover, $\hat{X}$ is complete exactly when X is complete, and $\hat{X}$ is regular since X is a regular surface. *Thus we can construct a regular minimal surface $\hat{X} : \hat{\Omega} \to \mathbb{R}^3$ of parabolic or hyperbolic type whose spherical image is $S^2 \setminus E$, where $E = \{a_1, \dots, a_{n+1}\}$ is an arbitrarily prescribed set of points on S^2.*

Are the surfaces $\hat{X}$ constructed in this way complete surfaces? As we shall see, this is true if and only if $n \le 4$, i.e., if and only if the exceptional set E contains at most four points.

To this end we consider a curve $\gamma : [0, 1) \to \Omega$ in the parameter domain of X and the corresponding curve $\Gamma = X \circ \gamma$ on the minimal surface X. In order to show that X is complete we have to prove that the length

$$L(\Gamma) = \int_\Gamma ds = \int_\gamma \lambda(w)|dw| = \frac{1}{2} \int_\gamma |\mu|(1 + |\nu|^2)|dw|$$

of Γ is infinite if Γ is a divergent curve on X. Because of (4) we then have to show that

$$L(\Gamma) = \frac{1}{2} \int_\gamma (1 + |w|^2) \prod_{k=1}^{n} |w - w_k|^{-1} |dw| \tag{5}$$

is infinite if $\Gamma = X \circ \gamma$ is a divergent path on X.

For any $R > 0$ there is a number $\varepsilon = \varepsilon(R) > 0$ such that

$$\frac{1}{2}(1 + |w|^2) \prod_{k=1}^{n} |w - w_k|^{-1} \ge \varepsilon \quad \text{for all } w \in B_R(0). \tag{6}$$

Hence, if $\gamma(t) \in \Omega \cap B_R(0)$ for all $t \in [0, 1)$, we obtain

$$L(\Gamma) \ge \varepsilon l(\gamma)$$

where $l(\gamma) := \int_\gamma |dw|$ denotes the Euclidean length of γ. We then conclude that a divergent path $\Gamma = X \circ \gamma$ can have finite length $L(\Gamma)$ only if $l(\gamma) < \infty$; but this assumption would imply that $\gamma(t)$ converges to some point $w_0 \in \mathbb{C}$ as $t \to 1 - 0$, and since Γ is divergent, we obtain that $w_0 \notin \Omega$. We then arrive at $w_0 \in \sigma(E \setminus \{a_{n+1}\}) = \{w_1, \dots, w_n\}$, and therefore $L(\Gamma) = \infty$ on account of (4). Thus we see that a divergent path $\Gamma = X \circ \gamma$ has infinite length if $\gamma([0, 1))$ is contained in a bounded set of $\mathbb{C}$.

Suppose now that $\Gamma = X \circ \gamma$ is a divergent path such that γ is not contained in a bounded set of $\mathbb{C}$. Then either $\lim_{t \to 1-0} |\gamma(t)| = \infty$ or there are two sequences $\{t_j\}, \{t'_j\}$ of points $t_j, t'_j \in [0,1)$ such that $\lim_{j\to\infty} |\gamma(t_j)| = \infty$, whereas the sequence of points $\gamma(t'_j)$ remains bounded. In the first case, the integral (4) diverges for $n \leq 3$ while it converges if $n \geq 4$. In the second case we find that $L(\Gamma) = \infty$ since γ must cross some annulus $A := \{w \in \mathbb{C}\colon R' < |w| < R\}$ infinitely often, and we have a bound of the kind (6) on A.

Let us resume the main result of this example.

Proposition 2. *For any set E consisting of four or less points of S^2 there exists a regular, complete minimal surface $X : \Omega \to \mathbb{R}^3$ of parabolic or hyperbolic type whose Gauss map omits exactly the points of E.*

The preceding construction suggests that in general the Gauss map of a complete regular minimal surface cannot omit more than four points. Although the construction given in $\boxed{4}$ is not conclusive as there might be other choices of μ and ν leading to a complete minimal surface with the desired omission property, the result is nevertheless true and will now be stated as the main result of this section.

Theorem 1. *If $\mathcal{X} : M \to \mathbb{R}^3$ is a complete regular minimal surface such that $\mathcal{X}(M)$ is not a plane, then the Gauss map of $\mathcal{X}$ can omit at most four points.*

This result is due to Fujimoto [3]. The proof given below was found by Mo and Osserman [1] (cf. also Osserman [24]). Weaker results were earlier obtained by Osserman, Ahlfors-Osserman, and Xavier.

Before we prove Fujimoto's theorem we shall derive another result that was conjectured by Nirenberg and proved by Osserman [1]. Although it is weaker than Theorem 1, it already provides a considerable sharpening of Bernstein's theorem stated in Section 2.4.

Theorem 2. *Let $\mathcal{X} : M \to \mathbb{R}^3$ be a regular complete minimal surface such that $\mathcal{X}(M)$ is not a plane. Then the image of the Gauss map of $\mathcal{X}$ is dense in S^2.*

We remark that in this theorem the assumption of regularity can be replaced by the weaker requirement that $\mathcal{X}$ has only finitely many branch points provided that M is assumed to be simply connected. However, the result does not remain true if we admit arbitrary minimal surfaces as we can see from the following example.

$\boxed{5}$ *There exist complete nonplanar minimal surfaces the spherical images of which lie in an arbitrarily small neighborhood of the south pole of S^2.* This can be seen as follows. We set $\nu(w) = \varepsilon w$ for some $\varepsilon > 0$, and choose a holomorphic function $\mu : B \to \mathbb{C}$ of the unit disk such that

$$\int_\gamma |\mu(w)||dw| = \int_0^1 |\mu(\gamma(t))||\dot\gamma(t)|\, dt = \infty$$

holds for every divergent path $\gamma : [0, 1) \to B$. Defining $X : B \to \mathbb{R}^3$ by formula (2) we obtain a complete minimal surface whose spherical image is contained in an arbitrarily small neighborhood of the south pole provided that $\varepsilon > 0$ is sufficiently small. For the construction of such functions $\mu(w)$ we refer to Osserman's thesis [25] where it is shown that the images of the functions μ are precisely those Riemann surfaces of class A which are of hyperbolic type. In the last section of his thesis, Osserman gave a number of examples for such surfaces which, consequently, lead to implicit examples of functions μ described above.

An explicit example, pointed out by Osserman, is provided by $\mu := J' \circ F$ where J is the elliptic modular function and F a conformal map of the unit disk B onto the upper halfplane. In particular, μ maps B onto a hyperbolic Riemann surface of class A with no boundary points at finite distance.

Note that Bernstein's theorem is an immediate corollary of Theorem 2, as a nonparametric minimal surface $\mathcal{X}(x, y) = (x, y, z(x, y))$ defined for all $(x, y) \in M = \mathbb{R}^2$ is a complete regular minimal surface. Since the Gauss map of $\mathcal{X}$ maps $\mathbb{R}^2$ into a hemisphere of S^2, the set $\mathcal{X}(M)$ has to be a plane, and then a straightforward computation yields that $z(x, y)$ is an affine function, i.e.,

$$z(x, y) = ax + by + c$$

for suitable constants $a, b, c \in \mathbb{R}$.

The *proof of Theorem* 2 will be based on the following

Lemma 2. *If $f : B \to \mathbb{C}$ is a holomorphic function with at most finitely many zeros, then there is a divergent path $\gamma : [0, 1) \to B$ of class C^∞ such that*

$$\int_\gamma |f(w)||dw| < \infty.$$

Proof. If $f(w) \neq 0$, then the holomorphic mapping $F : B \to \mathbb{C}$ defined by

$$F(w) := \int_0^w f(\zeta)\, d\zeta$$

is invertible in a neighborhood of the origin in B. Let $G(z)$ be the local inverse of F around $z = 0$ which is defined on some disk $B_R(0)$, and be

$$G(z) = a_1 z + a_2 z^2 + \cdots$$

the Taylor expansion of G. We can assume R to be its radius of convergence; it could be infinite as, for instance, it is the case for $f(w) \equiv 1$. Let us introduce the set I of all $\rho \in (0, R]$ such that $G(B_\rho(0)) \subset B$ and that the mapping

$$G : B_\rho(0) \to \Omega_\rho := G(B_\rho(0))$$

is bijective. By Liouville's theorem the number

$$r := \sup I$$

is finite since G is nonconstant.

We claim that there is a point $z_0 \in \partial B_r(0)$ such that

$$\lim_{t \to 1-0} |G(tz_0)| = 1$$

which would then imply that the path

$$\gamma(t) := G(tz_0), \quad 0 \le t < 1,$$

is divergent in B, but

$$\int_\gamma |f(w)||dw| = \int_\gamma |F'(w)||dw| = \int_{F(\gamma)} |dz| = |z_0| = r < \infty$$

and the assertion of the lemma were proved.

If we could not find some $z_0 \in \partial B_r(0)$ as claimed, then for any $z_0 \in \partial B_r(0)$ we could select a sequence $\{t_n\}$ of numbers $t_n \in (0,1)$ such that $t_n \to 1-0$ and that $G(t_n z_0)$ converges to some point $w_0 \in B$. Since $F'(w_0) \neq 0$, there is a neighborhood $\mathcal{V}$ of w_0 where F is invertible. Let $\hat{G}$ be the inverse of $F|_{\mathcal{V}}$. Since

$$F(w_0) = \lim_{n\to\infty} F(G(t_n z_0)) = \lim_{n\to\infty} t_n z_0 = z_0,$$

the intersection $F(\mathcal{V}) \cap B_r(0)$ is nonempty, and $\hat{G}$ must be an extension of G to some neighborhood of z_0. By a compactness argument we infer that G admits a holomorphic extension to some disk $B_{\rho'}(0)$ such that $r < \rho' < R$ and $G(B_{\rho'}(0)) \subset B$. By the principle of unique continuation we infer that G is bijective on $B_{\rho'}(0)$ since $F(G(z)) = z$ for $z \in B_\rho(0)$ if $0 < \rho < r$. However, the existence of such a ρ' would contradict the definition of r. Thus the lemma is proved if $f(w) \neq 0$ on B.

If $f(w)$ has finitely many zeros $w_1, \dots, w_n \in B$ of order $\nu_1, \dots, \nu_n$, then the function

$$\tilde{f}(w) := f(w) \prod_{k=1}^{n} \left(\frac{1 - \bar{w}_k w}{w - w_k} \right)^{\nu_k}$$

does not vanish on B. For any $a \in B$, the transformation $w \mapsto \frac{w-a}{1-\bar{a}w}$ provides a conformal mapping of B onto itself whence $|\tilde{f}(w)| \ge |f(w)|$ on B. The preceding argument implies that there is a divergent path $\gamma : [0,1) \to B$ such that $\int_\gamma |\tilde{f}(w)||dw| < \infty$ whence $\int_\gamma |f(w)||dw| < \infty$, and the lemma is proved in the general case. $\square$

Now we turn to the

Proof of Theorem 2. Passing to the universal covering of $\mathcal{X}$, we may assume that M is equal to $\mathbb{C}$ or to $B = \{w\colon |w| < 1\}$.

If $\mathcal{X}$ is of parabolic type (i.e., $M = \mathbb{C}$), and if the spherical image of $\mathcal{X}$ is not dense in S^2, then Proposition 1 yields that $\mathcal{X}(\mathbb{C})$ is contained in an affine plane of $\mathbb{R}^3$, and since $\mathcal{X}$ is complete, the set $\mathcal{X}(\mathbb{C})$ must be the whole plane.

Suppose now that $\mathcal{X}$ is of hyperbolic type (i.e., $M = B$), and that the spherical image of $\mathcal{X}$ is not dense in S^2. Then the Gauss map of $\mathcal{X}$ misses an open set which can be assumed to be a neighborhood of the north pole. Representing $\mathcal{X}(w) = X(w)$ by formula (2) we then infer that the function $\nu(w)$ is a bounded holomorphic function on B, and the branch points of X are precisely the zeros of the holomorphic function μ. We have assumed that there are no such zeros, but we could admit finitely many. By Lemma 2 there is a divergent path γ in B such that $\int_\gamma |\mu||dw| < \infty$. On the other hand, the length $L(\Gamma)$ of $\Gamma := X \circ \gamma$ is given by

$$L(\Gamma) = \int_\Gamma ds = \frac{1}{2}\int_\gamma |\mu|(1 + |\nu|^2)|dw|$$

whence

$$L(\Gamma) \le \text{const} \int_\gamma |\mu||dw| < \infty.$$

But this result is a contradiction to the completeness of the minimal surface X which requires that any divergent path on X is of infinite length. $\square$

Now we shall outline the

Proof of Theorem 1. Suppose that $\mathcal{X} : M \to \mathbb{R}^3$ is a complete regular minimal surface whose Gauss map omits at least five points $a_1, \dots, a_5 \in S^2$. We can assume that a_5 is the north pole. Then we pass to the universal covering X of $\mathcal{X}$ which we can assume to be defined on a simply connected domain of $\mathbb{C}$. On account of Proposition 1, the surface X must be of hyperbolic type, and thus we can suppose that its parameter domain is the unit disk $B = \{w \in \mathbb{C}\colon |w| < 1\}$. In other words, we are given a complete regular minimal surface $X : B \to \mathbb{R}^3$ which is represented on B by formula (1) where $\nu(w)$ is meromorphic, $\mu(w)$ and $\mu\nu^2$ are holomorphic, and $\mu(w) \not\equiv 0$, $\nu(w) \not\equiv 0$ on B. The meromorphic function ν is just the Gauss map of X followed by the stereographic projection $\sigma : S^2 \to \bar{\mathbb{C}}$. Consequently $\nu(w)$ omits the four points $w_k := \sigma(a_k)$, $1 \le k \le 4$, and the value $\infty = \sigma(a_5)$, i.e., ν is holomorphic. Since X is regular, we have $\mu(w) \ne 0$ for all $w \in B$.

Now we want to proceed in a similar way as in the proof of Lemma 2. We define a mapping $F : B \to \mathbb{C}$ by

$$F(w) := \int_0^w f(\zeta)\, d\zeta \tag{7}$$

where f has the properties stated in Lemma 2; a specific choice will be made later on. Let $G(z)$ be the inverse of F in a neighborhood of the origin, and let r be defined as in the proof of Lemma 2. Then we have $F(G(z)) = z$ for all $z \in B_r(0)$, and there is a point $z_0 \in \partial B_r(0)$ such that $|G(tz_0)| \to 1$ as $t \to 1-0$, and that G cannot be extended to a neighborhood of z_0 as a holomorphic function.

Let us introduce the curves γ^*, γ, and Γ by setting $\gamma^*(t) := tz_0$, $0 \le t \le 1$, $\gamma := G \circ \gamma^*$, and $\Gamma := X \circ \gamma$. Then the length

$$L(\Gamma) = \frac{1}{2}\int_\gamma |\mu|(1+|\nu|^2)|dw|$$

of Γ can be expressed in the form

$$(8) \qquad L(\Gamma) = \frac{1}{2}\int_{\gamma^*} |\mu \circ G|(1+|\nu \circ G|^2)\left|\frac{dw}{dz}\right||dz|$$

where

$$\frac{dw}{dz}(z) = \frac{1}{\frac{dz}{dw}(w)} = \frac{1}{f(w)}, \quad w = G(z).$$

Now we choose the function f in the form

$$(9) \qquad f(w) := \tfrac{1}{2}\mu(w)\varphi(w)$$

where $\varphi(w)$ is to be determined later. From (7) we then infer that

$$(10) \qquad L(\Gamma) = \int_{\gamma^*} \frac{1+|\nu(G(z))|^2}{|\varphi(G(z))|}|dz|.$$

We now want to choose φ in such a way that $L(\Gamma)$ becomes finite, and since Γ is by construction a divergent path on X (see the proof of Lemma 2), this would yield a contradiction to the completeness of $X : B \to \mathbb{R}^3$.

Note that $h := \nu \circ G$ is holomorphic in $B_r(0)$ and omits at least the four values w_1, w_2, w_3, w_4. Then, for any choice of the numbers ε and ε' satisfying $0 < \varepsilon < 1$ and $0 < \varepsilon' < \frac{\varepsilon}{4}$, there is a real number b depending only on $\varepsilon, \varepsilon'$ and the points w_j such that

$$(11) \qquad \{1+|h(z)|^2\}^{(1/2)(3-\varepsilon)} \prod_{j=1}^{4} |h(z)-w_j|^{\varepsilon'-1}|h'(z)| \le \frac{2br}{r^2-|z|^2}$$

holds true for all $z \in B_r(0)$.

For the moment we shall dispense with the proof of this inequality, and we proceed with the proof of the theorem by showing that $L(\Gamma) < \infty$ for a suitable choice of φ. Choose some $\varepsilon \in (0,1)$ and set $p := 2/(3-\varepsilon)$; then we have $\frac{2}{3} < p < 1$. Now we try to choose φ in such a way that

$$(12) \qquad (\varphi \circ G)(z) = \{h'(z)\}^{-p} \prod_{j=1}^{4} [h(z)-w_j]^{p(1-\varepsilon')}$$

is satisfied. On account of (11), this would imply the inequality

$$\text{(13)} \qquad \frac{1+|\nu(G(z))|^2}{|\varphi(G(z))|} \le \left(\frac{2br}{r^2-|z|^2}\right)^p = \frac{\kappa}{(r^2-|z|^2)^p} \quad \text{for } |z|<r$$

where $\kappa := (2br)^p$, and $\frac{2}{3} < p < 1$. Then (10) and (13) would yield the desired estimate $L(\Gamma) < \infty$.

However, we have defined G as the inverse of

$$F(w) = \frac{1}{2}\int_0^w \mu(\zeta)\varphi(\zeta)\,d\zeta.$$

Thus G is defined in terms of φ, and we cannot by rights use (12) for defining φ. To remove this difficulty, we transform in (12) everything from z to w using the relations $w = G(z)$, $h(z) = \nu(G(z)) = \nu(w)$ and $h'(z) = \nu'(w)\frac{dw}{dz} = \nu'(w)/\frac{dz}{dw}$. Then (12) can be expressed in the form

$$\left(\frac{dz}{dw}\right)^{1-p} = \frac{1}{2}\mu(w)\prod_{j=1}^{4}[\nu(w)-w_j]^{p(1-\varepsilon')}\{\nu'(w)\}^{-p},$$

that is,

$$\text{(14)} \quad f(w) = \left\{\frac{1}{2}\mu(w)\right\}^{1/(1-p)}\prod_{j=1}^{4}[\nu(w)-w_j]^{p(1-\varepsilon')/(1-p)}\{\nu'(w)\}^{-p/(1-p)}.$$

On the right-hand side of (14) we only have given quantities that do not involve φ, and therefore we can use (14) to define $f(w)$ for $w \in B$ provided that $\nu'(w) \neq 0$ in B. Then $F(w)$ will be defined by (6), and G is the inverse of F. We now derive from (14) that (12) holds whence we obtain (13) and then $L(\Gamma) < \infty$.

We still have to consider the case where $\nu'(w)$ vanishes on a nonempty set Σ in B. Since X is nonplanar we have $\nu(w) \not\equiv$ const, whence $\nu'(w) \not\equiv 0$. Thus Σ is either a finite set, or it consists of a sequence of points tending to the boundary of B. If we now define $f(w)$ for $w \in B\setminus\Sigma$ by (14), and then $F(w)$ by (6), we might obtain a multivalued function which, however, can be lifted to a single-valued function $\hat{F}$ on the universal covering surface $\hat{B}$ of $B\setminus\Sigma$. The surface $\hat{B}$ is conformally equivalent to the unit disk, and the reasoning of the proof of Lemma 2 leads again to a largest disk $B_r(0)$ where the inverse $\hat{G}$ of $\hat{F}$ is defined, and to a boundary point $z_0 \in \partial B_r(0)$ which is a singular point for $\hat{G}$. Now we define a mapping $G : B_r(0) \to B\setminus\Sigma$ by $G := \pi\circ\hat{G}$ where $\pi : \hat{B} \to B\setminus\Sigma$ is the canonical projection of the universal covering $\hat{B}$ onto $B\setminus\Sigma$. Defining γ^*, γ, and Γ as before we see that $L(\Gamma) < \infty$. To obtain a contradiction we have to verify that Γ is a divergent path on X. If this were not true, we could find a sequence of points $z_n = t_n z_0$ on γ^* with $t_n \to 1-0$ such that their images $w_n = G(z_n)$ on γ converge to an interior point w_0

of B. Then w_0 cannot be contained in $B \setminus \Sigma$ on account of the reasoning of Lemma 2, and therefore w_0 must be an element of Σ, i.e., $\nu'(w_0) = 0$. Thus we have the power series expansion

$$\nu'(w) = \alpha(w - w_0)^m + \cdots$$

for some $\alpha \neq 0$ and some integer $m \geq 1$ whence

$$\{\nu'(w)\}^{p/(1-p)} = \beta(w - w_0)^{mp/(1-p)} + \cdots \quad \text{as } w \to w_0$$

where we have set $p := 2/(3-\varepsilon)$ for some fixed $\varepsilon \in (0,1)$. Note that $p/(1-p) = 2/(1-\varepsilon) > 2$.

Case (i). Suppose that $\gamma(t) \to w_0$ as $t \to 1-0$. Then we arrive at the relations

$$r = \int_{\gamma^*} |dz| = \int_\gamma |f(w)||dw| \geq c \int_\gamma |w - w_0|^{-2}|dw|$$

with a positive constant $c > 0$. Since

$$\int_\gamma |w - w_0|^{-2}|dw| = \infty$$

we have found a contradiction.

Case (ii). If $\gamma(t)$ does not tend to w_0 as $t \to 1-0$, there is another accumulation point of $\gamma(t)$ in $B \setminus \Sigma$, and the reasoning of the proof of Lemma 2 leads to a contradiction.

Thus Γ is divergent but $L(\Gamma) < \infty$, and this contradicts the completeness of X.

It remains for us to verify the estimate (11). Let Ω be the domain

$$\mathbb{C} \setminus \{w_1, w_2, w_3, w_4\}.$$

Its universal covering is conformally equivalent to the unit disk B, and the standard Poincaré metric is pulled back to a conformally equivalent metric $ds = \rho(w)|dw|$ on Ω whose Gauss curvature is equal to -1. For $\rho(w)$ we have the asymptotic expansions

$$\rho(w) \sim \frac{C_j}{|w - w_j| \log |w - w_j|} \quad \text{as } w \to w_j, \quad 1 \leq j \leq 4 \tag{15}$$

and

$$\rho(w) \sim \frac{C_0}{|w| \log |w|} \quad \text{as } w \to \infty = w_5 \tag{16}$$

where C_0 and C_j are constants different from zero (see R. Nevanlinna [1], pp. 259–260 and 250).

Now consider the function

$$\psi(w) := (1 + |w|^2)^{(3-\varepsilon)/2} \rho(w)^{-1} \prod_{j=1}^{4} |w - w_j|^{\varepsilon' - 1}, \quad w \in \Omega, \tag{17}$$

which is positive and continuous on Ω. By (15) and (16) we have $\psi(w) \to 0$ as $w \to w_j$, $j = 1, \dots, 5$. Hence $\psi(w)$ has a positive maximum on Ω, the value b of which depends only on $\varepsilon, \varepsilon'$, and $w_1, \dots, w_4$, and we therefore obtain

$$\psi(w) \le b \quad \text{for all } w \in \Omega. \tag{18}$$

Now we consider an arbitrary holomorphic function $h(z)$ in $B_r(0)$ which omits the points $w_1, \dots, w_4$, say, the function $h = \nu \circ G$ that we considered before. We lift h to a conformal mapping H from $B_r(0)$ to B and apply the Schwarz–Pick lemma to $H \circ \tau$ where τ denotes a conformal rescaling mapping which maps B onto $B_r(0)$. This lemma states that holomorphic mappings of the unit disk B into itself decrease the noneuclidean length of an arc (cf. Ahlfors [6], p. 3, Carathéodory [5], vol. 2, pp. 14–20) which implies that

$$\rho(h(z))|h'(z)| \le \frac{2r}{r^2 - |z|^2} \quad \text{for } |z| < r. \tag{19}$$

From (17) and the two inequalities (18), (19) we infer the desired estimate (11). □

A detailed exposition of Fujimoto's work, in particular on the value distribution of the Gauss map of minimal surfaces, can be found in Fujimoto [5,8].

3.8 Scholia

1 Historical Remarks and References to the Literature

In Sections 3.1–3.6 we had a glimpse at the theory of minimal surfaces developed during the 19th century. The principal tools were methods of complex analysis, conformal mappings, the Gauss map and related differential geometric ideas, symmetry arguments and geometric intuition. Hence it is no surprise that this part of the theory of minimal surfaces has always been a preferred playground of differential geometers. During the last years this classical field has experienced a remarkable revival which is to no small extent the merit of computer graphics nowadays available. By the pioneering work of David Hoffman this amazing tool has become a useful working aid and a source of inspiration.[4] In former times it was rather difficult to visualize minimal surfaces in the large and, in fact, the classical treatises do not show many figures.

[4] See Callahan, Hoffman, and Hoffman [1], Hoffman [1–5], and Hoffman and Meeks [1,2,5, 8,9,11].

This absence of figures cannot only be explained by the dislike of some of the great French mathematicians for the old custom of supporting geometric reasoning by figures.[5] An exception from the rule was H.A. Schwarz who put much effort in the construction of permanent models of minimal surfaces (see also the figures at the end of vol. 1 of his *Abhandlungen* [2]). Also the work of Neovius (cf. in particular [5]) contains beautiful illustrations. In recent years crystallographers and chemists have discovered the use of minimal surfaces for the description of complicated crystalline structures, and, in addition to the use of computer graphics, they have developed various means of visualizing these surfaces by models.

A brief survey of the history of minimal surfaces until the time of Riemann's death can be found in the introduction to Riemann's paper [2]. It is missing in the reprint included in Riemann's *Gesammelten mathematischen Werken* [2] since the editor H. Weber had decided to omit it as it was written by Riemann's student Hattendorf.

Hattendorf begins his survey with the derivation of the minimal surface equation by Lagrange (1760/61), and he mentions that Lagrange found no other solution than the plane. Then he states the contributions of Meusnier (1776): The minimal surface equation is equivalent to $H = 0$ and has the catenoid and the helicoid as solutions. Moreover, he mentions the integration of the minimal surface equation by Monge (1784) and Legendre (1787) as well as a basic discovery by Dupin (1813): The asymptotic lines of a minimal surface are perpendicular to each other and enclose angles of 45 degrees with the lines of curvature.

The representation formulas of Monge and Legendre were, as Hattendorf remarks, not well suited for deriving other specific minimal surfaces besides the helicoid and the catenoid found by Meusnier. New surfaces were first derived by Scherk (in his prize-essay for the Jablonowski Society at Leipzig, 1831) by a kind of separation of variables. A similar approach was followed by Catalan (1858), and Hattendorf also mentions that, in two papers from 1842 and 1843, Catalan showed that the helicoid is the only ruled minimal surface (apart from the plane). Then Hattendorf discusses the solution of Björling's problem by Björling (*Grunert's Archive*, vol. 4, 1843) and later by Bonnet (*Comptes Rendus* 1853, 1855, 1856; *Liouville's Journal* 1860). He mentions that Bonnet investigated asymptotic lines, lines of curvature and geodesic lines on minimal surfaces and that he looked for those surfaces of zero mean curvature which satisfy certain geometric conditions. For instance, the surface might be generated by a curve via a screw motion, it might have plane lines of curvature, or it might pass through given lines. Of this latter problem, Bonnet

[5] Lagrange wrote in the preface to his *Mécanique analytique* (second edition, vol. 1, 1811): On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou mécaniques, mais seulement des opérations algébriques, assujéties à une marche regulière et uniforme. Ceux qui aiment l'Analyse, verront avec plaisir la Mécanique en devenir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine.

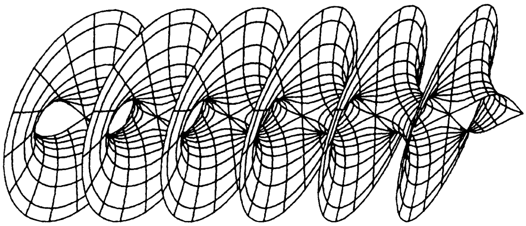


Fig. 1. Riemann's periodic minimal surface: an example with a translational symmetry. Courtesy of K. Polthier and M. Wohlgemuth

treated the problem of finding minimal surfaces containing a given strip or passing through two intersecting straight lines; the last question was also investigated by Serret (1855). Hattendorf closes his report with the remark that nothing more is known on minimal surfaces with given boundaries, and he states that Bonnet stopped at that point where the true problem begins, namely, the investigation of the limit and discontinuity properties. And then: *Diese Untersuchung gehört ihrem Wesen nach in die von Riemann geschaffene Theorie der Funktionen von complexen Variablen.*

A new period in the theory of minimal surfaces began in 1865 with the solution of Plateau's problem by H.A. Schwarz in the case that the boundary curve is a regular quadrilateral, and, in 1867, for the general quadrilateral (see Schwarz [2], vol. 1, pp. 1–91). These papers are based on the representation formulas for minimal surfaces derived in Section 3.3. Weierstrass had lectured on these formulas at the Mathematical Seminar of Berlin University as early as 1861, and he reported them to the Berlin Academy in 1866 (see Weierstrass [2–4]). Somewhat different representation formulas were stated by Enneper [1] in 1864 who used the lines of curvature as parameter lines $u = \text{const}$ and $v = \text{const}$ on a minimal surface. Other representation formulas were introduced by Weingarten (1863), Riemann (1866), Peterson (1866) and Beltrami (1868).[6]

Riemann's posthumous paper [2], published in 1867, treated minimal surfaces passing through one or several straight lines. In particular, it dealt with the following special boundaries: (i) Two infinitely long, skew straight lines. (ii) Three straight lines, two of which lie in a plane E and intersect; the third lies in a plane E' parallel to E. (iii) Three intersecting straight lines. (iv) A quadrilateral. (v) Two arbitrary circles which lie in parallel planes.

Already in 1866, Weierstrass [1] reported in a lecture to the Academy that he was able to solve Plateau's problem for an arbitrary [unknotted] polygonal boundary, but the details appeared only about thirty years later (cf. Weierstrass [4]).

The whole development can be studied in the first volume of Schwarz's *Abhandlungen* published in 1890 and exclusively dedicated to the study of

[6] For references, see R.v. Lilienthal: *Besondere Flächen.* Encyklopädie der Mathematischen Wissenschaften III.3, pp. 307–333, in particular pp. 310–315.

minimal surfaces. In several later supplements and annotations to his papers and to the whole volume Schwarz gives a very clear picture of what was known in his time. Particularly interesting is his report *Miscellen aus dem Gebiete der Minimalflächen* (see [2], pp. 168–189 and 325–333).

A comprehensive presentation of the whole field can be found in Darboux's *Leçons* [1]. (Especially relevant to the field of minimal surfaces are vols. 1 and 3.)

A brief but very readable description of the Schwarz–Riemann–Weierstrass approach to the solution of Plateau's problem for polygonal boundaries is given in chapters 14 and 15 of Bianchi's treatise [1]. The main topic of chapter 15 is the construction of Schwarz's minimal surface spanning a quadrilateral and a discussion of its properties and of its adjoint surface.

The knowledge available at the turn of this century is surveyed in Lilienthal's encyclopedia article [1].

An extensive presentation from the modern point of view can be found in Nitsche's treatise [28] (see also [37]); it is at the same time a rich source of bibliographic and historical references.

During the years 1900–1925 not much progress was made in the theory of parametric minimal surfaces apart from work of Neovius on periodic minimal surfaces which, however, is largely an extension of his earlier work carried out in the nineteenth century. The essential though indirect contributions of that period to the theory of minimal surfaces were the development of a powerful measure and integration theory by Lebesgue, of the direct methods by Hilbert, Lebesgue, Courant, Tonelli, and the foundation of functional analysis by Hilbert, F. Riesz, E. Schmidt, Fréchet, Hahn, and Banach. Moreover, the basic techniques of the theory of elliptic equations, regularity theorems and a priori estimates, were created by Korn, S. Bernstein, Lyapunov, Müntz and Lichtenstein in those years. The noteworthy results of S. Bernstein concern nonparametric minimal surfaces. Between 1925 and 1950 the theory of minimal surfaces sprang to new life; the following two chapters will give an impression of the achievements in that period. From then on boundary value problems for minimal surfaces have stood in the center of interest. In the sixties, DeGiorgi, Fleming, Federer, and Reifenberg developed the powerful tool of geometric measure theory which since then has become more and more important for the study of minimal surfaces.

For some time the Weierstrass–Schwarz theory of minimal surfaces moved into the background, and mainly the pioneering work of Osserman on complete minimal surfaces showed its usefulness and importance; in this respect we also mention the interesting contributions by Leichtweiß, Nitsche and Voss from that period, the main results of which are presented in Osserman's survey [10] which had a great influence on the subsequent development. We also refer to chapter 8 of Nitsche's *Vorlesungen* [28].

Thereafter, the interest in this area seemed more or less exhausted despite some interesting contributions by Gackstatter and the exciting discoveries of new triply periodic minimal surfaces by the physicist Alan Schoen (about

1970); their existence, however, seemed not to be sufficiently rigorously established. At the beginning of the 1880ties, the theory of complete and of periodic minimal surfaces gathered new speed. This is particularly the merit of Costa, D. Hoffman and Meeks who disproved a longstanding conjecture according to which the only complete embedded minimal surfaces in $\mathbb{R}^3$ of *finite topological type* are the *plane*, the *catenoid*, and the *helicoid*. This conjecture turned out to be false as there is a complete minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ defined on the square torus $\mathbb{C}/\mathbb{Z}^2$ with three points removed. This surface was discovered by Costa [1,2]. Its representation formula (7) in Section 3.3 uses the functions $\mu = \wp$ and $\nu = a/\wp'$ where $\wp$ is the Weierstrass p-function, $\wp'$ its derivative, and a denotes some constant $\neq 0$. Costa showed that $\mathcal{X}$ is a complete surface of genus one with three ends; Hoffman and Meeks proved that it is an embedded surface. Later on, many more similar surfaces were found, so that today a fascinating new theory is developing. We shall collect a few results in the next subsection. A second major achievement is the verification of A. Schoen's examples of triply periodic minimal surfaces by Karcher, see Section 3.5. However, many more beautiful and fascinating new examples of embedded minimal surfaces have recently been discovered, and the subject is still growing fast. Another 200–300 pages (or more) would be needed to do it justice. Thus we have to content ourselves with mentioning a few survey papers and some comprehensive presentations.

At an early stage, the development was documented in the lecture notes of Barbosa and Colares [1]. In his paper [1], Karcher showed how more embedded minimal surfaces can be derived from some of the Scherk examples, and in [2] he established the existence of Alan Schoen's triply periodic minimal surfaces. The reader should begin by studying Karcher's lecture notes [3] where he outlines devices to construct interesting examples of increasing topological complexity. Then we refer to the works of D. Hoffman and Meeks cited in our bibliography. Particularly, we mention Hoffman [1–5], Hoffman and Meeks [11], Meeks [6,7], Hoffman and Wohlgemuth [1], Wohlgemuth [1], and Polthier [1,2].

Lately crystallographers have showed much interest in triply periodic minimal surfaces, and they have very much stimulated recent developments. We especially refer the reader to the works of Sten Andersson, Blum, Bovin, Eberson, Ericsson, Fischer, Hyde, Koch, Larsson, Lidin, Nesper, Ninham, and v. Schnering—cited in our bibliography—where many beautiful surfaces are depicted.

The following collection of results is mainly drawn from the papers of Osserman, Karcher, Hoffman and Meeks quoted above.

2 Complete Minimal Surfaces of Finite Total Curvature and of Finite Topology

The first basic results on complete minimal surfaces of finite total curvature are due to Osserman; an excellent presentation is given in §9 of Osserman's survey [10].

Theorem 1. *Let M be a complete, orientable Riemannian two-manifold whose Gauss curvature K satisfies $K \le 0$ and $\int_M |K|\,dA < \infty$. Then there exist a compact Riemannian two-manifold $\tilde{M}$ and a finite number of points $p_1, \dots, p_k$ in $\tilde{M}$ such that M and $\tilde{M}' := \tilde{M} \setminus \{p_1, \dots, p_k\}$ are isometric. In other words, there is a length preserving diffeomorphism from M onto $\tilde{M}'$.*

As a consequence of this result we obtain

Theorem 2. *A complete regular minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ of finite total curvature $\int_M |K|\,dA$ defined on an orientable parameter manifold M is conformally equivalent to a compact Riemann surface $\mathcal{R}$ that has been punctured in a finite number of points.*

That means:

(K1) *Complete orientable minimal surfaces without branch points and of finite total curvature can be assumed to be parametrized on parameter domains $M = \mathcal{R} \setminus \{p_1, \dots, p_k\}$ which are compact Riemann surfaces $\mathcal{R}$ with k points removed $(k \ge 1)$.*

Definition 1. *A two-manifold is said to have finite topology if it is homeomorphic to a compact two-manifold from which finitely many points are removed. Correspondingly, a surface $\mathcal{X} : M \to \mathbb{R}^3$ is said to be of finite topology if its parameter manifold M has finite topology.*

Then property (K1) states that a complete minimal surface of finite total curvature has necessarily finite topology. However, the converse is not true as one can see from the helicoid. This minimal surface has the complete plane $\mathbb{C}$ as parameter domain which is conformally equivalent to the once punctured sphere. As the helicoid is periodic and not flat, its total curvature is infinite. (Note, however, that this example is somewhat artificial because of its periodicity, and a suitable, more stringent definition of finite topology dividing out the periodicities would remove the helicoid from the list of examples.) Meeks and Rosenberg [1,3] proved that the only complete, embedded, simply connected and periodic minimal surface is the helicoid.

Until recently, the plane, the catenoid, and the helicoid were the only known examples of complete embedded minimal surfaces with a finite topology. The first new example depicted in Fig. 20 (see also the frontispiece) is the Costa surface whose embeddedness was proved by Hoffman and Meeks. It is conformally a torus punctured in three points. More complicated examples of higher genus were discovered by D. Hoffman and Meeks. A sample is depicted in Plate II.

Let $\mathcal{R}$ be a compact Riemann surface (without boundary), and $p_1, \dots, p_k$ a finite number of points in $\mathcal{R}$. We consider a regular minimal surface $\mathcal{X} : M \to \mathbb{R}^3$ of finite topology, defined on $M := \mathcal{R} \setminus \{p_1, p_2, \dots, p_k\}$.

The image $E_j := \mathcal{X}(B_j')$ of a punctured disk neighborhood $B_j' = B_j \setminus \{p_j\}$ of p_j is called an *end* of the surface $\mathcal{X}$.

What can one say about the behavior of $\mathcal{X}$ at its ends? Some answers should be obtainable from information about the behavior of the Gauss map $\mathcal{N} : M \to S^2 \subset \mathbb{R}^3$ of $\mathcal{X}$ at the ends E_j, that is, from the meromorphic function $\nu := \sigma \circ \mathcal{N}$ obtained by composing $\mathcal{N}$ with the stereographic projection $\sigma : S^2 \to \bar{\mathbb{C}}$. Let η be the holomorphic 1-form on M associated with $\mathcal{X}$ which in local coordinates w is given by $\eta(w) = \mu(w)dw$ (here $\mu(w)$ is the function from the representation formula (7) in Section 3.3). Then we have the following basic information (see Osserman [5,10]):

Theorem 3. *Let $\mathcal{X} : M \to \mathbb{R}^3$ be a complete regular minimal surface of finite total curvature $\int_M K\,dA$; for the sake of brevity we call such a mapping a* (K1)*-surface. Then we have*:

(K2) *The meromorphic function $\nu : M \to \bar{\mathbb{C}}$ extends to a meromorphic function on $\mathcal{R}$ and the holomorphic* 1*-form η on M extends to a meromorphic* 1*-form on $\mathcal{R}$.*

(K3) *The number $m := \frac{1}{4\pi}\int_M K\,dA$ is an integer satisfying $m \leq -(\boldsymbol{g} + k - 1)$ where $\boldsymbol{g}$ is the genus of M and k is the number of puncturing points in $\mathcal{R}$.*

(K4) *The mapping $\mathcal{X} : M \to \mathbb{R}^3$ is proper* (*i.e., pre-images of compact sets in $\mathbb{R}^3$ are compact sets in M*)*.*

Further properties of (K1)-surfaces $\mathcal{X} : M \to \mathbb{R}^3$

(K5) *Set $S_j(R) := \{Q \in \mathbb{R}^3 : RQ \in E_j \text{ and } Q \in S^2\}$. Then $S_j(R)$ converges smoothly as $R \to \infty$ to a great circle on S^2 covered an integral number of times, say, d_j times. Moreover, we have*

$$\int_M K\,dA = 4\pi\left\{1 - \boldsymbol{g} - k - \sum_{j=1}^{k}(d_j - 1)\right\}, \quad \boldsymbol{g} = \text{genus}(M)$$

(see Jorge and Meeks [1], Gackstatter).

(K6) *Denote by $n(\mathcal{X}) := \sum_{j=1}^{k} d_j$ the total spinning of $\mathcal{X}$; clearly, $n(\mathcal{X}) \geq k$. Then we have*: $n(\mathcal{X}) = k \Leftrightarrow \int_M K\,dA = -4\pi(\boldsymbol{g} + k - 1) \Leftrightarrow$ *all of the ends of $\mathcal{X}$ are embedded* (*that is, for each $j = 1, \dots, k$, the map $\mathcal{X}$ embeds some punctured neighborhood of p_j*) (see Jorge and Meeks [1]).

(K7) *Let E_j be an embedded end corresponding to the puncture p_j. The Gauss map $\mathcal{N} : M \to S^2$ can be extended continuously from M to $\mathcal{R}$* (see (K2))*. Assume that $\mathcal{N}(p_j) = (0, 0, 1)$. Then outside of a compact set, the end E_j has the asymptotic behavior*

$$z(x, y) = \alpha \log r + \beta + r^{-2}(\gamma_1 x + \gamma_2 y) + O(r^{-2})$$

as $r = \sqrt{x^2 + y^2} \to \infty$ (see R. Schoen [3]).

We call the end E_j *flat* or *planar* if $\alpha = 0$; for $\alpha \neq 0$ we speak of a *catenoid end*. This means that, far out, all (K1)-surfaces look at their embedded ends either like planes or like half catenoids.

(K8) *If* $\mathcal{X} : M \to \mathbb{R}^3$ *is an embedded* (K1)*-surface of genus* $\boldsymbol{g}$ *with* k *ends, then we have*:

(i) *If* $\boldsymbol{g} = 0$, *then* $k \neq 3, 4, 5$ (*Jorge and Meeks* [1]). *In fact,* $\boldsymbol{g} = 0$ *implies that* $\mathcal{X}$ *is a plane* ($k = 1$) *or a catenoid* ($k = 2$) (*Lopez and Ros* [1]).

(ii) *If* $k = 1$, *then* $\mathcal{X}(M)$ *is a plane* (*see, e.g. Hoffman and Meeks* [8]).

(iii) *If* $k = 2$, *then* $\mathcal{X}(M)$ *is a catenoid* (*R. Schoen* [3]).

Property (ii) *follows from the strong halfspace theorem stated below.*

(K9) *The plane has total curvature* 0, *the catenoid* -4π; *all other embedded* (K1)*-surfaces have a total curvature of less than or equal to* -12π (Hoffman and Meeks [8]).

(K10) *The Costa surface* $\mathcal{X}$ *is an embedded* (K1)*-surface of genus* 1 *with three ends and total curvature* -12π. *One end is flat, the other two are catenoid ends. The function* $\nu = \sigma \circ \mathcal{N}$ *is of the form* $\nu = a/\wp'$ *where* $\wp$ *is the Weierstrass p-function and a is a constant. The Costa surface contains two straight lines intersecting perpendicularly; moreover, it can be decomposed into eight congruent pieces, each of which lies in a different octant and each of which is a graph* (Hoffman and Meeks [1]). *Generalizing the Costa example, Hoffman and Meeks were able to show that, for any genus* $\boldsymbol{g} \geq 1$, *there is an embedded* (K1)*-surface with one flat end and two catenoid ends. The total curvature* $\int_M K\,dA$ *of this surface is* $-4\pi(\boldsymbol{g}+2)$. *In fact, each of these examples belongs to a* 1*-parameter family of embedded minimal surfaces* (Hoffman [4], Hoffman and Meeks [7]).

A sample of a Hoffman–Meeks surface is depicted in Plate II.

We mention that the underlying Riemann surface $\mathcal{R}$ is the $(\boldsymbol{g}+1)$-fold covering of the sphere given by $\zeta^{\boldsymbol{g}+1} = w^{\boldsymbol{g}}(w^2 - 1)$ punctured at $w = \pm 1$ and $w = \infty$.

(K11) Callahan, Hoffman, and Meeks [3] constructed *examples of embedded* (K1)*-surfaces with four ends, two of which are flat, the others catenoidal.* Following a suggestion of Karcher, Wohlgemuth and Boix constructed many more examples of increasing complexity.

3 Complete Properly Immersed Minimal Surfaces

A very useful result proved by means of the maximum principle is the following

Halfspace Theorem (Hoffman and Meeks [4,10]**).** *A complete, properly immersed minimal surface* $\mathcal{X} : M \to \mathbb{R}^3$ *cannot be contained in a half space, except for a plane.*

(An immersed minimal surface is a surface without branch points, and properly means that the pre-image of any compact set on $\mathcal{X}(M)$ is a compact subset of M.)

Note that the assumption of properness cannot be omitted as Jorge and Xavier [1] exhibited examples of complete minimal surfaces $\mathcal{X} : M \to \mathbb{R}^3$ contained between two parallel planes; see also Rosenberg and Toubiana [1].

A strengthening of the previous result is the **strong halfspace theorem** (Hoffman and Meeks [4,10]): *Two complete, properly immersed minimal surfaces* $\mathcal{X} : M \to \mathbb{R}^3$ *must intersect if they are not parallel planes.*

4 Construction of Minimal Surfaces

The material of this subsection is essentially drawn from Karcher's excellent lecture notes [3] to which the reader is referred for details. We adjust our notation from Chapter 3 to that of Karcher [3] so that we can immediately use Karcher's formulas. A very detailed presentation of the following material and of related topics is given in the encyclopaedia article by D. Hoffman and H. Karcher [1]; see [EMS].

Let us recall the representation formula (7) of Section 3.3 for a minimal surface $X : \Omega \to \mathbb{R}^3$ by means of a holomorphic function $\mu(w)$ and a meromorphic function $\nu(w)$ on Ω:

$$X(w) = X(w_0) + \mathrm{Re} \int_{w_0}^{w} \psi'(\zeta)\, d\zeta \tag{1}$$

where ψ is defined by

$$\psi' = \left(\frac{1}{2}\mu(1-\nu^2), \frac{i}{2}\mu(1+\nu^2), \mu\nu\right). \tag{2}$$

If we introduce the two meromorphic functions g and h by

$$g := \nu, \quad h' := \mu\nu, \tag{3}$$

we have

$$dh = \mu\nu\, d\zeta,$$

and we can write (2) as

$$d\psi = \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1\right) dh. \tag{4}$$

Clearly, the functions ψ and h are multiple-valued while the 1-forms $d\psi$ and dh are single-valued on Ω, and mutatis mutandis Ω can be replaced by a domain on a Riemann surface.

The Gauss map $N : \Omega \to S^2$ associated with X is given by

$$N = \frac{1}{|g|^2 + 1}(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1). \tag{5}$$

The line element ds of $X : \Omega \to \mathbb{R}^3$ can be written as

$$ds = \frac{1}{2}\left(|g| + \frac{1}{|g|}\right)|dh| \tag{6}$$

and the Gauss curvature K has now the form

$$K = -16\left(|g| + \frac{1}{|g|}\right)^{-4}\left|\frac{dg}{g}\right|^2 |dh|^{-2}. \tag{7}$$

For $w = u + iv$ and for a tangent vector $W \in T_w\Omega = \mathbb{C}$, the second fundamental form $\mathrm{II}(W, W)$ can be written as

$$\mathrm{II}(W, W) = \operatorname{Re}\left\{\frac{dg}{g}(W) \cdot dh(W)\right\}. \tag{8}$$

Moreover, W *describes an asymptotic direction exactly if* $\frac{dg}{g}(W) \cdot dh(W) \in i\mathbb{R}$, *and* W *is a principal curvature direction if and only if* $\frac{dg}{g}(W) \cdot dh(W) \in \mathbb{R}$.

The **reflection principles** yield: *If a straight line or a planar geodesic lies on a complete minimal surface, then the* $180°$*-rotation around the straight line or the reflection at the plane of the planar geodesic respectively is a congruence of the minimal surface.*

This observation has the following useful *application*: If there is a line $\gamma : I \to \Omega$ such that the stereographic projection $g \circ \gamma : I \to \mathbb{C}$ of its Gauss image is contained in the stereographic projection of a meridian or of the equator of S^2, and if also $h' \circ \gamma$ is contained in the stereographic projection of a meridian of S^2, then analytic reflection at γ does not change the values of $|g| + \frac{1}{|g|}$ and of $|h'|$, nor does it change the Euclidean metric $|dw|$. Therefore this reflection is a Riemannian isometry for the metric (6) and, consequently, the curve γ defines a geodesic $c := X \circ \gamma$ on the minimal surface. Moreover, $g \circ \gamma$ corresponds either to a meridian of S^2 or to its equator.

The following constructions will be based on Osserman's results described in Subsection 2 of these Scholia. The guiding idea is to describe meromorphic Weierstrass data g and h on Riemann surfaces M which are punctured Riemann surfaces $\mathcal{R}$, i.e., $M = \mathcal{R} \setminus \{p_1, p_2, \dots, p_k\}$.

A translational symmetry of the minimal surface generated by integrating its Weierstrass data around a homotopically nontrivial loop on M is called a **period of the Weierstrass data**. Integration of the Weierstrass data leads to a single-valued minimal surface $X(w) = \operatorname{Re}\psi(w)$ if all periods $P = (P_1, P_2, P_3)$ vanish or, more generally, if the components of all periods are purely imaginary (i.e., $P \in i\mathbb{R}^3$).

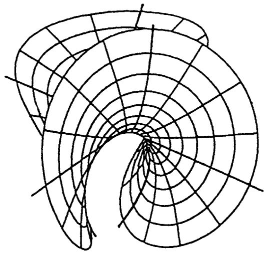


Fig. 2. (A1) Enneper's surface: $g(w) = w$. Courtesy of K. Polthier

Proposition 1. *If a line of symmetry L passes through a puncture, then we can consider closed curves around the puncture p which are symmetric with respect to L. The integrated curve on the minimal surface then consists of two congruent parts which are symmetric either with respect to a reflection plane E or with respect to the axis A of a $180°$-rotation. The period P is the difference vector between the two pieces of the curve; thus it is perpendicular either to E or to A.*

This observation can sometimes be used to show without computation that some punctures cause no periods, for instance, if two nonparallel symmetry planes pass through the punctures.

A very useful tool for proving embeddedness of surfaces is the following theorem presented at the end of Section 3.3:

Theorem of R. Krust. *If an embedded minimal surface $X : B \to \mathbb{R}^3$ can be written as a graph over a convex domain of a plane, then the corresponding adjoint surface $X^* : B \to \mathbb{R}^3$ is also a graph.*

Now we turn to the discussion of specific examples.

A. Minimal Surfaces Parametrized on Punctured Spheres

(A1) *Enneper's surface*. Here we have

$$
\begin{aligned}
&g(w) = w, \quad dh = w\,dw, \quad w \in \mathbb{C}, \\
&\psi(w) = \tfrac{1}{2}\big(w - \tfrac{1}{3}w^3, i\big(w + \tfrac{1}{3}w^3\big), w^2\big).
\end{aligned}
$$

Reflections in straight lines through 0 are Riemannian isometries for the corresponding metric

$$
ds = \frac{1}{2}\left(|w| + \frac{1}{|w|}\right)|w||dw|.
$$

All these radial lines are therefore geodesics, and rotation about the origin is an isometry group. Moreover, $\mathbb{R}$ and $i\mathbb{R}$ are planar symmetry lines, and the $45°$-meridians are straight lines on Enneper's surface. The Riemannian

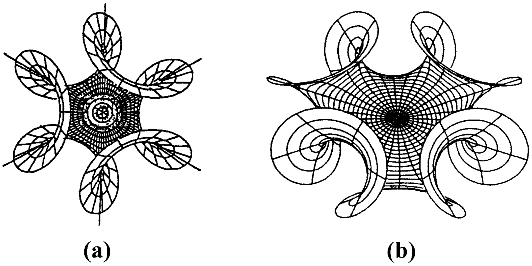


Fig. 3. (A2) Higher order Enneper surfaces. **(a)** $g(w) = w^2$. With courtesy of K. Polthier. **(b)** $g(w) = w^3$. Courtesy of J. Hahn and K. Polthier

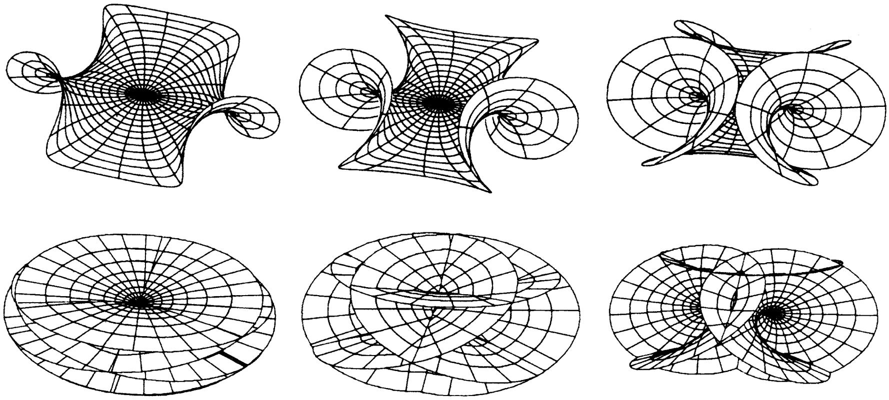


Fig. 4. A view of increasing parts of a higher order Enneper surface ($g(w) = w^2$) from an increasing distance. Courtesy of J. Hahn and K. Polthier

metric ds is complete on $M := \mathbb{C} \cong S^2 \setminus \{\text{north pole}\}$ and nondegenerate, i.e., Enneper's surface is a regular minimal surface. Moreover, all associate surfaces of Enneper's surface are congruent. Circles $\gamma(\varphi) = Re^{i\varphi}$ of sufficiently large radius R are mapped to curves $c(\varphi) = \mathrm{Re}\,\psi(\gamma(\varphi))$ which wind three times about the z-axis. Therefore the end of Enneper's surface is not embedded, but $d = 3$.

(A2) *Higher order Enneper surfaces* are defined by

$$g(w) = w^n, \quad dh = w^n dw, \quad w \in \mathbb{C}, \quad n = 1, 2, 3, \ldots,$$

and they allow the same reasoning. However, we have more symmetry lines, and the end winds $(2n + 1)$-times about the z-axis ($d = 2n + 1$).

Interesting deformations can be obtained in the form

$$g(w) = w^n + tp(w), \quad dh = g(w)dw, \quad w \in \mathbb{C},$$

where $t \in \mathbb{R}$, and $p(w)$ is a polynomial of degree $\leq n - 1$. These surfaces are regular and have the same behavior at their ends as the corresponding higher order Enneper surfaces given by $t = 0$.

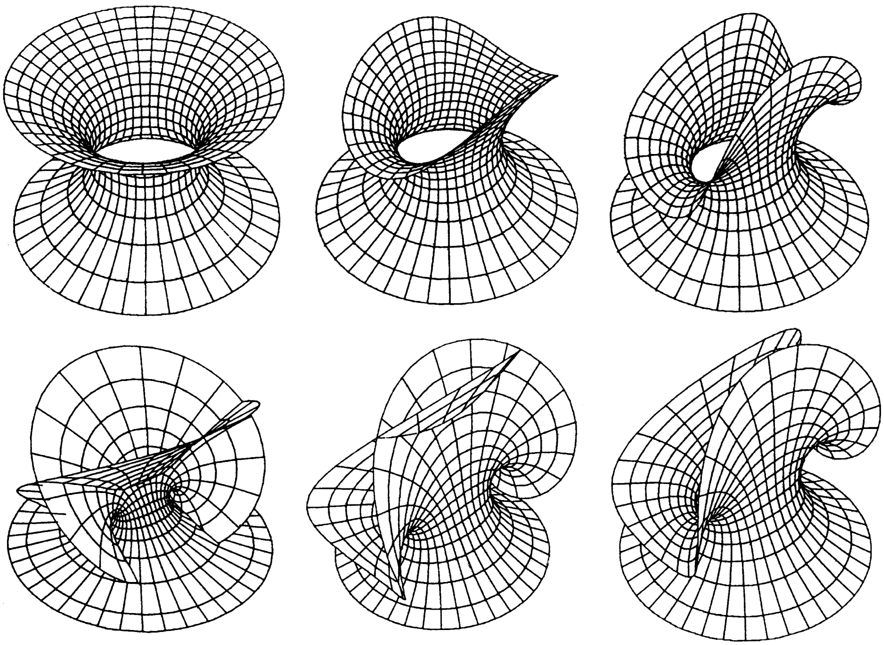


Fig. 5. Deformation of a catenoidal end into an Enneper end. Courtesy of K. Polthier and M. Wohlgemuth

The simplest minimal immersions of higher genus such as the *Chen–Gackstatter surface* (see (B2)) can be obtained from Weierstrass data which have the same behavior at their end as an Enneper surface.

(A3) *The catenoid* is given by

$$g(w) = w, \quad dh = \frac{dw}{w},$$

$w \in \mathbb{C} \setminus \{0\} \cong S^2 \setminus \{p_1, p_2\}, p_1 = \text{north pole}, p_2 = \text{south pole}$. Integration of the Weierstrass data once around 0 adds the period $P = (0, 0, 2\pi i)$ to ψ. Hence the catenoid is defined on $\mathbb{C} \setminus \{0\}$ whereas its adjoint, the helicoid, lives on the universal cover of $S^2 \setminus \{p_1, p_2\}$, and its symmetry group is a screw motion.

(A4) *examples with one planar end* can be obtained by the data

$$g(w) = w^{n+1}, \quad dh = w^{n-1}\, dw$$

for $w \in \mathbb{C} \setminus \{0\} \cong$ twice punctured sphere $= M$. Hence we have

$$\psi(w) = \left(\frac{1}{2}\left(-\frac{1}{w} - \frac{w^{2n+1}}{2n+1}\right), \frac{i}{2}\left(-\frac{1}{w} + \frac{w^{2n+1}}{2n+1}\right), \frac{w^n}{n}\right)$$

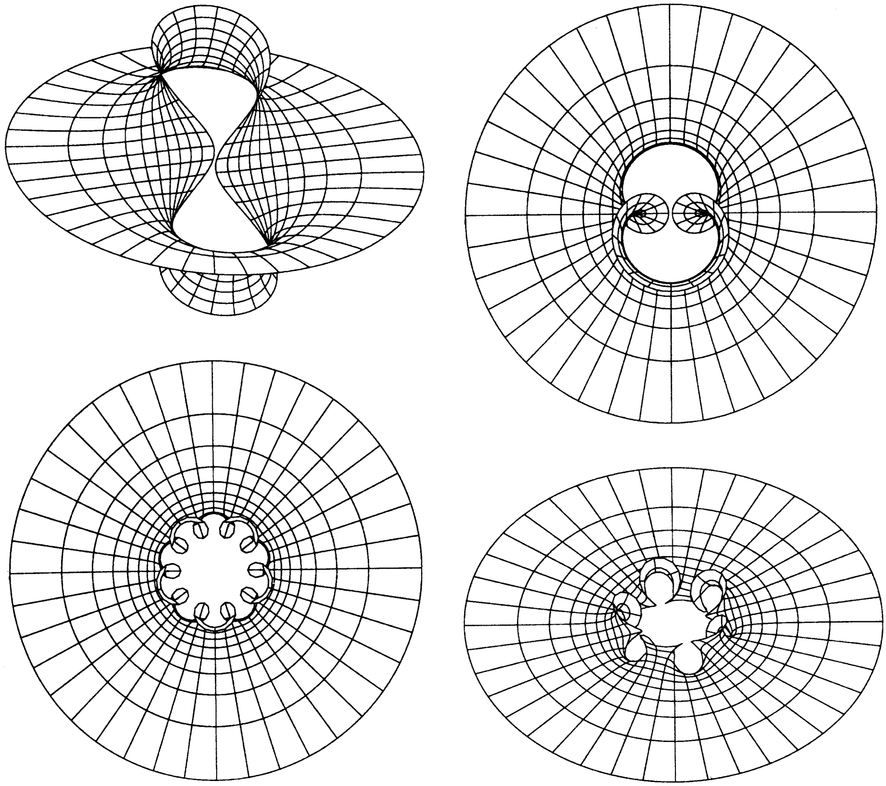


Fig. 6. Minimal surfaces with one planar end. Courtesy of K. Polthier

and

$$ds = (|w|^{2n} + |w|^{-2})|dw|.$$

This metric is complete on M. Reflections in all meridians define Riemannian isometries. The end at $w = \infty$ winds $(2n+1)$-times around the z-axis just as in the case of the higher order Enneper surfaces. The end at $w = 0$ is embedded and turns out to be a flat end which is asymptotic to the x, y-plane.

(A5) *Scherk's saddle tower* (*Scherk's fifth surface*) is given by the Weierstrass data

$$g(w) = w, \quad dh = \frac{1}{w^2 + w^{-2}} \frac{dw}{w}, \quad w \in M,$$

where $M = \bar{\mathbb{C}} \setminus \{\pm 1, \pm i\}$ is conformally the four times punctured sphere. The line element of Scherk's fifth surface $X = \operatorname{Re} \psi$ is given by

$$ds = \frac{|w| + |w|^{-1}}{|w^2 + w^{-2}|} \left| \frac{dw}{w} \right|.$$

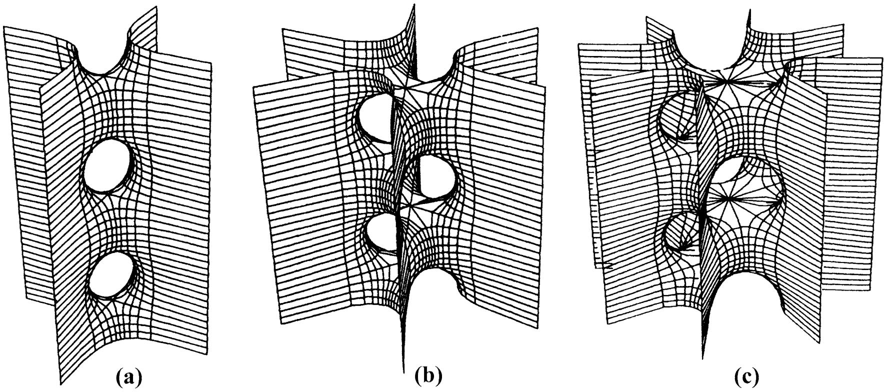


Fig. 7. Saddle towers, **(a)** Scherk's saddle tower (A5): $g(w) = w$. This surface is also called Scherk's fifth surface. It can be described by the equation $\sin z = \sinh x \sinh y$. **(b)**, **(c)** Higher order saddle towers (A6): **(b)** $g(w) = w^2$, **(c)** $g(w) = w^3$. Parts **(a), (b)** with courtesy of K. Polthier and part **(c)** with courtesy of J. Hahn and K. Polthier

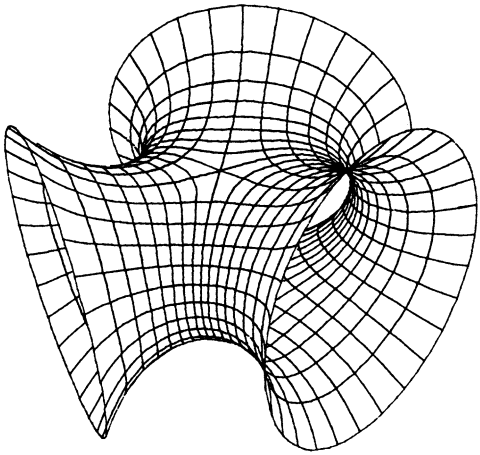


Fig. 8. The Jorge–Meeks 3-noid ($g(w) = w^2$). It can be viewed as limit of saddle towers. Courtesy of J. Hahn and K. Polthier

The corresponding metric is complete. The unit circle S^1 in $\mathbb{C}$, the axes $\mathbb{R}$, $i\mathbb{R}$ and the 45°-meridians allow Riemannian reflections. In particular we have a horizontal symmetry line (corresponding to S^1) through all four punctures whence all periods are vertical (and equal up to sign). Hence, on the open unit disk B, the mapping $X : B \to \mathbb{R}^3$ defines a regular minimal surface bounded by four horizontal symmetry lines which lie in only two parallel planes. Extension by reflection in these planes yields a complete minimal surface with one vertical period, and this surface is embedded if the fundamental piece is embedded. In fact, it turns out to be a graph. By Krust's theorem, the adjoint

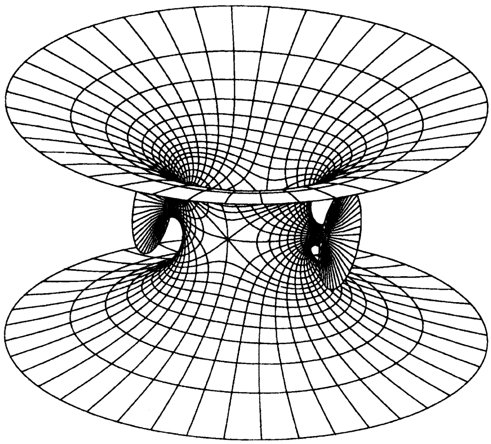


Fig. 9. A 4-noid with two orthogonal symmetry planes through each puncture. Courtesy of K. Polthier and M. Wohlgemuth

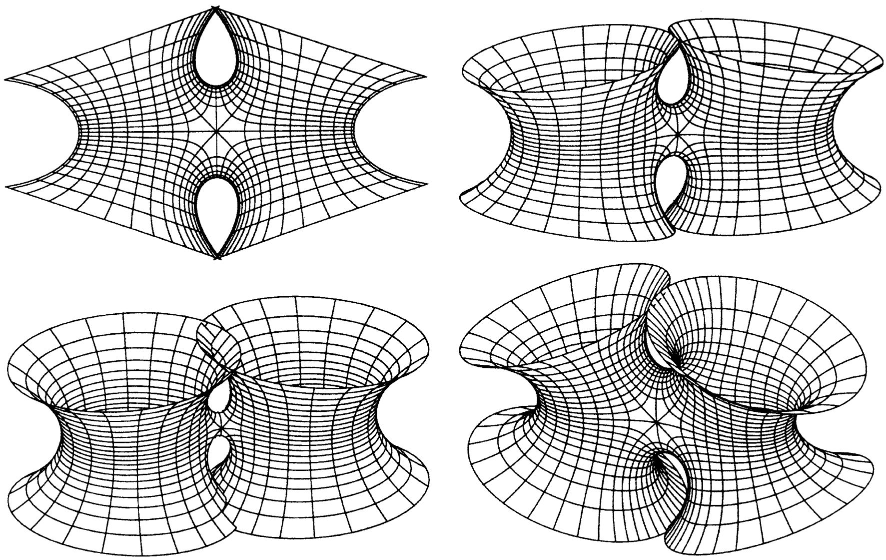


Fig. 10. Several 4-noids. Courtesy of K. Polthier

surface is also embedded; it is *Scherk's doubly periodic minimal surface.* Its Weierstrass data are

$$g(w) = w, \quad dh = \frac{i}{w^2 + w^{-2}} \frac{dw}{w},$$

(A6) *Higher order saddle towers* (Karcher) are defined by the Weierstrass data

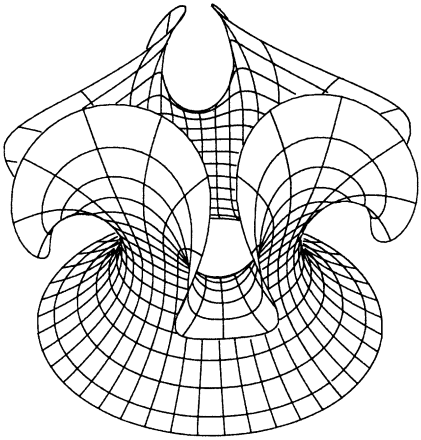


Fig. 11. An Enneper catenoid (corresponding to $g(w) = w^{-1} + w^3$). Courtesy of J. Hahn and K. Polthier

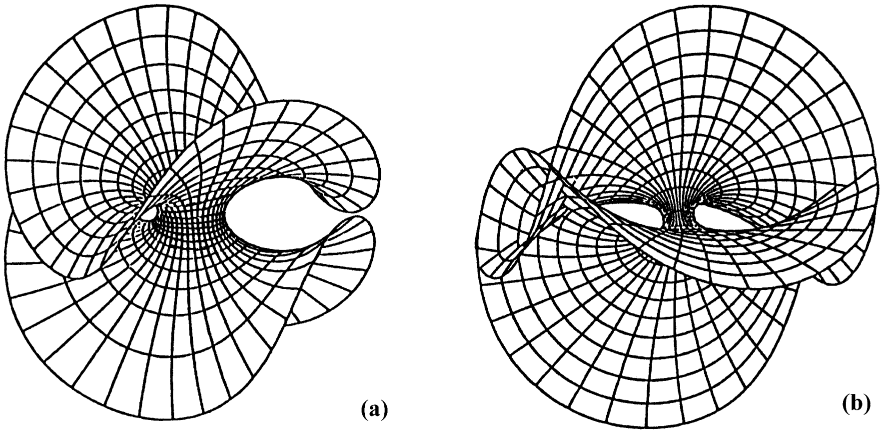


Fig. 12. Doubled Enneper surfaces. **(a)** without symmetry planes (rotated ends), **(b)** with symmetry planes. Courtesy of K. Polthier

$$g(w) = w^{n-1}, \quad dh = \frac{1}{w^n + w^{-n}} \frac{dw}{w}$$

which are defined on $M = \bar{\mathbb{C}} \setminus \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}\}$ where ε_j are the $(2n)$-th roots of 1; M is conformally the $2n$-times punctured sphere.

(A7) *Less symmetric saddle towers* are obtained from

$$g(w) = w^{n-1}, \quad dh = (w^n + w^{-n} - 2\cos n\varphi)^{-1} \frac{dw}{w}$$

$w \in M$, where M is $\bar{\mathbb{C}}$ punctured at $w = e^{\pm i\varphi} e^{2\pi i l/n}$, $l = 0, 1, \ldots, n-1$, and φ is a real parameter restricted by $0 < \varphi \leq \frac{\pi}{2n}$.

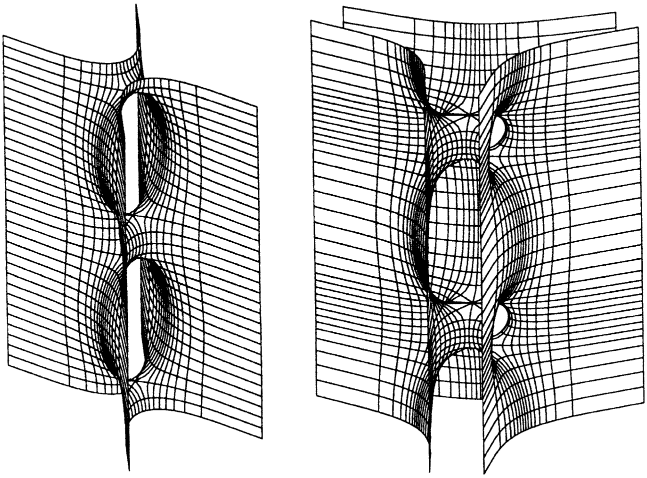


Fig. 13. (A7) Less symmetric saddle towers. Courtesy of K. Polthier

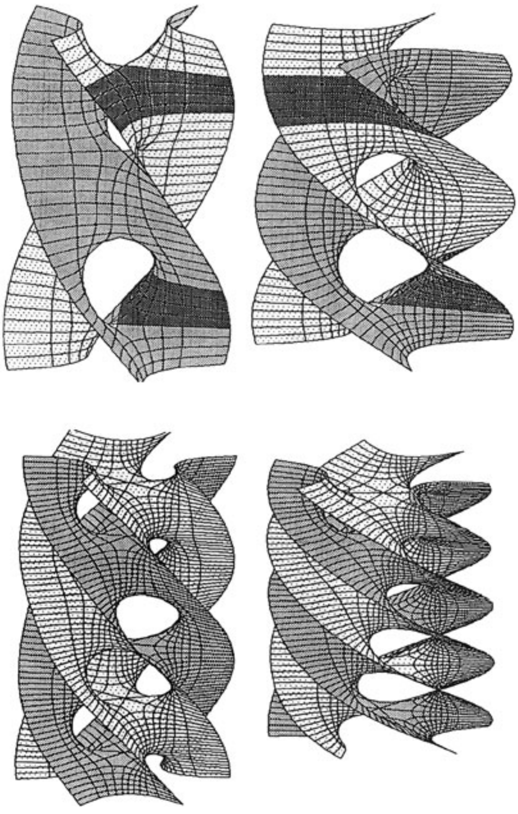


Fig. 14. Helicoidal saddle towers: Deformed Scherk surfaces constructed by Karcher. Courtesy of H. Karcher

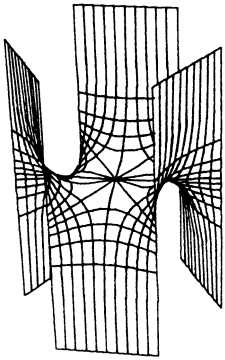


Fig. 15. The Jenkins–Serrin theorem for the hexagon ($n = 3$). Courtesy of J. Hahn and K. Polthier

If $\varphi \to 0$, then the punctures move pairwise together to become double poles of dh, and their images lie on vertical symmetry planes. Hence the punctures have no periods, and their ends turn out to be embedded catenoidal ends. In fact, the surfaces with $\varphi = 0$ are the *n-noids of Jorge–Meeks* which are not embedded.

We remark that the saddle towers as well as the n-noids allow deformations which are again complete minimal surfaces. For more details, see Karcher [1, 3], and also Figs. 7–14.

Moreover, the construction of *embedded saddle towers* can be obtained from a result by Jenkins and Serrin [2] by passing to the adjoint of the *Jenkins–Serrin surface* and by applying the reflection principle and Krust's theorem; see Karcher [3].

Theorem of Jenkins and Serrin. *Let Ω be a convex $2n$-gon with all edges of the same length and alternatingly marked $\infty, -\infty, \infty, -\infty, \dots$. Then there is a uniquely determined nonparametric minimal surface $z = u(x, y)$, $x, y \in \Omega$, over Ω which converges to ∞ or $-\infty$ respectively as it approaches the marked edges of Ω. The graph of u is a minimal surface bounded by the vertical lines over the vertices of $\partial\Omega$ which has finite total curvature.*

B. Minimal Surfaces Parametrized on Punctured Tori

While the examples (A) were constructed by Weierstrass data which are rational functions on the punctured sphere, we shall now use meromorphic maps $T^2 \to \bar{\mathbb{C}}$ on the torus T^2, that is, doubly periodic functions (or: elliptic functions). Karcher [3] effectively operates with a doubly periodic function $\gamma : T^2 \to \bar{\mathbb{C}}$ which, by reflection, is built from a biholomorphic map $\gamma : B \to D$ of a rectangle B with the corners a, b, c, d onto the quarter circle D with the vertices $0, 1, i$. The mapping γ is obtained by Riemann's mapping theorem. Using the 3-point-condition $\gamma(a) = i, \gamma(b) = 0, \gamma(c) = 1$, we define an angle

$\alpha \in (0, \frac{\pi}{2})$ by $\gamma(d) = e^{i\alpha}$; this angle is called the *conformal parameter* of γ. One obtains

$$\left(\frac{\gamma'}{\gamma}\right)^2 = \kappa(\gamma^2 + \gamma^{-2} - 2\cos\alpha)$$

where κ is a positive constant. As γ turns out to be a degree-two elliptic function, there is a close connection to the geometric p-function. In fact, we have

$$\gamma^2 = \frac{-\tan\alpha - \cot\alpha}{p - \frac{1}{p} + \tan\alpha - \cot\alpha}$$

and

$$p'\gamma = \kappa^* p \quad (\kappa^* = \text{positive constant}).$$

Note that, in section B, the geometric p-function is not the usual Weierstrass $\wp$-function, but the one that has been modified linearly such that it has a double zero in the middle, and that the product of the two finite branch values is -1. Another useful elliptic function f is defined as extension by reflection of the biholomorphic mapping from a rectangle B to the quarter disk D such that b, c, d are mapped into $0, 1, i$ respectively whereas a goes to $i\ \tan\frac{\alpha}{2}$. The functions γ, p and f are linked by

$$f\gamma = \frac{p}{\cos\alpha - p\sin\alpha}.$$

(B1) *A fence of catenoids* (*Hoffman–Karcher*). One can construct a periodic surface with a translational symmetry as depicted in Fig. 16. Dividing out the symmetry, we obtain a torus with two embedded catenoidal ends. The stereographic projection g of the Gauss map of this surface turns out to be γ whereas f determines dh:

$$g = \gamma, \quad dh = f dw.$$

The symmetries of f and γ yield that reflections in the expected symmetry lines are Riemannian isometries for the metric

$$ds = \left(|\gamma| + \frac{1}{|\gamma|}\right)|f||dw|$$

of the fence.

(B2) *The Chen–Gackstatter surface* was the first minimal surface without periods or branch points defined on a punctured torus that was discovered. It has one puncture and therefore one end. Thus it is the direct relative of Enneper's surface, only that it possesses a handle (see Fig. 17). The Weierstrass data are given as

$$g = r\gamma, \quad dh = p'\,dw$$

where the parameter $r \in \mathbb{R}^+$ has to be chosen in such a way that the periods vanish. The removal of the periods is one of the difficulties in this and other examples.

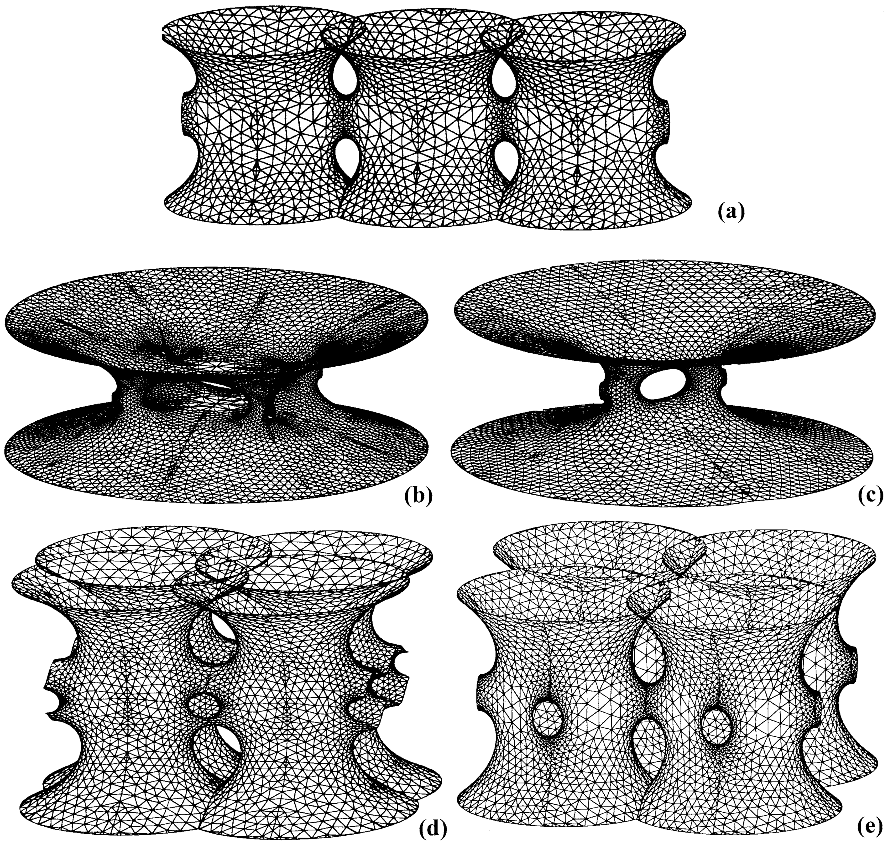


Fig. 16. Construction of higher genus minimal surfaces by growing handles out of a catenoid. **(a)** A fence of catenoids (B1), **(b)**–**(e)** More catenoids with handles. Courtesy of E. Boix, J. Hoffman, and M. Wohlgemuth

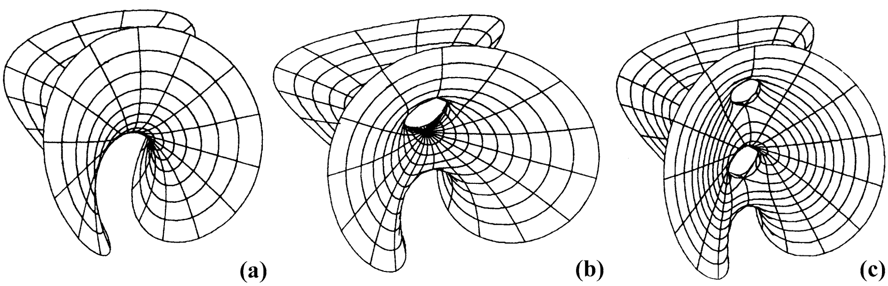


Fig. 17. **(a)** Enneper's surface (A1): no handle. Courtesy of K. Polthier. **(b)** Chen–Gackstatter surface (B2): one handle. Courtesy of J. Hahn and K. Polthier. **(c)** Chen–Gackstatter surface with two handles. Courtesy of K. Polthier and M. Wohlgemuth

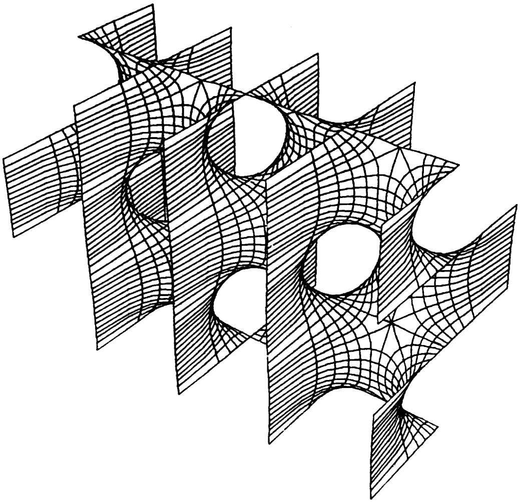


Fig. 18. A fence of Scherk towers—a doubly periodic toroidal surface (B3). Courtesy of K. Polthier

(B3) *Doubly periodic examples* are depicted in Figs. 18 and 19.

(B4) *Riemann's minimal surface* is a simply periodic embedded minimal surface defined on a twice punctured rectangular torus and with one period. Its two ends are flat. A careful discussion can be found in Nitsche's treatise [28]. The corresponding Weierstrass data are

$$g = p, \quad dh = dw = \frac{dp}{p'}.$$

In fact, there is a 1-parameter family of Riemann examples, two for each rectangular torus. The adjoint surface of a Riemann example is another Riemann example which is not congruent to the first, except in the special case of a square torus.

(B5) *Costa's surface* is an embedding of the three times punctured square torus (i.e., without periods). In Karcher's description [3], its Weierstrass data are

$$g = rp' = r\frac{p}{\gamma},$$

$$dh = \gamma\, dw = \frac{\gamma}{\gamma'}\, d\gamma = \frac{2}{1-p^2}\, dp.$$

Again, the parameter $r \in \mathbb{R}^+$ is used to remove all periods.

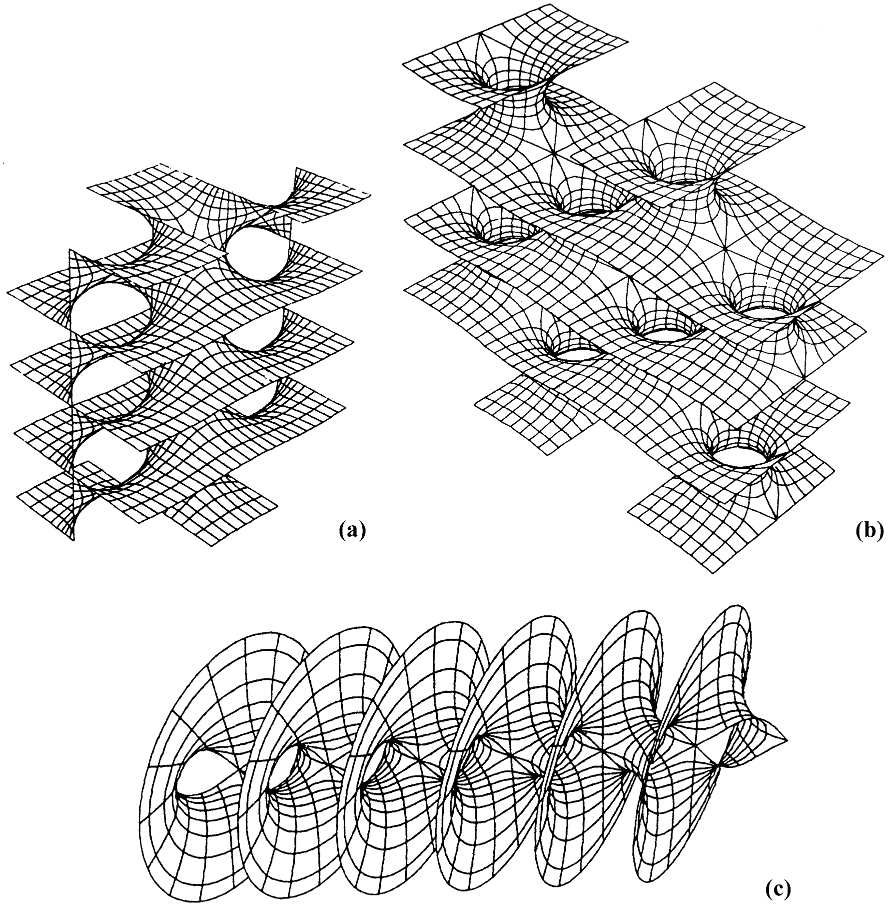


Fig. 19. **(a)** and **(b)** A conjugate pair of embedded doubly periodic minimal surfaces (B3). Part **(a)** with courtesy of K. Polthier and M. Wohlgemuth and part **(b)** with courtesy of K. Polthier. **(c)** Riemann's periodic minimal surface (B4) can be viewed as a limit of **(b)** under deformation. Courtesy of K. Polthier and M. Wohlgemuth

5 Triply Periodic Minimal Surfaces

Five surfaces of this type were already known to H.A. Schwarz (see [2], vol. 1, pp. 1–125, 136–147; cf. also Figs. 21–27 of this section, Figs. 37–39 of Section 3.5, and Plates II–VII). They were obtained by spanning a disk-type minimal surface $X : B \to \mathbb{R}^3$ into a polygon Γ and then reflecting this surface at the edges of Γ. In 1891, A. Schoenflies (see [1,2]) proved that in this way exactly six different periodic minimal surfaces can be obtained from (skew) quadrilaterals, whereas Schwarz had erroneously claimed that there existed exactly five surfaces of this type (see [2], vol. 1, pp. 221–222). All of these

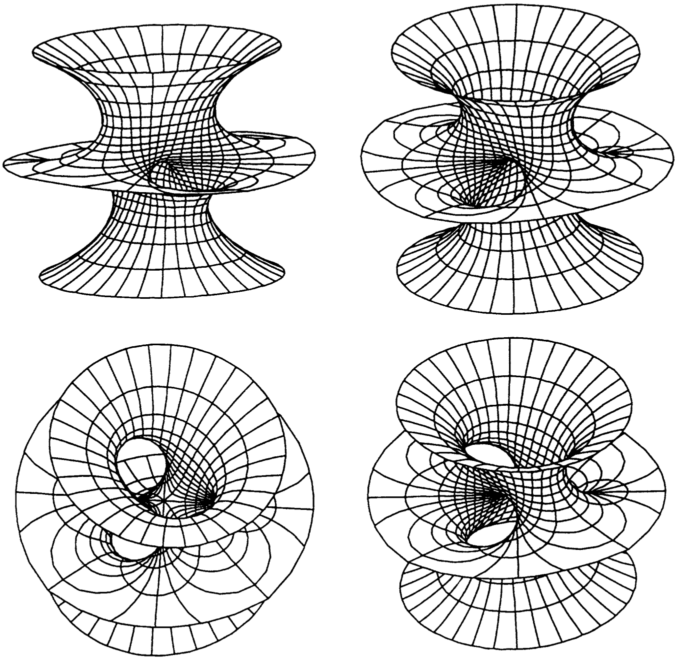


Fig. 20. The Costa surface (B5). Courtesy of K. Polthier

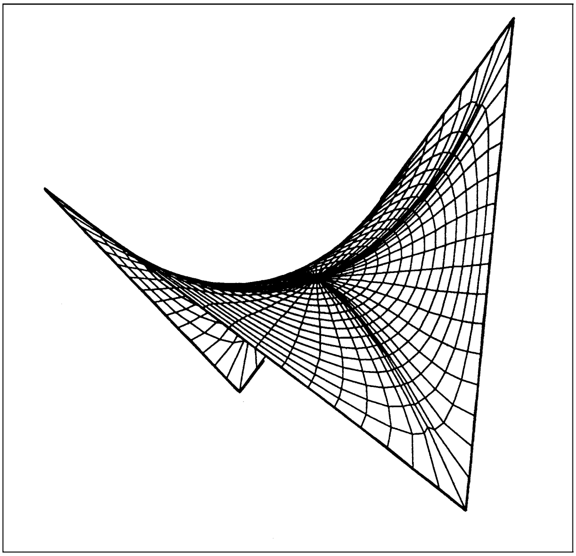


Fig. 21. Schwarz's surface. Courtesy of O. Wohlrab

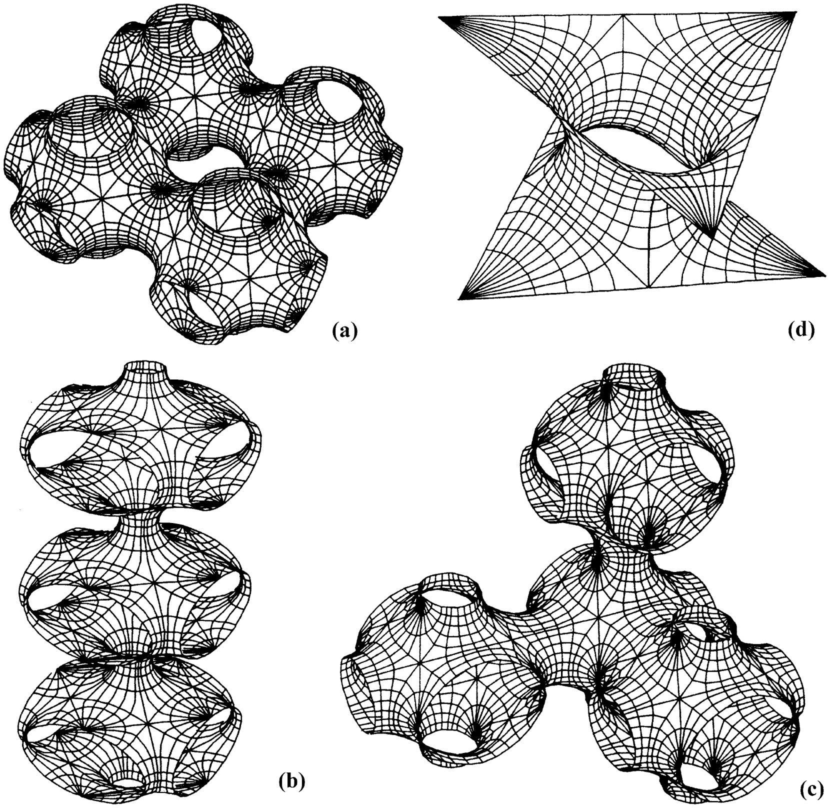


Fig. 22. (a) Schwarz's P-surface and **(b)**, **(c)** deformations thereof. **(d)** This annulus bounded by two triangles is part of the adjoint of the Schwarzian P-surface if the ratio of edge length to height is $2\sqrt{3}$. Courtesy of K. Polthier

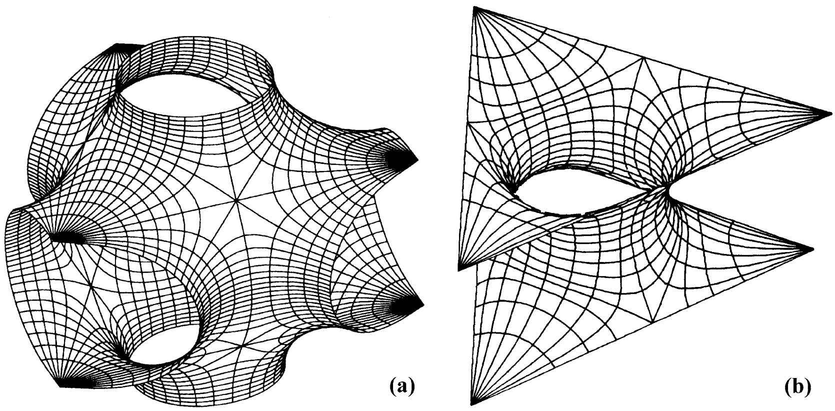


Fig. 23. (a) A part of Schwarz's H-surface. **(b)** An annulus-type minimal surface bounded by two triangles which is part of the H-surface. Courtesy of K. Polthier

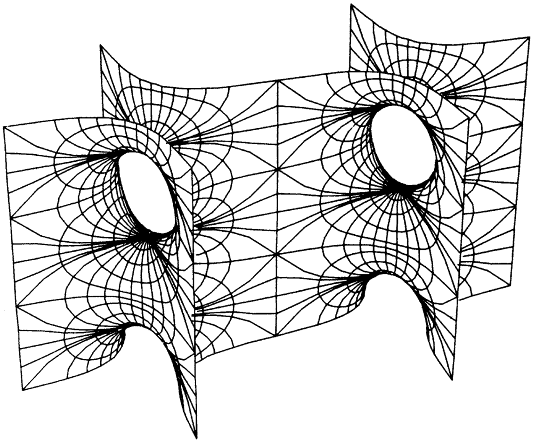


Fig. 24. Schwarz's *CLP*-surface. Courtesy of K. Polthier

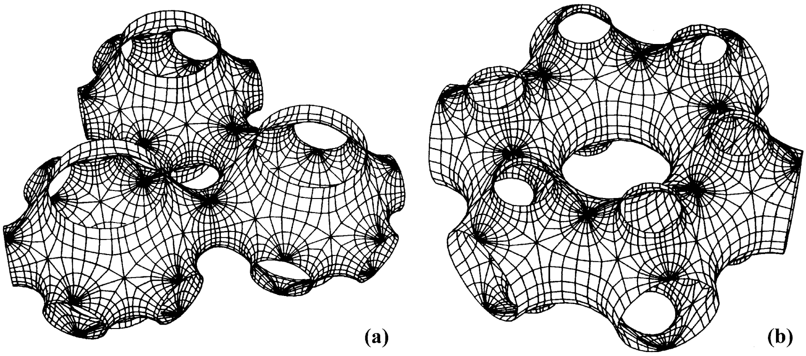


Fig. 25. Alan Schoen's H'–T-surface: **(a)** in a trigonal cell, **(b)** in the dual hexagonal cell. Courtesy of K. Polthier

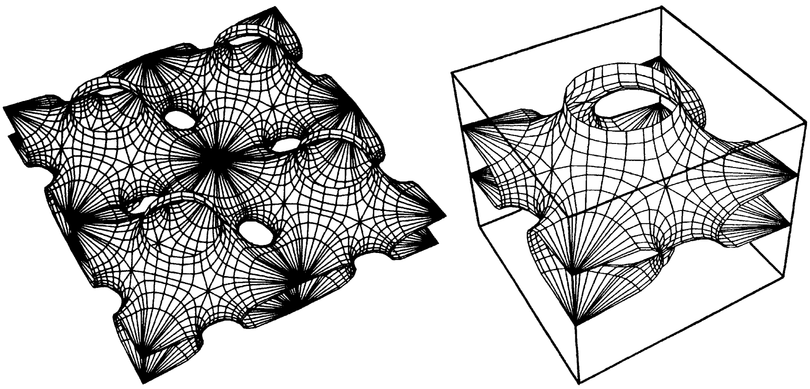


Fig. 26. Alan Schoen's S'–S''-surface. This part solves a free boundary problem with regard to the faces of a cube. Courtesy of K. Polthier

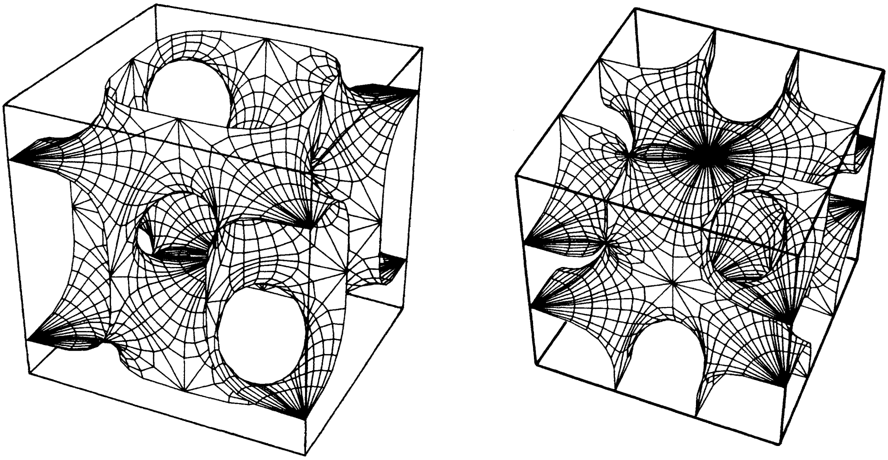


Fig. 27. Two views of A. Schoen's I–Wp-surface. Both parts sit in a cube and meet its faces at a right angle. Courtesy of K. Polthier

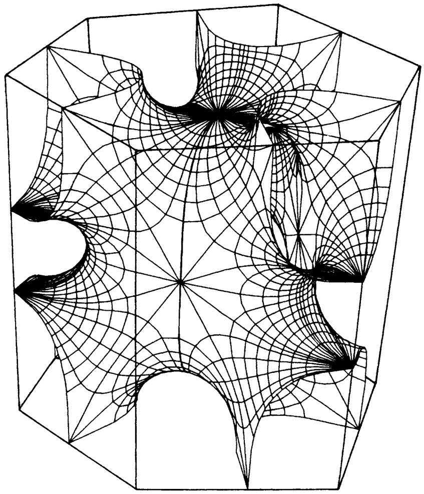


Fig. 28. An analogue to A. Schoen's I–Wp-surface found by Karcher; it sits in a hexagonal cell and meets the faces of this cell perpendicularly. Courtesy of K. Polthier

periodic minimal surfaces were described in detail by Steßmann [1]; one of them was discovered by Neovius.

Clearly one can try to obtain other triply periodic minimal surfaces by spanning pieces of minimal surfaces as stationary points of the area functional into a general Schwarzian chain $\langle \Gamma_1, \dots, \Gamma_k, S_1, \dots, S_l \rangle$ and then reflecting them at the edges Γ_j and the planar faces S_j. In this way, Neovius, Nicoletti, Marty, Tenius, Stenius and Wernick generated more triply periodic minimal

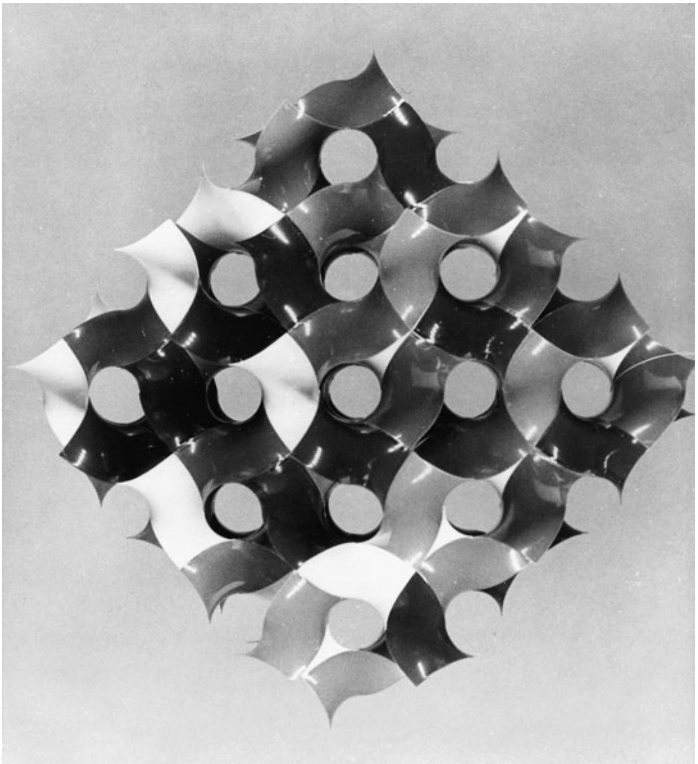


Fig. 29. Alan Schoen's gyroid, an associate to Schwarz's surface, is an embedded triply periodic minimal surface. Courtesy of A. Schoen

surfaces. We refer to Nitsche's treatise [28], § 818, pp. 664–665 for pertinent references. After Steßmann's paper, the subject was at rest for more than 30 years until the physicist and crystallographer Alan Schoen [1,2] revived it. He discovered many new triply periodic minimal surfaces, and he built marvelous models of enormous size which stunned everyone who had a chance to see them (a few are depicted in Hildebrandt and Tromba [1]). However, Schoen's reports were a bit sketchy and thus, among mathematicians, there remained some doubts whether all details could be filled in, whereas Schoen's work became very popular among crystallographers and chemists. Schoen's remarkable geometric intuition proved to be correct; H. Karcher established the existence of all of Schoen's surfaces, and he found *triply periodic constant mean curvature companions* to them (see Karcher [2] and also [3]). By solving conjugate Plateau problems, Karcher and his students found many more triply periodic embedded minimal surfaces and even whole families of them. The strategy for finding such examples is lucidly described in Section 4 of Karcher's lecture notes [3].

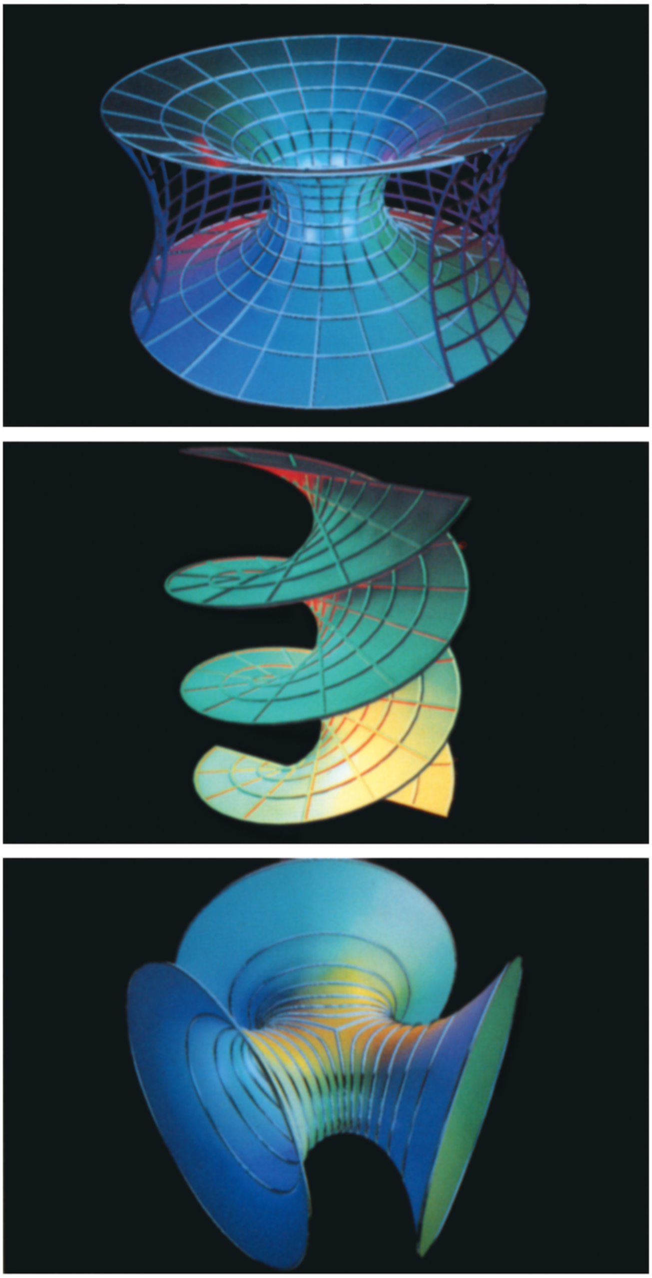
6 Structure of Embedded Minimal Disks

In a series of papers (cf. bibliography), T.H. Colding and W.P. Minicozzi investigated the structure of embedded minimal disks, i.e. of minimal surfaces $X : \overline{B} \to \mathbb{R}^3$ defined on closed disks $\overline{B} \subset \mathbb{R}^2$ being embeddings. (In particular such surfaces are free of branch points.) One of their main results states that every embedded minimal disk can either be modeled by a minimal graph or by a piece of the helicoid depending on whether the supremum of the Gauss curvature is small or not. Together with a Heinz-type curvature estimate which is also due to Colding & Minicozzi, Meeks and Rosenberg [GTMS] proved that the plane and the helicoid are the only complete, properly embedded, simply-connected minimal surfaces in $\mathbb{R}^3$.

7 Complete Minimal Surfaces and the Plateau Problem

One might think that a complete minimal surface "extends" to infinity, i.e. cannot be contained in a compact set. The question whether or not this is true had been raised by E. Calabi in the 1960ies, and in 1996 N. Nadirashvili [1] found a surprising answer: He constructed a complete minimal surface in $\mathbb{R}^3$ which is contained in a ball. Even more surprising is a result obtained by Martín and Nadirashvili [1] in 2007: *There exists a minimal surface $X : B \to \mathbb{R}^3$ on the unit disk of $\mathbb{R}^2$ which is complete and possesses a continuous extension to $\overline{B}$ such that $X|_{\partial B} : \partial B \to \mathbb{R}^3$ provides a nonrectifiable Jordan curve Γ of dimension* 1. Such curves Γ are not rare: *For any Jordan curve Γ_0 in $\mathbb{R}^3$ and any $\epsilon > 0$ one can find a Jordan curve Γ such that the Hausdorff distance of Γ and Γ_0 satisfies $\delta^H(\Gamma, \Gamma_0) < \epsilon$, and that Γ is the boundary of a complete minimal surface $X : B \to \mathbb{R}^3$ in the sense described above.* (Concerning the Plateau problem we refer to Sections 4.1–4.5 and 4.12.) We note that these surfaces have infinite area, and they cannot be embedded on account of work by Colding and Minicozzi.

Color Plates

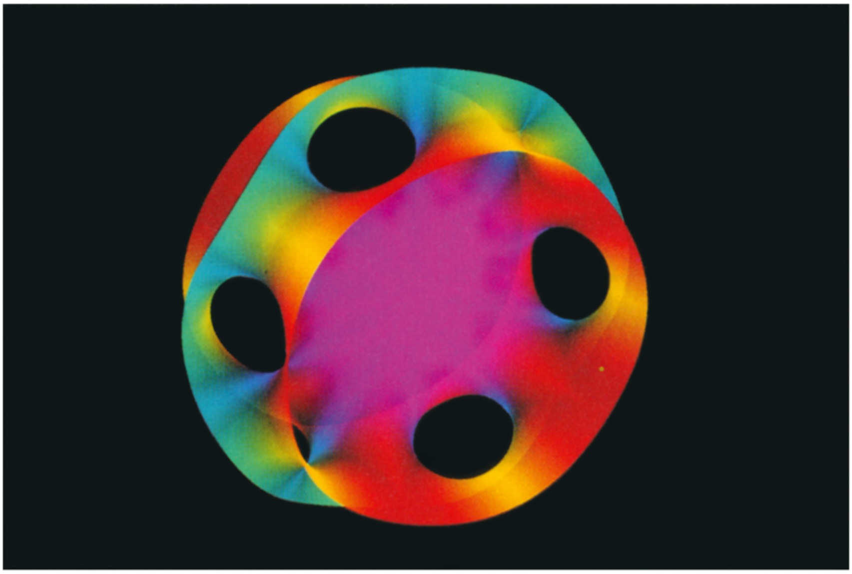


(a)

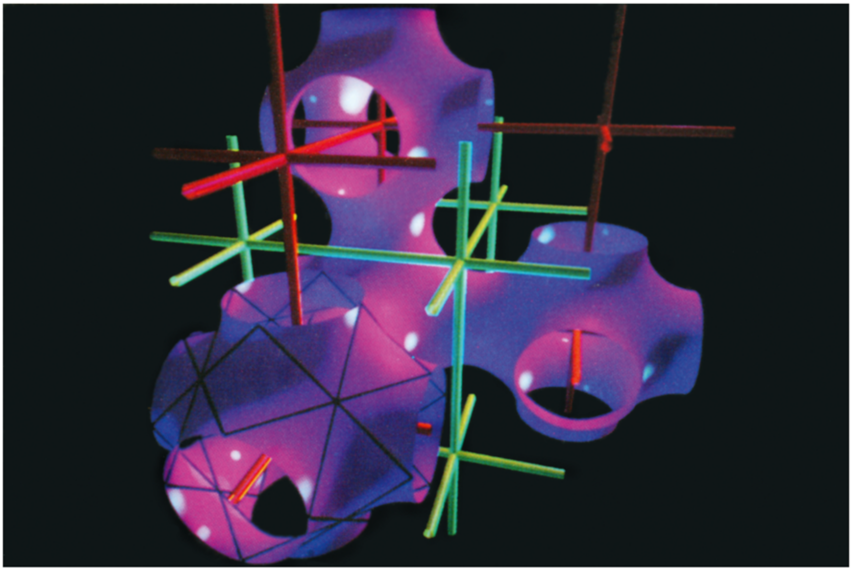
(b)

(c)

Plate I. **(a)** Stable and unstable catenoid, **(b)** helicoid and double helix, **(c)** Jorge–Meeks surface. Courtesy of K. Polthier

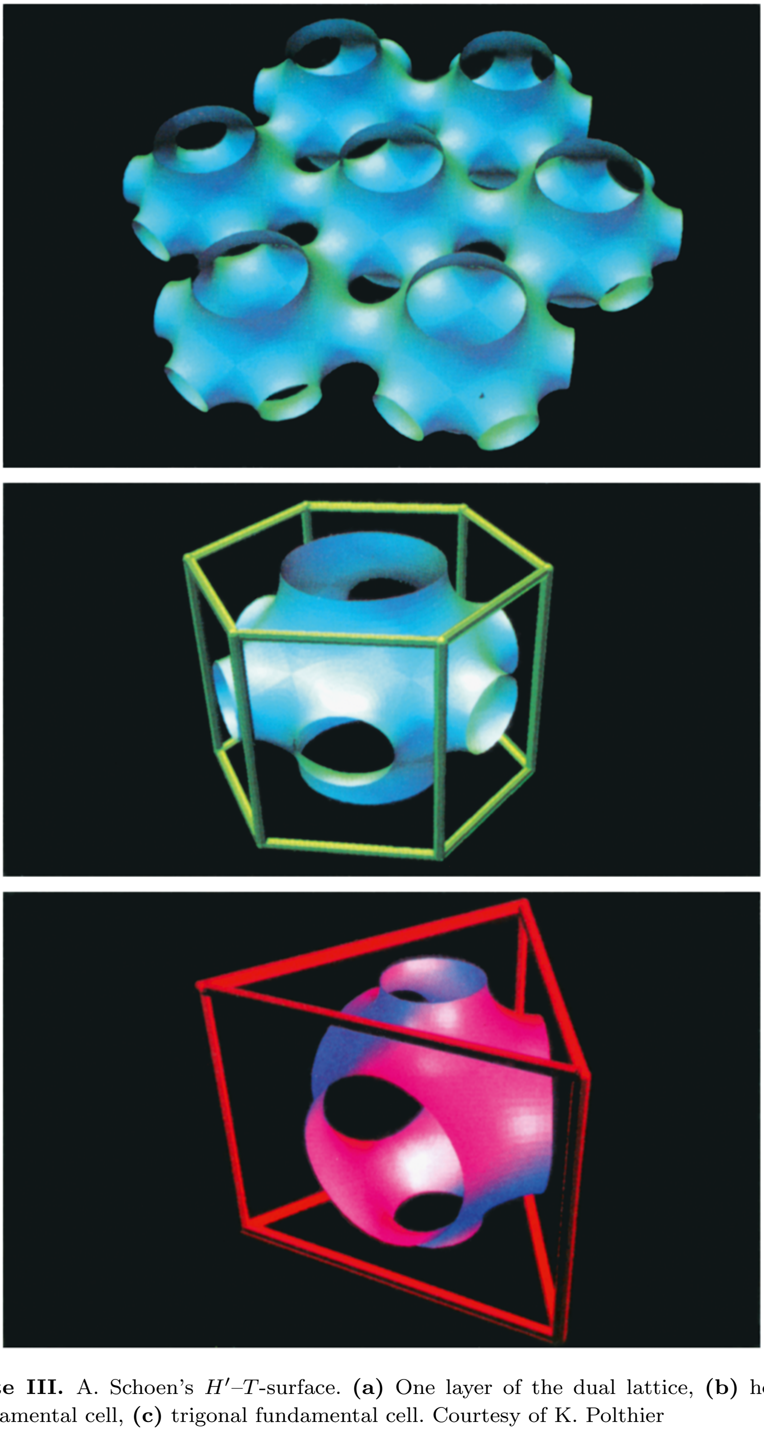


(a)



(b)

Plate II. (a) A Hoffman–Meeks surface, **(b)** part of Schwarz's P-surface. Courtesy of D. Hoffman and K. Polthier



(a)

(b)

(c)

Plate III. A. Schoen's H'–T-surface. **(a)** One layer of the dual lattice, **(b)** hexagonal fundamental cell, **(c)** trigonal fundamental cell. Courtesy of K. Polthier

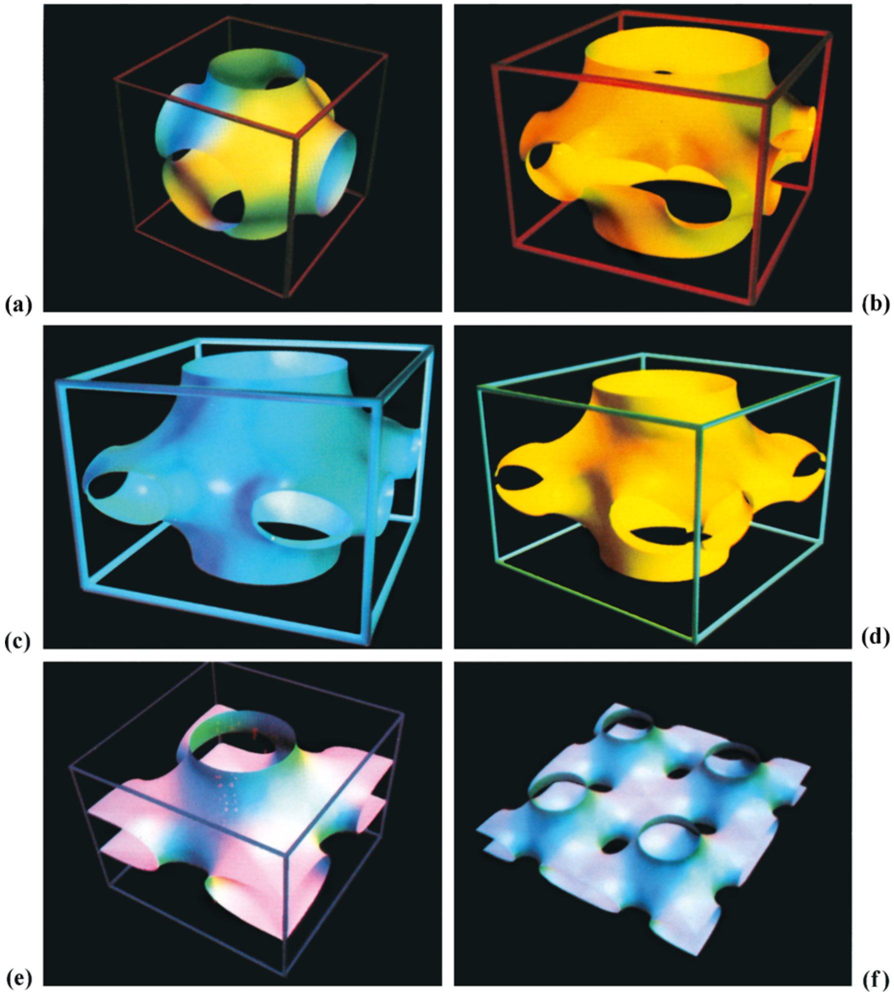
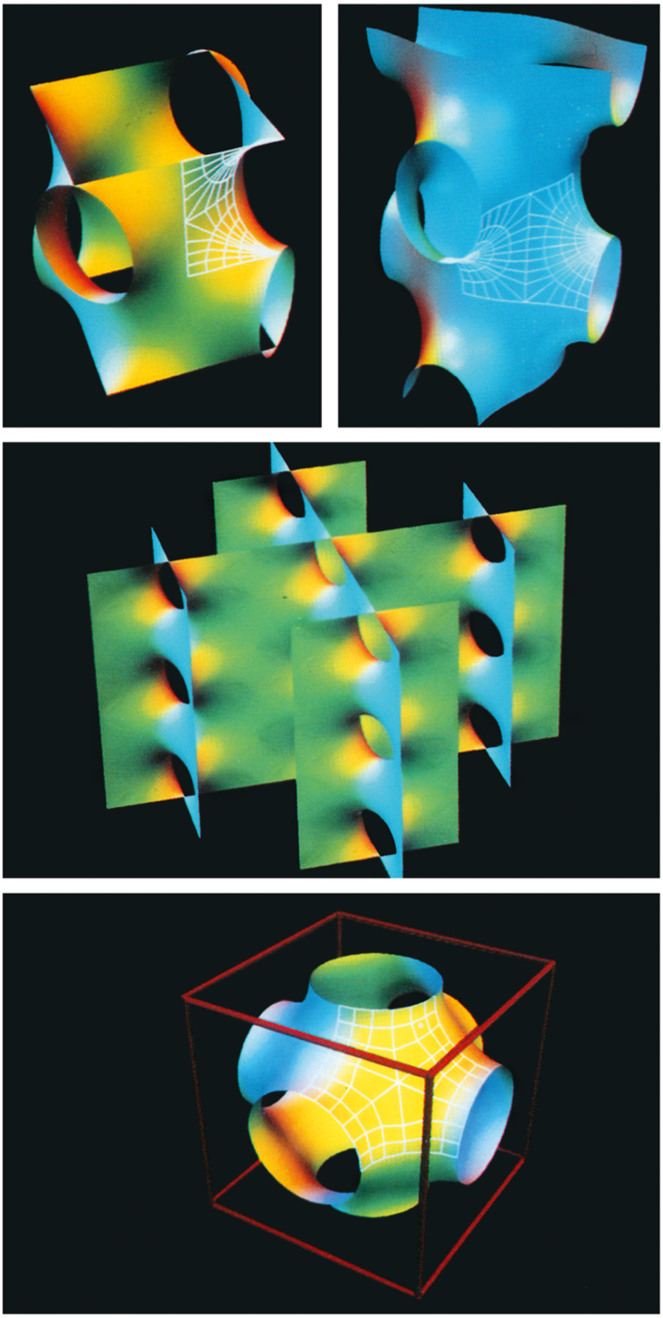


Plate IV. (a)–(e) The Karcher process of handle growing demonstrated by the transition from Schwarz's P-surface to Schoen's S'–S''-surface, **(f)** Schoen's S'–S''-surface. Courtesy of K. Polthier



(a) (b) (c) (d)

Plate V. **(a)–(c)** Schwarz's *CLP*-surface, **(d)** Schwarz's *P*-surface. Courtesy of K. Polthier

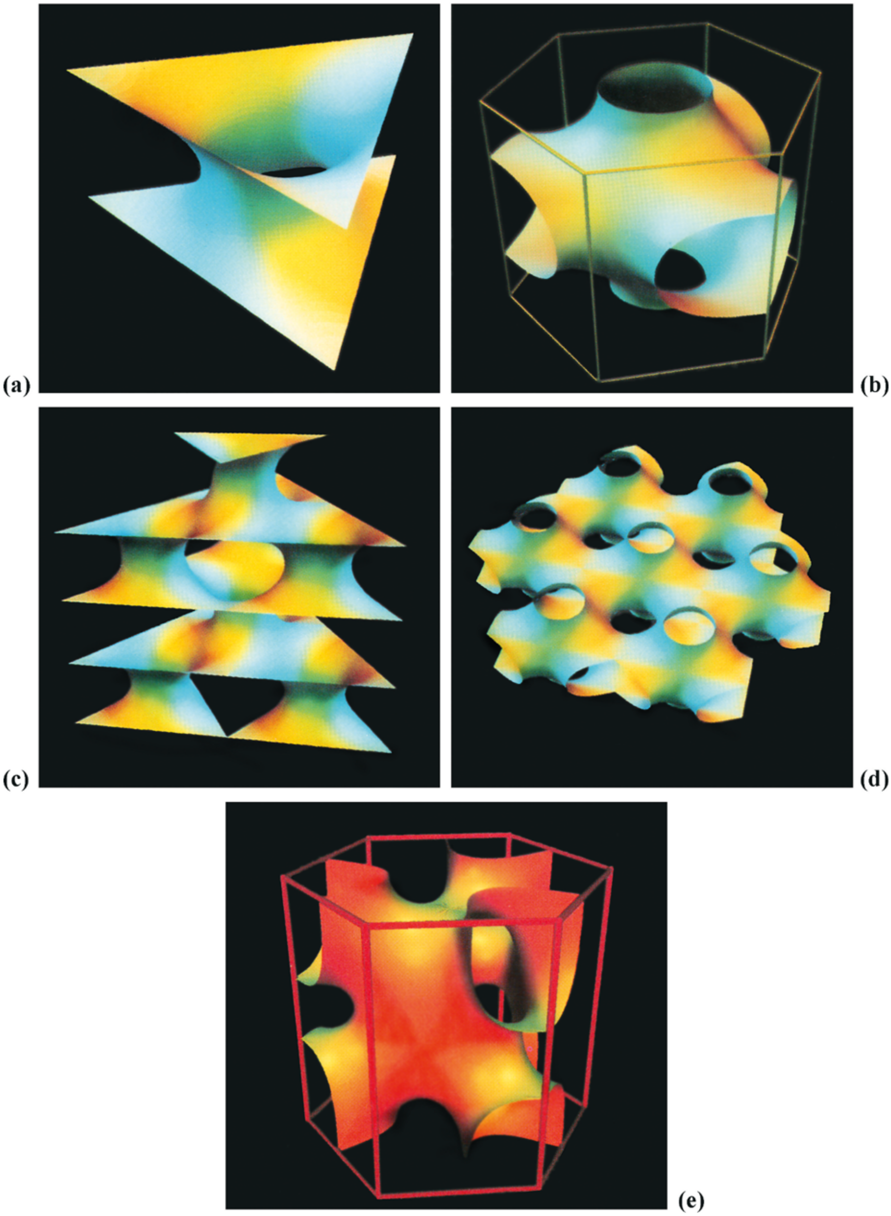
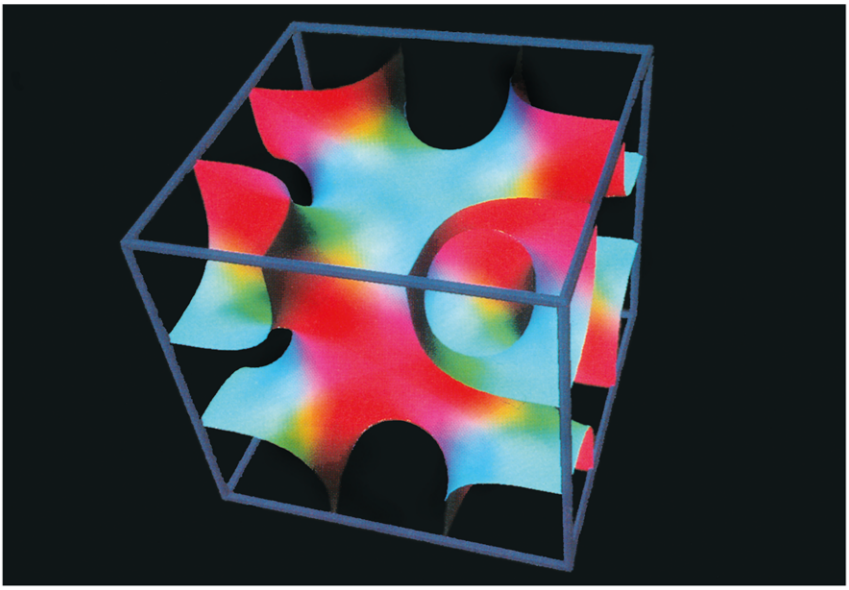
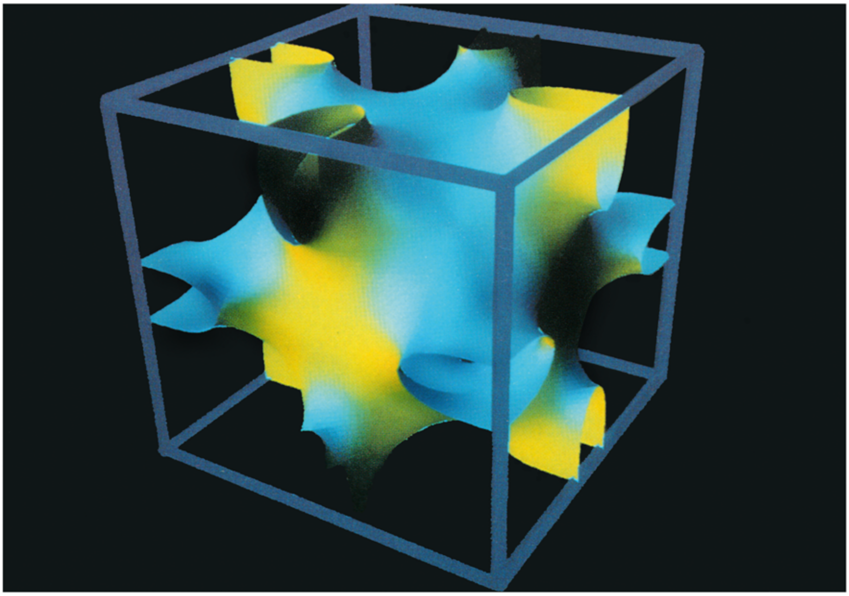


Plate VI. **(a)–(d)** Schwarz's H-surface, **(e)** Karcher's T–WP-surface. Courtesy of K. Polthier



(a)



(b)

Plate VII. Fundamental cells. **(a)** A. Schoen's I–WP-surface, **(b)** Neovius surface. Courtesy of K. Polthier

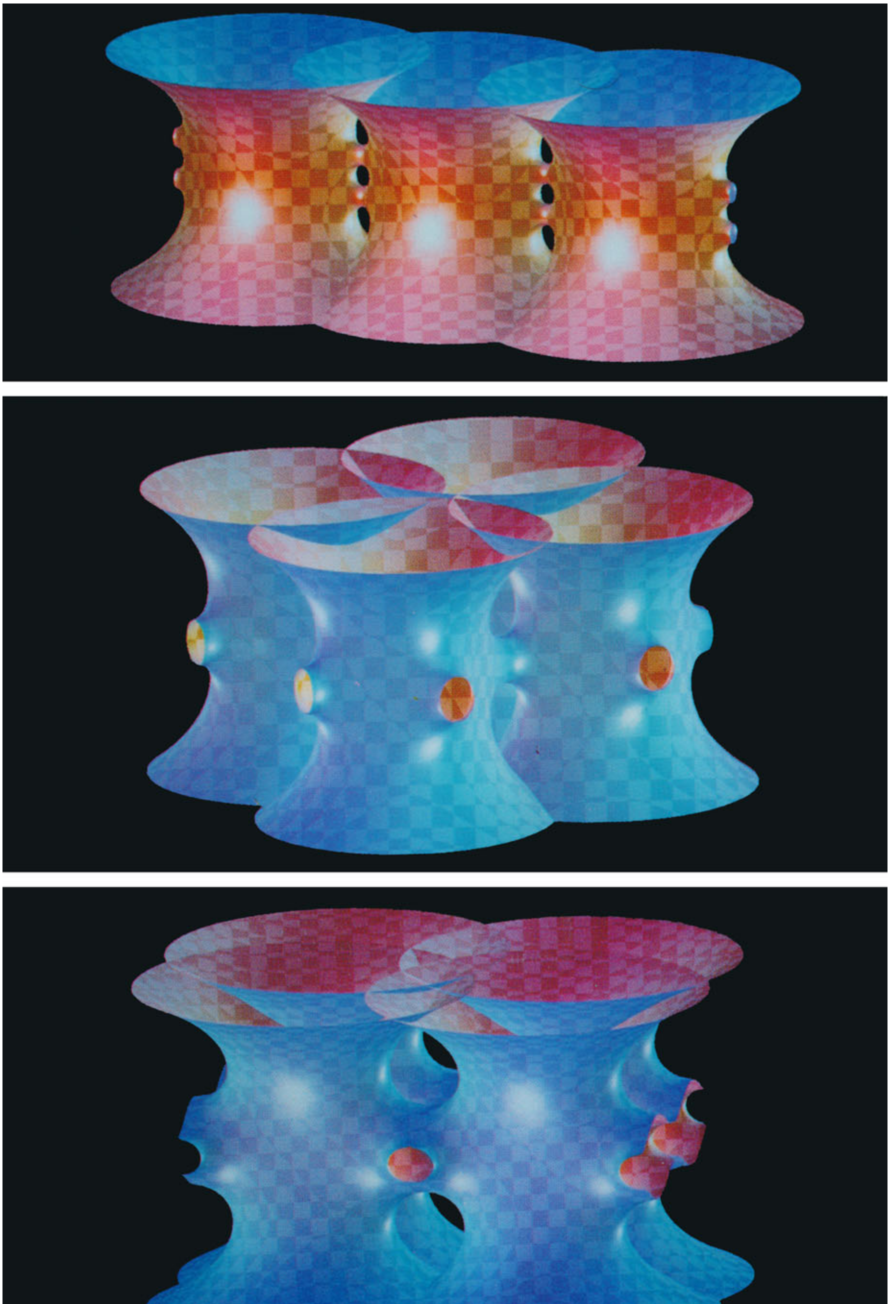


Plate VIII. Fences of catenoids. Courtesy of E. Boix, J. Hoffman, and M. Wohlgemuth

Part II

Plateau's Problem

Chapter 4

The Plateau Problem and the Partially Free Boundary Problem

The remainder of this book is essentially devoted to boundary value problems for minimal surfaces. The simplest of such problems was named *Plateau's problem*, in honor of the Belgian physicist J.A.F. Plateau, although it had been formulated much earlier by Lagrange, Meusnier, and other mathematicians. It is the question *of finding a surface of least area spanned by a given closed Jordan curve Γ.*

In his treatise *Statique expérimentale et théorétique des liquides soumis aux seules forces moléculaires* from 1873, Plateau described a multitude of experiments connected with the phenomenon of capillarity. Among other things, Plateau noted that every contour consisting of a single closed wire, whatever be its geometric form, bounds at least one soap film. Now the mathematical model of a thin wire is a closed Jordan curve of finite length. Moreover, the mathematical objects modeling soap films are two-dimensional surfaces in $\mathbb{R}^3$. To every such surface, the phenomenological theory of capillarity, due to Gauss, attaches a potential energy that is proportional to its surface area. Hence, by Johann Bernoulli's principle of virtual work, *soap films in stable equilibrium correspond to surfaces of minimal area.*

Turning this argument around, it stands to reason that every rectifiable closed Jordan curve bounds at least one surface of least area and that all possible solutions to Plateau's problem can be realized by soap film experiments. However, as R. Courant [15] has remarked, *empirical evidence can never establish mathematical existence—nor can the mathematician's demand for existence be dismissed by the physicist as useless rigor. Only a mathematical existence proof can ensure that the mathematical description of a physical phenomenon is meaningful.*

The mathematical question that we have formulated above as Plateau's problem was a great challenge to mathematicians. It turned out to be a formidable task. During the nineteenth century, Plateau's problem was solved for many special contours Γ, but a sufficiently general solution was only obtained in 1930 by J. Douglas [11,12] and simultaneously by T. Radó [17,18].

U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 339,
DOI 10.1007/978-3-642-11698-8_4,

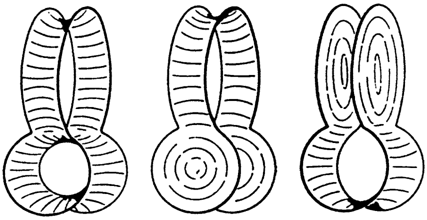


Fig. 1. A Jordan contour bounding two disk-type minimal surfaces **(b)**, **(c)** and a minimal surface of genus one **(a)**

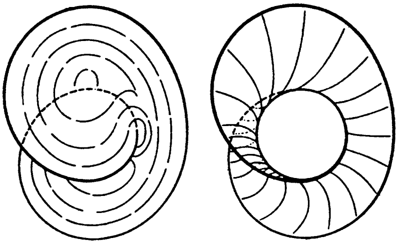


Fig. 2. A Jordan curve bounding **(a)** a disk-type minimal surface and **(b)** a minimal Möbius strip

A considerable simplification of their methods was found by R. Courant [4, 5] and, independently, by L. Tonelli [1]. In the present chapter we want to describe the Courant–Tonelli approach to Plateau's problem.

Recall that regular surfaces of least area are minimal surfaces, in the sense that their mean curvature vanishes throughout. Thus we can formulate a somewhat more general version of Plateau's problem: *Given a closed rectifiable Jordan curve Γ, find a minimal surface spanned by Γ*. Then the least area problem for Γ is more stringent than the Plateau problem: the first question deals with the (absolute or relative) minimizers of area, whereas the second is concerned with the stationary points of the area functional.

Note that for a fixed boundary contour Γ the solutions to Plateau's problem are by no means uniquely determined. Moreover, there may exist solutions of different genus within the same boundary curve, and there may exist both orientable and non-orientable minimal surfaces within the same boundary frame. This is illustrated by the minimal surfaces depicted in Fig. 2.

Even if we fix the topological type of the solutions to Plateau's problem, the unique solvability is, in general, not ensured. For instance, Figs. 1 and 4 depict some boundary configurations which can span several minimal surfaces of the topological type of the disk. In Section 4.9, we shall give a survey of what is known about the number of disk-type solutions to Plateau's problem. In the Scholia (Section 4.15) as well as in Chapters 5 and 7, the reader will find more

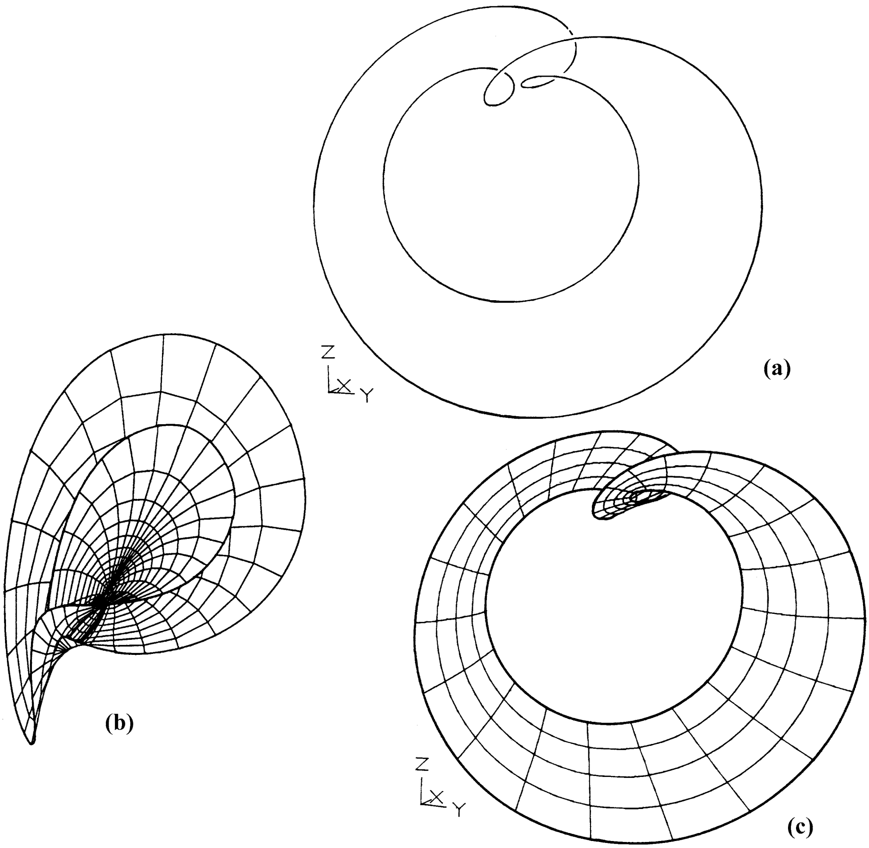


Fig. 3. A closed Jordan curve **(a)**, bounding a disk-type minimal surface **(b)**, as well as a Möbius strip **(c)**

examples and further results on the number of solutions of Plateau's problem, and we shall also discuss the question whether solutions are immersed or even embedded.

Other boundary value problems for minimal surfaces will be considered in Chapter 8 and in Vols. 2 and 3. For example, the last chapter of this volume as well as Chapter 4 of Vol. 3 deal with solutions of the *general Plateau problem* (also called *Douglas problem*) where one has to find a minimal surface of possibly higher topological type spanned by a frame consisting of one or several curves.

We begin the present chapter by having a closer look at Plateau's problem. First we compare Dirichlet's integral with the area functional, and we shall explain why it seems to be more profitable to minimize the Dirichlet integral rather than the area. Then, in Section 4.2, we set up Plateau's problem in a form that we shall deal with in Sections 4.3–4.5. In Section 4.2, we

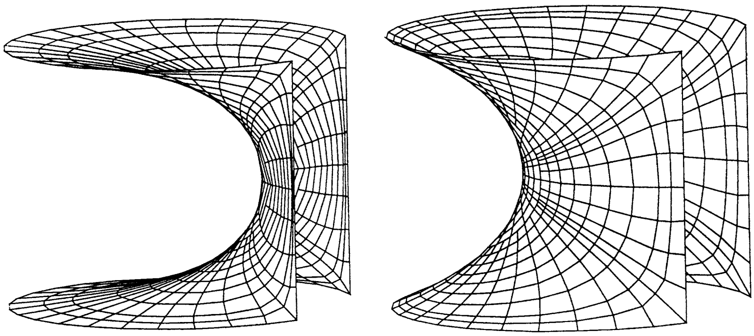


Fig. 4. Another Jordan curve spanned by two disk-type minimal surfaces

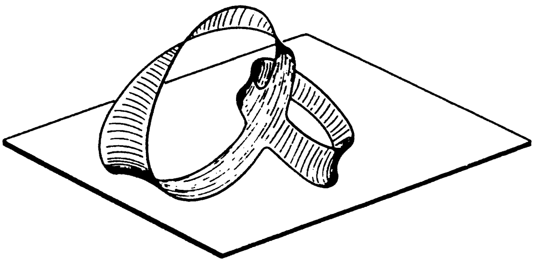


Fig. 5. A Jordan curve bounding a one-sided minimal surface of higher topological type

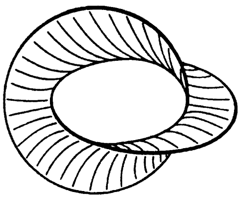


Fig. 6. Two interlocked Jordan curves spanned by an annulus-type minimal surface

describe the minimization procedure that will lead to a solution of Plateau's problem, and in Section 4.3, we prove the uniform convergence of a suitably chosen minimizing sequence to a harmonic mapping. This is achieved with the aid of the Courant–Lebesgue lemma proved in Section 4.4. In Section 4.5 we use variations of the independent variables for establishing a variational formula, from which we can derive that the minimizer $X(u, v)$, constructed in Section 4.3, also satisfies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0.$$

Hence it follows that X actually is a minimal surface solving Plateau's problem for the prescribed boundary curve Γ. Finally we shall see why X is also

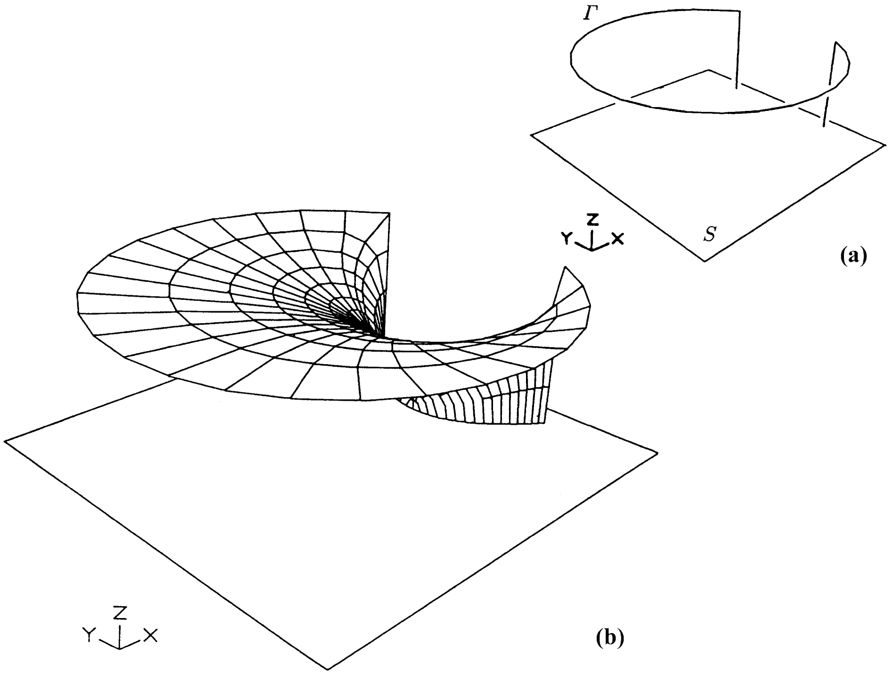


Fig. 7. (a) A configuration consisting of a planar surface S and a Jordan arc Γ. **(b)** Solution of the partially free boundary value problem corresponding to the configuration $\langle \Gamma, S \rangle$, computed by a finite-element method

a solution of the least area problem, using Morrey's lemma on ϵ-conformal mappings. A self-contained proof of this result is presented in Section 4.10; it is described below.

A slight modification of Courant's approach, given in Section 4.6, will lead to the solution of the *partially free boundary problem*.

A few results concerning the boundary behavior of minimal surfaces with rectifiable boundaries are collected in Section 4.7. They will in particular be needed in Chapter 5 of Vol. 2.

Reflection principles for minimal surfaces will be formulated in Section 4.8. Essentially we shall prove again two results from Section 3.4, without using Schwarz's solution to Björling's problem.

In 4.9 we give a survey on some results concerning the uniqueness and nonuniqueness of solutions to Plateau's problem; in particular Radó's uniqueness result is proved. Generalizations of Radó's theorem to free boundary problems are studied in Chapters 1 and 2 of Vol. 3.

Another approach to Plateau's problem, presented in 4.10, proceeds by minimizing the convex combination $(1-\epsilon)A+\epsilon D$ of the area functional A and the Dirichlet integral D for any $\epsilon \in (0,1]$ in $\mathcal{C}(\Gamma)$. It turns out that any minimizer yields a conformally parametrized solution of the problem "$A \to$ min

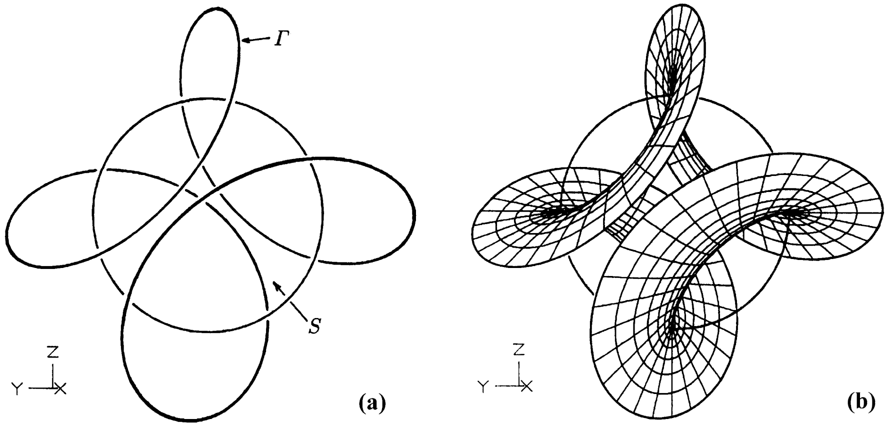


Fig. 8. (a) A boundary configuration $\langle \Gamma, S\rangle$ consisting of a disk S and of a closed Jordan curve Γ disjoint from S. **(b)** An annulus-type minimal surface which is stationary in $\langle \Gamma, S\rangle$

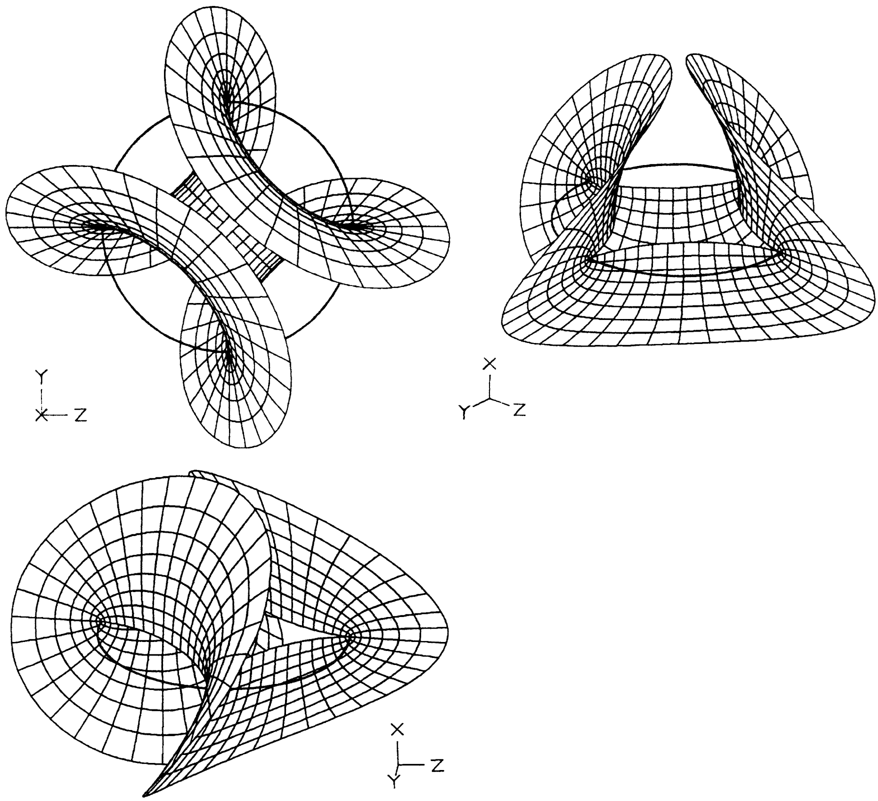


Fig. 9. Three more views of the minimal surface described in Fig. 8

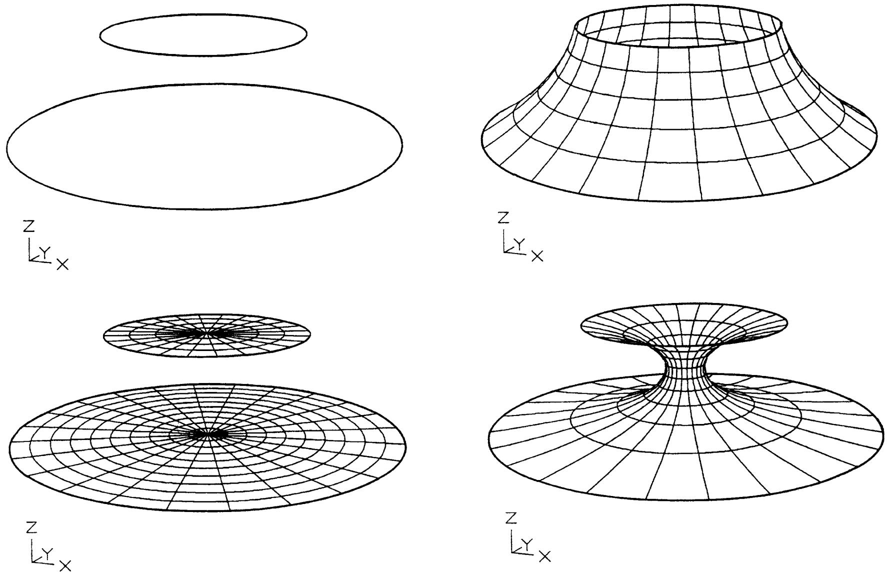


Fig. 10. The general Plateau problem consists in finding minimal surfaces spanning several closed Jordan curves. Here we show two parallel coaxial circles bounding three minimal surfaces of rotation

in $\mathcal{C}(\Gamma)$" which also minimizes D in $\mathcal{C}(\Gamma)$. This way we arrive at another proof of Theorem 4 in 4.5 and in particular of the relation (40) in 4.5 stating that $\overline{a}(\Gamma) = \overline{e}(\Gamma)$. This new approach only applies methods developed in the present chapter and completely avoids Morrey's Lemma on ϵ-conformal mappings (see 4.5). Thus no results on quasiconformal mappings nor on conformal representations of surfaces are needed for solving the minimal-area problem. Actually, the underlying idea of 4.10 can be used to obtain conformal representations of surfaces or of two-dimensional Riemannian metrics. This will be carried out in 4.11 where we show that the solution of Plateau's problem for planar contours provides a proof of the *Riemann mapping theorem*. This way we also verify that planar solutions to Plateau's problem are area-minimizing, free of branch points, and uniquely determined (up to a conformal reparametrization).

In a similar manner we derive other mapping theorems such as Lichtenstein's mapping theorem.

Nonrectifiable Jordan curves in $\mathbb{R}^3$ no longer need to bound a disk-type surface of finite Dirichlet integral. Nevertheless J. Douglas proved that any closed Jordan curve in $\mathbb{R}^3$ bounds a continuous disk-type minimal surface. A proof of this fact is presented in Section 4.12.

In Section 4.13 it is proved that every oriented closed, rectifiable Jordan curve bounds a continuous and conformally parametrized disk-type surface of finite area that minimizes an arbitrarily given regular Cartan functional, i.e. a given regular two-dimensional and parameter invariant variational integral $\mathcal{F}(X) = \int_B F(X, X_u \wedge X_v)\, du\, dv$. Here no general regularity theory for the corresponding Euler equation is available; therefore the existence proof is based on a variational method that resembles the technique of Section 4.10.

Thereafter we derive the basic isoperimetric inequality for disk-type minimal surfaces. Generalizations of this inequality are studied in Chapter 6 and in Chapter 4 of Vol. 2.

Finally the Scholia in Section 4.15 give a brief survey of the history of Plateau's problem as well as references to the literature. Moreover some basic results on the nonexistence of branch points for minimizers are described. In addition we discuss the question as to whether a contour bounds embedded solutions, the problem of uniqueness and nonuniqueness, index theorems, generic finiteness, and Morse-theoretic results. These topics will also (and in more detail) be treated in Chapter 6 and in Vol. 3. Thereafter we review some results on solutions to obstacle problems, a detailed presentation of which is given in Chapter 4 of Vol. 2. At last, some results on systems of minimal surfaces are described.

4.1 Area Functional Versus Dirichlet Integral

If one tries to formulate and to solve Plateau's problem, cumbersome difficulties may turn up. Among other problems one has to face the fact that there exist mathematical solutions to Plateau's problem which cannot be realized in experiment by soap films. This is, of course, to be expected for merely stationary solutions which are not minimizing, because they correspond to unstable soap films, and these will be destroyed by the tiniest perturbation of the soap lamellae caused by, say, a slight shaking of the boundary frame or by a breath of air.

However it can also happen that (mathematical) solutions of Plateau's problem have branch points, and that they have self-intersections. Both phenomena are unrealistic in the physical sense because Plateau has discovered the following rule for a stable configuration of soap films:

Three adjacent minimal surfaces of an area-minimizing system of surfaces, corresponding to a stable system of soap films, meet in a smooth line at an angle of $120°$. *Only four such lines, each being the soul of three soap films, can meet at a common point. At such a vertex, each pair of liquid edges forms an angle* φ *of* $109°28'16''$ *or, more precisely, of* $\cos\varphi = -1/3$.

Figure 11 in Section 4.15 shows a system of soap films exhibiting these features.

Solutions of Plateau's problem, which are absolute minimizers of area, cannot have interior branch points according to a result by Osserman–Gulliver–Alt. Their proof of this result is rather difficult and lengthy; thus it will only be sketched in Sections 1.9 and 5.3 of Vol. 2 (see also the Scholia 4.15 of the

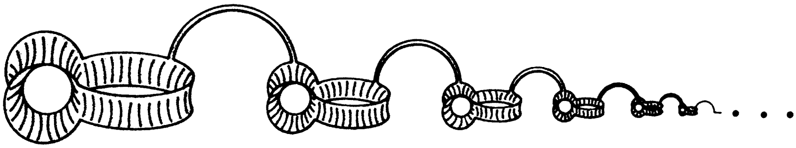


Fig. 1. The monster surface: a minimal surface of infinite genus

present chapter and the Scholia 6.7 of Vol. 2). A new approach leading to this result is described in Chapter 6 of Vol. 2.

Yet, despite the absence of branch points for minimizers, self-intersections of (mathematical) solutions are still conceivable, and so far only a few positive results are known, for instance:

If Γ is a closed Jordan curve that lies on a convex surface, then Γ bounds a disk-type minimal surface without self-intersections.

Another positive result, due to Ekholm, White, and Wienholtz [1] is the following:

If Γ is a closed Jordan curve in $\mathbb{R}^3$ with total curvature less or equal to 4π, then any minimal surface—independently of its topological type—is embedded up to and including the boundary, with no interior branch points.

A brief survey on the existence of embedded solutions of Plateau's problem is given in the Scholia 4.15, Subsection 3.

To solve Plateau's problem we would like to use the classical approach, which consists in minimizing area among surfaces given as mappings from a two-dimensional parameter domain into $\mathbb{R}^3$, this way fixing the topological type of the admissible surfaces. However, as we have already seen, it is by no means clear what the topological type of the surface of least area in a given configuration Γ will be. In fact, there may be rectifiable boundaries for which the area-minimizing solution of Plateau's problem is of infinite genus. An example for this phenomenon is depicted in Fig. 1.

Let us now restrict ourselves to surfaces $X \in C^0(\bar{B}, \mathbb{R}^3)$ which are parametrized on the closure of the unit disk $B = \{w \in \mathbb{C} \colon |w| < 1\}$, and which map the circle ∂B topologically onto a prescribed closed Jordan curve Γ in $\mathbb{R}^3$. Such a surface is said to be a solution of Plateau's problem for Γ if its restriction to B is a minimal surface. Since minimal surfaces are the critical points of the area functional

$$A_B(X) = \int_B |X_u \wedge X_v| \, du \, dv,$$

one is tempted to look for solutions of Plateau's problem by minimizing $A_B(X)$ in the class of all surfaces $X \in C^0(\bar{B}, \mathbb{R}^3)$ mapping ∂B homeomorphically onto Γ. But this method will produce literally hair-raising solutions. This can be seen as follows. Suppose that Γ is a circle in $\mathbb{R}^3$ contained in the x, y-plane, say

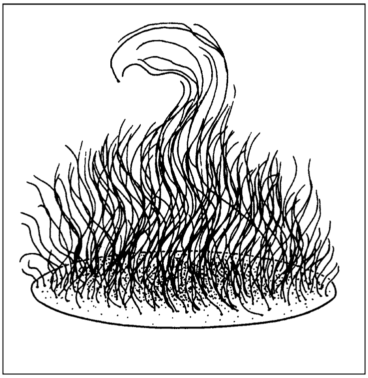


Fig. 2. A hairy disk

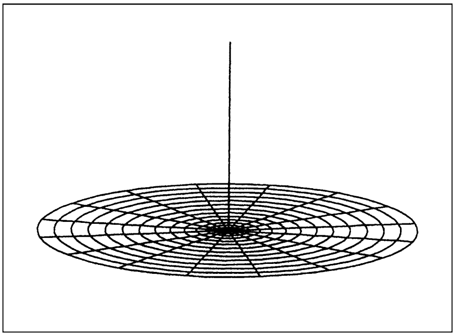


Fig. 3. A hair C^∞-grown on a disk

$$\Gamma = \{(x, y, z) \colon x^2 + y^2 = 1, z = 0\},$$

and let $K(\Gamma) = \{(x, y, z) \colon x^2 + y^2 \le 1, z = 0\}$ be the disk which is bounded by Γ. On account of the maximum principle, the only minimal surfaces X of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which map ∂B topologically onto Γ and satisfy

(1) $$\Delta X = 0,$$

(2) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

in B, are regular conformal mappings of $\bar{B}$ onto $K(\Gamma)$ (cf. Section 4.11).

On the other hand, among the minimizers of the area functional $A_B(X)$, there are mappings $X : \bar{B} \to \mathbb{R}^3$ which parametrize sets $K^*(\Gamma)$ which may be viewed as *hairy disks* bounded by Γ (see Fig. 2). They occur as additional, though nonregular, minimizers of A_B since hairs do not contribute to surface area. For example, *let us raise just one hair on the disk* $K(\Gamma)$. To this end, we consider the set

$$K^*(\Gamma) = K(\Gamma) \cup H$$

consisting of the disk $K(\Gamma)$ and the hair

$$H = \{(x, y, z) \colon x = y = 0,\ 0 \le z \le 1\}$$

attached to the center of $K(\Gamma)$. Then $K^*(\Gamma)$ can be parametrized by the following mapping $X(u, v)$ of class $C^\infty(\bar{B}, \mathbb{R}^3)$:

$$x(u, v) = y(u, v) := 0, \quad z(u, v) := \varphi(r) \quad \text{for } 0 \le r \le \tfrac{1}{2},$$

where $r = \sqrt{u^2 + v^2}$, and

$$x(u, v) := \psi(r) \cos\theta, \quad y(u, v) := \psi(r) \sin\theta, \quad z(u, v) := 0 \quad \text{for } \tfrac{1}{2} \le r \le 1.$$

Here, the functions $\varphi(r)$ and $\psi(r)$ are defined by

$$\varphi(r) := \exp 4\biggl(1 - \frac{1}{1 - 4r^2}\biggr), \quad \psi(r) := \exp 4\biggl(\frac{1}{3} - \frac{1}{4r^2 - 1}\biggr).$$

Note that the surface $X(u, v)$ is irregular for $0 \le r \le \frac{1}{2}$ which is also evident from the fact that the whole disk $B_{1/2} = \{(u, v) \colon u^2 + v^2 < \frac{1}{4}\}$ is mapped into the hair H (cf. Fig. 3).

Consequently, if we would use the variational problem

$$A_B(X) \to \min,$$

we would have to cope with a host of nasty solutions. In order to derive a reasonable solution satisfying equations (1) and (2), we would have to cut off all the hairs from a hairy solution.[1] This is fairly easy in the setting of geometric measure theory since a two-dimensional measure neglects hairs as sets of measure zero, whereas in the context of mappings the regularization of solutions requires quite an elaborate procedure.

In order to avoid this difficulty, we shall proceed similarly as in Riemannian geometry where one studies the one-dimensional Dirichlet instead of the length functional, using the fact that the critical points of Dirichlet's integral are also critical points of the length functional which are parametrized proportionally to the arc length, and vice versa. An analogous relation holds between the stationary surfaces of the two-dimensional *Dirichlet integral*

$$D_B(X) = \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2)\, du\, dv \tag{3}$$

and the area functional $A_B(X)$. This can be seen as follows: For arbitrary vectors $p, q \in \mathbb{R}^3$ we have

$$|p \wedge q| \le |p||q|,$$

[1] When David Hilbert had established Dirichlet's principle, Felix Klein wrote: "Hilbert schneidet den Flächen die Haare ab" (cf. D. Hilbert, Gesammelte Abhandlungen, Vol. 3, p. 409).

and therefore

$$|p \wedge q| \le \tfrac{1}{2}|p|^2 + \tfrac{1}{2}|q|^2. \tag{4}$$

The equality sign in (4) holds if and only if $p \perp q$ and $|p| = |q|$. Suppose now that $X \in C^1(B, \mathbb{R}^3)$ has a finite Dirichlet integral $D_B(X)$. Then we obtain the inequality

$$A_B(X) \le D_B(X), \tag{5}$$

and the equality sign is satisfied if and only if the conformality relations (2) are fulfilled on B. In other words, *area functional and Dirichlet integral coincide exactly on the conformally parametrized surfaces X, and, in general, the Dirichlet integral furnishes a majorant for the area functional.*

Moreover, every smooth regular surface $X : B \to \mathbb{R}^3$ can, by Lichtenstein's theorem, be reparametrized by a regular change $\tau : B \to B$ of parameters such that $Y := X \circ \tau$ satisfies the conformality relations

$$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0,$$

and we obtain

$$D_B(Y) = A_B(Y) = A_B(X).$$

This observation makes it plausible that, within a class $\mathcal{C}$ of surfaces which is invariant with respect to parameter changes, minimizers of $D_B(X)$ will also be minimizers of $A_B(X)$, and more generally, that stationary points of $D_B(X)$ will be stationary points of $A_B(X)$.

Certainly the class $\mathcal{C}$ defined by Plateau's boundary condition $X : \partial B \to \Gamma$ has this invariance property. Thus we are led to the idea that we should minimize Dirichlet's integral instead of the area functional since we would also obtain a minimizer for $A_B(X)$.

We will presently dispense with putting this idea on solid ground by making the above reasoning rigorous. Instead we shall simply use the following idea: Minimize $D_B(X)$ instead of $A_B(X)$, and justify it a posteriori by proving that, in suitable classes $\mathcal{C}$, the stationary points of $D_B(X)$ are in fact minimal surfaces.

The use of Dirichlet's integral in the minimizing procedure is a advantageous for several reasons:

(i) It is not advisable to carry out the minimization among regular surfaces only, because the class of such surfaces is not closed with respect to uniform convergence of $\bar{B}$ or to $H_2^1(B)$-convergence, and a better convergence of minimizing sequences will be difficult (or even impossible) to obtain. However, if we admit general surfaces for minimization, the hairy monsters will also turn up as minimizers when $A_B(X)$ is minimized. They are excluded if we instead minimize $D_B(X)$.

(ii) Minimizing sequences of $D_B(X)$ have better compactness properties than those of $A_B(X)$.

The basic reason for (i) and (ii) is that the expression $|p|^2 + |q|^2$ only vanishes if $p = 0$ and $q = 0$ holds, whereas $|p \wedge q|$ is zero for any pair of collinear vectors p and q. Moreover, $A_B(X)$ is invariant with respect to arbitrary reparametrizations of X, while $D_B(X)$ remains unchanged only under conformal parameter transformations.

Keeping these ideas in mind, we will now proceed to formulate a minimum problem, the solution of which will turn out to be a solution of Plateau's problem.

Notational convention: Occasionally we shall write $D(X, B)$ and $A(X, B)$ instead of $D_B(X)$ and $A_B(X)$, and, for two mappings X, Y, we denote by $D_B(X, Y)$ the polarization of the Dirichlet integral:

$$(6)\quad D_B(X,Y) := \frac{1}{2}\int_B (\langle X_u, Y_u\rangle + \langle X_v, Y_v\rangle)\, du\, dv = \frac{1}{2}\int_B \langle \nabla X, \nabla Y\rangle\, du\, dv.$$

4.2 Rigorous Formulation of Plateau's Problem and of the Minimization Process

Set

$$B := \{w \in \mathbb{C} \colon |w| < 1\}$$

and

$$C := \{w \in \mathbb{C} \colon |w| = 1\} = \partial B.$$

A *closed Jordan curve Γ in* $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ which is homeomorphic to ∂B. By distinguishing some fixed homeomorphism $\gamma : C \to \Gamma$ from C onto Γ we equip Γ with an *orientation*, and we say that Γ is *oriented* (by γ).

Definition 1. *Given a closed Jordan curve Γ in $\mathbb{R}^3$, we say that $X : \bar{B} \to \mathbb{R}^3$ is a solution of Plateau's problem for the boundary contour Γ (or: a minimal surface spanned in Γ) if it fulfills the following three conditions*:

(i) $X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$;

(ii) *The surface X satisfies in B the equations*

$$(1)\qquad \Delta X = 0,$$

$$(2)\qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0;$$

(iii) *The restriction $X|_C$ of X to the boundary C of the parameter domain B is a homeomorphism of C onto Γ.*

If it is necessary to be more precise, we shall denote a minimal surface X described in this definition as *disk-type solution of Plateau's problem for the contour Γ.*

Condition (iii) is equivalent to the assumption that $X|_C$ is a continuous, strictly monotonic (i.e. injective) mapping of C onto Γ.

Clearly this condition is not closed with respect to uniform convergence on C since uniform limits of strictly monotonic functions can be merely weakly monotonic, that is, they may have arcs of constancy on C. To be precise, we give the following

Definition 2. *Suppose that Γ is a closed Jordan curve in $\mathbb{R}^3$, which is oriented by a homeomorphism $\gamma : C \to \Gamma$ from C onto Γ. Then a continuous mapping $\varphi : C \to \Gamma$ of C onto Γ is said to be weakly monotonic if there is a nondecreasing continuous function $\tau : [0, 2\pi] \to \mathbb{R}$ with $\tau(2\pi) = \tau(0) + 2\pi$ such that*

$$\varphi(e^{i\theta}) = \gamma(e^{i\tau(\theta)}) \quad \text{for } 0 \le \theta \le 2\pi. \tag{3}$$

In other words, φ is weakly monotonic if the image points $\varphi(w)$ traverse Γ in a constant direction when w moves along C in a constant direction. The image points may stand still but never move backwards if w moves monotonically on C, and $\varphi(w)$ moves once around Γ if w travels once around C.

Introducing the mapping $\mathcal{E} : [0, 2\pi] \to C$ by $\mathcal{E}(\theta) := e^{i\theta}$, we can write (3) as

$$\varphi \circ \mathcal{E} = \gamma \circ \mathcal{E} \circ \tau$$

whence we arrive at

$$\mathcal{E} \circ \tau = \gamma^{-1} \circ \varphi \circ \mathcal{E}. \tag{4}$$

From this formula we obtain at once:

Lemma 1. *Let $\{\varphi_n\}$ be a sequence of weakly monotonic, continuous mappings of C onto a closed Jordan curve Γ, and suppose that the mappings φ_n converge uniformly on C to some mapping $\varphi : C \to \mathbb{R}^3$. Then φ is a weakly monotonic continuous mapping of C onto Γ.*

Remark. The assertion of Lemma 1 remains true if we assume that the mappings ψ_n are weakly monotonic, continuous mappings of C onto closed Jordan arcs Γ_n which converge in the sense of Fréchet to some Jordan arc Γ. That means, there are homeomorphisms γ_n and γ of C onto Γ_n and Γ respectively, such that γ_n tends uniformly to γ as $n \to \infty$.

Now we want to set up the variational problem that will lead us to a solution of Plateau's problem. First we define the *class* $\mathcal{C}(\Gamma)$ *of admissible functions*. We have exactly two essentially different orientations of Γ. Correspondingly there will be exactly two possibilities to define $\mathcal{C}(\Gamma)$ if Γ is not oriented, while $\mathcal{C}(\Gamma)$ will be uniquely defined for an oriented contour Γ.

Recall that every function $X \in H_2^1(B, \mathbb{R}^3)$ has a trace $X|_C$ on the boundary $C = \partial B$ which is of class $L_2(C, \mathbb{R}^3)$.

Definition 3. *Given a closed Jordan curve Γ in $\mathbb{R}^3$, a mapping $X : B \to \mathbb{R}^3$ is said to be of class $\mathcal{C}(\Gamma)$ with respect to a fixed orientation $\gamma : C \to \Gamma$ of Γ if $X \in H_2^1(B, \mathbb{R}^3)$ and if its trace $X|_C$ can be represented by a weakly monotonic, continuous mapping $\varphi : C \to \Gamma$ of C onto Γ (i.e., every $L_2(C)$-representative of $X|_C$ coincides with φ except for a subset of zero 1-dimensional Hausdorff measure).*

Let

$$D(X) = D_B(X) := \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2)\, du\, dv \tag{5}$$

be the Dirichlet integral of a mapping $X \in H_2^1(B, \mathbb{R}^3)$. Then we define the variational problem $\mathcal{P}(\Gamma)$ associated with Plateau's problem for the oriented curve Γ as the following task:

Minimize Dirichlet's integral $D(X)$, defined by (5), in the class $\mathcal{C}(\Gamma)$.

In other words, setting

$$e(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}(\Gamma)\}, \tag{6}$$

we have to find a surface $X \in \mathcal{C}(\Gamma)$ such that

$$D(X) = e(\Gamma) \tag{7}$$

is satisfied.

In order to solve the minimum problem $\mathcal{P}(\Gamma)$, we shall have to find a minimizing sequence $\{X_n\}$ whose boundary values $X_n|_C$ contain a subsequence which is uniformly convergent on C. The selection of such a minimizing sequence will be achieved by the following artifice:

Fix three different points w_1, w_2, w_3 on C, an orientation $\gamma : C \to \Gamma$ of Γ, and three different points Q_1, Q_2, Q_3 on Γ such that $\gamma(w_k) = Q_k$, $k = 1, 2, 3$. Let $\mathcal{C}(\Gamma)$ be defined with respect to the orientation γ of Γ, and consider those mappings $X \in \mathcal{C}(\Gamma)$ which satisfy the *three-point condition*

$$X(w_k) = Q_k, \quad k = 1, 2, 3. \tag{8}$$

The set of such mappings X will be denoted by $\mathcal{C}^*(\Gamma)$. Set

$$e^*(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}^*(\Gamma)\}. \tag{9}$$

We clearly have

$$e(\Gamma) \le e^*(\Gamma).$$

Moreover, if $X \in \mathcal{C}(\Gamma)$, then there exist three different points $\zeta_1, \zeta_2, \zeta_3$ on C such that

$$X(\zeta_k) = Q_k, \quad k = 1, 2, 3.$$

Let σ be a strictly conformal mapping of $\bar{B}$ onto itself with the property that

$$\sigma(w_k) = \zeta_k, \quad k = 1, 2, 3.$$

Then the mapping $Y := X \circ \sigma$ is of class $\mathcal{C}^*(\Gamma)$ and satisfies $D(Y) = D(X)$, because of the conformal invariance of the Dirichlet integral. Hence we even obtain

$$e(\Gamma) = e^*(\Gamma). \tag{10}$$

Consequently, any solution X of the *restricted minimum problem*

$$\mathcal{P}^*(\Gamma)\colon \quad \textit{Minimize } D(X) \textit{ in the class } \mathcal{C}^*(\Gamma) \tag{11}$$

is also a solution of the original minimum problem $\mathcal{P}(\Gamma)$. Hence we shall try to solve $\mathcal{P}^*(\Gamma)$ instead of $\mathcal{P}(\Gamma)$, in this way obtaining a convenient compactness property of the boundary values of any minimizing sequence, as we shall see.

Before we can start with our minimizing process, one final difficulty remains to be solved. Since $\mathcal{P}^*(\Gamma)$ would not have a solution if $\mathcal{C}^*(\Gamma)$ were empty, let us now study under which circumstances $\mathcal{C}^*(\Gamma)$ or, equivalently, $\mathcal{C}(\Gamma)$ is certainly nonempty.

Let $\varphi : C \to \Gamma$ be a homeomorphism representing Γ, and let

$$\varphi(e^{i\theta}) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \{A_n \cos n\theta + B_n \sin n\,\theta\} \tag{12}$$

be its Fourier expansion, $A_n, B_n \in \mathbb{R}^3$, which is convergent in $L_2([0, 2\pi], \mathbb{R}^3)$. We can assume that φ satisfies the prescribed three-point condition, i.e.,

$$\varphi(w_k) = Q_k, \quad k = 1, 2, 3.$$

Let ρ, θ be polar coordinates about the origin of the w-plane, that is,

$$w = \rho e^{i\theta},$$

and set

$$X(w) := \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta). \tag{13}$$

Since $|A_n|$ and $|B_n|$ are bounded by $2 \sup_C |\varphi|$, the series on the right-hand side converges uniformly on every compact subset of B, and a well-known computation shows that its limit is nothing but Poisson's integral for the boundary values $\varphi(e^{i\theta})$, i.e.,

$$X(w) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\psi}) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\theta - \psi)}\, d\psi \tag{14}$$

for $w = \rho e^{i\theta}$, $\rho < 1$. By the classical result of H.A. Schwarz, the mapping $X(w)$ is harmonic in B and satisfies $X(w) \to \varphi(w_0)$ as $w \to w_0$, $w \in B$, for every $w_0 \in \partial B$. Hence X can be extended to a continuous function on $\bar{B}$ with the boundary values φ on $C = \partial B$. A straight-forward computation yields

$$D(X) = \frac{\pi}{2} \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2). \tag{15}$$

Consequently the map $X : \bar{B} \to \mathbb{R}^3$ belongs to the class $H_2^1(B, \mathbb{R}^3)$ if and only if

$$\sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2) < \infty. \tag{16}$$

If this is true, then $\mathcal{C}^*(\Gamma)$ is nonempty.

Condition (16) is satisfied if and only if $\phi(\theta) := \varphi(e^{i\theta})$ has half a derivative which is square-integrable. This is, for example, true if the representation $\varphi : C \to \Gamma$ of the Jordan curve Γ is Lipschitz continuous. Such a representation of Γ exists if and only if Γ has finite length. Hence, *for any rectifiable Jordan curve Γ, neither $\mathcal{C}(\Gamma)$ nor $\mathcal{C}^*(\Gamma)$ are empty.* Note, however, that the rectifiability of Γ is only sufficient but not necessary for $\mathcal{C}(\Gamma)$ to be nonempty.

Remark. Since D is invariant under strictly conformal as well as under anticonformal mappings of B, its infimum $e(\Gamma)$ in $\mathcal{C}(\Gamma)$ is independent of the chosen orientation of Γ. The same holds for the generalized Dirichlet integral (34) in Section 4.5, whereas the infimum of the integral (36) in 4.5 may depend on the orientation of Γ, and the same holds for "Cartan functionals", as considered in Section 4.13. Thus for conformally invariant integrals in the general sense, such as D, we may neglect the orientation of the boundary contour Γ; both orientations lead to the same solutions of $\mathcal{P}(\Gamma)$; in the noninvariant cases we might obtain different solutions for opposite orientations.

Convention. *It goes without saying that $\mathcal{C}(\Gamma)$ always is defined with respect to a fixed orientation of Γ.*

4.3 Existence Proof, Part I: Solution of the Variational Problem

Let Γ be a closed oriented Jordan curve in $\mathbb{R}^3$, and let $\mathcal{C}(\Gamma)$ be the class of admissible surfaces bounded by Γ which we have defined in Section 4.2. The aim of this section is to find a solution of the minimum problem

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \text{in the class } \mathcal{C}(\Gamma).$$

We are going to prove the following

Theorem 1. *If $\mathcal{C}(\Gamma)$ is nonempty, then the minimum problem $\mathcal{P}(\Gamma)$ has at least one solution which is continuous on $\overline{B}$ and harmonic in B. In particular, $\mathcal{P}(\Gamma)$ has such a solution for every rectifiable curve Γ.*

Proof. As we have seen in Section 4.2, the class $\mathcal{C}(\Gamma)$ is nonempty for every closed Jordan curve of finite length. Hence it suffices to prove the first part of the assertion. Recall that we only have to find a solution of

$$\mathcal{P}^*(\Gamma): \quad D(X) \to \min \quad \text{in the class } \mathcal{C}^*(\Gamma),$$

where $\mathcal{C}^*(\Gamma)$ denotes the set of surfaces $X \in \mathcal{C}(\Gamma)$ satisfying a fixed three-point condition

$$X(w_k) = Q_k, \quad k = 1, 2, 3. \tag{1}$$

Here, w_1, w_2, w_3 are three different points on $C = \partial B$, and Q_1, Q_2, Q_3 denote three different points on Γ.

Choose a sequence $\{X_n\}$ of mappings $X_n \in \mathcal{C}^*(\Gamma)$ such that

$$\lim_{n\to\infty} D(X_n) = e^*(\Gamma) \tag{2}$$

holds. We can assume without loss of generality that X_n is a surface of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which satisfies

$$\Delta X_n = 0 \quad \text{in } B,$$

$n = 1, 2, 3, \dots$. (Otherwise we replace X_n by the solution Z_n of the boundary value problem

$$\begin{aligned} \Delta Z_n &= 0 \quad \text{in } B, \\ Z_n &= X_n \quad \text{on } C \end{aligned}$$

which is continuous on $\bar{B}$ and of class $C^2(B, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$. It is well known that this problem has exactly one solution. This solution minimizes $D(X)$ among all $X \in H_2^1(B, \mathbb{R}^3)$ with $X - X_n \in \mathring{H}_2^1(B, \mathbb{R}^3)$. Consequently, $D(Z_n) \le D(X_n)$, and by construction we have $Z_n \in \mathcal{C}^*(\Gamma)$ whence $e^*(\Gamma) \le D(Z_n)$. Thus we obtain

$$e^*(\Gamma) \le D(Z_n) \le D(X_n) \to e^*(\Gamma),$$

and therefore

$$\lim_{n\to\infty} D(Z_n) = e^*(\Gamma).$$

Hence we have found a minimizing sequence $\{Z_n\}$ for $\mathcal{P}^*(\Gamma)$ consisting of harmonic mappings Z_n which are continuous on $\bar{B}$.)

We now claim that the boundary values $X_{n|C}$ of the terms of any minimizing sequence $\{X_n\}$ for $\mathcal{P}^*(\Gamma)$ are equicontinuous on C. The key to this crucial result is the so-called Courant–Lebesgue lemma. We defer its proof to the next section so as not to interrupt our reasoning.

Courant–Lebesgue lemma. *Let X be of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$ and suppose that*

$$D(X) \le M \tag{3}$$

for some M with $0 \le M < \infty$. Then, for every $z_0 \in C$ and for each $\delta \in (0,1)$, there exists a number $\rho \in (\delta, \sqrt{\delta})$ such that the distance of the images $X(z), X(z')$ of the two intersection points z and z' of C with the circle $\partial B_\rho(z_0)$ can be estimated by

$$|X(z) - X(z')| \le \left\{ \frac{4M\pi}{\log 1/\delta} \right\}^{1/2}. \tag{4}$$

This lemma will be applied as follows: Since Γ is the topological image of C, there exists, for every $\varepsilon > 0$, a number $\lambda(\varepsilon) > 0$ with the following property:

Any pair of points $P, Q \in \Gamma$ with

$$0 < |P - Q| < \lambda(\varepsilon) \tag{5}$$

decomposes Γ into two arcs $\Gamma_1(P,Q)$ and $\Gamma_2(P,Q)$ such that

$$\operatorname{diam} \Gamma_1(P,Q) < \varepsilon \tag{6}$$

holds. Hence, if $0 < \varepsilon < \varepsilon_0 := \min_{j \ne k} |Q_j - Q_k|$, then $\Gamma_1(P,Q)$ can contain at most one of the points Q_j appearing in the three-point condition (1).

Let now X be an arbitrary mapping in $\mathcal{C}^*(\Gamma)$ that fulfills the assumptions of the Courant–Lebesgue lemma, and let $\delta_0 \in (0,1)$ be a fixed number with

$$2\sqrt{\delta_0} < \min_{j \ne k} |w_j - w_k| \tag{7}$$

where w_1, w_2, w_3 appear in (1).

For an arbitrary $\varepsilon \in (0, \varepsilon_0)$, we choose some number $\delta = \delta(\varepsilon) > 0$ such that

$$\left\{ \frac{4\pi M}{\log 1/\delta} \right\}^{1/2} < \lambda(\varepsilon) \tag{8}$$

and

$$\delta < \delta_0. \tag{9}$$

Consider an arbitrary point z_0 on C, and let $\rho \in (\delta, \sqrt{\delta})$ be some number such that the images $P := X(z), Q := X(z')$ of the two intersection points z, z' of C and $\partial B_\rho(z_0)$ satisfy

$$|P - Q| \le \left\{ \frac{4M\pi}{\log 1/\delta} \right\}^{1/2}.$$

Then we infer from (8) that $|P - Q| < \lambda(\varepsilon)$, whence

$$\operatorname{diam} \varGamma_1(P, Q) < \varepsilon$$

holds on account of (6). Because of $\varepsilon < \varepsilon_0$ the arc $\varGamma_1(P, Q)$ contains at most one of the points Q_j. On the other hand, it follows from $X \in \mathcal{C}^*(\varGamma)$ and from (1), (7), (9) that $X(C \cap \overline{B_\rho(z_0)})$ contains at most one of the points Q_j and must therefore coincide with the arc $\varGamma_1(P, Q)$:

$$\varGamma_1(P, Q) = X(C \cap \overline{B_\rho(z_0)}).$$

Consequently we have

$$|X(w) - X(w')| < \varepsilon \quad \text{for all } w, w' \in C \cap B_\rho(z_0).$$

This implies

$$(10) \qquad |X(w) - X(w')| < \varepsilon \quad \text{for all } w, w' \in C \text{ with } |w - w'| < \delta.$$

Consider now the minimizing sequence $\{X_n\}$. By (2), there is some number $M > 0$ such that

$$D(X_n) \le M$$

holds for all $n \in \mathbb{N}$. Thus we can apply (10) to $X = X_n, n = 1, 2, \ldots,$ and we conclude that the functions $X_n|_C$ are equicontinuous. Moreover, we infer from $X_n(C) = \varGamma$ that the functions $X_n|_C$ are uniformly bounded. Hence, by the theorem of Arzelà–Ascoli, we can assume that the $X_n|_C$ tend to some mapping $\varphi \in C^0(C, \mathbb{R}^3)$ as $n \to \infty$, uniformly on C, and that φ is a weakly monotonic mapping of C onto $\varGamma$. Since the functions X_n are continuous on $\bar{B}$ and harmonic in B, it follows that X_n tends uniformly on $\bar{B}$ to some function X, which is continuous on $\bar{B}$, harmonic in B, satisfies (1), and has the boundary values φ. Consequently, X is of class $\mathcal{C}^*(\varGamma)$, and therefore

$$e^*(\varGamma) \le D(X).$$

Moreover, a classical result for harmonic functions implies that grad X_n tends to grad X as $n \to \infty$, uniformly on every $B' \subset\subset B$, whence

$$\lim_{n \to \infty} D_{B'}(X_n) = D_{B'}(X)$$

and therefore

$$\liminf_{n \to \infty} D_B(X_n) \ge D_{B'}(X) \quad \text{if } B' \subset\subset B.$$

Thus we finally obtain

$$e^*(\varGamma) = \lim_{n \to \infty} D(X_n) \ge D(X) \ge e^*(\varGamma),$$

or

$$D(X) = e^*(\varGamma).$$

Therefore $X \in \mathcal{C}^*(\varGamma)$ is a minimizer of the Dirichlet integral $D(X)$ within the class $\mathcal{C}(\varGamma)$. $\square$

In the previous theorem we have obtained at least one harmonic minimizer of $D(X)$ in the class $\mathcal{C}(\Gamma)$. Now we want to show that every solution of $\mathcal{P}(\Gamma)$ is a harmonic mapping. In fact, we have

Theorem 2. *Every minimizer X of the Dirichlet integral within the class $\mathcal{C}(\Gamma)$ is continuous in $\bar{B}$ and harmonic in B.*

Proof. Let $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ be an arbitrary test function of class $C_c^\infty(B, \mathbb{R}^3)$. Then we have $X + \varepsilon\varphi \in \mathcal{C}(\Gamma)$ for every $\varepsilon \in \mathbb{R}$. On account of the minimum property of X, the quadratic polynomial

$$f(\varepsilon) := D(X + \varepsilon\varphi) = D(X) + 2\varepsilon D(X, \varphi) + \varepsilon^2 D(\varphi), \quad \varepsilon \in \mathbb{R},$$

has an absolute minimum at $\varepsilon = 0$, whence $f'(0) = 0$, or

$$D(X, \varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(B, \mathbb{R}^3). \tag{11}$$

By a classical result for harmonic functions (Weyl's lemma), we obtain from (11) that X is harmonic in B. Since $X \in H_2^1(B, \mathbb{R}^3)$ and $X|_C \in C^0(C, \mathbb{R}^3)$, it also follows that $X \in C^0(\bar{B}, \mathbb{R}^3)$. □

By the same reasoning that led to Theorem 1, we also obtain the following results (cf. Section 4.2, Lemma 1):

Theorem 3. *Let $\{\Gamma_n\}$ be a sequence of closed (oriented) Jordan curves in $\mathbb{R}^3$ which converge in the sense of Fréchet to some closed (oriented) Jordan curve Γ (notation: $\Gamma_n \to \Gamma$ as $n \to \infty$), and let $\{X_n\}$ be a sequence of mappings $X_n \in \mathcal{C}(\Gamma_n)$ with uniformly bounded Dirichlet integral, i.e.,*

$$D(X_n) \le M, \quad n \in \mathbb{N}. \tag{12}$$

Then their boundary values $\varphi_n := X_n|_C$ are equicontinuous if they satisfy a uniform three-point condition

$$\varphi_n(w_j) = Q_j^{(n)}, \quad j = 1, 2, 3, \tag{13}$$

with some points $w_j \in C$ and $Q_j^{(n)} \in \Gamma_n$, $j = 1, 2, 3$, such that $\lim_{n\to\infty} Q_j^{(n)} = Q_j$ holds, where Q_1, Q_2, Q_3 denote three different points on the limit curve Γ.

If, moreover, the mappings X_n are continuous on $\bar{B}$ and harmonic in B, then we can extract a subsequence $\{X_{n_p}\}$ that converges uniformly on $\bar{B}$ to some mapping $X \in \mathcal{C}(\Gamma)$ which is continuous on $\bar{B}$ and harmonic in B.

Remark. For minimal surfaces X_n, the isoperimetric inequality (cf. Section 4.14) implies that

$$D(X_n) \le \frac{1}{4\pi} L^2(\Gamma_n) \tag{14}$$

holds, where $L(\Gamma_n)$ denotes the lengths of the curves Γ_n. Hence condition (12) is satisfied by every sequence of minimal surfaces $X_n \in \mathcal{C}(\Gamma_n)$, $n = 1, 2, \ldots,$ spanned by closed Jordan curves Γ_n of uniformly bounded lengths,

$$L(\Gamma_n) \le l \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

Theorem 4. *Let $\Gamma, \Gamma_1, \Gamma_2, \dots$ be closed (oriented) Jordan curves in $\mathbb{R}^3$ with $\Gamma_n \to \Gamma$ as $n \to \infty$ (Fréchet convergence) and $\lim_{n\to\infty} e(\Gamma_n) = e(\Gamma)$. Furthermore, let $X_n \in \mathcal{C}(\Gamma_n)$ be a sequence of solutions for $\mathcal{P}(\Gamma_n)$ whose boundary values $\varphi_n = X_n|_C$ satisfy a uniform three-point condition such as in Theorem 3. Then we can extract a subsequence $\{X_{n_p}\}$ which converges uniformly on $\bar{B}$ to some solution X of $\mathcal{P}(\Gamma)$ as $p \to \infty$, and*

$$\lim_{n\to\infty} D(X_n) = D(X). \tag{16}$$

4.4 The Courant–Lebesgue Lemma

We now want to supply a proof for the Courant–Lebesgue lemma that was used in the previous section. In fact, this lemma will be an immediate consequence of the next proposition.

Let us introduce the following notations:

$$B := \{w \colon |w| < 1\}, \quad C := \partial B,$$

$$S_r(z_0) := B \cap B_r(z_0), \quad C_r(z_0) := \bar{B} \cap \partial B_r(z_0).$$

If $z_0 \in C$, then we can write

$$C_r(z_0) = \{z_0 + re^{i\theta} \colon \theta_1(r) \le \theta \le \theta_2(r)\}$$

with

$$0 < \theta_2(r) - \theta_1(r) < \pi.$$

Proposition 1. *Suppose that X is of class $C^0(\bar{B}, \mathbb{R}^n) \cap C^1(B, \mathbb{R}^n)$, $n \in \mathbb{N}$, and satisfies $D(X) < \infty$. Let z_0 be any point on C, and set $Z(r,\theta) := X(z_0 + re^{i\theta})$ where r, θ denote polar coordinates about z_0. Then, for every $\delta \in (0, R^2)$, $0 < R < 1$, there is a number $\rho \in (\delta, \sqrt{\delta})$ such that, for every pair θ, θ' with $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, we obtain the estimate*

$$\int_\theta^{\theta'} |Z_\theta(\rho, \theta)|\, d\theta \le \eta(\delta, R)|\theta - \theta'|^{1/2} \tag{1}$$

with

$$\eta(\delta, R) := \left\{ \frac{2}{\log(1/\delta)} \int_{S_R(z_0)} |\nabla X|^2\, du\, dv \right\}^{1/2}, \tag{2}$$

and in particular

$$|Z(\rho, \theta) - Z(\rho, \theta')| \le \eta(\delta, R)|\theta - \theta'|^{1/2}. \tag{3}$$

Remark. The assumption $z_0 \in C$ is not essential as we shall see from the proof. We shall leave it to the reader to formulate a corresponding result in other situations.

We begin the proof of Proposition 1 by verifying the following

Lemma 1. *Let X satisfy the assumptions of Proposition 1, and set*

$$Z(r,\theta) := X(z_0 + re^{i\theta}), \quad z_0 \in C,$$

and

$$p(r) := \int_{\theta_1(r)}^{\theta_2(r)} |Z_\theta(r,\theta)|^2 \, d\theta. \tag{4}$$

Moreover, let $\mathfrak{I}$ be a measurable subset of $(0,1)$, and suppose that both

$$0 < \int_{\mathfrak{I}} \frac{dr}{r} < \infty \quad \textit{and} \quad \int_{\mathfrak{I}} \frac{p(r)}{r} \, dr \le M < \infty \tag{5}$$

are satisfied. Then the set $\mathfrak{I}_M := \{\rho \in \mathfrak{I} \colon p(\rho) \int_{\mathfrak{I}} \frac{dr}{r} \le M\}$ has a positive 1-dimensional Lebesgue measure,

$$\mathcal{L}^1(\mathfrak{I}_M) > 0, \tag{6}$$

and for every $\rho \in \mathfrak{I}_M$ and all θ, θ' with $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$ we obtain the inequality

$$\int_\theta^{\theta'} |Z_\theta(\rho,\theta)| \, d\theta \le \left\{ M \Big/ \int_{\mathfrak{I}} \frac{dr}{r} \right\}^{1/2} |\theta - \theta'|^{1/2}. \tag{7}$$

Proof. (i) If $\mathcal{L}^1(\mathfrak{I}_M) = 0$, then we would obtain

$$p(\rho) > M \Big/ \int_{\mathfrak{I}} \frac{dr}{r} \quad \text{for almost all } \rho \in \mathfrak{I}.$$

Multiplying by $1/\rho$, and integrating over $\mathfrak{I}$ with respect to ρ, we would arrive at the inequality

$$\int_{\mathfrak{I}} \frac{p(\rho)}{\rho} \, d\rho > M$$

which is a contradiction to (5). Hence we see that $\mathcal{L}^1(\mathfrak{I}_M) > 0$.

(ii) Let $\rho \in \mathfrak{I}_M$ and $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$. Then it follows that

$$\begin{aligned} \int_\theta^{\theta'} |Z_\theta(\rho,\theta)| \, d\theta &\le \left\{ \int_\theta^{\theta'} |Z_\theta(\rho,\theta)|^2 \, d\theta \right\}^{1/2} |\theta - \theta'|^{1/2} \\ &\le \left\{ M \Big/ \int_{\mathfrak{I}} \frac{dr}{r} \right\}^{1/2} |\theta - \theta'|^{1/2}. \end{aligned}$$

□

Proof of Proposition 1. Let $p(r)$ be the function defined by (4). Then we obtain

$$\begin{aligned}\int_0^r \frac{p(\rho)}{\rho}\,d\rho &\le \int_0^r \int_{\theta_1(\rho)}^{\theta_2(\rho)} \left\{|Z_\rho(\rho,\theta)|^2 + \frac{1}{\rho^2}|Z_\theta(\rho,\theta)|^2\right\}\rho\,d\theta\,d\rho \\ &= 2D(X, S_r(z_0)).\end{aligned}$$

For $M := 2D(X, S_R(z_0))$ and $\mathcal{I} = (\delta, \sqrt{\delta})$, we infer from Lemma 1 that there is some ρ with $\delta < \rho < \sqrt{\delta} \le R$ such that

$$\begin{aligned}\int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta &\le \left\{M\Big/\int_\delta^{\sqrt{\delta}} \frac{dr}{r}\right\}^{1/2} |\theta - \theta'|^{1/2} \\ &= \left\{4D(X, S_R(z_0))\frac{1}{\log 1/\delta}\right\}^{1/2} |\theta - \theta'|^{1/2} \\ &= \eta(\delta, R)|\theta - \theta'|^{1/2},\end{aligned}$$

and from

$$Z(\rho,\theta') - Z(\rho,\theta) = \int_\theta^{\theta'} Z_\theta(\rho,\theta)\,d\theta$$

we infer that

$$|Z(\rho,\theta') - Z(\rho,\theta)| \le \int_\theta^{\theta'} |Z_\theta(\rho,\theta)|\,d\theta \le \eta(\theta, R)|\theta - \theta'|^{1/2}. \qquad \square$$

There is a generalization of Proposition 1 which holds for functions $X(w)$ of class $H_2^1(B, \mathbb{R}^n)$; see e.g. Morrey [8], Theorem 3.1.2(g). Recall the following property of such functions:

If $Z(r,\theta) := X(z_0 + re^{i\theta})$ is the transformation of X into polar coordinates r, θ about some point $z_0 \in C$, then there is representation of Z, again denoted by Z, such that $Z(r,\theta)$ is absolutely continuous with respect to θ for almost all $r \in (0,2)$, and that $Z(r,\theta)$ is absolutely continuous with respect to $r \in (r_0, 2)$, for any $r_0 > 0$ and for almost all θ. Moreover, the partial derivatives Z_r, Z_θ of Z with respect to r and θ coincide almost everywhere on $\{(r,\theta)\colon 0 < r < 2, \theta_1(r) < \theta < \theta_2(r)\}$ with the corresponding distributional derivatives. Consequently, the function

$$p(r) = \int_{\theta_1(r)}^{\theta_2(r)} |Z_\theta(r,\theta)|^2\,d\theta$$

is defined for almost all $r \in (0,2)$. Moreover, $p(r)$ is measurable on $(0,2)$, and $\int_0^2 \frac{p(r)}{r}\,dr < \infty$. Instead of Lemma 1, we now obtain

Lemma 2. *Let $\mathcal{I}$ be a measurable subset of* $(0,1)$ *such that*

$$0 < \int_{\mathcal{I}} \frac{dr}{r} < \infty \quad \textit{and} \quad \int_{\mathcal{I}} \frac{p(r)}{r}\,dr \le M < \infty.$$

Then the set $\mathcal{I}_M := \{\rho \in \mathcal{I}\colon p(\rho)\int_{\mathcal{I}} \frac{dr}{r} \le M\}$ satisfies $\mathcal{L}^1(\mathcal{I}_M) > 0$, and for almost all $\rho \in \mathcal{I}_M$ and all θ, θ' with $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$ we obtain

$$\int_{\theta}^{\theta'} |Z_\theta(\rho,\theta)|\, d\theta \le \left\{ M \Big/ \int_{\mathcal{J}} \frac{dr}{r} \right\}^{1/2} |\theta - \theta'|^{1/2}.$$

Consequently we arrive at the following analogue of Proposition 1:

Proposition 2. *Every* $X \in H_2^1(B, \mathbb{R}^n)$ *possesses a representative* $Z(r,\theta)$ *of* $X(z_0 + re^{i\theta}), z_0 \in C$, *which is absolutely continuous with respect to* θ *for a.a.* $r \in (0,2)$ *and which has the following property*:

For every $\delta \in (0, R^2), 0 < R < 1$, *there is a measurable subset* $\mathcal{J}$ *of the interval* $(\delta, \sqrt{\delta})$ *with* $\mathcal{L}^1(\mathcal{J}) > 0$ *such that*

$$|Z(\rho,\theta) - Z(\rho,\theta')| \le \int_{\theta}^{\theta'} |Z_\theta(\rho,\theta)|\, d\theta \le \eta(\delta, R)|\theta - \theta'|^{1/2}$$

holds for a.a $\rho \in \mathcal{J}$ *and* $\theta_1(\rho) \le \theta \le \theta' \le \theta_2(\rho)$, *where*

$$\eta(\delta, R) := \left\{ \frac{4}{\log 1/\delta} D(X, S_R(z_0)) \right\}^{1/2}.$$

This and other versions of the Courant–Lebesgue lemma are quite useful for many purposes, in particular for the treatment of free boundary value problems.

4.5 Existence Proof, Part II: Conformality of Minimizers of the Dirichlet Integral

In this section, we want to prove that the solutions $X(u,v)$ of the minimum problem $\mathcal{P}(\Gamma)$ satisfy the conformality relations

$$(1) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

To this end, we exploit the minimum property of X by changing the independent variables u, v in direction of arbitrarily prescribed vector fields $\lambda(u,v) = (\mu(u,v), \nu(u,v))$ on $\bar{B}$. Such variations of X will be called *inner variations*.

In order to make this variational technique precise, we start with an arbitrary vector field $\lambda = (\mu, \nu)$ on $\bar{B}$ which is of class $C^1(\bar{B}, \mathbb{R}^2)$. Without restriction we can assume that λ is defined on all of $\mathbb{R}^2$ and is of class $C^1(\mathbb{R}^2, \mathbb{R}^2)$. With λ we associate some 1-parameter family of mappings $\tau_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies

$$(2) \qquad \tau_\varepsilon(w) = \tau(w, \varepsilon) = w - \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

$w = (u,v)$. For instance, we could take $\tau_\varepsilon(w) = w - \varepsilon\lambda(w)$. The function $\tau(w,\varepsilon)$ is of class C^1 on $\mathbb{R}^2 \times \mathbb{R}$. Choose some open set B_0 with $B \subset\subset B_0$. Then it is easy to see that $\tau_\varepsilon : B_0 \to \tau_\varepsilon(B_0)$ furnishes an orientation preserving

C^1-diffeomorphism of B_0 onto its image $\tau_\varepsilon(B_0)$ provided that $|\varepsilon| < \varepsilon_0$, for some sufficiently small $\varepsilon_0 > 0$, because $\tau_\varepsilon(w)$ is just a perturbation of the identity map $\tau_0(w) = w$.

Clearly the inverse mappings $\sigma_\varepsilon = \tau_\varepsilon^{-1}$ exist on a common domain of definition Ω satisfying $B_\varepsilon^* \subset\subset \Omega \subset\subset B_0$, where we have set $B_\varepsilon^* := \tau_\varepsilon(B)$. We write $\omega = \tau_\varepsilon(w) = \tau(w, \varepsilon)$ and $w = \sigma_\varepsilon(\omega) = \sigma(\omega, \varepsilon)$. The function $\sigma(\omega, \varepsilon)$ is of class C^1 on $\Omega \times (-\varepsilon_0, \varepsilon_0)$ and satisfies both

$$\sigma(\omega, \varepsilon) = \omega + \varepsilon\lambda(\omega) + o(\varepsilon) \tag{3}$$

and

$$\tau(\sigma(\omega, \varepsilon), \varepsilon) = \omega \tag{4}$$

for all $(\omega, \varepsilon) \in \Omega \times (-\varepsilon_0, \varepsilon_0)$.

Restricting the region of definition of $\tau_\varepsilon = \tau(\cdot, \varepsilon)$ and $\sigma_\varepsilon = \sigma(\cdot, \varepsilon)$ to $\bar{B}$ and $\bar{B}_\varepsilon^*$, respectively, the mapping τ_ε is a diffeomorphism of $\bar{B}$ onto $\bar{B}_\varepsilon^*$, with the inverse σ_ε, and we have in particular

$$B_0^* = B, \quad \sigma(w, 0) = w, \quad \frac{\partial}{\partial\varepsilon}\sigma(w, \varepsilon)\bigg|_{\varepsilon=0} = \lambda(w) \quad \text{for } w \in \bar{B}. \tag{5}$$

Moreover, the Jacobian of the mapping $\tau_\varepsilon(w)$ is given by

$$\det D\tau_\varepsilon = \begin{vmatrix} 1 - \varepsilon\mu_u + o(\varepsilon) & -\varepsilon\mu_v + o(\varepsilon) \\ -\varepsilon\nu_u + o(\varepsilon) & 1 - \varepsilon\nu_v + o(\varepsilon) \end{vmatrix} = 1 - \varepsilon(\mu_u + \nu_v) + o(\varepsilon)$$

whence

$$\frac{\partial}{\partial\varepsilon} \det D\tau_\varepsilon\bigg|_{\varepsilon=0} = -(\mu_u + \nu_v) = -\operatorname{div}\lambda. \tag{6}$$

Consider now an arbitrary function $X \in C^1(\bar{B}, \mathbb{R}^3)$. We embed X into the family of functions

$$Z_\varepsilon := X \circ \sigma_\varepsilon, \quad \sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}, \tag{7}$$

which are obtained from X by the inner variations σ_ε. Let us compute the rate of change of the Dirichlet integral $D(Z_\varepsilon, B_\varepsilon^*)$ at $\varepsilon = 0$. Since we later may want to carry out the same computation for other variational integrals $\mathcal{F}(X)$ of the type

$$\mathcal{F}_B(X) = \mathcal{F}(X, B) := \int_B F(X, X_u, X_v)\, du\, dv \tag{8}$$

with a C^1-Lagrangian $F(x, p, q)$, we shall compute the derivative $f'(0)$ of the function

$$f(\varepsilon) := \mathcal{F}(Z_\varepsilon, B_\varepsilon^*) = \int_{B_\varepsilon^*} F\left(Z_\varepsilon, \frac{\partial}{\partial\alpha} Z_\varepsilon, \frac{\partial}{\partial\beta} Z_\varepsilon\right) d\alpha\, d\beta \tag{9}$$

where we have set $w = (u, v), \omega = (\alpha, \beta)$. By applying the transformation theorem to this integral and to the mapping $\tau_\varepsilon : \bar{B} \to \overline{B_\varepsilon^*}$, we obtain

$$f(\varepsilon) = \int_B F\left(X, \left(\frac{\partial}{\partial\alpha} Z_\varepsilon\right) \circ \tau_\varepsilon, \left(\frac{\partial}{\partial\beta} Z_\varepsilon\right) \circ \tau_\varepsilon\right) |\det D\tau_\varepsilon|\, du\, dv. \tag{10}$$

Set

$$\sigma_\varepsilon(\omega) = \sigma_\varepsilon(\alpha, \beta) = (\sigma_\varepsilon^1(\alpha, \beta), \sigma_\varepsilon^2(\alpha, \beta)).$$

From

$$Z_\varepsilon(\alpha, \beta) = X(\sigma_\varepsilon^1(\alpha, \beta), \sigma_\varepsilon^2(\alpha, \beta))$$

we infer that

$$\begin{aligned}
\frac{\partial}{\partial\alpha} Z_\varepsilon(\alpha, \beta) &= X_u(\sigma_\varepsilon(\omega)) \frac{\partial\sigma_\varepsilon^1}{\partial\alpha}(\omega) + X_v(\sigma_\varepsilon(\omega)) \frac{\partial\sigma_\varepsilon^2}{\partial\alpha}(\omega),\\
\frac{\partial}{\partial\beta} Z_\varepsilon(\alpha, \beta) &= X_u(\sigma_\varepsilon(\omega)) \frac{\partial\sigma_\varepsilon^1}{\partial\beta}(\omega) + X_v(\sigma_\varepsilon(\omega)) \frac{\partial\sigma_\varepsilon^2}{\partial\beta}(\omega).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\partial}{\partial\alpha} Z_\varepsilon\right)(\tau_\varepsilon(w)) &= X_u(w) \frac{\partial\sigma_\varepsilon^1}{\partial\alpha}(\tau_\varepsilon(w)) + X_v(w) \frac{\partial\sigma_\varepsilon^2}{\partial\alpha}(\tau_\varepsilon(w)),\\
\left(\frac{\partial}{\partial\beta} Z_\varepsilon\right)(\tau_\varepsilon(w)) &= X_u(w) \frac{\partial\sigma_\varepsilon^1}{\partial\beta}(\tau_\varepsilon(w)) + X_v(w) \frac{\partial\sigma_\varepsilon^2}{\partial\beta}(\tau_\varepsilon(w)).
\end{aligned} \tag{11}$$

Moreover, we have

$$\left.\begin{aligned}
\sigma_\varepsilon^1(\alpha, \beta) &= \alpha + \varepsilon\mu(\alpha, \beta) + o(\varepsilon)\\
\sigma_\varepsilon^2(\alpha, \beta) &= \beta + \varepsilon\nu(\alpha, \beta) + o(\varepsilon)
\end{aligned}\right. \quad \text{as } \varepsilon \to 0,$$

and therefore

$$\begin{aligned}
\frac{\partial}{\partial\alpha} \sigma_\varepsilon^1(\alpha, \beta) &= 1 + \varepsilon \frac{\partial}{\partial\alpha} \mu(\alpha, \beta) + o(\varepsilon),\\
\frac{\partial}{\partial\beta} \sigma_\varepsilon^1(\alpha, \beta) &= \varepsilon \frac{\partial}{\partial\beta} \mu(\alpha, \beta) + o(\varepsilon),\\
\frac{\partial}{\partial\alpha} \sigma_\varepsilon^2(\alpha, \beta) &= \varepsilon \frac{\partial}{\partial\alpha} \nu(\alpha, \beta) + o(\varepsilon),\\
\frac{\partial}{\partial\beta} \sigma_\varepsilon^2(\alpha, \beta) &= 1 + \varepsilon \frac{\partial}{\partial\beta} \nu(\alpha, \beta) + o(\varepsilon).
\end{aligned} \tag{12}$$

Replacing α and β by

$$\alpha = u - \varepsilon\mu(u, v) + o(\varepsilon), \quad \beta = v - \varepsilon\nu(u, v) + o(\varepsilon),$$

differentiating (12) with respect to ε, and setting $\varepsilon = 0$, we arrive at

$$\frac{\partial}{\partial \varepsilon}\frac{\partial \sigma_\varepsilon^1}{\partial \alpha}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \mu_u(u,v), \quad \frac{\partial}{\partial \varepsilon}\frac{\partial \sigma_\varepsilon^1}{\partial \beta}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \mu_v(u,v),$$
$$\frac{\partial}{\partial \varepsilon}\frac{\partial \sigma_\varepsilon^2}{\partial \alpha}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \nu_u(u,v), \quad \frac{\partial}{\partial \varepsilon}\frac{\partial \sigma_\varepsilon^2}{\partial \beta}(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} = \nu_v(u,v).$$

On account of (11), we then conclude that

$$(13)\qquad \begin{aligned} \frac{\partial}{\partial \varepsilon}\left(\frac{\partial}{\partial \alpha}Z_\varepsilon\right)(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} &= X_u(w)\mu_u(w) + X_v(w)\nu_u(w),\\ \frac{\partial}{\partial \varepsilon}\left(\frac{\partial}{\partial \beta}Z_\varepsilon\right)(\tau_\varepsilon(w))\bigg|_{\varepsilon=0} &= X_u(w)\mu_v(w) + X_v(w)\nu_v(w). \end{aligned}$$

Combining formulas (6) and (10)–(13), we finally obtain

$$(14)\qquad \begin{aligned} f'(0) = \int_B \big\{ &\langle F_p(X, X_u, X_v), X_u\mu_u + X_v\nu_u\rangle \\ &+ \langle F_q(X, X_u, X_v), X_u\mu_v + X_v\nu_v\rangle \\ &- F(X, X_u, X_v)[\mu_u + \nu_v]\big\}\, du\, dv. \end{aligned}$$

Following Giaquinta and Hildebrandt [1], we denote $\partial \mathcal{F}_B(X,\lambda) := f'(0)$ as (first) *inner variation of the functional* $\mathcal{F}_B$ *at* X *in direction of the vector field* $\lambda = (\mu,\nu)$*, that is,*

$$(15)\qquad \begin{aligned} \partial \mathcal{F}_B(X,\lambda) := \int_B \{ &\langle F_p, X_u\mu_u + X_v\nu_u\rangle + \langle F_q, X_u\mu_v + X_v\nu_v\rangle \\ &- F[\mu_u + \nu_v]\}\, du\, dv \end{aligned}$$

where the arguments of F, F_p, F_q are to be taken as X, X_u, X_v.

Collecting the previous results, we obtain the following

Proposition 1. *If* $\{\tau_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ *is a* C^1*-family of* C^1*-diffeomorphisms* $\tau_\varepsilon : \bar{B} \to \overline{B_\varepsilon^*}$ *with the inverses* $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$*, such that* $B_0^* = B$ *holds and that* $\sigma(w,\varepsilon) := \sigma_\varepsilon(w)$ *satisfies*

$$(16)\qquad \sigma(w,0) = w, \quad \frac{\partial \sigma}{\partial \varepsilon}(w,0) = \lambda(w), \quad \text{and} \quad \lambda \in C^1(\bar{B}, \mathbb{R}^2),$$

then, for every $X \in C^1(\bar{B}, \mathbb{R}^3)$*, we obtain*

$$(17)\qquad \frac{d}{d\varepsilon}\mathcal{F}(X\circ\sigma_\varepsilon, B_\varepsilon^*)\bigg|_{\varepsilon=0} = \partial\mathcal{F}_B(X,\lambda)$$

where $\partial\mathcal{F}_B(X,\lambda)$ *is defined by* (15).

Moreover, given any vector field $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$*, we can find a* 1*-parameter family of diffeomorphisms* σ_ε *with the above stated properties and, in particular, with the property* (16).

Let us now consider two important cases:

Examples 1. For the Dirichlet integral

$$D(X) = \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2)\,du\,dv$$

and for any vector field $\lambda = (\mu, \nu) \in C^1(\bar{B}, \mathbb{R}^2)$, the first inner variation $\partial D(X, \lambda)$ is given by

$$2\partial D(X,\lambda) = \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\,du\,dv, \tag{18}$$

where a and b denote the functions

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v\rangle. \tag{19}$$

Note that the expression $\partial D(X,\lambda)$ is not only defined for surfaces $X \in C^1(\bar{B}, \mathbb{R}^3)$, but also for surfaces $X \in H_2^1(B, \mathbb{R}^3)$. In fact, a closer inspection of the previous computations yields the following result:

Proposition 2. *If $\mathcal{F}_B(X) = D(X)$, then the assertion of Proposition 1 holds for every $X \in H_2^1(B,\mathbb{R}^3)$, and the inner variation $\partial D(X,\lambda)$ of the Dirichlet integral at X in direction of any $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$ is given by formulas (18) and (19).*

Examples 2. For the generalized Dirichlet integral

$$E(X) = \frac{1}{2}\int_B g_{jk}(X)\{X_u^j X_u^k + X_v^j X_v^k\}\,du\,dv \tag{20}$$

and for any $\lambda = (\mu,\nu) \in C^1(\bar{B}, \mathbb{R}^2)$, we obtain

$$2\partial E(X,\lambda) = \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\,du\,dv$$

with

$$\begin{aligned} a &:= g_{jk}(X)X_u^j X_u^k - g_{jk}(X) X_v^j X_v^k, \\ b &:= 2 g_{jk}(X) X_u^j X_v^k. \end{aligned} \tag{21}$$

Again we can prove a generalization of Proposition 1 which is similar to Proposition 2 and holds for E and $X \in H_2^1(B, \mathbb{R}^3)$.

Now we are in a position to prove the main results of this section.

Theorem 1. *Let $X(u,v)$ be a surface of class $H_2^1(B,\mathbb{R}^3)$ such that*

$$\partial D(X,\lambda) = 0 \quad \textit{for all } \lambda \in C^1(\bar{B}, \mathbb{R}^2) \tag{22}$$

is satisfied. Then X fulfills the conformality relations (1) a.e. in B. Conversely, if (1) holds a.e. in B for some $X \in H_2^1(B, \mathbb{R}^3)$, then the relation (22) is satisfied.

Proof. Choose arbitrary functions $\rho, \sigma \in C_c^\infty(B)$ and determine functions $h, k \in C^\infty(\bar{B})$ with

$$\begin{aligned} \Delta h = \rho, \quad \Delta k &= \sigma \quad \text{on } B, \\ h = 0, \qquad k &= 0 \quad \text{on } \partial B. \end{aligned}$$

(This is possible on account of well known results of potential theory, cf. Gilbarg and Trudinger [1].)

Then the functions

$$\mu := h_u + k_v, \quad \nu := -h_v + k_u$$

are of class $C^\infty(\bar{B})$ and satisfy

$$\mu_u - \nu_v = \rho, \quad \mu_v + \nu_u = \sigma.$$

We now infer from assumption (22) in conjunction with (18) and (19) that

$$\int_B \{a\rho + b\sigma\}\, du\, dv = 0$$

holds for all $\rho, \sigma \in C_c^\infty(B)$. By the fundamental lemma of the calculus of variations we conclude that

$$a = 0 \quad \text{and} \quad b = 0$$

a.e. on B.

It is a trivial conclusion from (18) and (19) that, conversely, the conformality relations (1) imply (22). □

Corollary 1. *If $X \in H_2^1(B, \mathbb{R}^3)$ is harmonic in B and satisfies (22), then X is a minimal surface.*

Theorem 2. *Every solution X of the variational problem*

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \textit{in the class } \mathcal{C}(\Gamma)$$

is of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and satisfies

$$\Delta X = 0,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

in B, that is, X is a minimal surface.

Proof. By virtue of Section 4.3, Theorem 2, we only have to verify the conformality relations. Let $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$ be a family of inner variations as described in Proposition 1, and set $Z_\varepsilon := X \circ \sigma_\varepsilon$, where X is a minimizer of $D_B(\cdot)$ in the class $\mathcal{C}(\Gamma)$. Clearly we have $Z_\varepsilon \in H_2^1(B_\varepsilon^*, \mathbb{R}^3)$. Since $\bar{B}$ and $\overline{B_\varepsilon^*}$ are diffeomorphic, $|\varepsilon| < \varepsilon_0$, there is a conformal mapping $\kappa_\varepsilon : B \to B_\varepsilon^*$ of B onto

B_ε^*, by virtue of Riemann's mapping theorem. Moreover, a classical result in function theory yields that κ_ε can be extended to a homeomorphism of $\bar{B}$ onto $\overline{B_\varepsilon^*}$ since ∂B_ε^* is a Jordan curve. It follows that $Y_\varepsilon := Z_\varepsilon \circ \kappa_\varepsilon$ is of class $\mathcal{C}(\Gamma)$, whence

$$D(X, B) \le D(Y_\varepsilon, B) \quad \text{for } |\varepsilon| < \varepsilon_0, \tag{23}$$

because of the minimum property of X.

A straightforward computation shows that the Dirichlet integral is invariant with respect to conformal mappings. Therefore we have

$$D(Y_\varepsilon, B) = D(Z_\varepsilon \circ \kappa_\varepsilon, B) = D(Z_\varepsilon, B_\varepsilon^*),$$

and in conjunction with (23), we arrive at

$$D(X, B) \le D(Z_\varepsilon, B_\varepsilon^*), \quad |\varepsilon| < \varepsilon_0. \tag{24}$$

Set $f(\varepsilon) := D(Z_\varepsilon, B_\varepsilon^*)$ and note that $X = Z_0$. Then we can write (24) as

$$f(0) \le f(\varepsilon), \quad |\varepsilon| < \varepsilon_0,$$

and we obtain

$$0 = f'(0) = \partial D(X, \lambda)$$

for every $\lambda \in C^1(\bar{B}, \mathbb{R}^2)$, on account of (9) and of Proposition 2. Then the conformality relations (1) are a consequence of Theorem 1. $\square$

Theorem 3. *Every solution of $\mathcal{P}(\Gamma)$ and, more generally, every minimal surface of class $\mathcal{C}(\Gamma)$ yields a topological mapping of C onto Γ.*

Proof. Let $X \in \mathcal{C}(\Gamma)$ be continuous in $\bar{B}$, harmonic in B, and suppose that (1) holds in B. It suffices to prove that X provides a one-to-one mapping of C onto Γ. Suppose that this were not true. Since $X|_C$ is weakly monotonic, we could then find an arc $C_0 = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ which is mapped onto a single point $P \in \mathbb{R}^3$:

$$X(e^{i\theta}) = P \quad \text{for all } \theta \in (\theta_1, \theta_2). \tag{25}$$

By Schwarz's reflection principle we could extend $X(w)$ as a harmonic mapping across C_0. Differentiating (25) in the tangential direction, we would then obtain

$$\frac{\partial}{\partial \theta} X(e^{i\theta}) = 0$$

and, applying the conformality relations, it would follow that grad X vanishes identically on C_0. This would imply grad $X \equiv 0$ on B, or $X(w) \equiv P$, a contradiction to $X \in \mathcal{C}(\Gamma)$. $\square$

Combining Theorems 1–3 with the results of Section 4.3, we have found the following

Main Theorem. *Let Γ be a closed curve in $\mathbb{R}^3$ and suppose that $\mathcal{C}(\Gamma)$ is nonempty. Then the minimum problem*

$$\mathcal{P}(\Gamma): \quad D(X) \to \min \quad \textit{in the class } \mathcal{C}(\Gamma)$$

has at least one solution. Every solution X of $\mathcal{P}(\Gamma)$ is continuous on $\bar{B}$, harmonic in B, satisfies the conformality relations (1) in B, and maps C topologically onto Γ. In particular, every closed rectifiable curve Γ in $\mathbb{R}^3$ spans at least one minimal surface of the type of the disk.

Obviously the proof of Theorem 3 does not use the fact that $D(X) < \infty$, and so we have also

Corollary 2. *Let $X : B \to \mathbb{R}^3$ be a minimal surface which is continuous on $\overline{B}$ and maps $C = \partial B$ in a weakly monotonic way onto Γ (as defined in 4.2, Definition 2). Then X yields a homeomorphism from C onto Γ.*

Supplementary Remarks.

1. For the proof of Theorem 2 we have used the Riemann mapping theorem. This can be avoided as we shall presently see. The advantage of this different proof is that the Main Theorem above can be used to provide an independent approach to Riemann's mapping theorem; see Section 4.11. Let us use the complex notation $w = u + iv$, and consider the variations

$$\omega = \tau_\varepsilon(w) = we^{i\varepsilon\varphi(r,\theta)} \tag{26}$$

with $\varphi(r,\theta) = \psi(w), w = re^{i\theta}$, where $\psi(u,v)$ denotes an arbitrary function of class $C^1(\bar{B})$. Writing

$$\tau_\varepsilon(w) = w - \varepsilon\lambda(w) + o(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{27}$$

we obtain

$$\lambda(w) = \mu(u,v) + i\nu(u,v) = -iw\varphi(r,\theta). \tag{28}$$

Clearly, the mappings τ_ε define diffeomorphisms of $\bar{B}$ onto itself, provided that $|\varepsilon|$ is sufficiently small. Hence, if we set $\sigma_\varepsilon := \tau_\varepsilon^{-1}$ and $Z_\varepsilon := X \circ \sigma_\varepsilon$ for some solution X of $\mathcal{P}(\Gamma)$, then the functions Z_ε are of class $\mathcal{C}(\Gamma)$, and we obtain

$$D(Z_\varepsilon) \geq D(X) \quad \text{for } |\varepsilon| \ll 1.$$

As in the proof of Theorem 2, we now conclude that

$$\int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv = 0 \tag{29}$$

holds for

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v\rangle.$$

One easily verifies the Cauchy–Riemann equations

$$a_u = -b_v, \quad a_v = b_u$$

in B, using the relation $\Delta X = 0$ which holds for every solution X of $\mathcal{P}(\Gamma)$. Consequently the mapping $\Phi : B \to \mathbb{C}$ defined by $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u + iv \in B$, and only this fact is used in the sequel. Suppose first that we had $X \in C^1(\bar{B}, \mathbb{R}^3)$. Then, by employing $\Delta a = \Delta b = 0$, we could transform the left-hand side of (29) into a line integral over $C = \partial B$, thus obtaining

$$\text{(30)} \qquad \operatorname{Im} \int_C \lambda(w)\Phi(w)\, dw = 0.$$

On account of (28), we then arrive at

$$\text{(31)} \qquad \operatorname{Im} \int_0^{2\pi} \varphi(1,\theta) w^2 \Phi(w)\, d\theta = 0, \quad w = e^{i\theta}.$$

Let $H(r,\theta) := \operatorname{Im} w^2\Phi(w)$, $w = re^{i\theta}$, and choose

$$\text{(32)} \qquad \varphi(r,\theta) := \zeta(r;\rho) K(r,\theta;\rho,\theta')$$

where $w' = \rho e^{i\theta'}$ is some fixed point in $B, \zeta(r;\rho)$ is a function of class $C^\infty(\mathbb{R})$ with respect to r which satisfies $\zeta(r;\rho) = 0$ for $0 \le r \le \rho'$, and $\zeta(r;\rho) = 1$ for $\rho'' < r$, where the numbers ρ', ρ'' satisfy $\rho < \rho' < \rho'' < 1$, and K denotes the Poisson kernel for the disk $B_r(0)$:

$$K(r,\theta;\rho,\theta') := \frac{1}{2\pi} \frac{r^2 - \rho^2}{r^2 - 2\rho r\cos(\theta - \theta') + \rho^2}.$$

Then we infer from (31) that

$$\int_0^{2\pi} K(1,\theta;\rho,\theta') H(1,\theta)\, d\theta = 0,$$

and Poisson's formula yields

$$H(\rho,\theta') = 0$$

for every $\rho \in (0,1)$ and $0 \le \theta' \le 2\pi$, or $\operatorname{Im} w^2\Phi(w) = 0$. Hence, $\operatorname{Re} w^2\Phi(w)$ is constant in B, whence

$$w^2\Phi(w) \equiv c$$

or

$$\Phi(w) \equiv \frac{c}{w^2}.$$

Since $\Phi(w)$ is holomorphic in B, we infer that $c = 0$ or $\Phi(w) \equiv 0$, that is, $a = 0$ and $b = 0$.

In general, however, we only know that $X \in C^1(B, \mathbb{R}^3)$. Thus we have to modify our proof slightly. Let

$$B_R := \{w \colon |w| < R\}, \quad C_R := \partial B_R, \quad 0 < R < 1.$$

Then we infer from (29) that

$$\int_{B_R} [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv \to 0 \quad \text{as } R \to 1 - 0.$$

Performing the same integration by parts as before, we obtain instead of (30) the relation

$$\operatorname{Im} \int_{C_R} \lambda(w)\Phi(w)\, dw \to 0 \quad \text{as } R \to 1 - 0$$

whence

$$\lim_{R \to 1-0} \operatorname{Im} \int_0^{2\pi} \varphi(R, \theta) w^2 \Phi(w)\, d\theta = 0, \quad w = \mathrm{Re}^{i\theta}. \tag{33}$$

If we choose $\varphi(r, \theta)$ as in (32) and assume that $\rho < \rho' < \rho'' < R < 1$, then Poisson's formula yields

$$\operatorname{Im} \int_0^{2\pi} \varphi(R, \theta) w^2 \Phi(w)\, d\theta = \int_0^{2\pi} K(R, \theta; \rho, \theta') H(R, \theta)\, d\theta = H(\rho, \theta')$$

and (33) implies $\lim_{R \to 1-0} H(\rho, \theta') = 0$, or $H(\rho, \theta') = 0$. The rest of the proof is the same as before.

2. Results that are similar to Theorems 1–3 can be obtained for the generalized Dirichlet integral

$$E_B(X) = \frac{1}{2} \int_B g_{jk}(X) \{X_u^j X_u^k + X_v^j X_v^k\}\, du\, dv, \tag{34}$$

where $X = (X^1, X^2, \ldots, X^n)$. The conformality relations for the minimizers of $E_B(X)$ in $\mathcal{C}(\Gamma)$, which will replace (1), are now of the form

$$g_{jk}(X) X_u^j X_u^k = g_{jk}(X) X_v^j X_v^k, \quad g_{jk}(X) X_u^j X_v^k = 0. \tag{35}$$

Using the complex notation $w = u + iv$, we can express (35) by the single complex equation

$$g_{jk}(X) X_w^j X_w^k = 0.$$

3. Other functionals $\mathcal{F}_B(X)$ which can be handled in the same way as $D_B(X)$ or $E_B(X)$ are expressions of the type

$$\mathcal{F}_B(X) = E_B(X) + V_B(X) \tag{36}$$

where $V(X)$ is invariant with respect to diffeomorphisms of the parameter domain B which have a positive Jacobian. In fact, if $\sigma_\varepsilon : \overline{B_\varepsilon^*} \to \bar{B}$ is such a family of diffeomorphisms from $\overline{B_\varepsilon^*}$ onto $\bar{B}$, then the property

$$V_B(X) = V_{B_\varepsilon^*}(X \circ \sigma_\varepsilon)$$

implies that

$$\mathcal{F}_{B_\varepsilon^*}(X \circ \sigma_\varepsilon) - \mathcal{F}_B(X) = E_{B_\varepsilon^*}(X \circ \sigma_\varepsilon) - E_B(X).$$

Hence, a minimum property of X with respect to $\mathcal{F}_B$ can be translated into a minimum property with respect to E, and we are in the previously considered situation. Under suitable assumptions we shall therefore obtain the conformality relations (35).

If, for instance, $V_B(X)$ denotes a volume functional of the type

$$V_B(X) = \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv \tag{37}$$

where $Q = (Q^1, Q^2, Q^3)$ is a C^1-vector field defined on $\mathbb{R}^3$ and $X = (X^1, X^2, X^3)$, then the Euler equations of the functional $\mathcal{F}_B(X) = E_B(X) + V_B(X)$ are given by

$$\Delta X^l + \Gamma^l_{jk}(X)[X^j_u X^k_u + X^j_v X^k_v] = \operatorname{div} Q(X)[X_u \wedge X_v]_m g^{lm}(X). \tag{38}$$

Here $(g_{jk}(x))$ is assumed to be a positive definite 3×3-matrix, and $(g^{jk}(x))$ denotes its inverse. Moreover, Γ_{jkl} and Γ^k_{jl} denote the Christoffel symbols of first and second kind:

$$\begin{aligned} \Gamma_{jkl} &= \tfrac{1}{2}\{g_{jk,l} + g_{kl,j} - g_{jl,k}\}, \\ \Gamma^l_{jk} &= g^{lm}\Gamma_{jmk} \end{aligned}$$

where $g_{jk,l}$ stands for the derivative g_{jk,x^l}. Finally, we have used the notation

$$\operatorname{div} Q = Q^1_{x^1} + Q^2_{x^2} + Q^3_{x^3}.$$

If X is conformal, then the equations (38) express that X is a surface of mean curvature

$$H(X) = \frac{1}{2\sqrt{g(X)}} \operatorname{div} Q(X), \quad g := \det(g_{jk}), \tag{39}$$

in the Riemannian manifold $(\mathbb{R}^3, ds^2)$ with the line element $ds^2 = g_{jk}(x)\, dx^j \, dx^k$. In Chapter 4 of Vol. 2 we give a survey on results concerning the Plateau problem for functionals $\mathcal{F} = D + V$ and present some of the proofs. The Plateau problem for the general definite parametric integral (= Cartan functional) $\mathcal{F}$ is treated in Section 4.13.

So far we have proved that every closed rectifiable curve Γ in $\mathbb{R}^3$ bounds at least one minimal surface X of class $\mathcal{C}(\Gamma)$, and this solution of the Plateau problem has been obtained by minimization of the Dirichlet integral among all (disk-type) surfaces of class $\mathcal{C}(\Gamma)$. Since any minimizer X is automatically continuous on $\bar{B}$, the solution of Plateau's problem can as well be achieved by minimizing $D(X)$ within the class

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3).$$

Although every minimizer X satisfies

$$D(X) = A(X),$$

it is by no means clear that a minimizer of the Dirichlet integral in $\bar{\mathcal{C}}(\Gamma)$ also minimizes the area functional among all surfaces in $\bar{\mathcal{C}}(\Gamma)$. For this we need to know that

$$\bar{a}(\Gamma) = \bar{e}(\Gamma), \tag{40}$$

where $\bar{a}(\Gamma)$ and $\bar{e}(\Gamma)$ denote the infimum of $A(X)$ and $D(X)$ respectively, among all $X \in \overline{\mathcal{C}}(\Gamma)$. However, the inequality

$$A(X) \le D(X)$$

only implies that

$$\bar{a}(\Gamma) \le \bar{e}(\Gamma).$$

In fact, the proof of the equality sign is not a trivial matter. Usually it is based on the fact proved by Carathéodory that polyhedral surfaces can be represented conformally (in the generalized sense). Equivalently one can apply a basic result on "ε-conformal mappings" due to C.B. Morrey which is derived by means of quasiconformal mappings; a somewhat weaker version was already stated by T. Radó [21]. We only quote Morrey's lemma without proving it, because we shall later present a self-contained proof of (40) that uses only fairly elementary tools (see Section 4.10). Roughly speaking, Morrey's lemma says that one can introduce nearly conformal parameters on every reasonable surface X. To be precise, we need the following

Lemma on ε-conformal mappings. *Let X be a mapping $\overline{B} \to \mathbb{R}^3$ of class $C^0(\bar{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$. Then, for every $\varepsilon > 0$, there exists a homeomorphism τ_ε of $\bar{B}$ onto itself which is of class H_2^1 on $\bar{B}$ and satisfies both*

$$Z_\varepsilon := X \circ \tau_\varepsilon \in C^0(\bar{B}, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$$

and

$$D(Z_\varepsilon) \le A(X) + \varepsilon.$$

(For a proof, we refer to Morrey [1], pp. 141–143, and [3], pp. 814–815.)

Let us turn to the *proof of* (40): Let X be an arbitrary surface in $\overline{\mathcal{C}}(\Gamma)$. Then, by Morrey's lemma, we can find homeomorphisms τ_n of $\bar{B}$ onto itself such that $Z_n := X \circ \tau_n \in \overline{\mathcal{C}}(\Gamma)$ and

$$D(Z_n) \le A(X) + \frac{1}{n}, \quad n = 1, 2, \dots .$$

Since

$$\bar{e}(\Gamma) \le D(Z_n) \quad \text{for all } n \in \mathbb{N},$$

we obtain

$$\bar{e}(\Gamma) \le A(X)$$

and therefore

$$\bar{e}(\Gamma) \le \bar{a}(\Gamma).$$

Thus the relation (40) is proved.

We notice that (40) implies the conformality relations (40). In fact, if X minimizes the Dirichlet integral in $\mathcal{C}(\Gamma)$, then $X \in C^0(\bar{B}, \mathbb{R}^3)$, and (40) yields $A(X) = D(X)$. As we have observed in Section 4.1, this equality can only hold if (1) is satisfied. □

Thus we have proved:

Theorem 4. *Every solution $X \in \mathcal{C}(\Gamma)$ of the minimum problem $\mathcal{P}(\Gamma)$ is a surface of least area in $\mathcal{C}(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$.*

Another, completely self-contained proof of this result will be given in Section 4.10, which even shows

$$e(\Gamma) = \overline{e}(\Gamma) = a(\Gamma) = \overline{a}(\Gamma), \tag{41}$$

where $a(\Gamma)$ and $\overline{a}(\Gamma)$ denote the infima of A over $\mathcal{C}(\Gamma)$ and $\overline{\mathcal{C}}(\Gamma)$ respectively, while $e(\Gamma)$ and $\overline{e}(\Gamma)$ are the corresponding infima of D. Note that the relation $e(\Gamma) = \overline{e}(\Gamma)$ follows from Theorem 2 whereas $a(\Gamma) = \overline{a}(\Gamma)$ is not immediately obvious.

4.6 Variant of the Existence Proof. The Partially Free Boundary Problem

In this section we want to give another existence proof for the minimum problem $\mathcal{P}(\Gamma)$ which is of a more functional-analytic nature and can easily be modified to handle other boundary value problems for minimal surfaces, for instance, the partially free problem. The Courant–Lebesgue lemma will once again play an essential role. We shall use it in the following form:

Proposition 1. *Let Γ be a closed (oriented) Jordan curve in $\mathbb{R}^3$, and let $\mathcal{C}^*(\Gamma)$ be the class of surfaces bounded by Γ and normalized by a fixed three-point condition as defined in Section 4.2. Then $\mathcal{C}^*(\Gamma)$ is a weakly sequentially closed subset of $H_2^1(B, \mathbb{R}^3)$.*

Proof. Let $\{X_n\}$ be a sequence of surfaces $X_n \in \mathcal{C}^*(\Gamma)$ which converge weakly in $H_2^1(B, \mathbb{R}^3)$ to some element $X \in H_2^1(B, \mathbb{R}^3)$. Then the norms of X_n are uniformly bounded,

$$|X_n|_{H_2^1(B)} \le c, \quad n = 1, 2, \dots, \tag{1}$$

and Rellich's theorem yields both

$$|X_n - X|_{L_2(B)} \to 0 \quad \text{as } n \to \infty$$

and

$$|\phi_n - \phi|_{L_2(C)} \to 0 \quad \text{as } n \to \infty \tag{2}$$

where ϕ_n and ϕ denote the L_2(C)-traces of X_n and X on C.

By (1) and Theorem 3 in Section 4.3, the functions ϕ_n, $n \in \mathbb{N}$, are equicontinuous on C, and $\phi_n(C) = \Gamma$ implies

$$\sup_C |\phi_n| \le \text{const}, \quad n = 1, 2, \dots. \tag{3}$$

Thus the functions ϕ_n are compact in $C^0(C, \mathbb{R}^3)$, and we can extract a subsequence $\{\phi_{n_p}\}$ which converges uniformly on C to some $\phi' \in C^0(C, \mathbb{R}^3)$ as $p \to \infty$. From (2) we infer that $\phi' = \phi$, and a well-known reasoning yields that $\{\phi_n\}$ itself converges to ϕ as $n \to \infty$. Moreover, Lemma 1 of Section 4.2 implies that ϕ is a weakly monotonic mapping of C onto Γ which satisfies the same three-point condition as the ϕ_n. Consequently, X is contained in $\mathcal{C}^*(\Gamma)$, and the assertion is proved. □

Now we shall give a new proof of the following result:

Theorem 1. *The minimum problem $\mathcal{P}(\Gamma)$ has at least one solution. Any solution of $\mathcal{P}(\Gamma)$ is harmonic in B, continuous on $\overline{B}$, and satisfies*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B.$$

Proof. We proceed in four steps:

(i) *First we show that there is a minimizing sequence $\{X_n\}$ for $\mathcal{P}(\Gamma)$, $X_n \in \mathcal{C}^*(\Gamma)$, which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$.*

In fact, let $\{X_n\}$ be a sequence of surface $X_n \in \mathcal{C}^*(\Gamma)$ which satisfy

$$\lim_{n \to \infty} D(X_n) = e(\Gamma) := \inf\{D(X) \colon X \in \mathcal{C}(\Gamma)\}. \tag{4}$$

Then we have

$$D(X_n) \le \text{const}, \quad n = 1, 2, \ldots, \tag{5}$$

and the boundary values $\phi_n := X_n|_C$ satisfy (3). A suitable variant of Poincaré's inequality, together with (3) and (5), yields

$$|X_n|_{H_2^1(B)} \le \text{const}, \quad n = 1, 2, \ldots. \tag{6}$$

In Hilbert space, any closed ball is weakly sequentially compact. Thus there is a subsequence $\{X_{n_p}\}$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$, and clearly

$$\lim_{p\to\infty} D(X_{n_p}) = e(\Gamma).$$

Renumbering the X_{n_p} and writing X_n instead of X_{n_p}, the assertion (i) is proved.

(ii) *The Dirichlet integral is weakly lower semicontinuous in* $H_2^1(B, \mathbb{R}^3)$. To verify this, we consider any sequence of elements $X_1, X_2, \ldots \in H_2^1(B, \mathbb{R}^3)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$. Since

$$\mathcal{F}(Z) := D(X, Z)$$

is a bounded linear functional on $H_2^1(B, \mathbb{R}^3)$, we obtain

$$\lim_{n\to\infty} \mathcal{F}(X_n) = \mathcal{F}(X),$$

and therefore

$$\begin{aligned} D(X_n) &= D(X - X_n) + 2D(X, X_n) - D(X) \\ &\ge 2D(X, X_n) - D(X) \to D(X). \end{aligned}$$

That is,

$$\liminf_{n\to\infty} D(X_n) \ge D(X), \tag{7}$$

and (ii) is verified.

(iii) *The set* $\mathcal{C}^*(\Gamma)$ *is a weakly (sequentially) closed subset of* $H_2^1(B, \mathbb{R}^3)$.

This assertion is the statement of Proposition 1.

Combining (i)–(iii), we obtain that X is a solution of $\mathcal{P}(\Gamma)$. In fact, (i) and (ii) imply

$$D(X) \le \lim_{n\to\infty} D(X_n) = e(\Gamma),$$

and (i) and (iii) yield $X \in \mathcal{C}^*(\Gamma)$, whence

$$e(\Gamma) \le D(X),$$

and therefore

$$D(X) = e(\Gamma).$$

(iv) Finally, if X is a solution of $\mathcal{P}(\Gamma)$, it follows from Weyl's lemma that X is harmonic in B, and then a well-known reasoning yields that X is continuous on $\overline{B}$. The conformality relations for X were derived in the previous section. □

Let us apply this method to another boundary value problem for minimal surfaces, the *semi-free* (or: *partially free*) *problem.*

Consider a boundary configuration $\langle \Gamma, S \rangle$ consisting of a closed set S in $\mathbb{R}^3$ (e.g., a smooth surface S with or without boundary, or something more exotic, see Figs. 4–7), and a Jordan curve Γ the endpoints P_1 and P_2 of which lie on $S, P_1 \neq P_2$, but all other points of Γ are disjoint from S.

Let us denote the arcs of ∂B lying in the half-planes $\{\operatorname{Im} w \geq 0\}$ and $\{\operatorname{Im} w \leq 0\}$ by C and I respectively. The class $\mathcal{C}(\Gamma, S)$ of admissible surfaces for the semi-free problem is the set of all maps $X \in H_2^1(B, \mathbb{R}^3)$ whose L_2-traces on C and I satisfy

(i) $X(w) \in S$ for $\mathcal{H}^1$-almost all $w \in I$;
(ii) $X|_C$ maps C continuously and in a weakly monotonic way onto Γ such that $X(1) = P_1$ and $X(-1) = P_2$.

We orient Γ and $\mathcal{C}(\Gamma, S)$ by taking P_1 as the initial point and P_2 as the endpoint of Γ.

The corresponding variational problem $\mathcal{P}(\Gamma, S)$ reads:

$$D(X) \to \min \quad \text{in the class } \mathcal{C}(\Gamma, S).$$

Again, as in the study of the Plateau problem, it is desirable to introduce a three-point-condition. Since we have already fixed the images of two boundary points, the image of only one more point needs to be prescribed: Let P_3 be some point of Γ different from P_1 and P_2, and let $\mathcal{C}^*(\Gamma, S)$ denote the class of all those surfaces $X \in \mathcal{C}(\Gamma, S)$ mapping $i = \sqrt{-1} \in C$ to P_3. The corresponding variational problem $\mathcal{P}^*(\Gamma, S)$ then requires:

$$D(X) \to \min \quad \text{in } \mathcal{C}^*(\Gamma, S).$$

Theorem 2. *If $\mathcal{C}(\Gamma, S)$ is nonempty, then there exists a solution of the minimum problem $\mathcal{P}(\Gamma, S)$. Moreover, every solution X of $\mathcal{P}(\Gamma, S)$ is of class $C^0(B \cup C', \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ for every arc C' contained in the interior of C, and satisfies both*

$$\Delta X = 0 \quad \textit{in } B$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

in B. Finally, the class $\mathcal{C}(\Gamma, S)$ is nonempty if Γ is rectifiable and if there exists a rectifiable arc in S which connects P_1 and P_2.

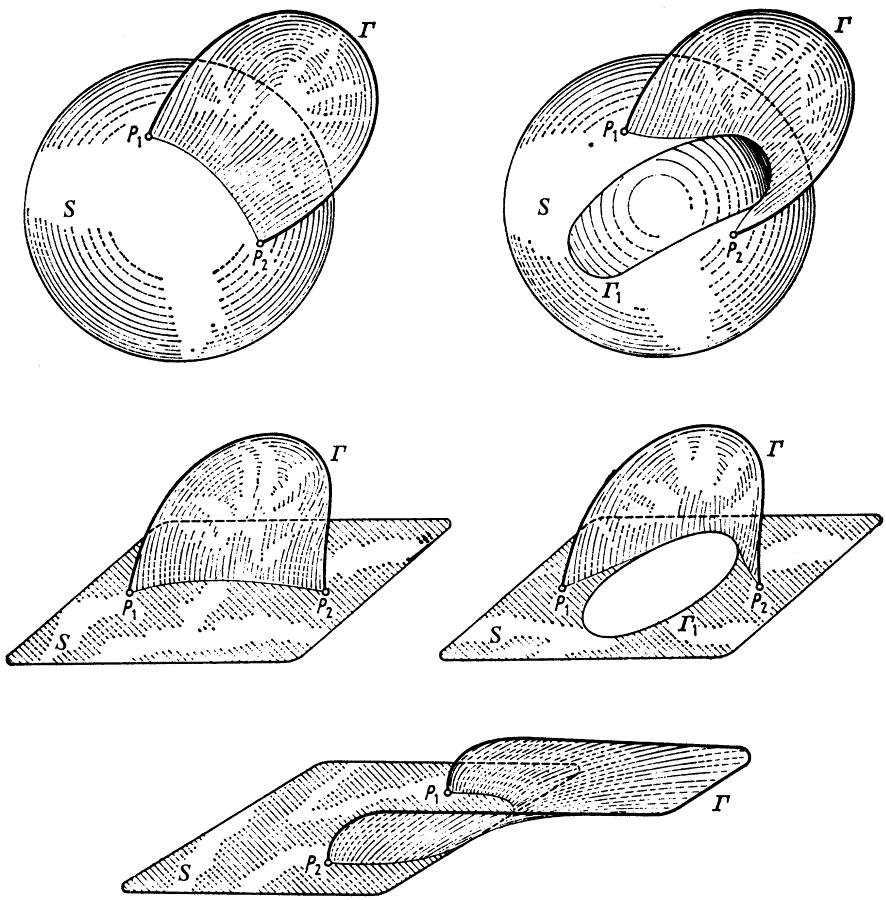


Fig. 1. Partially free problems and area minimizing solutions. From S. Hildebrandt and J.C.C. Nitsche [3]

Proof. The existence of a minimizer can be established more or less in the same way as for Theorem 1. The steps (i), (ii) and (iv) can be carried out in the same manner, whereas (iii) is to be replaced by:

(iii′) *The class $\mathcal{C}^*(\Gamma, S)$ is closed with respect to weak convergence of sequences in $H_2^1(B, \mathbb{R}^3)$.*

In fact, if $\{X_n\}$ is a sequence of surfaces $X_n \in \mathcal{C}^*(\Gamma, S)$ which converge weakly in $H_2^1(B, \mathbb{R}^3)$ to some element $X \in H_2^1(B, \mathbb{R}^3)$, then the norms of X_n are uniformly bounded, and we have

$$\lim_{n \to \infty} |\phi_n - \phi|_{L_2(\partial B)} = 0$$

for $\phi = X|_{\partial B}, \phi_n = X_n|_{\partial B}$. Hence there is a subsequence $\{\phi_{n_p}\}$ such that

$$\phi_{n_p}(w) \to \phi(w) \quad \text{as } p \to \infty,$$

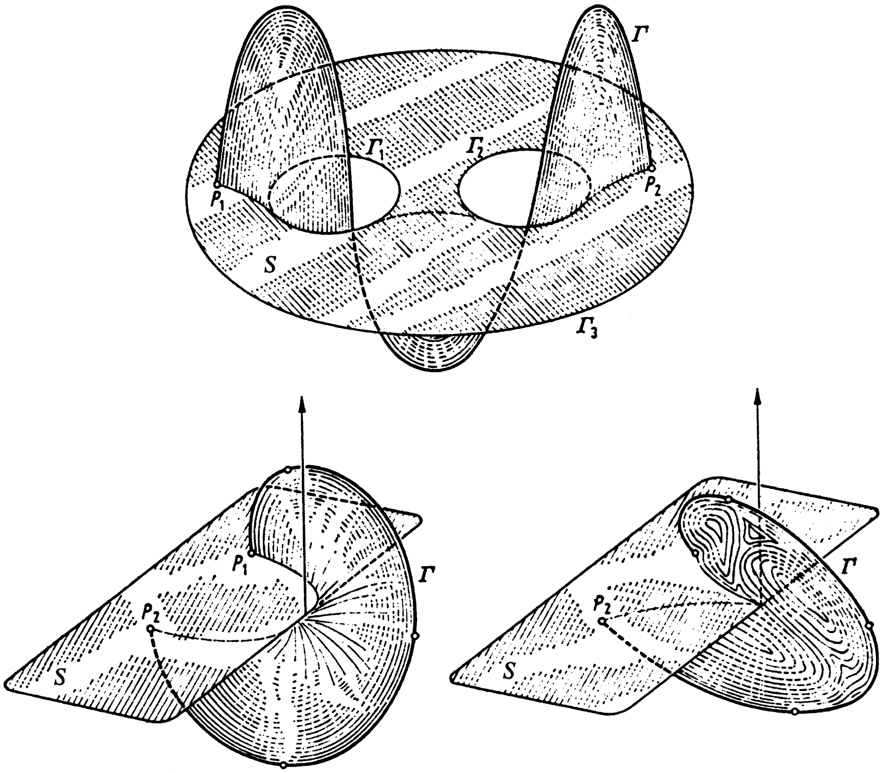


Fig. 2. Other partially free problems and area minimizing solutions. From S. Hildebrandt and J.C.C. Nitsche [3]

for $\mathcal{H}^1$-almost all $w \in \partial B$. Since

$$X_n(w) \in S \quad \text{for } \mathcal{H}^1\text{-almost all } w \in I,$$

we thus obtain that also

$$X(w) \in S \quad \text{for } \mathcal{H}^1\text{-almost all } w \in I.$$

Furthermore, a similar reasoning as in the proof of Proposition 1 yields that $X|_C$ maps C continuously and weakly monotonically onto Γ and satisfies the 3-point condition

$$X(1) = P_1, \quad X(i) = P_3, \quad X(-1) = P_2,$$

that is, $X \in \mathcal{C}^*(\Gamma, S)$.

In fact, all we have to prove is that the mappings $\phi_n|_C$ are equicontinuous on C. By the Courant–Lebesgue lemma, the ϕ_n are equicontinuous on every closed subarc C' lying in the interior of C. Thus we have to investigate how the

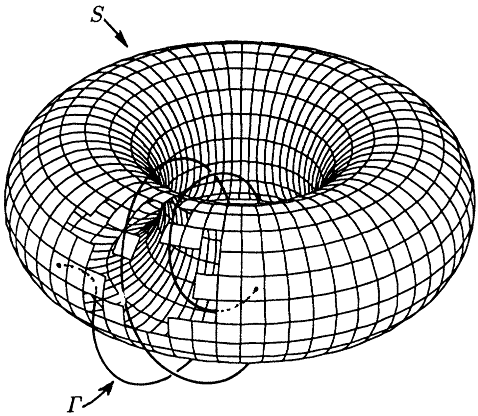


Fig. 3. An irregular support surface for the semifree boundary problem

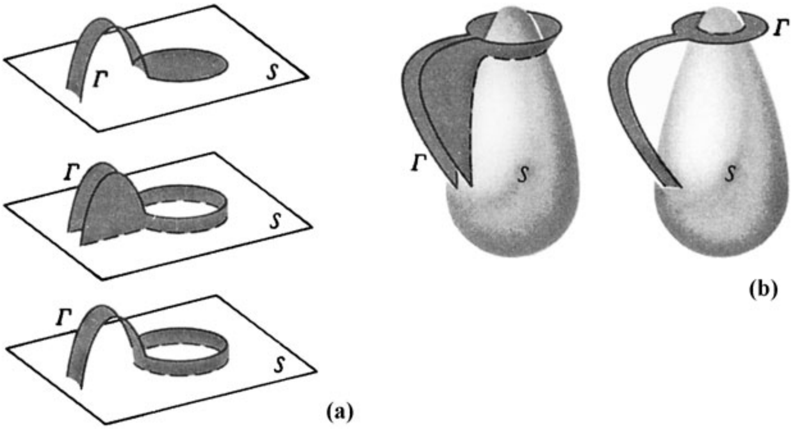


Fig. 4. Partially free problems can have several solutions

functions $\phi_n(e^{i\theta}), 0 \le \theta \le \pi$, behave for $\theta \to +0$ or $\theta \to \pi - 0$. To this end we use the assumption that Γ and S have only the points P_1 and P_2 in common. Let Γ_1 and Γ_2 be the subarcs of Γ with the endpoints P_1, P_3 and P_2, P_3 respectively. We conclude that, for every $\varepsilon > 0$, there is a number $\Delta(\varepsilon) > 0$ such that $|P - P_1| < \varepsilon$ holds true for every $P \in \Gamma_1$ with $\operatorname{dist}(P, S) < \Delta(\varepsilon)$, and that $|P_2 - P| < \varepsilon$ is fulfilled for every $P \in \Gamma_2$ with $\operatorname{dist}(P, S) < \Delta(\varepsilon)$.

Moreover, applying the Courant–Lebesgue lemma to the surfaces X_n (or, to be precise, Proposition 2 of Section 4.4 to $X = X_n$ and $z_0 = \pm 1$), we obtain sequences $\{w_n'\}, \{w_n''\}$ of points $w_n', w_n'' \in C$ with $w_n' \to 1, w_n'' \to -1$ as $n \to \infty$ such that $\operatorname{dist}(X_n(w_n'), S) \to 0$, $\operatorname{dist}(X_n(w_n''), S) \to 0, X_n(w_n') \in \Gamma_1, X_n(w_n'') \in \Gamma_2$. As each X_n furnishes a weakly monotonic map of C onto Γ, this implies the equicontinuity of the mappings X_n on C. $\square$

4.7 Boundary Behavior of Minimal Surfaces with Rectifiable Boundaries

So far we have considered (disk-type) minimal surfaces X of class $\mathcal{C}(\Gamma)$. They have continuous boundary values on $C = \partial B$ which are continuously assumed by $X(w), w \in B$, as w tends to some boundary point. In this section we want to prove that the first derivatives of X assume boundary values of class $L_1(C)$ on C if Γ is rectifiable, and that we can establish a general formula for integration by parts.

Throughout this section we shall only make the following

General assumption. *Let $X : \bar{B} \to \mathbb{R}^3$ be a surface of class $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which has boundary values of finite variation, i.e.,*

(1) $$L(X) := \int_C |dX| < \infty,$$

and which satisfies in B the equations $X(w) \not\equiv \text{const}$ and

(2) $$\Delta X = 0,$$

(3) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Moreover let X^* be any adjoint minimal surface to X, defined by the Cauchy–Riemann equations

(4) $$X_u = X_v^*, \quad X_v = -X_u^* \quad \text{in } B.$$

Clearly, the *general assumption* will be satisfied by the solutions of Plateau's problem in $\mathcal{C}(\Gamma)$, but it will be fulfilled in many other situations as well.

The main goal of this section is the following

Theorem 1. *If the minimal surface X satisfies the general assumption and if X^* is an adjoint surface to X, then we have*:

(i) *X^* admits a continuous extension to all of $\bar{B}$, and the boundary values $X^*|_C$ are likewise rectifiable and satisfy*

(5) $$\int_C |dX| = \int_C |dX^*|.$$

(ii) *The boundary values $X|_C$ and $X^*|_C$ are absolutely continuous functions on C.*

(iii) *Set $X(r,\theta) := X(re^{i\theta})$ and $X^*(r,\theta) := X^*(re^{i\theta})$. Then the partial derivatives $X_r(r,\theta), X_\theta(r,\theta), X_r^*(r,\theta), X_\theta^*(r,\theta)$, considered as periodic functions of $\theta \in [0, 2\pi]$, tend to limits in $L_1([0, 2\pi], \mathbb{R}^3)$ as r increases to 1, both in the L_1-norm on $[0, 2\pi]$ and pointwise almost everywhere on $[0, 2\pi]$. The limits of X_θ and X_θ^* coincide a.e. on ∂B with the pointwise derivatives of the boundary values $X(e^{i\theta})$ and $X^*(e^{i\theta})$. Moreover, these derivatives vanish at most on a subset of C of 1-dimensional Hausdorff measure zero.*

An essential step in the proof of the theorem is the following

Proposition 1. *The function* $\lambda(r)$, $0 \le r \le 1$, *defined by*

$$\lambda(r) := L(X|_{C_r}) = \int_{C_r} |dX|, \quad C_r := \{re^{i\theta} : 0 \le \theta \le 2\pi\},$$

increases monotonically and is bounded from above by $L(X|_C)$. *Consequently, we also have* $\lambda(1) = \lim_{r\to 1-0} \lambda(r)$.

Proof. We have to show that, if $0 \le r < R \le 1$, then $\lambda(r) \le \lambda(R)$. It suffices, however, to assume that $R = 1$, because the general case will then follow by considering the minimal surface $X(\frac{w}{R}) : B_R \to \mathbb{R}^3$.

Since X is continuous on $\bar{B}$, Poisson's formula yields that $X(r,\theta) := X(re^{i\theta})$ satisfies

$$X(r,\theta) = \int_0^{2\pi} K(r, \varphi - \theta) X(1,\varphi)\, d\varphi, \tag{6}$$

where

$$K(r,\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2} = \frac{1}{2\pi} \frac{1 - |w|^2}{|1 - w|^2}, \quad \text{if } w = re^{i\alpha}.$$

Hence

$$\begin{aligned} X_\theta(r,\theta) &= \int_0^{2\pi} K_\theta(r,\varphi-\theta) X(1,\varphi)\, d\varphi = -\int_0^{2\pi} K_\varphi(r,\varphi-\theta) X(1,\varphi)\, d\varphi \\ &= \int_0^{2\pi} K(r,\varphi-\theta)\, dX(1,\varphi). \end{aligned}$$

The integration by parts is justified since the total variation of $X|_{\partial B}$, i.e. the length of $X|_{\partial B}$, is finite (cf. Natanson [1], Chapter VIII). In addition, $K(r,\alpha)$ is positive throughout; thus

$$|X_\theta(r,\theta)| \le \int_0^{2\pi} K(r,\varphi-\theta) |\, dX(1,\varphi)|,$$

whence

$$\begin{aligned} \lambda(r) &= \int_0^{2\pi} |X_\theta(r,\theta)|\, d\theta \\ &\le \int_0^{2\pi}\int_0^{2\pi} K(r,\varphi-\theta)\, d\theta |\, dX(1,\varphi)| \le \lambda(1) \end{aligned}$$

because of

$$\int_0^{2\pi} K(r,\alpha)\, d\alpha = 1.$$

As $\lambda(r)$ is lower semicontinuous, we obtain $\lambda(r) \to \lambda(1)$ as $r \to 1 - 0$. $\square$

(Note that neither here nor in the proof of the next result the conformality relations (3) are used.)

Proposition 2. *If we write* $X(r, \theta) := X(re^{i\theta})$, *then we obtain*

$$\int_0^1 |X_r(r, \theta)| \, dr \leq \frac{1}{2} \int_C |dX| \tag{7}$$

for every $\theta \in [0, 2\pi]$.

Proof. It suffices to prove the inequality for $\theta = 0$. Applying (6) and an integration by parts, we obtain

$$X(r, 0) = \int_0^{2\pi} K(r, \varphi) X(1, \varphi) \, d\varphi = X(1, 0) - \int_0^{2\pi} h(r, \varphi) \, dX(1, \varphi)$$

where

$$h(r, \varphi) := \int_0^{\varphi} K(r, \alpha) \, d\alpha = \frac{\varphi}{2\pi} + \frac{1}{2\pi i} \log \frac{1 - \bar{w}}{1 - w}, \quad w = re^{i\varphi}.$$

Then it follows that

$$X_r(r, 0) = -\int_0^{2\pi} h_r(r, \varphi) \, dX(1, \varphi)$$

and therefore

$$|X_r(r, 0)| \leq \int_0^{\pi} h_r(r, \varphi) |\, dX(1, \varphi)| - \int_{\pi}^{2\pi} h_r(r, \varphi) |\, dX(1, \varphi)|$$

since

$$h_r(r, \varphi) = \frac{1}{\pi} \frac{\sin \varphi}{1 - 2r \cos \varphi + r^2}$$

is positive for $0 < \varphi < \pi$ and negative for $\pi < \varphi < 2\pi$. Consequently,

$$\begin{aligned} \int_0^1 |X_r(r, 0)| \, dr \leq & \int_0^{\pi} \{h(1, \varphi) - h(0, \varphi)\} |\, dX(1, \varphi)| \\ & + \int_{\pi}^{2\pi} \{h(0, \varphi) - h(1, \varphi)\} |\, dX(1, \varphi)|. \end{aligned}$$

As $|h(1, \varphi) - h(0, \varphi)| \leq \frac{1}{2}$, we arrive at the desired inequality.

Proposition 3. *The conjugate surface* X^* *can be extended continuously to* $\bar{B}$. *Moreover, both* X *and* X^* *are contained in* $H_2^1(B, \mathbb{R}^3)$, *and we obtain*

$$\int_C |dX| = \int_C |dX^*|, \quad D_B(X) = D_B(X^*), \tag{8}$$

and

$$\int_0^1 |X_r^*(r, \theta)| \, dr \leq \frac{1}{2} \int_C |dX^*|. \tag{9}$$

Proof. (i) Similar to Proposition 2, we have used the notations $X(r,\theta) = X(re^{i\theta})$ and $X^*(r,\theta) = X^*(re^{i\theta})$. The Cauchy–Riemann equations read

$$rX_r = X^*_\theta, \quad rX^*_r = -X_\theta, \tag{10}$$

and the conformality relations are equivalent to

$$r^2|X_r|^2 = |X_\theta|^2, \quad \langle X_r, X_\theta\rangle = 0. \tag{11}$$

Therefore we also have

$$|X_r| = |X^*_r|, \quad |X_\theta| = |X^*_\theta|, \tag{12}$$

and it follows that

$$\begin{aligned} |X^*(r_2,\theta) - X^*(r_1,\theta)| &\le \int_{r_1}^{r_2} |X^*_r(r,\theta)|\,dr \\ &= \int_{r_1}^{r_2} |X_r(r,\theta)|\,dr \le \frac{1}{2}\int_C |dX| \end{aligned} \tag{13}$$

for $0 < r_1 < r_2 < 1$, on account of Proposition 2. Hence,

$$\int_0^1 |X^*_r(r,\theta)|\,dr < \infty,$$

and the convergence of this integral implies that $\lim_{r\to 1-0} X^*(r,\theta)$ exists for all $\theta \in [0, 2\pi]$.

Consider now points $w_j = e^{i\theta_j}, 0 \le j \le n$, on C with

$$0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi.$$

Then

$$\begin{aligned} \sum_{j=1}^n |X^*(w_j) - X^*(w_{j-1})| &= \lim_{r\to 1}\sum_{j=1}^n |X^*(rw_j) - X^*(rw_{j-1})| \\ &\le \lim_{r\to 1}\int_0^{2\pi} |X^*_\theta(r,\theta)|\,d\theta \\ &= \lim_{r\to 1}\int_0^{2\pi} |X_\theta(r,\theta)|\,d\theta = \int_C |dX|, \end{aligned}$$

and we infer that

$$\int_0^{2\pi} |dX^*(1,\theta)| \le \int_0^{2\pi} |dX(1,\theta)| < \infty.$$

In other words, $X^*(1,\theta)$ is a function of bounded variation for $0 \le \theta \le 2\pi$.

(ii) From $X \in C^0(\bar B, \mathbb{R}^3)$ we infer that $\sup_B |X| < \infty$. Moreover, (13) implies

$$|X^*(w)| \le |X^*(0)| + |X^*(w) - X^*(0)| \le |X^*(0)| + \frac{1}{2}\int_C |dX|$$

whence also $\sup_B |X^*| < \infty$.

Moreover, the Cauchy–Riemann equations (4) yield

$$D_B(X) = D_B(X^*).$$

Hence, in order to prove, $X, X^* \in H_2^1(B, \mathbb{R}^3)$, we only have to verify that $D_B(X) < \infty$. Let $B_R = \{re^{i\theta} : 0 \le r < R\}$. Then an integration by parts leads to

$$\begin{aligned}
\int_{B_R} |\nabla X|^2 \, du \, dv &= \int_{\partial B_R} \langle X, X_r \rangle \, ds \le \int_{\partial B_R} |X||X_r| \, ds \\
&= \int_0^{2\pi} |X(R,\theta)||X_\theta(R,\theta)| \, d\theta \\
&\le \sup_B |X| \cdot \int_0^{2\pi} |X_\theta(R,\theta)| \, d\theta \\
&\le \sup_B |X| \cdot \int_C |dX|,
\end{aligned}$$

and for $R \to 1 - 0$ we obtain

$$\int_B |\nabla X|^2 \, du \, dv \le \sup_B |X| \cdot \int_C |dX| < \infty. \tag{14}$$

(iii) As we have shown that $X^*(1,\theta)$ is a function of bounded variation with respect to θ, these boundary values can have only denumerably many discontinuities, and, for every $\theta_0 \in \mathbb{R}$, both one-sided limits

$$\lim_{\theta \to \theta_0 - 0} X^*(1,\theta), \qquad \lim_{\theta \to \theta_0 + 0} X^*(1,\theta)$$

exist. Because of $D(X^*) < \infty$ and of the Courant–Lebesgue lemma (cf. Section 4.4, Proposition 2), we then conclude that $\lim_{\theta \to \theta_0} X^*(1,\theta)$ exists for all $\theta_0 \in \mathbb{R}$, and therefore $X^*(1,\theta)$ depends continuously on θ. Hence we can apply Proposition 2 to X^* instead of X, and we then obtain

$$\int_0^1 |X_r^*(r,\theta)| \, dr \le \frac{1}{2} \int_C |dX^*|.$$

Finally, Proposition 1, applied to both X and X^*, yields

$$\begin{aligned}
\lim_{r \to 1-0} \int_0^{2\pi} |X_\theta(r,\theta)| \, d\theta &= \int_C |dX|, \\
\lim_{r \to 1-0} \int_0^{2\pi} |X_\theta^*(r,\theta)| \, d\theta &= \int_C |dX^*|,
\end{aligned}$$

and both limits coincide because of (12), whence

$$\int_C |dX| = \int_C |dX^*|.$$

□

Now we turn to the

Proof of Theorem 1. Let us introduce the holomorphic function $f : B \to \mathbb{C}^3$ by

$$f(w) := X(w) + iX^*(w)$$

with the complex derivative $f'(w)$. By the conformality relations (3), we infer that

$$|f'(w)| = \sqrt{2}|X_r(w)| = r^{-1}\sqrt{2}\,|X_\theta(w)|, \quad w = re^{i\theta},$$

and Proposition 1 implies that the (increasing) Hardy function

$$\mu(r) := \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|\,d\theta = \frac{1}{\pi r\sqrt{2}}\int_0^{2\pi} |X_\theta(re^{i\theta})|\,d\theta$$

of $f'(w)$ satisfies

$$\lim_{r\to 1-0} \mu(r) \le \int_C |dX| < \infty.$$

Thus the holomorphic function $f'(w)$, $w \in B$, belongs to the Hardy class $\mathcal{H}_1$, and a well known theorem by F. Riesz [1] ensures the existence of a function $g(\theta)$ of class $L_1([0, 2\pi], \mathbb{C}^3)$ such that both

$$\lim_{r\to 1-0}\int_0^{2\pi} |f'(re^{i\theta}) - g(\theta)|\,d\theta = 0$$

and

$$\lim_{r\to 1-0} f'(re^{i\theta}) = g(\theta) \quad \text{a.e. on } [0, 2\pi].$$

If we write

$$f(re^{i\theta}) = X(r,\theta) + iX^*(r,\theta),$$

we see that

$$(15)\qquad \begin{aligned} X_r(r,\theta) + iX_r^*(r,\theta) &= \frac{\partial}{\partial r} f(re^{i\theta}) = e^{i\theta} f'(re^{i\theta}) \to e^{i\theta} g(\theta),\\ X_\theta(r,\theta) + iX_\theta^*(r,\theta) &= \frac{\partial}{\partial \theta} f(re^{i\theta}) = ire^{i\theta} f'(re^{i\theta}) \to ie^{i\theta} g(\theta) \end{aligned}$$

as $r \to 1 - 0$.

For any $r \in (0, 1)$ and for $0 \le \theta_1 \le \theta_2 \le 2\pi$, we can write

$$f(re^{i\theta_2}) - f(re^{i\theta_1}) = \int_{\theta_1}^{\theta_2} ire^{i\theta} f'(re^{i\theta})\,d\theta.$$

Letting $r \to 1 - 0$, it follows that

$$f(e^{i\theta_2}) - f(e^{i\theta_1}) = \int_{\theta_1}^{\theta_2} ie^{i\theta} g(\theta)\,d\theta$$

for any $\theta_1, \theta_2 \in [0, 2\pi]$, where $g \in L_1([0, 2\pi], \mathbb{C}^3)$. This implies that

$$f(e^{i\theta}) = X(1, \theta) + iX^*(1, \theta)$$

is an absolutely continuous function of $\theta \in [0, 2\pi]$ whose derivative

$$X_\theta(1, \theta) + iX_\theta^*(1, \theta) = \frac{\partial}{\partial\theta} f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta}) \tag{16}$$

exists a.e. on $[0, 2\pi]$. Comparing formulas (15) and (16), we obtain the assertions of (ii) and (iii), except for the fact that $X_\theta(1, \theta) \neq 0$ and $X_\theta^*(1, \theta) \neq 0$ a.e. on $[0, 2\pi]$.

Taking $X(w) \not\equiv$ const on B into account, it follows that $f(w) \not\equiv$ const on B, and a well known theorem by F. and M. Riesz [1] implies that the boundary values of $f'(w)$ can only vanish on a subset of C of measure zero.

Finally the assertion of (i) follows from Proposition 3. □

Notice that the proof of Proposition 3 and in particular formula (14) yield the following result:

Proposition 4. *If the minimal surface X is contained in a ball*

$$K_R(P_0) := \{P \in \mathbb{R}^3 \colon |P - P_0| \leq R\}$$

of radius R, then

$$A_B(X) = D_B(X) \leq R/2 \cdot L(X|_C). \tag{17}$$

Local versions of Theorem 1 are of course available. For instance, one has

Theorem 1′. *If C' is an open subarc of $C = \partial B$, and if $X \in H_2^1(B, \mathbb{R}^3)$ is a minimal surface in B which is continuous and has rectifiable boundary values on C', i.e.,*

$$L(X|_{C'}) = \int_{C'} |dX| < \infty,$$

then $X|_{C''}$ is absolutely continuous on any subarc $C'' \subset\subset C'$, and the tangential derivative X_θ of $X|_{C''}$ is nonzero a.e. on C''.

The proof can be reduced to the previous case by using the Courant–Lebesgue lemma (see Proposition 2 of Section 4.4) and suitable conformal reparametrizations.

Theorem 2 (Integration by parts). *If the minimal surface X satisfies the general assumption of this section, and if Y is an arbitrary function of class $L_\infty \cap H_2^1(B, \mathbb{R}^3)$, then we have*

$$\int_B \langle \nabla X, \nabla Y \rangle \, du \, dv = \int_{\partial B} \left\langle Y, \frac{\partial}{\partial\nu} X \right\rangle ds \tag{18}$$

where the line integral on the right-hand side is to be taken with positive orientation of ∂B, and $\frac{\partial}{\partial\nu} X$ denotes the normal derivative of X with respect to the exterior normal ν to ∂B.

Remark. An analogous result holds if the minimal surface X is parametrized on an arbitrary parameter domain B with piecewise smooth boundary.

Proof of the theorem. Let $0 < R < 1$ and $B_R = \{w\colon |w| < R\}$. Since

$$X \in C^1(\overline{B}_R, \mathbb{R}^3),$$

we have the classical formula

$$\int_{B_R} \langle \nabla X, \nabla Y \rangle \, du \, dv = \int_{\partial B_R} \langle X_r, Y \rangle \, ds.$$

Letting $R \to 1 - 0$, the left-hand side obviously tends to $\int_B \langle \nabla X, \nabla Y \rangle \, du \, dv$, whereas the right-hand side converges to $\int_C \langle X_r, Y \rangle \, ds$, on account of Theorem 1 and of Lebesgue's theorem on dominated convergence. $\square$

4.8 Reflection Principles

In this section Ω denotes a domain in the complex plane which is symmetric with respect to the real axis, i.e., $w \in \Omega$ if and only if $\overline{w} \in \Omega$. Set

$$\begin{aligned} \Omega^+ &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w > 0\}, \\ \Omega^- &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w < 0\}, \\ \mathfrak{I} &:= \Omega \cap \{w \in \mathbb{C}\colon \operatorname{Im} w = 0\}, \end{aligned}$$

where $\mathfrak{I}$ is an open subset of $\mathbb{R}$.

We want to prove two reflection principles for minimal surfaces which generalize the well known reflection principle for harmonic functions due to H.A. Schwarz.

Theorem 1. *Suppose that X is of class $C^0(\Omega^+ \cup \mathfrak{I}, \mathbb{R}^3) \cap C^2(\Omega^+, \mathbb{R}^3)$ and satisfies both*

$$\Delta X = 0 \tag{1}$$

and

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{2}$$

in Ω^+. Assume also that X maps $\mathfrak{I}$ into a straight line L_0. Then X can be extended across $\mathfrak{I}$ onto all of Ω by reflection in L_0, and the extended surface X satisfies (1) and (2) on Ω. To be precise, the extension of X to Ω^- is defined by

$$X(w) := (X(\overline{w}))^*, \quad w \in \Omega^-,$$

where, for $P \in \mathbb{R}^3$, we denote by P^* the reflection image of P in L_0.

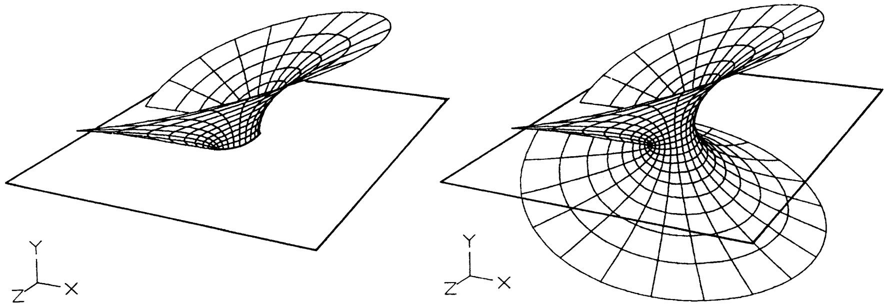


Fig. 1. Reflection of a minimal surface in a plane (Catalan's surface)

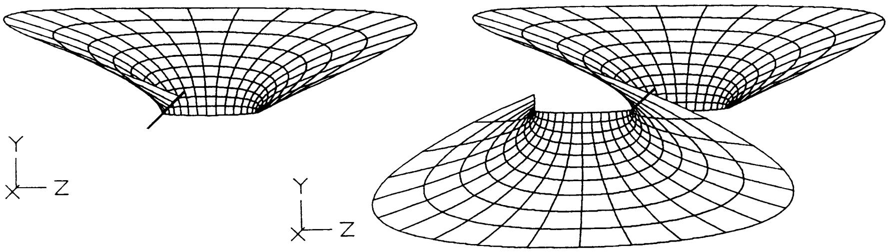


Fig. 2. Reflection of a minimal surface in a straight line (Catalan's surface)

Theorem 2. *Suppose that* X *is of class* $C^1(\Omega^+ \cup \mathfrak{I}, \mathbb{R}^3) \cap C^2(\Omega^+, \mathbb{R}^3)$ *and satisfies both* (1) *and* (2) *in* Ω^+. *Assume also that* X *maps* $\mathfrak{I}$ *into a plane* S *such that* X *is perpendicular to* S *along* $\mathfrak{I}$ (*i.e.,* $X_v(w) \perp S$ *for all* $w \in \mathfrak{I}$). *Then* X *can be extended across* $\mathfrak{I}$ *as a minimal surface on all of* Ω *if we reflect* X *in* S. *To be precise, the extension of* X *to* Ω^- *is defined by*

$$X(w) := (X(\overline{w}))^*, \quad w \in \Omega^-,$$

where P^* *denotes the mirror image in* S *of any point* $P \in \mathbb{R}^3$.

Note that these two reflection principles are more or less the same as those formulated in Section 3.4, only that we a priori require less regularity than before. To solve Björling's problem, we needed real analyticity of X along $\mathfrak{I}$ whereas here it suffices to assume $X \in C^0$ and $X \in C^1$ respectively along $\mathfrak{I}$. In fact, we shall prove that, under the assumptions of Theorems 1 and 2, X must be real analytic on $\mathfrak{I}$. Thus both theorems provide special cases of boundary regularity results. In Chapter 2 of Vol. 2 we shall treat the question of boundary regularity of minimal surfaces in some generality.

Proof of Theorem 1. Let us introduce Cartesian coordinates x, y, z in $\mathbb{R}^3$ such that L_0 becomes the z-axis, and set $X(w) = (x(w), y(w), z(w))$, $w = (u, v) = u + iv$, $\bar{w} = (u, -v) = u - iv$. Then we have

(3) $$x(w) = 0 \quad \text{and} \quad y(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Now, by Schwarz's reflection principle, we can extend x and y as harmonic functions to all of Ω if we set

(4) $$x(w) := -x(\overline{w}) \quad \text{and} \quad y(w) := -y(\overline{w}) \quad \text{for } w \in \Omega^-.$$

Moreover (3) implies that

(5) $$x_u(w) = 0, \quad y_u(w) = 0 \quad \text{for } w \in \mathfrak{I}.$$

Let $\{w_n\}$ be some sequence of points $w_n \in \Omega^+$ such that $w_n \to w_0 \in \mathfrak{I}$ as $n \to \infty$. We obtain from (2), (5) and $x, y \in C^\infty(\Omega)$ that

(6) $$\lim_{n\to\infty} z_u(w_n) z_v(w_n) = 0,$$

and from

$$\begin{aligned} z_u^2(w_n) &= |X_u(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \\ &= |X_v(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \\ &\geq |z_v(w_n)|^2 - x_u^2(w_n) - y_u^2(w_n) \end{aligned}$$

together with (5) and (6) we infer that

(7) $$\lim_{n\to\infty} z_v(w_n) = 0$$

for all sequences $\{w_n\}$, $w_n \in \Omega^+$, with $w_n \to w_0 \in \mathfrak{I}$. Hence the harmonic function $z_v(w)$, $w \in \Omega^+$, is continuous on $\Omega^+ \cup \mathfrak{I}$ and satisfies

(8) $$z_v(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Schwarz's reflection principle yields that we can extend $z(w)$ as harmonic function across $\mathfrak{I}$ to Ω by setting

(9) $$z(w) := z(\overline{w}) \quad \text{for } w \in \Omega^-.$$

Then X is harmonic in Ω, and formulas (4) and (9) together with (2) show that X fulfills the conformality relations on all of Ω. □

Proof of Theorem 2. We now introduce Cartesian coordinates x, y, z in $\mathbb{R}^3$ such that S is described by the equation $z = 0$. Then the minimal surface

$$X(w) = (x(w), y(w), z(w))$$

satisfies

(10) $$z(w) = 0 \quad \text{for all } w \in \mathfrak{I}.$$

Moreover, $X_u(w)$ is tangential to S for all $w \in \mathcal{I}$, and X_v is perpendicular to X_u. Since we have assumed that $X(w)$ meets S along $\mathcal{I}$ at a right angle, it follows that $X_v(w)$ is orthogonal to the two vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0)$$

spanning S, for all $w \in \mathcal{I}$, whence we conclude that

$$(11) \qquad x_v(w) = 0, \quad y_v(w) = 0 \quad \text{for all } w \in \mathcal{I}.$$

Applying Schwarz's reflection principle for harmonic functions, we infer from (10) and (11) that $x(w), y(w), z(w)$ can be continued to Ω as harmonic functions, by setting

$$(12) \qquad x(w) = x(\overline{w}), \quad y(w) = y(\overline{w}), \quad z(w) = -z(\overline{w}) \quad \text{for } w \in \Omega^-.$$

One easily checks that the harmonic vector $X(w)$, $w \in \Omega$, satisfies the conformality relations on all of Ω. □

Recently, Choe [4] proved that a minimal surface can also be analytically extended across a plane S if it meets this plane at a constant angle θ with $0 < \theta < \pi$, and the extension is again carried out by reflection in S.

4.9 Uniqueness and Nonuniqueness Questions

How many minimal surfaces can be spanned in a given closed Jordan curve? The answer to this question is not known in general, not even if we fix the topological type of the solutions of Plateau's problem. As we have considered only disk-type minimal surfaces, we want to consider the more modest question of:

How many minimal surfaces of the type of the disk can be spanned in a given closed Jordan curve Γ?

The situation would be simple if we could prove that Γ bounds only one disk-type minimal surface $X \in \mathcal{C}(\Gamma)$ (up to reparametrizations $X \circ \tau$ of X by conformal mappings $\tau : B \to B$ of the parameter domain B onto itself; such reparametrizations would not be counted as different from X and could be excluded by fixing a three-point condition for the surfaces $X \in \mathcal{C}(\Gamma)$ which are prospective solutions; in other words: *Uniqueness of the solution of Plateau's problem in $\mathcal{C}(\Gamma)$ actually means 'uniqueness in $\mathcal{C}^*(\Gamma)$'*).

However, examples (cf. Figs. 1 and 4 of the Introduction) warn us not to expect uniqueness for disk-type solutions of Plateau's problem. Thus we may ask whether additional geometric conditions for Γ are known which ensure this uniqueness. Essentially, we know three results:

1. **Theorem of Radó** [16]: *If Γ has a one-to-one parallel projection onto a planar convex curve γ, then Γ bounds at most one disk-type minimal surface.*

This result will be proved in the sequel (cf. Theorem 1). By the same technique, Radó [20] was able to ensure uniqueness in the case of Γ admitting a one-to-one *central projection* onto a planar convex curve γ. (See also Nitsche [28], pp. 360–362.)

For the sake of completeness we mention a result of Tromba [3] which looks like a corollary to Radó's theorem but, at closer inspection, turns out not to be included. Actually it is proved in a completely different way.

Tromba's observation. *If Γ is C^2-close to a planar curve γ of class C^2, then Γ bounds a unique minimal surface of the type of the disk.*

2. **Theorem of Nitsche** [26]: *If Γ is regular, real analytic and has a total curvature less than or equal to 4π, then Γ bounds only one disk-type minimal surface.*

A proof of this result is given in Section 5.6. It is based on a "field embedding". We shall establish this by using a technique due to H.A. Schwarz, modified by J.C.C. Nitsche. The third uniqueness theorem, due to F. Sauvigny, is described in Section 7.2.

For polygonal Γ of total curvature less than 4π, this result was earlier conjectured by R. Schneider [2] whose sketch of a proof contained some of the ideas used in Nitsche's proof.

In general, however, nothing is known about the number of solutions of Plateau's problem which are of class $\mathcal{C}(\Gamma)$. Actually, the situation seems to be rather unpromising on account of the following remarkable result due to Böhme [6]:

For each positive integer N and for each $\varepsilon > 0$, there exists a regular real analytic Jordan curve Γ in $\mathbb{R}^3$ with total curvature less than $4\pi + \varepsilon$ which bounds at least N minimal surfaces of class $\mathcal{C}^*(\Gamma)$, i.e., of the type of the disk.

One does not even know whether the number of solutions $X \in \mathcal{C}(\Gamma)$ of Plateau's Problem for the curve Γ is finite or not. There are suggestive examples of P. Levy [2] and Courant [15] which indicate that there might be rectifiable Jordan curves Γ bounding non-denumerably many minimal surfaces. The validity of these examples, however, depends strictly on the validity of the *strong bridge theorem* which recently was rigorously proved by B. White. For the construction principle of the Levy–Courant examples and for comments on the bridge principle we refer the reader to the Scholia.

Whatever may be the case, we have two satisfactory partial answers to the *finiteness question*:

1. **Theorem of Böhme–Tromba** [1]. *Generically, the number of disk-type solutions of Plateau's problem is finite.*

For a proof of this result, see Vol. 3.

2. **Theorem of Tomi** [6]. *There are only finitely many disk-type solutions of Plateau's problem which are absolute minimizers of the area functional in* $\mathcal{C}^*(\Gamma)$ *provided that* Γ *is a regular real-analytic Jordan curve.*

3. **Theorem of Nitsche** [31]. *If the regular contour* $\Gamma \in C^{3,\alpha}$ *is either extreme or real analytic and has a total curvature of less than* 6π*, then there exist only finitely many immersions* $X : \overline{B} \to \mathbb{R}^3$ *of class* $\mathcal{C}^*(\Gamma)$*.*

A proof of the results 2 and 3 can be found in Section 5.7.

Doubtless, the *number-of-solutions problem* is the most exciting and most challenging question that can be raised in connection with Plateau's problem.

Now we want to discuss Radó's result.

Theorem 1. *If* Γ *possesses a one-to-one parallel projection onto a plane convex Jordan curve* γ*, then* Γ *bounds at most one minimal surface except, of course, for conformal reparametrizations. It has no branch points, and it admits a non-parametric representation.*

As an example, let us consider an arbitrary quadrilateral Γ in $\mathbb{R}^3$. By this we mean a Jordan curve which is a polygon with four edges and four vertices. If Γ is a planar curve, then it bounds exactly one (planar) minimal surface on account of the maximum principle. On the other hand it is easy to verify that any nonplanar quadrilateral admits a one-to-one orthogonal projection onto a convex plane quadrilateral. Applying Radó's theorem, we then obtain:

Every quadrilateral bounds a uniquely determined minimal surface of the type of the disk.

For the proof of Theorem 1 we need the following

Lemma 1 (Monodromy principle). *Let* Ω *be a simply connected, bounded domain in* $\mathbb{C}$ *and let* $f \in C^0(\bar{\Omega}, \mathbb{C}) \cap C^1(\Omega, \mathbb{C})$ *be a mapping whose Jacobian* $\det Df$ *vanishes nowhere in* Ω *so that* f *is an open mapping of* Ω *onto the domain* $\Omega' = f(\Omega)$*. Then* f *is injective if at least one of the following conditions is satisfied*:

(i) f *maps* $\partial\Omega$ *into a closed Jordan curve* γ *in* $\mathbb{C}$;
(ii) f *maps* Ω *into a simply connected domain* $\hat{\Omega}$ *and* $\partial\Omega$ *into* $\partial\hat{\Omega}$*.*

Proof. First we will show that $\partial\Omega' \subset f(\partial\Omega)$. In fact, for an arbitrary point $z \in \partial\Omega'$ we can find a sequence of points $z_n \in \Omega'$ converging to z, and another sequence of points $w_n \in \Omega$ such that $f(w_n) = z_n$ and $w_n \to w$ for some $w \in \bar{\Omega}$. Since f is continuous on $\bar{\Omega}$, we obtain $f(w) = z$, and this implies $w \in \partial\Omega$ as the mapping f is open. Thus we have proved that $\partial\Omega' \subset f(\partial\Omega)$.

Let us now assume that (i) holds true. Then $\mathbb{C} \setminus \gamma$ consists of two components, the simply connected interior $\hat{\Omega}$ of γ, and the unbounded exterior $\tilde{\Omega}$.

Because of

$$\partial\Omega' \subset f(\partial\Omega) \subset \gamma = \partial\tilde{\Omega}$$

we obtain

$$\overline{\Omega'} \cap \tilde{\Omega} = \Omega' \cap \tilde{\Omega},$$

and this implies $\Omega' \cap \tilde{\Omega} = \emptyset$, whence $f(\Omega) = \Omega' \subset \hat{\Omega}$ and $f(\partial\Omega) \subset \gamma = \partial\hat{\Omega}$. Thus we are in the situation described by (ii). Let us now consider this case. Repeating the previous reasoning, we get $\overline{\Omega'} \cap \hat{\Omega} = \Omega' \cap \hat{\Omega}$, and we conclude that $f(\Omega) = \Omega' = \hat{\Omega}$, and therefore $\partial f(\Omega) = \partial\hat{\Omega}$.

The injectivity of f can now be justified by a standard monodromy argument. Suppose that two points w_1 and w_2 in Ω were mapped onto the same image point $z \in \hat{\Omega}$. Any arc α in Ω joining w_1 and w_2 will be mapped by f onto a closed curve β in $\hat{\Omega}$ since $f(w_1) = f(w_2) = z$. By some homotopy in the simply connected domain $\hat{\Omega}$ we can shrink β to the point z. Since f is a local diffeomorphism in Ω, each curve of the homotopy is the image of an arc in Ω which joins w_1 and w_2, and this curve must be closed as soon as its image lies in a sufficiently small neighborhood of z. □

For a more detailed proof of the monodromy principle in $\mathbb{C}$ we refer the reader to a suitable text book of complex analysis such as Ahlfors [5] or Bieberbach [2]. (More general versions of this principle in algebraic topology can for instance be found in Greenberg [1].)

The next two lemmata contain the essential ideas needed for the proof of the theorem.

We shall again encounter the reasoning employed in the proof of the following lemma in Chapters 1 and 2 of Vol. 3 where similar uniqueness theorems for surfaces with semifree boundaries will be proved.

Lemma 2 (Radó's lemma). *If $f : \bar{B} \to \mathbb{R}$ is harmonic in B, continuous on $\bar{B}$, not identically zero, and if its derivatives of orders $0, 1, \ldots, m$ vanish at some point $w_0 \in B$, then f changes its sign on ∂B at least $2(m+1)$ times.*

Proof. The function f is the real part of a holomorphic function $F : B \to \mathbb{C}$ whose power series expansion close to w_0 is given by

$$F(w) = i\beta_0 + a_n(w - w_0)^n + O(|w - w_0|^{n+1})$$

for $|w - w_0| \to 0$, where $n \geq m + 1, a_n \neq 0$, and β_0 is real. Consequently the set $\{w \in B\colon f(w) = 0\}$ divides a neighborhood of w_0 into $2n$ open sectors $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$ by means of $2n$ analytic arcs emanating from w_0 such that f is positive on $\sigma_1, \sigma_3, \ldots, \sigma_{n-1}$ and negative on $\sigma_2, \sigma_4, \ldots, \sigma_{2n}$, cf. Fig. 1.

The set $\{w \in B\colon f(w) \neq 0\}$ is open, therefore it has at most denumerably many connected components. Let $Q_1, Q_2, \ldots, Q_{2n}$ be the components containing the sectors $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$ respectively. We claim that no two of them coincide.

Suppose for example that $Q_{2k} = Q_{2l}, k \neq l$. Then we can construct a (piece-wise linear) closed Jordan curve γ starting at w_0, running first through

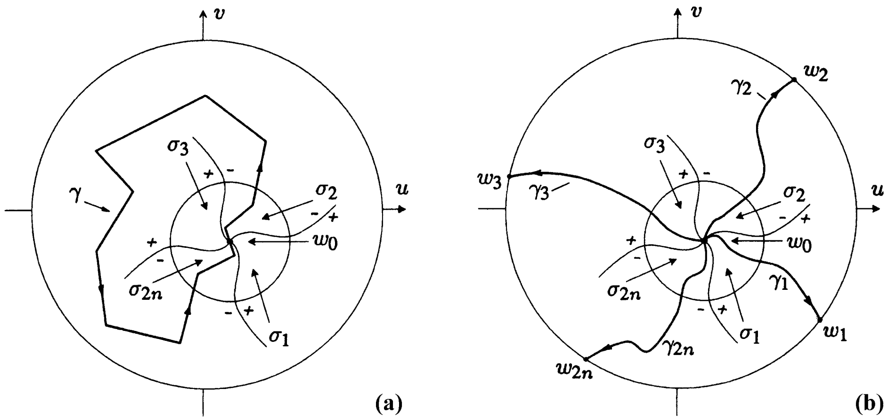


Fig. 1. Rado's lemma: Sectors

the sector σ_{2k} and finally traversing σ_{2l} before it returns to w_0. Now either the sector σ_{2k-1} or σ_{2k+1} belongs to the bounded component Ω of $\mathbb{C} \setminus \gamma$. The function f is non-positive on $\gamma = \partial\Omega$ but positive on σ_{2k-1} and on σ_{2k+1}. The maximum principle applied to the harmonic function $f : \Omega \to \mathbb{R}$ yields the desired contradiction, and the remaining cases are excluded similarly.

Another application of the maximum principle shows that each of the components Q_j, $j = 1, \dots, 2n$, has a boundary point $w_j \in \partial Q_j$ lying on ∂B such that $f(w_j)$ is positive for $j = 1, 3, \dots, 2n-1$ and negative for $j = 2, 4, \dots, 2n$. Moreover, for any of these w_j we can construct a path γ_j in Q_j starting in the sector σ_j and ending at w_j. Since these paths γ_j do not intersect, the pattern of the points w_j on ∂B reflects the one of sectors σ_j close to w_0. Thus between any w_j and its successor w_{j+1} the continuous function $f|_{\partial B}$ has a zero. □

The third lemma is a variant of Lemma 2 and a consequence of the monodromy principle. The conclusions are the same, but the assumptions are different. This result is known as **Kneser's lemma** (cf. T. Radó [5], H. Kneser [1]).

Lemma 3. *Suppose that $\varphi : \bar{B} \to \mathbb{R}^2$ is a transformation which is harmonic in B, continuous in $\bar{B}$, and which maps ∂B in a weakly monotonic manner onto the boundary $\partial\Omega$ of a convex domain $\Omega \subset \mathbb{R}^2$. Then φ is a diffeomorphism from B onto Ω. If in addition $\varphi : \partial B \to \partial\Omega$ is a homeomorphism, then so is $\varphi : \bar{B} \to \bar{\Omega}$.*

Proof. This lemma will follow immediately from the monodromy principle, if we can show that the Jacobian det $D\varphi$ of the transformation φ has no zeros in B.

First of all, the maximum principle for harmonic functions implies that $\varphi(B)$ lies in Ω. Now, if det $D\varphi(w_0) = 0$ for some $w_0 \in B$, then the rows of the

Jacobi matrix $D\varphi(w_0)$ are linearly dependent, i.e., there are real constants a and b, at least one of which is nonzero and such that

$$ax_u(w_0) + by_u(w_0) = 0$$

and

$$ax_v(w_0) + by_v(w_0) = 0$$

where x and y are the real and the imaginary parts of φ respectively. Moreover there is a real number c such that

$$ax(w_0) + by(w_0) + c = 0,$$

which means that $\varphi(w_0)$ lies on the straight line

$$L(w_0) = \{x + iy\colon ax + by + c = 0\}$$

which intersects $\partial\Omega$ in exactly two points P_1 and P_2. Let us now inspect the harmonic function

$$f(w) = ax(w) + by(w) + c$$

which is continuous on $\bar{B}$. Note that φ maps ∂B onto $\partial\Omega$. Then, for any $w \in \partial B$, the function $f(w)$ vanishes if and only if $\varphi(w)$ lies on the intersection of $\partial\Omega$ with the straight line $L(w_0)$. Furthermore, since φ maps ∂B in a weakly monotonic manner onto $\partial\Omega$, the pre-image

$$\varphi^{-1}\{P_1, P_2\} = \varphi^{-1}(\partial\Omega \cap L(w_0)) = f^{-1}_{|\partial B}\{0\}$$

consists of two closed connected subarcs of ∂B. On the other hand, Radó's lemma implies that f has at least four zeros in ∂B which are separated by points where f does *not* vanish. This contradiction shows that the assumption $\det D\varphi(w_0) = 0$ is impossible. □

Now we turn to the

Proof of Theorem 1. After a rotation of coordinates we may suppose that the parallel projection mentioned in the theorem is the orthogonal projection onto the xy-plane. Then Γ possesses a 1–1 orthogonal projection $\overline{\gamma}$ which is a convex curve contained in the plane $\{z = 0\}$. Replacing γ by $\overline{\gamma}$, we may assume that Γ lies as a graph above a plane convex curve γ which is contained in the plane $\{z = 0\}$. Therefore the preceding lemma shows that the first two components x and y of any minimal surface $X = (x, y, z) \in \mathcal{C}(\Gamma)$ which solves Plateau's problem for Γ determine a diffeomorphism φ from B onto the convex domain Ω enclosed by γ. Denoting the inverse of φ by $(u(x, y), v(x, y))$, the function

$$Z(x, y) := z(u(x, y), v(x, y))$$

defines a nonparametric representation of the surface $X(B)$. Of course, X has no branch points since its first two components define a diffeomorphism. Consequently $X(B)$ is a regular embedded surface whose mean curvature vanishes.

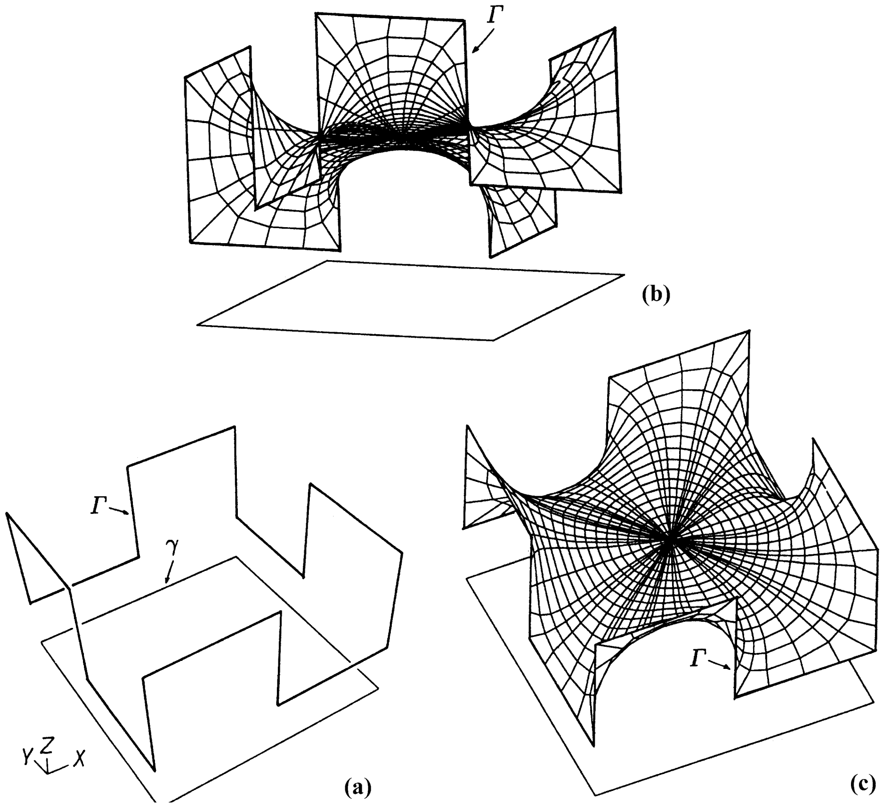


Fig. 2. Rado's uniqueness theorem (strengthened): The Jordan curve Γ is a generalized graph over a plane convex curve, the square shown in **(a)**. The solution to Plateau's problem for Γ is therefore unique **(b)**, **(c)**

Thus, as we have seen in Section 2.2, $Z(x, y)$ is a solution of the minimal surface equation with bounded, but not necessarily continuous boundary values.

Suppose now that X and $\hat{X}$ are two solutions of Plateau's problem for Γ, and denote their corresponding non-parametric representations by $Z(x, y)$ and $\hat{Z}(x, y)$ respectively. If the projection of Γ onto γ is one-to-one, then φ is a homeomorphism from $\bar{B}$ onto $\bar{\Omega}$. Consequently, since X and $\hat{X}$ are continuous on $\bar{B}$, the functions Z and $\hat{Z}$ are continuous on $\bar{\Omega}$, and so is the difference $Z - \hat{Z}$, which vanishes on $\partial\Omega$. Moreover $Z - \hat{Z}$ satisfies a second order linear equation in Ω for which the maximum principle holds true (cf. Gilbarg and Trudinger [1], p. 208). This implies that Z and $\hat{Z}$ coincide in $\bar{\Omega}$ so that we have in particular $X(B) = \hat{X}(B)$.

Now since X and $\hat{X}$ are conformal and invertible, $X^{-1} \circ \hat{X}$ is a conformal mapping from B onto itself. Thus a three-point-condition guarantees that X is equal to $\hat{X}$. $\square$

Remark to Theorem 1. The uniqueness result of Theorem 1 remains true under somewhat weaker assumptions on Γ. Instead of requiring the existence of a 1-to-1 parallel projection of Γ onto a plane convex curve γ, we can allow vertical segments for Γ which are mapped onto single points of γ. For the proof of this more general fact one needs a sharpening of the maximum principle provided by Nitsche [11]; see also Nitsche [28], §401 and §586. This reasoning is essentially based on an extension of Theorem 6 in Section 7.3.

Concluding this section, we want to draw some further results from Radó's lemma.

Theorem 2. *If $w_0 \in B$ is an interior branch point of the minimal surface $X \in \mathcal{C}(\Gamma)$, then each plane Π through the point $X(w_0)$ intersects Γ in at least four distinct points.*

Proof. Let $\nu \in S^2$ be a vector normal to Π. Then we have

$$\Pi = \{x \in \mathbb{R}^3 \colon \langle x - X(w_0), \nu \rangle = 0\}.$$

Consider the function $f : \overline{B} \to \mathbb{R}^3$ defined by

$$f(w) := \langle X(w) - X(w_0), \nu \rangle, \quad w \in \overline{B},$$

which is continuous on $\overline{B}$, harmonic in B, and satisfies

$$f(w_0) = 0, \quad f_u(w_0) = 0, \quad f_v(w_0) = 0.$$

On account of Lemma 2 it follows that f has at least four zeros on ∂B. $\square$

Corollary 1. *If there is a straight line $\mathcal{L}$ in $\mathbb{R}^3$ such that each plane Π through $\mathcal{L}$ intersects Γ in at most three points, then any minimal surface $X \in \mathcal{C}(\Gamma)$ is free of interior branch points.*

An immediate consequence of this result is

Corollary 2. *A minimal surface $X \in \mathcal{C}(\Gamma)$ has no interior branch points if Γ possesses a one-to-one parallel or central projection onto a star-shaped planar curve.*

4.10 Another Solution of Plateau's Problem by Minimizing Area

In this section we want to present a solution of the *minimal area problem* for disk-type surfaces which is obtained by minimizing the functional $A^\epsilon := (1-\epsilon)A + \epsilon D$ in the class $\mathcal{C}(\Gamma)$. This will lead to a direct solution of the *simultaneous problem* of finding a minimal surface of class $\mathcal{C}(\Gamma)$ that minimizes both the area functional

$$A(X) = \int_B |X_u \wedge X_v| \, du \, dv$$

and the Dirichlet integral

$$D(X) = \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2) \, du \, dv$$

among all admissible surfaces $X \in \mathcal{C}(\Gamma)$. Thereby we obtain another proof of Theorem 4 and of relation (40) in Section 4.5.

We begin by recalling that D is (sequentially) weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$; cf. 4.6. It turns out that A has the same property:

Lemma 1. *Let $\{X_n\}$ be a sequence in $H_2^1(B, \mathbb{R}^3)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$. Then*

$$A(X) \le \liminf_{n \to \infty} A(X_n). \tag{1}$$

Proof. First we note the identity

$$A(Z) = \sup \left\{ \int_B \phi \cdot (Z_u \wedge Z_v) \, du \, dv \colon \phi \in C_c^\infty(B, \mathbb{R}^3), \ |\phi| \le 1 \right\} \tag{2}$$

which holds for any $Z \in H_2^1(B, \mathbb{R}^3)$; it can easily be verified.

We claim that for proving (1) it suffices to show

$$\lim_{n \to \infty} \int_B \phi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv = \int_B \phi \cdot (X_u \wedge X_v) \, du \, dv \tag{3}$$

for any $\phi \in C_c^\infty(B, \mathbb{R}^3)$ satisfying $|\phi| \le 1$. In fact, equations (2) and (3) imply

$$\begin{aligned}
\int_B \phi \cdot (X_u \wedge X_v) \, du \, dv &= \lim_{n \to \infty} \int_B \phi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv \\
&\le \liminf_{n \to \infty} \left[\sup \left\{ \int_B \psi \cdot (X_{n,u} \wedge X_{n,v}) \, du \, dv \colon \psi \in C_c^\infty(B, \mathbb{R}^3), \ |\psi| \le 1 \right\} \right] \\
&= \liminf_{n \to \infty} A(X_n).
\end{aligned}$$

Taking the supremum over all ϕ in $C_c^\infty(B, \mathbb{R}^3)$ with $|\phi| \le 1$ we then arrive at (1).

Thus it suffices to verify (3). Let Z be of class $C^2(B, \mathbb{R}^3)$; then for $\phi \in C_c^\infty(B, \mathbb{R}^3)$ an integration by parts yields

$$\int_B \phi \cdot (Z_u \wedge Z_v) \, du \, dv = -\frac{1}{2} \int_B \big[\phi_u \cdot (Z \wedge Z_v) + \phi_v \cdot (Z_u \wedge Z) \big] \, du \, dv. \tag{4}$$

Using a suitable approximation device, this identity can as well be established for arbitrary $Z \in H_2^1(B, \mathbb{R}^3)$.

Suppose now that $X_n \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. By Rellich's theorem we obtain $X_n \to X$ in $L_2(B, \mathbb{R}^3)$, and so (3) follows from (4). $\square$

Next we define the functionals $A^\epsilon : H_2^1(B, \mathbb{R}^3) \to \mathbb{R}$ by

$$A^\epsilon := (1-\epsilon)A + \epsilon D, \quad 0 \le \epsilon \le 1.$$

Since A and D are weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$ also A^ϵ has this property, i.e. we have

Lemma 2. *If $X_n \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$ then*

$$A^\epsilon(X) \le \liminf_{n \to \infty} A^\epsilon(X_n)$$

for any $\epsilon \in [0, 1]$.

Our goal is now to find a conformally parametrized minimizer of A in $\mathcal{C}(\Gamma)$. As A is a somewhat singular functional we take a detour by first considering the *modified variational problem*

$$A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma) \tag{5}$$

for an arbitrary $\epsilon \in (0, 1]$. As A^ϵ is conformally invariant we can find a minimizing sequence $\{X_n\}$ for A^ϵ in $\mathcal{C}(\Gamma)$ that satisfies a fixed three-point condition, i.e.

$$A^\epsilon(X_n) \to \alpha(\epsilon) := \inf_{\mathcal{C}(\Gamma)} A^\epsilon = \inf_{\mathcal{C}^*(\Gamma)} A^\epsilon$$

and $X_n \in \mathcal{C}^*(\Gamma)$ if we use the notation of 4.3. Then

$$(1-\epsilon)A(X_n) + \epsilon D(X_n) = A^\epsilon(X_n) \le \alpha(\epsilon) + 1 \quad \text{for } n \gg 1,$$

whence

$$D(X_n) \le \text{const} \quad \text{for all } n \in \mathbb{N}.$$

Now we can proceed as in the proof of Theorem 1 in 4.6: We obtain a subsequence $\{X_{n_p}\}$ of $\{X_n\}$ that tends weakly in $H_2^1(B, \mathbb{R}^3)$ to some X^ϵ which is contained in $\mathcal{C}^*(\Gamma)$ as this set is weakly (sequentially) closed in $H_2^1(B, \mathbb{R}^3)$. It follows that

$$\alpha(\epsilon) \le A^\epsilon(X^\epsilon) \le \lim_{p \to \infty} A^\epsilon(X_{n_p}) = \alpha(\epsilon),$$

and so $A^\epsilon(X^\epsilon) = \alpha(\epsilon)$. Thus, for any $\epsilon > 0$, we have found a minimizer $X^\epsilon \in \mathcal{C}^*(\Gamma)$ of A^ϵ in $\mathcal{C}(\Gamma)$. As in 4.5 this minimum property implies

$$\partial A^\epsilon(X^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1(\overline{B}, \mathbb{R}^2). \tag{6}$$

Since A is parameter invariant it follows that

$$\partial A^\epsilon(X^\epsilon, \lambda) = \epsilon \partial D(X^\epsilon, \lambda),$$

and so we obtain

$$\partial D(X^\epsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1(\overline{B}, \mathbb{R}^2). \tag{7}$$

By Theorem 1 of 4.5 we see that X^ϵ satisfies the conformality relations

$$|X^\epsilon_u|^2 = |X^\epsilon_v|^2, \quad \langle X^\epsilon_u, X^\epsilon_v\rangle = 0. \tag{8}$$

Before we proceed we remark the following: In proving relation (6) we have used the *Riemann mapping theorem*. This can be avoided by using the method presented in the Supplementary Remark 1 of 4.5: Using vector fields $\lambda \in C^1(\overline{B}, \mathbb{R}^3)$ such that $\lambda(w)$ is tangential to ∂B for any $w \in \partial B$ we construct diffeomorphisms τ_ϵ of $\overline{B}$ onto itself which are of the form $\tau_\epsilon(w) = w - \epsilon\lambda(w) + o(\epsilon)$ as $\epsilon \to 0$. Let us denote the class of these vector fields by $C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$. Then we arrive at

$$\partial A^\epsilon(X^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$$

without employing the Riemann mapping theorem. This leads to

$$\partial D(X^\epsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

and by the formulae derived in Example 1 of 4.5 we arrive at

$$\int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv = 0 \quad \text{for all } \lambda = (\mu, \nu) \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2) \tag{9}$$

where

$$a := |X^\epsilon_u|^2 - |X^\epsilon_v|^2, \quad b := 2\langle X^\epsilon_u, X^\epsilon_v\rangle.$$

We claim that a, b satisfy the Cauchy–Riemann equations

$$a_u = -b_v, \quad a_v = b_u \quad \text{on } B \tag{10}$$

whence $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u + iv$ in B. Since we do not yet know that X^ϵ is harmonic in B, we cannot derive (10) as in the Supplementary Remark 1 of 4.5. Instead we apply (9) to vector fields λ of the form $\lambda = \mathcal{S}_\delta \eta$ with $\eta = (\eta^1, \eta^2) \in C^\infty_c(B', \mathbb{R}^2)$ with $B' \subset\subset B$, where $\mathcal{S}_\delta$ is a smoothing operator with a symmetric kernel k_δ, $0 < \delta \ll 1$, i.e. $\mathcal{S}_\delta \eta = k_\delta * \eta$. Set

$$a^\delta := \mathcal{S}_\delta a, \quad b^\delta := \mathcal{S}_\delta b.$$

Then we obtain

$$\begin{aligned}
0 &= \int_B \big\{a[(\mathcal{S}_\delta\eta^1)_u - (\mathcal{S}_\delta\eta^2)_v] + b[(\mathcal{S}_\delta\eta^1)_v + (\mathcal{S}_\delta\eta^2)_u]\big\}\, du\, dv \\
&= \int_B \big\{a[\mathcal{S}_\delta(\eta^1_u) - \mathcal{S}_\delta(\eta^2_v)] + b[\mathcal{S}_\delta(\eta^1_v) + \mathcal{S}_\delta(\eta^2_u)]\big\}\, du\, dv \\
&= \int_B \big\{a^\delta(\eta^1_u - \eta^2_v) + b^\delta(\eta^1_v + \eta^2_u)\big\}\, du\, dv \\
&= \int_B \big\{-(a^\delta_u + b^\delta_v)\eta^1 + (a^\delta_v - b^\delta_u)\eta^2\big\}\, du\, dv
\end{aligned}$$

since $\mathcal{S}_\delta$ commutes with $\partial/\partial u$ and $\partial/\partial v$ and

$$\int_B f \cdot \mathcal{S}_\delta \varphi \, du \, dv = \int_B \mathcal{S}_\delta f \cdot \varphi \, du \, dv$$

for $f \in L_1(B)$ and $\varphi \in C_c^\infty(B')$, $B' \subset\subset B$, $0 < \delta \ll 1$. By the fundamental theorem of the calculus of variations it follows that

$$a_u^\delta + b_v^\delta = 0 \quad \text{and} \quad a_v^\delta - b_u^\delta = 0 \quad \text{in } B' \subset\subset B.$$

In other words: For any fixed $B' \subset\subset B$ the function $\Phi^\delta(w) := a^\delta(u,v) - ib^\delta(u,v)$ is holomorphic for $w = u+iv \in B'$ if $0 < \delta < \delta_0(B')$ where $\delta_0(B') > 0$ is a sufficiently small number depending on B'. Since

$$\|a - a^\delta\|_{L_1(B')} \to 0 \quad \text{and} \quad \|b - b^\delta\|_{L_1(B')} \to 0 \quad \text{as } \delta \to +0$$

we obtain

$$\int_{B'} |\Phi - \Phi^\delta| \, du \, dv \to 0 \quad \text{as } \delta \to +0.$$

Since the L_1-limit of holomorphic functions is holomorphic we infer that Φ is holomorphic in $B' \subset\subset B$, and so it is holomorphic in B. Thus we have verified (10), and from now on we can proceed as in the Supplementary Remark 1 of 4.5 obtaining $\Phi(w) \equiv 0$ in B, i.e. $a(u,v) \equiv 0$ and $b(u,v) \equiv 0$ on B. Therefore we have verified the conformality relations

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad \langle X_u^\epsilon, X_v^\epsilon \rangle = 0 \quad \text{in } B$$

for any $\epsilon \in (0,1]$, which imply $A(X^\epsilon) = D(X^\epsilon)$, and we obtain

$$A^\epsilon(X^\epsilon) = A(X^\epsilon) = D(X^\epsilon) \quad \text{for } 0 < \epsilon \le 1.$$

On the other hand we infer from $A \le D$ and the minimum property of X^ϵ that

$$A^\epsilon(X^\epsilon) \le A^\epsilon(X) = (1-\epsilon)A(X) + \epsilon D(X) \le D(X)$$

holds for any $X \in \mathcal{C}(\Gamma)$ and any $\epsilon \in (0,1]$. Choosing $X = X^{\epsilon'}$ we arrive at

$$D(X^\epsilon) \le D(X^{\epsilon'}) \quad \text{for any } \epsilon, \epsilon' \in (0,1]$$

whence

$$D(X^\epsilon) = A(X^\epsilon) = A^\epsilon(X^\epsilon) \equiv \text{const} =: c \quad \text{for } 0 < \epsilon \le 1. \tag{11}$$

Set

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then, for arbitrary $Z \in \mathcal{C}(\Gamma)$ and any $\epsilon, \epsilon' \in (0,1]$ we obtain

$$a(\Gamma) \le A(X^\epsilon) = A^\epsilon(X^\epsilon) = A^{\epsilon'}(X^{\epsilon'}) \le A^{\epsilon'}(Z)$$

and

$$e(\Gamma) \le D(X^\epsilon) = A^\epsilon(X^\epsilon) \le A^\epsilon(Z) \le D(Z).$$

Letting $\epsilon' \to +0$ the first set of inequalities yields

$$a(\Gamma) \le A(X^\epsilon) \le A(Z),$$

and the second furnishes

$$e(\Gamma) \le D(X^\epsilon) \le D(Z)$$

for all $Z \in \mathcal{C}(\Gamma)$. This implies

$$a(\Gamma) \le A(X^\epsilon) \le a(\Gamma) \quad \text{and} \quad e(\Gamma) \le D(X^\epsilon) \le e(\Gamma)$$

whence

$$a(\Gamma) = A(X^\epsilon) = D(X^\epsilon) = e(\Gamma) \quad \text{for all } \epsilon \in (0,1].$$

Set $\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$ and

$$\overline{a}(\Gamma) = \inf_{\overline{\mathcal{C}}(\Gamma)} A, \quad \overline{e}(\Gamma) := \inf_{\overline{\mathcal{C}}(\Gamma)} D.$$

Then we know that every minimizer X of D in $\mathcal{C}(\Gamma)$ lies in $\overline{\mathcal{C}}(\Gamma)$, and so

$$a(\Gamma) \le \overline{a}(\Gamma) \le A(X) \le D(X) = e(\Gamma) = a(\Gamma)$$

and

$$e(\Gamma) \le \overline{e}(\Gamma) \le D(X) = e(\Gamma).$$

Thus we have $a(\Gamma) = \overline{a}(\Gamma) = A(X) = D(X) = e(\Gamma) = \overline{e}(\Gamma)$. In addition, every conformally parametrized minimizer X of A in $\mathcal{C}(\Gamma)$ satisfies $a(\Gamma) = A(X) = D(X)$. So we have proved

Theorem 1. *For any rectifiable curve Γ in $\mathbb{R}^3$ one has*

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D, \tag{12}$$

and any minimizer of Dirichlet's integral in $\mathcal{C}(\Gamma)$ is simultaneously a minimizer of area in $\mathcal{C}(\Gamma)$, and conversely every conformally parametrized minimizer of area in $\mathcal{C}(\Gamma)$ is as well a minimizer of Dirichlet's integral in $\mathcal{C}(\Gamma)$.

Remark 1. Starting from (11) we alternatively could have argued in the following way: Applying the reasoning of 4.6 we obtain a sequence of positive numbers ϵ_j with $\epsilon_j \to 0$ and an $X \in \mathcal{C}^*(\Gamma)$ such that $X^{\epsilon_j} \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. Then

$$\begin{aligned} a(\Gamma) &\le A(X) \le \liminf_{j\to\infty} A(X^{\epsilon_j}) = \lim_{\epsilon\to 0} A^\epsilon(X^\epsilon) = c \\ &\le \lim_{\epsilon\to 0} A^\epsilon(Z) = A(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma) \end{aligned}$$

and so $a(\Gamma) \le A(X) \le a(\Gamma)$, i.e. $A(X) = a(\Gamma)$. Therefore the weak limit X of the X^{ϵ_j} is a minimizer of A in $\mathcal{C}(\Gamma)$. By (11) and the minimum property of X^ϵ we have

$$c = A^\epsilon(X^\epsilon) \le A^\epsilon(X) \quad \text{for } 0 < \epsilon \le 1,$$

and by $\epsilon \to +0$ we get

$$c \le A(X) \le D(X) \le \liminf_{j\to\infty} D(X^{\epsilon_j}) = c,$$

and so $c = A(X) = D(X)$, which implies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v\rangle = 0$$

as well as $a(\Gamma) = e(\Gamma)$.

In other words, one can—by the detour via A^ϵ—solve the variational problem "$A \to$ min in $\mathcal{C}(\Gamma)$" thereby simultaneously solving the problem "$D \to$ min in $\mathcal{C}(\Gamma)$" by a minimal surface $X \in \mathcal{C}(\Gamma)$.

Remark 2. We have proved Theorem 1 without using Riemann's mapping theorem. Therefore it is no circulus vitiosus if we try to prove this theorem by using the solution of Plateau's problem that is provided by Theorem 1. This idea will be carried out in the next section.

4.11 The Mapping Theorems of Riemann and Lichtenstein

First we want to show that the solution of Plateau's problem applied to planar curves provides a proof of **Riemann's mapping theorem**, which states the following:

Suppose that Ω is a simply connected domain in $\mathbb{C}$ bounded by a closed Jordan curve Γ. Then there is a homeomorphism φ from $\overline{\Omega}$ onto $\overline{B}$ which is holomorphic in Ω and provides a conformal mapping of Ω onto B, i.e. $\varphi'(z) \neq 0$ for all $z \in \Omega$.

We prove an equivalent assertion:

Theorem 1. *Let Ω be a simply connected domain in $\mathbb{C}$ bounded by a closed Jordan curve Γ. Then there exists a homeomorphism f from $\overline{B}$ onto $\overline{\Omega}$, $B := \{w \in \mathbb{C} : |w| < 1\}$, which is holomorphic in B and satisfies $f'(w) \neq 0$ for all $w \in B$.*

Proof. (i) Firstly we prove the assertion under the additional assumption that the contour Γ is rectifiable. We identify $\mathbb{C}$ with the x^1, x^2-plane $\mathbb{R}^2$ and consider a minimal surface $X = (X^1, X^2, X^3)$ of class $\mathcal{C}(\Gamma)$ which is continuous on $\overline{B}$. Since Γ lies in the x^1, x^2-plane we obtain $X^3(w) \equiv 0$ on account of the maximum principle. Thus the conformality relations for $w = u + iv \in B$ read as

$$|X_u^1|^2 + |X_u^2|^2 = |X_v^1|^2 + |X_v^2|^2, \tag{1}$$

$$X_u^1 X_v^1 + X_u^2 X_v^2 = 0. \tag{2}$$

Equation (2) implies that

$$X_v^1 = -\lambda X_u^2, \quad X_v^2 = \lambda X_u^1 \tag{3}$$

holds for some function $\lambda : B \to \mathbb{R}$, and (1) yields that $\lambda(u, v) = \pm 1$ on $B \setminus \Sigma$ where Σ denotes the set of branch points of X in B. On Σ equation (3) is satisfied for any choice of λ. Since the points of Σ are isolated in B it follows that either $\lambda(u, v) \equiv 1$ or $\lambda(u, v) \equiv -1$. In the first case we set $\lambda(u, v) := 1$ on Σ, and $\lambda(u, v) := -1$ in the second. Thus either X^1, X^2 or $X^1, -X^2$ satisfy the Cauchy–Riemann equations on B. By applying the reflection $z = x^1 + ix^2 \mapsto \overline{z} = x^1 - ix^2$ we can assume that the equations

$$X_u^1 = X_v^2, \quad X_v^1 = -X_u^2 \tag{4}$$

hold in B, and so $f(w) := X^1(u, v) + iX^2(u, v)$, $w = u + iv$, is holomorphic in B and continuous on $\overline{B}$. Furthermore, $f|_{\partial B}$ yields a homeomorphism from ∂B onto Γ. Therefore the loop $\varphi : [0, 2\pi] \to \mathbb{C}$ defined by $\varphi(t) := f(e^{it})$ has the winding numbers

$$W(\varphi, z) := W(\varphi - z) = \begin{cases} 1 & \text{for } z \in \Omega, \\ 0 & \text{for } z \in \mathbb{C} \setminus \Omega. \end{cases} \tag{5}$$

For $0 < r < 1$ and $\varphi_r(t) := f(re^{it})$ we have

$$\max_{[0,2\pi]} |\varphi(t) - \varphi_r(t)| \to 0 \quad \text{as } r \to 1 - 0.$$

Hence for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(t) - \varphi_r(t)| < \epsilon \quad \text{for all } t \in [0, 2\pi], \text{ provided that } 1 - \delta < r < 1.$$

Then for any $z \in \mathbb{C}$ with $\text{dist}(z, \Gamma) > \epsilon$ we obtain

$$W(\varphi_r, z) = W(\varphi, z). \tag{6}$$

Since φ_r is real analytic we on the other hand have

$$W(\varphi_r, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\dot{\varphi}_r(t)}{\varphi_r(t) - z} \, dt. \tag{7}$$

This equation can be written as

$$\begin{aligned} W(\varphi_r, z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(re^{it})}{f(re^{it}) - z} ire^{it} \, dt \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{f'(w)}{f(w) - z} \, dz \end{aligned} \tag{8}$$

where C_r denotes the positively oriented circle $\{re^{it}\colon 0 \le t \le 2\pi\}$ bounding the disk $B_r(0) := \{w \in \mathbb{C}\colon |w| < r\}$. By Rouché's formula we know that

$$\frac{1}{2\pi i}\int_{C_r} \frac{f'(w)}{f(w) - z}\, dz = n(f, B_r(0), z) \tag{9}$$

where $n(f, B_r(0), z)$ is the number of zeros of the function $f - z$ in $B_r(0)$ counted with respect to their multiplicities. From (5)–(9) we infer the following: For any $z \in \mathbb{C} \setminus \Gamma$ there is a $\delta \in (0,1)$ such that

$$n(f, B_r(0), z) = \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{if } z \notin \overline{\Omega}, \end{cases} \quad \text{provided that } 1 - \delta < r < 1.$$

This implies for $z \in \mathbb{C} \setminus \Gamma$ that

$$n(f, B, z) = \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

In other words, the equation $f(w) = z$ has no solution $w \in B$ if $z \in \mathbb{C} \setminus \overline{\Omega}$, and exactly one solution $w \in B$ if $z \in \Omega$; this solution is a zero of order 1 for the function $f - z$. Thus f yields a 1–1 mapping of B onto Ω such that $f'(w) \neq 0$ for all $w \in B$, i.e. f is a conformal mapping from B onto Ω. Moreover, f maps ∂B one-to-one onto Γ (see 4.5, Theorem 3), and so f provides a bijective mapping of $\overline{B}$ onto $\overline{\Omega}$. Since f is continuous on $\overline{B}$ it finally follows that f is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and so the assertion is proved in case that Γ is rectifiable.

(ii) If Γ is not rectifiable we choose a sequence of rectifiable Jordan curves Γ_j that converge to Γ in the sense of Fréchet as $j \to \infty$. Let Ω_j be the bounded component of $\mathbb{C} \setminus \Gamma_j$. On account of (i) there is for every $j \in \mathbb{N}$ a homeomorphism of $\overline{B}$ onto $\overline{\Omega}_j$ which maps B conformally onto Ω_j.

Now we proceed as in the proof of Theorem 3 in Section 4.3. Since we did only sketch this proof we shall now fill in the details for the convenience of the reader.

We can assume that the f_j satisfy three-point conditions

$$f_j(w_k) = z_{k,j}, \quad k = 1, 2, 3,\ j \in \mathbb{N},$$

where w_1, w_2, w_3 are three different points on ∂B, and $z_{1,j}, z_{2,j}, z_{3,j}$ are three different points on Γ_j converging to three different points z_1, z_2, z_3 on Γ : $z_{k,j} \to z_k$ as $j \to \infty$.

Any pair of points P_j, Q_j on Γ_j divides Γ_j into two subarcs Γ_j' and Γ_j''. There is a $\sigma_0 > 0$ such that one of the two arcs contains at most one of the three points $z_{1,j}, z_{2,j}, z_{3,j}$ if $|P_j - Q_j| < \sigma_0$; let this arc be Γ_j'. Since $\Gamma_j \to \Gamma$ in the sense of Fréchet, there is a uniform estimate of the moduli of continuity of the Jordan curves Γ_j, i.e.: For every $\epsilon > 0$ there is a number $\sigma(\epsilon)$ with

$0 < \sigma(\epsilon) < \sigma_0$ such that $\operatorname{diam} \Gamma'_j < \epsilon$ holds for any "short" subarc Γ'_j of Γ_j provided that its endpoints P_j, Q_j satisfy $|P_j - Q_j| < \sigma(\epsilon)$.

Moreover, there is a constant $M > 0$ such that $\operatorname{meas} \Omega_j \leq M$ for all $j \in \mathbb{N}$. This implies

$$D(f_j) = A(f_j) = \operatorname{meas} \Omega_j \leq M \quad \text{for all } j \in \mathbb{N}.$$

For $0 < r < 1$ and $w_0 \in \partial B$ we define the two-gon

$$S_r(w_0) := B \cap B_r(w_0)$$

which is bounded by the two closed circular arcs C'_r and C''_r with common endpoints ζ'_r and ζ''_r on ∂B and $C''_r \subset \partial B$. By the Courant–Lebesgue lemma we obtain: For every $\delta \in (0,1)$ there is a number $\rho_j \in (\delta, \sqrt{\delta})$ such that the oscillation of f_j on C_{ρ_j} is estimated by

$$\operatorname{osc}(f_j, C'_{\rho_j}) \leq \left\{ \frac{8\pi M}{\log 1/\delta} \right\}^{1/2} \quad \text{for all } w_0 \in \partial B.$$

For a given $\epsilon > 0$ we can find a number $\tau(\epsilon) > 0$ such that for $0 < \delta < \tau(\epsilon)$ the arc $C''_{\sqrt{\delta}}$ contains at most one of the points z_k (and so f_j maps C''_{ρ_j} onto the short arc Γ''_j with the endpoints $f_j(\zeta'_{\rho_j})$ and $f_j(\zeta''_{\rho_j})$), and secondly that

$$\operatorname{osc}(f_j, C'_{\rho_j}(w_0)) < \sigma(\epsilon).$$

It follows that

$$\operatorname{osc}(f_j, C''_{\rho_j}(w_0)) < \epsilon \quad \text{for all } w_0 \in \partial B \text{ and } j \in \mathbb{N}.$$

Since f_j maps ∂B homeomorphically onto Γ, we conclude that

$$\operatorname{osc}(f_j, C''_{\delta}(w_0)) < \epsilon \quad \text{for all } w_0 \in \partial B \text{ and } j \in \mathbb{N},$$

provided that $0 < \delta < \tau(\epsilon)$. Furthermore $f_j(\partial B) = \Gamma_j \to \Gamma$ implies

$$\max_{\partial B} |f_j| \leq \text{const} \quad \text{for all } j \in \mathbb{N},$$

and so $\{f_j|_{\partial B}\}$ is compact in $C^0(\partial B, \mathbb{C})$ equipped with the sup-norm on ∂B. Thus, after renumbering, we may assume that $\{f_j|_{\partial B}\}$ converges uniformly on ∂B to some continuous function. By virtue of Weierstrass's theorem we obtain $f_j \rightrightarrows f$ for some $f \in C^0(\overline{B}, \mathbb{C})$, and $f \in \mathcal{C}^*(\Gamma)$ as $f_j \in \mathcal{C}^*(\Gamma_j)$ and $\Gamma_j \to \Gamma$, where the $*$ denotes the corresponding three-point conditions $f_j(w_k) = z_{k,j}$ and $f(w_k) = z_k$ with $z_{k,j} \to z_k$ as $j \to \infty$. The uniform limit of holomorphic functions is holomorphic. Therefore f is holomorphic in B, continuous on $\overline{B}$, and non-constant as f is of class $\mathcal{C}^*(\Gamma)$. By a theorem of Hurwitz the uniform limit of injective holomorphic maps is injective, provided that this limit is nonconstant. Consequently the holomorphic mapping $f|_B$ is injective, and so it maps B conformally onto the open set $f(B)$ and ∂B continuously and weakly monotonically onto Γ. Since f is open it follows that $f(B)$ is the inner domain Ω of the Jordan contour Γ, and Theorem 3 of 4.5 yields that $f|_{\partial B}$ maps ∂B one-to-one onto Γ. Thus f is a continuous bijective mapping from $\overline{B}$ onto $\overline{\Omega}$, and so it is a homeomorphism. $\square$

Remark 1. A lucid presentation of the properties of the winding number can be found in Sauvigny [15], Vol. 1, III.1.

Remark 2. The mapping f in Theorem 1 is essentially unique. In fact, if f_1 and f_2 are two (strictly) conformal mappings of B onto Ω then $f_1^{-1} \circ f_2$ is a (strictly) conformally automorphism τ of B, i.e.

$$f_2 = f_1 \circ \tau \quad \text{with } \tau(w) = e^{i\varphi} \frac{w-a}{1-\overline{a}w}, \; a \in B, \; 0 \le \varphi < 2\pi.$$

Remark 3. We now want to sketch another proof of Theorem 1 for a rectifiable contour which in essence describes the approach to proving Lichtenstein's theorem that will follow next. So let us return to the mapping $f := X^1 + iX^2$ which we can assume to be holomorphic in B. Moreover, f is continuous on $\overline{B}$, and $f|_{\partial B}$ provides a homeomorphism from ∂B onto Γ. Hence $f(w) \not\equiv \text{const}$ on B, and therefore f is an open mapping from B onto the open set $f(B)$. Furthermore, $f(\overline{B})$ is compact, and we conclude that $f(\partial B) = \partial f(B) = \Gamma$ and $\Omega = \operatorname{int} f(\overline{B}) = f(B)$. The set of zeros of f' in B coincides with the set Σ of branch points of X in B. We claim that Σ is empty and f is univalent in B. In fact if $f'(w_0) = 0$ and $z_0 := f(w_0)$ for some $w_0 \in B$ it follows by Rouché's theorem in connection with Theorem 1 of 4.7 that for any $z \in \Omega$ the function $f(w) - z$ has at least two zeros w_1 and w_2 in B, except if z is the image of a branch point. Since Σ is at most denumerable it follows that

$$N(f, B, z) \ge 2 \quad \text{for almost all } z \in \Omega,$$

where $N(f, B, z)$ denotes the number of different solutions $w \in B$ for the equation $f(w) = z$ with $z \in \Omega$. Since the area of X is given by

$$A(X) = \int_B |f'(w)|^2 \, du \, dv$$

the area formula yields for $z = x^1 + ix^2$ that

$$A(X) = \int_\Omega N(f, B, z) \, dx^1 \, dx^2 \ge 2 \operatorname{meas} \Omega. \tag{10}$$

Moreover we may assume that X minimizes A, taking Theorem 1 of 4.10 into account. Since $f(B) = \Omega$, inequality (10) contradicts the minimizing property of X, and so we obtain $N(f, B, z) = 1$ for all $z \in \Omega$. Consequently $f|_B$ is injective and Σ is empty. Now one concludes as before that f is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$ that maps B conformally onto Ω.

Now we shall use the approach of Section 4.10, combined with the ideas described in the preceding Remark 3 to give a proof of *Lichtenstein's theorem* (cf. 1.4). As we shall nowhere base the reasoning of this book onto this result we may use some regularity results that are proved only later in Chapters 7 and 8.

Let B again be the standard unit disk $\{w \in \mathbb{R}^2 \colon |w| < 1\}$, $w = (u, v)$, equipped with the Euclidean metric

$$ds_e^2 := du^2 + dv^2,$$

and Ω be a simply connected, open set in $\mathbb{R}^2$, bounded by a closed rectifiable Jordan curve Γ. We assume that $\overline{\Omega}$ carries a Riemannian metric

$$ds^2 := g_{jk}(x)\, dx^j\, dx^k, \quad x = (x^1, x^2).$$

For mappings $\tau \in H_2^1(B, \mathbb{R}^2)$ we define the "Gauss functions" $\mathcal{E}(\tau), \mathcal{F}(\tau)$, $\mathcal{G}(\tau) : B \to \mathbb{R}$ by

$$\mathcal{E}(\tau) := g_{jk}(\tau)\tau_u^j \tau_u^k, \quad \mathcal{G}(\tau) := g_{jk}(\tau)\tau_v^j\, \tau_v^k, \quad \mathcal{F}(\tau) := g_{jk}(\tau)\tau_u^j\, \tau_v^k.$$

We call τ *weakly conformal* if τ satisfies the *conformality relations*

$$\mathcal{E}(\tau) = \mathcal{G}(\tau), \quad \mathcal{F}(\tau) = 0. \tag{11}$$

Definition 1. *A* **conformal mapping** *from $\overline{B}$ onto Ω is a diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$ satisfying the conformality relations (11).*

The pull-back $\tau^* \, ds^2$ of ds^2 by a diffeomorphism $\tau : \overline{B} \to \overline{\Omega}$ from $\overline{\Omega}$ to $\overline{B}$ is given by

$$\tau^*\, ds^2 = \mathcal{E}(\tau)\, du^2 + 2\mathcal{F}(\tau)\, du\, dv + \mathcal{G}(\tau)\, dv^2.$$

For a conformal mapping $\tau : \overline{B} \to \overline{\Omega}$ we have

$$\lambda := \mathcal{E}(\tau) = \mathcal{G}(\tau) > 0 \quad \text{on } \overline{B}$$

and

$$\tau^*\, ds^2 = \lambda(u, v) \cdot (du^2 + dv^2).$$

It follows from (11) that the components τ^1, τ^2 of a conformal mapping $\tau(u, v) = (\tau^1(u, v), \tau^2(u, v))$, satisfy the *Beltrami equations*

$$\begin{aligned} \sqrt{g(\tau)}\tau_v^1 &= -\rho[g_{12}(\tau)\tau_u^1 + g_{22}(\tau)\tau_u^2], \\ \sqrt{g(\tau)}\tau_v^2 &= \rho[g_{11}(\tau)\tau_u^1 + g_{12}(\tau)\tau_u^2] \end{aligned} \tag{12}$$

where

$$g(x) := \det(g_{jk}(x))$$

and either $\rho(u, v) \equiv 1$ or $\rho(u, v) \equiv -1$. From (12) it follows that

$$\sqrt{g(\tau)}\,\det D\tau = \rho\,\mathcal{E}(\tau).$$

Thus τ is orientation preserving or reversing if $\rho = 1$ or $\rho = -1$ respectively. The Riemannian analogue of the area functional is

$$A(\tau) := \int_B \sqrt{\mathcal{E}(\tau)\mathcal{G}(\tau) - \mathcal{F}^2(\tau)}\,du\,dv = \int_B \sqrt{g(\tau)}|\det D\tau|\,du\,dv$$

and the corresponding Dirichlet integral is defined as

$$D(\tau) := \frac{1}{2}\int_B [\mathcal{E}(\tau) + \mathcal{G}(\tau)]\,du\,dv.$$

We now state the following global version of **Lichtenstein's theorem**:

Theorem 2. *Suppose that $\Gamma \in C^{m,\alpha}$ and $g_{jk} \in C^{m-1,\alpha}(\overline{\Omega})$ for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there is a conformal mapping τ from $\overline{B}$ onto $\overline{\Omega}$ which is of class $C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$.*

Proof. We extend (g_{jk}) to all of $\mathbb{R}^2$, in such a way that $g_{jk}(x) = \delta_{jk}$ for $|x| \gg 1$ and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$. Then there are numbers $0 < m_1 \le m_2$ such that

$$m_1|\xi|^2 \le g_{jk}(x)\xi^j\xi^k \le m_2|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^2.$$

For any $\tau \in H_2^1(B, \mathbb{R}^2)$ the functions $\mathcal{E}(\tau)$, $\mathcal{F}(\tau)$, $\mathcal{G}(\tau)$ are of class $L_1(B)$, and so A and D are well-defined on $H^{1,2}(B, \mathbb{R}^2)$. Analogous to Definition 3 in 4.2 we define $\mathcal{C}(\Gamma)$ as the class of mappings $\tau \in H_2^1(B, \mathbb{R}^2)$ whose trace $\tau|_{\partial B}$ can be represented by a weakly monotonic, continuous mapping from ∂B onto Γ, and $\mathcal{C}^*(\Gamma)$ is the subclass of mappings $\tau \in \mathcal{C}(\Gamma)$ satisfying a fixed three-point condition.

Now we define the functionals $A^\epsilon : H_2^1(B, \mathbb{R}^2) \to \mathbb{R}$ by

$$A^\epsilon := (1-\epsilon)A + \epsilon D, \quad 0 \le \epsilon \le 1.$$

As in 4.10 we have the following lower semicontinuity property: *If $\tau_n \rightharpoonup \tau$ in $H_2^1(B, \mathbb{R}^2)$ then*

$$A^\epsilon(\tau) \le \liminf_{n\to\infty} A^\epsilon(\tau_n) \quad \text{for any } \epsilon \in [0,1].$$

Unfortunately the simple proof of Lemma 1 in Section 4.10 does not seem to work in the present situation; therefore we refer the reader to the general lower semicontinuity theorem in Acerbi and Fusco [1] which contains the above stated property as a special case.

Consider the variational problem "$A^\epsilon \to \min$ in $\mathcal{C}(\Gamma)$" for an arbitrary $\epsilon \in (0,1]$. By the same reasoning as in 4.10 we see that there is a minimizer $\tau^\epsilon \in \mathcal{C}^*(\Gamma)$ satisfying

$$\partial A^\epsilon(\tau^\epsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2)$$

and so

$$\partial D(\tau^\epsilon, \lambda) = 0 \quad \text{for all } \lambda = (\mu, \nu) \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

where

$$\begin{aligned} \partial D(\tau^\epsilon, \lambda) &= \int_B [a(\mu_u - \nu_v) + b(\mu_v + \nu_u)]\, du\, dv, \\ a &:= \mathcal{E}(\tau^\epsilon) - \mathcal{G}(\tau^\epsilon), \quad b := 2\mathcal{F}(\tau^\epsilon). \end{aligned}$$

It follows that $\Phi(w) := a(u,v) - ib(u,v)$ is a holomorphic function of $w = u + iv$ in B, and then $\Phi(w) \equiv 0$ using the Supplementary Remark 1 of 4.5; see 4.10. Thus we have

$$\mathcal{E}(\tau^\epsilon) = \mathcal{G}(\tau^\epsilon), \quad \mathcal{F}(\tau^\epsilon) = 0 \quad \text{for any } \epsilon \in (0, 1]$$

whence $A(\tau^\epsilon) = D(\tau^\epsilon)$ and so

$$A^\epsilon(\tau^\epsilon) = A(\tau^\epsilon) = D(\tau^\epsilon) \quad \text{for } 0 < \epsilon \le 1.$$

On the other hand we infer from $A \le D$ and the minimum property of τ^ϵ that

$$A^\epsilon(\tau^\epsilon) \le A^\epsilon(\tau) = (1-\epsilon)A(\tau) + \epsilon D(\tau) \le D(\tau)$$

holds for any $\tau \in \mathcal{C}(\Gamma)$ and $0 < \epsilon \le 1$. Choosing $\tau = \tau^{\epsilon'}$ we obtain

$$D(\tau^\epsilon) \le D(\tau^{\epsilon'}) \quad \text{for all } \epsilon, \epsilon' \in (0, 1],$$

and so

$$D(\tau^\epsilon) = A(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) \equiv \text{const} =: c \quad \text{for } 0 < \epsilon \le 1.$$

Set

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then, for arbitrary $\tau \in \mathcal{C}(\Gamma)$ and $\epsilon, \epsilon' \in (0, 1]$, we have

$$\begin{aligned} a(\Gamma) &\le A(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) = A^{\epsilon'}(\tau^{\epsilon'}) \le A^{\epsilon'}(\tau), \\ e(\Gamma) &\le D(\tau^\epsilon) = A^\epsilon(\tau^\epsilon) \le A^\epsilon(\tau) \le D(\tau). \end{aligned}$$

Letting $\epsilon' \to +0$ we arrive at

$$a(\Gamma) \le A(\tau^\epsilon) \le A(\tau), \quad e(\Gamma) \le D(\tau^\epsilon) \le D(\tau) \quad \text{for any } \tau \in \mathcal{C}(\Gamma),$$

which implies

$$a(\Gamma) \le A(\tau^\epsilon) \le a(\Gamma), \quad e(\Gamma) \le D(\tau^\epsilon) \le e(\Gamma)$$

whence

$$a(\Gamma) = A(\tau^\epsilon) = D(\tau^\epsilon) = e(\Gamma) \quad \text{for all } \epsilon \in (0, 1].$$

In particular we obtain

$$a(\Gamma) = A(\tau) = D(\tau) = e(\Gamma)$$

for $\tau := \tau^1$, that is, the minimizer τ of Dirichlet's integral D in $\mathcal{C}(\Gamma)$ minimizes also the area functional A in $\mathcal{C}(\Gamma)$. From $D(\tau) = e(\Gamma)$ it follows that τ is a minimal surface in the two-dimensional Riemannian manifold $(\mathbb{R}^2, ds^2)$, provided that $m \geq 2$ and $\alpha \in (0,1)$, and $\tau \in C^{m,\alpha}(\overline{B}, \mathbb{R}^2)$; in particular, τ satisfies (11). Furthermore, if w_0 is a branch point of τ, i.e. $\mathcal{E}(\tau)(w_0) = 0$, then there is an $a \in \mathbb{C}^2$, $a \neq 0$, and a number $\nu \in \mathbb{N}$ such that the Wirtinger derivative τ_w has the expansion

$$\tau_w(w) = a(w - w_0)^\nu + o(|w - w_0|)^\nu) \quad \text{as } w \to w_0.$$

These results are derived in Chapters 2 and 3 of Vol. 2 for the Euclidean case. In the Riemannian case the statements at the boundary are verified in the same way, and the interior results are even easier to prove than the boundary results. (We also refer to Morrey [8], Chapter 9; Tomi [1], and Heinz and Hildebrandt [1].) Integrating the asymptotic expansion of τ_w we obtain for $0 < |x - \tau(w_0)| \ll 1$ and $x \in \mathbb{R}^2$ that the indicatrix

$$N(\tau, \overline{B}, x) := \#\{w \in \overline{B}, \tau(w) = x\}$$

satisfies

$$N(\tau, \overline{B}, x) \geq \begin{cases} 2 & \text{if } w_0 \in B, \\ 1 & \text{if } w_0 \in \partial B \end{cases} \tag{13}$$

in case that w_0 is a branch point of τ.

Since τ maps ∂B weakly monotonically and continuously onto Γ and $\tau \in C^0(\overline{B}, \mathbb{R}^2)$, a topological argument yields $\overline{\Omega} \subset \tau(\overline{B})$. Therefore we also have

$$N(\tau, \overline{B}, x) \geq 1 \quad \text{for all } x \in \overline{\Omega}. \tag{14}$$

Let τ_0 be a conformal mapping of B onto Ω, $\tau_0 \in \mathcal{C}(\Gamma)$. Then

$$A(\tau_0) = \int_\Omega \sqrt{g(x)}\, dx^1\, dx^2 = \int_{\overline{\Omega}} \sqrt{g(x)}\, dx^1\, dx^2$$

since $\mathcal{L}^2$-meas $\Gamma = 0$ for a rectifiable curve Γ. Since τ minimizes A in $\mathcal{C}(\Gamma)$ we obtain

$$A(\tau) \leq A(\tau_0),$$

and the *area formula* yields

$$A(\tau) = \int_{\mathbb{R}^2} N(\tau, \overline{B}, x) \sqrt{g(x)}\, dx^1\, dx^2.$$

Thus,

$$\int_{\mathbb{R}^2} N(\tau, \overline{B}, x)\sqrt{g(x)}\,dx^1\,dx^2 \le \int_{\Omega} \sqrt{g(x)}\,dx^1\,dx^2. \tag{15}$$

On account of (13)–(15) it firstly follows that τ has no branch points on $\overline{B}$, whence $\nabla\tau(w) \neq 0$ for all $w \in \overline{B}$. Thus $\tau|_{\partial B}$ is 1–1, and so it yields a homeomorphism from ∂B onto Γ. Secondly, $\tau|_B$ is open; hence it follows from (14) and (15) that $N(\tau, \overline{B}, x) = 1$ for $x \in \overline{\Omega}$ and $N(\tau, \overline{B}, x) = 0$ for $x \in \mathbb{R}^2 \setminus \overline{\Omega}$. Consequently τ is a conformal mapping from $\overline{B}$ onto $\overline{\Omega}$ satisfying the Beltrami equations (12).

If we merely assume $\Gamma \in C^{1,\alpha}$ and $g_{jk} \in C^{0,\alpha}$, τ turns out to be a conformal mapping of class $C^{1,\alpha}(\overline{B}, \mathbb{R}^2)$ from $\overline{B}$ onto $\overline{\Omega}$. This one obtains from the preceding result ($m \ge 2$) by approximating Γ and g_{jk} by C^∞-data Γ_n, g^n_{jk}, and applying a priori estimates for the corresponding mappings τ_n and their inverses τ_n^{-1} which satisfy similar Beltrami equations as the τ_n (see e.g. Schulz [1], Chapter 6; Jost [17], Chapter 3; or Morrey [8], pp. 373–374). □

A slight modification of the preceding reasoning combined with a suitable approximation argument yields the following analog of Theorem 1:

Theorem 3. *If Γ is a closed Jordan curve with the inner domain Ω and $g_{jk} \in C^{m-1,\alpha}(\mathbb{R}^2)$ for some $m \in \mathbb{N}$ and $\alpha \in (0,1)$, then there is a homeomorphism τ from $\overline{B}$ onto $\overline{\Omega}$ which yields a conformal mapping of class $C^{m,\alpha}(B, \mathbb{R}^2)$ from B onto Ω.*

As a corollary of Theorem 2 we obtain the following version of the original Lichtenstein theorem:

Theorem 4. *If $X : \overline{B} \to \mathbb{R}^n$, $n \ge 2$, is an immersed surface of class $C^{m,\alpha}$, $m \in \mathbb{N}$, $\alpha \in (0,1)$, then there exists an equivalent representation $Y = X \circ \tau$ which is conformally parametrized, i.e. $|Y_u|^2 = |Y_v|^2$, $\langle Y_u, Y_v\rangle = 0$.*

Proof. $X(x^1, x^2)$ with $(x^1, x^2) \in \overline{B}$ induces on $\overline{B}$ the Riemannian metric $ds^2 = g_{jk}(x)\,dx^j\,dx^k$ with

$$g_{jk} := \langle X_{x^j}, X_{x^k}\rangle \in C^{m-1,\alpha}(\overline{B}).$$

If we now determine a conformal mapping τ from $(\overline{B}, ds_e)$ onto $(\overline{B}, ds)$ as in Theorem 2 then $Y := X \circ \tau$ has the desired property. □

4.12 Solution of Plateau's Problem for Nonrectifiable Boundaries

A general closed Jordan curve Γ need not bound any surface $X : B \to \mathbb{R}^3$ with a finite Dirichlet integral. In fact, $\mathcal{C}(\Gamma)$ is nonempty if and only if Γ possesses a representation of class $H_2^{1/2}([0,2\pi],\mathbb{R}^3)$. Nevertheless J. Douglas proved that every closed Jordan curve Γ in $\mathbb{R}^3$ spans a continuous disk-like minimal surface. To see this we approximate Γ by sequences of rectifiable Γ_n, each of which bounds a minimal surface X_n of finite area. There is a subsequence $\{X_{n_p}\}$ that uniformly converges to a minimal surface $X \in C^2(B,\mathbb{R}^3)$ on every $\Omega' \subset\subset B$. Yet it is not obvious that the limit X is continuous and that it maps ∂B onto Γ in the sense of 4.2, Definition 2. Namely, as $A(X_n)$ may tend to infinity, one cannot derive a uniform bound for the moduli of continuity of the boundary values $X_n|_{\partial B}$ by means of the Courant–Lebesgue lemma, and so, at first, it only follows that $X|_{\partial B}$ yields a weakly monotonic mapping from ∂B into Γ which might have denumerably many jump discontinuities. The crucial part of the proof consists in showing that these discontinuities do not appear.

We use a result on sequences of monotonic functions that in essence is due to Helly; a proof can be derived from A. Wintner [1].

Lemma 1. *Let $\{\tau_n\}$ be a sequence of increasing functions $\tau_n \in C^0(\mathbb{R})$ with $\tau_n(0) = 0$ and $\tau_n(\theta + 2\pi) = \tau_n(\theta)$. Then there is a function $\tau : \mathbb{R} \to \mathbb{R}$ and a subsequence $\{\tau_{n_k}\}$ with the following properties*:

(i) *τ is nondecreasing and continuous except for at most denumerably many jump discontinuities.*
(ii) *If τ is continuous at θ then $\tau_{n_k}(\theta) \to \tau(\theta)$ as $k \to \infty$.*
(iii) *Because of* (i) *the one-sided limits $\tau(\theta_0 - 0)$ and $\tau(\theta_0 + 0)$ exist at any $\theta_0 \in \mathbb{R}$, and we can redefine τ by $\tau(\theta_0) := \frac{1}{2}[\tau(\theta_0 - 0) + \tau(\theta_0 + 0)]$ without changing* (i) *and* (ii)*. Set*

$$\sigma(\theta_0) := \tfrac{1}{2}[\tau(\theta_0 + 0) - \tau(\theta_0 - 0)].$$

(iv) *For any $\delta > 0$ there exist numbers $\eta(\delta) > 0$ and $N_0(\delta) \in \mathbb{N}$ such that for all $\theta_0 \in \mathbb{R}$ the following holds*:

$$\begin{aligned} |\tau(\theta) - \tau(\theta_0)| &\le \sigma(\theta_0) + \delta \quad \textit{if } |\theta - \theta_0| < \eta, \\ |\tau_{n_k}(\theta) - \tau(\theta_0)| &\le \sigma(\theta_0) + \delta \quad \textit{if } |\theta - \theta_0| < \eta \textit{ and } k > N_0. \end{aligned}$$

Now we can state the main result.

Theorem 1. *For any closed Jordan curve Γ in $\mathbb{R}^3$ there is a minimal surface $X : B \to \mathbb{R}^3$ of class $C^0(\overline{B},\mathbb{R}^3)$ which maps ∂B homeomorphically onto Γ.*

Proof. Let Γ be represented by $\gamma \in C^0(\mathbb{R},\mathbb{R}^3)$ which is monotonic and 2π-periodic such that $\Gamma = \gamma([0,2\pi])$. We approximate Γ by rectifiable Jordan curves Γ_n (say, by simple closed polygons) with continuous, monotonic, 2π-periodic representations $\gamma_n : \mathbb{R} \to \mathbb{R}^3$, $\Gamma_n = \gamma_n([0,2\pi])$, such that γ_n converges uniformly to $\gamma : \gamma_n(t) \rightrightarrows \gamma(t)$ on $\mathbb{R}$ as $n \to \infty$. For any n there is a minimal surface $X_n \in \mathcal{C}(\Gamma_n) \cap C^0(\overline{B},\mathbb{R}^3)$ that maps ∂B homeomorphically onto Γ_n. If we choose the orientation of Γ_n appropriately and require that $X_n(e^{i\theta})$ respects this orientation, we can write

$$(1) \qquad X_n(e^{i\theta}) = \gamma_n(\tau_n(\theta)) \quad \text{for } \theta \in \mathbb{R} \text{ and } n \in \mathbb{N}$$

where the τ_n are increasing functions of class $C^0(\mathbb{R})$ with $\tau_n(\theta+2\pi) = \tau_n(\theta)+2\pi$. As one can impose an arbitrarily chosen three-point condition on any X_n we may also assume that $\tau_n(0) = 0$, $\tau_n(1) = 1$, $\tau_n(2) = 2$ for any $n \in \mathbb{N}$. Passing to a suitable subsequence of $\{X_n\}$ and renumbering it we obtain $X_n(w) \rightrightarrows X(w)$ on $\Omega \subset\subset B$ where $X : B \to \mathbb{R}^3$ is a minimal surface. On account of Lemma 1 we can furthermore assume that there is a nondecreasing, possibly discontinuous function $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau_n(\theta) \to \tau(\theta)$ as $n \to \infty$, provided that τ is continuous at θ, and for any $\delta > 0$ there are numbers $\eta(\delta) > 0$ and $N_0(\delta) \in \mathbb{N}$ such that

$$(2) \qquad |\tau(\theta) - \tau(\theta_0)| \le \sigma(\theta_0) + \delta \quad \text{for } |\theta - \theta_0| < \eta(\delta)$$

and

$$(3) \qquad |\tau_n(\theta) - \tau(\theta_0)| \le \sigma(\theta_0) + \delta \quad \text{for } |\theta - \theta_0| < \eta(\delta) \text{ and } n > N_0(\delta)$$

where $\sigma(\theta_0) := \frac{1}{2}[\tau(\theta_0 + 0) - \tau(\theta_0 - 0)]$ and τ is redefined as

$$\tau(\theta_0) = \tfrac{1}{2}[\tau(\theta_0 + 0) + \tau(\theta_0 - 0)].$$

First we will prove that

$$(4) \qquad \lim_{w \to w_0} X(w) = \gamma(\tau(\theta_0)) \quad \text{for } w_0 = e^{i\theta_0} \in \partial B,$$

provided that τ is continuous at θ_0. So let us assume that

$$(5) \qquad \sigma(\theta_0) = 0$$

for some fixed θ_0, and choose some $\epsilon > 0$. Since γ is uniformly continuous on $\mathbb{R}$ there is some $\delta_1(\epsilon) > 0$ such that

$$(6) \qquad |\gamma(t) - \gamma(t')| < \epsilon \quad \text{for } |t - t'| < \delta_1(\epsilon).$$

Because of $\gamma_n(t) \rightrightarrows \gamma(t)$ on $\mathbb{R}$ there is an $N_1(\epsilon) \in \mathbb{N}$ such that

$$(7) \qquad |\gamma(t) - \gamma_n(t)| < \epsilon \quad \text{for } n > N_1(\epsilon) \text{ and all } t \in \mathbb{R}.$$

Furthermore, by (3) and (5) we obtain

$$(8) \qquad |\tau_n(\theta_0) - \tau(\theta_0)| < \delta_1(\epsilon) \quad \text{for } n > N_0(\delta_1(\epsilon)).$$

On account of (6)–(8) and

$$\begin{aligned} \left|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\right| \le{}& \left|\gamma(\tau(\theta_0)) - \gamma(\tau_n(\theta_0))\right| + \left|\gamma(\tau_n(\theta_0)) - \gamma_n(\tau_n(\theta_0))\right| \\ &+ \left|\gamma_n(\tau_n(\theta_0)) - X_n(re^{i\theta})\right| \end{aligned}$$

we see that

$$
\begin{aligned}
(9)\qquad & \big|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\big| < 2\epsilon + \big|\gamma_n(\tau_n(\theta_0)) - X_n(re^{i\theta})\big| \\
& \quad \text{for } n > N(\epsilon) := \max\big\{N_1(\epsilon), N_0(\delta_1(\epsilon))\big\}.
\end{aligned}
$$

For $0 \le r < 1$ Poisson's integral formula and (1) yield

$$
\begin{aligned}
(10)\qquad & \big|X_n(re^{i\theta}) - \gamma_n(\tau_n(\theta_0))\big| \\
& \le \int_{\theta_0-\pi}^{\theta_0+\pi} K(r, \varphi - \theta)\big|\gamma_n(\tau_n(\varphi)) - \gamma_n(\tau_n(\theta_0))\big|\, d\varphi \\
& = \int_{I_1} \cdots + \int_{I_2} \cdots
\end{aligned}
$$

with $I_1 := \{\varphi \colon |\varphi - \theta_0| < \eta(\delta_1(\epsilon))\}$, $I_2 := [\theta_0 - \pi, \theta_0 + \pi] \setminus I_1$, and

$$
K(r, \alpha) := \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}.
$$

On account of (3) and (6)–(8) we obtain for $n > N(\epsilon)$, $|\varphi - \theta_0| < \eta(\delta_1(\epsilon))$ and $|\theta - \theta_0| < \eta(\delta_1(\epsilon))$ that

$$
\begin{aligned}
\big|\gamma_n(\tau_n(\varphi)) - \gamma_n(\tau_n(\theta_0))\big| \le\ & \big|\gamma(\tau_n(\varphi)) - \gamma_n(\tau_n(\varphi))\big| + \big|\gamma(\tau_n(\theta_0)) - \gamma_n(\tau_n(\theta_0))\big| \\
& + \big|\gamma(\tau_n(\varphi)) - \gamma(\tau(\theta_0))\big| + \big|\gamma(\tau(\theta_0)) - \gamma(\tau_n(\theta_0))\big| \\
<\ & \epsilon + \epsilon + \epsilon + \epsilon = 4\epsilon,
\end{aligned}
$$

whence

$$
(11)\qquad \left|\int_{I_1} \cdots\right| < 4\epsilon \int_{I_1} K(r, \varphi - \theta)\, d\varphi \le 4\epsilon \int_0^{2\pi} K(r, \alpha)\, d\alpha = 4\epsilon.
$$

Because of $\gamma_n \rightrightarrows \gamma$ there is a constant c_0 such that

$$
|\gamma_n(t)| + |\gamma(t)| \le c_0 \quad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{N},
$$

and so

$$
\left|\int_{I_2} \cdots\right| \le c_0 \int_{I_2} K(r, \varphi - \theta)\, d\theta.
$$

Hence there is a constant $\delta_2(\epsilon) > 0$ such that

$$
\begin{aligned}
(12)\qquad & \left|\int_{I_2} \cdots\right| < \epsilon \quad \text{for } 0 < 1 - r < \delta_2(\epsilon), \\
& \qquad\qquad |\theta - \theta_0| < \delta_3(\epsilon) := \tfrac{1}{2}\eta(\delta_1(\epsilon)) \text{ and all } n \in \mathbb{N}.
\end{aligned}
$$

By (9)–(12) it follows that

$$
\begin{aligned}
(13)\qquad & \big|\gamma(\tau(\theta_0)) - X_n(re^{i\theta})\big| < 2\epsilon + 4\epsilon + \epsilon = 7\epsilon \\
& \quad \text{for } n > N(\epsilon),\ 0 < 1 - r < \delta_2(\epsilon) \text{ and } |\theta - \theta_0| < \delta_3(\epsilon).
\end{aligned}
$$

With $n \to \infty$ we obtain

$$\left|\gamma(\tau(\theta_0)) - X(re^{i\theta})\right| \le 7\epsilon \quad \text{for } 0 < 1 - r < \delta_2(\epsilon) \text{ and } |\theta - \theta_0| < \delta_3(\epsilon). \tag{14}$$

This implies

$$\lim_{w \to w_0} X(w) = \gamma(\tau(\theta_0)) \quad \text{for } w_0 = e^{i\theta_0},$$

provided that $\tau(\theta)$ is continuous at $\theta = \theta_0$.

Now we want to show by reductio ad absurdum that τ is everywhere continuous. To this end, suppose that τ is discontinuous at θ_0; then it is no loss of generality if we assume that $\theta_0 = 0$. Set $\tau^+ := \tau(+0)$, $\tau^- := \tau(-0)$, and $X^+ := \gamma(\tau^+)$, $X^- := \gamma(\tau^-)$. Since $\tau^+ = \lim_{\theta \to +0} \tau(\theta)$ we have: For any $\delta > 0$ there is a number $\eta^*(\delta) > 0$ such that

$$\tau^+ \le \tau(\theta) \le \tau^+ + \delta/4 \quad \text{for } 0 < \theta < \eta^*(\delta).$$

By virtue of (3) we may therefore even assume that

$$|\tau_n(\theta) - \tau^+| < \delta \quad \text{for } 0 < \theta < \eta^*(\delta) \text{ and } n > N_0(\delta).$$

Choose some $\epsilon > 0$ and set $\delta := \delta_1(\epsilon)$. Then the same reasoning as before yields for $\delta_4(\epsilon) := \eta^*(\delta_1(\epsilon))$ the following: For any point $\tilde{w} = e^{i\varphi}$ with $0 < \varphi < \delta_4(\epsilon)$ there is an open neighborhood $U(\varphi)$ of $\tilde{w}$ in B such that

$$|X^+ - X(w)| < 7\epsilon \quad \text{for } w \in U(\varphi),\ 0 < \varphi < \delta_4(\epsilon), \tag{15}$$

and correspondingly we can achieve

$$|X^- - X(w)| < 7\epsilon \quad \text{for } w \in U(\varphi), -\delta_4(\epsilon) < \varphi < 0. \tag{16}$$

Now we are going to derive a contradiction to the assumption $\tau^+ \ne \tau^-$ by proving that $X^+ \ne X^-$ is impossible. To this end we consider the conformal automorphisms f_a of $\overline{B}$ which are defined by

$$z = f_a(w) := \frac{w - a}{1 - aw} \quad \text{with } a \in \mathbb{R},\ 0 < a < 1.$$

We have $f_a(1) = 1$, $f_a(0) = -a$, $f_a(-1) = -1$ and $\overline{f_a(w)} = f_a(\overline{w})$ whence $f_a(\mathbb{R}) = \mathbb{R}$ and $f_a(C^+) = C^+$, $f_a(C^-) = C^-$ for $C^+ := \{w \in \partial B \colon \operatorname{Im} w > 0\}$, $C^- := \{w \in \partial B \colon \operatorname{Im} w < 0\}$. Moreover,

$$\lim_{a \to 1-0} f_a(w) = -1 \quad \text{for any } w \in \overline{B} \setminus \{1\}.$$

Hence, for $a \in (0, 1)$ sufficiently close to 1, we see that f_a maps the arc $C_0^+ := \{e^{i\varphi} \colon 0 < \varphi < \delta_4(\epsilon)\}$ onto an arc $f_a(C_0^+)$ which contains $C_1^+ := \{e^{i\psi} \colon 1 \le \psi \le 2\}$ in its interior. Then $C_0^- := \{e^{i\varphi} \colon -\delta_4(\epsilon) < \varphi < 0\}$ is mapped onto $f_a(C_0^-)$ which contains $C_1^- := \{e^{i\psi} \colon -2 \le \psi \le -1\}$ in its interior. Set

$$Y(z) := X(f_a^{-1}(z)) \quad \text{for } z \in \overline{B}.$$

Clearly, $Y|_B$ is again a minimal surface. Let

$$U^+ := \bigcup_{0<\varphi<\delta_4(\epsilon)} U(\varphi), \quad U^- := \bigcup_{-\delta_4(\epsilon)<\varphi<0} U(\varphi).$$

By the choice of a, the image $f_a(U^+)$ covers a whole strip $\Sigma^+(\epsilon)$ along C_1^+ in B, and $f_a(U^-)$ covers a strip $\Sigma^-(\epsilon)$ along C_1^- in B where $\Sigma^+(\epsilon)$ and $\Sigma^-(\epsilon)$ are of the form

$$\Sigma^+(\epsilon) = \{\rho e^{i\psi} \colon 1 - \delta_5(\epsilon) < \rho < 1, 1 \le \psi \le 2\},$$
$$\Sigma^-(\epsilon) = \{\rho e^{i\psi} \colon 1 - \delta_5(\epsilon) < \rho < 1, -2 \le \psi \le -1\},$$

and $\delta_5(\epsilon)$ is some positive number depending on $\epsilon > 0$. Then we infer from (15) and (16) that

$$(17) \quad |Y(z) - X^+| < 7\epsilon \quad \text{for } z \in \Sigma^+(\epsilon), \quad |Y(z) - X^-| < 7\epsilon \quad \text{for } z \in \Sigma^-(\epsilon).$$

We choose a sequence of numbers $\epsilon_j > 0$ with $\epsilon_j \to 0$, thereafter a sequence of radii ρ_j with $1 - \delta_5(\epsilon_j) < \rho_j < 1$, and then we set $Z_j(z) := Y(\rho_j z)$ for $z \in \overline{B}$. The mappings Z_j are minimal surfaces of class $C^0(\overline{B}, \mathbb{R}^3)$ which satisfy

$$(18) \quad \begin{aligned} |Z_j(e^{i\psi}) - X^+| &< 7\epsilon \quad \text{for } 1 \le \psi \le 2, \\ |Z_j(e^{i\psi}) - X^-| &< 7\epsilon \quad \text{for } -2 \le \psi \le -1. \end{aligned}$$

Moreover,

$$(19) \quad |Z_j(e^{i\psi}) - X^+|, |Z_j(e^{i\psi}) - X^-| \le c_0 \quad \text{for } \psi \in \mathbb{R} \text{ and } j \in \mathbb{N}.$$

From Poisson's integral formula we get

$$\begin{aligned} |Z_j(re^{i\theta}) - X^+| &\le \int_0^{2\pi} K(r, \psi - \theta)|Z_j(e^{i\psi}) - X^+| \, d\psi \\ &= \int_{|\psi-\theta|<\frac{1}{4}} \cdots + \int_{\frac{1}{4}\le|\psi-\theta|\le\pi} \cdots . \end{aligned}$$

If we restrict θ by $\frac{5}{4} \le \theta \le \frac{7}{4}$, then for $|\psi - \theta| < \frac{1}{4}$ we have $1 < \psi < 2$, and so it follows from (18) and (19) that

$$|Z_j(re^{i\theta}) - X^+| < 7\epsilon_j + c_0 p(r) \quad \text{if } \tfrac{5}{4} \le \theta \le \tfrac{7}{4}$$

with

$$p(r) := \int_{\frac{1}{4}\le|\alpha|\le\pi} K(r, \alpha) \, d\alpha \to 0 \quad \text{as } r \to 1 - 0.$$

Because of $Z_j(z) = Y(\rho_j z)$ we then obtain for $j \to \infty$ that

$$|Y(re^{i\theta}) - X^+| \le c_0 p(r) \quad \text{for } 0 < r < 1,\ \tfrac{5}{4} < \theta < \tfrac{7}{4},$$

and similarly,

$$|Y(re^{i\theta}) - X^-| \le c_0 p(r) \quad \text{for } 0 < r < 1,\ -\tfrac{7}{4} < \theta < -\tfrac{5}{4}.$$

Thus Y assumes the constant boundary values X^+ on $C_2^+ := \{e^{i\theta}\colon \frac{5}{4} < \theta < \frac{7}{4}\}$ and the constant boundary values X^- on $C_2^- := \{e^{i\theta}\colon -\frac{7}{4} < \theta < -\frac{5}{4}\}$. By the reasoning used in the proof of Theorem 3 in Section 4.5 it follows that $Y(z) \equiv \text{const}$ on $B \cup C_2^+ \cup C_2^-$ which is a contradiction to $Y|_{C_2^+} = X^+$, $Y|_{C_2^-} = X^-$, $X^+ \ne X^-$.

Therefore τ is continuous on $\mathbb{R}$, and so X is continuous on $\overline{B}$ and yields a weakly monotonic mapping from ∂B onto Γ. By virtue of Corollary 2 in Section 4.5 we see that $X|_{\partial B}$ is a homeomorphism from ∂B onto Γ. □

Remark 1. Another proof of Theorem 1 can be found in Nitsche's treatise [28], pp. 269–271. The above proof is a slight modification of the approach used by H. Werner [2], which also works for surfaces X of constant mean curvature H provided that $|X| \le 1$ and $|H| < \frac{1}{2}$. The general case $|H| \le 1$ is apparently not yet treated. Similarly Theorem 1 has not been carried over to surfaces of prescribed variable mean curvature $H(x)$ or to minimal surfaces in a Riemannian manifold.

4.13 Plateau's Problem for Cartan Functionals

Now we want to solve Plateau's problem for regular Cartan functionals. Here a *Cartan functional* means a two-dimensional variational integral

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du\, dv \tag{1}$$

with a continuous Lagrangian $F(x, z)$, $(x, z) \in \mathbb{R}^3 \times \mathbb{R}^3$, that is positively homogeneous of first degree in z, i.e.

$$F(x, tz) = tF(x, z) \quad \text{for } t > 0 \text{ and } (x, z) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{H}$$

As before we assume that B is the unit disk $\{w = (u, v)\colon u^2 + v^2 < 1\}$ in $\mathbb{R}^2$.

A Cartan functional $\mathcal{F}$ is said to be *regular* if its Lagrangian $F(x, z)$ is *definite* and *weakly elliptic*. The first assumption means that there are constants m_1, m_2 with $0 < m_1 \le m_2$ such that

$$m_1 \le F(x, z) \le m_2 \quad \text{for } (x, z) \in \mathbb{R}^3 \times S^2$$

with $S^2 := \{z \in \mathbb{R}^3\colon |z| = 1\}$. Because of (H) the *assumption of definiteness* means that

(D) $$m_1|z| \le F(x,z) \le m_2|z| \quad \text{for } (x,z) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Secondly, *weak ellipticity* of $F(x,z)$ is defined as *convexity* of $F(x,z)$ in z for any $x \in \mathbb{R}^3$, i.e. we assume

(C) $$F(x, tz_1 + (1-t)z_2) \le tF(x,z_1) + (1-t)F(x,z_2)$$
$$\text{for } t \in [0,1] \text{ and } x, z_1, z_2 \in \mathbb{R}^3.$$

Because of (D), a regular Cartan functional $\mathcal{F}$, given by (1), is well-defined for any $X \in H_2^1(B, \mathbb{R}^3)$, and by (H) it follows that $\mathcal{F}(X \circ \tau) = \mathcal{F}(X)$ for any orientation preserving C^1-diffeomorphism of $\overline{B}$ onto itself, i.e. a Cartan functional is a *parameter invariant (two-dimensional) variational integral*. The notation "Cartan functional" is derived from Elie Cartan's memoir [1] where he introduced a geometry based on an "angular metric" that is defined by means of an integral (1) as

$$ds^2 = g_{jk}\, dx^j\, dx^k, \quad (g_{jk}) = (g^{jk})^{-1}, \quad g^{jk} := a^{-1/2} a^{jk},$$
$$a := \det(a^{jk}), \quad a^{jk} := \left(\frac{1}{2}F^2\right)_{z^j z^k} = F F_{z^j z^k} + F_{z^j} F_{z^k}.$$

This is a generalization of Finsler's geometry which is based on one-dimensional integrals $\mathcal{F}(X) = \int_0^1 F(X, \dot{X})\, dt$ with a Lagrangian $F(x,z)$ satisfying (H), (D), and (C).

Note that an F satisfying (H) and (D) cannot be of class $C^1(\mathbb{R}^3 \times \mathbb{R}^3)$, but it may very well be of class C^s on $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$. The prototype of a regular Cartan functional is the area integral $A(X) = \int_B |X_u \wedge X_v|\, du\, dv$ with the Lagrangian $F(x,z) = |z|$. If $F \in C^2(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}))$ and $F(x,z)$ is convex in z, then $F_{zz}(x,z) \ge 0$ for $z \ne 0$, but we never have $F_{zz}(x,z) > 0$ since Euler's relation implies $F_{zz}(x,z)z = 0$ because of (H). Thus the best we can hope for is: $F_{zz}(x,z) > 0$ on $\{z\}^\perp$, which is equivalent to

$$\zeta \cdot |z| F_{zz}(x,z)\zeta \ge \lambda[|\zeta|^2 - |z|^{-2}(\zeta \cdot z)^2] \quad \text{for } z \ne 0$$

and some constant $\lambda > 0$, i.e. to $F_\lambda(x,z) := F(x,z) - \lambda|z|$ being convex in z.

Let Γ be a closed, rectifiable Jordan curve in $\mathbb{R}^3$ which is oriented, and denote by $\mathcal{C}(\Gamma)$ the class of surfaces $X \in H_2^1(B, \mathbb{R}^3)$ bounded by Γ (see Section 4.2, Definitions 2 and 3); then $\mathcal{C}(\Gamma)$ is nonempty. We want to solve the variational problem

(2) $$\mathcal{F} \to \min \quad \text{in } \mathcal{C}(\Gamma),$$

which we denote as *Plateau problem for the Cartan functional* $\mathcal{F}$. This will be achieved by a method that is similar to the reasoning used in Section 4.10 for solving the problem "$A \to \min$ in $\mathcal{C}(\Gamma)$".

Theorem 1. *For any regular Cartan functional* (1) *the minimum problem* (2) *has a solution $X \in \mathcal{C}(\Gamma)$ which is conformally parametrized in the sense that*

(3) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Proof. Instead of (2) we first consider the modified minimum problems

$$\mathcal{F}^\varepsilon \to \min \quad \text{in } \mathcal{C}(\Gamma), \tag{4}$$

$\varepsilon \in (0,1]$, where the auxiliary functionals $\mathcal{F}^\varepsilon$ are defined for $0 < \varepsilon \le 1$ as

$$\mathcal{F}^\varepsilon := \mathcal{F} + \varepsilon D. \tag{5}$$

We can write

$$\mathcal{F}^\varepsilon(X) = \int_B f^\varepsilon(X, \nabla X)\, du\, dv$$

where the Lagrangian

$$f^\varepsilon(x,p) := F(x, p_1 \wedge p_2) + \frac{\varepsilon}{2}\,|p|^2$$

is *polyconvex* in $p = (p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and satisfies

$$m_1|p_1 \wedge p_2| + \frac{\varepsilon}{2}\,|p|^2 \le f^\varepsilon(x,p) \le \frac{1}{2}(m_2 + \varepsilon)|p|^2$$

because of $2|p_1 \wedge p_2| \le |p_1|^2 + |p_2|^2$. By a theorem of Acerbi and Fusco [1] the functional $\mathcal{F}^\varepsilon$ is (sequentially) weakly lower semicontinuous (w.l.s.) on $H_2^1(B, \mathbb{R}^3)$. Let $X_j \in \mathcal{C}(\Gamma)$ be a minimizing sequence for the problem (4), i.e.

$$\mathcal{F}^\varepsilon(X_j) \to d(\varepsilon) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F}^\epsilon.$$

We can assume that all X_j satisfy a uniform three-point condition $X_j(w_k) = Q_k$, $k = 1, 2, 3$, with $w_k \in \partial B$ and $Q_k \in \Gamma$, i.e. $X_j \in \mathcal{C}^*(\Gamma)$ in the sense of Section 4.2. From (5) we infer

$$D(X_j) \le \varepsilon^{-1}\mathcal{F}^\epsilon(X_j) \le \text{const} \quad \text{for all } j \in \mathbb{N} \text{ and fixed } \varepsilon > 0,$$

and the "boundary values" (= Sobolev traces) ϕ_j of X_j on ∂B satisfy $\sup_{\partial B} |\phi_j| \le \text{const}$. Then a suitable variant of Sobolev's inequality yields

$$|X_j|_{H_2^1(B,\mathbb{R}^3)} \le \text{const} \quad \text{for all } j \in \mathbb{N}.$$

Passing to an appropriate subsequence of $\{X_j\}$ which (by renumbering) is again called $\{X_j\}$ we obtain $X_j \rightharpoonup X^\varepsilon$ in $H_2^1(B, \mathbb{R}^3)$ for some $X^\varepsilon \in H_2^1(B, \mathbb{R}^3)$ whence

$$\mathcal{F}^\varepsilon(X^\varepsilon) \le \lim \mathcal{F}^\varepsilon(X_j) = d(\varepsilon).$$

On the other hand $\mathcal{C}^*(\Gamma)$ is a weakly sequentially closed subset of $H_2^1(B, \mathbb{R}^3)$ (cf. 4.6, Proposition 1), and so $X^\varepsilon \in \mathcal{C}^*(\Gamma)$, whence $d(\varepsilon) \le \mathcal{F}^\varepsilon(X^\varepsilon)$. This implies

$$\mathcal{F}^\varepsilon(X^\varepsilon) = d(\varepsilon), \tag{6}$$

i.e. X^ε is a solution of (4). As in Section 4.10 we obtain

$$\partial \mathcal{F}^\varepsilon(X^\varepsilon, \lambda) = 0 \quad \text{for any } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

therefore

$$\partial D(X^\varepsilon, \lambda) = 0 \quad \text{for all } \lambda \in C^1_{\text{tang}}(\overline{B}, \mathbb{R}^2),$$

and now the reasoning of 4.10 yields

$$(7) \qquad |X^\varepsilon_u|^2 = |X^\varepsilon_v|^2, \quad \langle X^\varepsilon_u, X^\varepsilon_v \rangle = 0.$$

This is equivalent to

$$(8) \qquad A(X^\varepsilon) = D(X^\varepsilon) \quad \text{for } \varepsilon \in (0, 1].$$

On the other hand assumption (D) implies $m_1 A \le \mathcal{F}$, and so we infer from (5) and (8) that

$$(m_1 + \varepsilon) D(X^\varepsilon) \le \mathcal{F}^\varepsilon(X^\varepsilon).$$

Furthermore,

$$\mathcal{F}^\varepsilon \le (m_2 + \epsilon) D$$

by $\mathcal{F} \le m_2 A$ and $A \le D$, and we also have

$$\mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma)$$

on account of (6). Consequently,

$$(m_1 + \varepsilon) D(X^\varepsilon) \le (m_2 + \varepsilon) D(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma).$$

Since

$$\frac{m_2 + \varepsilon}{m_1 + \varepsilon} < \frac{m_2}{m_1} \quad \text{for any } \epsilon > 0,$$

we arrive at

$$(9) \qquad D(X^\varepsilon) \le (m_2/m_1) \cdot e(\Gamma) \quad \text{for all } \varepsilon \in (0, 1]$$

with

$$e(\Gamma) := \inf_{\mathcal{C}(\Gamma)} \mathcal{D},$$

and by the same reasoning as above it follows that

$$|X^\varepsilon|_{H^1_2(B, \mathbb{R}^3)} \le \text{const} \quad \text{for all } \varepsilon \in (0, 1].$$

Hence there is an $X \in \mathcal{C}^*(\Gamma)$ and a sequence of numbers $\varepsilon_j > 0$ with $\varepsilon_j \to 0$ such that $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$. Since also $\mathcal{F}$ is sequentially w.l.s. by Acerbi and Fusco [1], it follows that

$$d(0) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F} \le \mathcal{F}(X) \le \liminf_{j \to \infty} \mathcal{F}(X^{\varepsilon_j}).$$

As $d(\varepsilon)$ is nondecreasing, $\lim_{\varepsilon\to+0} d(\varepsilon)$ exists, and by

$$d(\varepsilon) = \mathcal{F}^\varepsilon(X^\varepsilon) = \mathcal{F}(X^\varepsilon) + \varepsilon D(X^\varepsilon)$$

we infer from (9) that

$$\lim_{\varepsilon\to+0} d(\varepsilon) = \lim_{\varepsilon\to+0} \mathcal{F}^\varepsilon(X^\varepsilon) = \lim_{\varepsilon\to+0} \mathcal{F}(X^\varepsilon).$$

Thus we have

$$d(0) \le \mathcal{F}(X) \le \lim_{\varepsilon\to+0} d(\varepsilon). \tag{10}$$

On the other hand,

$$d(\varepsilon) = \mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(Z) = \mathcal{F}(Z) + \varepsilon D(Z) \quad \text{for any } Z \in \mathcal{C}(\Gamma).$$

Hence $\lim_{\varepsilon\to+0} d(\varepsilon) \le \mathcal{F}(Z)$, and so $\lim_{\varepsilon\to+0} d(\varepsilon) \le d(0)$. By virtue of (10) we arrive at

$$\mathcal{F}(X) = d(0) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F},$$

and so $X \in \mathcal{C}(\Gamma)$ is a solution of (2), i.e. a minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$.

We still have to prove (3) which does not immediately follow from (7) because the X^{ε_j} merely converge weakly to X in $H_2^1(B,\mathbb{R}^3)$. However (3) follows from (7) as soon as we have proved the strong convergence $X^{\varepsilon_j} \to X$ in $H_2^1(B,\mathbb{R}^3)$. For this it suffices to prove

$$\lim_{\varepsilon_j\to 0} D(X^{\varepsilon_j}) = D(X), \tag{11}$$

which will be verified as follows: Since X^ε minimizes $\mathcal{F}^\varepsilon$ in $\mathcal{C}(\Gamma)$, we have $\mathcal{F}^\varepsilon(X^\varepsilon) \le \mathcal{F}^\varepsilon(X)$, i.e.

$$\mathcal{F}(X^\varepsilon) + \varepsilon D(X^\varepsilon) \le \mathcal{F}(X) + \varepsilon D(X),$$

and we also have

$$\mathcal{F}(X) \le \mathcal{F}(X^\varepsilon)$$

as X minimizes $\mathcal{F}$ in $\mathcal{C}(\Gamma)$. Therefore

$$\varepsilon D(X^\varepsilon) \le \varepsilon D(X), \quad \varepsilon \in (0,1],$$

and so

$$D(X^\varepsilon) \le D(X) \quad \text{for } \varepsilon \in (0,1],$$

whence

$$\limsup_{j\to\infty} D(X^{\varepsilon_j}) \le D(X).$$

On the other hand, $X^{\varepsilon_j} \rightharpoonup X$ in $H^{1,2}(B,\mathbb{R}^3)$ implies

$$D(X) \le \liminf_{j\to\infty} D(X^{\varepsilon_j})$$

and so we obtain (11). □

Theorem 2. *Every minimizer X of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ that satisfies (3) is Hölder continuous in B and continuous on $\overline{B}$.*

Proof. Fix some $w_0 \in B$ and set $R := 1 - |w_0| > 0$. For $0 < r < R$ we define $H \in H_2^1(B_r(w_0), \mathbb{R}^3)$ as the solution of

$$\Delta H = 0 \quad \text{in } B_r(w_0), \quad H - X \in \mathring{H}_2^1(B_r(w_0), \mathbb{R}^3),$$

and then we set $Y(w) := H(w)$ for $w \in B_r(w_0)$ and $Y(w) := X(w)$ for $w \in B \setminus B_r(w_0)$. Since $Y \in \mathcal{C}(\Gamma)$ it follows that

$$\mathcal{F}(X) \le \mathcal{F}(Y),$$

whence

$$\mathcal{F}_{B_r(w_0)}(X) \le \mathcal{F}_{B_r(w_0)}(Y).$$

Here and in the following the index $B_r(w_0)$ means that the corresponding integrals are to be taken over the set $B_r(w_0)$. By (D) and (3) we have

$$m_1 D_{B_r(w_0)}(X) = m_1 A_{B_r(w_0)}(X) \le \mathcal{F}_{B_r(w_0)}(X),$$

and (3) together with $A \le D$ and $Y = H$ on $B_r(w_0)$ yields

$$\mathcal{F}_{B_r(w_0)}(Y) = \mathcal{F}_{B_r(w_0)}(H) \le m_2 A_{B_r(w_0)}(H) \le m_2 D_{B_r(w_0)}(H).$$

Thus

$$D_{B_r(w_0)}(X) \le \frac{m_2}{m_1} D_{B_r(w_0)}(H),$$

that is,

$$\Phi(r) := \int_{B_r(w_0)} |\nabla X|^2 \, du \, dv \le \frac{m_2}{m_1} \int_{B_r(w_0)} |\nabla H|^2 \, du \, dv. \tag{12}$$

Let us introduce polar coordinates ρ, θ around w_0 by $w = w_0 + \rho e^{i\theta}$; we write

$$X(w) = X(w_0 + \rho e^{i\theta}) =: X^*(\rho, \theta).$$

Then

$$\Phi(r) = \int_0^r \int_0^{2\pi} \{|X_\rho^*(\rho,\theta)|^2 + \rho^{-2} |X_\theta^*(\rho,\theta)|^2\} \rho \, d\rho \, d\theta.$$

Since

$$|X_\rho^*|^2 = \rho^{-2} |X_\theta^*|^2, \quad \langle X_\rho^*, X_\theta^* \rangle = 0$$

we have

$$\Phi(r) = 2 \int_0^r \rho^{-1} \left(\int_0^{2\pi} |X_\theta^*(\rho,\theta)|^2 \, d\theta \right) d\rho.$$

We can find a representative $X^*(\rho, \theta)$ that is absolutely continuous in θ for almost all $\rho \in (0, R)$ and $\int_0^{2\pi} |X_\theta^*(\rho,\theta)|^2 \, d\theta < \infty$ for these ρ. The function $\Phi(r)$ is absolutely continuous on $[0, R]$, and

$$\Phi'(r) = 2r^{-1} \int_0^{2\pi} |X_\theta^*(r,\theta)|^2 \, d\theta \quad \text{for } r \in (0,R) \setminus \mathcal{N}$$

where $\mathcal{N}$ is a one-dimensional null set.

Furthermore we have

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv \le \int_0^{2\pi} |X_\theta^*(r,\theta)|^2 \, d\theta$$

(see e.g. Vol. 2, Section 2.5, (18)), and so

$$\int_{B_r(w_0)} |\nabla H|^2 \, du \, dv \le \frac{1}{2} r\Phi'(r) \quad \text{for } r \in (0,R) \setminus \mathcal{N}.$$

By virtue of (12) we arrive at

$$\Phi(r) \le \frac{1}{2} \frac{m_2}{m_1} r\Phi'(r) \quad \text{a.e. on } (0,R).$$

Setting $\mu := m_1/m_2$ we have

$$2\mu\Phi(r) \le r\Phi'(r) \quad \text{a.e. on } (0,R)$$

whence

$$\Phi(r) \le (r/R)^{2\mu}\Phi(R) \quad \text{for } r \in (0,R), \tag{13}$$

and then it follows that $X \in C^{0,\mu}(B,\mathbb{R}^3)$ on account of Morrey's "Dirichlet growth theorem" (see Morrey [8], p. 79).

It remains to prove that $X \in C^0(\overline{B},\mathbb{R}^3)$. To this end we introduce polar coordinates ρ, ϑ around the origin and write

$$X(w) = X(\rho e^{i\vartheta}) =: X^*(\rho,\vartheta).$$

Set

$$\varepsilon(X,h) := \int_{1-2h}^1 \int_0^{2\pi} \left[|X_\rho^*(\rho,\vartheta)|^2 + |X_\vartheta^*(\rho,\vartheta)|^2\right] d\rho \, d\vartheta$$

for $0 < h < 1/4$; then

$$\frac{1}{2}\,\varepsilon(X,h) \le \int_{1-2h}^1 \int_0^{2\pi} (|X_\rho^*|^2 + \rho^{-2}|X_\vartheta^*|^2)\rho \, d\rho \, d\vartheta \le 2\varepsilon(X,h).$$

It follows as in Morrey [8], Theorem 3.5.2, that there is a number $c_0(\mu)$ depending only on μ such that

$$|X^*(1-h,\theta) - X^*(1-h,\theta')| \le c_0(\mu)\varepsilon(X,h)h^{-\mu}|\theta-\theta'|^\mu \le c_0(\mu)\varepsilon(X,h)$$

for all $\theta' \in \mathbb{R}$ with $|\theta - \theta'| \le h < 1/4$.

Let $\xi(\theta)$ be the continuous Sobolev trace $X^*(1,\theta)$ of X^* on $\rho = 1$, and set

$$\omega(\xi, h) := \sup\{|\xi(\theta') - \xi(\theta'')| \colon \theta', \theta'' \in \mathbb{R}, |\theta' - \theta''| < h\}.$$

Then $\omega(\xi, h) \to 0$ as $h \to +0$.

Furthermore, for any $\theta \in \mathbb{R}$ there is a θ_1 with $|\theta - \theta_1| \leq h$ such that $X(\rho, \theta_1)$ is absolutely continuous in $\rho \in [1/2, 1]$, $X_\rho(\cdot, \theta_1) \in L_2([1/2, 1], \mathbb{R}^3)$ and

$$|X^*(\rho, \theta_1) - \xi(\theta_1)| \to 0 \quad \text{as } \rho \to 1 - 0$$

as well as

$$\int_{1-h}^{1} |X_\rho^*(\rho, \theta_1)|^2 d\rho \leq h^{-1} \varepsilon^2(X, h).$$

It follows that

$$\begin{aligned}
|\xi(\theta_1) - X^*(1-h, \theta_1)| &\leq \int_{1-h}^{1} |X_\rho^*(\rho, \theta_1)|\, d\rho \\
&\leq \sqrt{h} \cdot \left\{ \int_{1-h}^{1} |X_\rho^*(\rho, \theta_1)|^2\, d\rho \right\}^{1/2} \leq \varepsilon(X, h).
\end{aligned}$$

Given θ_0 and θ with $|\theta - \theta_0| \leq h' < 1/4$ we choose θ_1 as above. Because of

$$\begin{aligned}
&|X^*(1-h, \theta) - \xi(\theta_0)| \\
&\quad \leq |X^*(1-h, \theta) - X^*(1-h, \theta_1)| + |X^*(1-h, \theta_1) - \xi(\theta_1)| \\
&\qquad + |\xi(\theta_1) - \xi(\theta)| + |\xi(\theta) - \xi(\theta_0)|
\end{aligned}$$

we then obtain

$$|X^*(1-h, \theta) - \xi(\theta_0)| \leq [1 + c_0(\mu)]\varepsilon(X, \theta, h) + \omega(\xi, h) + \omega(\xi, h').$$

This proves $X^*(\rho, \theta) \to \xi(\theta_0)$ as $\rho \to 1 - 0$ and $\theta \to \theta_0$. Hence $X \in C^0(\overline{B}, \mathbb{R}^3)$. $\square$

Remark 1. So far no general results concerning higher regularity of solutions to (2) are known. For a special class of Cartan functionals it was proved that the minimizers X of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ satisfy $X \in H_2^2(B, \mathbb{R}^3) \cap C^{1,\alpha}(\overline{B}, \mathbb{R}^3)$ for some $\alpha \in (0, 1)$ provided that $F \in C^2$ on $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ and $\Gamma \in C^4$; see Hildebrandt and von der Mosel [1–7].

4.14 Isoperimetric Inequalities

Now we want to derive the isoperimetric inequality for disk-type surfaces $X : B \to \mathbb{R}^3$ of class $C^1(\overline{B}, \mathbb{R}^3)$ or, more generally, for $X \in H_2^1(B, \mathbb{R}^3)$ with the parameter domain

$$B = \{w \in \mathbb{C} \colon |w| < 1\},$$

the boundary of which is given by

$$C = \partial B = \{w \in \mathbb{C} \colon |w| = 1\}.$$

Recall that any $X \in H_2^1(B, \mathbb{R}^3)$ has boundary values $X|_C$ of class $L_2(C, \mathbb{R}^3)$. Denote by $L(X)$ the length of the boundary trace $X|_C$, i.e.,

$$L(X) = L(X|_C) := \int_C |dX|.$$

We recall a result that, essentially, has been proved in Section 4.7.

Lemma 1. (i) *Let $X : B \to \mathbb{R}^3$ be a minimal surface with a finite Dirichlet integral*

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv$$

and with boundary values $X|_C$ of finite total variation

$$L(X) = \int_C |dX|.$$

Then X is of class $H_2^1(B, \mathbb{R}^3)$ and has a continuous extension to $\overline{B}$, i.e., $X \in C^0(\overline{B}, \mathbb{R}^3)$. Moreover, the boundary values $X|_C$ are of class $H_1^1(C, \mathbb{R}^3)$. Setting $X(r, \theta) := X(re^{i\theta})$, we obtain that, for any $r \in (0, 1]$, the function $X_\theta(r, \theta)$ vanishes at most on a set of θ-values of one-dimensional Hausdorff measure zero, and that the limits

$$\lim_{r \to 1-0} X_r(r, \theta) \quad \text{and} \quad \lim_{r \to 1-0} X_\theta(r, \theta)$$

exist, and that

$$\frac{\partial}{\partial \theta} X(1, \theta) = \lim_{r \to 1-0} X_\theta(r, \theta) \quad \text{a.e. on } [0, 2\pi]$$

holds true. Finally, setting $X_r(1, \theta) := \lim_{r \to 1-0} X_r(r, \theta)$, it follows that

$$\int_B \langle \nabla X, \nabla \phi \rangle \, du \, dv = \int_C \langle X_r, \phi \rangle \, d\theta \tag{1}$$

is satisfied for all $\phi \in H_2^1 \cap L_\infty(B, \mathbb{R}^3)$. Moreover, we have

$$\lim_{r \to 1-0} \int_0^{2\pi} |X_\theta(r, \theta)| r \, d\theta = \int_0^{2\pi} |dX(1, \theta)|. \tag{2}$$

(ii) *If $X : B \to \mathbb{R}^3$ is a minimal surface with a continuous extension to $\overline{B}$ such that $L(X) := \int_C |dX| < \infty$, then we still have (2).*

Proof. Since $L(X) < \infty$, the finiteness of $D(X)$ is equivalent to the relation $X \in H_2^1(B, \mathbb{R}^3)$, on account of Poincaré's inequality. Hence X has an $L_2(C)$-trace on the boundary C of ∂B which, by assumption, has a finite total variation $\int_C |dX|$. Consequently, the two one-sided limits

$$\lim_{\theta \to \theta_0 - 0} X(1, \theta) \quad \text{and} \quad \lim_{\theta \to \theta_0 + 0} X(1, \theta)$$

exist for every $\theta_0 \in \mathbb{R}$. In conjunction with the Courant–Lebesgue lemma, we obtain that $X(1, \theta)$ is a continuous function of $\theta \in \mathbb{R}$ whence $X \in C^0(\overline{B}, \mathbb{R}^3)$ (cf. Section 4.7, part (iii) of the proof of Proposition 3). The rest of the proof follows from Theorems 1 and 2 in Section 4.7. $\square$

Lemma 2 (Wirtinger's inequality). *Let $Z : \mathbb{R} \to \mathbb{R}^3$ be an absolutely continuous function that is periodic with the period $L > 0$ and has the mean value*

$$P := \frac{1}{L} \int_0^L Z(t)\, dt. \tag{3}$$

Then we obtain

$$\int_0^L |Z(t) - P|^2\, dt \le \left(\frac{L}{2\pi}\right)^2 \int_0^L |\dot{Z}(t)|^2\, dt, \tag{4}$$

and the equality sign holds if and only if there are constant vectors A_1 and B_1 in $\mathbb{R}^3$ such that

$$Z(t) = P + A_1 \cos\left(\frac{2\pi}{L} t\right) + B_1 \sin\left(\frac{2\pi}{L} t\right) \tag{5}$$

holds for all $t \in \mathbb{R}$.

Proof. We first assume that $L = 2\pi$ and $\int_0^{2\pi} |\dot{Z}|^2\, dt < \infty$. Then we have the expansions

$$\begin{aligned} Z(t) &= P + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt), \\ \dot{Z}(t) &= \sum_{n=1}^{\infty} n(B_n \cos nt - A_n \sin nt) \end{aligned}$$

of Z and $\dot{Z}$ into Fourier series with $A_n, B_n \in \mathbb{R}^3$, and

$$\begin{aligned} \int_0^{2\pi} |Z - P|^2\, dt &= \pi \sum_{n=1}^{\infty} (|A_n|^2 + |B_n|^2), \\ \int_0^{2\pi} |\dot{Z}|^2\, dt &= \pi \sum_{n=1}^{\infty} n^2 (|A_n|^2 + |B_n|^2). \end{aligned} \tag{6}$$

Consequently it follows that

$$\int_0^{2\pi} |Z - P|^2 \, dt \le \int_0^{2\pi} |\dot{Z}|^2 \, dt, \tag{7}$$

and the equality sign holds if and only if all coefficients A_n and B_n vanish for $n > 1$. Thus we have verified the assertion under the two additional hypotheses. If $\int_0^{2\pi} |\dot{Z}|^2 \, dt = \infty$, the statement of the lemma is trivially satisfied, and the general case $L > 0$ can be reduced to the case $L = 2\pi$ by the scaling transformation $t \mapsto (2\pi/L)t$. $\square$

Now we shall state the isoperimetric inequality for minimal surfaces in its simplest form.

Theorem 1. *Let $X \in C^2(B, \mathbb{R}^3)$ with $B = \{w\colon |w| < 1\}$ be a minimal surface, i.e. X be nonconstant and satisfy*

$$\Delta X = 0,$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Assume also that X is either of class $H_2^1(B, \mathbb{R}^3)$ or of class $C^0(\overline{B}, \mathbb{R}^3)$, and that $L(X) = \int_C |dX| < \infty$. Then $D(X)$ is finite, and we have

$$D(X) \le \frac{1}{4\pi} L^2(X). \tag{8}$$

Moreover, the equality sign holds if and only if $X : B \to \mathbb{R}^3$ represents a (simply covered) disk.

Remark 1. Note that for every minimal surface $X : B \to \mathbb{R}^3$ the area functional $A(X)$ coincides with the Dirichlet integral $D(X)$. Thus (8) can equivalently be written as

$$A(X) \le \frac{1}{4\pi} L^2(X). \tag{8'}$$

Proof of Theorem 1. (i) Assume first that X is of class $H_2^1(B, \mathbb{R}^3)$, and that P is a constant vector in $\mathbb{R}^3$. Because of $L(X) < \infty$, the boundary values $X|_C$ are bounded whence X is of class $L_\infty(B, \mathbb{R}^3)$ (this follows from the maximum principle in conjunction with a suitable approximation device). Thus we can apply formula (1) to $\phi = X - P$, obtaining

$$\begin{aligned} \int_B \langle \nabla X, \nabla X \rangle \, du \, dv & \\ = \int_B \langle \nabla X, \nabla (X - P) \rangle \, du \, dv & \\ = \int_C \langle X_r, X - P \rangle \, d\theta \le \int_C |X_r| |X - P| \, d\theta & \\ = \int_C |X_\theta| |X - P| \, d\theta = \int_0^{2\pi} |X_\theta(1, \theta)| |X(1, \theta) - P| \, d\theta. & \end{aligned} \tag{9}$$

Introducing $s = \sigma(\theta)$ by

$$\sigma(\theta) := \int_0^\theta |X_\theta(1,\theta)| \, d\theta,$$

we obtain that $\sigma(\theta)$ is a strictly increasing and absolutely continuous function of θ, and $\dot\sigma(\theta) = |X_\theta(1,\theta)| > 0$ a.e. on $\mathbb{R}$. Hence $\sigma : \mathbb{R} \to \mathbb{R}$ has a continuous inverse $\tau : \mathbb{R} \to \mathbb{R}$. Let us introduce the reparametrization

$$Z(s) := X(1, \tau(s)), \quad s \in \mathbb{R},$$

of the curve $X(1,\theta)$, $\theta \in \mathbb{R}$. Then, for any $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, the numbers $\theta_1 := \tau(s_1)$, $\theta_2 := \tau(s_2)$ satisfy $\theta_1 < \theta_2$ and

$$(10) \qquad \int_{s_1}^{s_2} |dZ| = \int_{\theta_1}^{\theta_2} |dX| = \sigma(\theta_2) - \sigma(\theta_1) = s_2 - s_1,$$

whence

$$|Z(s_2) - Z(s_1)| \le s_2 - s_1.$$

Consequently, the mapping $Z : \mathbb{R} \to \mathbb{R}^3$ is Lipschitz continuous and therefore also absolutely continuous, and we obtain from (10) that

$$(11) \qquad \int_{s_1}^{s_2} |Z'(s)| \, ds = s_2 - s_1$$

($' = \frac{d}{ds}$), whence

$$(12) \qquad |Z'(s)| = 1 \quad \text{a.e. on } \mathbb{R}.$$

In other words, the curve $Z(s)$ is the reparametrization of $X(1,\theta)$ with respect to the parameter s of its arc length.

As the mapping $\sigma : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, it maps null sets onto null sets, and we derive from

$$\frac{\tau(s_2) - \tau(s_1)}{s_2 - s_1} = \frac{1}{\frac{\sigma(\theta_2) - \sigma(\theta_1)}{\theta_2 - \theta_1}}$$

and from $\dot\sigma(\theta) > 0$ a.e. on $\mathbb{R}$ that

$$(13) \qquad \tau'(s) = \frac{1}{\dot\sigma(\tau(s))} > 0 \quad \text{a.e. on } \mathbb{R}.$$

On account of

$$\dot\sigma(\theta) = |X_\theta(1,\theta)| \quad \text{a.e. on } \mathbb{R}$$

it then follows that

$$(14) \qquad |X_\theta(1, \tau(s))| \frac{d\tau}{ds}(s) = 1 \quad \text{a.e. on } \mathbb{R},$$

and thus we obtain

$$\int_0^{2\pi} |X_\theta(1,\theta)||X(1,\theta) - P| \, d\theta = \int_0^L |Z(s) - P| \, ds. \tag{15}$$

We now infer from (9) and (15) that

$$\int_B \langle \nabla X, \nabla X \rangle \, du \, dv \le \int_0^L |Z(s) - P| \, ds. \tag{16}$$

By Schwarz's inequality, we have

$$\int_0^L |Z(s) - P| \, ds \le \sqrt{L} \left\{ \int_0^L |Z(s) - P|^2 \, ds \right\}^{1/2}, \tag{17}$$

and Wirtinger's inequality (4) together with (12) implies that

$$\left\{ \int_0^L |Z(s) - P|^2 \, ds \right\}^{1/2} \le \frac{L^{3/2}}{2\pi} \tag{18}$$

if we choose P as the barycenter of the closed curve $Z : [0, L] \to \mathbb{R}^3$, i.e., if

$$P := \frac{1}{L} \int_0^L Z(s) \, ds.$$

By virtue of (16)–(18), we arrive at

$$\int_B |\nabla X|^2 \, du \, dv \le \frac{1}{2\pi} L^2 \tag{19}$$

which is equivalent to the desired inequality (8).

Suppose that equality holds true in (8) or, equivalently, in (19). Then equality must also hold in Wirtinger's inequality (18), and by Lemma 2 we infer

$$Z(s) = P + A_1 \cos\left(\frac{2\pi}{L} s\right) + B_1 \sin\left(\frac{2\pi}{L} s\right).$$

Set $R := L/(2\pi)$ and $\varphi = s/R$. Because of $|Z'(s)| \equiv 1$, we obtain

$$R^2 = |A_1|^2 \sin^2 \varphi + |B_1|^2 \cos^2 \varphi - 2\langle A_1, B_1 \rangle \sin\varphi \cos\varphi.$$

Choosing $\varphi = 0$ or $\frac{\pi}{2}$, respectively, it follows that

$$|A_1| = |B_1| = R,$$

and therefore

$$\langle A_1, B_1 \rangle = 0.$$

Then the pair of vectors $E_1, E_2 \in \mathbb{R}^3$, defined by

$$E_1 := \frac{1}{R} A_1, \quad E_2 := \frac{1}{R} B_1,$$

is orthonormal, and we have

$$Z(R\varphi) = P + R\{E_1 \cos\varphi + E_2 \sin\varphi\}.$$

Consequently $Z(R\varphi)$, $0 \le \varphi \le 2\pi$, describes a simply covered circle of radius R, centered at P, and the same holds true for the curve $X(1,\theta)$ with $0 \le \theta \le 2\pi$. Hence $X : \overline{B} \to \mathbb{R}^3$ represents a (simply covered) disk of radius R, centered at P. This can be seen as follows: We may assume that the circle $\Gamma := \{X(1,\theta) \colon 0 \le \theta \le 2\pi\}$ lies in the x, y-plane and is given by

$$\Gamma = \{(x,y,z) \colon x^2 + y^2 = R^2,\ z = 0\}.$$

Then the maximum principle implies that X has the form

$$X = (X^1, X^2, 0) \quad \text{with } |X^1(w)|^2 + |X^2(w)|^2 \le R^2 \text{ for } w \in \overline{B}$$

since $\Delta X^3 = 0$ and $\Delta(|X^1|^2 + |X^2|^2) \ge 0$. Using the conformality relation it follows that either $f(w) = X^1(w) + iX^2(w)$ or $\overline{f(w)}$ is holomorphic and, in fact, conformal on B (for details, we refer to the proof of Theorem 1 in Section 4.11).

Conversely, if $X : \overline{B} \to \mathbb{R}^3$ represents a simply covered disk, then the equality sign holds true in (8′) and, therefore also in (8).

Thus the assertion of the theorem is proved under the assumption that $X \in H_2^1(B, \mathbb{R}^3)$.

(ii) Suppose now that X is of class $C^0(\overline{B}, \mathbb{R}^3)$. Then we introduce nonconstant minimal surfaces $X_k : B \to \mathbb{R}^3$ of class $C^\infty(\overline{B}, \mathbb{R}^3)$ by defining

$$X_k(w) := X(r_k w) \quad \text{for } |w| \le 1,\ r_k := \frac{k}{k+1}.$$

We can apply (i) to each of the surfaces X_k, thus obtaining

$$4\pi D(X_k) \le \left\{ \int_0^{2\pi} |dX_k(1,\theta)| \right\}^2. \tag{20}$$

For $k \to \infty$, we have $r_k \to 1 - 0$, $D(X_k) \to D(X)$, and part (ii) of Lemma 1 yields

$$\lim_{k\to\infty} \int_0^{2\pi} |dX_k(1,\theta)| = \int_0^{2\pi} |dX(1,\theta)|.$$

Thus we infer from (20) that

$$4\pi D(X) \le L^2(X)$$

which implies in particular that X is of class $H_2^1(B, \mathbb{R}^3)$. For the rest of the proof, we can now proceed as in part (i). $\square$

If the boundary of a minimal surface X is very long in comparison to its "diameter", then another estimate of $A(X) = D(X)$ might be better which depends only linearly on the length $L(X)$ of the boundary of X. We call this estimate *the linear isoperimetric inequality*. It reads as follows:

Theorem 2. *Let X be a nonconstant minimal surface with the parameter domain $B = \{w : |w| < 1\}$, and assume that X is either continuous on $\overline{B}$ or of class $H_2^1(B, \mathbb{R}^3)$. Moreover, suppose that the length $L(X) = \int_C |dX|$ of its boundary is finite, and let $\mathcal{K}_R(P)$ be the smallest ball in $\mathbb{R}^3$ containing $X(\partial B)$ and therefore also $X(\overline{B})$. Then we have*

$$D(X) \leq \tfrac{1}{2} R L(X). \tag{21}$$

Equality holds in (21) if and only if $X(B)$ is a plane disk.

Proof. By Theorem 1 it follows that $D(X) < \infty$ and $X \in H_2^1(B, \mathbb{R}^3)$, and formula (9) implies

$$2D(X) \leq \int_C |X_\theta||X - P| \, d\theta \leq RL(X) \tag{22}$$

whence we obtain (21).

Suppose now that

$$D(X) = \tfrac{1}{2} R L(X). \tag{23}$$

Then we infer from (9) and (22) that

$$\int_C \langle X_r, X - P \rangle \, d\theta = \int_C |X_r||X - P| \, d\theta$$

is satisfied; consequently we have

$$\langle X_r, X - P \rangle = |X_r||X - P|$$

a.e. on C, that is, the two vectors X_r and $X - P$ are collinear a.e. on C.

Secondly we infer from (22) and (23) that

$$|X - P| = R \quad \text{a.e. on } C.$$

Hence the H_1^1-curve Σ defined by $X : C \to \mathbb{R}^3$ lies on the sphere $S_R(P)$ of radius R centered at P, and the side normal X_r of the minimal surface X at Σ is proportional to the radius vector $X - P$. Thus $X_r(1, \theta)$ is perpendicular to $S_R(P)$ for almost all $\theta \in [0, 2\pi]$. Hence the surface X meets the sphere $S_R(P)$ orthogonally a.e. along Σ. As in the proof of Theorem 1 in Section 5.4 we can show that X is a stationary surface with a free boundary on $S_R(P)$ and that X can be viewed as a stationary point of Dirichlet's integral in the class $\mathcal{C}(S_R(P))$. By Theorems 1 and 2 of Vol. 2, Section 2.8, the surface X is real analytic on the closure $\overline{B}$ of B. Then it follows from the Theorem in Vol. 2, Section 1.7 that $X(\overline{B})$ is a plane disk.

Conversely, if $X : B \to \mathbb{R}^3$ represents a plane disk, then (23) is fulfilled. $\square$

A more general version of the isoperimetric inequality (8′) will be proved in Vol. 2, Section 6.5. We also refer to Section 6.4 of this volume where the isoperimetric inequality of Morse–Tompkins for harmonic surfaces is derived.

4.15 Scholia

1 Historical Remarks and References to the Literature

Although Plateau's problem is one of the classical questions in geometry and analysis, progress in solving it was very slow. The problem was already formulated by Lagrange in his *Essai d'une nouvelle méthode* ... [1]: *trouver la surface qui est la moindre de toutes celles qui ont un même périmètre donné*, but neither he nor Euler were able to solve the question. It seemed even difficult to find solutions of the minimal surface equation, not to speak of the corresponding boundary value problem. In the early 19th century, Gergonne [1] drew the attention of his contemporaries again to this and related boundary value problems, but still Jacobi was unable to tackle them. *In his Lectures on the Calculus of Variations* at Königsberg, 1837/38, he said: *Es haben sich in der neuesten Zeit die ausgezeichnetsten Mathematiker wie Poisson und Gauß mit der Auffindung der Variation des Doppelintegrals beschäftigt, die wegen der willkürlichen Funktionen unendliche Schwierigkeiten macht. Dennoch wird man durch ganz gewöhnliche Aufgaben darauf geführt, z.B. durch das Problem: unter allen Oberflächen, die durch ein schiefes Viereck im Raum gelegt werden können, diejenige anzugeben, welche den kleinsten Flächeninhalt hat. Es ist mir nicht bekannt, daß schon irgend jemand daran gedacht hätte, die zweite Variation solcher Doppelintegrale zu untersuchen; auch habe ich, trotz vieler Mühe, nur erkannt, daß der Gegenstand zu den allerschwierigsten gehört.*

The problem mentioned by Jacobi, namely to span a minimal surface in a general quadrilateral of $\mathbb{R}^3$, was first solved by H.A. Schwarz and, independently and at about the same time, by Riemann. Riemann's paper appeared posthumously in 1867, the same year that Schwarz's prize-essay was sent to the Berlin Academy. Later on, Plateau's problem was solved for other polygonal boundaries and, more generally, also free and partially free boundary problems for so-called Schwarzian chains were tackled. In particular, we mention the work of Weierstraß [4], Tallquist [2], and Neovius [1–5]. An outline of the techniques used by these authors can be found in the treatise of Bianchi [1]; a very extensive presentation is given in volume 1 of Darboux's *Leçons* [1].

The first general existence proof for the nonparametric Plateau problem was given by A. Haar [3] in 1927, with important supplements by Radó concerning the regularity of minimizers. The contributions of Haar and Radó were major mathematical achievements; for the first time, the program envisioned by Hilbert in his problems 19 and 20 had been carried out for a fundamental variational problem with nonlinear Euler equations.

A first solution of the Plateau problem for a general contour was published by R. Garnier [2] in 1928. By a limit procedure he obtained a solution for unknotted and piecewise smooth Jordan curves from a penetrating analysis of the Plateau problem for polygonal boundaries. However, Garnier's long paper was apparently seldom read if it was read at all (see Nitsche [28], p. 251),[2] and it was soon superseded by the convincing proofs of J. Douglas [11,12] and T. Radó [17,18] published about 1930. Douglas began to publish on Plateau's problem in 1927, and he announced a solution as early as 1929 (see Douglas [5] and [17]). Still, his first papers were apparently not convincing to everyone (see Constance Reid [1], pp. 173–174), and the long list of Douglas's announcements prior to 1931 might indicate that Douglas himself did not think he had found the best possible presentation, see Douglas [1–11].

Douglas based his approach to Plateau's problem on the functional

$$A_0(X) := \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|\mathcal{X}(\theta)-\mathcal{X}(\varphi)|^2}{4\sin^2\frac{1}{2}(\theta-\varphi)}\,d\theta\,d\varphi,$$

$\mathcal{X}(\theta) := X(e^{i\theta}) = X(\cos\theta, \sin\theta)$, which, for harmonic mappings

$$X : B = \{w \colon |w| < 1\} \to \mathbb{R}^N,$$

coincides with Dirichlet's integral $D(X)$ (cf. Section 6.4). The Douglas functional $A_0(X)$ has certain advantages as it only takes the boundary values $\mathcal{X}(\theta)$ of a harmonic mapping $X : B \to \mathbb{R}^N$ into account, but the Dirichlet integral is more natural and easier to handle. In the case of the general Plateau problem, the Dirichlet integral can still be used while the Douglas functional has to be replaced by a rather unwieldy expression, and also for free boundary problems the Dirichlet integral seems to be the natural tool.

Radó's method to attack Plateau's problem is much closer to the approach used in the present chapter than the method of Douglas. Radó runs through several approximation steps. First he treats the case of a polygonal boundary Γ where one can find a sequence of polyhedra P_n whose areas approach the infimum $a(\Gamma)$ of areas of surfaces within Γ. As polyhedra admit conformal representations $Z_n : B \to \mathbb{R}^3$, the Dirichlet integrals of these representations approach the infimum value $e(\Gamma)$ of Dirichlet's integral for surfaces $X : B \to \mathbb{R}^3$ within Γ, and we have $e(\Gamma) = a(\Gamma)$. Replacing the Z_n by harmonic maps $X_n : B \to \mathbb{R}^3$ with the same boundary values as Z_n, we obtain $D(X_n) \to e(\Gamma) = a(\Gamma)$ as $n \to \infty$. A standard selection theorem for harmonic maps implies that we can extract a subsequence from $\{X_n\}$, again denoted by $\{X_n\}$, which converges uniformly on any $B' \subset\subset B$ to some harmonic map $X : B \to \mathbb{R}^3$, and whose derivatives converge uniformly on $B' \subset\subset B$ to the derivatives of X. Then we obtain

$$\int_{B'} (|D_u X_n| - |D_v X_n|)^2\,du\,dv \to \int_{B'} (|D_u X| - |D_v X|)^2\,du\,dv,$$
$$\int_{B'} |\langle D_u X_n, D_v X_n\rangle|\,du\,dv \to \int_{B'} |\langle D_u X, D_v X\rangle|\,du\,dv$$

[2] However, note the recent work of L. Desideri; cf. p. 364.

as $n \to \infty$. On the other hand, the choice of the X_n together with a simple estimation yields that the integrals on the left-hand side tend to zero as $n \to \infty$ since the X_n are approximate solutions for the Plateau problem to Γ. This implies that X is a minimal surface, i.e., a harmonic map satisfying the conformality conditions $|X_u| = |X_v|$, $\langle X_u, X_v \rangle = 0$. Moreover, a sophisticated *approximation theorem* yields that X is continuous on $\bar{B}$, and that $X|_{\partial B}$ gives a parametrization of Γ. Thus Plateau's problem is solved for polygons.

In the next step, a rectifiable curve Γ is approximated by polygons Γ_n in the sense of Fréchet. Solving the Plateau problem for any of the Γ_n by a minimal surface X_n, another application of the approximation theorem together with a suitable compactness result for sequences of harmonic maps yields a solution of Plateau's problem minimizing area.

An admirably clear and short presentation of the results of Haar, Douglas and Radó is given in the report [21] by Radó.

We note that the methods of Douglas and Radó yield area-minimizing minimal surfaces spanned into Γ if $a(\Gamma) < \infty$ whereas Garnier's solutions might only be stationary. Moreover, Douglas was able to solve Plateau's problem even in the case when $a(\Gamma) = \infty$. The essential simplification achieved in the proofs of R. Courant [4] and L. Tonelli [1] presented in this chapter follows from the Courant–Lebesgue lemma which is also of use in many other situations. The method of deriving the conformality conditions by a variation of the independent variables is due to Radó (cf. [21], pp. 87–89). The efficient variational formula generalizing Radó's idea was stated by Courant [15].

Another solution of Plateau's problem was found by McShane [1,2] in 1933 who directly attacked the problem of minimizing area. Using ideas of Lebesgue he showed: (i) One can find a minimizing sequence of Lebesgue monotone surfaces. (ii) Each of these surfaces can be replaced by a (weakly) conformally parametrized Lebesgue monotone surface. (iii) The minimizing sequence obtained by (i), (ii) is compact in $C^0(\overline{B}, \mathbb{R}^3)$. A detailed presentation of McShane's approach is given in Nitsche [28], pp. 414–430.

The approach of Section 4.10 is due to S. Hildebrandt and H. von der Mosel [1–7]; it leads to another solution of Plateau's problem by minimizing area. Contrary to all other methods this approach does not use any results on conformal or quasiconformal reparametrizations of a given surface such as the theorems of Lichtenstein or of Carathéodory, and so it establishes an *elementary proof* of the fact that the minimizers of Dirichlet's integral in the class of disk-type surfaces bounded by a given rectifiable Jordan contour are as well area minimizing. This was thought to be impossible; see Courant [15], pp. 116–118. Moreover, a modification of the method is used in 4.11 to derive the *global Lichtenstein theorem* by a variational method (cf. Hildebrandt and von der Mosel [6,7]). Another variational proof of this theorem was earlier given by J. Jost [6] and [17], rectifying the original approach by C.B. Morrey (see [8], Chapter 9) which contains a gap.

The partially free problem was originally treated by Courant using some of the ideas described in Chapter 1 of Vol. 2. The simplified version of Section 4.6 is due to Morrey [8]. Courant's original approach can be studied in his monograph [15] and also in Nitsche's treatise [28].

A solution of Plateau's problem for minimal surfaces $X : B \to M$ in Riemannian manifolds M of great generality was given by C.B. Morrey [3], with later supplements by L. Lemaire [1] and J. Jost [6]. Extensions to surfaces of constant or prescribed mean curvature H (=H-surfaces) are due to E. Heinz [2], H. Werner [1,2], S. Hildebrandt [4–10], H. Wente [1–5], K. Steffen [1–6], R. Gulliver [1,3], Gulliver and Spruck [1,2], Hildebrandt and Kaul [1], Brezis and Coron [1–4], M. Struwe [5,7,11,12,14], J. Jost [17], U. Dierkes [2], and Duzaar and Steffen [6,7]. The presentation given by Morrey in Chapter 9 of his treatise [8] is not quite correct but can be rectified. This was carried out by Jost in his paper [6] and also in his monograph [17] where one finds a complete theory of two-dimensional geometric variational problems comprising the theory of conformal and harmonic mappings, Teichmüller theory, minimal surfaces of disk-type as well as of higher topological type, Plateau's problem, and free boundary problems. We also refer to Sections 4.10–4.13 above.

Detailed presentations of the results concerning Plateau's problem can be found in the survey of Radó [21], Courant's monograph [15] and, most complete of all, in Nitsche's *Lectures* [28,37]. Beautiful recent surveys, also covering results on H-surfaces, were written by M. Struwe [11] and J. Jost [17].

In solving Plateau's problem, it is essential that Γ is a Jordan curve, i.e., a continuous embedding of the unit circle S^1 into $\mathbb{R}^3$, in other words, that Γ is not allowed to have selfintersections. Nevertheless one can pose the problem of minimizing area among surfaces bounded by a rectifiable closed curves Γ with selfintersections if one enlarges the notion of admissible surfaces. For example, if Γ is the "figure eight" in $\mathbb{R}^2$, it bounds a surface of minimal area that splits into two minimal disks. Still one can write it as a continuous mapping. Using the Lebesgue notion of area, J. Hass [2] proved:

Any closed rectifiable curve Γ in $\mathbb{R}^3$ bounds a "disk of least area" which is a smooth immersion away from the boundary. This means: There is a mapping $X \in C^0(\overline{B}, \mathbb{R}^3)$ of the unit disk $\overline{B} \subset \mathbb{R}^2$ into $\mathbb{R}^3$ bounded by Γ such that X yields a smooth immersion of $B \setminus X^{-1}(\Gamma)$. The phrase "$X$ is bounded by Γ" means: If $\gamma : S^1 \to \mathbb{R}^3$ is a continuous representation of Γ and $\gamma' = X|_{\partial B}$, then γ' is a continuous mapping $S^1 \to \mathbb{R}^3$ with $d_F(\gamma, \gamma') = 0$ where $d_F(\gamma, \gamma')$ is the "Fréchet distance" of γ and γ', i.e.

$$d_F(\gamma, \gamma') = \inf\Big\{\sup_{S^1} |\gamma - \gamma' \circ \varphi| \colon \varphi \in \mathrm{Hom}(S^1)\Big\},$$

where $\mathrm{Hom}(S^1)$ denotes the set of homeomorphisms $\varphi : S^1 \to S^1$. Here the splitting phenomenon is expressed by the fact that $X^{-1}(\Gamma)$ can be larger than ∂B, i.e. $B \cap X^{-1}(\Gamma)$ can be nonempty.

Another approach to the splitting (or bubbling) problem is contained in the work of E. Kuwert [5–7], operating with Dirichlet's integral; cf. Vol. 2, Scholia to Chapter 1.

2 Branch Points

In Section 3.2, Proposition 1 we have derived asymptotic expansions for minimal surfaces $X : B \to \mathbb{R}^3$ and their complex derivative X_w. Analogous expansions can be established at boundary branch points as we shall see in Section 2.10 of Vol. 2. The basic tool for proving such asymptotic formulas is a method due to Hartman and Wintner which is described in Chapter 3 of Vol. 2.

It was a long-standing question whether the area-minimizing solution of Plateau's problem obtained by Douglas and Radó is a *regular* surface, that is, an immersion. This was eventually confirmed in a series of papers by R. Osserman [12], R. Gulliver [2], H.W. Alt [1,2], and Gulliver, Osserman, and Royden [1]. The break-through was achieved by Osserman [12] who, by an ingenious idea, was able to rule out the existence of *true branch points* for minimizers. A true branch point of a minimal surface $X : B \to \mathbb{R}^3$ is characterized by the fact that there are several geometrically different sheets of the surface lying over the tangent plane at w_0. These sheets intersect transversally along smooth curves in $\mathbb{R}^3$ emanating from $X(w_0)$. A false (interior) branch point is a singular point $w_0 \in B$ which has a neighborhood U in B such that $X(U)$ turns out to be (the trace of) an embedded surface. In other words, false branch points cannot be detected by looking at the image of a minimal surface; they are just the result of a false parametrization.

Osserman's reasoning did not rule out the existence of false branch points for a Douglas–Radó solution. This second part of the regularity proof was, more or less simultaneously, achieved by Gulliver and Alt in the papers cited above. Another treatment can be found in the paper of Gulliver–Osserman–Royden. It is still an open problem whether there can be branch points at the boundary ∂B; however, Gulliver and Lesley [1] indicated that the Douglas–Radó solution is free of boundary branch points if Γ is a regular, real-analytic Jordan curve. Thus we now have the following sharpened version of the

Fundamental existence theorem. *Every closed rectifiable Jordan curve Γ in $\mathbb{R}^3$ bounds an area minimizing surface $X : B \to \mathbb{R}^3$ of the type of the disk, and all solutions of this type are regular surfaces, i.e., they are free of branch points $w_0 \in B$. If Γ is regular and real analytic, then they have no branch points on ∂B, either.*

So far, all known proofs excluding the existence of branch points of area minimizing solutions of Plateau's problem were quite involved; thus we have abstained from presenting them. However, on two occasions we have used the opportunity to sketch the basic ideas. At the end of Section 5.3 in Vol. 2 we have outlined Osserman's idea of how to exclude true branch points at the boundary, and in Section 1.9 of Vol. 2 we have indicated how false branch points can be excluded.

A. Tromba has recently developed a method to exclude true interior branch points for minimizers of A, which is technically simple and applies in many

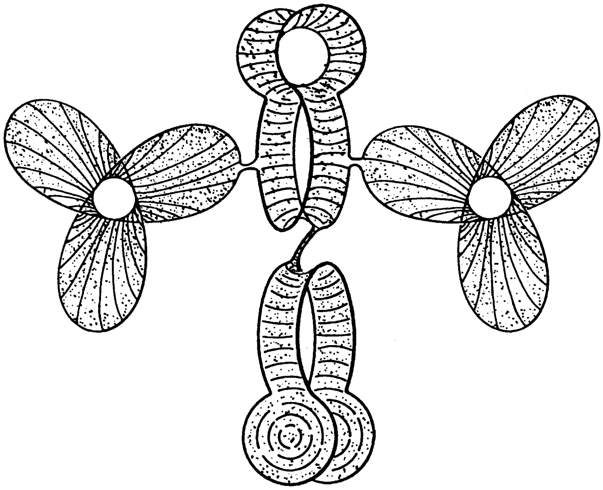


Fig. 1. A knotted curve bounding an embedded minimal surface of higher topological type

cases also to weak minimizers of D. This approach is presented in Chapter 6 of Vol. 2.

We should like to mention that Gulliver and Alt have ruled out interior branch points for other surfaces such as for minimal surfaces in Riemannian manifolds or for surfaces of prescribed mean curvature which satisfy Plateau-type boundary conditions and minimize a suitable functional. However, all these results only hold in $\mathbb{R}^3$ or, more generally, in a three-dimensional manifold, and they become false if $n \geq 4$, i.e., if the codimension exceeds one. For instance, let $z = x + iy$ and set $X(x,y) = (x, y, \operatorname{Re} z^4, \operatorname{Im} z^4)$. Then $X(z)$, $z \in B_R(0)$, describes a nonparametric minimal surface in $\mathbb{R}^4$ with a singular point at $z = 0$. The surface S given by $X : B_R(0) \to \mathbb{R}^4$ is bounded by a Jordan curve, and a simple differential-form argument similar to the one used in Section 2.8 shows that S is in fact area minimizing. The branch-point result is one of the very few basic results mentioned in our notes which only holds true for codimension-one surfaces. The same remark applies to Nitsche's uniqueness theorem, cf. Section 5.6. We also mention that Steffen and Wente [1] have excluded the existence of branch points for minimizers of Dirichlet's integral (as well as of more general functionals) subject to a volume constraint.

3 Embedded Solutions of Plateau's Problem

The absence of branch points does not mean that a minimal surface is free of selfintersections. However, selfintersecting minimal surfaces can never be realized as soap films, i.e., they are unrealistic from the physical point of view. Soap films either appear as surfaces of higher topological type (see Fig. 1), thereby avoiding selfintersections which necessarily have to appear

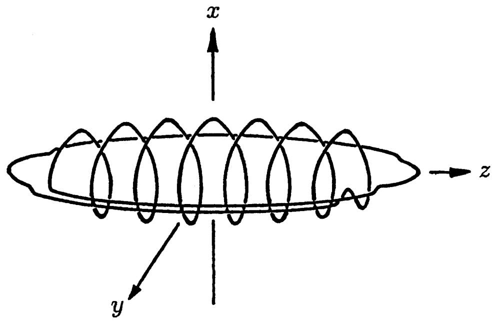


Fig. 2. An example of Almgren and Thurston

for disk-type surfaces spanned into knotted curves, or they arrange themselves as systems forming the characteristic 120-degrees angle at their common liquid edges (with a Y-shaped cross-cut; see No. 7 of these Scholia), but true selfintersections can never be seen. Thus it is of interest to see whether a given boundary curve Γ can be spanned by an embedded minimal disk (i.e., by an injective mapping $X : B \to \mathbb{R}^3$). For topological reasons, this cannot be the case for knotted boundaries Γ, and we therefore have to look among unknotted curves for promising candidates.

Let us begin with an interesting example of an unknotted closed curve Γ described by Almgren and Thurston [1] (see Fig. 2) which can only bound an oriented and embedded surface S lying in the convex hull of Γ if S has at least three handles. (By stretching in the z-direction with a suitably large factor, one can even achieve that the total curvature of Γ does not exceed the value $4\pi + \varepsilon$ where ε is an arbitrarily given positive number.) Hence no minimal disk spanned by Γ can be an embedding since, by the maximum principle, its image in $\mathbb{R}^3$ is necessarily contained in the convex hull of Γ. Similar constructions lead to boundaries Γ spanning only embedded surfaces S with $S \subset$ convex hull of Γ if the genus of S is at least p where p is an arbitrarily prescribed positive integer.

Another example, which is simpler than that of Almgren–Thurston, but shows the same phenomenon, was somewhat later given by J.H. Hubbard [1].

Generally speaking, the classical mapping-approach to minimal surfaces pursued in our notes has the disadvantage that one a priori fixes the topological type of the geometric object. Thus it is much more difficult to decide in this setting whether an area minimizing surface is *geometrically* regular. In geometric measure theory this and other disadvantages have been overcome by the introduction of generalized objects called currents and varifolds.

Simply speaking, an n-current $T \in \mathcal{D}_n(U)$ is a continuous linear functional on the space $\mathcal{D}^n(U)$ of n-forms with compact support in a domain U of $\mathbb{R}^m$. Then each n-dimensional oriented submanifold M of $\mathbb{R}^{n+k}$ (with locally finite n-dimensional Hausdorff measure $\mathcal{H}^n$) represents an n-current in the following way: Let $\tau_1, \dots, \tau_n$ be an adapted orthonormal frame of the tangent space

T_xM, and let $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$ be an orientation on T_xM. Then we define the current $[M]$ by

$$[M](\omega) := \int_M \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^n(x)$$

for $\omega \in \mathcal{D}^n(U)$.

Conversely, the currents which are representable by a manifold (or, more precisely, by a rectifiable n-varifold with integer multiplicity) are of basic importance. They are called *locally rectifiable* (in the terminology of Federer and Fleming), or they are said to be *integer multiplicity currents* (Simon [8], p. 146). To be precise, T is of integer multiplicity if it is representable as

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \theta(x) \, d\mathcal{H}^n(x), \quad \omega \in \mathcal{D}^n(U),$$

where M is an $\mathcal{H}^n$-measurable, countably n-rectifiable subset of U, θ is a locally $\mathcal{H}^n$-integrable positive integer-valued function, and $\xi(x)$ is an $\mathcal{H}^n$-measurable orientation for the approximate tangent space T_xM (see Simon [8] for details). The *mass of a current* T in U is defined as

$$\mathbf{M}_U(T) := \sup\{T(\omega) \colon \|\omega\| \le 1, \ \omega \in \mathcal{D}^n(U)\}.$$

Using the tools of geometric measure theory, Hardt and Simon [1] answered the question of embeddedness in the following way.

Theorem 1. *Each closed Jordan curve $\Gamma \subset \mathbb{R}^3$ of class $C^{1,\alpha}$ bounds at least one embedded orientable minimal surface.*

However, note that, because of the semicontinuity of mass with respect to weak convergence, one has no control over the topological type of the minimal surface except for an upper bound on its genus. In fact, in the limit, cancellation of several parts of currents (with opposite orientation) may produce higher connectivity of the minimizing current. On the other hand it seems plausible that under suitable geometric assumptions on Γ one might obtain embedded minimal surfaces of prescribed topological type which are bounded by Γ. In a sequence of papers starting with Gulliver and Spruck [3], the following was proved by Tomi and Tromba [1], Almgren and Simon [1], and Meeks and Yau [3]:

Theorem 2. *Let K be a strictly convex body in $\mathbb{R}^3$ whose boundary ∂K is of class C^2, and suppose that Γ is a closed rectifiable Jordan curve contained in ∂K. Then there exists an embedded minimal surface of the type of the disk which is bounded by Γ.*

All the papers cited above use different methods. Gulliver and Spruck gave the first proof with the additional requirement that the total curvature of Γ be not larger than 4π. Tomi and Tromba used methods from global analysis, while Almgren and Simon minimized area in the class of embedded disks, thereby obtaining in the limit a certain varifold which corresponds to the minimal embedded disk. Finally, Meeks and Yau proved that the minimizing surface of the type of the disk is embedded.

Moreover, in their paper [4] Meeks and Yau established a connection between the problem of embeddedness and the problem of uniqueness. First they gave a generalization of Theorem 2.

Theorem 3. *Let M be a compact region in $\mathbb{R}^3$ whose boundary is C^2-smooth and has nonnegative mean curvature with respect to the inward normal. Secondly, let Γ be a closed rectifiable Jordan curve contained in ∂M. Then any (in $\mathcal{C}(\Gamma)$) area minimizing minimal surface of disk type which is contained in M and bounded by Γ has to be embedded.*

Another result of Meeks and Yau is the following

Theorem 4. *Let $X : \Sigma \to M \subset \mathbb{R}^3$ be a minimal surface defined on a compact Riemann surface Σ with boundary, and suppose that M satisfies the assumptions of Theorem 3. Assume also that $X|_{\partial\Sigma}$ is a regular smooth embedding of $\partial\Sigma$ into ∂M which decomposes ∂M into components Σ_j, and that $X|_{\partial\Sigma}$ is homotopically trivial in the component of $M \setminus X(\Sigma)$ which contains Σ_j. Then each such component $\Gamma := X(\partial\Sigma)$ bounds an embedded stable minimal surface which is disjoint from $X(\Sigma)$ unless $X(\Sigma)$ is an embedded stable disk.*

As a consequence of this result one obtains:

Theorem 5. *If ∂M is a C^2-surface homeomorphic to S^2 and if the mean curvature of ∂M with respect to the inward normal is nonnegative, then every smooth Jordan curve Γ on ∂M either bounds at least two distinct embedded minimal disks in M, or the only immersed minimal surface $X : \Sigma \to \mathbb{R}^3$ bounded by Γ (with no restriction on the genus of Σ) is a uniquely determined stable, embedded minimal surface of the type of the disk.*

Suppose that Γ is a regular, real analytic, closed Jordan curve which lies on the boundary ∂K of a convex body, and suppose that the total curvature of Γ is less than 4π. Then Theorem 5 in conjunction with Nitsche's uniqueness theorem implies that the only minimal surface $X : \Sigma \to \mathbb{R}^3$ is a unique area minimizing disk. Clearly, this result is a considerable refinement of Nitsche's uniqueness theorem. It is unknown whether one can omit the assumption that Γ lies on a convex surface. (*Remark*: Meeks and Yau indicate that in the above conclusion ∂K need not be smooth.)

Further work in this connection has been done by F.H. Lin [3].

We also mention the fundamental paper by Ekholm, White, and Wienholtz [1] on the embeddedness of minimal surfaces. The main result of this article is

Theorem 6. *Let Γ be a closed Jordan curve in $\mathbb{R}^n$, $n \geq 3$, with total curvature $\leq 4\pi$, and let $X : \Sigma \to \mathbb{R}^n$ be a minimal surface with boundary Γ where Σ is* *a compact, 2-dimensional C^∞-manifold with boundary (i.e. $X \in C^0(\Sigma, \mathbb{R}^n)$ is harmonic and conformal in* $\operatorname{int}\Sigma$*, and $X|_{\partial\Sigma}$ maps $\partial\Sigma$ homeomorphically onto Γ). Then X is an embedding of M up to and including the boundary, with no interior branch points. If Γ is regular and of class $C^{s,\alpha}$, $s \geq 1$, $0 < \alpha < 1$, then $X \in C^{s,\alpha}(\Sigma, \mathbb{R}^n)$ is smoothly embedded and therefore has no boundary branch points.*

Furthermore, the authors point out that there are closed Jordan curves in $\mathbb{R}^3$ with total curvature $<4\pi$ that bound "minimal Möbius strips", and they make the following interesting

Conjecture. *Let Γ be a smooth, closed Jordan curve in $\mathbb{R}^3$ with total curvature $\leq 4\pi$. Then, in addition to a unique minimal disk, Γ bounds either* (i) *no other minimal surface, or* (ii) *exactly one minimal Möbius strip and no other minimal surfaces, or* (iii) *exactly two minimal Möbius strips and no other minimal surfaces.*

Returning to geometric measure theory, we denote by $\mathcal{R}_n^{(\mathrm{loc})}(U)$ for an open set U in $\mathbb{R}^m$ the set of all currents in U which locally are of integer multiplicity.

A fact of central importance concerning the Plateau problem in arbitrary dimensions and codimensions is the following compactness theorem which was first proved by Federer and Fleming [1]:

Theorem 7. *If $T_j \in \mathcal{D}_n(U)$, $j = 1, 2, \dots,$ is a sequence of integer multiplicity currents with*

$$\sup_{j\geq 1}(\mathbf{M_W}(\mathbf{T}_j) + \mathbf{M}(\partial\mathbf{T}_j)) < \infty \quad \textit{for all } \mathbf{W} \subset\subset \mathbf{U},$$

then there is a current $T \in \mathcal{R}_n^{\mathrm{loc}}(U)$ and a subsequence $\{T_{j'}\}$ converging weakly to T in U.

(The nontrivial part in the proof is to show that the limit is, in fact, of integer multiplicity.)

Employing the lower semicontinuity of mass under weak convergence of currents, one concludes by means of Theorem 7 the following existence result:

Theorem 8. *Let $S \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ be of integer multiplicity, of compact support* $\operatorname{supp} S$ *and with $\partial S = 0$. Then there is a current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with $\partial T = S$ such that* $\operatorname{supp} T$ *is compact and $\mathbf{M}(T) \leq \mathbf{M}(R)$ for all $R \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with compact support and with $\partial R = S$.*

(Here the boundary current ∂T is defined by the relation

$$\partial T(\omega) = T(d\omega) \quad \text{for all } \omega \in \mathcal{D}^{n-1}(U),$$

in analogy with Stokes's theorem.)

The next step is to examine the regularity of a minimizing current. One sets

$$\begin{aligned}\operatorname{Reg}(T) := \{x \in \operatorname{supp} T\colon & \text{ there is a neighborhood } U(x) \\ & \text{such that } \operatorname{supp} T \cap U(x) \text{ is an embedded} \\ & n\text{-dimensional submanifold } M \text{ of } \mathbb{R}^{n+k}\}\end{aligned}$$

and

$$\operatorname{Sing}(T) := \operatorname{supp} T \setminus \operatorname{Reg}(T),$$

to denote the regular and the singular part of the support of T respectively. In codimension one, the following basic regularity result was proved by Fleming [2] $(n-2)$, Almgren [1] $(n=3)$, Simons [1] $(n=4,5,6)$, and Federer [3]:

Theorem 9. *Let $U \subset \mathbb{R}^{n+1}$ be open, $T \in \mathcal{R}_n(U)$ with $\mathbf{M}(T) \le \mathbf{M}(R)$ for all R with $\operatorname{supp}(T \setminus R) \subset\subset U$. Then $\operatorname{Sing}(T \cap U)$ is empty for $n \le 6$, locally finite for $n = 7$, and $\mathcal{H}^{n-7+\alpha}$ $(\operatorname{Sing}(T \cap U)) = 0$ for all $\alpha > 0$ and $n > 7$.*

Bombieri, de Giorgi, and Giusti [1] proved that the seven-dimensional cone in $\mathbb{R}^8$ given by $\{x \in \mathbb{R}^8\colon x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$ is mass-minimizing which proves the sharpness of Theorem 8.

If the codimension is greater than one, we have the following result of Almgren [6]:

Theorem 10. *An n-dimensional, area minimizing integer multiplicity current in $\mathbb{R}^{n+k}$ is in the interior a smooth embedded manifold, except for a singular set whose Hausdorff dimension is at most $n-2$.*

This result is again sharp.

Finally the question of boundary regularity in codimension one was completely settled by Hardt and Simon [1]:

Theorem 11. *In the setting of Theorem 4, let $T \in \mathcal{R}_n(\mathbb{R}^{n+1})$ be area minimizing with an $(n-1)$-dimensional oriented submanifold S of class $C^{1,\alpha}$ as boundary. Then, near S, the support of T is an embedded C^1-manifold with boundary.*

Note that in Theorem 11 there is no restriction on the dimension n.

4 More on Uniqueness and Nonuniqueness

Let us begin with a classical example, Enneper's surface

$$X(w) = \operatorname{Re}\left(w - \frac{w^3}{3}, iw + i\frac{w^3}{3}, w^2\right), \quad w = u + iv,$$

and define the closed curve Γ_r by

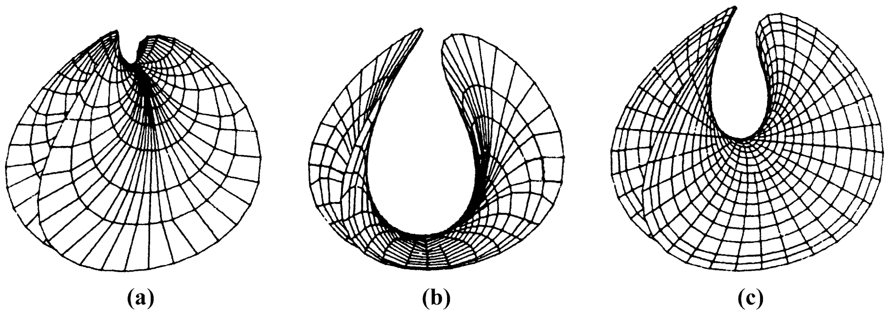


Fig. 3. A closed curve bounding a part of Enneper's surface **(c)** as well as two other minimal surfaces of the type of the disk: see **(a), (b)**. Courtesy of O. Wohlrab

$$\Gamma_r := \{X(w) \colon |w| = r\}.$$

Nitsche [14] has proved that Γ_r bounds at least two distinct minimal surfaces of the type of the disk provided that $1 < r < \sqrt{3}$, and that it bounds at least three disk-type solutions if $r_0 < r < \sqrt{3}$ where the value of r_0 is about 1.681475 (see Fig. 3). For $0 < r < 1/\sqrt{3}$ the orthogonal projection of Γ_r onto the x, y-plane is convex and one-to-one whence one concludes that Γ_r bounds exactly one disk-type surface. By a sharpened version of Nitsche's uniqueness theorem, Ruchert [1] proved uniqueness for $0 < r \le 1$. Thereafter, Beeson and Tromba [1] showed that a bifurcation occurs at $r = 1$ which is of the type of the cusp catastrophe (in Thom's morphogenesis) and that there is a number $\delta_0 > 0$ such that Γ_r bounds at least three disk-type surfaces if $1 < r < 1 + \delta_0$. By means of the estimates of Chapter 2 of Vol. 2 one can then show that Γ_r bounds *exactly* three disk-type surfaces if $1 < r < 1 + \delta_0$.

The bifurcation of minimal surfaces was also studied in a remarkable paper by Büch [1]. Starting with Weierstrass's representation formula (27) of Section 3.3 he was able to establish conditions on the Weierstrass function $\mathfrak{F}(\omega)$ which imply the appearance of bifurcations of the type of the *fold*, the *cusp*, and of the *swallow tail* (of Thom's list).

Although it is not easy to find curves which bound only one disk-type solution, the opposite problem is complicated as well, namely to verify by a rigorous mathematical proof that a given curve bounds at least two minimal surfaces. Therefore the following result of Quien and Tomi [1] might be of interest:

There exist Jordan curves Γ which are arbitrarily close to a plane and which bound (at least) a given number of geometrically distinct immersed minimal surfaces of the type of the disk.

Let us outline the *proof*. Suppose that $\varphi : S^1 = \partial B \to \mathbb{R}^2$ is an immersion of the unit circle. We begin by looking at the question as to whether φ can be extended to an immersion $f : \bar{B} \to \mathbb{R}^2$ with $f|_{\partial B} = \varphi$ and, if so, how many nonequivalent such extensions will exist (two immersions f and g are

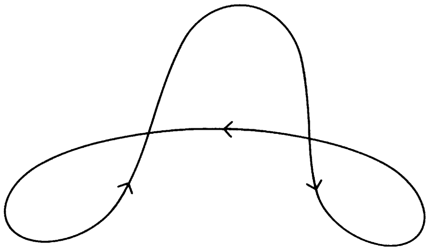
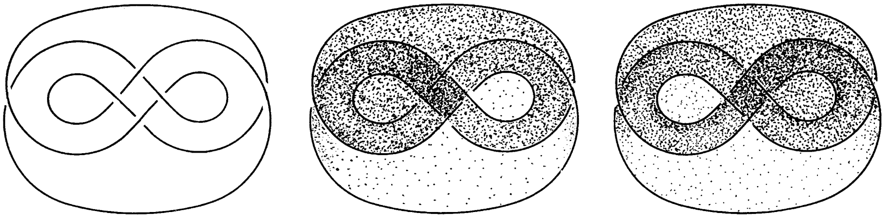
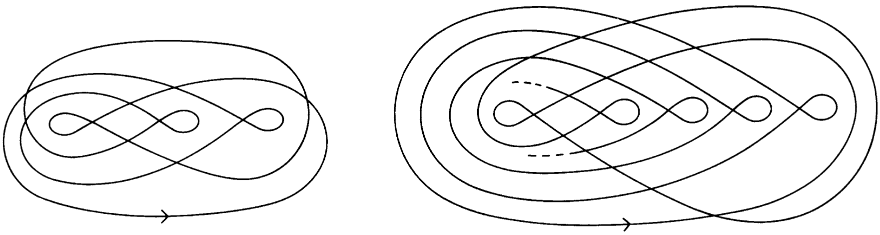


Fig. 4. An immersion $\varphi : S^1 \to \mathbb{R}^2$ which cannot be extended as an immersion $f : \bar{B} \to \mathbb{R}^2$ of the disk into $\mathbb{R}^2$



(a) **(b)**

Fig. 5. **(a)** A Milnor curve $\varphi : S^1 \to \mathbb{R}^2$ and its two extensions $f : \bar{B} \to \mathbb{R}^2$ which are immersions of the disk. **(b)** The leaves of two extensions to Milnor's curve



(a) **(b)**

Fig. 6. Milnor curves admitting **(a)** three extensions, **(b)** n extensions

equivalent if there is a diffeomorphism σ of $\bar{B}$ onto itself such that $f = g \circ \sigma$). For instance, the immersion $\varphi : S^1 \to \mathbb{R}^2$ depicted in Fig. 4 cannot be extended while Fig. 5a depicts an example due to Milnor which allows two extensions, the leaves of which are depicted in Fig. 5b. Then, in Fig. 6 we exhibit a curve with three different extensions which can inductively be improved to a curve $\varphi : S^1 \to \mathbb{R}^2$ allowing n extensions (see Fig. 6b). For a proof of these results we refer to Poénaru [1].

Let us now consider an immersion $\varphi : S^1 \to \mathbb{R}^2$ which allows n different extensions f of class $C^3(\bar{B}, \mathbb{R}^2)$. By the Lichtenstein mapping theorem we can assume that $f(u, v) = (f^1(u, v), f^2(u, v))$ is conformally parametrized, i.e., we have

$$|f_u|^2 = |f_v|^2 =: \Lambda, \quad \langle f_u, f_v \rangle = 0.$$

Next we choose a perturbation function $\psi \in C^{2,\beta}(\partial B)$, $0 < \beta < 1$, such that $F := (f^1, f^2, \psi)$ defines a Jordan curve $F : \partial B \to \mathbb{R}^3$ in $\mathbb{R}^3$. This can be achieved by a function ψ with arbitrarily small C^2-norm. Now we consider the class $\mathcal{C}$ of functions

$$Z(u,v) = (f^1(u,v), f^2(u,v), z(u,v)), \quad (u,v) \in B,$$

such that $z \in \mathrm{Lip}(\bar{B})$ and $z|_{\partial B} = \psi|_{\partial B}$. The area of $Z \in \mathcal{C}$ is given by

$$A(z) := \int_B |Z_u \wedge Z_v| \, du \, dv = \int_B \Lambda \sqrt{1 + \Lambda^{-1} |\nabla z|^2} \, du \, dv.$$

This functional is strictly convex whence there can exist at most one stationary point $x(u,v)$ of A, and the corresponding surface $X = (f^1, f^2, x)$ would be the absolute minimum of A within $\mathcal{C}$. The Euler equation of A is

$$\mathcal{L}(x) := a^{\alpha\beta} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} + b = 0$$

where we have set

$$\begin{aligned} a^{\alpha\beta} &:= (1 + \Lambda^{-1}) |\nabla x|^2 \delta^{\alpha\beta} - \Lambda^{-1} \frac{\partial x}{\partial u^\alpha} \frac{\partial x}{\partial u^\beta}, \\ b &:= -\frac{1}{2} |\nabla x|^2 \frac{\partial}{\partial u^\alpha} \Lambda^{-1} \frac{\partial x}{\partial u^\alpha}. \end{aligned}$$

For $\Lambda = 1$, the equation $\mathcal{L}(x) = 0$ is the classical minimal surface equation.

We will show that the boundary value problem

$$\mathcal{L}(x) = 0 \quad \text{in } B, \quad x = \psi \quad \text{on } \partial B$$

can be solved for boundary values ψ with a sufficiently small C^2-norm. We only have to establish a gradient estimate along ∂B for any solution since then a priori bounds for x and ∇x follow from standard estimates for scalar problems (cf. Gilbarg and Trudinger [1], Chapters 9 and 14). To derive the desired estimate we consider barrier functions of the type

$$c^{\pm}(w) := \psi(w) \pm \varepsilon(1 - |w|^2), \quad w = u + iv,$$

where $|\psi|_{C^2(\bar{B})} < \varepsilon \le 2/\sqrt{27M}, M := \max_B |\nabla \Lambda^{-1}|$. Then a brief computation will show that $\mathcal{L}(c^-) \ge 0$, and similarly we obtain $\mathcal{L}(c^+) \le 0$. Consequently ∇x can be estimated along ∂B by means of the maximum principle. This shows that, for every equivalence class $[f]$, we find a minimal immersion $X = (f^1, f^2, x)$ which is bounded by $\Gamma = F(\partial B), F = (f^1, f^2, \psi)$. □

It is still unknown whether a smooth regular Jordan curve can bound infinitely many minimal surfaces of the type of the disk (or, more generally, of

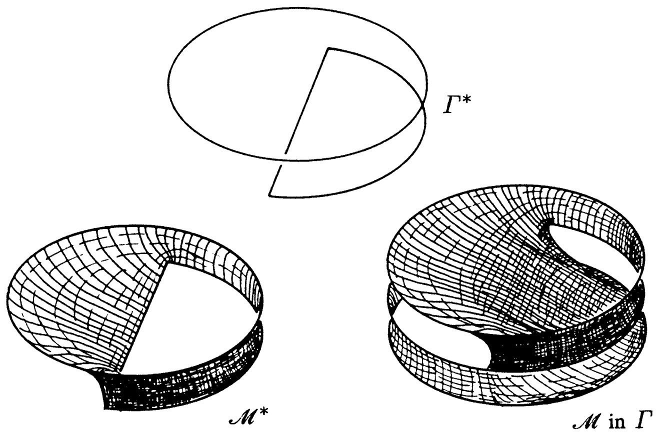


Fig. 7. Construction of a boundary configuration Γ bounding a one-parameter family of (congruent) minimal surfaces of genus zero. The rotationally symmetric configuration Γ consists of three coaxial circles $\Gamma_0, \Gamma_1, \Gamma_{-1}$

the same topological type). Note, however, that one can find boundary configurations consisting of several closed curves which even bound one-parameter families of distinct minimal surfaces of the same topological type. In fact, one can construct rotationally symmetric configurations $\Gamma = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_n \rangle$ consisting of n coaxial circles $\Gamma_1, \dots, \Gamma_n$ bounding one-parameter families of solutions. The first example of this kind was given by Morgan [3] for $n = 4$. In the paper [1] of Gulliver and Hildebrandt an example working with three circles is exhibited which will be described below. Note that $n = 3$ is the minimum number of circles for which such examples can be found since R. Schoen [3] proved that, for $n = 2$, each immersed minimal surface bounded by two coaxial circles Γ_1 and Γ_2 is either a pair of disks or a piece of a catenoid.

Now we are going to describe the construction of a rotationally symmetric 1-parameter family of minimal surfaces of genus zero which are bounded by three coaxial circles which lie in parallel planes cf. Fig. 7.

To this end we consider a configuration Γ consisting of three circles $\Gamma_0, \Gamma_1, \Gamma_{-1}$ described by the equations $x^2 + y^2 = 1$ and $z = 0, \lambda$ and $-\lambda$ respectively, $\lambda > 0$, and a second configuration Γ^* which consists of the circle Γ_1 and another closed curve γ that lies in the same plane as Γ_0, and is formed by the semicircle $\Gamma_0' = \Gamma_0 \cap \{x \geq 0\}$ and by the interval $I = \{x = 0,\ z = 0,\ -1 < y < 1\}$ on the y-axis. For small λ there is a minimal surface $\mathcal{M}^*$ of the type of an annulus bounded by Γ^* (see below). By Schwarz's reflection principle, we can extend $\mathcal{M}^*$ as a minimal surface across the straight segment I. For this purpose we rotate $\mathcal{M}^*$ by 180° about the y-axis to form a second minimal surface $\mathcal{M}^{**}$. Their union $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is a minimal surface with boundary Γ having genus zero. The segment I has become part of the interior of $\mathcal{M}$, and the surface $\mathcal{M}$ can be described by a harmonic mapping

$X : B \to \mathbb{R}^3$ given in conformal coordinates of a triply connected planar domain B. Since $\mathcal{M}^*$ is not symmetric under rotations about the z-axis, also $\mathcal{M}$ has to be rotationally nonsymmetric.

We still have to find a connected minimal surface $\mathcal{M}^*$ which is bounded by the configuration Γ^*. By virtue of J. Douglas's theorem (cf. Chapter 8), there exists an area minimizing minimal surface $\mathcal{M}^*$ which is defined on an annulus and has Γ^* as boundary, provided that λ is small enough. In fact, the existence of Douglas' solution is ascertained under the hypothesis that

$$a(\Gamma^*) < a(\gamma) + a(\Gamma_1), \tag{1}$$

where $a(\Gamma^*)$ is the greatest lower bound of area for surfaces of the type of the annulus with boundary $\Gamma^* = \gamma \cup \Gamma_1$, and $a(\gamma)$ and $a(\Gamma_1)$ are the corresponding lower bounds for disk-type surfaces bounded by γ and Γ_1 respectively. Clearly,

$$a(\gamma) = \pi/2, \quad a(\Gamma_1) = \pi,$$

and $a(\Gamma^*)$ is smaller than the area $A(S)$ of the surface S that consists of the cylinder surface between Γ_0 and Γ_1 and of the half-disk $\{x^2 + y^2 \le 1,\ x \le 0, z = 0\}$, that is,

$$a(\Gamma^*) < 2\pi\lambda + \pi/2.$$

Thus Douglas's condition (1) is satisfied for $\lambda \le 1/2$. A somewhat more complicated comparison surface S, consisting of half of a catenoid, half of a cone, and two triangles shows that even the condition $\lambda \le 0.7$ suffices to ensure the existence of a Douglas solution $\mathcal{M}^*$ within the frame Γ^*. Moreover, hypothesis (1) implies that the surface $\mathcal{M}^*$ is an immersion (cf. Gulliver [7], Theorem 10.5). By the maximum principle, the interior of $\mathcal{M}^*$ lies between two planes $z = 0$ and $z = \lambda$. Therefore the interior of $\mathcal{M}^*$ does not meet the interior of $\mathcal{M}^{**}$ where $\mathcal{M}^{**}$ is the reflection of $\mathcal{M}^*$ at the y-axis. Thus also $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is immersed. Since $\mathcal{M}$ is not rotationally symmetric, we have shown:

The configuration Γ consisting of three coaxial unit circles in parallel planes at a distance $\lambda \le 0.7$ bounds a continuum of congruent immersed minimal surfaces of genus zero.

We also note that $\mathcal{M}$ cannot have branch points on the boundary since its boundary lies on a strictly convex set, a cylinder (see Section 2 of Vol. 2).

Let us now discuss examples of rectifiable Jordan curves bounding infinitely many minimal surfaces of the type of the disk. Such examples were first described by P. Lévy [2] and R. Courant [15]; they are based on the so-called *bridge-theorem*. This is a very convincing heuristic reasoning which, in essence, amounts to the following (see Fig. 8):

Let Γ_1 and Γ_2 be two disjoint Jordan curves in $\mathbb{R}^3$. Then construct a new Jordan curve Γ by connecting Γ_1 and Γ_2 by a bridge β consisting of two arcs γ_1 and γ_2 which look like two parallel lines, and by omitting two pieces of Γ_1 and Γ_2. Suppose also that the two arcs γ_1 and γ_2 have a small distance $\varepsilon > 0$.

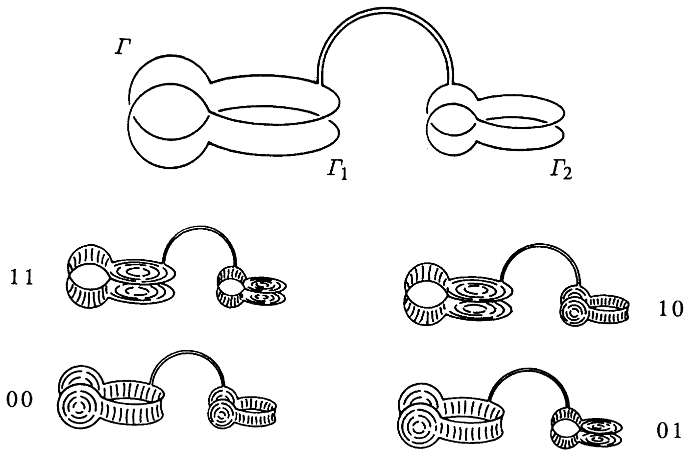


Fig. 8. Application of the bridge principle

Claim. *If X_1 and X_2 are two disk-type minimal surfaces bounded by Γ_1 and Γ_2 respectively, then there exists a disk-type surface X spanned into Γ which is close to the surface Z formed by X_1, X_2 and a small strip σ spanned into the bridge β. As ε tends to zero, the surface X converges to a geometric figure consisting of X_1, X_2 and an arc γ connecting Γ_1 and Γ_2.*

A few remarks might be appropriate:

(i) It is unlikely that the claim holds if X_1 and X_2 are unstable solutions since a very tiny perturbation of the boundary might completely destroy them. Thus one probably has to assume that X_1 and X_2 are local minimizers of area within the classes $\mathcal{C}(\Gamma_1)$ and $\mathcal{C}(\Gamma_2)$ respectively, or at least stable minimal surfaces. Even then the assertion might not be true as it stands since it is unknown if minimizers are *isolated* or not. It is conceivable that there exist *blocks* of minimizers, and therefore it might occur that, for $\varepsilon \to 0$, the surface X in the unified contour Γ approaches surfaces $\tilde{X}_1$ and $\tilde{X}_2$ in the contours Γ_1 and Γ_2 which belong to the same blocks as X_1 and X_2 but are different from these surfaces.

(ii) Very likely one has to impose restrictions on the positions of Γ_1 and Γ_2 if the bridge theorem is to hold. For instance, if Γ_1 and Γ_2 are two circles of radius 1 and 2 respectively which have the same center and lie in the same plane Π, and if β is a bridge consisting of two parallel lines joining Γ_1 and Γ_2, then there is no bridge-solution X in the joint Γ. To remove this difficulty we could, for instance, assume that the convex hulls of Γ_1 and Γ_2 are disjoint. Another option is to leave suitable freedom in the choice of the bridge and not to insist on a given pair of bridging curves γ_1 and γ_2. It might even be necessary to leave freedom for the whole curve Γ in the sense that Γ should merely be a curve close to the joint formed of Γ_1, Γ_2 and the bridge β, that is, we might have to wiggle the joint a little bit. Possibly we might also have to smoothen the corners to make the procedure work.

We know of several (published and unpublished) attempts to establish a rigorous version of the bridge theorem. Neither Courant nor Levy indicated how to go about this task. The first paper containing such a proof was written by Courant's student M. Kruskal [1]; however, his reasoning turned out to be incomplete. Another very promising attack was carried out by Meeks and Yau in their paper [4] dealing with the connection between uniqueness and embeddedness on which we have reported in Subsection 3 of these Scholia. However, we are not able to follow all of their arguments, and we think that possibly a more detailed discussion might be needed to establish a good bridge theorem that will imply the existence of curves bounding infinitely many minimal surfaces of the type of the disk. We have to mention that N. Smale [1] gave a satisfactory proof of a bridge principle; however, his result is of no use for the construction of contours bounding many or even infinitely many solutions of Plateau's problem because he constructs Γ not only in dependence on Γ_1 and Γ_2 but also in dependence on two (stable) minimal surfaces X_1 and X_2 within Γ_1 and Γ_2. That means that, if we pick different surfaces $\tilde{X}_1$ and $\tilde{X}_2$ in Γ_1 and Γ_2, N. Smale's construction will lead to another joint $\tilde{\Gamma}$ which, in general, will differ from the joint Γ. At last the matter was settled by B. White [21,22] who proved fairly general versions of the bridge principle.

Let us now turn to the (heuristic) Levy–Courant construction. We take a contour Γ_1 which bounds at least two stable disk-type minimal surfaces such as in Fig. 8. Next we consider a sequence $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ of curves of the same kind, selected in such a way that Γ_2 is half the size of Γ_1, the curve Γ_3 is half the size of Γ_2, and so on (see Fig. 9). Then we join Γ_1 and Γ_2 by a bridge β_1, Γ_2 and Γ_3 by a bridge β_2 etc. such that a rectifiable Jordan arc Γ is formed. Each Γ_j spans two stable surfaces which we say to be of type 0 or 1. Pick for each Γ_j one of these two numbers. Then we obtain a sequence $A = \{a_j\}$ of digits $a_j = 0$ or 1, and to any such sequence there corresponds a stable disk-type minimal surface X_A bounded by Γ which in Γ_j is close to a surface of the type a_j. Hence $A \neq A'$ implies that $X_A \neq X_{A'}$, and we have found a bijective mapping $\tau : A \to X_A$ of all binary representations of the interval $[0, 1]$ onto the set of geometrically different minimal surfaces bounded by Γ. In other words, if we are willing to accept a strong bridge principle applying to infinitely many curves, the above reasoning yields the following result (see Fig. 9):

There exist rectifiable Jordan curves Γ which bound nondenumerably many minimal surfaces of the type of the disk.

In fact, the construction seems to imply that one can choose Γ as a regular C^∞-curve except for a single kink.

It would be very interesting to make the Levy–Courant construction precise with the aid of B. White's versions of the bridge principle.

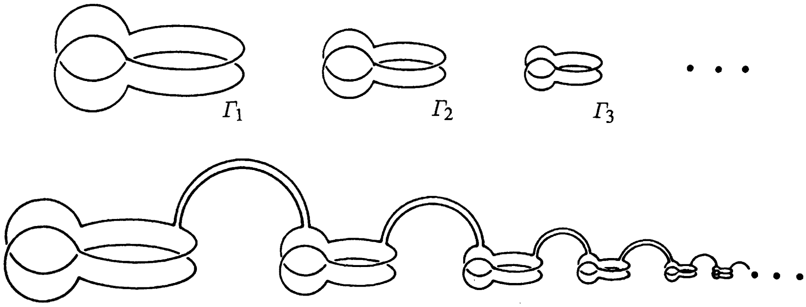


Fig. 9. Construction of a curve Γ bounding nondenumerably minimal surfaces of the type of the disk

The *finiteness question* is a truly fundamental problem. J.C.C. Nitsche conjectured that every reasonable (i.e., smooth, analytic, ...) curve Γ bounds only finitely many minimal surfaces of disk-type. Despite the generic finiteness result of Böhme–Tromba mentioned in Section 4.9, this question is completely open. It would be very desirable to obtain *upper and lower bounds for the number of solutions of Plateau's problem.*

According to J.C.C. Nitsche [31,32], *a regular, real analytic Jordan curve Γ bounds only finitely many minimal surfaces of disk-type if its total curvature does not exceed* 6π*, and if every disk-solution for Γ is free of branch points* (cf. also Beeson [5]).

Nitsche indicated that instead of $\Gamma \in C^\omega$ the assumption $\Gamma \in C^{3,\alpha}$ is sufficient. A version of the 6π-theorem is proved in Section 5.7 (cf. 5.7, Theorem 3 and Remark 10).

Important contributions to the finiteness problem were also given in the papers [3,4] of M. Beeson. In this context, we mention the papers of R. Böhme [1,5], and of Böhme and Tomi [1] who started to investigate the structure of the space of solutions for Plateau's problem. Major progress in this direction was achieved in Böhme–Tromba's papers [1] and [2] where a fundamental *index theorem* was derived. This index theorem has in the meantime been carried over to various cases of the general Plateau problem (cf. Thiel [1–3], Schüffler [6], Schüffler and Tomi [1], and finally Tomi and Tromba [6]).

In this respect we also have to mention the work on *unstable minimal surfaces in a given contour.* In particular, we refer to the work of Courant which is described in Chapter 6 of his treatise [15], and the generalizations of his work given by E. Heinz [13,14], G. Ströhmer [1–4], and F. Sauvigny [3–6]. In Chapter 6, we present a version of Courant's approach to unstable minimal surfaces that also uses ideas due to E. Heinz. In the Scholia to Chapter 6 as well as in Vol. 3, Chapter 6, further results concerning the existence of unstable minimal surfaces will be described, in particular the work of M. Struwe.

Here we mention the following uniqueness theorem by Sauvigny [3]: *Let Γ be a polygon of total curvature less than* 4π *which lies on the boundary of* *a bounded convex set of $\mathbb{R}^3$. Then Γ bounds exactly one disk-type minimal surface, and this solution is free of branch points up to the boundary.*

Interestingly, Sauvigny could generalize his uniqueness result to $\mathbb{R}^n$, $n \geq 4$, under the assumption that the total curvature of Γ is less than $10\pi/3$. This generalization was possible since Sauvigny did not work with a field construction but with the so-called *Courant function* $d(\tau)$ and with the *Marx–Shiffman function* $\theta(\tau)$. The function $d(\tau)$ was introduced by Courant in his monograph [15], pp. 223–236, where it plays an important role in his treatment of unstable minimal surfaces with polygonal boundaries. On the other hand, Heinz in his subsequent basic work [19–24] emphasized the role of the *Marx-Shiffman function* $\theta(\tau)$. The functions $d(\tau)$ and $\theta(\tau)$ are defined as follows. Let Γ be a polygon with $N+3$ vertices e_j, $1 \leq j \leq N+3$. Consider mappings $X : \bar{B} \to \mathbb{R}^3$ of class $C^\circ(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ such that $X(-1) = e_{N+1}$, $X(-i) = e_{N+2}$, $X(1) = e_{N+3}$ and $X(e^{i\tau_j}) = e_j$, $1 \leq j \leq N$, where $\tau = (\tau_1, \dots, \tau_n)$ is an N-tupel of parameter values τ_j satisfying $0 < \tau_1 < \tau_2 < \cdots \tau_N < \pi$. Let $\mathcal{F}(\tau)$ be the class of such mappings which map the arc $C_k := \{e^{i\theta} \colon \tau_k \leq \theta \leq \tau_{k+1}\}$ into the straight line Γ_k through the points e_k and e_{k+1}, whereas $\mathcal{F}'(\tau)$ denotes the subset of mappings $X \in \mathcal{F}(\tau)$ which map C_k weakly monotonically onto the interval $[e_k, e_{k+1}]$ on $\Gamma_k(\tau_j = \tau_k, e_j = e_k$ if $j \equiv k \bmod N+3)$. We set

$$
\begin{aligned}
d(\tau) &:= \inf\{D(X) \colon X \in \mathcal{F}'(\tau)\},\\
\theta(\tau) &:= \inf\{D(X) \colon X \in \mathcal{F}(\tau)\}.
\end{aligned}
$$

Then we clearly have $d(\tau) \geq \theta(\tau)$, and simple examples show that we can have $d(\tau) > \theta(\tau)$ for certain values of τ (see F. Lewerenz [1]). The function $d(\tau)$ is of class C^1, and its critical points correspond bijectively to the solutions of Plateau's problem of disk-type bounded by the polygon Γ. In this way, Plateau's problem for polygonal boundaries is connected with the critical points of a function of finitely many variables. Unfortunately it is unknown whether $d(\tau)$ is of class C^2; therefore Courant's function is not suited to develop a Morse theory. The situation is much better for the function $\theta(\tau)$. Heinz [20,23] proved that $\theta(\tau)$ is real analytic and that its critical points correspond to solutions of a *generalized Plateau problem* for Γ (generalized means: the solution X can overshoot the vertices, and we only know that $X(C_k) \subset \Gamma_k$). The *Morse index* of such generalized solutions was computed by Sauvigny [4], by studying the second derivative of the function θ. Note that the two functions $d(\tau)$ and $\theta(\tau)$ are closely connected as they coincide in the critical points of $d(\tau)$.

We have presented some of the results by Courant and Shiffman as well as extensions by Heinz, Sauvigny, and Jakob in Chapter 6 (note that there the functions d and θ are denoted by Θ and Θ^* respectively).

Uniqueness theorems and finiteness questions for minimal surfaces in Riemannian manifolds and for H-surfaces were discussed by Ruchert [2], Koiso [1,4,6], and Quien [1].

5 Index Theorems, Generic Finiteness, and Morse-Theory Results

In this subsection we sketch some results for minimal surfaces which in Sections 5 and 6 of Vol. 3 are developed in detail.

Let B be the unit disk and $S^1 = \partial B$. For integers r and s, $r \geq 2s+4$, define

$$\mathcal{D} = \mathcal{D}^s = \{u : S^1 \to S^1 \colon \deg u = 1 \text{ and } u \in H^s(S^1, \mathbb{C})\},$$

where H^s denotes the Sobolev space of s-times differentiable functions with values in $\mathbb{C}$; set

$$\mathcal{A} = \{\alpha : S^1 \to \mathbb{R}^n \colon \alpha \in H^r(S^1, \mathbb{R}^n),\ \alpha \text{ an embedding}\}$$

(i.e. α is one-to-one and $\alpha'(\xi) \neq 0$ for all $\xi \in S^1$), and let the total curvature of $\Gamma^\alpha = \alpha(S^1)$ be bounded by $\pi(s-2)$.

Denote by $\pi : \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ the projection map onto the first factor. A minimal surface $X : \bar{B} \to \mathbb{R}^n$ spanning $\alpha \in \mathcal{A}$ can be viewed as an element of $\mathcal{A} \times \mathcal{D}$, since X is harmonic and therefore determined by its boundary values

$$X|_{S^1} = \alpha \circ u, \quad \text{where } (\alpha, u) \in \mathcal{A} \times \mathcal{D}.$$

The classical approach to minimal surfaces is to understand the set of minimal surfaces spanning a given fixed wire α; that is, the set of minimal surfaces in $\pi^{-1}(\alpha)$. The approach of Böhme–Tomi–Tromba is to first understand the structure of the subset of minimal surfaces in the bundle $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ viewed as a fiber bundle over $\mathcal{A}$, and then to attack the question of the set of minimal surfaces in the fiber $\pi^{-1}(\alpha)$ in terms of the singularities of the projection map π restricted to a suitable subvariety of $\mathcal{N}$. This is in the spirit of Thom's original approach to unfoldings of singularities.

Let us say that a minimal surface $X \in \mathcal{A} \times \mathcal{D}$ has branching type (λ, ν), $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{Z}^p$, $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{Z}^q$, $\lambda_i, \nu_i \geq 0$ if X has p distinct but arbitrarily located interior branch points $w_1, \ldots, w_p$ in B of integer orders $\lambda_1, \ldots, \lambda_p$ and q distinct boundary branch points $\xi_1, \ldots, \xi_q$ in S^1 of (even) integer orders $\nu_1, \ldots, \nu_q$. In a formal sense, the subset $\mathcal{M}$ of minimal surfaces in $\mathcal{N}$ is an algebraic subvariety of $\mathcal{N}$ which is a stratified set, stratified by branching types. To be more precise, let $\mathcal{M}^\lambda_\nu$ denote the minimal surfaces of branching type (λ, ν). Then we have the following index result of Böhme and Tromba [2].

Index theorem for disk surfaces. *The set $\mathcal{M}^\lambda_0$ is a C^{r-s-1}-submanifold of $\mathcal{N}$, and the restriction π^λ of π to $\mathcal{M}^\lambda_0$ is of class C^{r-s-1}. Moreover, π^λ is a Fredholm map of index $I(\lambda) = 2(2-n)|\lambda| + 2p + 3$, where $|\lambda| = \sum \lambda_i$.*

Moreover, locally, for $\nu \neq 0$, we have $\mathcal{M}^\lambda_\nu \subset \mathcal{W}^\lambda_\nu$ where $\mathcal{W}^\lambda_\nu$ is a submanifold of $\mathcal{N}$ and where the restriction π^λ_ν of π to $\mathcal{W}^\lambda_\nu$ is Fredholm of index $I(\lambda,\nu) = 2(2-n)|\lambda| + (2-n)|\nu| + 2p + q + 3$, $|\nu| = \sum \nu_i$. The number 3 comes from the equivariance of the problem under the action of the three dimensional conformal group of the disk.

Ursula Thiel [3] has shown that if one uses weighted Sobolev spaces as a model, the sets $\mathcal{M}_\nu^\lambda$ can indeed be given a manifold structure with the index of $\pi_\nu^\lambda := \pi|\mathcal{M}_\nu^\lambda$ being $I(\lambda,\nu)$.

These stratification and index results are the basis to prove the generic finiteness and stability of minimal surfaces of the type of the disk as discussed in Böhme and Tromba [2]: There exists an open dense subset $\hat{\mathcal{A}} \subset \mathcal{A}$ such that if $\alpha \in \hat{\mathcal{A}}$, then there exists only a finite number of minimal surfaces bounded by α, and these minimal surfaces are stable under perturbations of α. If $n > 3$, they are nondegenerate critical points of Dirichlet's integral. The open set $\hat{\mathcal{A}}$ will be the set of regular values of the map π. Moreover we have the following

Remark. If $n > 3$, the minimal surfaces spanning $\alpha \in \hat{\mathcal{A}}$ are all immersed up to the boundary, and if $n = 3$, they are at most simply branched.

Schüffler [1–4,6,8], Schüffler and Tomi [1], and Thiel [1,2] have extended the index theorem in various directions. Tomi and Tromba [6] have obtained an index theorem for higher genus minimal surfaces employing the Teichmüller theory; cf. Vol. 3, Chapters 4 and 5.

Finally, these results are also essential for a Morse theory for disk surfaces.

Let $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ be the bundle over $\mathcal{A}, \alpha \in \mathcal{A}$, and let $\Gamma^\alpha = \alpha(S^1)$ be the image of such an embedding. Consider the manifold of maps $H^s(S^1, \Gamma^\alpha)$. In A. Tromba [5] it is shown that $H^s(S^1,\Gamma^\alpha)$ is a C^{r-s}-submanifold of $H^2(S^1,\mathbb{R}^n)$. Let $\mathcal{N}(\alpha)$ denote the component of $H^s(S^1,\Gamma^\alpha)$ determined by α. We can identify $\mathcal{N}(\alpha)$ with the set of mappings $X \in C^0(\bar{B},\mathbb{R}^n)$ which are harmonic in B and whose boundary values $X|_{\partial B}$ yield a parametrization of Γ^α. Then the Dirichlet functional $E_\alpha : \mathcal{N}(\alpha) \to \mathbb{R}$ is defined by

$$E_\alpha(X) = \frac{1}{2}\int_B |\nabla X|^2 \, du \, dv.$$

We know by the index theorem that *there exists an open dense set of contours $\hat{\mathcal{A}} \subset \mathcal{A}$, $\mathcal{A} \subset H^r(S^1,\mathbb{R}^n)$, $n \geq 4$, such that if $\alpha \in \hat{\mathcal{A}}$, there are only a finite number of nondegenerate minimal surfaces $X_1,\dots,X_m$ spanning α. Let $D^2E_\alpha(X_i) : T_{X_i}\mathcal{N}(\alpha) \times T_{X_i}\mathcal{N}(\alpha) \to \mathbb{R}$ denote the Hessian of Dirichlet's functional at X_i, and be λ_i the dimension of the maximal subspace on which $D^2E_\alpha(X_i)$ is negative definite. Then A. Tromba* [11] *proved the Morse equality*

$$\sum_i (-1)^{\lambda_i} = 1. \tag{1}$$

A version of this formula which holds in $\mathbb{R}^3$ was developed by A. Tromba in his papers [10,11]. The theory leading to these results is presented in Chapter 6 of Vol. 3.

The full Morse inequalities in the case $n \geq 4$ were established by Struwe [4], who proved

$$\sum_{\lambda=0}^{l} (-1)^{l-\lambda} m_\lambda \geq (-1)^l$$

and

$$m_0 \geq 1$$

where m_λ is the number of minimal surfaces of Morse index λ.

However, the case $n = 3$ remains open since the generic nondegeneracy assumption is known not to hold (see Böhme and Tromba [2]). Here only Tromba's version of formula (1) is known.

6 Obstacle Problems

The minimization procedure can also be used to solve *obstacle problems*, that is, to find surfaces of minimal area (or of a minimal Dirichlet integral) which are spanning a prescribed boundary configuration and avoid certain open sets (obstacles). In other words, the competing surfaces X of the variational problem are confined to certain closed subsets of $\mathbb{R}^3$ (or, more generally, to closed subsets of the target manifold M of the mappings $X : B \to M$). Problems of this kind were treated by F. Tomi [2–4], S. Hildebrandt [12,13], and Hildebrandt and Kaul [1]. One can also consider obstacle problems where the obstacle is *thin*. (In elasticity theory these problems are called *Signorini problems*.) In the context of minimal surfaces such problems occur naturally if we consider free or partially free boundary problems with a supporting surface S. If S has a nonempty boundary, then we can view S as part of a larger surface S_0 without boundary, and the part $S_0 \setminus S$ can be considered as an obstacle since the boundary values of the competing surfaces X are confined to S. The existence theory for such boundary problems with a thin obstacle can be carried along the lines of Chapters 4 and 5, and no additional difficulties will arise. The boundary behavior of solutions of such problems will be investigated in the Vols. 2 and 3.

Presently we shall confine our attention to *thick obstacles* in $\mathbb{R}^3$ (or M) which are to be avoided by the admissible surfaces. To describe some of the results, consider the functionals

$$\mathcal{F}_B(X) := E_B(X) + V_B(X)$$

where

$$E_B(X) := \frac{1}{2} \int_B g_{jk}(X)(X_u^j X_u^k + X_v^j X_v^k)\, du\, dv,$$

$$V_B(X) := \int_B \langle Q(X), X_u \wedge X_v \rangle \, du\, dv$$

that is,

$$\mathcal{F}_B(X) = \int_B e(X, \nabla X)\, du\, dv$$

with the Lagrangian

$$e(x, p) = \tfrac{1}{2} g_{jk}(x)(p_1^j p_1^k + p_2^j p_2^k) + \langle Q(x), p_1 \wedge p_2 \rangle$$

where $x \in \mathbb{R}^3$ and $p = (p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Furthermore let $\mathcal{C}^*$ denote one of the classes $\mathcal{C}^*(\Gamma)$ or $\mathcal{C}^*(\Gamma, S)$, i.e., the set of surfaces bounded by Γ or $\langle \Gamma, S \rangle$ respectively which are normalized by a three point condition, see Sections 4.2 and 4.6. Suppose that $K \subset \mathbb{R}^3$ is a closed set; then we put $C = C(K, \mathcal{C}^*) := \mathcal{C}^* \cap H_2^1(B, K)$, where $H_2^1(B, K)$ denotes the subset of functions $f \in H_2^1(B, \mathbb{R}^3)$ which map almost all of B into K. We consider the variational problem $\mathcal{P}(\mathcal{F}, C)$ given by

$$\mathcal{F} \to \min \quad \text{in } C.$$

Theorem. *Suppose that $Q \in C^0(K, \mathbb{R}^3)$, $g_{ij} \in C^0(K, \mathbb{R})$, $g_{ij} = g_{ji}$, $i, j \in \{1, 2, 3\}$, and let $0 < m_0 \le m_1$ be numbers with the property*

$$m_0 |p|^2 \le e(x, p) \le m_1 |p|^2 \quad \text{for all } (x, p) \in K \times \mathbb{R}^6. \tag{1}$$

Moreover assume that K is a closed set in $\mathbb{R}^3$ such that $C = C(K, \mathcal{C}^*)$ is nonempty. Then the variational problem $\mathcal{P}(\mathcal{F}, C)$ has (at least) one solution in $C(K, \mathcal{C}^*)$.

Proof. The following three statements have to be verified:

(i) The class $C(K, \mathcal{C}^*)$ is a weakly closed subset of $H_2^1(B, \mathbb{R}^3)$.
(ii) There exists a minimizing sequence $X_n \in C(K, \mathcal{C}^*)$ for $\mathcal{P}(\mathcal{F}, C)$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ to some $X \in H_2^1(B, \mathbb{R}^3)$.
(iii) The functional $\mathcal{F}_B(\cdot)$ is weakly lower semicontinuous in $H_2^1(B, \mathbb{R}^3)$.

(i)–(iii) immediately imply that X is an element of C furnishing a solution of $\mathcal{P}(\mathcal{F}, C)$; in fact, (iii) yields

$$\mathcal{F}_B(X) \le \liminf_{n \to \infty} \mathcal{F}_B(X_n) = e = \inf_C \mathcal{F}_B,$$

and hence $\mathcal{F}_B(X) = e$.

Property (i) follows from the weak closedness of $\mathcal{C}^*$, from a theorem of Rellich and from the fact that one can extract from any L_2-convergent sequence a subsequence which converges pointwise almost everywhere, cf. Theorem 2 in Section 4.6.

Statement (ii) is a consequence of the fact that for any minimizing sequence of surfaces $X_n \in C(K, \mathcal{C}^*)$ there holds an estimate

$$\|X_n\|_{H_2^1(B)} \le \text{const}$$

which follows from the ellipticity condition (1) and from a suitable Poincaré inequality. Finally (iii) is a special case of a general lower semicontinuity result of Serrin, see Morrey [8], Theorem 1.8.2. □

In addition to the preceding theorem we have the following result concerning *conformal parameters*.

Proposition 12. *Any solution $X \in C(K, \mathcal{C}^*)$ of the variational problem $\mathcal{P}(\mathcal{F}, C)$ satisfies almost everywhere in B the conformality relations*

$$g_{ij} X_u^i X_u^j = g_{ij} X_v^i X_v^j \quad \text{and} \quad g_{ij} X_u^i X_v^j = 0.$$

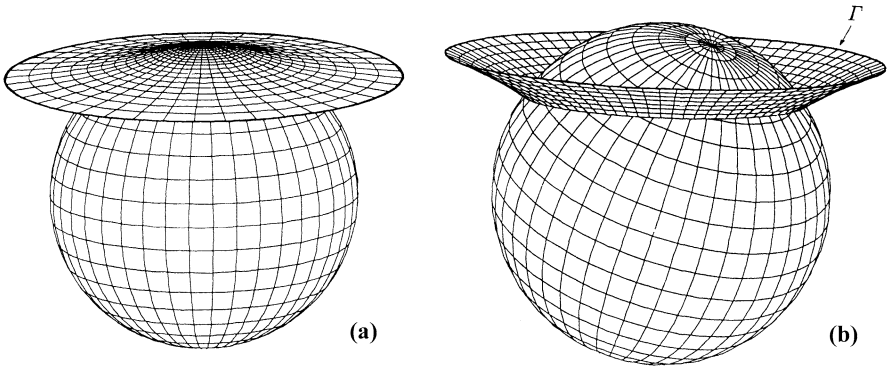


Fig. 10. **(a)** The obstacle problem is to find a surface of least Dirichlet integral which is bounded by a Jordan curve Γ and which remains outside given solid bodies. The minimizers among the disk-type surfaces are minimal surfaces away from the obstacle, but where they touch it they may have non-zero mean curvature. **(b)** If multiply connected surfaces with free boundaries on the obstacle are also admitted as comparison surfaces, then smaller Dirichlet integrals can be achieved and the minimizers will be perpendicular to the obstacle along their free boundaries

The proof of this result is obtained by a suitable adaptation of the argument given in Sections 4.5 and 4.10.

In the special case where $g_{ij} = \delta_{ij}$ and $Q = 0$ we conclude from the above theorems the existence of a minimal surface X bounded by Γ or $\langle \Gamma, S \rangle$ respectively which is spanned over the obstacle ∂K. Note that, in general, the coincidence set $\mathfrak{T} = \{w \in B \colon X(w) \in \partial K\}$ will be a nonempty subset of B. If $\mathfrak{T}$ is nonempty, then a soap film corresponding to X touches the surface ∂K of the obstacle. If one allows the film to change its topological type by, say, admitting a number of holes, it can slide down on ∂K, thereby reducing its area (see Fig. 10). The corresponding surfaces $X : \Omega \to \mathbb{R}^3$ will then be defined on a multiply connected parameter domain $\Omega \subset \mathbb{C}$ and have free boundaries on ∂K. This phenomenon was treated by Tolksdorf in his paper [1] where he proved the existence of a minimum X for the functional

$$\tilde{D}(X) = \int_B |\tilde{\nabla} X|^2 \, du \, dv$$

with

$$\tilde{\nabla} X(u,v) := \begin{cases} \nabla X(u,v) & \text{if } X(u,v) \notin \partial K, \\ 0 & \text{if } X(u,v) \in \partial K \end{cases}$$

in a suitably chosen class of comparison functions. For details we refer the reader to Tolksdorf's paper.

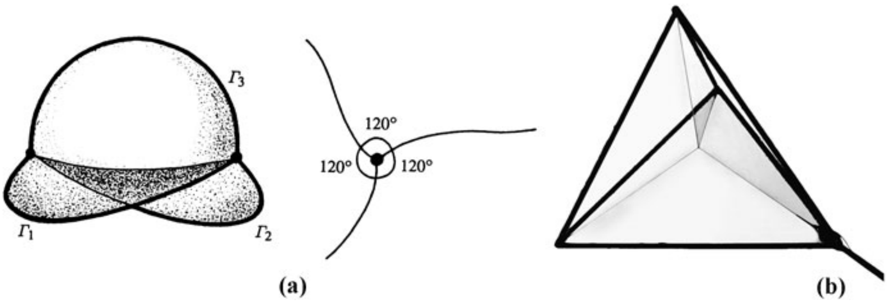


Fig. 11. (a) Rule 1, **(b)** Rule 2 demonstrated by a system of 6 soap-films in tetrahedron

7 Systems of Minimal Surfaces

Usually one encounters soap films and soap bubbles in the shape of *foam*. Roughly speaking, foam is a system of soap films and soap bubbles which are attached to each other and meet at common *liquid edges*. More than a hundred years ago Plateau observed in experiments that such systems obey two simple rules which he stated in his treatise [1]. Let us formulate these rules simply for systems of minimal surfaces ($H = 0$), neglecting systems of bubbles ($H = \text{const} \neq 0$) and mixed systems.

A *system of minimal surfaces* is a connected set which is a finite union of smooth regular manifolds of zero mean curvature which sit in a given frame and meet each other at free boundary curves called liquid edges. These liquid edges form the singular part of the minimal surface system.

Rule 1. *At each liquid edge meet exactly three minimal surfaces of the system, and any two of them enclose an angle of* 120 *degrees.*

Rule 2. *Liquid edges can meet at supersingular points p. Each supersingular point is the meeting point of exactly four liquid edges. Any two adjacent edges form an angle* $\varphi = 109°28'16''$ (*precisely speaking,* $\cos\varphi = -1/3$).

These two principles are illustrated by Fig. 11.

The first rigorous proof for the two rules governing systems of minimal surfaces was given by J. Taylor [2] using the means of geometric measure theory. Let us briefly outline her arguments. Consider a system S of minimal surfaces which is bounded by a closed system Γ of Jordan arcs $\Gamma_1, \Gamma_2, \dots,$ and assume that S minimizes area within all other systems bounded by Γ. In the first step, a *monotonicity formula* is employed to prove the existence of tangent cones T_pS at each point p of S. Moreover, it is verified that each tangent cone (which in general is not known to be unique) is again area minimizing for the frame formed by the intersection of the cone with the unit sphere S^2 centered at p. Such a frame is a system of arcs on S^2, and each arc is part of a great circle. At any vertex of such a system only three arcs

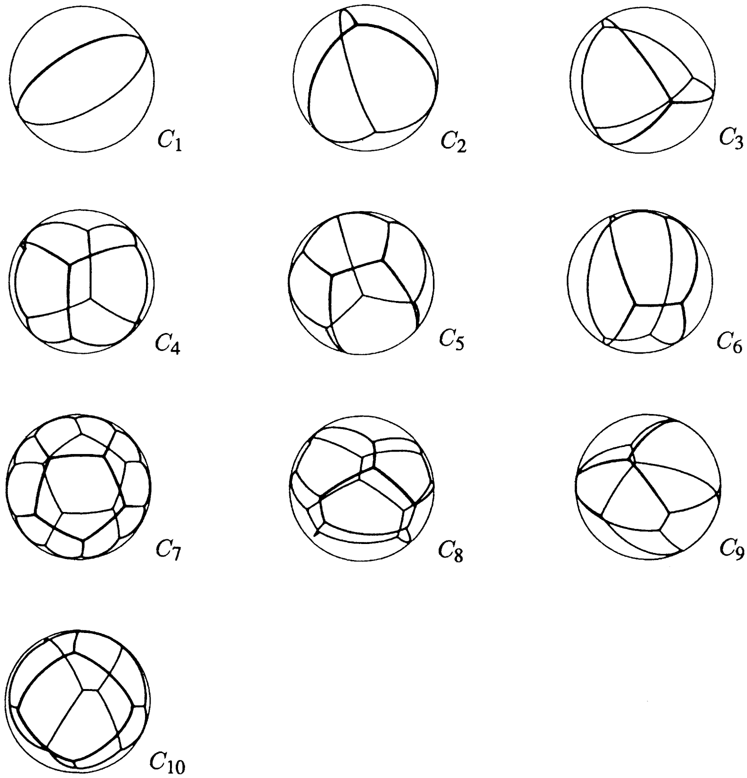


Fig. 12. The ten geodesic nets on S^2

can run together, and any two of them form an angle of 120°. A frame on S^2 with these properties is called a *geodesic net*. Thus, in order to classify all area minimizing cones, one first looks at the simpler question of determining all (equiangular) geodesic nets on S^2. It turns out that exactly ten different such nets exist. This classification was already carried out by Lamarle who, however, missed one net. The complete list, depicted in Fig. 12, was given by Heppes [1]. According to Lamarle and Heppes, the ten (equiangular) geodesic nets $C_1, \dots, C_{10}$ can be described as follows:

(a) C_1 is a great circle;
(b) C_2 consists of three halves of great circles with common endpoints;
(c) C_3 is a spherical tetrahedron;
(d) C_4 is a spherical cube;
(e) C_5 consists of 15 arcs forming the 1-skeleton of the prism over the regular pentagon;
(f) C_6 is a prism over a regular triangle and consists of 9 arcs;
(g) C_7 is a spherical dodecahedron made of 30 arcs;

(h) C_8 consists of 24 arcs forming two regular quadrilaterals and 8 congruent pentagons;
(i) C_9 is formed by 18 arcs which determine 4 equal pentagons and 4 equal quadrilaterals;
(j) C_{10} consists of 21 arcs forming three regular quadrilaterals and six congruent pentagons.

Now that we know $C_1, \dots, C_{10}$, the crucial question of determining all area minimizing tangent cones is reduced to the problem of finding out which of the cones over C_j with their vertex at p are area minimizing. Jean Taylor proved that C_1, C_2 and C_3 are minimizers whereas the cones over $C_4, \dots, C_{10}$ are not even stable. The mathematical proof is rather elaborate whereas the physical demonstration of this fact is easily provided by a soap film experiment which is depicted in Hildebrandt and Tromba [1], pp. 128–129. The pictures show that the area minimizing soap films in C_1, C_2 and C_3 are cones but not those in $C_4, \dots, C_{10}$. Thus we are led to the following

Theorem of J. Taylor. *Let S be a system of minimal surfaces which is bounded by a closed system of Jordan arcs and minimizes area within its boundary. Then the following holds true*:

(i) *At each point $p \in S$ there exists a unique tangent cone which is congruent to one of the cones* (a), (b) *or* (c) *in Fig.* 12.
(ii) *Let $\mathcal{R}(S) := \{p \in S$: the tangent cone to S at p is congruent to* (a)$\}$ *denote the regular part of S. Then $\mathcal{R}(S)$ is a two-dimensional manifold in $\mathbb{R}^3$. Each component of $\mathcal{R}(S)$ has mean curvature zero.*
(iii) *Let $\Sigma(S) := \{p \in S$: the tangent cone to S at p is congruent to* (b)$\}$ *denote the set of singular points in S. Then $\Sigma(S)$ is a one-dimensional $C^{1,\alpha}$-manifold in $\mathbb{R}^3$ for some $\alpha \in (0,1)$. There exists a neighborhood $\mathcal{U}(p)$ for each $p \in \Sigma(S)$ and a conformal $C^{1,\alpha}$-diffeomorphism f of $\mathbb{R}^3$ onto itself such that $\mathcal{U} \cap S$ is the image of* (b) *under f.*
(iv) *Let $\sigma(S) := \{p \in S$: the tangent cone to S at p is congruent to* (c)$\}$ *denote the set of supersingular points in S. Then $\sigma(S)$ consists of isolated points. Furthermore, for each $p \in \sigma(S)$ there exists a neighborhood $\mathcal{U}(p)$ in $\mathbb{R}^3$ and a conformal $C^{1,\alpha}$-diffeomorphism f of $\mathbb{R}^3$ onto itself such that $\mathcal{U}(p) \cap S$ is the image of (c) under f.*
(v) *The system S decomposes into $S = \mathcal{R}(S) \cup \Sigma(S) \cup \sigma(S)$.*

For the proof of this result we refer to J. Taylor [2]. The above theorem also extends to systems of surfaces of constant mean curvature as well as to systems of surfaces which are extremals of some functional which is close to the area functional in a suitable sense.

Nitsche [33] proved that the singular part $\Sigma(S)$ is a union of regular C^∞-curves, and Kinderlehrer, Nirenberg, and Spruck [1] even showed that $\Sigma(S)$ is a union of real analytic curves.

It is fairly easy to prove the existence of area minimizing systems in a given frame Γ by means of geometric measure theory.

We finally note that, under rather restrictive symmetry assumptions on the boundary Γ, the existence of area minimizing systems S and the regularity of their singular parts had earlier been established by A. Solomon [1,2].

8 Isoperimetric Inequalities

Historical remarks on this topic and further comments can be found in the Scholia to Chapter 4 of Vol. 2.

9 Plateau's Problem for Infinite Contours

It is also of interest to determine minimal surfaces which are not bounded by one or several loops, but by one or several arcs, finite or infinite. Actually, already Riemann [2] developed a method to construct minimal surfaces which are simply connected and have straight line segments as boundaries. As examples he studied two infinite straight lines which are not contained in a plane ([1], §15), two infinite half lines meeting at a common endpoint and an infinite straight line parallel to the plane of the first two ([1], §16), three pairwise skew lines ([1], §17). As a main idea to solve these three problems as well as Plateau's problem for the skew quadrilateral ([1], §18), Riemann used the fact that the surface normal of a solution maps any straight segment of the boundary onto an arc of a great circle on S^2. This work was generalized by E. Neovius [1–5]. In this context we also refer to the treatises of Darboux [1] and Bianchi [1,2].

The problem of determining minimal surfaces with prescribed unbounded contours was anew taken up by López and Wei [1], López and Martín [1], and Ferrer and Martín for unbounded polygonal boundaries. For a fairly general class of unbounded contours the problem was recently solved by F. Tomi [13]. The curves Γ considered by Tomi are described by $\Gamma = \xi(\mathbb{R})$, where ξ provides a noncompact proper embedding of $\mathbb{R}$ into $\mathbb{R}^3$ which is piecewise of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and satisfies $\xi(0) = 0$ and $|\xi'(s)| = 1$ as well as the following conditions:

(i) There is a constant $\delta > 0$ such that $|p - q| \leq \delta$ for all p, q contained in different components of $\Gamma \setminus \Gamma_1$ where Γ_1 is the connected components of $\Gamma \cap B_1$ containing 0.

(ii) Let $\gamma(s) := |\xi(s)|^{-1}\xi(s)$ for $s \neq 0$. Then

$$|\langle \gamma(x), \xi'(x)\rangle| \to 1 \quad \text{as } |s| \to \infty.$$

(iii) $\int_{\Gamma\setminus\Gamma_1} |\xi(s)|^{-1}\sqrt{1 - \langle \gamma(s), \xi'(s)\rangle^2}\, ds < \infty.$

Then Tomi's theorem reads as follows:

There exists a proper mapping $X \in C^0(\overline{H}, \mathbb{R}^3) \cap C^\infty(H, \mathbb{R})$ *of a closed half-plane* H *in* $\mathbb{R}^2$ *which is an immersed minimal surface on* H *and maps* ∂H *in a strictly monotonic way onto* Γ.

Tomi's class of admissible curves Γ contains all properly embedded curves with polynomial ends. The main idea of the proof is to work with surfaces whose area in a ball of radius R growth at most quadratically in R.

10 Plateau's Problem for Polygonal Contours

(Added in Proof, May 2010)

In her recent thesis (Dec. 4, 2009) Laura Desideri [1] has rectified and supplemented Garnier's approach to Plateau's problem for polygonal boundaries. She proved the following beautiful theorem:

Let Γ *be a polygon in* $\mathbb{R}^3 \cup \{\infty\}$ *with* $n+3$ *sides in "generic position", possibly with one of its vertices lying at infinity. Then* Γ *bounds an immersed minimal surface* $X : \mathbb{C}_+ \to \mathbb{R}^3$ *defined on the upper halfplane* $\mathbb{C}_+$ *in the sense that* Γ *is the boundary of the image* $X(\mathbb{C}_+)$. *If* Γ *has a vertex at infinity, then the immersion* X *has a helicoidal end at this vertex.*

This result contributes also to the problems discussed in No. 2 and No. 9. Furthermore, Desideri has proved an analog of the above theorem for the Plateau problem in Minkowski space.

Chapter 5

Stable Minimal- and H-Surfaces

Solving Plateau's problem in the preceding chapter we concentrated our attention to a solution X of this problem, and we somewhat neglected its Gauss mapping N, the surface normal of X. However, the mapping N turns out to be continuous even in case of a branched solution X, and so it is seen to be a real analytic surface of constant mean curvature one. As it will be very useful to study the pair (X, N) together and not X alone, we are invited to enlarge our spectrum and to investigate directly surfaces of prescribed mean curvature. This will enable us in Chapter 7 to solve the *nonparametric equation of prescribed mean curvature* via the solution of Plateau's problem for parametric surfaces of prescribed mean curvature. Using and extending the ideas presented in Chapter 4, this more general Plateau problem for H-surfaces will be solved in Vol. 2, Chapter 4. In order to shorten the presentation of this chapter we shall strongly rely on the treatise of F. Sauvigny [16] as well as on Vol. 2. Especially the control of the boundary regularity will be indispensable for our considerations.

In Section 5.1 we derive the basic equation for the Gauss map N of an H-surface $X : B \to \mathbb{R}^3$ and prove that N is a classical—and in particular continuous solution of this equation. In Section 5.2 we study a substitute for the Weingarten mapping S introduced in Section 1.2, namely *Bonnet's mapping* $R : T_w X \to T_w X$, which leads to the definition of *Bonnet's surface* $Y : B \to \mathbb{R}^3$ for a constant mean curvature surface (= cmc-surface). This surface again is a cmc-surface of mean curvature one provided that not all points of X are umbilical points, and it might give more information than N if properly exploited.

The *stability* of H-surfaces is discussed in Section 5.3 by means of the second variation $\delta^2 F(X, \varphi N)$, $\varphi \in C_c^\infty(B)$, of a functional F defined by

$$(1) \qquad F(X) = A(X) + 2V(X), \quad V(X) := \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv$$

where the associated vector field $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the equation

U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*,
Grundlehren der mathematischen Wissenschaften 339,
DOI 10.1007/978-3-642-11698-8_5,

(2) $$\operatorname{div} Q(x) = H(x).$$

Although Q is not uniquely determined by H, the stability condition depends only on H: X is *stable* if

(3) $$\int_B |\nabla\varphi|^2 \, du \, dv \geq \mu \int_B p\varphi^2 \, du \, dv \quad \text{for all } \varphi \in C_c^\infty(B)$$

and $\mu = 2$ with

(4) $$p := \Lambda[2H^2(X) - K - \langle H_x(X), N\rangle], \quad \Lambda := |X_u \wedge X_v|,$$

while *strict stability* of X means that (3) holds true with $\mu > 2$.

The central result of this section states that the stability of X together with the monotonicity condition

(5) $$\frac{\partial H}{\partial e} = \langle H_x, e\rangle \geq 0 \quad \text{for some } e \in S^2$$

and the boundary condition $\langle N, e\rangle > 0$ on ∂B implies $\langle N, e\rangle > 0$ on $\overline{B}$.

In Section 5.4 a kind of converse is proved for immersed cmc-surfaces satisfying $\langle N, e\rangle > 0$ on $\overline{B}$ for some $e \in S^2$ as they prove to be strictly stable. Furthermore a cmc-surface X is strictly stable if its density function $p = 2H^2 - K$ satisfies

(6) $$\int_B (2H^2 - K)\Lambda \, du \, dv < 2\pi.$$

For minimal surfaces ($H = 0,\ K \leq 0$) this condition means

(7) $$\int_B |K| \, dA < 2\pi.$$

Finally Gulliver's estimate

(8) $$A(X) \leq \frac{2\mu}{2\mu - 1}\pi r^2$$

is established for any μ-stable, immersed cmc-surface $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$, $\mu > 1/2$, representing a geodesic disk $K_r(x_0)$ of radius r. This leads to the curvature estimate

(9) $$\kappa_1^2(0) + \kappa_2^2(0) \leq c(h_0) r^{-2}$$

with a universal constant $c(h_0)$ proved in Theorems 1 and 2 of Section 5.5. This estimate holds for all stable, immersed cmc-surfaces X with $X(0) = 0$ and $|H| \leq h_0$ which represent a geodesic disk $K_r(0)$ of radius r around the origin. The estimate (9) implies a "Bernstein-type" result that was first stated by do Carmo and Peng [1] and by Fischer-Colbrie and Schoen [1].

In Section 5.6 the uniqueness theorem of J.C.C. Nitsche is proved, after establishing the perturbation equation for a field embedding and constructing the field immersion of a strictly stable immersed minimal surface that can be slightly extended beyond its boundary.

5.1 H-Surfaces and Their Normals

In Theorem 1 of Section 2.6 we have seen that a regular (i.e. immersed) surface $X \in C^2(\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^2$, that satisfies the conformality relations

$$(1) \qquad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0$$

is a surface of mean curvature $\mathcal{H}(u,v)$ at $(u,v) \in \Omega$ if and only if X satisfies *Rellich's equation*

$$(2) \qquad \Delta X = 2\mathcal{H} X_u \wedge X_v.$$

Suppose now that $H(x)$ is a prescribed scalar-valued function of $x \in \mathbb{R}^3$ which is of class $C^{0,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$. Then a C^2-solution X of (2) with $\mathcal{H} := H \circ X$ and satisfying (1) will be called a *surface of prescribed mean curvature H in* $\mathbb{R}^3$. As for minimal surfaces we will also consider *branched surfaces* of this kind, i.e. we allow points $w \in \Omega$ where the function Λ, defined by

$$(3) \qquad \Lambda := |X_u|^2 = |X_v|^2 = |X_u \wedge X_v| = \mathcal{W},$$

is vanishing. Such points are again called **branch points** of X. As usual we write $H(X)$ for the composed function $H \circ X$. Summarizing we give the following

Definition 1. *A nonconstant solution $X \in C^2(\Omega, \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^2$, of*

$$(4) \qquad \Delta X = 2H(X) X_u \wedge X_v,$$

satisfying the conformality relations (1)*, will be called an H-surface. We speak of an immersed H-surface X if Λ given by*

$$|dX|^2 = \Lambda \cdot (du^2 + dv^2)$$

satisfies

$$(5) \qquad \Lambda(u,v) > 0 \quad \textit{for all points } (u,v) \in \Omega.$$

If $H(x) \equiv$ const we may address X as a constant H-surface; *also the notation cmc-surface is common.*

All notions of these definitions pertain to the class $C^2(\overline{\Omega}, \mathbb{R}^3)$. Usually we shall investigate *disk-type H-surfaces*, i.e. the parameter domain Ω will in most cases be the unit disk

$$B := \{w = (u,v) \in \mathbb{R}^2 \colon |w| < 1\},$$

and often the complex notation $w = u + iv \in \mathbb{C}$ is used.

Remark 1. Suppose that $H \in C^{r,\alpha}(\mathbb{R}^3)$, $r \geq 0$, $\alpha \in (0,1)$. Then any solution $X \in C^2(\Omega, \mathbb{R}^3)$ of (4) is of class $C^{r+2,\alpha}(\Omega, \mathbb{R}^3)$. This result follows from elliptic theory; see e.g. Sauvigny [16], Chapter IX, §4.

Remark 2. Let $H \in C^{0,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$, and suppose that X is an H-surface of class $C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ such that $X(\partial B)$ lies on a regular Jordan curve Γ of class $C^{2,\alpha}$. Then $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$. If $H \in C^{r-2,\alpha}(\mathbb{R}^3)$ and $\Gamma \in C^{r,\alpha}$, $r \geq 2$, it follows that $X \in C^{r,\alpha}(\overline{B}, \mathbb{R}^3)$. For a proof see Vol. 2, Section 7.3.

Remark 3. Let X be an H-surface of class $C^{2,\alpha}(B, \mathbb{R}^3)$ or $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ respectively. Then, for each point $w_0 \in B$ or $\overline{B}$, there is a vector $A \in \mathbb{C}^3$ with $A \neq 0$ and $\langle A, A \rangle = 0$, and a nonnegative integer $n = n(w_0)$ such that

$$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n) \quad \text{as } w \to w_0. \tag{6}$$

A proof of this fact by means of the Hartman-Wintner technique is given in Vol. 2, Section 2.10 (using Section 3.1 of Vol. 2). Another proof can be found in Sauvigny [16], Chapter XII, §10 which is based on the theory of "generalized analytic functions" (see Sauvigny [16], Chapter IV). The point w_0 is a branch point of X if and only if $n(w_0) \geq 1$, and $n(w_0) \geq 1$ is called the **order of the branch point** $w_0 \in B$ (or $\overline{B}$ respectively). The point w_0 is a regular point of X if and only if $n(w_0) = 0$.

Formula (6) implies that branch points w_0 of an H-surface $X \in C^{2,\alpha}(B, \mathbb{R}^3)$ or $\in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ are isolated in B or $\overline{B}$ respectively. In the first case there are at most finitely many branch points in any $\Omega \subset\subset B$, and in the second case there are at most finitely many branch points $w_1, \dots, w_{k+\ell} \in \overline{B}$, say, $w_1, \dots, w_k \in B$ and $w_{k+1}, \dots, w_{k+\ell} \in \partial B$. The points $w_1, \dots, w_k$ are the **inner branch points** of X, and $w_{k+1}, \dots, w_{k+\ell}$ the **boundary branch points** of the H-surface X.

The first fundamental form ds^2 of an H-surface X is given by

$$\begin{aligned} ds^2 &= \langle dX, dX \rangle = \Lambda(du^2 + dv^2) \\ &= 2\langle X_w, X_{\overline{w}} \rangle (du^2 + dv^2) = 2|X_w|^2(du^2 + dv^2). \end{aligned} \tag{7}$$

The **set of regular points of** $X \in C^2(\overline{B}, \mathbb{R}^3)$, denoted by $\overline{B}'$, is given by

$$\overline{B}' = \{w \in \overline{B} \colon \Lambda(w) > 0\} = \overline{B} \setminus \{w_1, \dots, w_{k+\ell}\}. \tag{8}$$

An important tool to cope with branch points analytically is the subsequent

Proposition 1. *There exists a sequence* $\{\chi_n\}$ *of functions* $\chi_n \in C_c^\infty(\overline{B}')$ *with* $0 \leq \chi_n \leq 1$ *satisfying*

$$\lim_{n\to\infty} \chi_n(w) = 1 \quad \textit{for all } w \in \overline{B}' \quad \textit{and} \quad \lim_{n\to\infty} D(\chi_n) = 0. \tag{9}$$

Proof. For $0 < r < R$ we define the functions $\phi(w)$ by $\phi(w) := 1$ for $|w| \le r$, $\phi(w) := 0$ for $|w| \ge R$, and

$$\phi(w) := \frac{\log|w| - \log R}{\log r - \log R} \quad \text{for } r < |w| < R.$$

Then $0 \le \phi \le 1$ and, for any $R > 0$,

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 \, du \, dv = \frac{2\pi}{\log(R/r)} \to 0 \quad \text{as } r \to +0.$$

By mollifying these functions we can construct a sequence $\{\phi_n\}$ of functions $\phi_n \in C^\infty(\mathbb{R}^2)$ with $0 \le \phi_n \le 1$ such that $\phi_n(w) = 0$ for $|w| \ge R_n$, $R_n \to 0$, $\phi_n(w) = 1$ for $|w| \le r_n$, $0 < r_n < R_n$, and $\int_{\mathbb{R}^2} |\nabla \phi_n|^2 \, du \, dv \to 0$ as $n \to \infty$. Furthermore we have $\phi_n(w) \to 0$ for all $w \ne 0$.

Finally we define $\chi_n \in C^\infty(\overline{B})$ for $n \in \mathbb{N}$ by

$$\chi_n(w) := \prod_{\nu=1}^{k+\ell} [1 - \phi_n(w - w_\nu)], \quad w \in \overline{B}.$$

Obviously the sequence $\{\chi_n\}$ possesses the desired properties. □

Now we consider the normal N of an H-surface X near a branch point $w_0 \in \overline{B}$. A straight-forward calculation yields

$$N = \Lambda^{-1} X_u \wedge X_v = \frac{-i}{\langle X_w, X_{\overline{w}} \rangle} X_w \wedge X_{\overline{w}}. \tag{10}$$

Inserting the asymptotic expansion (6) with $A = a - ib$, $a, b \in \mathbb{R}^3$, $|a| = |b|$, $\langle a, b \rangle = 0$, we obtain

$$N(w) \to |a|^{-2} a \wedge b \in S^2 \quad \text{for } w \in \overline{B}' \text{ with } w \to w_0. \tag{11}$$

Therefore the normal N can be extended continuously from $\overline{B}'$ into the branch points of X, i.e. $N \in C^0(\overline{B}, \mathbb{R}^3)$ with $N(\overline{B}) \subset S^2$. Furthermore, N is of class $C^{2,\alpha}$ on B' and $C^{1,\alpha}$ on $\overline{B}'$. F. Sauvigny [1,2] proved that N is even of class $C^{2,\alpha}$ on B and established the following

Theorem 1. *The normal N to an H-surface $X \in C^{3,\alpha}(B, \mathbb{R}^3)$ is of class $C^{2,\alpha}(B, \mathbb{R}^3)$ and satisfies the differential equation*

$$\Delta N + 2pN = -2\Lambda H_x(X) \tag{12}$$

with

$$p := 2\Lambda H^2(X) - \Lambda K - \Lambda \langle H_x(X), N \rangle. \tag{13}$$

For the term ΛK involving the Gaussian curvature K of X we have

$$\Lambda K \in C^{1,\alpha}(B).$$

Proof. (i) Equation (12) with (13) is derived on $B' := B \setminus \{w_1, \ldots, w_k\}$ in Sauvigny [16], Chapter XII, §9 (see Proposition 2). If the reader wants to check it, he finds the necessary formulae from classical differential geometry in Chapter 1 above, particularly in Section 1.3.

Using the Weingarten equations one obtains on $\overline{B}'$:

$$
\begin{aligned}
|\nabla N|^2 &= |N_u|^2 + |N_v|^2 = \Lambda^{-1}(\mathcal{L}^2 + 2\mathcal{M}^2 + \mathcal{N}^2) \\
&= \Lambda^{-1}[(\mathcal{L} + \mathcal{N})^2 - 2(\mathcal{L}\mathcal{N} - \mathcal{M}^2)] \\
&= 4\Lambda H^2(X) - 2\Lambda K = 2[2\Lambda H^2(X) - \Lambda K].
\end{aligned}
\tag{14}
$$

Invoking the evident orthogonal expansion

$$
\Lambda H_x(X) = \langle H_x(X), X_u\rangle X_u + \langle H_x(X), X_v\rangle X_v + \Lambda\langle H_x(X), N\rangle N
\tag{15}
$$

we transform the differential equation (12) into the following equivalent form:

$$
\Delta N + N|\nabla N|^2 + f(X, \nabla X) = 0 \quad \text{in } B',
\tag{16}
$$

with

$$
f(X, \nabla X) := 2[\langle H_x(X), X_u\rangle X_u + \langle H_x(X), X_v\rangle X_v].
\tag{17}
$$

(ii) By the Gauss–Bonnet theorem (see Vol. 2, Section 2.11, Theorem 1 and in particular Remark 2) it follows that $\int_B |K|\, dA = \int_B \Lambda|K|\, du\, dv$ is finite. Then (14) implies that

$$
\int_B |\nabla N|^2 \, du\, dv < \infty.
\tag{18}
$$

With the aid of a "smoothing sequence" $\{\chi_n\}$ from Proposition 1 we now derive a weak differential equation for N in B, using (16). To this end we choose an arbitrary test function $\phi \in C_c^\infty(B, \mathbb{R}^3)$, multiply (16) by $\phi \cdot \chi_n$, and perform an integration by parts. Then

$$
\begin{aligned}
&\int_B \langle \nabla N, \nabla(\phi \cdot \chi_n)\rangle \, du\, dv \\
&\quad = \int_B \langle N, \phi\rangle |\nabla N|^2 \chi_n \, du\, dv + \int_B \langle f(X, \nabla X), \phi\rangle \chi_n \, du\, dv.
\end{aligned}
\tag{19}
$$

In this identity we want to let n tend to infinity. First we consider the left-hand side; we have

$$
\begin{aligned}
&\int_B \langle \nabla N, \nabla(\phi \cdot \chi_n)\rangle \, du\, dv \\
&\quad = \int_B \langle \nabla N, \nabla \phi\rangle \chi_n \, du\, dv + \int_B \langle \nabla N, \phi \nabla \chi_n\rangle \, du\, dv,
\end{aligned}
$$

and Schwarz's inequality yields

$$\begin{aligned}
&\left|\int_B \langle \nabla N, \phi \nabla \chi_n \rangle \, du \, dv\right| \\
&\quad \le \left\{\int_B |\nabla N|^2 \, du \, dv\right\}^{1/2} \left\{\int_B |\phi \nabla \chi_n|^2 \, du \, dv\right\}^{1/2} \\
&\quad \le 2 \sup_B |\phi| \sqrt{D(N)} \sqrt{D(\chi_n)} \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

on account of (18) and $D(\chi_n) \to 0$. Furthermore, $\chi_n(w) \to 1$ on B'. Then (18) and Lebesgue's convergence theorem imply

$$\int_B \langle \nabla N, \nabla \phi \rangle \chi_n \, du \, dv \to \int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv$$

whence

$$\int_B \langle \nabla N, \nabla(\phi \cdot \chi_n) \rangle \, du \, dv \to \int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv.$$

For the same reason the right-hand side of (19) tends to

$$\int_B \langle N, \phi \rangle |\nabla N|^2 \, du \, dv + \int_B \langle f(X, \nabla X), \phi \rangle \, du \, dv$$

as $n \to \infty$, and so we infer from (19) that

$$\int_B \langle \nabla N, \nabla \phi \rangle \, du \, dv = \int_B \{\langle N, \phi \rangle |\nabla N|^2 + \langle f(X, \nabla X), \phi \rangle\} \, du \, dv. \tag{20}$$

Since N is already known to be continuous on B (and even on $\overline{B}$), and $f(X, \nabla X) \in C^\alpha(B, \mathbb{R}^3)$, a regularity result by Ladyzhenskaya and Uraltseva [1,2] implies $N \in C^{2,\alpha}(B, \mathbb{R}^3)$; for a simple proof of this fact see F. Tomi [1].

Finally, equation (14) leads to

$$-\Lambda K = \tfrac{1}{2} |\nabla N|^2 - 2\Lambda H^2(X), \tag{21}$$

and therefore $\Lambda K \in C^{1,\alpha}(B)$. □

Remark 4. Although we know that $N \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{2,\alpha}(\overline{B} \setminus \Sigma', \mathbb{R}^3)$, $\Sigma' = \{w_{k+1}, \dots, w_{k+\ell}\}$ = set of boundary branch points, we do not know whether $\lim_{w \to w'} \nabla N(w)$ or even $\lim_{w \to w'} \nabla^2 N(w)$ exist for $w' \in \Sigma'$. An answer to this question seems to be complicated but valuable.

5.2 Bonnet's Mapping and Bonnet's Surface

In this section we briefly want to discuss *Bonnet's fundamental form* associated with any H-surface, and the *Bonnet surface* associated with a cmc-surface. The Bonnet surface provides valuable information on the umbilical points of a cmc-surface and can serve as a useful substitute for the Gauss mapping N. It might prove to be useful in further investigations.

For the notations to be used in the sequel we refer to Sections 1.1 and 1.2, and also to the brief introduction to the differential-geometric formulae given in Sauvigny [16], Chapter XI, §1.

Let $S(w) : T_w X \to T_w X$ be the *Weingarten mapping* associated with an arbitrary H-surface $X : \overline{B} \to \mathbb{R}^3$. At each regular point $w \in \overline{B}$ (i.e. for $w \in \overline{B}'$) this mapping is a selfadjoint linear mapping of the tangent space $T_w X$ of X corresponding to w (or, less precisely, the tangent space of the surface X at the point $X(w)$). Secondly, let

$$I(w) : T_w X \to T_w X \quad \text{with } I(w)V = V \text{ for } V \in T_w X$$

be the identity on $T_w X$.

Definition 1. *Let $X : \overline{B} \to \mathbb{R}^3$ be an H-surface of class $C^{2,\alpha}$. Then, for any regular point $w \in \overline{B}$ of X, we define the* **Bonnet mapping**

$$R(w) : T_w X \to T_w X$$

by

$$R(w) := H(X(w))I(w) - S(w). \tag{1}$$

Remark 1. Clearly the Bonnet mapping $R(w)$ is a selfadjoint linear operator on $T_w X$ with the two eigenvalues $\lambda_1(w)$ and $\lambda_2(w)$, given by

$$\lambda_1(w) = H(X(w)) - \kappa_1(w), \quad \lambda_2(w) = H(X(w)) - \kappa_2(w),$$

where $\kappa_1(w)$ and $\kappa_2(w)$ are the principal curvatures of X at $w \in \overline{B}'$. Since $2H(X(w)) = \kappa_1(w) + \kappa_2(w)$, we obtain

$$\lambda_1(w) = \tfrac{1}{2}[\kappa_2(w) - \kappa_1(w)], \quad \lambda_2(w) = \tfrac{1}{2}[\kappa_1(w) - \kappa_2(w)]. \tag{2}$$

Therefore the Bonnet mapping has a vanishing trace,

$$\operatorname{tr} R(w) = 0 \quad \text{for all } w \in \overline{B}', \tag{3}$$

and from

$$\det R = \lambda_1 \lambda_2 = -\tfrac{1}{4}[\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2] = -\left[\tfrac{1}{4}(\kappa_1 + \kappa_2)^2 - \kappa_1\kappa_2\right]$$

it follows for $w \in \overline{B}'$ that

$$\det R(w) = -[H^2(X(w)) - K(w)] = -\lambda_1^2(w) = -\lambda_2^2(w). \tag{4}$$

Since $\lambda_1(w) = -\lambda_2(w)$, the Bonnet map $R(w)$ is either indefinite or the zero mapping. Clearly $R(w) = 0$ if and only if $\kappa_1(w) = \kappa_2(w)$, that is, $R(w)$ vanishes exactly at the umbilical points $w \in \overline{B}'$ of the H-surface X. Furthermore, $R^*R = \lambda_1^2 I$ since $\lambda_1^2 = \lambda_2^2$, and so we obtain the fundamental identity

$$R^*(w)R(w) = [H^2(X(w)) - K(w)]I(w) \quad \text{for all } w \in \overline{B}' \tag{5}$$

with

$$H^2(X(w)) - K(w) = \lambda_1^2(w) = \lambda_2^2(w) \geq 0. \tag{6}$$

Since $SX_u = -N_u$, $SX_v = -N_v$, one obtains

$$RX_u = N_u + H(X)X_u, \quad RX_v = N_v + H(X)X_v.$$

Set $M := (X_u, X_v)$ and multiply (5) from the right by M and from the left by M^*. Then the right-hand side becomes

$$[H^2(X) - K] \cdot \begin{pmatrix} |X_u|^2 & \langle X_u, X_v\rangle \\ \langle X_u, X_v\rangle & |X_v|^2 \end{pmatrix} = \Lambda[H^2(X) - K] \cdot I$$

whereas the left-hand side becomes

$$M^*R^*RM = \begin{pmatrix} \mu & \tau \\ \tau & \nu \end{pmatrix}$$

with

$$\begin{aligned} &\mu := |N_u + H(X)X_u|^2, \quad \nu := |N_v + H(X)X_v|^2, \\ &\tau := \langle N_u + H(X)X_u, N_v + H(X)X_v\rangle. \end{aligned} \tag{7}$$

Thus

$$\mu = \nu = \Lambda \cdot [H^2(X) - K], \quad \tau = 0. \tag{8}$$

Definition 2. *With any H-surface $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ we associate the quadratic form*

$$d\sigma^2 = \langle N_{u^\alpha} + H(X)X_{u^\alpha}, N_{u^\beta} + H(X)X_{u^\beta}\rangle\, du^\alpha\, du^\beta, \tag{9}$$

$u^1 := u$, $u^2 := v$, which is called **Bonnet's fundamental form**.

The formulae (7)–(9) imply that Bonnet's fundamental form is conformal to the first fundamental form $|dX|^2 = \Lambda(du^2 + dv^2)$. More precisely, since $X, N \in C^1(B, \mathbb{R}^3)$, we obtain:

Theorem 1. *Bonnet's fundamental form $d\sigma^2$ of an H-surface X with the Gauss curvature K can be written as*

$$d\sigma^2 = \Lambda[H^2(X) - K](du^2 + dv^2), \quad \Lambda = |X_u|^2. \tag{10}$$

This quadratic form is positive semidefinite and vanishes exactly at those points $w \in B$ which are either umbilical or branch points of X.

Definition 3. *With any H-surface $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ of constant mean curvature H and any vector $Y_0 \in \mathbb{R}^3$ we associate a new surface, the* **Bonnet surface** *$Y \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{2,\alpha}(B, \mathbb{R}^3)$ of X, which is defined by*

$$Y(w) := N(w) + HX(w) + Y_0 \quad \textit{for } w \in \overline{B}. \tag{11}$$

As a consequence of Theorem 1 we obtain the following result:

Corollary 1. *The Bonnet surface $Y = N + HX + Y_0$ of any cmc-surface $X \in C^2(B, \mathbb{R}^3)$ satisfies*

$$|dY|^2 = \Lambda(H^2 - K)(du^2 + dv^2) = d\sigma^2 \quad \textit{in } B \tag{12}$$

whence in particular

$$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0 \quad \textit{in } B \tag{13}$$

and

$$|Y_u \wedge Y_v| = \Lambda \cdot (H^2 - K). \tag{14}$$

Remark 2. The Bonnet surface Y of a cmc-surface X degenerates exactly on the set Σ of branch points of X in B and the set Σ^* of umbilical points of X in B. Whereas the points of Σ are isolated, the set Σ^* might have nonisolated points. Even $\operatorname{int} \Sigma^*$ can be nonvoid as in the case of a planar surface or a spherical cap. Note however that, by regularity theory, each cmc-surface X is real analytic in B, and so $\operatorname{int} \Sigma^* \neq \emptyset$ implies $H^2 - K(w) \equiv 0$ on $B \setminus \Sigma$, i.e. all points $w \in B \setminus \Sigma$ are umbilical points. This implies that X is either planar or a spherical surface.

Theorem 2. *The Bonnet surface Y of a cmc-surface X is either a constant mapping or a cmc-surface of mean curvature one. In the first case all points of X are umbilical, i.e. X is either planar or a spherical surface, while in the second case X has only isolated umbilical points in B, and the normal $\tilde{N}$ of Y coincides with $-N$ where N is the normal of X.*

Proof. On account of Theorem 1 in Section 5.1 we have

$$\Delta N = -4\Lambda H^2 N + 2\Lambda K N \quad \text{in } B,$$

and furthermore

$$\Delta X = 2H\Lambda N \quad \text{in } B.$$

This implies

$$\Delta Y = -2\Lambda(H^2 - K)N \quad \text{in } B. \tag{15}$$

In addition, the relations (4) and (6) imply

$$\det R(w) = -(H^2 - K(w)) \le 0.$$

Thus, by (4) and $Y_u = RX_u$, $Y_v = RX_v$ it follows that

$$Y_u \wedge Y_v = -(H^2 - K) X_u \wedge X_v$$

whence

$$Y_u \wedge Y_v = -\Lambda(H^2 - K)N \quad \text{in } B. \tag{16}$$

From (15) and (16) one finally infers

$$\Delta Y = 2Y_u \wedge Y_v \quad \text{in } B, \tag{17}$$

and the formulae (13) of Corollary 1 state that

$$|Y_u|^2 = |Y_v|^2, \quad \langle Y_u, Y_v \rangle = 0 \quad \text{in } B.$$

Then the Hartman–Wintner theorem states that either (i) $Y(w) \equiv \text{const}$ in B, or (ii) $Y(w)$ is nowhere locally constant in B, and the branch points of Y are isolated. In case (i) the surface is either planar or spherical, while in case (ii) the surface X has at most isolated umbilical points, and Y is a cmc-surface of mean curvature one. Moreover, in this case the surface normal $\tilde{N}$ of Y is defined by

$$\tilde{N} := \frac{1}{|Y_u \wedge Y_v|}(Y_u \wedge Y_v) \quad \text{on } B \setminus (\Sigma \cup \Sigma^*)$$

and can be extended continuously to all of B.

Note also that (14) and (16) imply

$$Y_u \wedge Y_v = -|Y_u \wedge Y_v| N,$$

whence $\tilde{N} = -N$ on B. □

Remark 3. For any cmc-surface X, its Bonnet surface Y "realizes" the Bonnet fundamental form $d\sigma^2$ of X via the formula (12). For an H-surface X with variable H one cannot expect to find a similar realization of its $d\sigma^2$ since the set of umbilical points of X might be very general.

Remark 4. For a cmc-surface X with $H \neq 0$, the associated Bonnet surface Y provides a suitable substitute for the Gauss map N of X.

Remark 5. Let Y be the Bonnet map of a cmc-surface X with $H \neq 0$, and set

$$Z := X + \frac{1}{H} N. \tag{18}$$

Then $Y = HZ$, and it follows that

$$\Delta Z = 2HZ_u \wedge Z_v, \quad |Z_u|^2 = |Z_v|^2, \quad \langle Z_u, Z_v \rangle = 0.$$

Therefore we obtain O. Bonnet's result that (18) *defines a second H-surface parallel to X*, except for a spherical X when Z reduces to a point.

5.3 The Second Variation of F for H-Surfaces and Their Stability

As already mentioned in No. 3 of the *Supplementary Results* to Section 4.5, H-surfaces are closely related to certain functionals $E := D + 2V$ that generalize Dirichlet's integral D. In fact if H is a given scalar function on $\mathbb{R}^3$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field on $\mathbb{R}^3$,

$$Q(x) = (Q^1(x), Q^2(x), Q^3(x)), \quad x = (x^1, x^2, x^3) \in \mathbb{R}^3,$$

such that

$$\text{(1)} \qquad \operatorname{div} Q = H, \quad \text{i.e.} \quad Q^1_{x^1} + Q^2_{x^2} + Q^3_{x^3} = H,$$

then any H-surface $X : B \to \mathbb{R}^3$ is a stationary point of the functional $E = D + 2V$ where

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv$$

is the *Dirichlet integral* of X and V denotes a *volume integral* defined by

$$\text{(2)} \qquad V(X) = \int_B \langle Q(X), X_u \wedge X_v \rangle \, du \, dv.$$

Introducing the trilinear product

$$[a, b, c] = \det(a, b, c) = a \cdot (b \wedge c) = b \cdot (c \wedge a) = c \cdot (a \wedge b)$$

we can write V as

$$\text{(3)} \qquad V(X) = \int_B [Q(X), X_u, X_v] \, du \, dv.$$

In Vol. 2, Chapter 4, we shall construct H-surfaces within a prescribed boundary contour Γ by minimizing the functional[1]

$$\text{(4)} \qquad E(X) := \int_B \left\{ \frac{1}{2} |\nabla X|^2 + 2[Q(X), X_u, X_v] \right\} du \, dv$$

in a subset of the class $\mathcal{C}(\Gamma)$ defined in Section 4.2.

Closely related to $E = D + 2V$ is the functional $F := A + 2V$ where A is the usual area functional

$$A(X) = \int_B |X_u \wedge X_v| \, du \, dv = \int_B \sqrt{|X_u|^2 |X_v|^2 - \langle X_u, X_v \rangle^2} \, du \, dv,$$

that is,

$$\text{(5)} \qquad F(X) := \int_B \{ |X_u \wedge X_v| + 2[Q(X), X_u, X_v] \} \, du \, dv.$$

[1] However, we shall write $E = D + V$ which changes (1) to $\operatorname{div} Q = 2H$.

We have

$$F(X) \le E(X) \tag{6}$$

and the equality sign holds if and only if

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0.$$

Clearly, $V(X)$ (and therefore also $E(X)$ and $F(X)$) are well-defined if $X \in H_2^1(B, \mathbb{R}^3)$ and either $\sup_{\mathbb{R}^3} |Q| < \infty$ or $X \in L^\infty(B, \mathbb{R}^3)$.

In Sections 2.1 and 2.8 we have already derived the first variation $\delta A(X, Y)$ and the second variation $\delta^2 A(X, Y)$ of a regular C^2-surface $X : B \to \mathbb{R}^3$ in normal direction $Y = \varphi N$, $\varphi \in C_c^\infty(B)$, N being the normal of X. Now we want to admit also branched surfaces X; for the sake of simplicity we assume that X is an H-surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, $0 < \alpha < 1$, that is regular (i.e. immersed) in $\overline{B}'$ as in 5.1, $\overline{B}' = \overline{B} \setminus \{\text{branch points of } X\}$. For an arbitrary test function $\varphi \in C_c^\infty(B')$ with $B' = \overline{B}' \cap B$ we consider the *normal variation* $Z : \overline{B} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^3$, $\epsilon_0 > 0$, which is defined by

$$Z(w, t) := X(w) + t\varphi(w) N(w), \quad w \in B, \ |t| < \epsilon_0, \tag{7}$$

where N is the normal of X. From formula (15) in Section 2.8 we obtain the following expansion at all regular points $w \in B$ of X:

$$\begin{aligned} &|Z_u(w,t) \wedge Z_v(w,t)| \\ &\quad = \Lambda(w) - 2t\Lambda(w) H(X(w))\varphi(w) \\ &\qquad + \tfrac{1}{2} t^2 [|\nabla\varphi(w)|^2 + 2\Lambda(w) K(w) \varphi^2(w)] + O(w, t^3) \end{aligned} \tag{8}$$

where $\Lambda = |X_u|^2$ and K is the Gauss curvature of X. The error term $O(w, t^3)$ vanishes outside of $\operatorname{supp}\varphi$, and we have

$$|O(w, t^3)| \le \text{const} \cdot t^3 \quad \text{for all } w \in \operatorname{supp}\varphi \subset\subset B'. \tag{9}$$

For $\varphi \in C_c^\infty(B')$, this implies

$$\frac{d}{dt} A(Z(\cdot, t))\bigg|_{t=0} = -\int_B 2\Lambda H(X)\varphi \, du \, dv \tag{10}$$

and

$$\frac{d^2}{dt^2} A(Z(\cdot, t))\bigg|_{t=0} = \int_B \{|\nabla\varphi|^2 + 2\Lambda K \varphi^2\} \, du \, dv. \tag{11}$$

By Theorem 1 of 5.1, the right-hand sides of (10) and (11) can continuously be extended onto functions $\varphi \in C_c^\infty(B)$. Therefore we take (10) and (11) as definitions of the first two derivatives of $A(Z(\cdot, t))$ at $t = 0$, i.e. for $\delta A(X, \varphi N)$ and $\delta^2 A(X, \varphi N)$, if $\varphi \in C_c^\infty(B)$. In order to compute $\frac{d}{dt} V(Z(\cdot, t))|_{t=0}$ and

$\frac{d^2}{dt^2} V(Z(\cdot, t))|_{t=0}$ for $\varphi \in C_c^\infty(B)$, we introduce $P(w,t) := Q(Z(w,t))$, $w \in B$, $|t| < \epsilon_0$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}[P, Z_u, Z_v] &= [P_t, Z_u, Z_v] + [P, (\varphi N)_u, Z_v] + [P, Z_u, (\varphi N)_v] \\
&= [P_t, Z_u, Z_v] + [P, \varphi N, Z_v]_u + [P, Z_u, \varphi N]_v \\
&\quad - [P_u, Z_t, Z_v] - [P_v, Z_u, Z_t] \\
&= [P_u, Z_v, Z_t] + [P_v, Z_t, Z_u] + [P_t, Z_u, Z_v] \\
&\quad + \{[P, \varphi N, Z_v]_u + [P, Z_u, \varphi N]_v\} \\
&= [Q_x(Z) Z_u, Z_v, Z_t] + [Q_x(Z) Z_v, Z_t, Z_u] + [Q_x(Z) Z_t, Z_u, Z_v] \\
&\quad + \{\dots\} \\
&= [Z_u, Z_v, Z_t] \cdot (\operatorname{div} Q)(Z) + \{\dots\} \\
&= [Z_u, Z_v, Z_t] \cdot H(Z) + \{\dots\}.
\end{aligned}$$

The divergence theorem implies $\int_B \{\dots\}\, du\, dv = 0$ since $\operatorname{supp} \varphi \subset B$, and so

$$\frac{d}{dt} V(Z) = \int_B H(Z) Z_t \cdot (Z_u \wedge Z_v)\, du\, dv. \tag{12}$$

We have

$$\begin{aligned}
Z_u \wedge Z_v &= (X_u + t\varphi_u N + t\varphi N_u) \wedge (X_v + t\varphi_v N + t\varphi N_v) \\
&= X_u \wedge X_v + t\{\varphi_v X_u \wedge N + \varphi_u N \wedge X_v + \varphi(X_u \wedge N_v + N_u \wedge X_v)\} \\
&\quad + t^2(\varphi\varphi_u N \wedge N_v + \varphi\varphi_v N_u \wedge N + \varphi^2 N_u \wedge N_v).
\end{aligned}$$

Multiplication by $Z_t = \varphi N$ yields

$$Z_t \cdot (Z_u \wedge Z_v) = \Lambda\varphi - 2\Lambda H(X)\varphi^2 t + \Lambda K \varphi^3 t^2.$$

Then formula (12) and Theorem 1 of 5.1 imply

$$\frac{d}{dt} V(Z) = \int_B \{\Lambda H(Z)\varphi - 2\Lambda H(X) H(Z)\varphi^2 t + \Lambda K H(Z)\varphi^3 t^2\}\, du\, dv. \tag{13}$$

Therefore

$$\left.\frac{d}{dt} V(Z)\right|_{t=0} = \int_B \Lambda H(X)\varphi\, du\, dv. \tag{14}$$

Furthermore we infer from (13) that

$$\left.\frac{d^2 V(Z)}{dt^2}\right|_{t=0} = \int_B \{\langle H_x(X), N\rangle - 2H^2(X)\}\varphi^2 \Lambda\, du\, dv. \tag{15}$$

Since $F = A + 2V$, we infer from (10) and (14) that

$$\frac{d}{dt}F(Z(\cdot,t))\bigg|_{t=0} = \int_B \{-2\Lambda H(X)\varphi + 2\Lambda H(X)\varphi\}\,du\,dv = 0,$$

and from (11) and (15) that

$$\frac{d^2}{dt^2}F(Z(\cdot,t))\bigg|_{t=0} = \int_B \{|\nabla\varphi|^2 + 2\Lambda(K - 2H^2(X) + \langle H_x(X), N\rangle)\varphi^2\}\,du\,dv.$$

Set

$$\delta F(X,\varphi N) := \frac{d}{dt}F(Z)\bigg|_{t=0}, \quad \delta^2 F(X,\varphi N) := \frac{d^2}{dt^2}F(Z)\bigg|_{t=0}. \tag{16}$$

Thus we have proved:

Theorem 1. *The first variation $\delta F(X,\varphi N)$ of $F = A + 2V$ with $\operatorname{div} Q = H$ at an H-surface $X \in C^{3,\alpha}(\overline{B},\mathbb{R}^3)$ in the normal direction $Y = \varphi N$ with $\varphi \in C_c^\infty(B)$ vanishes, and for the second variation $\delta^2 F(X,\varphi N)$ we have*

$$\delta^2 F(X,\varphi N) = \int_B \{|\nabla\varphi|^2 - 2p\varphi^2\}\,du\,dv, \tag{17}$$

where the density function p associated with X is defined by

$$p := \Lambda \cdot [2H^2(X) - K - \langle H_x(X), N\rangle]. \tag{18}$$

If $Q \in C^{2,\alpha}(\mathbb{R}^3,\mathbb{R}^3)$, then $p \in C^{0,\alpha}(B)$. Note that p is the same function as in Section 5.1, Theorem 1, formula (13).

Definition 1. *An H-surface $X \in C^{3,\alpha}(\overline{B},\mathbb{R}^3)$ is called* **stable** *if it satisfies the* **stability inequality**

$$\delta^2 F(X,\varphi N) \ge 0 \quad \text{for all } \varphi \in C_c^\infty(B) \tag{19}$$

which can be written as

$$\int_B |\nabla\varphi|^2\,du\,dv \ge 2\int_B p\varphi^2\,du\,dv \quad \text{for all } \varphi \in C_c^\infty(B). \tag{20}$$

Remark 1. By means of Proposition 1 in 5.1 it follows easily that the stability condition (19) is equivalent to

$$\delta^2 F(X,\varphi N) \ge 0 \quad \text{for all } \varphi \in C_c^\infty(B')$$

where $B' := B \setminus \{\text{branch points of } X\}$.

We note also that it suffices to assume $X \in C^{3,\alpha}(B,\mathbb{R}^3)$ in Definition 1 and in Theorem 1 since we only consider φ with compact support in B.

Remark 2. When an H-surface is **nonstable** we can find some $\varphi \in C_c^\infty(B)$ such that $\delta^2 F(X, \varphi N) < 0$. Obviously, global and local minimizers of F are stable.

On the other hand, an H-surface is said to be **unstable**, if it does not constitute a strong local minimum of F, i.e. in any $C^0(\overline{B}, \mathbb{R}^3)$-neighborhood of X one can find a surface $\tilde{X}$ with $F(\tilde{X}) < F(X)$. A nonstable surface is necessarily unstable while the converse need not be true.

In the next section we shall define the notions **μ-stable** for $\mu > 0$ and **strictly stable** ($\mu > 2$).

Remark 3. The vector field Q is not uniquely determined by the equation $\operatorname{div} Q = H$, neither is the functional F. Nevertheless the notions "stable and nonstable" are uniquely defined since in $\delta^2 F(X, \varphi N)$ only the expressions H and H_x enter. *Occasionally one prefers the notation $\delta^2 F(X, \varphi)$ which means the same as $\delta^2 F(X, \varphi N)$.*

The following central result for stable H-surfaces was found by F. Sauvigny [1,2]. It is used to prove that under certain assumptions a stable surface is in fact "nonparametric", that is, a graph of a function which is defined on a domain of the x^1, x^2-plane.

Theorem 2. *Suppose that the prescribed mean curvature $H(x) = H(x^1, x^2, x^3)$ is of class $C^{1,\alpha}(\mathbb{R}^3)$ and satisfies the monotonicity condition*

$$H_{x^3}(x) \geq 0 \quad \text{for } x \in \mathbb{R}^3. \tag{21}$$

Furthermore let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, $0 < \alpha < 1$, be a stable H-surface the normal $N = (N^1, N^2, N^3)$ of which satisfies the boundary condition

$$N^3(w) > 0 \quad \text{for all } w \in \partial B. \tag{22}$$

Then it follows that $N^3(w) > 0$ for all $w \in \overline{B}$.

Proof. Let $e_3 = (0, 0, 1)$ be the unit vector in x^3-direction and set

$$f := N^3 = \langle N, e_3 \rangle. \tag{23}$$

Multiplying both sides of equation (12) in 5.1 by e_3 and noting $-2\Lambda H_{x^3}(X) \leq 0$, it follows

$$\Delta f + 2pf \leq 0 \quad \text{in } B. \tag{24}$$

Since, by assumption, $f(w) > 0$ for $w \in \partial B$ holds true, Proposition 1 below yields $f(w) > 0$ for all $w \in \overline{B}$. $\square$

Remark 4. The geometrical content of Theorem 2 is the following: If a stable H-surface constitutes a positively oriented, branched graph over the x^1, x^2-plane at the boundary, then the same property holds true in the interior.

Now we establish the result that was used in the proof of Theorem 2. It is of independent interest; a similar reasoning will be applied when we treat partially free boundary value problems for minimal surfaces (cf. Vol. 3, Section 3).

Proposition 1. *Suppose that* $p \in C^{0,\alpha}(B)$ *satisfies the* **stability inequality**

$$\int_B |\nabla\varphi|^2 \, du\, dv \geq 2 \int_B p\varphi^2 \, du\, dv \quad \text{for all } \varphi \in C_c^\infty(B) \tag{25}$$

and let $f \in C^0(\overline{B}) \cap C^2(B)$ *be a solution of the boundary value problem*

$$\Delta f + 2pf \leq 0 \quad \text{in } B, \qquad f(w) > 0 \quad \text{on } \partial B. \tag{26}$$

Then one has $f(w) > 0$ *for all* $w \in \overline{B}$.

Remark 5. The assertion would already follow from (26) alone if one had $p(w) \leq 0$ on B, as one could apply the maximum principle. The gist of Proposition 1 is that the assumption $p \leq 0$ can be replaced by (25). Note that even for minimal surfaces one has $p = -\Lambda K \geq 0$.

Proof of Proposition 1. We first show that $f(w) \geq 0$ on B. To this end we consider the "negative part" f^- of f, defined by

$$f^-(w) := \min\{f(w), 0\} \quad \text{for } w \in \overline{B},$$

which is of the class $H_2^1(B)$ with compact support in B and satisfies

$$\nabla f^-(w) = \begin{cases} 0 & \text{for almost all } w \in B \text{ with } f(w) \geq 0, \\ \nabla f(w) & \text{for all } w \in B \text{ with } f(w) < 0. \end{cases}$$

Then

$$\int_B |\nabla f^-|^2 \, du\, dv = -\int_B f^- \Delta f \, du\, dv$$

on account of a generalized version of the divergence theorem (see e.g. Sauvigny [16], Chapter VIII, §9, Propositions 1 and 2), and by (26):

$$-\int_B f^- \Delta f \, du\, dv \leq 2 \int_B p f f^- \, du\, dv = 2 \int_B p |f^-|^2 \, du\, dv.$$

Therefore,

$$\int_B |\nabla f^-|^2 \, du\, dv \leq 2 \int_B p|f^-|^2 \, du\, dv. \tag{27}$$

Next, with $\psi \in C_c^\infty(B)$, we insert $\varphi := f^- + \epsilon\psi \in \mathring{H}_2^1(B)$ into (25), which even holds for test functions of class $\mathring{H}_2^1(B)$, $|\epsilon| \leq \epsilon_0$, $\epsilon_0 > 0$, thus obtaining

$$\begin{aligned} &\int_B |\nabla f^-|^2 \, du\, dv + 2\epsilon \int_B \nabla f^- \cdot \nabla\psi \, du\, dv + \epsilon^2 \int_B |\nabla\psi|^2 \, du\, dv \\ &\quad \geq 2 \int_B p|f^-|^2 \, du\, dv + 4\epsilon \int_B p f^- \psi \, du\, dv + 2\epsilon^2 \int_B p\psi^2 \, du\, dv. \end{aligned}$$

With the aid of (27) we arrive at

$$2\epsilon \int_B (\nabla f^- \cdot \nabla \psi - 2pf^- \psi)\, du\, dv + \epsilon^2 \int_B (|\nabla \psi|^2 - 2p\psi^2)\, du\, dv \geq 0$$

for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, whence we obtain the weak differential equation

$$\int_B (\nabla f^- \cdot \nabla \psi - 2pf^- \psi)\, du\, dv = 0 \quad \text{for all } \psi \in C_c^\infty(B). \tag{28}$$

Applying Moser's inequality (see Gilbarg–Trudinger [1], or Sauvigny [16], Chapter X, §5, Theorem 1) we infer from $f^-(w) \equiv 0$ near ∂B, that $f^-(w) \equiv 0$ in $\overline{B}$, and therefore $f(w) \geq 0$.

Finally, (26) implies

$$\int_B (\nabla f \cdot \nabla \varphi - 2pf\varphi)\, du\, dv \geq 0 \quad \text{for all } \varphi \in C_c^\infty(B) \text{ with } \varphi \geq 0,$$

and we have $f \geq 0$. Invoking once more Moser's inequality (see loc. cit. above) and recalling the assumption $f(w) > 0$ on ∂B we arrive at the desired inequality $f(w) > 0$ for $w \in \overline{B}$. □

5.4 On μ-Stable Immersions of Constant Mean Curvature

The density function p associated with an H-surface X might even change its sign if $H(x)$ is variable. This phenomenon is excluded for constant H since in this case

$$p = \Lambda \cdot (2H^2 - K) = \tfrac{1}{2}\Lambda \cdot (\kappa_1^2 + \kappa_2^2) \geq 0. \tag{1}$$

Assumption. *In this section we consider immersed cmc-surfaces $X : \overline{B} \to \mathbb{R}^3$ of class $C^{2,\alpha}$, i.e.*

$$X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3), \quad 0 < \alpha < 1, \quad \textit{and} \quad \Lambda(w) > 0 \quad \textit{on } \overline{B}.$$

Then X is real analytic on B, $H \equiv \text{const}$, $K \in C^{0,\alpha}(\overline{B})$ and the density function p associated with X is of class $C^{0,\alpha}(\overline{B})$; in particular, p is continuous up to the boundary ∂B.

Definition 1. *An immersed cmc-surface X is called* **μ-stable** *with $\mu > 0$ if*

$$\int_B |\nabla \varphi|^2\, du\, dv \geq \mu \int_B p\varphi^2\, du\, dv \quad \textit{for all } \varphi \in C_c^\infty(B) \tag{2}$$

holds true; if even $\mu > 2$, the surface X is said to be **strictly stable**.

Remark 1. Since $p \in L^\infty(B)$, relation (2) is equivalent to

$$\int_B |\nabla\varphi|^2\,du\,dv \ge \mu \int_B p\varphi^2\,du\,dv \quad \text{for all } \varphi \in \mathring{H}_2^1(B).$$

Remark 2. The 2-stable, immersed cmc-surfaces X are stable in the sense of Section 5.3.

Let us begin with the following instructive result which for minimal surfaces is due to H.A. Schwarz.

Theorem 1. *If the immersed cmc-surface X with the surface normal $N = (N^1, N^2, N^3)$ satisfies*

$$N^3(w) > 0 \quad \textit{for all } w \in \overline{B}, \tag{3}$$

then X is strictly stable.

Proof. We solve the variational problem

$$D(\varphi) \to \min \quad \text{in } \left\{\varphi \in \mathring{H}_2^1(B)\colon \int_B p\varphi^2\,du\,dv = 1\right\}. \tag{4}$$

Its solution φ_0 is an eigenfunction to the least eigenvalue $\mu > 0$ of the eigenvalue problem

$$-\Delta\varphi_0 = \mu p\varphi_0 \quad \text{in } B, \quad \varphi_0 = 0 \quad \text{on } \partial B, \tag{5}$$

where $\varphi_0 \in \mathring{H}_2^1(B)$. Elliptic theory yields $\varphi_0 \in C^{2,\alpha}(\overline{B})$.

Let $e_3 := (0,0,1)$ and set $\psi := N^3 = \langle N, e_3\rangle \in C^{1,\alpha}(\overline{B})$. The function ψ is real analytic on B and satisfies $\psi(w) > 0$ on $\overline{B}$. In order to compare the eigenfunction φ_0 with the auxiliary function ψ, we first note that ψ is a solution of

$$-\Delta\psi = 2p\psi \quad \text{in } B, \tag{6}$$

taking equation (12) of Section 5.1 into account. Obviously we can find a number $\lambda \in \mathbb{R}$ such that the further auxiliary function

$$\chi := \psi + \lambda\varphi_0 \tag{7}$$

satisfies

$$\chi \ge 0 \quad \text{in } B, \quad \chi > 0 \quad \text{on } \partial B, \quad \chi(w_0) = 0 \quad \text{for at least one } w_0 \in B. \tag{8}$$

From

$$\begin{aligned} -\Delta\chi &= -\Delta\psi - \lambda\Delta\varphi_0 = 2p\psi + \mu p\lambda\varphi_0 \\ &= (2-\mu)p\psi + \mu p \cdot (\psi + \lambda\varphi_0) \end{aligned}$$

we infer

$$\Delta\chi + \mu p\chi = (\mu - 2)p\psi. \tag{9}$$

Suppose now that X were not strictly stable. Then we had $\mu \le 2$, and (9) would yield the differential inequality

$$\Delta\chi + \mu p\chi \le 0 \quad \text{in } B. \tag{10}$$

Applying the same reasoning as in the proof of Proposition 1 of Section 5.3 we infer $\chi(w) \equiv 0$ in B, which evidently contradicts (8). Therefore X must be strictly stable. □

The following profound result will be used in Section 5.6 to prove a uniqueness result for Plateau's problem.

Theorem 2. *Let X be an immersed cmc-surface whose density function $p = (2H^2 - K)\Lambda$ satisfies*

$$\int_B (2H^2 - K)\Lambda \, du\, dv < 2\pi. \tag{11}$$

Then X is strictly stable.

Proof. (i) On the northern hemisphere $S_r^+ := \{x \in \mathbb{R}^3 \colon |x| = r,\ x^3 > 0\}$ of radius r with the area $2\pi r^2$ we consider the eigenvalue problem for the Laplace–Beltrami operator with zero boundary values on the equator $\partial S_r^+ = \{x \in \mathbb{R}^3 \colon |x| = r,\ x^3 = 0\}$. The least eigenvalue $\lambda_1(S_r^+)$ can explicitly be determined as

$$\lambda_1(S_r^+) = 2/r^2 \tag{12}$$

in the following way: Via stereographic projection we construct a conformal mapping

$$Z : \overline{B} \to \overline{S}_r^+ \quad \text{with } Z(\partial B) = \partial S_r^+, \tag{13}$$

which is necessarily a cmc-surface of mean curvature $1/r$. Then the auxiliary function $\varphi := Z^3$ satisfies

$$\varphi > 0 \quad \text{in } B \quad \text{and} \quad \varphi = 0 \quad \text{on } \partial B. \tag{14}$$

From the system

$$\Delta Z = \frac{2}{r} Z_u \wedge Z_v = -\frac{2}{r^2}|Z_u|^2 Z \quad \text{in } B,$$

which is satisfied by the $\frac{1}{r}$-surface Z,we obtain the equation

$$-\Delta\varphi = \frac{2}{r^2}\varphi \quad \text{on } B \tag{15}$$

where Δ is the Laplace–Beltrami operator Δ_Z on S_r^+. From (14) and (15) we infer that $\lambda_1(S_r^+) = \frac{2}{r^2}$, as stated in (12).

(ii) Now we invoke Theorem 2 from Section 5.2. Accordingly the Bonnet surface $Y = N + HX$ associated with X is either a constant surface or else a cmc-surface of mean curvature one. Moreover we have $Y(w) \equiv \text{const}$ on $\overline{B}$ if and only if $H^2 - K(w) \equiv 0$ on $\overline{B}$. In this case it follows trivially for all $r_1 > 0$ that

$$\int_B |\nabla\phi|^2\,du\,dv \ge \frac{2}{r_1^2}\int_B (H^2 - K)\Lambda\phi^2\,du\,dv \quad \text{for all } \phi \in C_c^\infty(B).$$

If $H^2 - K(w) \not\equiv 0$ on $\overline{B}$ it makes sense to study the eigenvalue problem for the Laplace–Beltrami operator on the surface Y with respect to zero boundary values. Its smallest eigenvalue

$$\lambda_1(|dY|^2) = \inf\left\{2D(\phi) \colon \phi \in \mathring{H}_2^1(B) \text{ with } \int_B (H^2 - K)\Lambda\phi^2\,du\,dv = 1\right\}$$

can be compared with that of all surfaces of equal area, whose Gaussian curvature is bounded from above by a constant greater than or equal to one. The smallest eigenvalue is assumed on the spherical cap $S_{r_1}^+$ of radius $r_1 > 0$ with the area $\int_B (H^2 - K)\Lambda\,du\,dv = 2\pi r_1^2$. This yields the estimate

$$\int_B |\nabla\phi|^2\,du\,dv \ge \frac{2}{r_1^2}\int_B (H^2 - K)\Lambda\phi^2\,du\,dv \quad \text{for all } \phi \in C_c^\infty(B). \tag{16}$$

Consequently X is strictly stable if $H = 0$.

(iii) In case that $H \ne 0$ we additionally consider the cmc-surface $\tilde{Y} := HX$ with the area $\int_B H^2\Lambda\,du\,dv = 2\pi r_2^2$. By the same arguments as in (ii) we find that

$$\int_B |\nabla\phi|^2\,du\,dv \ge \frac{2}{r_2^2}\int_B H^2\Lambda\phi^2\,du\,dv \quad \text{for all } \phi \in C_c^\infty(B) \tag{17}$$

is valid.

(iv) From (11) we infer

$$2\pi > \int_B (2H^2 - K)\Lambda\,du\,dv = 2\pi(r_1^2 + r_2^2)$$

whence $0 < r_1^2 < r_2^2 < 1$. Addition of (16) and (17) yields

$$(r_1^2 + r_2^2)\int_B |\nabla\phi|^2\,du\,dv \ge 2\int_B (2H^2 - K)\Lambda\phi^2\,du\,dv \quad \text{for all } \phi \in C_c^\infty(B).$$

Thus the H-surface X is μ-stable with the value

$$\mu := \frac{2}{r_1^2 + r_2^2} > 2. \qquad \square$$

Remark 3. The reasoning used in part (ii) of the preceding proof depends on isoperimetric inequalities and symmetrization techniques in the class of surfaces with bounded Gaussian curvature from above. For the methods that cope with branch points in these surfaces we refer to the paper by Barbosa and do Carmo [4], especially Proposition (3.13) in Section 3. Here the authors prove the following result: Let p be a nonnegative C^2-function on B vanishing only at isolated points, and denote by λ_1 the first eigenvalue of the problem

$$\Delta f + \lambda p f = 0 \quad \text{in } B, \quad f \in \mathring{H}_2^1(B).$$

Furthermore, suppose that the Gaussian curvature $\hat{K}$ of the manifold $(B, d\sigma^2)$ with the singular metric $d\sigma^2 = p\, ds^2$, $ds^2 = du^2 + dv^2$ the standard metric on B, satisfies $\hat{K} \le K_0$ for some constant $K_0 \in [0, \infty)$. Then we have the inequality $\lambda_1 \ge \tilde{\lambda}_1(B_0)$ where B_0 denotes a geodesic disk in the 2-dimensional space of constant Gaussian curvature K_0, and $\tilde{\lambda}_1(B_0)$ is the smallest eigenvalue of the Laplace–Beltrami operator on B_0 corresponding to this metric.

Remark 4. For minimal surfaces it is advantageous to operate with the normal N whose image might yield a multiple covering on the sphere. In this case the original condition of Barbosa and do Carmo [1], namely that the spherical image $N(B)$ be contained in a spherical domain of area less than 2π, is considerably weaker than the inequality (11).

Remark 5. With the aid of H. Hopf's quadratic differential, H. Ruchert [1] established the above result alternatively without using the Bonnet surface.

The following area estimate constitutes the first step to prove a curvature estimate and subsequent Bernstein results for stable minimal surfaces. The estimate to be presented here even pertains to nonstable H-surfaces. Applied to geodesic disks of radius r on complete minimal submanifolds we see that their areas grow at most quadratically in r as $r \to \infty$.

Theorem 3 (R. Gulliver [15]). *Let $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ be an immersed, μ-stable cmc-surface with $\mu > \frac{1}{2}$, and suppose that $X(\overline{B}) = K_r(x_0)$, where $K_r(x_0)$ denotes a geodesic disk of radius r and center $x_0 := X(0)$ as described in (19)–(21) below. Then we have the estimate*

$$A(X) \le \frac{2\mu}{2\mu - 1} \pi r^2 \tag{18}$$

for the area of X.

Proof. (i) We represent the geodesic disk $K_r(x_0) = X(\overline{B})$ with respect to geodesic polar coordinates ρ, φ by the mapping

$$Z : [0, r] \times [0, 2\pi] \to \mathbb{R}^3, \quad Z(0,0) = x_0, \tag{19}$$
$$X(\overline{B}) = \{Z(\rho,\varphi)\colon 0 \le \rho \le r,\ 0 \le \varphi \le 2\pi\},$$

with the first fundamental form

$$ds^2 = |dZ|^2 = d\rho^2 + P(\rho,\varphi)\, d\varphi^2 \tag{20}$$

(i.e. $|Z_\rho|^2 = 1$, $\langle Z_\rho, Z_\varphi\rangle = 0$, $|Z_\varphi|^2 = P$). Here the function $P(\rho,\varphi) > 0$ in $(0, r] \times [0, 2\pi)$ satisfies the asymptotic conditions

$$\lim_{\rho\to+0} P(\rho,\varphi) = 0 \quad \text{and} \quad \lim_{\rho\to+0} \frac{\partial}{\partial\rho}\sqrt{P(\rho,\varphi)} = 1 \quad \text{for } 0 \le \varphi \le 2\pi. \tag{21}$$

According to Minding's formula for the geodesic curvature $\kappa_g(\rho,\varphi)$ of the curve $\Gamma_\rho := \{Z(\rho,\varphi)\colon 0 \le \varphi < 2\pi\}$ we obtain

$$\frac{\partial}{\partial\rho}\sqrt{P(\rho,\varphi)} = \kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)} \quad \text{for } 0 < \rho \le r,\ 0 \le \varphi < 2\pi, \tag{22}$$

cf. Section 1.3, and W. Blaschke [1], §83, formula (127).

(ii) We introduce the length of Γ_ρ by

$$L(\rho) := \int_0^{2\pi} \sqrt{P(\rho,\varphi)}\, d\varphi, \quad 0 < \rho \le r. \tag{23}$$

Differentiating $L(\rho)$ with the aid of (22) and applying the Gauss–Bonnet theorem, we obtain

$$\begin{aligned} L'(\rho) &= \int_0^{2\pi} \kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)}\, d\varphi \\ &= 2\pi - \int_0^{\rho}\int_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\, d\tau\, d\varphi \end{aligned} \tag{24}$$

and consequently

$$L''(\rho) = -\int_0^{2\pi} K(\rho,\varphi)\sqrt{P(\rho,\varphi)}\, d\varphi. \tag{25}$$

(iii) In order to apply the stability condition (2) we choose the test function $\varphi(w)$ as

$$\varphi(w) = \eta(\rho) := 1 - \rho/r \quad \text{for } 0 \le \rho \le r \text{ if } w \leftrightarrow (\rho,\varphi).$$

By (23) we have

$$\int_0^r |\eta'(\rho)|^2 L(\rho)\, d\rho = \int_0^r \int_0^{2\pi} |\eta'(\rho)|^2 \sqrt{P(\rho,\varphi)}\, d\rho\, d\varphi =: J.$$

Now we use the invariant first Beltrami operator

$$\|\nabla\phi\|^2 := (\mathcal{E}\mathcal{G} - \mathcal{F}^2)^{-1}(\mathcal{G}\phi_u^2 - 2\mathcal{F}\phi_u\phi_v + \mathcal{E}\phi_v^2)$$

for the metric $ds^2 = \mathcal{E}\, du^2 + 2\mathcal{F}\, du\, dv + \mathcal{G}\, dv^2$. Especially for the geodesic metric $ds^2 = d\sigma^2 + P(\rho,\varphi)\, d\varphi^2$ the stability inequality (2) yields

$$\begin{aligned} J = \int_0^r \int_0^{2\pi} \frac{P\eta_\rho^2 + 1\cdot\eta_\varphi^2}{P}\sqrt{P}\, d\rho\, d\varphi &\geq \mu \int_0^r \int_0^{2\pi} (2H^2 - K)\eta^2 \sqrt{P}\, d\varphi\, d\rho \\ &\geq -\mu \int_0^r \eta^2 \left(\int_0^{2\pi} K\sqrt{P}\, d\varphi \right) d\rho. \end{aligned}$$

Taking (25) into account, we arrive at

$$\begin{aligned} J &\geq \mu \int_0^r L''(\rho)\eta^2(\rho)\, d\rho \\ &= \mu[L'(\rho)\eta^2(\rho)]_{+0}^r - 2\mu \int_0^r L'(\rho)\eta(\rho)\eta'(\rho)\, d\rho \end{aligned}$$

after an integration by parts. Since $\eta(0) = 1$, $\eta(r) = 0$, and $L'(+0) = 2\pi$, it follows that

$$J \geq -2\pi\mu - 2\mu \int_0^r L'\eta\eta'\, d\rho$$

and

$$\int_0^r L'\eta\eta'\, d\rho = [L\eta\eta']_{+0}^r - \int_0^r [L(\eta')^2 + L\eta\eta'']\, d\rho = -\int_0^r L(\eta')^2\, d\rho$$

since $\eta'' = 0$, $L(+0) = 0$, and $\eta(r) = 0$. Thus we obtain

$$\int_0^r L(\eta')^2\, d\rho \geq -2\pi\mu + 2\mu \int_0^r L(\eta')^2\, d\rho$$

whence, by $\eta'(\rho) = -\frac{1}{r}$ it follows that

$$\frac{1}{r^2}\int_0^r L(\rho)\, d\rho \leq \frac{2\pi\mu}{2\mu - 1}$$

and finally

$$A(X) = A(Z) = \int_0^r \int_0^{2\pi} \sqrt{P(\rho,\varphi)}\, d\rho\, d\varphi \leq \frac{2\pi\mu}{2\mu-1} r^2. \qquad \square$$

5.5 Curvature Estimates for Stable and Immersed cmc-Surfaces

The basic result of this section is the following

Theorem 1 (F. Sauvigny [7,8]). *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be a stable, immersed cmc-surface with $X(0) = 0$ whose mean curvature H is bounded by a constant $h_0 \geq 0$, i.e. $|H| \leq h_0$. Suppose also that X represents a geodesic disk $K_1(0)$ of radius 1 about $X(0) = 0$ such that $X(\overline{B}) = K_1(0)$. Furthermore let κ_1 and κ_2 be the principal curvatures of X. Then there is a universal constant $c(h_0)$ depending only on the parameter value h_0 such that*

$$\kappa_1^2(0) + \kappa_2^2(0) \leq c(h_0). \tag{1}$$

Proof. (i) Since X is 2-stable, Gulliver's estimate yields

$$\int_B |\nabla X|^2 \, du \, dv = 2A(X) \leq \frac{8\pi}{3} \tag{2}$$

(cf. Section 5.4, Theorem 3). In order to effectively use the Courant–Lebesgue lemma, we fix the number

$$\nu_0 := \tfrac{1}{3} \exp\bigl(-\tfrac{32}{3}\pi^2\bigr) \in \bigl(0, \tfrac{1}{3}\bigr) \tag{3}$$

and claim the following

Preliminary Statement. *There exists a point $w_* = \rho_0 e^{i\varphi_0} \in B$ with $|w_*| \leq 1 - 3\nu_0$ such that the radial derivative X_ρ of X satisfies*

$$|X_\rho(w_*)| \geq \lambda_0 := \tfrac{1}{2} \cdot (1 - 3\nu_0)^{-1} > 0. \tag{4}$$

To verify this claim, we introduce the set $\Gamma(B)$ of continuous and piecewise regular curves $\gamma : [0,1] \to \overline{B}$ with $\gamma(0) = 0$ and $\gamma(1) \in \partial B$. From the properties of the geodesic disk $K_1(0) = X(\overline{B})$ we infer

$$\inf_{\gamma \in \Gamma(B)} \int_0^1 \left| \frac{d}{dt} X(\gamma(t)) \right| dt = 1. \tag{5}$$

We fix a point $w_1 \in \partial B$, and set $\delta := 3\nu_0$. By (2) and the Courant–Lebesgue lemma there is a number $\delta^* \in (\delta, \sqrt{\delta})$ such that for

$$C_{\delta^*}(w) := \{w \in B \colon |w - w_1| = \delta^*\}$$

we can estimate

$$\int_{C_{\delta^*}(w_1)} |dX| \leq 2 \left\{ \pi \cdot \left(\frac{8}{3}\pi \right) \frac{1}{\log \frac{1}{\delta}} \right\}^{\frac{1}{2}} = 2 \left\{ \frac{8}{3}\pi^2 \cdot \frac{1}{\frac{32}{3}\pi^2} \right\}^{\frac{1}{2}} = 1. \tag{6}$$

Denote by $\gamma_1 : [0, 1 - \delta^*] \to B$ the path

$$\gamma_1(t) := tw_1, \quad 0 \le t \le 1 - \delta^*,$$

from the origin to the point $w_2 := (1 - \delta^*)w_1$ on the circle $\partial B_{1-\delta^*}(0)$. For $\epsilon = \pm 1$ we additionally consider the paths

$$\gamma_2(t) := w_1 + (w_2 - w_1)e^{i\epsilon t}, \quad 0 \le t \le t_2(\delta^*),$$

leading within B on the circle $C_{\delta^*}(w_1)$ from w_2 to the boundary ∂B. On account of (6) we conclude that either for $\epsilon = 1$ or for $\epsilon = -1$ the inequality

$$\int_0^{t_2(\delta^*)} \left| \frac{d}{dt} X(\gamma_2(t)) \right| dt \le \frac{1}{2} \tag{7}$$

holds true. We combine γ_1 and γ_2 to a path $\gamma \in \Gamma(B)$. By means of (5) and (7) it follows that

$$\begin{aligned} 1 \le \int_0^1 |d(X \circ \gamma)| &= \int_0^{1-\delta^*} \left| \frac{d}{dt} X(\gamma_1) \right| dt + \int_0^{t_2(\delta^*)} \left| \frac{d}{dt} X(\gamma_2) \right| dt \\ &\le \int_0^{1-\delta^*} \left| \frac{d}{dt} X(\gamma_1) \right| dt + \frac{1}{2}. \end{aligned}$$

Hence there is a value $t^* \in [0, 1 - \delta^*]$ such that

$$\left| \frac{d}{dt} X(\gamma_1(t^*)) \right| \ge \frac{1}{2(1-\delta)}.$$

This proves the desired "preliminary statement".

(ii) Now we choose a test function $\varphi \in C_c^\infty(B)$ with $\varphi(w) \equiv 1$ for $|w| \le 1 - \nu_0$ and $|\nabla \varphi| \le 2/\nu_0$ in B, which will be inserted into the stability condition. By formula (14) of 5.1 we also have

$$\tfrac{1}{2} |\nabla N|^2 = (2H^2 - K)\Lambda,$$

and so

$$\begin{aligned} \int_{|w| \le 1 - \nu_0} |\nabla N|^2 \, du \, dv &= 2 \int_{|w| \le 1 - \nu_0} (2H^2 - K)\Lambda \, du \, dv \\ &\le 2 \int_B (2H^2 - K)\Lambda \varphi^2 \, du \, dv \\ &\le \int_B |\nabla \varphi|^2 \, du \, dv \le 4\pi \nu_0^{-2}. \end{aligned}$$

Hence we have found the *universal bound*

$$\int_{B_{1-\nu_0}(0)} |\nabla N|^2 \, du \, dv \le 4\pi \nu_0^{-2} \tag{8}$$

for the energy of the unit normal N of X on the disk $B_{1-\nu_0}(0)$ of radius $1-\nu_0$ about the origin.

With the aid of the Courant–Lebesgue lemma we then find a universal constant δ_1 with $0 < \delta_1 < \sqrt{\delta_1} \le 2\nu_0$, such that to each point $w_0 \in B_{1-3\nu_0}(0)$ there exists a radius $\delta^* = \delta^*(w_0, X) \in (\delta_1, \sqrt{\delta_1})$ satisfying

$$\text{(9)} \qquad \int_{C_{\delta^*}(w_0)} |dN| \le \pi \quad \text{for } C_{\delta^*}(w_0) := \{w \in B\colon |w - w_0| = \delta^*\}.$$

From this we infer the following result: There is a universal constant $\tau > 0$ with the property that for any $w_0 \in B_{1-3\nu_0}(0)$ there exists a "pole vector" $e_0 = e_0(w_0) \in S^2$ such that

$$\langle N(w), e\rangle > 0 \quad \text{for all } w \in C_{\delta^*}(w_0) \text{ and all } e \in S^2 \text{ with } |e - e_0| \le \tau.$$

Then one derives from Theorem 2 of Section 5.3 the basic

Auxiliary Statement. *There is a universal constant τ with the property that for any $w_0 \in B_{1-3\nu_0}(0)$ there is a "pole vector" $e_0 \in S^2$ such that*

$$\text{(10)} \quad \langle N(w), e\rangle > 0 \quad \textit{for all } w \in B_{\delta_1}(w_0) \textit{ and all } e \in S^2 \textit{ with } |e - e_0| \le \tau.$$

(iii) The auxiliary statement means geometrically that $B_{\delta_1}(w_0)$ is mapped by N into a geodesic disk on S^2, i.e. into a spherical cap, with a universal geodesic radius smaller than $\pi/2$ (= geodesic radius of a hemisphere), and that the center of this disk depends on the point $w_0 \in B_{1-3\nu_0}(0)$. Therefore the set $N(B_{\delta_1}(w_0))$ is contained in a closed 3-dimensional ball of a fixed radius $M \in (0, 1)$. Especially at the origin we find a vector $N_0 \in \mathbb{R}^3$, such that

$$\text{(11)} \qquad |N(w) - N_0| \le M \quad \text{for all } w \in B_{\delta_1}(0)$$

holds true with a universal constant $M \in (0, 1)$.

Furthermore the formulae (16) and (17) of Section 5.1 imply that

$$\text{(12)} \qquad \Delta N = -N|\nabla N|^2 \quad \text{in } B.$$

From the gradient estimate of E. Heinz we infer that there is an a priori constant $c_1 > 0$ such that

$$\text{(13)} \qquad |\nabla N(0)| \le c_1$$

holds true (cf. Vol. 2, Section 2.2, Proposition 1, or F. Sauvigny [16], Chapter XII, §2, Theorem 1).

(iv) For an arbitrary point $w_0 \in B_{1-3\nu_0}(0)$ we can achieve that

$$\text{(14)} \qquad X(w_0) = 0 \quad \text{and} \quad e_0 = e_3 := (0, 0, 1)$$

applying a suitable translation and rotation in $\mathbb{R}^3$. Consider the planar mapping $f : B_{\delta_1}(w_0) \to \mathbb{R}^2$ defined by

$$(15) \qquad f(w) := (X^1(w), X^2(w)), \quad w \in B_{\delta_1}(w_0).$$

By the "auxiliary statement" its Jacobian J_f satisfies

$$(16) \qquad J_f := \frac{\partial(X^1, X^2)}{\partial(u,v)} > 0 \quad \text{in } B_{\delta_1}(w_0),$$

and (2) implies

$$(17) \qquad \int_{B_{\delta_1}(w_0)} |\nabla f|^2 \, du \, dv \le \frac{8\pi}{3}.$$

From $X_w \cdot X_w = 0$ it follows that $|\nabla X^3|^2 \le |\nabla f|^2$ whence

$$(18) \qquad \tfrac{1}{2}|\nabla X|^2 \le |\nabla f|^2 \le |\nabla X|^2.$$

Thus any bound on $|\nabla f|$ is equivalent to a bound on $|\nabla X|$. Furthermore

$$|\Delta f| \le |\Delta X| = 2|H| \cdot |X_u \wedge X_v| \le h_0 |\nabla X|^2,$$

and so we infer from (18) that

$$(19) \qquad |\Delta f| \le 2h_0 |\nabla f|^2 \quad \text{in } B_{\delta_1}(w_0).$$

With the aid of the Courant–Lebesgue lemma we obtain a further universal constant δ_2 with $0 < \delta_2 < \sqrt{\delta_2} \le \delta_1$ and an "individual" constant $\delta^{**} = \delta^{**}(w_0, X) \in (\delta_2, \sqrt{\delta_2})$ satisfying

$$(20) \qquad 4h_0 \int_{C_{\delta^{**}}(w_0)} |df| \le 1 \quad \text{for } C_{\delta^{**}}(w_0) := \{w \in B \colon |w - w_0| = \delta^{**}\}.$$

Therefore, $f(C_{\delta^{**}}(w_0))$ is contained in a closed plane disk of radius $(8h_0)^{-1}$. Since f has a positive Jacobian J_f in $B_{\delta_1}(w_0)$ and $\overline{B}_{\delta^{**}}(w_0) \subset B_{\delta_1}(w_0)$, the mapping f is not allowed to protrude from this disk. Taking $f(w_0) = 0$ into account, we arrive at the inequality

$$(21) \qquad |f(w)| \le \frac{1}{4h_0} \quad \text{for all } w \in B_{\delta_2}(w_0).$$

(v) We set $\nu := \frac{1}{2}\delta_2$; then $\nu \in (0, \nu_0)$. Recalling that $f : B_{2\nu}(w_0) \to \mathbb{R}^2$ provides an open mapping of $\overline{B}_{2\nu}(w_0)$ onto its image which satisfies

$$|\Delta f| \le 2h_0 |\nabla f|^2 \quad \text{and} \quad |f| \le \frac{1}{4h_0} \quad \text{on } \overline{B}_{2\nu}(w_0),$$

we are now in the position to apply an inequality of E. Heinz that is based on the theory of pseudoholomorphic functions (see Sauvigny [16], Chapter XII, §5, Theorem 2). Thus we obtain a priori constants $c'(h_0)$ and $c''(h_0)$ with $0 < c' \le c''$ such that

$$(22) \quad c'(h_0)|\nabla f(w_0)|^5 \le |\nabla f(w)| \le c''(h_0)|\nabla f(w_0)|^{\frac{1}{5}} \quad \text{for all } w \in \overline{B}_\nu(w_0).$$

Furthermore, by virtue of (18), the surface element $\Lambda = \frac{1}{2}|\nabla X|^2$ of X satisfies

$$\tfrac{1}{2}|\nabla f|^2 \le \Lambda \le |\nabla f|^2 \quad \text{in } B_{2\nu}(w_0),$$

and so (22) yields the following

Intermediate Statement. *There exists a universal constant $\Theta = \Theta(h_0) \in (0,1)$ such that the surface element Λ satisfies the distortion estimate*

$$(23) \qquad \Theta(h_0)\Lambda^5(w) \le \Lambda(w_0) \quad \textit{for all } w \in B_\nu(w_0)$$

holds true for any $w_0 \in B_{1-3\nu_0}(0)$.

(vi) In order to estimate $\Lambda(0)$ from below, we apply the "preliminary statement" and pick a point $w_* \in B_{1-3\nu_0}(0)$ satisfying

$$(24) \qquad \Lambda(w_*) \ge \lambda_0^2 > 0,$$

cf. (4). Then we choose $n \in \mathbb{N}$ in such a way that

$$1 - 3\nu \le n\nu < 1 - 2\nu$$

is fulfilled and introduce the points

$$w_j := \frac{j}{n} w_* \quad \text{for } j = 0, 1, \dots, n.$$

Then we have

$$|w_j| = \frac{j}{n}|w_*| \le |w_*| \le 1 - 3\nu_0 \quad \text{for } j = 0, 1, \dots, n$$

and

$$|w_{j+1} - w_j| = \frac{1}{n}|w_*| \le \frac{1 - 3\nu_0}{n} \le \frac{1 - 3\nu}{n} \le \nu \quad \text{for } j = 0, 1, \dots, n-1.$$

Applying repeatedly (23) and (24) we obtain

$$\begin{aligned} \Lambda(0) &= \Lambda(w_0) \ge \Theta\Lambda^5(w_1) \ge \Theta^{1+5}\Lambda^{5^2}(w_2) \ge \Theta^{1+5+5^2}\Lambda^{5^3}(w_3) \\ &\ge \cdots \ge \Theta^{1+5+5^2+\cdots+5^{n-1}}\Lambda^{5^n}(w_n) \ge \Theta^{5^n}\lambda_0^{2\cdot 5^n} =: c_2(h_0) \end{aligned}$$

that is,

$$(25) \qquad \Lambda(0) \ge c_2(h_0).$$

(vii) We have

$$\kappa_1^2 + \kappa_2^2 = 4H^2 - 2K = \frac{|\nabla N|^2}{\Lambda}$$

on account of formula (14) in Section 5.1. Setting

$$c(h_0) := c_1^2(h_0)c_2^{-1}(h_0),$$

the estimates (13) and (25) yield the desired inequality

$$\kappa_1^2(0) + \kappa_2^2(0) \le c(h_0). \qquad \square$$

By a scaling argument we immediately obtain the interesting

Theorem 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ denote a stable, immersed minimal surface representing a geodesic disk $K_r(x_0)$ of radius $r > 0$ centered at $x_0 := X(0)$, briefly: $X(\overline{B}) = K_r(x_0)$. Then the principal curvatures κ_1 and κ_2 of X satisfy*

$$\kappa_1^2(0) + \kappa_2^2(0) \le \frac{c(0)}{r^2} \tag{26}$$

where $c(0)$ is the universal constant $c(h_0)$ of Theorem 1 for $h_0 = 0$.

Proof. Consider the scaled minimal surface

$$Y := \frac{1}{r}(X - x_0), \quad r > 0.$$

The normals of X and Y coincide whereas the Weingarten mapping $\tilde{S}$ of Y differs from the Weingarten mapping S of X by the factor r. Hence the principal curvatures of Y are $r\kappa_1$ and $r\kappa_2$ if κ_1, κ_2 are the principal curvatures of X, and $Y(\overline{B}) = K_1(0)$. Then formula (1) of Theorem 1 yields

$$r^2(\kappa_1^2(0) + \kappa_2^2(0)) \le c(0),$$

which is the desired estimate (26). $\square$

As a corollary of Theorem 2 we obtain the following "Bernstein-type" result proved by do Carmo and Peng [1] and Fischer-Colbrie and Schoen [1].

Theorem 3. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^3$ represent a regular and embedded minimal surface which is geodesically complete and stable (that is, stable on each geodesic disk). Then Y represents a plane in $\mathbb{R}^3$.*

Proof. The set $\mathcal{M} := Y(\mathbb{R}^2)$ is a complete Riemannian manifold of dimension two, the Gauss curvature of which is nonpositive. A theorem by Hadamard implies that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2$. Thus, for each $r > 0$ and for any point $x_0 \in \mathcal{M}$, there is a geodesic disk $K_r(x_0)$ on $\mathcal{M}$ about the center x_0. If Y is not already conformal, then we introduce conformal parameters on $K_r(x_0)$, obtaining a harmonic mapping X from $\overline{B}$ onto $K_r(x_0)$ such that $X(0) = x_0$. By Theorem 2, the principal curvatures κ_1 and κ_2 of $\mathcal{M}$ at x_0, i.e. the principal curvatures of X at $w = 0$, are zero, if we let r tend to ∞. Since x_0 can be chosen arbitrarily on $\mathcal{M}$, it follows that Y represents a plane in $\mathbb{R}^3$. $\square$

5.6 Nitsche's Uniqueness Theorem and Field-Immersions

In this section we prove a uniqueness theorem, due to J.C.C. Nitsche [26], for minimal surfaces solving Plateau's problem. This result was already stated in Section 4.9.

Proposition 1. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ denote an immersed minimal surface with the normal N. For any function $\zeta \in C^{3,\alpha}(\overline{B})$ we consider the* varied surface $Y : \overline{B} \to \mathbb{R}^3$ *defined by*

$$Y := X + \zeta N. \tag{1}$$

Then Y represents an immersed, but not necessarily conformal, surface of zero mean curvature if and only if ζ satisfies the perturbation equation

$$L\zeta = \Phi(\zeta) \quad \text{in } B \tag{2}$$

where L denotes the Schwarzian operator L *associated with the minimal surface X, which is defined by*

$$L\zeta := -\Delta\zeta + 2\Lambda K\zeta. \tag{3}$$

Here Λ is the area element of X, and K is its Gaussian curvature. The right-hand side Φ in (2) is a sum of homogeneous terms of second till fifth order in $\zeta, \nabla\zeta$ and $\nabla^2\zeta$, the coefficient-functions of which depend on $X, \nabla X, \nabla^2 X, \nabla^3 X$ and on $1/\Lambda$. Furthermore, there is a constant $c_1 > 0$ depending only on $\|X\|_{C^{3,\alpha}(\overline{B},\mathbb{R}^3)}$ and $\|1/\Lambda\|_{C^0(\overline{B})}$ such that Φ satisfies

$$\|\Phi(\zeta) - \Phi(\eta)\|_{C^{0,\alpha}(\overline{B})} \le c_1[\|\zeta\|_{C^{2,\alpha}(\overline{B})} + \|\eta\|_{C^{2,\alpha}(\overline{B})}]\|\zeta - \eta\|_{C^{2,\alpha}(\overline{B})} \tag{4}$$

for all $\zeta, \eta \in C^{2,\alpha}(\overline{B})$ whose $C^{2,\alpha}(\overline{B})$-norms are bounded by one.

Proof. Differentiating (1) we obtain

$$Y_u = X_u + \zeta_u N + \zeta N_u, \quad Y_v = X_v + \zeta_v N + \zeta N_v \tag{5}$$

and

$$\begin{aligned} Y_{uu} &= X_{uu} + \zeta_{uu} N + 2\zeta_u N_u + \zeta N_{uu}, \\ Y_{uv} &= X_{uv} + \zeta_{uv} N + (\zeta_u N_v + \zeta_v N_u) + \zeta N_{uv}, \\ Y_{vv} &= X_{vv} + \zeta_{vv} N + 2\zeta_v N_v + \zeta N_{vv}. \end{aligned} \tag{6}$$

We write the first fundamental form of Y as

$$\langle dY, dY \rangle = \mathcal{E}^* \, du^2 + 2\mathcal{F}^* \, du \, dv + \mathcal{G}^* dv^2 \tag{7}$$

and, using (5), evaluate $\mathcal{E}^*, \mathcal{F}^*, \mathcal{G}^*$. Recall that the coefficients of the second fundamental form of X are given by

$$\begin{aligned}\mathcal{L} &= -\langle X_u, N_u\rangle = \langle X_{uu}, N\rangle, \quad \mathcal{N} = -\langle X_v, N_v\rangle = \langle X_{vv}, N\rangle,\\ \mathcal{M} &= -\langle X_u, N_v\rangle = -\langle X_v, N_u\rangle = \langle X_{uv}, N\rangle.\end{aligned}$$

From (5) we infer

$$\frac{\mathcal{E}^*}{\Lambda} = 1 - 2\zeta\frac{\mathcal{L}}{\Lambda} + \cdots, \quad \frac{\mathcal{F}^*}{\Lambda} = -2\zeta\frac{\mathcal{M}}{\Lambda} + \cdots, \quad \frac{\mathcal{G}^*}{\Lambda} = 1 - 2\zeta\frac{\mathcal{N}}{\Lambda} + \cdots \tag{8}$$

where $+\cdots$ stands for terms which are quadratic in $\zeta, \zeta_u, \ldots, \zeta_{vv}$.

The surface Y has zero mean curvature if and only if

$$0 = \left\langle \frac{\mathcal{E}^*}{\Lambda}Y_{vv} - 2\frac{\mathcal{F}^*}{\Lambda}Y_{uv} + \frac{\mathcal{G}^*}{\Lambda}Y_{uu}, \frac{1}{\Lambda}(Y_u \wedge Y_v)\right\rangle. \tag{9}$$

From (5), (6), and (8) we obtain the differential equation (2) with the Schwarzian operator L and a right-hand side Φ that has the properties described. Let us sketch the necessary computations: We write (9) as

$$\text{Linear expression in } \zeta, \zeta_u, \ldots, \zeta_{vv} + \Phi(\zeta) = 0$$

where $\Phi(\zeta)$ consists of all nonlinear ζ-terms. Now $\Phi(\zeta)$ is a polynomial of fifth degree in $\zeta, \zeta_u, \ldots, \zeta_{vv}$ with coefficients depending on $1/\Lambda, X, \nabla X, \nabla^2 X, N, \nabla N, \nabla^2 N$. Obviously we can estimate these coefficients in the $C^{0,\alpha}$-norm using a bound for $\|X\|_{C^{3,\alpha}}$ and $\|1/\Lambda\|_{C^{0,\alpha}}$ on $\overline{B}$. The terms of $\Phi(\zeta)$ are at least quadratic in ζ and its derivatives up to second order.

Furthermore,

$$\frac{1}{\Lambda}Y_u \wedge Y_v = N + (\text{terms in } \zeta, \zeta_u, \ldots, \zeta_{vv} \text{ of at least first order}), \tag{10}$$

and (6) and (8) imply

$$\begin{aligned}&\frac{\mathcal{E}^*}{\Lambda}Y_{vv} - 2\frac{\mathcal{F}^*}{\Lambda}Y_{uv} + \frac{\mathcal{G}^*}{\Lambda}Y_{uu}\\ &\quad = \Delta X + \Delta\zeta \cdot N + 2\zeta_u N_u + 2\zeta_v N_v + \zeta\Delta N\\ &\qquad - 2\frac{\zeta}{\Lambda}(\mathcal{L}X_{vv} - 2\mathcal{M}X_{uv} + \mathcal{N}X_{uu}) + \cdots\\ &\quad = (\Delta\zeta + 2\Lambda K\zeta)N + 2(\zeta_u N_u + \zeta_v N_v)\\ &\qquad - 2\frac{\zeta}{\Lambda}(\mathcal{L}X_{vv} - 2\mathcal{M}X_{uv} + \mathcal{N}X_{uu}) + \cdots\end{aligned} \tag{11}$$

where $+\cdots$ stands again for terms that are quadratic in $\zeta, \ldots, \zeta_{vv}$.

Here we have used the equation $\Delta N = 2\Lambda K N$, cf. (11) and (12) of 5.1. From (9), (10) and (11) it follows that

$$0 = \Delta\zeta + 2\Lambda K\zeta - 2\frac{\zeta}{\Lambda}(\mathcal{L}\mathcal{N} - 2\mathcal{M}^2 + \mathcal{N}\mathcal{L}) + \cdots.$$

Furthermore, by formula (32) of Section 1.3 we have

$$\Lambda^2 K = \mathcal{L}\mathcal{N} - \mathcal{M}^2$$

and so we arrive at

$$0 = \Delta\zeta - 2\Lambda K\zeta + \cdots .$$

Therefore equation (9) is equivalent to

$$-\Delta\zeta + 2\Lambda K\zeta = \Phi(\zeta) \tag{12}$$

as it was claimed.

The nonlinearity $\Phi(\zeta)$ consists of finitely many terms of the form

$$a(X)\partial^{i_1}\zeta \dots \partial^{i_k}\zeta$$

with $2 \le k \le 5$ where ∂^{i_ℓ} denotes a partial derivative of order i_ℓ with $0 \le i_\ell \le 2$, and $\|a(X)\|_{C^\alpha(\overline{B})}$ can be estimated by $\|X\|_{C^{3,\alpha}(\overline{B},\mathbb{R}^3)}$ and $\|1/\Lambda\|_{C^0(\overline{B})}$. We leave it as an easy exercise to the reader to verify the "condition of contraction" (4). □

With the aid of Schauder's theory we will now show the fundamental result that a strictly stable, immersed minimal surface $X : \overline{B} \to \mathbb{R}^3$ can be embedded into a field of surfaces of zero mean curvature provided that X is extendable beyond ∂B.

Proposition 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be a strictly stable, immersed minimal surface which can be extended as a minimal surface to a larger disk $\Omega := B_{1+\delta}(0)$ with $\delta > 0$. Then there is a one-parameter family*

$$Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$$

of zero mean curvature surfaces $Z(\cdot, t)$ (not necessarily conformally parametrized) which is of class $C^{2,\alpha}(\overline{B} \times [-t_0, t_0], \mathbb{R}^3)$, $|t| \le t_0$, and has the following properties:

(a) $Z(w, 0) = X(w)$ *for* $w \in \overline{B}$;
(b) $J_Z := \frac{\partial(Z^1,Z^2,Z^3)}{\partial(u,v,t)} > 0$ *on* $\overline{B} \times [-t_0, t_0]$;
(c) *If $N^*(\cdot, t)$ denotes the normal to the surface $Z(\cdot, t)$, one has*

$$Z_t(w, t) = \rho(w, t) N^*(w, t) \quad \textit{for } w \in \overline{B} \textit{ and } |t| < t_0$$

with $\rho(w, t) > 0$ on $\overline{B} \times [-t_0, t_0]$.

Definition 1. *A mapping Z as described in Proposition 2 is called a* **field immersion** *of the minimal surface X.*

Proof of Proposition 2. (i) It is easily seen that also the extension $X : \Omega \to \mathbb{R}^3$ is strictly stable and immersed for $0 < \delta \ll 1$. The strict stability of this extension implies that the boundary value problem

$$L\zeta = 0 \quad \text{in } \Omega, \quad \zeta = 0 \quad \text{on } \partial\Omega$$

has only the trivial solution $\zeta(w) \equiv 0$ on $\overline{\Omega}$. Then Schauder's theory implies that there is a uniquely determined solution $\xi \in C^{2,\alpha}(\overline{\Omega})$ of the boundary value problem

$$(13) \qquad L\xi = 0 \quad \text{in } \Omega, \quad \xi = 1 \quad \text{on } \partial\Omega$$

(see e.g. Sauvigny [16], Chapter IX, §6, Theorem 5).

By virtue of Proposition 1 in 5.3 it follows that $\xi(w) > 0$ for all $w \in \overline{\Omega}$. Set

$$C_0^{2,\alpha}(\overline{\Omega}) := \{\eta \in C^{2,\alpha}(\overline{\Omega}) \colon \eta(w) = 0 \text{ for all } w \in \partial\Omega\}$$

and note that the operator

$$(14) \qquad L_0 := L|_{C_0^{2,\alpha}(\overline{\Omega})} : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$$

is an invertible mapping satisfying

$$(15) \qquad \|L_0^{-1} f\|_{2,\alpha} \le c_2 \|f\|_\alpha \quad \text{for all } f \in C^{0,\alpha}(\overline{\Omega})$$

with an a priori constant $c_2 > 0$. Here we have used the abbreviating notation

$$\|\cdot\|_{m,\alpha} := \|\cdot\|_{C^{m,\alpha}(\overline{\Omega})}.$$

Finally set

$$c_3 := \|\xi\|_{2,\alpha}.$$

(ii) For sufficiently small $t_1 > 0$ we now want to solve the nonlinear Dirichlet problem

$$(16) \qquad L\zeta(\cdot,t) = \Phi(\zeta(\cdot,t)) \quad \text{in } \Omega, \quad \zeta(\cdot,t) = t \quad \text{on } \partial\Omega$$

by $\zeta(\cdot,t) \in C^{2,\alpha}(\overline{\Omega})$ and for parameter values with $|t| \le t_1$. To this end we make the “Ansatz”

$$(17) \qquad \zeta(w,t) := \eta(w,t) + t\xi(w) \quad \text{for } w \in \overline{\Omega}, \quad |t| < t_1,$$

where $\eta(\cdot,t) \in C_0^{2,\alpha}(\overline{\Omega})$ is to be determined as solution of

$$(18) \qquad L\eta(\cdot,t) = \Phi(\eta(\cdot,t) + t\xi) \quad \text{in } \Omega.$$

This is equivalent to finding a solution $\eta(\cdot,t) \in C_0^{2,\alpha}(\overline{\Omega})$ of the fixed point equation

$$(19) \qquad \eta(\cdot,t) = L_0^{-1}\Phi^t(\eta(\cdot,t)) \quad \text{with } \Phi^t(\zeta) := \Phi(\zeta + t\xi).$$

In order to solve (19) by Banach's fixed point theorem we introduce the balls

$$\mathcal{B}(t) := \{\zeta \in C_0^{2,\alpha}(\overline{\Omega}) \colon \|\zeta\|_{2,\alpha} \le |t|^{3/2}\}$$

with $|t| \le t_1 \ll 1$. For $\zeta \in \mathcal{B}(t)$ and $|t| \ll 1$ we have $\|\zeta + t\xi\|_{2,\alpha} \le 1$. Then by (15) and (4) ($B$ replaced with Ω) we obtain for $\zeta \in \mathcal{B}(t)$ that

$$\begin{aligned} \|L_0^{-1}\Phi^t(\zeta)\|_{2,\alpha} &\le c_2\|\Phi^t(\zeta)\|_\alpha = c_2\|\Phi(\zeta + t\xi)\|_\alpha \\ &\le c_2c_1\|\zeta + t\xi\|^2_{2,\alpha} \le 2c_1c_2\{\|\zeta\|^2_{2,\alpha} + t^2\|\xi\|^2_{2,\alpha}\} \\ &\le 2c_1c_2\{|t|^3 + c_3^2t^2\} = 2c_1c_2\{|t|^{3/2} + c_3^2|t|^{1/2}\}|t|^{3/2}. \end{aligned}$$

For $|t| \le t_1 \ll 1$ it follows that

$$\|L_0^{-1}\Phi^t(\zeta)\|_{2,\alpha} \le |t|^{3/2},$$

and the operator $L_0^{-1}\Phi^t$ maps $\mathcal{B}(t)$ into itself.

Secondly, for $\zeta, \eta \in \mathcal{B}(t)$ with $|t| \ll 1$ we have $\|\zeta + t\xi\|_{2,\alpha} \le 1$ and also $\|\eta + t\xi\|_{2,\alpha} \le 1$, whence

$$\begin{aligned} \|L_0^{-1}\Phi^t(\zeta) - L_0^{-1}\Phi^t(\eta)\|_{2,\alpha} &= \|L_0^{-1}(\Phi^t(\zeta) - \Phi^t(\eta))\|_{2,\alpha} \\ \le c_2\|\Phi^t(\zeta) - \Phi^t(\eta)\|_\alpha &= c_2\|\Phi(\zeta + t\xi) - \Phi(\zeta + t\eta)\|_\alpha \\ \le c_2c_1|t| \cdot \|\zeta - \eta\|_{2,\alpha} &\le \tfrac{1}{2}\|\zeta - \eta\|_{2,\alpha} \quad \text{for } |t| \ll 1, \end{aligned}$$

and so

$$\begin{aligned} &\|L_0^{-1}\Phi^t(\zeta) - L_0^{-1}\Phi^t(\eta)\|_{2,\alpha} \le \tfrac{1}{2}\|\zeta - \eta\|_{2,\alpha} \\ &\quad \text{for } \zeta, \eta \in \mathcal{B}(t) \text{ and } |t| < t_1 \text{ with } 0 < t_1 \ll 1. \end{aligned}$$

Therefore the mapping $L_0^{-1}\Phi^t : \mathcal{B}(t) \to \mathcal{B}(t)$ is contracting, and so it possesses a uniquely determined fixed point $\eta(\cdot, t) \in C_0^{2,\alpha}(\overline{\Omega})$ for $0 < |t| < t_1$ with $0 < t_1 \ll 1$, and for $t = 0$ we have $L_0^{-1}\Phi^0(0) = 0$, i.e. $\eta(\cdot, 0) = 0$. A slight modification of the proof shows that $\eta(\cdot, t)$ is differentiable with respect to t and that even $\eta \in C^{2,\alpha}(\overline{\Omega} \times [-t_1, t_1])$ holds true (see e.g. Giaquinta and Hildebrandt [1], vol. 1, Chapter 6). Moreover, the choice of $\mathcal{B}(t)$ shows that

$$\eta_t(w, 0) = 0 \quad \text{for } w \in \overline{\Omega},$$

and so the superposition (17) yields a solution $\zeta \in C^2(\overline{\Omega} \times [-t_1, t_1])$ of (16), satisfying

$$\zeta_t(w, 0) = \xi(w) > 0 \quad \text{for all } w \in \overline{\Omega}.$$

For $0 < t_1 \ll 1$ we then obtain

$$\zeta_t(w, t) > 0 \quad \text{for } w \in \overline{\Omega} \text{ and } |t| \le t_1. \tag{20}$$

Hence the family $Y : \overline{\Omega} \times [-t_1, t_1] \to \mathbb{R}^3$ constitutes a field of surfaces

$$Y(\cdot, t) = X + \zeta(\cdot, t)N \tag{21}$$

of zero mean curvature surfaces in $\mathbb{R}^3$ with $Y(\cdot, 0) = X$. Finally by reparametrizing Y via their orthogonal trajectories we obtain for some $t_0 \in [0, t_1]$ a family

$$Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3 \tag{22}$$

of zero mean curvature surfaces $Z(\cdot, t)$ satisfying (a)–(c). (The reparametrization is left to the reader as an exercise in ordinary differential equations.) □

From Section 2.8 we already know that those minimal surfaces that can be embedded into a foliation of simply covering surfaces of zero mean curvature furnish a relative minimum of the area functional. Now we are confronted with the more intricate problem to prove a similar property for immersed minimal surfaces that can be embedded into a field of surfaces with $H = 0$ that might have selfintersections.

Let Γ be an oriented Jordan curve in $\mathbb{R}^3$, and denote by $\mathcal{C}(\Gamma)$ the class of surfaces $X \in H_2^1(B, \mathbb{R}^3)$ bounded by Γ in the sense of Section 4.2. Set

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3),$$

and define

$$\overline{\mathcal{C}}^*(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon X(w_j) = Q_j,\ j = 1, 2, 3\}$$

where Q_1, Q_2, Q_3 are three fixed points on Γ and $w_j = \exp(\frac{2\pi i}{3} j)$, $j = 1, 2, 3$.

Proposition 3. *Let $X \in \mathcal{C}(\Gamma)$ be a minimal surface that satisfies the assumptions of Proposition 2. Then there is a number $\epsilon = \epsilon(X) > 0$ such that*

$$D(X) < D(Y) \quad \textit{for all } Y \in \overline{\mathcal{C}}^*(\Gamma) \textit{ with } 0 < \sup_B |Y(w) - X(w)| < \epsilon.$$

Proof. (i) We embed X in a field immersion $Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$ as described in Proposition 2 and consider the corresponding surface element

$$\mathcal{W}(w, t) := |Z_u(w, t) \wedge Z_v(w, t)| = [N^*(w, t), Z_u(w, t), Z_v(w, t)].$$

Using the orthogonality condition and (c) we obtain

$$\begin{aligned} \mathcal{W}_t &= [N_t^*, Z_u, Z_v] + [N^*, Z_{tu}, Z_v] + [N^*, Z_u, Z_{tv}] \\ &= 0 + [N^*, (\rho N^*)_u, Z_v] + [N^*, Z_u, (\rho N^*)_v] \\ &= \rho\{[N^*, N_u^*, Z_v] + [N^*, Z_u, N_v^*]\} \\ &= \rho\langle N^*, N_u^* \wedge Z_v + Z_u \wedge N_v^* \rangle. \end{aligned}$$

Furthermore, Theorem 2 of Section 2.5 yields

$$N_u^* \wedge Z_v + Z_u \wedge N_v^* = 0.$$

Thus we have

$$\mathcal{W}_t(w, t) = 0 \quad \text{for } w \in \overline{B} \text{ and } |t| \le t_0. \tag{23}$$

(ii) The field immersion of Proposition 2 is constructed even on a larger disk Ω_0 with $B \subset\subset \Omega_0 \subset\subset \Omega$. This implies that $Z : \overline{\Omega}_0 \times [-t_0, t_0] \to \mathbb{R}^3$ furnishes a local diffeomorphism provided that $0 < t_0 \ll 1$, but globally the inverse of Z need not exist. For compactness reasons the local inverse Z^{-1} is defined on domains of uniform size. Consequently there is an $\epsilon = \epsilon(X) > 0$ such that all admissible $Y \in \overline{\mathcal{C}}^*(\Gamma)$ with $\sup_B |Y - X| < \epsilon$ can be written as

$$Y(w) = Z(f(w), \tau(w)), \quad w \in \overline{B}, \tag{24}$$

with a continuous mapping f from $\overline{B}$ into $\mathbb{R}^2$ and a continuous function $\tau : \overline{B} \to \mathbb{R}$ such that

$$f \text{ maps } \partial B \text{ monotonically onto itself}, \quad f(w_j) = w_j, \quad j = 1, 2, 3, \tag{25}$$

and $f|_{\partial B}$ is positive-oriented with respect to B, and

$$\tau(w) = 0 \quad \text{for } w \in \partial B \quad \text{and} \quad |\tau(w)| \leq t_0 \quad \text{on } \overline{B}. \tag{26}$$

On account of Dirichlet's principle, harmonic functions are unique minimizers of D for given boundary values; thus it suffices to consider Y, f, τ that are real analytic on B and of class C^0 on $\overline{B}$.

(iii) Assume for the moment that f furnishes a diffeomorphism from B onto B with the inverse g, and set $\sigma := \tau \circ g$ as well as

$$\tilde{Y}(w) := Y(g(w)) = Z(w, \sigma(w)). \tag{27}$$

Then it follows that

$$\begin{aligned} \tilde{Y}_u(w) \wedge \tilde{Y}_v(w) &= Y_u(g(w)) \wedge Y_v(g(w)) J_g(w) \\ &= [(Z_u + Z_t \sigma_u) \wedge (Z_v + Z_t \sigma_v)](w, \sigma(w)) \\ &= [Z_u \wedge Z_v + \sigma_u Z_t \wedge Z_v + \sigma_v Z_u \wedge Z_t](w, \sigma(w)). \end{aligned}$$

Multiplication by $N^*(w, \sigma(w))$ yields by virtue of (23) that

$$\begin{aligned} &\langle N^*(w, \sigma(w)), Y_u(g(w)) \wedge Y_v(g(w)) \rangle J_g(w) \\ &\quad = \mathcal{W}(w, \sigma(w)) = \mathcal{W}(w, 0) = |X_u(w) \wedge X_v(w)|. \end{aligned}$$

Integration over B then leads to the *Schwarz comparison formula*:

$$\int_B \langle N^*(f(w), \tau(w)), Y_u(w) \wedge Y_v(w) \rangle \, du \, dv = \int_B |X_u \wedge X_v| \, du \, dv. \tag{28}$$

(As demonstrated in Section 2.8, this formula can be seen as a precursor of Hilbert's independent integral.) The relation (28) implies $A(Y) \geq A(X)$, and the equality sign holds if and only if $Y_u \wedge Y_v$ points in the direction of $N^*(f, \tau)$. This implies $Y = X$, and the result is proved.

(iv) In the sequel we have to verify this result even if f is not a global diffeomorphism of B onto B. We have to deal with the possibility that $f(B)$

might "overshoot" B, and that $f(B)$ could cover B in several layers. We note first that, in general, the critical values of the real analytic mapping $f : B \to B$ constitute a Lebesgue null set $\mathcal{N}$ in $\mathbb{R}^2$ (see Sauvigny [16], Chapter III, §4). Combining this observation with arguments using the winding number, we come to the following

Conclusion. There exist sequences $\{G_\ell\}$ and $\{H_\ell\}$ of subdomains of B such that

$$(29) \qquad f_\ell := f|_{G_\ell} : G_\ell \to \mathbb{R}^2 \quad \text{is a positively oriented diffeomorphism from } G_\ell \text{ onto } H_\ell;\ \tau_\ell := \tau|_{G_\ell};$$

and

$$(30) \qquad G_\ell \cap G_k = \emptyset, \quad H_\ell \cap H_k = \emptyset \quad \text{for } \ell \neq k;$$
$$B \Big\backslash \bigcup_{\ell=1}^{\infty} H_\ell \text{ is a Lebesgue null set in } \mathbb{R}^2.$$

One obtains the G_ℓ and H_ℓ as follows: For $z_0 \in B \setminus \mathcal{N}$ one has pre-images $w_0, w_0', w_0'', \ldots$ such that $J_f(w_0) \neq 0$, $J_f(w_0') \neq 0$, $J_f(w_0'') \neq 0, \ldots$. Since $f|_{\partial B}$ is positive oriented, at least one of these numbers has to be positive, say, $J_f(w_0) > 0$. Then there is a neighborhood G of w_0 such that $f|_G$ is a positively oriented diffeomorphism of G onto a neighborhood H of z_0. A repeated application of this argument leads to the selection of diffeomorphisms $f_\ell : G_\ell \to H_\ell$ with the properties (29) and (30). Note that $B \setminus \bigcup G_\ell$ might have positive measure. This means that we have omitted multiple coverings of B by $f(B)$ as well as parts of B that are mapped onto $f(B) \setminus (B)$.

When we apply the arguments of part (iii) to these individual diffeomorphisms, the Schwarz comparison formula (28) implies

$$(31) \qquad \begin{aligned} D(Y) \geq A(Y) &= \int_B |Y_u \wedge Y_v| \, du \, dv \\ &\geq \sum_{\ell=1}^{\infty} \int_{G_\ell} |Y_u \wedge Y_v| \, du \, dv \\ &\geq \sum_{\ell=1}^{\infty} \int_{G_\ell} \langle N^*(f_\ell, \tau_\ell), Y_u \wedge Y_v \rangle \, du \, dv \\ &= \sum_{\ell=1}^{\infty} \int_{H_\ell} |X_u \wedge X_v| \, du \, dv = A(X) = D(X). \end{aligned}$$

(v) If $D(X) = D(Y)$ then all inequalities in (31) turn into equalities, and so we have in particular that $\mathcal{S} := B \setminus \bigcup G_\ell$ is a two-dimensional null set. Otherwise we would have $\nabla Y(w) = 0$ for $w \in \mathcal{S}$ with meas $\mathcal{S} > 0$, and therefore $\nabla Y(w) \equiv 0$ on B since Y is real analytic. This implies $Y(w) \equiv \text{const}$

on B, a contradiction to $Y \in \mathcal{C}(\Gamma)$. Thus we obtain $J_f(w) = \frac{\partial(f^1,f^2)}{\partial(u,v)} > 0$ a.e. on B. Furthermore, the vectors

$$
\begin{aligned}
Y_u \wedge Y_v \;=\; & Z_u(f_\ell,\tau_\ell) \wedge Z_v(f_\ell,\tau_\ell) \frac{\partial(f_\ell^1, f_\ell^2)}{\partial(u,v)} + Z_v(f_\ell,\tau_\ell) \wedge Z_t(f_\ell,\tau_\ell) \frac{\partial(f_\ell^2,\tau_\ell)}{\partial(u,v)} \\
& + Z_t(f_\ell,\tau_\ell) \wedge Z_u(f_\ell,\tau_\ell) \frac{\partial(\tau_\ell, f_\ell^1)}{\partial(u,v)}
\end{aligned}
$$

have to point into the direction of the normals $N^*(f_\ell,\tau_\ell)$ on G_ℓ. Thus the two determinants

$$\frac{\partial(f_\ell^2,\tau_\ell)}{\partial(u,v)} \quad \text{and} \quad \frac{\partial(\tau_\ell, f_\ell^1)}{\partial(u,v)}$$

have to vanish on G_ℓ for $\ell = 1, 2, \ldots$, since $Z_v \wedge Z_t$ and $Z_t \wedge Z_u$ are two linearly independent vectors perpendicular to N^*, and so we obtain

$$\nabla\tau = 0 \quad \text{in} \; \bigcup_{\ell=1}^{\infty} G_\ell$$

because of $J_f = \frac{\partial(f^1,f^2)}{\partial(u,v)} > 0$ on G_ℓ. Since meas $\mathcal{S} = 0$ we obtain $\nabla\tau(w) \equiv 0$ in B whence $\tau(w) \equiv$ const in $\overline{B}$, and $\tau = 0$ on ∂B yields $\tau(w) = 0$ on $\overline{B}$. Thus we arrive at

$$Y(w) = Z(f(w), 0) = X(f(w)) \quad \text{for } w \in \overline{B}. \tag{32}$$

(vi) We have found: Any $Y \in \overline{\mathcal{C}}^*(\Gamma)$ with $\max_{\overline{B}} |Y - X| < \epsilon \ll 1$ satisfies $D(Y) \geq D(X)$, and $D(Y) = D(X)$ if and only if $Y = X \circ f$. It follows that $D(Y) \leq D(\tilde{Y})$ for all $\tilde{Y} \in \overline{\mathcal{C}}^*(\Gamma)$ with $\max_{\overline{B}} |\tilde{Y} - Y| < \tilde{\epsilon}$ for some $\tilde{\epsilon}$ with $0 < \tilde{\epsilon} \ll 1$. This implies that Y is conformally parametrized in the sense that $Y_w \cdot Y_w = 0$. Hence it follows from (32) that f is a mapping from $\overline{B}$ onto $\overline{B}$ which is conformal (in the generalized sense) in B, monotonic on ∂B with $f(\partial B) = \partial B$ and $f(w_j) = w_j$, $j = 1, 2, 3$. We conclude that $f(w) \equiv w$ on $\overline{B}$ and therefore $Y(w) = X(w)$ on $\overline{B}$. Thus we have proved

$$D(X) < D(Y) \quad \text{for } 0 < \sup_B |X - Y| < \epsilon. \qquad \square$$

We are now prepared to prove the following

Theorem 1 (J.C.C. Nitsche). *Let Γ be a real analytic, regular Jordan curve with a total curvature $\kappa(\Lambda)$ less or equal to 4π. Then there is exactly one disk-type minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$, i.e. exactly one solution $X \in \overline{\mathcal{C}}^*(\Gamma)$ solving Plateau's problem to the contour Γ. This solution is free of branch points up to the boundary and can be continued analytically across Γ as a minimal surface.*

Proof. (i) If Γ lies in the plane E, any minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ is contained in this plane as well, due to the convex hull property, and so it reduces to a strictly conformal or anticonformal mapping from B onto the interior of Γ in E which is uniquely determined by the three-point condition $X(w_j) = Q_j$, $j = 1, 2, 3$ (cf. Section 4.11). By the asymptotic expansion of X_w at $w_0 \in \partial B$ it turns out that there are no boundary branch points of X, because otherwise $X(\overline{B})$ would overshoot Γ into $\mathbb{R}^2 \setminus \overline{\Omega}$, where Ω is the interior domain of Γ. Thus the assertion of the theorem holds in this case even without the assumption $\kappa(\Gamma) \leq 4\pi$; actually it would suffice that $\Gamma \in C^{2,\alpha}$ (or even $\Gamma \in C^{1,\alpha}$) in order to prove the uniqueness of $X \in \overline{\mathcal{C}}^*(\Gamma)$.

(ii) Thus from now on we assume that Γ *is nonplanar*. By H. Lewy's regularity theorem [5] we know that any minimal surface $X \in \overline{\mathcal{C}}(\Gamma)$ can be continued analytically across Γ onto a larger disk $\Omega := B_{1+\delta}(0)$, cf. Vol. 2, Section 2.8. Furthermore, by the Gauss–Bonnet formula established in Vol. 2, Section 2.11, we have the following: Let $w_1, \dots, w_k \in B$ and $w_{k+1}, \dots, w_{k+\ell} \in \partial B$ be the finitely many branch points of a minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ with the orders $\nu_1, \dots, \nu_k$ and $\nu_{k+1}, \dots, \nu_{k+\ell}$ respectively (see Vol. 2, Section 2.10), $\nu_j \in \mathbb{N}$. Then

$$0 \leq \sum_{j=1}^{k} \nu_j + \frac{1}{2} \sum_{j=k+1}^{k+\ell} \nu_j = \frac{1}{2\pi} \left\{ \int_B K \Lambda \, du \, dv + \int_\Gamma \kappa_g \, ds - 2\pi \right\}. \tag{33}$$

In (iii) we shall see that the integral of the geodesic curvature κ_g of Γ on X is bounded by the total curvature of Γ, i.e.

$$\int_\Gamma \kappa_g \, ds \leq \int_\Gamma \kappa \, ds =: \kappa(\Gamma), \tag{34}$$

and by assumption we have $\kappa(\Gamma) \leq 4\pi$. Therefore we obtain

$$\int_\Gamma \kappa_g \, ds - 2\pi \leq 2\pi. \tag{35}$$

The Gaussian curvature K of X satisfies $K = \kappa_1 \kappa_2 = -\kappa_1^2 \leq 0$ in $B' = B \setminus \{w_1, \dots, w_k\}$ since $0 = 2H = \kappa_1 + \kappa_2$. If we had $K(w) \equiv 0$ in B', it would follow that the Weingarten mapping of X were everywhere trivial in B', i.e. $N(w) \equiv \text{const}$ in B' and then in B. This would imply that X and thus Γ were planar, which is excluded by the assumption above. Therefore $K(w) \not\equiv 0$ on B', and consequently

$$\int_B K \Lambda \, du \, dv < 0. \tag{36}$$

From (33), (35), and (36) it follows that

$$0 \leq \sum_{j=1}^{k} \nu_j + \frac{1}{2} \sum_{j=k+1}^{k+\ell} \nu_j < 1. \tag{37}$$

Furthermore the orders of the boundary branch points $w_{k+1}, \dots, w_{k+\ell}$ have to be even because of the monotonicity of the mapping $X|_{\partial B}$ (see Vol. 2, Section 2.10). Therefore (37) implies

$$\nu_j = 0 \quad \text{for } j = 1, \dots, k+\ell, \tag{38}$$

i.e. any minimal surface $X \in \mathcal{C}(\Gamma)$ with $\kappa(\Gamma) \le 4\pi$ is an immersion of $\overline{B}$ into $\mathbb{R}^3$, and (33) reduces to the classical Gauss–Bonnet theorem

$$-\int_B K\Lambda\, du\, dv = \int_\Gamma \kappa_g\, ds - 2\pi \tag{39}$$

for an immersed minimal surface X.

(iii) Let us parametrize Γ by the arc length parameter $s \in [0, L]$, $L =$ length of Γ, setting

$$Y(s) := X(\cos\varphi(s), \sin\varphi(s)), \quad 0 \le s \le L,$$

satisfying $|Y'(s)| \equiv 1$ for $0 \le s \le L$. Furthermore set

$$Z(s) := N(\cos\varphi(s), \sin\varphi(s)), \quad 0 \le s \le L.$$

Then $\kappa(s) = |Y''(s)|$ is the curvature of the arc Γ at the point $Y(s)$, and

$$\kappa(\Gamma) = \int_0^L \kappa(s)\, ds = \int_0^L |Y''(s)|\, ds$$

is the total curvature of Γ. Since the geodesic curvature satisfies

$$|\kappa_g(s)| = |[Y''(s), Z(s), Y'(s)]|$$

and the normal curvature κ_n of Y fulfills

$$|\kappa_n(s)| = |\langle Z(s), Y''(s)\rangle|,$$

we have the decomposition

$$\kappa^2(s) = \kappa_g^2(s) + \kappa_n^2(s)$$

whence indeed

$$|\kappa_g(s)| \le \kappa(s) \quad \text{for } 0 \le s \le L.$$

For $\int_0^L \kappa_g\, ds = \int_0^L \kappa\, ds$, then $\kappa_n(s) \equiv 0$ for $0 \le s \le L$. Note that with $w(s) := (\cos\varphi(s), \sin\varphi(s))$ we obtain

$$Y' = [X_u(w)(-\sin\varphi) + X_v(w)\cos\varphi]\varphi'$$

and

$$Y'' = [X_{uu}(w)(\sin^2\varphi) - 2X_{uv}(w)\sin\varphi\cos\varphi + X_{vv}(w)\cos^2\varphi]|\varphi'|^2 + \cdots$$

where $+\cdots$ stands for the neglected tangential terms.

From $\kappa_n(s) \equiv 0$ and $|\kappa_n(s)| = |\langle Z(s), Y''(s)\rangle|$ as well as $X_{uu} = -X_{vv}$ we infer that

$$(40) \qquad \begin{aligned} 0 &= \langle Z, X_{uu}(w)[\cos^2\varphi - \sin^2\varphi] + 2X_{uv}(w)\sin\varphi\cos\varphi\rangle \\ &= \mathcal{L}(w)[\cos^2\varphi - \sin^2\varphi] + 2\mathcal{M}\sin\varphi\cos\varphi \\ &= \ \mathrm{Re}\ \{[\mathcal{L}(w) - i\mathcal{M}(w)](\cos\varphi + i\sin\varphi)^2\}. \end{aligned}$$

In Section 1.3 we have seen via the Codazzi equations that

$$f(w) := [\mathcal{L}(w) - i\mathcal{M}(w)]w^2, \quad w \in \overline{B},$$

is holomorphic on B. Then (40) implies $\mathrm{Re}\, f|_{\partial B} = 0$ and therefore $\mathrm{Re}\, f(w) \equiv 0$ in $\overline{B}$, whence $f(w) \equiv$ const in $\overline{B}$. Since $f(0) = 0$ it follows that $f(w) \equiv 0$ in $\overline{B}$. Thus we arrive at

$$\mathcal{L}(w) \equiv 0, \quad \mathcal{M}(w) \equiv 0, \quad \mathcal{N}(w) \equiv 0 \quad \text{in } \overline{B}$$

whence $K(w) \equiv 0$ in B which contradicts (36). Thus $\kappa_n(s) \equiv 0$ is impossible, and (35) is strengthened into

$$\int_\Gamma \kappa_g\, ds < 4\pi.$$

Combining this with (39) it follows that

$$(41) \qquad -\int_B K\Lambda\, du\, dv < 2\pi.$$

Because of Theorem 2 in Section 5.4 we infer from (41) that X is strictly stable. According to Proposition 2 we can therefore embed X into a field immersion of minimal surfaces, and so Proposition 3 implies that any minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ furnishes a strict relative minimum for Dirichlet's integral D in $\overline{\mathcal{C}}^*(\Gamma)$.

Suppose now that two different minimal surfaces X_1 and X_2 existed in $\overline{\mathcal{C}}^*(\Gamma)$. Then both would furnish a strict relative minimum of D in $\overline{\mathcal{C}}^*(\Gamma)$. Then by Courant's "Mountain Pass Lemma", to be presented in the next chapter, there would exist a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which were unstable in the sense that it were not a local minimizer of D (cf. Theorem 2 in Section 6.7). The existence of such a surface X_3 is impossible as we have seen above, and so there cannot be two different minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$. However there is always one minimal surface X in $\overline{\mathcal{C}}^*(\Gamma)$, which proves the theorem. □

Remark 1. The unique solution in Nitsche's theorem actually is not only immersed, but even *embedded*, according to the following remarkable result due to T. Ekholm, B. White, and D. Wienholtz [1]:

Theorem 2. *Let Γ be a closed Jordan curve in $\mathbb{R}^n$ with total curvature $\leq 4\pi$, and let $X : \overline{B} \to \mathbb{R}^n$ be a minimal surface in $\mathcal{C}(\Gamma)$. Then X is embedded up to and including the boundary, with no interior branch points.*

In fact Theorem 2 even holds for minimal surfaces $X : M \to \mathbb{R}^n$ defined on a compact 2-manifold M with boundary ∂M which is mapped homeomorphically onto Γ.

5.7 Some Finiteness Results for Plateau's Problem

For Plateau's problem the most challenging question is: "How many minimal surfaces of the type of the disk, or of general topological type, are bounded by a preassigned 'well-behaved' closed Jordan curve Γ?" The Courant–Levy examples (cf. No. 4 of Section 4.15) show that Γ may bound infinitely many solutions even if it is regular and smooth *except for one point*. Thus a reasonable answer can only be expected if we interpret the attribute "well-behaved" in a suitably restricted way, say as *regular and real analytic*, or as *regular and of class C^k for some $k \geq 1$*, or as *piecewise linear* (i.e. Γ is a *polygon*). Moreover it is interesting to find *upper or lower bounds* for the number of solutions bounded by a well-behaved contour Γ. However, even the decision whether or not a well-behaved Γ spans only finitely many disk-type minimal surfaces is still open.

We shall prove in this section that stable, immersed surfaces of the type of the disk bounded by a real analytic, regular contour Γ are isolated; hence only finitely many of them can be bounded by such a Γ. The possibility to estimate quantitatively a suitable neighborhood, where no further solution exists, seems to be out of reach. The pioneering contribution towards isolatedness of stable solutions for Plateau's problem is due to F. Tomi [6].

We begin our discussion with the following *local uniqueness theorem* that is already contained in the considerations of the last section. To this end we need the *perturbation equation* $L\zeta = \Phi(\zeta)$ defined in 5.6, Proposition 1, which is associated with a given immersed minimal surface X. We have the following result:

Proposition 1. *Let X be an immersed, strictly stable minimal surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with the normal N. Then there is a number $\epsilon(X) > 0$ such that all solutions $\zeta \in C^{2,\alpha}(\overline{B})$ of*

$$L(\zeta) = \Phi(\zeta) \quad \text{in } B, \quad \zeta = 0 \quad \text{on } \partial B, \tag{1}$$

satisfying $|\zeta(w)| < \epsilon(X)$ for all $w \in B$, are identically zero. Consequently, if $Y \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ is an immersed zero mean curvature surface with $Y(w) = X(w)$ for $w \in \partial B$ and

$$|Y(w) - X(w)| < \epsilon(X) \quad \text{for } w \in B \tag{2}$$

which can be written as

$$Y(w) = X(w) + \zeta(w)N(w) \quad \textit{for } w \in \overline{B}, \tag{3}$$

ζ *as above, then* $Y = X$.

Proof. In Proposition 2 of Section 5.6 we have constructed a one-parameter family $\eta \in C^{2,\alpha}(\overline{B} \times [-t_0, t_0])$ of functions $\eta(w,t)$, $w \in \overline{B}$, $t \in [-t_0, t_0]$, $t_0 > 0$, solving

$$L\eta(\cdot,t) = \Phi(\eta(\cdot,t)) \quad \text{in } B, \quad \eta(\cdot,t) = t \quad \text{on } \partial B, \tag{4}$$

such that the family of surfaces

$$Z(w,t) := X(w) + \eta(w,t)N(w), \quad w \in \overline{B}, \ |t| \le t_0 \tag{5}$$

yields a field immersion of X, as $\eta(w,0) \equiv 0$ on $\overline{B}$. (Note that in 5.6 the function η was called ζ.) Then there is a number $\epsilon = \epsilon(X) > 0$ such that any Y of the form (3) with $\zeta \in C_0^{2,\alpha}(\overline{B})$ satisfying $L\zeta = \Phi(\zeta)$ and $|\zeta(w)| < \epsilon(X)$ for $w \in \overline{B}$ is covered by the field (5). Then we can write

$$\zeta(w) = \eta(w, \tau(w)) \quad \text{for } w \in \overline{B} \tag{6}$$

where the "height function" τ is of class $C^{2,\alpha}(\overline{B})$ and satisfies $|\tau(w)| \le t_0$ for $w \in \overline{B}$ as well as

$$\tau(w) = 0 \quad \text{on } \partial B. \tag{7}$$

Now we prove $\tau(w) \equiv 0$ on $\overline{B}$ which is turn implies

$$\zeta(w) = \eta(w,0) \equiv 0 \quad \text{on } \overline{B}$$

whence $Y = X$.

In fact, suppose that $\tau(w) \not\equiv 0$. Then there is a point $w_0 \in B$ such that $\tau(w_0) = t_0$ with

$$|t_0| = \max\{|\tau(w)| \colon w \in \overline{B}\} > 0.$$

Then the minimal immersion Y of the form (3), satisfying (1) and (2), touches the minimal immersion $Z(\cdot, t_0)$ at the interior point $x_0 := X(w_0)$. We represent both Y and $Z(\cdot,t)$ locally as minimal graphs over the same plane in a neighborhood of x_0. Applying the maximum principle to the difference of the two equations for these graphs we conclude that the two graphs coincide. Repeating this reasoning, a continuity argument yields $Y(w) \equiv Z(w,t_0)$ for $w \in \overline{B}$, whence $\zeta(w) \equiv \eta(w,t_0)$ for all $w \in \overline{B}$, and therefore

$$\zeta(w) = t_0 \quad \text{for all } w \in \partial B. \tag{8}$$

Since $t_0 \neq 0$, this contradicts the assumption $\zeta|_{\partial B} = 0$, and so we have verified $\tau(w) \equiv 0$ on $\overline{B}$. $\square$

Next we modify the reasoning used to prove Proposition 2 of Section 5.6. This will lead to the following central result due to F. Tomi [6] and J.C.C. Nitsche [26].

Proposition 2. *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be an immersed, stable minimal surface, and suppose that $\{\zeta_j\}$ is a sequence of functions $\zeta_j \in C^{2,\alpha}(\overline{B})$ satisfying*

$$L\zeta_j = \Phi(\zeta_j) \quad \text{in } B, \quad \zeta_j = 0 \quad \text{on } \partial B \tag{9}$$

and

$$0 < \|\zeta_j\|_{2,\alpha} \to 0 \quad \text{for } j \to \infty \tag{10}$$

(where $\|\cdot\|_{2,\alpha}$ is the $C^{2,\alpha}(\overline{B})$-norm). Then X is weakly stable, and there exists a real analytic one-parameter family

$$\zeta : \overline{B} \times [-t_0, t_0] \to \mathbb{R}, \quad t_0 > 0,$$

of solutions $\zeta(\cdot, t) \in C^{2,\alpha}(\overline{B})$ of

$$L\zeta(\cdot, t) = \Phi(\zeta(\cdot, t)) \quad \text{in } B, \quad \zeta(w, t) = 0 \quad \text{for } w \in \partial B, \tag{11}$$

$|t| \le t_0$, satisfying

$$\frac{\partial}{\partial t}\zeta(w,t)\bigg|_{t=0} > 0 \quad \text{for all } w \in B. \tag{12}$$

Proof. Since the stable minimal immersion is not isolated, we infer from Proposition 1 that X is only "weakly stable" in the sense that the Schwarzian operator L has zero as its lowest eigenvalue with respect to zero boundary values. Equivalently this means: There exists a function $\xi \in C^{2,\alpha}(\overline{B})$ satisfying

$$L\xi = 0 \quad \text{in } B, \quad \xi|_{\partial B} = 0, \quad \xi(w) > 0 \quad \text{for all } w \in B. \tag{13}$$

Consider the closed subspace

$$\tilde{\mathfrak{B}} := \left\{ \eta \in C_0^{2,\alpha}(\overline{B}) \colon \int_B \xi\eta \, du \, dv = 0 \right\}$$

of the Banach space $(C_0^{2,\alpha}(\overline{B}), \|\cdot\|_{2,\alpha})$, as well as the restriction

$$\tilde{L} := L|_{\tilde{\mathfrak{B}}} : \tilde{\mathfrak{B}} \to C^{0,\alpha}(\overline{B}). \tag{14}$$

Next we define the "projection" $\tilde{\Phi}^t$ by

$$\tilde{\Phi}^t(\eta) := \Phi(t\xi + \eta) - \left\{ \int_B \xi\Phi(t\xi + \eta)\, du\, dv \right\} \xi, \quad \eta \in \tilde{\mathfrak{B}} \tag{15}$$

for all $t \in \mathbb{R}$. Similarly as in the proof of Propositions 1 and 2 in Section 5.6 one can show that, for any t with $|t| \le t_0$ and $0 < t_0 \ll 1$ there is exactly one solution $\eta(\cdot, t) \in \tilde{\mathfrak{B}}$ of

$$\tilde{L}(\eta(\cdot, t)) = \tilde{\Phi}^t(\eta(\cdot, t)) \quad \text{in } B, \tag{16}$$

and the structure of the right-hand side in (16) yields a real analytic dependence of $\eta(\cdot, t)$ on the parameter $t \in [-t_0, t_0]$. From the assumptions (9) and (10) we infer the representations

$$\zeta_j = t_j \zeta + \eta(\cdot, t_j) \tag{17}$$

with

$$t_j \to 0 \quad \text{as } j \to \infty \quad \text{and} \quad t_j \ne 0 \tag{18}$$

for $j \gg 1$. Define the real analytic function $\psi : (-t_0, t_0) \to \mathbb{R}$ by

$$\psi(t) := \int_B \Phi(t\xi(u, v) + \eta(u, v, t))\xi(u, v)\, du\, dv. \tag{19}$$

With the aid of (9), (17), (13), (16) and (15) we obtain for $j \gg 1$ that

$$\begin{aligned} \Phi(\zeta_j) &= L(\zeta_j) = L(\eta(\cdot, t_j)) = \tilde{L}(\eta(\cdot, t_j)) \\ &= \tilde{\Phi}^{t_j}(\eta(\cdot, t_j)) = \Phi(t_j\xi + \eta(\cdot, t_j)) - \psi(t_j)\xi \\ &= \Phi(\zeta_j) - \psi(t_j)\xi. \end{aligned}$$

This implies

$$\psi(t_j) = 0 \quad \text{for } j \gg 1,\ t_j \to 0;$$

hence the real analytic function ψ satisfies

$$\psi(t) \equiv 0 \quad \text{on } (-t_0, t_0).$$

Finally we infer from (16) that

$$L\zeta(\cdot, t) = \Phi(\zeta(\cdot, t)) \quad \text{in } B \quad \text{for } |t| \le t_0$$

with the family of functions

$$\zeta(w, t) := t\xi(w) + \eta(w, t), \quad w \in \overline{B},\ |t| \le t_0$$

satisfying

$$\frac{\partial}{\partial t}\xi(w, 0) = \xi(w) + \frac{\partial}{\partial t}\eta(w, 0) > 0 \quad \text{for } w \in B$$

since

$$\frac{\partial}{\partial t}\eta(w, 0) = 0 \quad \text{for all } w \in \overline{B}$$

(see part (ii) of the proof of Proposition 2 in Section 5.6). $\square$

The Propositions 1 and 2 motivate the following

Definition 1. *An immersed minimal surface $X \in \mathcal{C}(\Gamma)$ is called* **weakly stable** *if it is stable, but not strictly stable.*

Remark 1. Let λ_1 be the smallest eigenvalue of the Schwarzian operator $L = -\Delta + 2\Lambda K$ of X on B with respect to zero boundary values, i.e. the smallest number $\lambda \in \mathbb{R}$ such that the boundary value problem

$$L\zeta = \lambda\zeta \quad \text{in } B, \quad \zeta = 0 \quad \text{on } \partial B$$

possesses a nontrivial solution ζ. It is well known that λ_1 is simple and that each eigenfunction ζ corresponding to λ_1 satisfies $\zeta(w) \neq 0$ for all $w \in B$. Thus the eigenspace to λ_1 is one-dimensional and will be spanned by an eigenfunction ζ satisfying $\zeta(w) > 0$ for all $w \in B$. Hence we have:

X is stable if and only if $\lambda_1 \geq 0$, weakly stable if and only if $\lambda_1 = 0$, strictly stable if and only if $\lambda_1 > 0$, nonstable if and only if $\lambda_1 < 0$.

Furthermore we have:

1. X is weakly stable if and only if there is a $\zeta \in C^{2,\alpha}(\overline{B})$ with $L\zeta = 0$ in B, $\zeta = 0$ on ∂B, and $\zeta(w) > 0$ for $w \in B$.
2. X is strictly stable if there is a $\zeta \in C^{2,\alpha}(\overline{B})$ with $L\zeta = 0$ in B and $\zeta > 0$ on $\overline{B}$.
3. X is nonstable if there is a subdomain Ω of B with $\Omega \neq B$ such that $B \setminus \overline{\Omega}$ is non-empty and "$L\zeta = 0$ in Ω" possesses a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ with $\zeta = 0$ on $\partial\Omega$ and $\zeta > 0$ on Ω.

Remark 2. Tomi's original proof of Proposition 2 did not use the "perturbation equation" $L\zeta = \Phi(\zeta)$, but was based on a real analytic version of the implicit function theorem in Banach spaces.

Proposition 3. *With the family $\zeta(\cdot, t)$, $|t| \leq t_0$ from Proposition 2 we define the real analytic one-parameter family of immersions*

$$Y(\cdot, t) := X + \zeta(\cdot, t)N, \quad |t| \leq t_0, \tag{20}$$

from $\overline{B}$ into $\mathbb{R}^3$ which have mean curvature zero and satisfy

$$Y(w, t) = X(w) \quad \textit{for } w \in \partial B$$

and

$$|Y_t(w, t)| = |\zeta_t(w, t)| > 0 \quad \textit{for } w \in B \textit{ and } |t| \leq t_0.$$

Furthermore, all surfaces $Y(\cdot, t)$ have the same area, i.e.

$$A(Y(\cdot, t)) \equiv \text{const} \quad \textit{for } |t| \leq t_0. \tag{21}$$

Proof. As at the end of the proof of Proposition 2 in Section 5.6, formula (22), we reparametrize the surfaces $Y(\cdot,t)$ via their orthogonal trajectories and obtain (possibly for some smaller $t_0 > 0$) a family $Z : \overline{B} \times [-t_0, t_0] \to \mathbb{R}^3$ of zero mean curvature surfaces $Z(\cdot,t)$, whose area elements $\mathcal{W}(u,v,t) := |Z_u(u,v,t) \wedge Z_v(u,v,t)|$ satisfy

$$\frac{\partial}{\partial t}\mathcal{W}(u,v,t) \equiv 0 \quad \text{on } \overline{B} \times [-t_0, t_0] \tag{22}$$

(cf. the proof of formula (23) in Section 5.6). This implies

$$A(Z(\cdot t)) \equiv \text{const} \quad \text{for } |t| \le t_0.$$

Since $Z(\cdot,t)$ is a reparametrization of $Y(\cdot,t)$, it follows that

$$A(Y(\cdot,t)) = A(Z(\cdot,t)) \quad \text{for } |t| \le t_0,$$

which in conjunction with the preceding identity implies (21). □

The above lense-shaped field of zero mean curvature surfaces $Y(\cdot,t)$ defined by (20) is defined in a similar way as a field of conjugate geodesics. This motivates

Definition 2. *A family $Y(\cdot,t) = X + \zeta(\cdot,t)N$, $|t| \le t_0$, of zero mean curvature immersions $\overline{B} \to \mathbb{R}^3$ and of constant area $A(Y(\cdot,t))$, as described in Propositions 2 and 3, is called* **conjugate field for** X. *We also say: X is embedded in the conjugate field $\{Y(\cdot,t)\}_{|t|\le t_0}$.*

This leads to the question whether a minimal immersion that is sufficiently close to a surface X and has the properties required in Propositions 2 and 3, can be "covered" by a conjugate field for X. This might not be the case if we interpret "close" in the sense of the $C^0(\overline{B},\mathbb{R}^3)$-norm. However, this property can be proved if we understand "close" in the $C^{2,\alpha}$-sense. This is a consequence of the following result due to R. Böhme and F. Tomi [1], §3, pp. 15–20. For the convenience of the reader we shall provide a proof.

Proposition 4. *Let $\{X_j\}$ be a sequence of immersions $\overline{B} \to \mathbb{R}^3$ with $X_j \in \mathcal{C}(\Gamma) \cap C^{3,\alpha}(\overline{B},\mathbb{R})$ and*

$$\lim_{j\to\infty} \|X_j - X\|_{C^{3,\beta}(\overline{B},\mathbb{R}^3)} = 0 \quad \textit{for } \beta \in (0,\alpha),$$

where the limit X is also an immersion $\overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}(\Gamma) \cap C^{3,\alpha}(\overline{B},\mathbb{R}^3)$. Then there are reparametrizations $Y_j \in \mathcal{C}(\Gamma) \cap C^{2,\alpha}(\overline{B},\mathbb{R}^3)$ of X_j which can be expressed as "generalized graphs above X" in the form

$$Y_j = X + \zeta_j N \quad \textit{with } \zeta_j \in C_0^{2,\alpha}(\overline{B})$$

and

$$\|\zeta_j\|_{2,\beta} \to 0 \quad \textit{as } j \to \infty \quad \textit{for } 0 < \beta < \alpha.$$

Proof. First we continue X to an immersion of class $C^{3,\alpha}(\Omega, \mathbb{R}^3)$ for some Ω with $\overline{B} \subset \Omega$ and consider the family of surfaces

$$Z(w,t) := X(w) + tN(w), \quad w \in \Omega, \ |t| < \epsilon,$$

for some ϵ with $0 < \epsilon \ll 1$. Then $Z \in C^{2,\alpha}(\Omega \times (-\epsilon, \epsilon), \mathbb{R}^3)$, and the Jacobian J_Z of Z is everywhere positive on $\Omega \times (-\epsilon, \epsilon)$. Thus Z is an open mapping of $\Omega \times (-\epsilon, \epsilon)$ into $\mathbb{R}^3$, and Z can locally be inverted. In conjunction with a monodromy argument it follows that, for $j \gg 1$, each X_j can be represented in the form

$$X_j(w) = Z(f_j(w), z_j(w)), \quad w \in \overline{B},$$

with a mapping $f_j : \overline{B} \to \mathbb{R}^3$ of the class $C^{2,\alpha}(\overline{B}, \mathbb{R}^2)$ such that $f_j|_{\partial B}$ maps ∂B monotonically onto itself, and a height function $z_j \in C_0^{2,\alpha}(\overline{B})$.

Setting $f(w) := w$ and $z(w) := 0$ for $w \in \overline{B}$ we can write

$$X(w) = Z(f(w), z(w)).$$

Then we infer from $X_j \to X$ in $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ and the fact that the local inverse of Z is of class $C^{2,\alpha}$:

$$f_j \to f \quad \text{in } C^{2,\alpha}(\overline{B}, \mathbb{R}^2), \quad z_j \to z = 0 \quad \text{in } C^{2,\alpha}(\overline{B}).$$

Since $f(w) \equiv w$ on $\overline{B}$, the mappings f_j satisfy

$$J_{f_j}(w) > 0 \quad \text{on } \overline{B} \quad \text{for } j \gg 1,$$

and so every $f_j|_B$ is an open mapping of B into $\mathbb{R}^2$, $j \gg 1$. Since $f_j \in C^0(\overline{B}, \mathbb{R}^2)$ and $f_j|_{\partial B}$ is a homeomorphism of ∂B onto ∂B, we infer $f_j(\overline{B}) = \overline{B}$ for $j \gg 1$; therefore the f_j are $C^{2,\alpha}$-diffeomorphisms of $\overline{B}$ onto $\overline{B}$ for $j \gg 1$. Setting $\zeta_j := z_j \circ f_j^{-1} \in C_0^{2,\alpha}(\overline{B})$ and $Y_j := X_j \circ f_j^{-1} = Z(\mathrm{id}_{\overline{B}}, \zeta_j)$ we obtain

$$Y_j(w) = X_j(f_j(w)) = X(w) + \zeta_j(w)N(w) \quad \text{for } w \in \overline{B}, \ j \gg 1,$$

with

$$\|\zeta_j\|_{C^{2,\alpha}(\overline{B})} \to 0 \quad \text{as } j \to \infty.$$

□

Remark 3. Let us interpret the preceding results in a geometric way. Proposition 1 states that a strictly stable, immersed $X \in \mathcal{C}(\Gamma)$ can be embedded into a field $\{Z(\cdot, t)\}_{|t| \le t_0}$ of minimal immersions such that every immersion $Y \in \mathcal{C}(\Gamma)$ with the mean curvature zero, given in the "normal form" $Y = X + \zeta N$ with $\zeta \in C_0^{2,\alpha}(\overline{B})$ satisfying $L\zeta = \Phi(\zeta)$, coincides with X if it is sufficiently close to X in the $C^0(\overline{B}, \mathbb{R}^3)$-norm. This means: *A strictly stable minimal immersion is isolated with respect to the C^0-norm compared with normal variations $Y = X + \zeta N$, ζ as above.*

Yet it is not clear whether every minimal immersions $\tilde{X} \in \mathcal{C}(\Gamma)$ that is C^0-close to X has a normal-form representation Y; but, by Proposition 4, such a reparametrization can be achieved if $\tilde{X}$ is $C^{2,\alpha}$-close to X. Thus we obtain: *Any strictly stable minimal immersion $X \in \mathcal{C}(\Gamma)$ is isolated in the $C^{2,\alpha}$-norm among all minimal immersions of class $\mathcal{C}(\Gamma)$.*

If, however, the stable immersion $X \in \mathcal{C}(\Gamma)$ is the $C^{2,\alpha}$-limit of stable immersions $X_j \in \mathcal{C}(\Gamma)$ with $X_j \neq X$, then X is weakly stable and can be embedded into a conjugate field $\{Y(\cdot,t)\}_{|t|\le t_0}$ which forms a regular, real analytic curve in $C^{2,\alpha}(B,\mathbb{R}^3)$.

Now we turn to Tomi's "finiteness result". We recall some definitions and formulate a compactness result.

The class $\mathcal{C}^*(\Gamma)$ consists of those $X \in \mathcal{C}(\Gamma)$ which satisfy a preassigned three-point condition $*$, and $\overline{\mathcal{C}}^*(\Gamma) := \mathcal{C}^*(\Gamma) \cap C^0(\overline{B},\mathbb{R}^3)$. For $X \in \mathcal{C}(\Gamma)$ the area $A(X)$ and Dirichlet's integral $D(X)$ are

$$A(X) = \int_B |X_u \wedge X_v|\, du\, dv, \quad D(X) = \frac{1}{2}\int_B |\nabla X|^2\, du\, dv.$$

We know that

$$a(\Gamma) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}^*(\Gamma)} A = \inf_{\overline{\mathcal{C}}^*(\Gamma)} D.$$

Proposition 5. *For any $\Gamma \in C^{k,\alpha}$ there is a constant $c(\Gamma,k,\alpha,*)$ such that each minimal surface $X \in \mathcal{C}^*(\Gamma)$ is of class $C^{k,\alpha}(\overline{B},\mathbb{R}^3)$ and satisfies*

$$\|X\|_{C^{k,\alpha}(\overline{B},\mathbb{R}^3)} \le c(\Gamma,k,\alpha,*), \quad k \in \mathbb{N},\ \alpha \in (0,1),$$

where $c(\Gamma,k,\alpha,*)$ is a constant which depends only on $\Gamma,k,\alpha,*$. Hence, from any sequence of minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$, we can extract a subsequence $X_{j_\nu} \to X$ in $C^{k,\beta}(\overline{B},\mathbb{R}^3)$ as $\nu \to \infty$ for any $\beta \in (0,\alpha)$, where $X \in \mathcal{C}^*(\Gamma) \cap C^{k,\alpha}(\overline{B},\mathbb{R}^3)$ is a minimal surface.

Proof. See Vol. 2, Chapter 2. □

Theorem 1 (F. Tomi [6]). *Let Γ be a closed Jordan curve in $\mathbb{R}^3$ of class $C^{3,\alpha}$, and suppose that every minimal surface X of class $\mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ is an immersion of $\overline{B}$ into $\mathbb{R}^3$, i.e. X be free both of interior and boundary branch points. Then Γ spans only finitely many minimal surfaces $X \in \mathcal{C}^*(\Gamma)$ which satisfy $A(X) = a(\Gamma)$, i.e. which are area minimizing in $\mathcal{C}(\Gamma)$.*

This immediately implies the following

Corollary 1. *If all minimal surfaces $X \in \mathcal{C}(\Gamma)$ with $\Gamma \in C^{3,\alpha}$ are immersed up to the boundary, i.e. have no branch points on $\overline{B}$, then there are only finitely many minimal surfaces $X \in \mathcal{C}^*(\Gamma)$ with $A(X) = a(\Gamma)$.*

Remark 4. In Section 4.9 we have exhibited conditions on Γ which ensure that any $X \in \mathcal{C}(\Gamma)$ is free of branch points, in which case Corollary 1 can be applied.

Remark 5. By the papers by R. Osserman, H.W. Alt, R. Gulliver, and Gulliver/Osserman/Royden it follows that any minimal surface $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ is free of interior branch points. Furthermore, R. Gulliver and F.D. Lesley [1] have stated that, in addition, every $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$ has no boundary branch point if Γ is a regular, real analytic Jordan curve. This result implies

Corollary 2. *If Γ is a regular, real analytic, closed Jordan curve, then there exist only finitely many $X \in \mathcal{C}^*(\Gamma)$ with $A(X) = a(\Gamma)$, and all of them are immersions.*

Proof of Theorem 1. Suppose that Γ bounds infinitely many X with $A(X) = a(\Gamma)$. By Proposition 5 there is a sequence $\{X_j\}$ of minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$ with $A(X_j) = a(\Gamma)$ and

$$0 < \|X_j - X\|_{C^{3,\beta}(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } j \to 0$$

for $\beta \in (0, \alpha)$, and the limit X is a minimal surface of class $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $A(X) = a(\Gamma)$.

By Propositions 2, 3, 4 we embed X into a conjugate field, with α replaced by $\beta \in (0, \alpha)$, i.e. there is a regular, real analytic curve $\{Y(\cdot, t)\}_{|t| \le t_0}$ with $Y(\cdot, 0) = X$ which lies in the level set

$$\mathcal{M}_c(\Gamma) := \{X \in \mathcal{C}^*(\Gamma) \colon D(X) = A(X) = c\}, \quad c := a(\Gamma).$$

We equip $\mathcal{M}_c(\Gamma)$ with the $C^2(\overline{B}, \mathbb{R}^3)$-norm and denote by $\mathcal{K}_c$ the closed, connected component of $\mathcal{M}_c(\Gamma)$ containing X. A continuity argument combined with the above reasoning yields: Through every $X_0 \in \mathcal{K}_c$ there is a real analytic, regular curve $\{Y(\cdot, t)\}_{|t| \le t_0}$ contained in $\mathcal{K}_c$ such that $Y(\cdot, 0) = X_0$.

Consider now the volume functional V on the "block" $\mathcal{K}_c$ which is defined by

$$V(X) := \frac{1}{3} \int_B [X, X_u, X_v] \, du \, dv. \tag{23}$$

Since $\mathcal{K}_c$ is a compact subset of $C^2(\overline{B}, \mathbb{R}^3)$ and V is continuous on $\mathcal{K}_c$, there is an $X_0 \in \mathcal{K}_c$ such that

$$V(X_0) = \max_{\mathcal{K}_c} V.$$

Let $\{Y(\cdot, t)\}_{|t| \le t_0}$ be a regular, real analytic arc with $Y(\cdot, 0) = X_0$. Then

$$\frac{d}{dt} V(Y(\cdot, t)) \bigg|_{t=0} = 0. \tag{24}$$

On the other hand we have

$$Y(\cdot, t) := X_0 + \zeta(\cdot, t) N_0,$$

$N_0 =$ normal of X_0, with $\xi = \zeta_t(\cdot, 0), \xi(w) > 0$ on B, and $Y_t(\cdot, 0) = \xi N_0$. Set

$$\Lambda_0 := |X_{0,u} \wedge X_{0,v}| = |X_{0,u}|^2$$

with $\Lambda_0(w) > 0$ on $\overline{B}$. The computations in 5.3 show that

$$\frac{d}{dt} V(Y(\cdot, t))\bigg|_{t=0} = \int_B \Lambda_0(w)\xi(w)\, du\, dv > 0 \tag{25}$$

if we take div $\frac{1}{3}x = 1$ into account. Clearly, (25) contradicts (24), and so the theorem is proved. □

Now we want to generalize Theorem 1 to stable solutions of Plateau's problem. So far we can carry out this program only for some special classes of boundaries, e.g. for *extreme curves.*

Definition 3. *A closed Jordan curve Γ in $\mathbb{R}^3$ is called* **extreme** *if for any point P of Γ there is a plane of support, that is, a plane Π such that Γ lies on one side of Π but is not completely contained in Π.*

Clearly, *Γ is extreme if and only if it lies on the boundary of a convex body. Equivalently we have*: *Γ is extreme if and only if it lies on the boundary of its convex hull.*

Proposition 6 (Compactness property of stable minimal immersions). *Let Γ be a closed regular Jordan curve of class $C^{3,\alpha}$ which is extreme, and suppose that $\{X_j\}$ is a sequence of stable minimal surfaces $X_j \in \mathcal{C}^*(\Gamma)$ free of branch points on $\overline{B}$. Then*:

(i) *We can extract a subsequence $\{X_{j_\nu}\}$ converging in $C^{3,\beta}(\overline{B}, \mathbb{R}^3)$ with $\beta \in (0, \alpha)$ to a minimal surface $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$.*
(ii) *The limit surface X is a stable minimal immersion of $\overline{B}$ into $\mathbb{R}^3$.*

Proof. Statement (i) follows from Proposition 5, and the limit X of the X_{j_ν} has no branch points on ∂B since Γ is extreme. It remains to prove that X is stable and has no branch points in B. To this end we consider the Gauss curvature K_ν and the surface element $\Lambda_\nu = |D_u X_{j_\nu}|^2$ of X_{j_ν} as well as the normal $N_\nu : \overline{B} \to S^2 \subset \mathbb{R}^3$ of X_{j_ν}. We have

$$\tfrac{1}{2}|\nabla N_\nu|^2 = -\Lambda_\nu K_\nu,$$

and the Gauss–Bonnet formula yields

$$-\int_B \Lambda_\nu K_\nu du\, dv = \int_{\partial B} (\kappa_g)_\nu\, ds - 2\pi \le \kappa(\Gamma) - 2\pi.$$

Thus the total curvature $\kappa(\Gamma)$ of Γ estimates the Dirichlet integrals $D(N_\nu) = \frac{1}{2}\int_B |\nabla N_\nu|^2\, du\, dv$ of the normals N_ν by

$$D(N_\nu) \le \kappa(\Gamma) - 2\pi \quad \text{for all } \nu \in \mathbb{N}. \tag{26}$$

Furthermore the isoperimetric inequality yields

$$D(X_{j_\nu}) \le \frac{1}{4\pi} L^2(\Gamma) \quad \text{for all } \nu \in \mathbb{N}. \tag{27}$$

Thus by the reasoning in (i), (ii), (iii) of the proof of Theorem 1 in Section 5.5 we conclude: For any $B' \subset\subset B$ there is a constant $c(B') > 0$ such that

$$|\nabla N_\nu(w)| \le c(B') \quad \text{for all } w \in B'. \tag{28}$$

Since $|N_\nu| \le 1$ we may assume that the subsequence $\{j_\nu\}$ also satisfies

$$N_\nu(w) \rightrightarrows N(w) \quad \text{for } w \in \overline{B} \text{ and for any } B' \subset\subset B. \tag{29}$$

(Actually it suffices to apply merely (i) and (ii) of the proof quoted above since in this way we already obtain a uniform modulus of continuity of the N_ν on any $B' \subset\subset B$.)

Suppose now that X had an interior branch point $w_0 \in B$; we may assume that $w_1 = 0$, $X(0) = 0$, $N(0) = e_3 = (0, 0, 1)$. Then the associated planar mapping $f : B \to \mathbb{C}$ with

$$f(w) := X^1(w) + iX^2(w), \quad w \in B,$$

has the asymptotic expansion

$$f(w) = aw^n + o(|w|^{n+1}) \quad \text{as } w \to 0, \quad a \in \mathbb{C} \setminus \{0\}$$

where $n \ge 2$. Thus the winding number $i(f, 0)$ of f about $w = 0$ is at least 2. On the other hand the planar mappings $f_\nu : B \to \mathbb{C}$ associated with X_{j_ν},

$$f_\nu(w) := X^1_{j_\nu}(w) + iX^2_{j_\nu}, \quad w \in B,$$

satisfy $f_\nu(w) \rightrightarrows f(w)$ for $|w| \ll 1$ as well as

$$f_\nu(w) = a_\nu w + o(|w|^2) \quad \text{for } |w| \le \delta,\ 0 < \delta \ll 1,\ a_\nu \in \mathbb{C} \setminus \{0\},\ \nu \gg 1,$$

since $X_{j_\nu}(w) \rightrightarrows X(w)$ and $N_\nu(w) \rightrightarrows N(w)$ as $\nu \to \infty$ for $|w| \le \delta$ with $0 < \delta \ll 1$. Hence the winding numbers $i(f_\nu, 0)$ of f_ν about 0 satisfy $i(f_\nu, 0) = 1$ for $\nu \gg 1$. Since $i(f_\nu, 0) \to i(f, 0)$ as $\nu \to \infty$, we obtain $i(f, 0) = 1$, a contradiction to $i(f, 0) \ge 2$. Thus X has no branch points in $\overline{B}$.

Then we conclude

$$\Lambda_\nu(w) K_\nu(w) \to \Lambda(w) K(w) \quad \text{as } \nu \to \infty \quad \text{for } w \in \overline{B}, \tag{30}$$

and Lebesgue's theorem on dominated convergence yields the stability of X.
□

When we combine the reasoning in the proof of Theorem 1 with Proposition 6, we obtain

Theorem 2. *An extreme, regular Jordan contour $\Gamma \in C^{3,\alpha}$ bounds at most finitely many stable minimal immersions $\overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$.*

Remark 6. The central reason why we can carry over the proof of Theorem 1 to the situation considered in Theorem 2 is the observation stated in Proposition 3 that all elements $Y(\cdot, t)$ of the regular, real analytic family (20) have the same area $A(Y(\cdot, t))$.

Remark 7. The arguments used for the proof of Theorem 2 and the subsequent Theorem 3 are based on Sauvigny's paper [10].

Remark 8. The same reasoning holds true if we replace the assumption that *Γ be extreme* by the property: *No minimal surface $X \in \mathcal{C}(\Gamma)$ has a boundary branch point on ∂B.*

In this context, J.C.C. Nitsche [31] has proved the following result:

Proposition 7. *Let Γ be a closed, regular, real analytic Jordan curve in $\mathbb{R}^3$ with the property that there is a straight line in $\mathbb{R}^3$ such that no plane through this line intersects Γ in more than two distinct points. Then every solution of Plateau's problem for Γ is free of branch points.*

We now present a modified version of the 6π-*finiteness theorem* by J.C.C. Nitsche [31] which considers also nonstable solutions of Plateau's problem.

Proposition 8. *Let $\Gamma \in C^{3,\alpha}$ be a closed, regular, extreme Jordan curve in $\mathbb{R}^3$ with a total curve $\kappa(\Gamma)$ less than 6π. Then from any sequence $\{X_j\}$ of minimal immersions $X_j : \overline{B} \to \mathbb{R}^3$ we can extract a subsequence $\{X_{j_\nu}\}$ converging in $C^{3,\beta}(\overline{B}, \mathbb{R}^3)$ for $0 < \beta < \alpha$ to a minimal immersion $X : \overline{B} \to \mathbb{R}^3$ of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$.*

Proof. We copy the reasoning used for proving Proposition 6, but we have to replace the stability condition with the subsequent *curvatura-integra condition* to achieve a uniform modulus of continuity for the normals N_ν of the converging subsequence $\{X_{j_\nu}\}$ of $\{X_j\}$. The estimate (26) yields

$$D(N_\nu) = A(N_\nu) \le \kappa(\Gamma) - 2\pi =: \omega \quad \text{with } 0 \le \omega < 4\pi. \tag{31}$$

If $\omega = 0$ then $N_\nu = \text{const}$ for all $\nu \in \mathbb{N}$, and thus the N_ν are certainly uniformly continuous. Hence we can assume that

$$0 < \omega < 4\pi.$$

With the aid of the Courant–Lebesgue lemma we obtain a universal radius $\rho > 0$ such that N_ν maps the circle $\partial B_\rho(w_0)$ contained in B into a spherical cap on S^2 with a "sufficiently small" geodesic radius. Since $N_\nu(w) \not\equiv \text{const}$ on $\overline{B}$, it follows that $N_\nu : B \to S^2$ is an open mapping (because the composition $\sigma \circ N_\nu$ with a stereographic projection $\sigma : S^2 \to \mathbb{C}$ is locally holomorphic, see Section 3.3). We then conclude that all spherical images $N_\nu(B_\rho(w_0))$ have to remain within this cap. Otherwise $N_\nu(B_\rho(w_0))$ would entirely cover the complementary cap, in contradiction to the integral condition (31). Thus we obtain a modulus of continuity for the mappings $N_\nu, \nu \in \mathbb{N}$, in the interior of B. $\square$

Now we present the following version of Nitsche's 6π-theorem:

Theorem 3. *Let $\Gamma \in C^{3,\alpha}$ be a closed, regular, extreme Jordan curve of the total curvature $\kappa(\Gamma) < 6\pi$. Then there exist only finitely many minimal immersions $X : \overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$.*

Proof. If there were infinitely many minimal immersions, Proposition 8 would yield a sequence of distinct minimal immersions $X_j : \overline{B} \to \mathbb{R}^3$ of class $\mathcal{C}^*(\Gamma)$ which converge in $C^{2,\beta}(\overline{B}, \mathbb{R}^3)$, $0 < \beta < \alpha$, to some minimal immersion $X \in \mathcal{C}^*(\Gamma)$ of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with

$$\Lambda(w) := \tfrac{1}{2}|\nabla X(w)|^2 > 0 \quad \text{in } \overline{B}$$

and

$$-\int_B K\Lambda \, du\, dv \le \omega < 4\pi. \tag{32}$$

By virtue of Proposition 4 we can represent the surfaces X_j as graphs Y_j over X in the form

$$\begin{gathered} Y_j(w) = X(w) + \zeta_j(w) N(w) \quad \text{for } w \in \overline{B} \\ \text{with } \zeta_j \in C_0^{2,\alpha}(\overline{B}) \text{ and } \|\zeta_j\|_{2,\beta} \to 0 \text{ for } 0 < \beta < \alpha. \end{gathered} \tag{33}$$

For $j \gg 1$ the ζ_j are solutions of

$$L\zeta_j = \Phi(\zeta_j) \quad \text{in } B \quad \text{with } \zeta_j = 0 \text{ on } \partial B, \tag{34}$$

where L is the Schwarzian operator for X. If $\lambda = 0$ were not an eigenvalue of L with respect to the boundary condition $\zeta = 0$ on ∂B, we would obtain

$$\zeta_j = L_0^{-1}\Phi(\zeta_j), \quad j \in \mathbb{N},$$

with $L_0 := L|_{C_0^{2,\beta}(\overline{B})}$. Since $L_0^{-1}\phi$ is contracting (see Proposition 2 of Section 5.6) we obtain a contradiction to the property $\|\zeta_j\|_{2,\beta} \to 0$ as $j \to \infty$. Thus $\lambda = 0$ is an eigenvalue of L.

If $\lambda = 0$ is the smallest eigenvalue of L then X is stable, and the arguments used in the proofs of the Theorems 1 and 2 lead to a contradiction.

Now we show that $\lambda = 0$ has to be the smallest eigenvalue of L. Otherwise there is a $\xi \in C_0^{2,\beta}(\overline{B})$ with

$$L\xi = 0 \quad \text{in } B$$

with

$$\int_B \xi(w) \cdot \xi_1(w)\, dw = 0,$$

where ξ_1 is an eigenfunction to the smallest eigenvalue of L, i.e.

$$L\xi_1 = \lambda_1 \xi_1 \quad \text{in } B, \quad \xi_1 = 0 \quad \text{on } \partial B,$$

satisfying $\xi_1(w) > 0$ in B. Then there are two disjoint and nonempty open subsets Ω_1 and Ω_2 of $\{w \in B\colon \xi(w) \neq 0\}$ such that

$$L\xi = 0 \quad \text{in } \Omega_j, \quad \xi = 0 \quad \text{on } \partial\Omega_j, \quad \xi(w) \neq 0 \quad \text{on } \Omega_j \quad \text{for } j = 1, 2. \tag{35}$$

Condition (32) implies that one of the domains Ω_j, say Ω_1, has the property

$$-\int_{\Omega_1} K\Lambda\, du\, dv < 2\pi. \tag{36}$$

In virtue of the stability theorem by Barbosa–do Carmo (see Section 5.4), property (36) implies that $X|_{\Omega_1}$ is strictly stable, which is a contradiction to (35) for $j = 1$.

Therefore, Γ bounds only finitely many minimal immersions of class $\mathcal{C}^*(\Gamma)$. □

Remark 9. The last theorem remains true under the weaker assumption $\kappa(\Gamma) \leq 6\pi$. To cover the case $\kappa(\Gamma) = 6\pi$ we refer to the proof of Theorem 1 in Section 5.6, estimating the total geodesic curvature by the total curvature.

Remark 10. It would be desirable to establish Theorem 3 for real analytic contours, renouncing the assumption that Γ be extreme. Nitsche's 6π-theorem in [31] states finiteness under the assumption that Γ be real analytic and that no minimal surface $X \in \mathcal{C}(\Gamma)$ has a branch point on $\overline{B}$. We also hint at the work of Beeson [3–5].

5.8 Scholia

H.A. Schwarz initiated the study of the second variation of area for immersed minimal surfaces in his celebrated memoir *Ueber ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung* (1885), dedicated to K. Weierstrass on occasion of his seventieth birthday (cf. Schwarz [2], Vol. I, pp. 223–269). The main purpose of that paper is to establish a criterion whether or not a given minimal surface furnishes a relative minimum of area among all surfaces bounded by the same contour. As Schwarz showed, a minimal surface is a local minimizer if it can be embedded in a field, i.e. a one-parameter foliation, of minimal surfaces. We have described this idea in Sections 2.7 and 2.8. When is such an embedding possible? To decide this question, Schwarz considered the spherical image Ω of the given surface and introduced the *Schwarz operator* L on this image. The desired embedding is possible if the equation $L\zeta = 0$ in Ω possesses a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ which is positive on $\overline{\Omega}$. In this connection, Schwarz connected the study of the operator $\mathfrak{L} = \Delta + p$, $p > 0$, with the minimum problem for the Rayleigh quotient $J_1(\zeta)/J_0(\zeta)$, where

$$J_1(\zeta) := \int_\Omega |\nabla\zeta|^2 \, dx \, dy, \quad J_0(\zeta) := \int_\Omega p\zeta^2 \, dx \, dy.$$

This led him to the minimum characterization of the smallest eigenvalue for Δ and $\mathfrak{L}$ respectively, which can be considered as the beginning of Hilbert's theory of eigenvalue problems in the form that later was developed by Courant. In this paper one also finds *Schwarz's inequality* (see Schwarz [2], Vol. I, p. 251) in the form

$$\left| \int_\Omega \varphi\psi \, dx \, dy \right| \le \sqrt{\int_\Omega \varphi^2 \, dx \, dy} \sqrt{\int_\Omega \psi^2 \, dx \, dy}.$$

These ideas were generalized by L. Lichtenstein, and later by many other mathematicians to study the corresponding minimum problem for general multiple integrals in the calculus of variations; see Giaquinta and Hildebrandt [1], Vol. 1, Chapter 6, in particular Section 4.

J.C.C. Nitsche [26] revived Schwarz's field construction to prove the celebrated uniqueness theorem presented in Section 5.6. Basic ingredients of Nitsche's proof are the results of Chapter 6 concerning the existence of unstable minimal surfaces, obtained by the mountain-pass lemma, and the stability theorem of J.L. Barbosa and M. do Carmo [1].

We also note that the renewed interesting and flourishing study of stable minimal surfaces was, in fact, initiated by the work of Barbosa and do Carmo.

A very careful and comprehensive description of results connected with the second variation of surface area and stable minimal surfaces can be found in J.C.C. Nitsche's treatises [28] and [37], §§98–119; in particular a lucid presentation of Schwarz's approach is given.

In Sections 5.1–5.5 we essentially followed the work of F. Sauvigny [1, 2,7–11]. We also mention prior work by R. Schoen [2], who generalized the fundamental curvature estimate by E. Heinz [1], presented in Section 2.4, to minimal immersions $X : B \to N$ in a three-dimensional oriented Riemannian manifold N. A special case of his Theorem 3 is the following result: *Let $M = X(B)$ be an immersed, stable surface in $\mathbb{R}^3$ which compactly contains a geodesic ball $B_{R_0}(P_0)$ for some $P_0 \in M$ and some $r_0 > 0$. Then there is an absolute constant $c > 0$ such that the second fundamental form A of M at P_0 is estimated by $|A|^2(P_0) \le c r_0^{-2}$.* The corresponding analogue for cmc-surfaces, due to F. Sauvigny [7,8], is given in Section 5.5, see Theorems 1 and 2. We also refer the reader to the interesting work of S. Fröhlich [1–5] on curvature estimates for immersions of mean curvature type, even with higher codimensions, where the notion of μ-stable extremals appears.

Nitsche's uniqueness result had a predecessor in an unpublished paper by R. Schneider, who formulated the following beautiful theorem (1968): *A closed polygon in* $\mathbb{R}^3$ *with a total curvature* $\kappa(\Gamma) < 4\pi$ *bounds only one disk-type minimal surface.* Moreover, he conjectured that every Jordan curve with a total curvature less than 4π spans only one disk-type minimal surface, and for any $\epsilon > 0$ he gave an example of a curve Γ with $\kappa(\Gamma) < 4\pi + \epsilon$ bounding at least two disk-type minimal immersions.

Schneider's Example (1968): Consider the minimal surface $X : \overline{\Omega} \to \mathbb{R}^3$ defined by

$$X(u, v) := (-v \sin u, v \cos u, u), \quad \Omega := \{(u, v) \colon |u| < \alpha\pi, |v| < R\}$$

for $R > 0$ and $0 < \alpha < 1$, which is part of the helicoid given by the equation $x + y \tan z = 0$. The boundary Γ of X consists of two straight segments Γ_1, Γ_2 and two parts Γ_3, Γ_4 of helices meeting Γ_1 and Γ_2 perpendicularly. The total curvature of Γ_3 as well of Γ_4 is $2\pi\alpha R(1 + R^2)^{-\frac{1}{2}}$. Adding the contributions of the four corners of Γ, one obtains

$$\kappa(\Gamma) = 2\pi[1 + 2\alpha R(1 + R^2)^{-\frac{1}{2}}].$$

Given $\epsilon \in (0, 2\pi)$ we choose $\alpha \in (\frac{1}{2}, 1)$ as $\alpha := \frac{1}{2} + \frac{\epsilon}{4\pi}$. Then

$$\kappa(\Gamma) = 2\pi + (2\pi + \epsilon)R(1 + R^2)^{-\frac{1}{2}}.$$

The right-hand side is an increasing function of $R \in [0, \infty)$ which tends to 1 as $R \to \infty$, thus $\kappa(\Gamma) < 4\pi + \epsilon$ for any $R > 0$. On the other hand, Schwarz showed in 1872 (see [1]; and [2], Vol. 1, pp. 161–163) that for $\alpha \in (\frac{1}{2}, 1)$ there is a value $R_0(\alpha) \in (\sqrt{3}, \infty)$ with $R_0(\alpha) \to \infty$ as $\alpha \to \frac{1}{2} + 0$, $R_0(\alpha) \to \sqrt{3}$ as $\alpha \to 1 - 0$, such that $X \circ \tau$ does not furnish a relative minimum of area in $\mathcal{C}(\Gamma)$ if $R \in (R_0(\alpha), \infty)$, where τ is a conformal mapping of the unit disk B onto Ω. We know however that there is a minimal surface $\tilde{X} \in \mathcal{C}(\Gamma)$ which minimizes area in $\mathcal{C}(\Gamma)$. This surface is an immersion for the following reason. Since Γ lies on the boundary of a compact convex set K, $\tilde{X}$ cannot have any boundary branch points. Furthermore, through every point of K there is a plane which intersects Γ only in two points. Hence no minimal surface of class $\mathcal{C}(\Gamma)$ has an interior branch point (see Radó [21], p. 35). *Thus* Γ *bounds at least two regular* (i.e. immersed) *minimal surfaces.* We recall that Böhme [6] later on showed that for any $\epsilon > 0$ and any $N \in \mathbb{N}$ there is a real analytic Jordan curve Γ with $\kappa(\Gamma) < 4\pi + \epsilon$ which bounds at least N disk-type minimal surfaces.

Schneider's paper was not published since it depended on fragmentary results by Marx and Shiffman (cf. Marx [1]) which in 1968 were considered to be unproved (Oberwolfach meeting on the "Calculus of Variations"). This desideratum stimulated E. Heinz to write a series of fundamental papers (cf. Heinz [19–24]) which rigorously dealt with the asymptotic behaviour of minimal surfaces in corners and led to the theory of quasi-minimal surfaces. Some of Heinz's results are described in the Scholia to Chapter 6. Using these results, F. Sauvigny [3–5] developed a theory of the second variation of the area for minimal surfaces bounded by polygons, and he rediscovered Schneider's unpublished result, thereby also establishing an analog for $\mathbb{R}^p$, $p > 3$. In addition, the "finiteness question" for certain polygonal boundaries was answered affirmatively by R. Jakob [9,10], building on Heinz's results.

We also mention a paper by H. Ruchert [2] where Nitsche's uniqueness theorem is carried over to "small" surfaces of constant mean curvature.

In a fundamental paper by R. Böhme and F. Tomi [1], the structure of the set of solutions to Plateau's problems was analyzed with the aid of semi-analytic sets. This in turn led to F. Tomi's seminal paper [6] about the finiteness of the number of absolute minimizers for Plateau's problem.

J.C.C. Nitsche proved the 6π-finiteness theorem in his paper [31]. The isolatedness of cmc-immersions solving the corresponding Plateau problem was investigated by F. Sauvigny [10]. His ideas are used in Section 5.7, especially for the compactness results concerning minimal immersions.

Chapter 6

Unstable Minimal Surfaces

In this chapter we want to show that the existence of two minimal surfaces in a closed rectifiable contour Γ, which are local minimizers of Dirichlet's integral D, guarantees the existence of a third minimal surface bounded by Γ, which is unstable, i.e. of non-minimum character. Results of this kind were first proved by M. Shiffman [2] and simultaneously by M. Morse and C. Tompkins [1,2]. Here we present an approach to the result stated above that is due to R. Courant [13] (a detailed presentation is given in his treatise [15], Chapter VI, Sections 7 and 8). Courant's method proceeds by reduction of the problem to a finite-dimensional one for a function $\Theta : T \to \mathbb{R}^n$ provided the boundary Γ is a closed polygon.[1] In Section 6.1 we describe *Courant's reduction method* in a modified version due to E. Heinz [13]. Then, in 6.2, we prove several results concerning the existence of unstable critical points for a function $f \in C^1(\Omega)$ defined on a bounded, open, connected set of $\mathbb{R}^n$. The prototype is the following theorem: *If f possesses two strict local minimizers $x_1, x_2 \in \Omega$ and satisfies $f(x) \to \infty$ as x tends to $\partial\Omega$, then there exists a third critical point x_3 which is of non-minimum type.* The proof of such a result uses a maximum-minimum principle that is nowadays called the *mountain pass lemma*.

In 6.3 this result is used to show that a polygonal contour bounds an unstable minimal surface if it bounds two surfaces which are *separated by a wall*, for instance if it spans two strict local minimizers with respect to the "strong norm"

$$\|X\|_{1,B} := \|X\|_{C^0(B,\mathbb{R}^3)} + \sqrt{D(X)}.$$

Shiffman [4] extended Courant's approach from polygons to general rectifiable contours using convergence results for the area functional A and for Dirichlet's integral D. These results are presented in Section 6.4. One kind of convergence employs the *Douglas functional* A_0 which is seen to coincide with D on harmonic mappings. The other kind of convergence uses computations and

[1] In Section 4.15, No. 5, the Courant function was denoted by d just as in Courant's original work.

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estimates which also lead to an *isoperimetric inequality for harmonic surfaces* H of class $\overline{\mathcal{C}}(\Gamma)$,

$$A(H) \leq \tfrac{1}{4} L^2(\Gamma),$$

where $L(\Gamma)$ is the length of the boundary contour Γ, or more generally,

$$A(H) \leq \frac{1}{4} \left(\int_{\partial B} |dX| \right)^2$$

for $H \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ if $X|_{\partial B}$ is not monotonic. Shiffman's passage to the limit from polygons to general rectifiable contours satisfying a chord-arc condition is worked out in Section 6.6, also using ideas due to Heinz [14] and a topological reasoning that we have taken from R. Jakob [1] and [2]; this part is presented in Section 6.5. It should be pointed out that in 6.6 we have to work with the weaker norm $\|X\|_{C^0(\overline{B}, \mathbb{R}^3)}$ since the convergence results of 6.4 do not suffice to carry over the results for polygons to the case of general boundaries in full strength.

6.1 Courant's Function Θ

Let Γ be a simple closed polygon (i.e. a piecewise linear and closed Jordan curve) in $\mathbb{R}^3$ with $N + 3$ (≥ 4) consecutive vertices

$$Q_0, A_1, \ldots, A_l, Q_1, A_{l+1}, \ldots, A_m, Q_2, A_{m+1}, \ldots, A_N, Q_0.$$

Set $\psi_k := \frac{2k\pi}{3}$ for $k = 0, 1, 2, 3$ and consider the set T of points $t = (t^1, \ldots, t^N) \in \mathbb{R}^N$ satisfying

$$\psi_0 < t^1 < \cdots < t^l < \psi_1 < t^{l+1} < \cdots < t^m < \psi_2 < t^{m+1} < \cdots < t^N < \psi_3.$$

Clearly T is a bounded, open, and convex subset of $\mathbb{R}^N$. We define $\overline{\mathcal{C}}(\Gamma)$ as in Chapter 4, and the subclass $\overline{\mathcal{C}}^*(\Gamma)$ is to consist of those $X \in \overline{\mathcal{C}}(\Gamma)$ which satisfy the three-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, with $w_k := e^{i\psi_k}$.

Definition 1. *With every $t \in T$ we associate the set*

$$(1) \qquad U(t) = \{X \in \overline{\mathcal{C}}^*(\Gamma) \colon X(e^{it^k}) = A_k,\ k = 1, \ldots, N\}.$$

Then we define the **Courant function** $\Theta : T \to \mathbb{R}$ *by*

$$(2) \qquad \Theta(t) := \inf\{D(X) \colon X \in U(t)\}.$$

Proposition 1. *For every $t \in T$ there is exactly one $X \in U(t)$ such that*

$$D(X) = \Theta(t).$$

This X is harmonic in B, continuous on $\overline{B}$, and the quadratic differential $dX \cdot dX$ is holomorphic, that is, the function $f := a - ib$ with $a := |X_u|^2 - |X_v|^2$, $b := 2\langle X_u, X_v \rangle$ is holomorphic in B.

Proof. (i) The existence of a solution of *Courant's minimum problem*

$$D \to \min \quad \text{in } U(t) \tag{3}$$

is proved in the same way as the existence of a solution to Plateau's problem "$D \to \min$ in $\overline{\mathcal{C}}^*(\Gamma)$", and also the regularity properties follow in the same manner. Since $D(X) = \Theta(t)$ implies

$$\partial D(X, \lambda) = 0 \quad \text{for any } \lambda \in C_c^\infty(B, \mathbb{R}^2)$$

the function $f = a - ib$ is holomorphic in B (see Chapter 4).

(ii) Suppose now that $X_1, X_2 \in U(t)$ are two solutions of (3), i.e.

$$D(X_1) = D(X_2) = \Theta(t).$$

Set $Y_1 := \frac{1}{2}(X_1 - X_2)$, $Y_2 := \frac{1}{2}(X_1 + X_2)$. Since $U(t)$ is evidently convex we have $Y_2 \in U(t)$, and so

$$\Theta(t) \le D(Y_2).$$

The parallelogram law yields

$$D(X_1) + D(X_2) - 2D(Y_2) = 2D(Y_1)$$

whence $D(X_1 - X_2) = 0$. This implies $\nabla(X_1 - X_2) = 0$, and so $X_1 - X_2 = \text{const}$. As $X_j(w_k) = Q_k$, $k = 0, 1, 2$, for both $j = 1$ and $j = 2$, we arrive at $X_1 = X_2$. □

Definition 2. *We introduce the* **Courant mapping** $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$ *as the mapping* $t \mapsto Z(t)$ *for* $t \in T$ *where* $Z(t)$ *is the uniquely determined element in* $U(t)$ *such that*

$$\Theta(t) = D(Z(t)). \tag{4}$$

For $w = u + iv \widehat{=} (u, v) \in \overline{B}$ *we write*

$$Z(t, u, v) = Z(t, w) := Z(t)(w).$$

There is a close connection between the Courant function Θ, the Courant map Z, and the minimal surfaces bounded by Γ. In fact we shall see that the minimal surfaces of class $\overline{\mathcal{C}}^*(\Gamma)$ are in one-to-one correspondence to the critical points t of Θ, and they are given by the values $Z(t)$ of Z at these t. Precisely speaking we shall prove:

Theorem 1. (i) *The Courant function* Θ *is of class* $C^1(T)$, *and* $\Theta(t)$ *tends to infinity if* t *approaches the boundary* ∂T.

(ii) *If* X *is a minimal surface of class* $\overline{\mathcal{C}}^*(\Gamma)$ *then* $X = Z(t)$ *for exactly one* $t \in T$.

(iii) *For* $t \in T$ *the harmonic surface* $Z(t)$ *is a minimal surface if and only if* t *is a critical point of* Θ. *Thus the set of minimal surfaces in* $\overline{\mathcal{C}}^*(\Gamma)$ *is in* 1–1 *correspondence to the set of critical points of* Θ.

To verify this result we proceed in several steps. We begin by proving

Lemma 1. *Suppose that* $X \in U(t)$ *is a minimal surface. Then* X *is real analytic on* $B' := \overline{B} \setminus \{e^{is_1}, \dots, e^{is_{N+4}}\}$ *where* $s_1 < s_2 < \cdots < s_{N+4}$ *stand for the* $N+4$ *parameters*

$$\psi_0 < t^1 < \cdots < t^l < \psi_1 < t^{l+1} < \cdots < t^m < \psi_2 < t^{m+1} < \cdots < t^N < \psi_3$$

corresponding to the vertices $Q_0, A_1, \dots, A_l, Q_1, \dots, A_N, Q_0$ *of* Γ*. Moreover, transforming* X *to polar coordinates* r, φ *around the origin by* $Y(r,\varphi) := X(re^{i\varphi})$ *we find for any* $j \in \{1, 2, \dots, N+3\}$ *an orthonormal triple of constant vectors* $p_1, p_2, p_3 \in \mathbb{R}^3$ *such that*

$$(5)\quad Y(r,\varphi) = Y(1,s_j) + \alpha_1(r,\varphi)p_1 + \alpha_2(r,\varphi)p_2 + \alpha_3(r,\varphi)p_3 \quad \text{for } re^{i\varphi} \in B'$$

and

$$(6)\qquad \alpha_1(1,\varphi) = 0, \quad a_2(1,\varphi) = 0, \quad a_{3,r}(1,\varphi) = 0 \quad \text{for } s_j < \varphi < s_{j+1}.$$

Proof. If X is a minimal surface of class $U(t)$, the assertions follow from the reflection principle, see Section 4.8, Theorem 1. □

Proposition 2. *Any minimal surface* X *of class* $U(t)$ *coincides with the minimizer* $Y := Z(t)$ *of* D *in* $U(t)$*.*

Proof. Consider the domain

$$\Omega := B \Big\backslash \bigcup_{j=1}^{N+3} B_{\varepsilon_j}(\tilde{w}_j), \quad \tilde{w}_j := e^{is_j},\ \varepsilon_j > 0,$$

with $s_1, \dots, s_{N+3}$ as in Lemma 1. For $0 < \varepsilon_j \ll 1$ the domain Ω is simply connected, and $\partial\Omega$ consists of subarcs γ_j of ∂B and of circular subarcs C_j of $\partial B_{\varepsilon_j}(\tilde{w}_j)$. For $\phi := Y - X$ we have

$$D_\Omega(Y) = D_\Omega(X) + D_\Omega(\phi) + 2D_\Omega(X,\phi),$$

and

$$D_\Omega(X,\phi) = \frac{1}{2}\int_{\partial\Omega} X_\nu \cdot \phi \, d\mathcal{H}^1, \quad \nu := \text{exterior normal to } \partial\Omega,$$

since $X \in C^2(\overline{\Omega}, \mathbb{R}^3)$, $X, \phi \in C^0(\overline{\Omega}, \mathbb{R}^3)$, and $\Delta X = 0$ in B. By Lemma 1 it follows that $X_\nu \cdot \phi = 0$ on $\gamma_1, \gamma_2, \dots, \gamma_{N+3}$ whence

$$D_\Omega(X,\phi) = \sum_{j=1}^{N+3} \int_{C_j} X_\nu \phi \, d\mathcal{H}^1$$

and therefore

$$|D_\Omega(X,\phi)| \le \text{const} \sum_{j=1}^{N+3} \int_{C_j} |X_\nu| \, d\mathcal{H}^1.$$

Choosing $\varepsilon_1, \dots, \varepsilon_{N+3}$ appropriately we can make the right hand side as small as we like (using the conformality relations and the Courant–Lebesgue Lemma, see Section 4.4), and so we arrive at

$$D(Y) = D(X) + D(Y - X) \ge D(X).$$

Since Y is assumed to be the uniquely determined minimizer of D in $U(t)$ we obtain $X = Y$. $\square$

Proof of part (ii) of Theorem 1. Since X is of class $\overline{\mathcal{C}}^*(\Gamma)$ it satisfies the 3-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, and $X|_{\partial B}$ is (weakly) monotonic. Thus there is an n-tuple $t \in T$ such that $X \in U(t)$. By Proposition 2 it follows that $X = Z(t)$. Suppose that there is another $t' \in T$ with $t' \ne t$ such that $X = Z(t')$. Then there exist values s and s' with $0 \le s < s' < 2\pi$ such that $\gamma := \{e^{i\varphi} \colon s < \varphi < s'\}$ lies in B' and $X(1,\varphi) \equiv \text{const}$ on γ whence $\nabla X \equiv 0$ on γ because of the conformality relations. As the branch points of a nonconstant minimal surface are isolated we obtain $X(w) \equiv \text{const}$ on $\overline{B}$ which contradicts the 3-point condition. Therefore $t = t'$. $\square$

This shows that the minimal surfaces within $\overline{\mathcal{C}}^*(\Gamma)$ are in one-to-one correspondence with a nonempty subset T_0 of T. We want to prove that T_0 is the set of critical points of Θ. This, in particular, requires to show that Θ is of class $C^1(T)$.

An important technical tool is a formula for the inner variation of Dirichlet's integral, for which we shall state a certain generalization. First we introduce an important class of diffeomorphisms $\sigma_\varepsilon = \sigma(\cdot, \varepsilon)$ of $\overline{B}$ onto itself that was already used in Chapter 4; see 4.5, Supplementary Remarks.

Lemma 2. *There exist two constants $\delta_0 > 0$ and $\kappa_0 > 0$ with the following properties*:

(i) *For every $\varepsilon \in (-2, 2)$ and any real-valued function $\mu \in C^1(\overline{B})$ with $|\mu|_{C^1(\overline{B})} < \delta_0$, the mapping $\tau_\varepsilon = \tau(\cdot, \varepsilon)$ of $\overline{B}$ into $\mathbb{R}^2$ defined by*

$$\tau_\varepsilon(w) = \tau(w, \varepsilon) := w e^{i\varepsilon\mu(w)}, \quad w \in \overline{B}, \tag{7}$$

is a C^1-diffeomorphism of $\overline{B}$ onto itself which maps any circle $C_r := \{w \in \mathbb{C} \colon |w| = r\}$, $0 < r \le 1$, onto itself, in particular $\sigma_\varepsilon(\partial B) = \partial B$. Denote by $\sigma_\varepsilon = \sigma(\cdot, \varepsilon) := \tau_\varepsilon^{-1}$ the inverse mapping to τ_ε. If we view $w \mapsto \sigma_\varepsilon(w)$ and $\omega \mapsto \tau_\varepsilon(w)$ as one-parameter families of diffeomorphisms from $\overline{B}$ onto itself, we have

$$\begin{aligned} &\tau_\varepsilon(w) = w - \varepsilon\lambda(w) + \rho(w, \varepsilon), \\ &\lambda(w) = -iw\mu(w), \quad \rho(w, \varepsilon) = \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} w i^n \mu^n(w). \end{aligned} \tag{8}$$

Writing $\lambda(w) = \lambda^1(w) + i\lambda^2(w)$ *in the real form* $\lambda(u,v) = (\lambda^1(u,v), \lambda^2(u,v))$ *we obtain*

$$\lambda^1(u,v) = v\mu(u,v), \quad \lambda^2(u,v) = -u\mu(u,v). \tag{9}$$

Clearly $\lambda(u,v)$ *is tangential to* ∂B *at* $w = (u,v)$.

(ii) *For* $X \in \mathcal{C}(\Gamma) \cap C^1(B, \mathbb{R}^3)$, $|\varepsilon| < 2$ *and* $|\mu|_{C^1(\overline{B})} < \delta_0$ *we can represent* $D(X \circ \sigma_\varepsilon)$ *in the following way*:

$$D(X \circ \sigma_\varepsilon) = D(X) + \varepsilon \partial D(X, \lambda) + \varepsilon^2 R(X, \mu) \tag{10}$$

with

$$\begin{aligned} \partial D(X,\lambda) = & \frac{1}{2} \int_B [a(\lambda^1_u - \lambda^2_v) + b(\lambda^1_v + \lambda^2_u)]\, du\, dv, \\ & a := |X_u|^2 - |X_v|^2, \ b := 2\langle X_u, X_v \rangle, \end{aligned} \tag{11}$$

and

$$|R(X,\mu)| \le \kappa_0 D(X) |\mu|^2_{C^1(\overline{B})}. \tag{12}$$

Furthermore we can write $\partial D(X,\lambda)$ *in the form*

$$\partial D(X,\lambda) = 4 \int_B \operatorname{Im}[w \mu_{\overline{w}} X_w \cdot X_w]\, du\, dv =: V(X,\mu). \tag{13}$$

Proof. Part (i) is fairly obvious and can be left to the reader. Assertion (ii) follows by the computations of Section 4.5. Note that the functions φ, μ, ν in 4.5, (26)–(28) have in (10)–(13) been replaced by μ, λ^1, λ^2 respectively. Thus we have

$$\begin{aligned} \tau_\varepsilon(w) = we^{i\varepsilon\mu(w)} = \sum_{n=0}^\infty \frac{\varepsilon^n}{n!} w i^n \mu^n(w) = w - \varepsilon\lambda(w) + \cdots \quad \text{as } \varepsilon \to 0, \\ \sigma_\varepsilon(\omega) = \omega + \varepsilon\lambda(\omega) + \cdots \quad \text{as } \varepsilon \to 0, \quad \lambda(w) = \lambda^1(w) + i\lambda^2(w) = -iw\mu(w), \end{aligned}$$

i.e.

$$\lambda^1(u,v) = v\mu(u,v), \quad \lambda^2(u,v) = -u\mu(u,v),$$

and so

$$\begin{aligned} \lambda^1_u &= v\mu_u, & \lambda^1_v &= \mu + v\mu_v, \\ \lambda^2_u &= -\mu - u\mu_u, & \lambda^2_v &= -u\mu_v, \end{aligned}$$

whence

$$a(\lambda^1_u - \lambda^2_v) + b(\lambda^1_v + \lambda^2_u) = a(v\mu_u + u\mu_v) + b(v\mu_v - u\mu_u)$$

and

$$2w\mu_{\overline{w}} = (u + iv)(\mu_u + i\mu_v) = (u\mu_u - v\mu_v) + i(v\mu_u + u\mu_v).$$

In conjunction with $4X_w \cdot X_w = a - ib$ the last equation yields

$$\begin{aligned} 8\,\mathrm{Im}[w\mu_{\overline{w}} X_w \cdot X_w] &= a(u\mu_v + v\mu_u) + b(v\mu_v - u\mu_u) \\ &= a(\lambda_u^1 - \lambda_v^2) + b(\lambda_v^1 + \lambda_u^2). \end{aligned}$$

Therefore equation (13) is equivalent to (11). $\square$

Proposition 3. *We have $\Theta \in C^0(T)$.*

Proof. For $t_p = (t_p^1, \dots, t_p^N) \in T$ with $t_p \to t \in T$ as $p \to \infty$, $t = (t^1, \dots, t^N)$ we have to show that $\Theta(t_p) \to \Theta(t)$. To this end we choose functions $\nu_n \in C^1(\overline{B})$, $n = 1, \dots, N$, satisfying

$$(14) \qquad \nu_n(w_k) = 0, \quad k = 0, 1, 2, \quad \text{and} \quad \nu_n(\zeta_j) = \delta_{nj}, \quad \zeta_j := e^{it^j},$$

δ_{nj} = Kronecker symbol, and set

$$(15) \qquad \mu_p(w) := \sum_{n=1}^N (t_p^n - t^n)\nu_n(w) \quad \text{for } w \in \overline{B}.$$

Then $\mu_p \in C^1(\overline{B})$ and $|\mu_p|_{C^1(\overline{B})} \to 0$ as $p \to \infty$, in particular

$$|\mu_p|_{C^1(\overline{B})} < \delta_0 \quad \text{for } p \gg 1.$$

Hence the mappings $\sigma_p := \tau_p^{-1}$ with τ_p defined by

$$(16) \qquad \tau_p(w) := we^{i\mu_p(w)}, \quad w \in \overline{B},$$

satisfy the assumptions of Lemma 2 for $p \gg 1$, and $\sigma_p(w_k) = w_k$ and $\sigma_p(\zeta_j^p) = \zeta_j$, $\zeta_j^p := e^{it_p^j}$, whence $X_p := Z(t) \circ \sigma_p \in U(t_p)$. Therefore

$$\Theta(t_p) \le D(X_p), \quad \Theta(t) = D(Z(t)),$$

and by Lemma 2 we obtain

$$(17) \qquad \Theta(t_p) \le \Theta(t) + \kappa D(Z(t))|\mu_p|^2_{C^1(\overline{B})}$$

for $p \gg 1$ and some constant $\kappa > 0$. Since $\Theta(t_p) = D(Z(t_p))$ it follows that

$$(18) \qquad D(Z(t_p)) \le \Theta(t)[1 + \kappa\delta_0^2] \quad \text{for } p \gg 1.$$

On the other hand, replacing μ_p by μ_p' with

$$\mu_p'(w) := \sum_{n=1}^N (t^n - t_p^n)\nu_n^p(w)$$

with $\nu_n^p(w_k) = 0$, $0 \le k \le 2$, $\nu_n^p(\zeta_j^p) = \delta_{nj}$ for $1 \le j, n \le N$ and $\nu_n^p \in C^1(\overline{B})$ as well as $|\nu_n^p|_{C^1(\overline{B})} \le c$ for some constant c and all $p \in \mathbb{N}$, $1 \le n \le N$, we consider $\sigma_p' := (\tau_p')^{-1}$ with

$$\tau_p'(w) := we^{i\mu_p'(w)} \quad \text{for } w \in \overline{B}.$$

Then $Y_p := Z(t_p) \circ \sigma_p' \in U(t)$ whence

$$\Theta(t) \le D(Y_p), \quad \Theta(t_p) = D(Z(t_p)),$$

and Lemma 2 yields

$$\Theta(t) \le \Theta(t_p) + \kappa D(Z(t_p))|\mu_p'|^2_{C^1(\overline{B})} \quad \text{for } p \gg 1 \tag{19}$$

with the same κ as in (17). On account of (17)–(19) we arrive at

$$|\Theta(t) - \Theta(t_p)| \le \kappa\Theta(t)\Big\{|\mu_p|^2_{C^1(\overline{B})} + [1 + \kappa\delta_0^2]|\mu_p'|^2_{C^1(\overline{B})}\Big\}$$

for $p \gg 1$. Since $|\mu_p|_{C^1(\overline{B})} \to 0$ and $|\mu_p'|_{C^1(\overline{B})} \to 0$ as $p \to \infty$ we obtain

$$\Theta(t_p) \to \Theta(t) \quad \text{as } t_p \to t. \tag{20}$$

□

Because of (18) and $|Z(t_p)|_{C^0(\overline{B},\mathbb{R}^3)} \le$ const for $p \in \mathbb{N}$, it follows that

$$|Z(t_p)|_{H_2^1(B,\mathbb{R}^3)} \le \text{const} \quad \text{for all } p \in \mathbb{N}.$$

Then, for any subsequence of $\{Z(t_p)\}$ we may extract another subsequence $\{Z(t_{p_k})\}$ such that $Z(t_{p_k}) \rightharpoonup Y$ in $H_2^1(B,\mathbb{R}^3)$ for some $Y \in H_2^1(B,\mathbb{R}^3)$. By the Courant–Lebesgue Lemma and the maximum principle we may also assume that

$$|Y - Z(t_{p_k})|_{C^0(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } k \to \infty.$$

This implies $Y \in U(t) \cap C^0(\overline{B},\mathbb{R}^3)$; therefore $\Theta(t) \le D(Y)$. On the other hand we infer from $Z(t_{p_k}) \rightharpoonup Y$ in $H_2^1(B,\mathbb{R}^3)$ that

$$D(Y) \le \varliminf_{k\to\infty} D(Z(t_{p_k})) = \lim_{k\to\infty} \Theta(t_{p_k}) = \Theta(t).$$

Thus we have $D(Y) = \Theta(t)$. In virtue of Proposition 1 and Definition 2 it follows that $Y = Z(t)$, whence $Z(t_{p_k}) \to Z(t)$ in $H_2^1(B,\mathbb{R}^3)$. By a standard reasoning we obtain

$$\big|Z(t) - Z(t_p)\big|_{H_2^1(B,\mathbb{R}^3)} + \big|Z(t) - Z(t_p)\big|_{C^0(\overline{B},\mathbb{R}^3)} \to 0 \quad \text{as } t_p \to t, \tag{21}$$

and well-known estimates for harmonic mappings yield

$$\big|Z(t) - Z(t_p)\big|_{C^s(\Omega,\mathbb{R}^3)} \to 0 \quad \text{as } t_p \to t \tag{22}$$

for any $\Omega \subset\subset B$ and any $s \in \mathbb{N}$. Thus we have found:

Proposition 4. *The Courant mapping* $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$ *is continuous in the sense of* (21) *and* (22).

Lemma 3. *Suppose that* $\mu \in C^1(\overline{B})$ *satisfies* $|\mu|_{C^1(\overline{B})} < \delta_0$ *and* $\mu(w_k) = 0$, $k = 0, 1, 2$, *where* δ_0 *is the constant from Lemma* 2. *Then for every* $t = (t^1, \dots, t^N) \in T$ *and* $u = (u^1, \dots, u^N)$ *with* $u^j := \mu(\zeta_j)$ *and* $\zeta_j = e^{it^j}$ *we have*

$$\Theta(t+u) \le \Theta(t) + V(Z(t), \mu) + R(Z(t), \mu) \tag{23}$$

with

$$\big|R(Z(t), \mu)\big| \le \kappa_0 D(Z(t)) |\mu|^2_{C^1(\overline{B})}. \tag{24}$$

Proof. Consider the diffeomorphism $\sigma = \tau^{-1}$ defined by $\tau(w) := we^{i\mu(w)}$, and let $Z(t)$ be the minimizer of D in $U(t)$. Then $Z' := Z(t) \circ \sigma \in U(t+u)$ and (10)–(13) implies

$$D(Z') = D(Z(t)) + V(Z(t), \mu) + R(Z(t), \mu)$$

where $R(Z(t), \mu)$ is estimated by (24), see Lemma 2. Furthermore $\Theta(t) = D(Z(t))$ and $\Theta(t+u) \le D(Z')$, and so we obtain (23). $\square$

Proposition 5. *For any* $t \in T$ *and any* $u \in \mathbb{R}^N$ *the directional derivative*

$$\frac{\partial}{\partial u} \Theta(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\Theta(t + \varepsilon u) - \Theta(t)] \tag{25}$$

exists and can be computed as follows: Choose some real-valued function $\mu \in C^1(\overline{B})$ *satisfying* $\mu(w_k) = 0$, $k = 0, 1, 2$, *and* $\mu(\zeta_j) = u^j$, $1 \le j \le N$, *for* $\zeta_j := \exp(it^j)$. *Then*

$$\frac{\partial}{\partial u} \Theta(t) = V(Z(t), \mu). \tag{26}$$

Setting $\lambda(w) := -iw\mu(w)$ *we can equivalently write*

$$\frac{\partial}{\partial u} \Theta(t) = \partial D(Z(t), \lambda). \tag{27}$$

Proof. Given $t \in T$ and $u \in \mathbb{R}^N$ we choose a function $\mu \in C^1(\overline{B})$ satisfying

(I) $\mu(w_k) = 0$, $k = 0, 1, 2$, and $\mu(\zeta_j) = u^j$ for $\zeta_j := e^{it^j}$, $1 \le j \le N$.
(II_r) There is some number $r > 0$ such that

$$\mu(w) \equiv \mu(\zeta_j) \quad \text{for } w \in \overline{B} \cap B_r(w_j),\ 1 \le j \le N.$$

Then we choose some $\varepsilon_0 > 0$ such that

$$\varepsilon_0 |\mu|_{C^1(\overline{B})} < \delta_0.$$

If we apply Lemma 3 to t and εu (instead of t and u) we obtain

$$\Theta(t+\varepsilon u) \le \Theta(t) + \varepsilon V(Z(t),\mu) + R(Z(t),\varepsilon\mu)$$

provided that $|\varepsilon| < \varepsilon_0$. Here we only have used (I). To establish the next inequality we employ (II$_r$). For this purpose we apply Lemma 3 to $t' := t+\varepsilon u$ and $-\varepsilon u$ instead of t and u respectively. Then

$$\Theta(t) \le \Theta(t+\varepsilon u) - \varepsilon V\big(Z(t+\varepsilon u),\mu) + R(Z(t+\varepsilon u), -\varepsilon\mu)$$

for $|\varepsilon| < \varepsilon_1(r)$ and some $\varepsilon_1(r) \in (0,\varepsilon_0)$. Thus, for $0<|\varepsilon|<\varepsilon_1(r)$,

$$\begin{aligned}&\big|\Theta(t+\varepsilon u) - \Theta(t) - \varepsilon V\big(Z(t),\mu)\big|\\ &\quad \le |\varepsilon|\big|V(Z(t+\varepsilon u),\mu) - V(Z(t),\mu)\big| + \big|R(Z(t),\varepsilon\mu\big| + \big|R(Z(t+\varepsilon u),-\varepsilon\mu)\big|.\end{aligned}$$

By (13) and (21) we see that

$$\big|V(Z(t+\varepsilon u),\mu) - V(Z(t),\mu)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Furthermore we have $|Z(t+\varepsilon u)|_{H_2^1(B,\mathbb{R}^3)} \le c = \text{const}$ for $|\varepsilon| \ll 1$, and so

$$|\varepsilon|^{-1}\big\{\big|R(Z(t),\varepsilon\mu)\big| + \big|R(Z(t+\varepsilon u),-\varepsilon\mu)\big|\big\} \le 2c|\mu|^2_{C^1(\overline{B})}\cdot|\varepsilon| \quad \text{for } |\varepsilon| \ll 1.$$

We then conclude that

$$\left|\frac{1}{\varepsilon}[\Theta(t+\varepsilon u) - \Theta(t)] - V(Z(t),\mu)\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we have proved (26) for functions $\mu \in C^1(\overline{B})$ satisfying (I) and (II$_r$). In order to show that (II$_r$) is superfluous we approximate a given $\mu \in C^1(\overline{B})$ satisfying (I) by functions $\mu_p \in C^1(\overline{B})$ satisfying (I) and (II$_{r_p}$) with $r_p \to +0$, as well as

$$|\mu - \mu_p|_{C^1(\Omega)} \to 0 \quad \text{for } p \to \infty \text{ and any } \Omega \subset\subset B.$$

By (13) we obtain

$$V(Z(t),\mu_p) \to V(Z(t),\mu) \quad \text{as } p \to \infty.$$

On the other hand we have already proved that $\frac{\partial}{\partial u}\Theta(t)$ exists and that

$$\frac{\partial}{\partial u}\Theta(t) = V(Z(t),\mu_p) \quad \text{for all } p \in \mathbb{N}.$$

Letting p tend to infinity we obtain (26) for any $\mu \in C^1(\overline{B})$ satisfying (I). □

Proposition 6. *The Courant function Θ is of class $C^1(T)$. Moreover, for any $t = (t^1,\dots,t^N) \in T$, $\zeta_j = \exp(it^j)$, and any $\mu \in C^1(\overline{B})$ satisfying $\mu(w_k) = 0$, $k = 0,1,2$, we have*

$$\sum_{j=1}^{N} \Theta_{t^j}(t)\mu(\zeta_j) = V(Z(t), \mu) \tag{28}$$

and, equivalently, for $\lambda(w) := -wi\mu(w)$:

$$\sum_{j=1}^{N} \Theta_{t^j}(t)\mu(\zeta_j) = \partial D(Z(t), \lambda). \tag{29}$$

Proof. Let $u = (1, 0, \ldots, 0)$ and choose $\mu, \mu' \in C^1(\overline{B})$ with

$$\mu(w_k) = 0, \quad \mu'(w_k) = 0, \quad k = 0, 1, 2,$$

and

$$\mu(\zeta_1) = \mu'(\zeta_1') = 1, \quad \mu(\zeta_j) = \mu'(\zeta_j') = 0 \quad \text{for } 2 \le j \le N,$$

where $\zeta_j = \exp(it^j)$, $\zeta_j' = \exp(it'^j)$. According to (26) we have

$$\frac{\partial \Theta}{\partial t^1}(t) = V(Z(t), \mu), \quad \frac{\partial \Theta}{\partial t^1}(t') = V(Z(t'), \mu'),$$

and Proposition 4 yields $|Z(t') - Z(t)|_{H_2^1(B, \mathbb{R}^3)} \to 0$ as $t' \to t$.

Furthermore we can construct μ' as a one-parameter family of functions $\mu'(t', \cdot) \in C^1(\overline{B})$, $|t' - t| \ll 1$, with

$$\lim_{t' \to t} \left| \mu'(t', \cdot) - \mu \right|_{C^1(\overline{B})} = 0.$$

Then we obtain

$$\lim_{t' \to t} \left| \Theta_{t^1}(t') - \Theta_{t^1}(t) \right| = 0,$$

i.e. $\Theta_{t^1} \in C^0(T)$, and similarly it follows that $\Theta_{t^2}, \ldots, \Theta_{t^N} \in C^0(T)$. Hence we have proved that $\Theta \in C^1(T)$, and this implies

$$\frac{\partial \Theta}{\partial u}(t) = \sum_{j=1}^{N} \Theta_{t^j}(t) u_j.$$

On account of Proposition 5, we now obtain (28) and (29). □

Proposition 7. *We have* $\Theta(t) \to \infty$ *as* $\operatorname{dist}(t, \partial T) \to 0$.

Proof. Otherwise there is a sequence $\{t_p\}$ of points $t_p \in T$ with $t_p \to t \in \partial T$ and $\Theta(t_p) = D(Z(t_p)) \le \text{const}$, and by the Courant–Lebesgue Lemma we may assume that the mappings $Z(t_p)|_{\partial B}$ are uniformly convergent on ∂B. This clearly contradicts the fact that $t_p \to t \in \partial T$, which means that at least one of the sequences of intervals

$$[\psi_0, t_p^1], \; [t_p^1, t_p^2], \; \ldots, \; [t_p^N, \psi_2], \quad p \in \mathbb{N},$$

shrinks to one point, whereas each of these intervals is mapped by $Z(t_p)|_{\partial B}$ onto one of the sides of Γ. □

Proof of Theorem 1. Part (i) of the assertion follows from Propositions 6 and 7, and Part (ii) is already proved. Thus it remains to prove Part (iii):

(I) If $Z(t)$ is a minimal surface we have $Z(t)_w \cdot Z(t)_w = 0$, and consequently $V(Z(t), \mu) = 0$ for any $\mu \in C^1(\overline{B})$, which implies $\nabla \Theta(t) = 0$ by virtue of Propositions 5 and 6 respectively.

(II) If $\nabla \Theta(t) = 0$ we infer from Proposition 6 that

$$V(Z(t), \mu) = 0 \quad \text{for any } \mu \in C^1(\overline{B}) \text{ with } \mu(w_k) = 0, \quad k = 0, 1, 2. \tag{30}$$

By Proposition 8 to be proved consequently we obtain $Z(t)_w \cdot Z(t)_w = 0$, and therefore $Z(t)$ is a minimal surface since $Z(t)$ is harmonic.

(III) By assertion (ii) of Theorem 1 we know that for every minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ there is exactly one $t \in T$ such that $X = Z(t)$. Hence the set of minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$ is in one-to-one correspondence to the set of critical points of Θ. □

Proposition 8. *Suppose that (30) is satisfied. Then*

$$V(Z(t), \mu) = 0 \quad \textit{for any } \mu \in C^1(\overline{B}), \tag{31}$$

and so $Z(t)$ satisfies the conformality relation

$$Z(t)_w \cdot Z(t)_w = 0, \tag{32}$$

i.e. $Z(t)$ is a minimal surface.

Proof. For the sake of brevity we set $X := Z(t)$. We have

$$V(X, \mu) = \lim_{r \to 1-0} V_r(X, \mu)$$

with

$$V_r(X, \mu) := 4 \operatorname{Im} \int_{B_r} w X_w \cdot X_w \mu_{\overline{w}} \, du \, dv,$$

$B_r := \{w \in B \colon |w| < r\}$, $0 < r < 1$. Set

$$f(w) := w X_w(w) \cdot X_w(w) \mu(w), \quad g(w) := w X_w(w) \cdot X_w(w) \mu_{\overline{w}}(w).$$

Since $X_w \cdot X_w$ is holomorphic we have

$$f_{\overline{w}} = g,$$

and Gauß's theorem yields

$$\int_{B_r} g(w) \, du \, dv = \frac{1}{2i} \int_{\partial B_r} f(w) \, dw = \frac{1}{2} \int_0^{2\pi} \tilde{f}(re^{i\varphi}) \mu(re^{i\varphi}) \, d\varphi$$

with

$$\tilde{f}(w) := w^2 X_w(w) \cdot X_w(w).$$

Therefore,

$$V_r(X, \mu) = \int_0^{2\pi} h(re^{i\varphi})\mu(re^{i\varphi})\, d\varphi$$

with

$$h(w) := 2 \operatorname{Im} \tilde{f}(w) = \sum_{k=2}^{\infty} (a_k w^k + \overline{a}_k \overline{w}^k).$$

Let

$$\mu_0(w) := \operatorname{Re}(a + bw + c\overline{w})$$

for arbitrarily chosen $a, b, c \in \mathbb{C}$. Then

$$\int_0^{2\pi} h(re^{i\varphi})\mu_0(re^{i\varphi})\, d\varphi = 0 \quad \text{for } 0 < r < 1$$

and therefore

$$V_r(X, \mu) = V_r(X, \mu - \mu_0).$$

With $r \to 1 - 0$ we arrive at

$$V(X, \mu) = V(X, \mu - \mu_0).$$

We can choose $a, b, c \in \mathbb{C}$ in such a way that $\mu(w_k) = \mu_0(w_k)$ for $k = 0, 1, 2$ whence $V(X, \mu - \mu_0) = 0$ on account of (30), and so $V(X, \mu) = 0$; i.e. we have verified (31).

Now we can argue as in Section 4.5, Supplementary Remark 1, to obtain $X_w \cdot X_w = 0$.

Another way to verify this equation is to apply the relation

$$0 = \lim_{r \to 1-0} \int_0^{2\pi} h(re^{i\varphi})\mu(re^{i\varphi})\, d\varphi, \quad h(w) = \sum_{k=2}^{\infty} (a_k w^k + \overline{a}_k \overline{w}^k),$$

to $\mu(w) := \frac{1}{2}(w^k + \overline{w}^k)$ as well as to $\mu(w) := \frac{1}{2i}(w^k - \overline{w}^k)$. This leads to $a_k + \overline{a}_k = 0$ and $a_k - \overline{a}_k = 0$, i.e. $a_k = 0$ for $k \geq 2$, and so $h(w) \equiv 0$ on B. Since

$$h(w) := 2 \operatorname{Im}[w^2 X_w(w) \cdot X_w(w)]$$

it follows that

$$w^2 X_w(w) \cdot X_w(w) \equiv \text{const} =: c.$$

Therefore $X_w(w) \cdot X_w(w) = cw^{-2}$ on $B \setminus \{0\}$, which implies $c = 0$ since the left-hand side is holomorphic in B. $\square$

6.2 Courant's Mountain Pass Lemma

In this section we want to prove several versions of the mountain pass lemma that can essentially be found in Courant's treatise [15], VI.7.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following two properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$;
(ii) *there are two distinct strict local minimizers* $x_1, x_2 \in \Omega$ *of* f.

Then f possesses a third critical point $x_3 \in \Omega$ that is "unstable" in the sense that x_3 is not a local minimizer of f. Furthermore x_3 has the following "saddle point property":

$$f(x_3) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x) =: c \tag{1}$$

where $\mathcal{P}$ denotes the set of all compact connected subsets $\boldsymbol{p}$ of Ω with $x_1, x_2 \in \boldsymbol{p}$ (i.e. the set of all "paths" in Ω connecting x_1 and x_2).

Proof. Because of (i) there is an $\varepsilon > 0$ such that

$$f(x) > c + 1 \quad \text{for all } x \in \Omega \text{ with } \operatorname{dist}(x, \partial\Omega) < \varepsilon. \tag{2}$$

We choose a sequence $\{\boldsymbol{p}_m\}$ of paths $\boldsymbol{p}_m \in \mathcal{P}$ with

$$c_m := \max_{\boldsymbol{p}_m} f \le c + 1 \quad \text{for all } m \in \mathbb{N} \quad \text{and} \quad \lim_{m \to \infty} c_m = c,$$

and then we set

$$\boldsymbol{p}_m^* := \text{closure}\,(\boldsymbol{p}_m \cup \boldsymbol{p}_{m+1} \cup \boldsymbol{p}_{m+2} \cup \cdots), \quad \boldsymbol{p}^* := \boldsymbol{p}_1^* \cap \boldsymbol{p}_2^* \cap \boldsymbol{p}_3^* \cap \cdots.$$

By (2), the compact sets $\boldsymbol{p}_m^*$ are contained in Ω, and $\boldsymbol{p}_1^* \supset \boldsymbol{p}_2^* \supset \boldsymbol{p}_3^* \supset \cdots$. Therefore $\boldsymbol{p}^*$ is a compact subset of Ω. Since x_1 and x_2 are contained in all $\boldsymbol{p}_m$ it follows that all sets $\boldsymbol{p}_m^*$ are connected, and so $\boldsymbol{p}^*$ is connected (see e.g. Alexandroff and Hopf [1], p. 118). Hence $\boldsymbol{p}^* \in \mathcal{P}$ and so

$$\max_{\boldsymbol{p}^*} f \ge \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f = c. \tag{3}$$

On the other hand, $\boldsymbol{p}^* = \limsup_{m \to \infty} \boldsymbol{p}_m :=$ set of all points $x \in \mathbb{R}^n$ with $x = \lim_{j \to \infty} z_j$ of points $z_j \in \boldsymbol{p}_{m_j}$ with $m_j \to \infty$. Thus any point $y \in \boldsymbol{p}^*$ is the limit of a sequence of points $z_j \in \boldsymbol{p}_{m_j}$ with $m_j \to \infty$. Therefore $f(z_j) \to f(y)$ and

$$f(z_j) \le \max_{\boldsymbol{p}_{m_j}} f = c_{m_j} \to c,$$

whence $f(y) \le c$, and consequently $\max_{\boldsymbol{p}^*} f \le c$. By virtue of (3) we obtain

$$\max_{\boldsymbol{p}^*} f = c \quad \text{with } c := \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x). \tag{4}$$

On account of (ii) we have also

$$\max\{f(x_1), f(x_2)\} < c. \tag{5}$$

Now we want to show that there is a critical point x_3 of f with $x_3 \in \boldsymbol{p}^*$ and $f(x_3) = c$. To prove this we consider the level set L_c in $\boldsymbol{p}^*$, defined by

$$L_c := \{x \in \boldsymbol{p}^* \colon f(x) = c\},$$

which is compact and nonvoid. We claim that $\nabla f(x_3) = 0$ for some $x_3 \in L_c$. Otherwise, $|\nabla f(x)| \geq 2\varepsilon > 0$ for all $x \in L_c$. Since $f \in C^1(\Omega)$ there would exist a number $\delta > 0$ such that

$$|\nabla f(x)| > \varepsilon \text{ in } \mathcal{U} := \{x \in \Omega \colon \operatorname{dist}(x, L_c) < \delta\} \subset\subset \Omega.$$

By virtue of (5) we can also choose $\delta > 0$ so small that $x_1, x_2 \notin \mathcal{U}$. Let $\mathcal{V}$ be an open subset of the open set $\mathcal{U}$ such that

$$L_c \subset\subset \mathcal{V} \subset\subset \mathcal{U} \subset\subset \Omega.$$

By Tietze's theorem there is a function $\eta \in C_c^0(\Omega)$ with $0 \leq \eta \leq 1$, $\eta(x) \equiv 1$ on $\mathcal{V}$, and $\operatorname{supp} \eta \subset \mathcal{U}$. Then we define $\varphi \in C^0(\Omega \times \mathbb{R}, \mathbb{R}^n)$ by

$$\varphi(x,t) := x - t\eta(x)\nabla f(x).$$

Clearly, $\varphi(x, \cdot) \in C^1(\mathbb{R}, \mathbb{R}^n)$ for any $x \in \Omega$. Since $\mathcal{U} \subset\subset \Omega$ and $\eta(x) = 0$ for $x \in \Omega \setminus \mathcal{U}$, there is a number $t_0 > 0$ such that $\varphi(x,t) \in \Omega$ for any $x \in \Omega$ and $|t| \leq t_0$. Thus $f \circ \varphi$ is defined on $\Omega \times [-t_0, t_0]$, and

$$\begin{aligned} \frac{d}{dt} f(\varphi(x,t)) &= -\eta(x)\langle \nabla f(\varphi(x,t)), \nabla f(x)\rangle \\ &= -\eta(x)|\nabla f(x)|^2 - \eta(x)\langle a(x,t) - \nabla f(x), \nabla f(x)\rangle \end{aligned}$$

with

$$a(x,t) := \nabla f(\varphi(x,t)).$$

By making $t_0 > 0$ sufficiently small we can achieve that

$$|a(x,t) - \nabla f(x)| \leq \frac{\varepsilon}{2} \leq \frac{1}{2}|\nabla f(x)|$$

for $x \in \overline{\mathcal{U}}$ and $|t| \leq t_0$ whence

$$-\eta(x)\langle a(x,t) - \nabla f(x), \nabla f(x)\rangle \leq \frac{\eta(x)}{2}|\nabla f(x)|^2$$

for $x \in \overline{\mathcal{U}}$ and $|t| \leq t_0$. Since $\eta(x) = 0$ for $x \in \Omega \setminus \mathcal{U}$ this inequality is also satisfied for $x \in \Omega \setminus \mathcal{U}$ and $|t| \leq t_0$, and so

$$\frac{d}{dt} f(\varphi(x,t)) \le -\frac{\eta(x)}{2} |\nabla f(x)|^2 \quad \text{for } x \in \Omega \text{ and } |t| \le t_0.$$

Let $\boldsymbol{p}_0$ be the compact, connected set

$$\boldsymbol{p}_0 := \varphi(\boldsymbol{p}^*, t_0) = \{\varphi(x, t_0) \colon x \in \boldsymbol{p}^*\}.$$

Since $x_1, x_2 \in \boldsymbol{p}^*$ and $x_1, x_2 \notin \mathcal{U}$ we obtain $x_1, x_2 \in \boldsymbol{p}_0$ on account of $\varphi(x,t) = x$ for $x \in \Omega \setminus \mathcal{U}$, and so we see that $\boldsymbol{p}_0 \in \mathcal{P}$. Consequently we have

$$\max_{\boldsymbol{p}_0} f \ge c. \tag{6}$$

Furthermore, for any $x \in \boldsymbol{p}^*$ and $z := \varphi(x, t_0)$ we can write

$$f(z) - f(x) = f(\varphi(x,t_0)) - f(\varphi(x,0)) = \int_0^{t_0} \frac{d}{dt} f(\varphi(x,t))\, dt$$

and therefore

$$f(z) \le f(x) - \frac{t_0}{2} \eta(x) |\nabla f(x)|^2.$$

For $x \in L_c$ we then obtain

$$f(z) \le f(x) - \frac{t_0}{2}\, \varepsilon^2 < f(x) = c,$$

and for $x \in \boldsymbol{p}_0 \setminus L_c$ we have $f(x) < c$ and therefore $f(z) \le f(x) < c$. This implies $f(z) < c$ for all $z \in \boldsymbol{p}_0$, whence $\max_{\boldsymbol{p}_0} f < c$ which is a contradiction to (6), and so we infer that $\min_{L_c} |\nabla f| = 0$. Hence L_c contains a critical point of f. Let K_c be the set of critical points of f contained in L_c, i.e.

$$K_c = \{x \in \boldsymbol{p}^* \colon f(x) = c \text{ and } \nabla f(x) = 0\}.$$

Clearly K_c is a closed subset of the compact set $\boldsymbol{p}^*$. Since x_1 and x_2 are contained in $\boldsymbol{p}^* \setminus L_c$ there is a boundary point x_3 of K_c (viewed as a subspace of the connected topological space $\boldsymbol{p}^*$). Then, in any neighborhood $\mathcal{N}$ of x_3 there is a point $y \in \boldsymbol{p}^* \setminus K_c$. By $f(x) \le c$ for all $x \in \boldsymbol{p}^*$ we either have $f(y) < c$ or $f(y) = c$. In the second case we have $\nabla f(y) \ne 0$; consequently there is a point $z \in \mathcal{N}$ with $f(z) < f(y) = c$. Thus any neighborhood $\mathcal{N}$ of x_3 contains a point x with $f(x) < f(x_3)$, i.e. x_3 is not a local minimizer of f. $\square$

The first part of the preceding proof yields the following result.

Proposition 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ satisfies $f(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ for $x \in \Omega$. Then, for any $x_1, x_2 \in \Omega$, there exists a compact, connected set $\boldsymbol{p}^* \subset \Omega$ with $x_1, x_2 \in \boldsymbol{p}^*$ such that*

$$\max_{\boldsymbol{p}^*} f = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f$$

where $\mathcal{P}$ denotes the set of all compact connected sets $\boldsymbol{p}$ in Ω with $x_1, x_2 \in \boldsymbol{p}$. We call $\boldsymbol{p}^*$ a **minimal path of** $\mathcal{P}$.

Remark 1. We note that the unstable critical point x_3 of f determined in the proof of Theorem 1 lies on a minimal path $\boldsymbol{p}^*$ of $\boldsymbol{p}$.

The result of Theorem 1 can be extended in the following way:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$.
(ii*) *There are two distinct points $x_1, x_2 \in \Omega$ such that*

$$\max_{\boldsymbol{p}} f > \max\{f(x_1), f(x_2)\} \quad \textit{for all } \boldsymbol{p} \in \mathcal{P}$$

where $\mathcal{P}$ denotes the set of all compact, connected subsets $\boldsymbol{p}$ of Ω containing x_1 and x_2.

Then f possesses a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}$ and an unstable critical point x_3 such that $x_3 \in \boldsymbol{p}^*$, $f(x_3) = \max_{\boldsymbol{p}^*} f$, and

$$f(x_3) = \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{x \in \boldsymbol{p}} f(x).$$

Proof. By Proposition 1 one shows that there is a "minimal path" $\boldsymbol{p}^*$ in $\mathcal{P}$ satisfying (4). Then (ii*) implies (5), and we can proceed as before. $\square$

Remark 2. If (ii*) holds we say that x_1 *and* x_2 *are separated by a wall.* This is, for instance, the case if there exist numbers c and $r > 0$ such that $|x_1 - x_2| > r$, $f(x) \geq c$ for $x \in \Omega$ with $|x - x_1| = r$, and $f(x_1), f(x_2) < c$.

Now we want to discuss the situation that f possesses two local minimizers which are not necessarily separated by a wall.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $f \in C^1(\Omega)$ has the following two properties*:

(i) $f(x) \to \infty$ *as* $\operatorname{dist}(x, \partial\Omega) \to 0$ *for* $x \in \Omega$;
(ii) *there are two distinct local minimizers $x_1, x_2 \in \Omega$ of f.*

Then either

(1°) *there is a compact connected set $\boldsymbol{p}^*$ in Ω containing x_1 and x_2 such that*

$$f(x_1) = f(x_2) =: c \quad \textit{and} \quad f(x) \equiv c, \quad \nabla f(x) \equiv 0 \quad \textit{for } x \in \boldsymbol{p}^*,$$

or else

(2°) *f possesses a third critical point $x_3 \in \Omega$ which is unstable.*

Proof. As before let $\mathcal{P}$ be the set of "paths" $\boldsymbol{p}$ containing x_1 and x_2 and set

$$c := \inf_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{p}} f.$$

We may assume that $f(x_1) \le f(x_2)$. By Proposition 1 we show that there is a minimal path $\boldsymbol{p}^* \in \mathcal{P}$ such that

$$\max_{\boldsymbol{p}^*} f = c.$$

If $f(x_2) < c$ we can proceed as before and obtain (2°). Therefore it suffices to consider the case $f(x_2) = c$. Since x_2 is a local minimizer of f, there is a $\delta > 0$ such that $f(x) \ge c$ on the ball $\mathcal{U}_\delta(x_2) := \{x \in \mathbb{R}^n \colon |x - x_2| < \delta\}$. Since $f(x) \le c$ for $x \in \boldsymbol{p}^*$ we have $f(x) \equiv c$ for $x \in \boldsymbol{p}^* \cap \mathcal{U}_\delta(x_2)$, which implies $\nabla f(x) \equiv 0$ for $x \in \boldsymbol{p}^* \cap \mathcal{U}_\delta(x_2)$. Set

$$L_c := \{x \in \boldsymbol{p}^* \colon f(x) = c\}, \quad K_c := \{x \in L_c \colon \nabla f(x) = 0\}.$$

If $K_c = \boldsymbol{p}^*$ we obtain assertion (1°), and we finally have to consider the case that $\boldsymbol{p}^* \setminus K_c$ is nonempty. Then there is a boundary point x_3 of K_c (viewed as a subspace of the connected topological space $\boldsymbol{p}^*$), and it follows as in the proof of Theorem 1 that x_3 is an unstable critical point of f. $\square$

6.3 Unstable Minimal Surfaces in a Polygon

Now we return to the situation considered in 6.1 where Γ is a simple closed polygon in $\mathbb{R}^3$ with $N+3$ vertices. As before $\overline{\mathcal{C}}^*(\Gamma)$ denotes the subset of surfaces X in $\overline{H}_2^1(B, \mathbb{R}^3) := H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ that map ∂B monotonically onto Γ (in the sense of 4.2, Definition 3) and fulfill a fixed 3-point condition $X(w_k) = Q_k$, $k = 0, 1, 2$, as described in 6.1. We equip $\overline{H}_2^1(B, \mathbb{R}^3)$ with the norm

$$\|X\|_{1,B} := \|X\|_{C^0(\overline{B}, \mathbb{R}^3)} + \sqrt{D(X)} \tag{1}$$

and the corresponding distance function

$$d_1(X, Y) := \|X - Y\|_{1,B}, \quad X, Y \in \overline{H}_2^1(B, \mathbb{R}^3). \tag{2}$$

Clearly $(\overline{H}_2^1(B, \mathbb{R}^3), \|\cdot\|_{1,B})$ is a Banach space, and $\overline{\mathcal{C}}^*(\Gamma)$ is a closed subset of this space. Therefore $(\overline{\mathcal{C}}^*(\Gamma), d_1)$ is a complete metric space.

In this section all topological concepts concerning subsets of $\overline{\mathcal{C}}^*(\Gamma)$ will refer to the metric d_1. (In 6.6 we shall change to a weaker metric, to d_0).

A *path* P in $\overline{\mathcal{C}}^*(\Gamma)$ is defined as a compact, connected subset of $\overline{\mathcal{C}}^*(\Gamma)$. We say that P *joins* (or *connects*) X_1 and X_2 if X_1, X_2 are contained in P, and $\mathbb{P}(X_1, X_2)$ denotes the *set of paths* in $\overline{\mathcal{C}}^*(\Gamma)$ joining X_1 and X_2.

Furthermore let $\mathcal{H}^*(\Gamma)$ be the subset of $X \in \overline{\mathcal{C}}^*(\Gamma)$ that are harmonic in B, and $\mathcal{W}^*(\Gamma)$ be the image of the bounded, open, convex subset T of $\mathbb{R}^N$ introduced in 6.1 under the Courant mapping $Z : T \to \overline{\mathcal{C}}^*(\Gamma)$. Then we have

(3) $$\mathcal{W}^*(\Gamma) := Z(T) \subset \mathcal{H}^*(\Gamma) \subset \overline{\mathcal{C}}^*(\Gamma).$$

For $t_1, t_2 \in T$ we denote by $\mathcal{P}(t_1, t_2)$ the set of paths $\boldsymbol{p}$ in T joining t_1, t_2, i.e. the set of compact, connected subsets $\boldsymbol{p}$ of T with $t_1, t_2 \in \boldsymbol{p}$. Moreover, $\mathbb{P}'(X_1, X_2)$ be the set of all paths $P \in \mathbb{P}(X_1, X_2)$ with $P \subset \mathcal{H}^*(\Gamma)$, and $\mathbb{P}''(X_1, X_2)$ be the set of paths $P \in \mathbb{P}(X_1, X_2)$ with $P \in \mathcal{W}^*(\Gamma)$.

The set T is connected, and Z is continuous according to 6.1, Proposition 4. Hence $\mathcal{W}^*(\Gamma)$ is connected, and the image $Z(\boldsymbol{p})$ of any path $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ is a path in $\mathcal{W}^*(\Gamma)$, i.e.

(4) $$Z(\boldsymbol{p}) \in \mathbb{P}''(X_1, X_2) \quad \text{for } \boldsymbol{p} \in \mathcal{P}(t_1, t_2) \text{ and } X_1 := Z(t_1),\ X_2 := Z(t_2).$$

For any $t \in T$ the set is convex. Hence, for any $X \in U(t)$ the mapping $R(t, X) : [0, 1] \to U(t)$, given by

(5) $$R(t, X)(\lambda) := \lambda Z(t) + (1 - \lambda)X, \quad 0 \le \lambda \le 1,$$

defines a continuous arc in $U(t)$ which connects X with $Z(t)$, and so the segment

(6) $$\Sigma(t, X) := \{R(t, X)(\lambda) \colon 0 \le \lambda \le 1\}$$

is a path in $\overline{\mathcal{C}}^*(\Gamma)$ joining X and $Z(t)$, i.e.

(7) $$\Sigma(t, X) \in \mathbb{P}(X, Z(t)) \quad \text{for } X \in U(t).$$

Lemma 1. *For any $X \in U(t)$ we have*

$$\max_{\Sigma(t,X)} D = D(X).$$

Proof. For $0 \le \lambda \le 1$ we set $Y(\lambda) := R(t, X)(\lambda)$, i.e.

$$Y(\lambda) = Z(t) + (1 - \lambda)\phi \quad \text{with } \phi := X - Z(t).$$

Then

$$D(Y(\lambda)) = D(Z(t)) + 2(1 - \lambda)D(Z(t), \phi) + (1 - \lambda)^2 D(\phi),$$

and consequently

(8) $$\frac{d}{d\lambda} D(Y(\lambda)) = -2D(Z(t), \phi) - 2(1 - \lambda)D(\phi) \quad \text{for } 0 \le \lambda \le 1.$$

On the other hand we have

$$D(Y(\lambda)) \ge D(Z(t)) = D(Y(1)) \quad \text{for } 0 \le \lambda \le 1$$

since $Y(\lambda) \in U(t)$ and $Z(t)$ is the minimizer of D in $U(t)$. It follows that

$$\frac{D(Y(1)) - D(Y(\lambda))}{1-\lambda} \leq 0 \quad \text{for } 0 \leq \lambda < 1,$$

whence

$$\frac{d}{d\lambda} D(Y(\lambda))\bigg|_{\lambda=1} \leq 0.$$

From (8) we infer for $\lambda = 1$ that

$$D(Z(t), \phi) \geq 0,$$

and so (8) yields

$$\frac{d}{d\lambda} D(Y(\lambda)) \leq 0 \quad \text{for } 0 \leq \lambda \leq 1.$$

Thus the function $\lambda \mapsto D(Y(\lambda))$ is decreasing for $0 \leq \lambda \leq 1$, whence $D(X) = D(Y(0)) \geq D(Y(\lambda))$ for $0 \leq \lambda \leq 1$. □

Lemma 2. *For any $X_1 \in U(t_1)$ and $X_2 \in U(t_2)$ there exists a path $P^* \in \mathbb{P}(X_1, X_2)$ such that*

$$\max_{X \in P^*} D(X) \leq \max\Big\{D(X_1), D(X_2), \max_{t \in \boldsymbol{p}} \Theta(t)\Big\} \tag{9}$$

holds for any $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$. Moreover, if $X_1, X_2 \in \mathcal{H}^*(\Gamma)$ then $P^* \in \mathbb{P}'(X_1, X_2)$.

Proof. By Lemma 1 we have

$$D(X) \leq D(X_j) \quad \text{for } X \in \Sigma(t_j, X_j) \text{ with } j = 1, 2,$$

and

$$\max_{X \in Z(\boldsymbol{p})} D(X) = \max_{t \in \boldsymbol{p}} \Theta(t) \quad \text{for } \boldsymbol{p} \in \mathcal{P}(t_1, t_2).$$

On account of 6.2, Proposition 1, there is a path $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$ such that

$$\max_{t \in \boldsymbol{p}^*} \Theta(t) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{t \in \boldsymbol{p}} \Theta(t).$$

Then $P^* := \Sigma(t_1, X_1) \cup Z(\boldsymbol{p}^*) \cup \Sigma(t_2, X_2)$ is a path in $\overline{\mathcal{C}}^*(\Gamma)$ joining X_1 and X_2 which, in addition, satisfies (9). Moreover, $P^* \subset \mathcal{H}^*(\Gamma)$ if $X_1, X_2 \in \mathcal{H}^*(\Gamma)$. □

Remark 1. Consequently $\mathbb{P}(X_1, X_2)$ is nonempty for any $X_1, X_2 \in \mathcal{C}^*(\Gamma)$ since there are points $t_1, t_2 \in T$ such that $X_1 \in U(t_1)$ and $X_2 \in U(t_2)$. Correspondingly, $\mathbb{P}'(X_1, X_2)$ is nonvoid for any $X_1, X_2 \in \mathcal{H}^*(\Gamma)$.

Now we want to establish the existence of unstable minimal surfaces spanning the polygon Γ using the results from 6.2. To this end we recall that Courant's function $\Theta := D \circ Z$ is of class $C^1(T)$ and satisfies $\Theta(t) \to \infty$ as $\operatorname{dist}(t, \partial T) \to 0$ for $t \in T$. Therefore, taking $\Omega := T$, $n := N$, and $f := \Theta$, we see that f satisfies assumption (i) of Theorem 1–3 in 6.2, which we will now apply to the present situation.

Definition 1. *A minimal surface $X \in \overline{\mathcal{C}}^*(\Gamma)$ is said to be unstable if for any $\rho > 0$ there is a mapping $Y \in \overline{\mathcal{C}}^*(\Gamma)$ such that $d_1(Y, X) < \rho$ and $D(Y) < D(X)$.*

Remark 2. Precisely speaking, a minimal surface X as in the preceding definition should be called D-*unstable*, whereas it could be called A-*unstable* if for any $\rho > 0$ there is a $Y \in \overline{\mathcal{C}}^*(\Gamma)$ such that $d_1(Y, X) < \rho$ and $A(Y) < A(X)$. We have: *Any D-unstable minimal surface X in $\overline{\mathcal{C}}^*(\Gamma)$ is also A-unstable.* In fact, the inequality $D(Y) < D(X)$ implies $A(Y) < A(X)$ because of

$$A(Y) \le D(Y) < D(X) = A(X).$$

Theorem 1. *Let X_1 and X_2 be two distinct minimal surfaces which are strict local minimizers of Dirichlet's integral on $(\overline{\mathcal{C}}^*(\Gamma), d_1)$. Then there exists an unstable minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$.*

Proof. By assumption there is an $\varepsilon_0 > 0$ such that

$$D(X_j) < D(X) \quad \text{for all } X \in \overline{\mathcal{C}}^*(\Gamma) \text{ with } 0 < d_1(X, X_j) < \varepsilon_0, \quad j = 1, 2.$$

Furthermore there are two points $t_1, t_2 \in T$ with $t_1 \ne t_2$ and $X_1 = Z(t_1)$, $X_2 = Z(t_2)$. Since Z is continuous there is a $\delta_0 > 0$ such that

$$d_1(Z(t), Z(t_j)) < \varepsilon_0 \quad \text{if } t \in T \text{ satisfies } |t - t_j| < \delta_0, \quad j = 1, 2.$$

Because of Theorem 1(ii) of 6.1 we have $Z(t) \ne Z(t_j)$ for $t \ne t_j$. It follows that

$$D(Z(t_j)) < D(Z(t)) \quad \text{for } t \in T \text{ with } 0 < |t - t_j| < \delta_0, \quad j = 1, 2,$$

which is equivalent to

$$\Theta(t_j) < \Theta(t) \quad \text{for } t \in T \text{ satisfying } 0 < |t - t_j| < \delta_0, \quad j = 1, 2.$$

Then, by 6.2, Theorem 1, there is an unstable critical point $t_3 \in T$ of Θ, i.e. $\nabla\Theta(t_3) = 0$, and for any $\delta > 0$ there is a point $t_\delta \in T$ with $|t_\delta - t_3| < \delta$ and $\Theta(t_\delta) < \Theta(t_3)$. Moreover, given $\rho > 0$, we have $d_1(Z(t), Z(t_3)) < \rho$ if $|t - t_3| < \delta \ll 1$. Setting $X_3 := Z(t_3)$ and $Y := Z(t_\delta)$ it follows that

$$D(Y) < D(X_3) \quad \text{and} \quad d_1(Y, X_3) < \rho.$$

By taking Theorem 1 of 6.1 into account we see that X_3 is an unstable minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$. □

On account of 6.2, formula (1), the unstable minimal surface X_3 of Theorem 1 has the following *saddle point property*:

Corollary 1. *If the two strict local minima X_1, X_2 of D are given by $X_1 = Z(t_1)$, $X_2 = Z(t_2)$, then*

$$D(X_3) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1,t_2)} \max_{t \in \boldsymbol{p}} D(Z(t)). \tag{10}$$

The preceding theorem can be generalized as follows:

Theorem 2. *Suppose that $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are "separated by a wall", i.e. it is assumed that $X_1 \neq X_2$ and*

$$\max_{X \in P} D(X) > \max\{D(X_1), D(X_2)\} \quad \textit{for all } P \in \mathbb{P}(X_1, X_2). \tag{11}$$

Then there exists an unstable minimal surface X_3 in $\overline{\mathcal{C}}^*(\Gamma)$.

Proof. There are two points $t_1, t_2 \in T$ with $X_1 \in U(t_1)$, $X_2 \in U(t_2)$. Here, t_1 and t_2 are not necessarily uniquely determined by X_1 and X_2 respectively. However, $t_1 \neq t_2$ since $t_1 = t_2$ would imply that $P := \Sigma(t_1, X_1) \cup \Sigma(t_2, X_2)$ is a path contained in $\mathbb{P}(X_1, X_2)$ such that $\max_{X \in P} D(X) = \max\{D(X_1), D(X_2)\}$ if we take Lemma 1 into account; but this were a contradiction to (11).

We claim that

$$\max_{\boldsymbol{p}} \Theta > \max\{\Theta(t_1), \Theta(t_2)\} \quad \text{for all } \boldsymbol{p} \in \mathcal{P}(t_1, t_2). \tag{12}$$

Otherwise we would have for all $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ that

$$\max_{\boldsymbol{p}} \Theta = \max\{\Theta(t_1), \Theta(t_2)\} \leq \max\{D(X_1), D(X_2)\},$$

since $\Theta(t_1) \leq D(X_1)$ and $\Theta(t_2) \leq D(X_2)$. Then it follows from Lemma 2 that there is a path $P^* \in \mathbb{P}(X_1, X_2)$ such that

$$\max_{X \in P^*} D(X) = \max\{D(X_1), D(X_2)\},$$

a contradiction to (11). Thus we have verified (12), and by 6.2, Theorem 2, there is an unstable critical point $t_3 \in T$ of the Courant function. Setting $X_3 := Z(t_3)$, we see as in the proof of Theorem 1 that X_3 is an unstable minimal surface in $\overline{\mathcal{C}}^*(\Gamma)$. $\square$

As before we obtain the saddle point property (9) for X_3:

Corollary 2. *If $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are separated by a wall, there is an unstable minimal surface $X = Z(\bar{t}) \in \overline{\mathcal{C}}^*(\Gamma)$ such that*

$$D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1,t_2)} \max_{t \in \boldsymbol{p}} D(Z(t))$$

for some critical point $\bar{t}$ of Θ.

Next we want to consider the case where $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ are two local minimizers of Dirichlet's integral which are not necessarily separated by a wall.

Theorem 3. *Suppose that X_1 and X_2 are two distinct minimal surfaces in $\overline{\mathcal{C}}^*(\Gamma)$ both of which are local minimizers of D on $\overline{\mathcal{C}}^*(\Gamma)$.*

Then either

1° *There is a path $P^* \in \mathbb{P}''(X_1, X_2)$ such that*

$$D(X) \equiv \text{const} =: c \quad \textit{for all } X \in P^*,$$

or else

2° *D possesses a third critical point X_3 in $\overline{\mathcal{C}}^*(\Gamma)$ which is an unstable minimal surface.*

Proof. There are uniquely determined points $t_1, t_2 \in T$ with $t_1 \neq t_2$ such that $X_1 = Z(t_1)$ and $X_2 = Z(t_2)$. The assumption of the theorem implies that t_1 and t_2 are distinct local minimizers of Θ. By virtue of 6.2, Theorem 3, there is a path $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$ such that either

$$\Theta(t) \equiv \text{const} =: c \quad \text{and} \quad \nabla\Theta(t) \equiv 0 \quad \text{for } t \in \boldsymbol{p}^*,$$

or else Θ possesses a third critical point $t_3 \in T$ which is unstable. In the first case we have 1° for $P^* = Z(\boldsymbol{p}^*) \in \mathbb{P}''(X_1, X_2)$, and in the second we obtain 2° for $X_3 := Z(t_3)$. □

In 6.6 we shall use the following variant of the preceding results.

Theorem 4. *For $t_1, t_2 \in T$ with $t_1 \neq t_2$ there is a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}(t_1, t_2)$ satisfying*

$$\max_{X \in Z(\boldsymbol{p}^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{X \in Z(\boldsymbol{p})} D(X). \tag{13}$$

If, in addition,

$$\max_{X \in Z(\boldsymbol{p}^*)} D(X) > \max\{D\big(Z(t_1)\big), D(Z(t_2))\} \tag{14}$$

then there is an unstable minimal surface X_3 in $\overline{\mathcal{C}}^*(\Gamma)$ such that $X_3 = Z(t_3)$ for $t_3 \in \boldsymbol{p}^*$, i.e. $X_3 \in Z(\boldsymbol{p}^*)$, and

$$D(X_3) = \max_{X \in Z(\boldsymbol{p}^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{X \in Z(\boldsymbol{p})} D(X).$$

Proof. The assertions follow immediately from 6.2, Proposition 1 and Theorem 2, if we take Theorem 1 of 6.1 into account. □

The next result will not be needed for the proof of the final theorems in Section 6.6; yet they are of independent interest.

Proposition 1. *If* $X_1 = Z(t_1)$ *and* $X_2 = Z(t_2)$ *for* $t_1, t_2 \in T$ *then*

$$d(X_1, X_2) := \inf_{P \in \mathbb{P}''(X_1, X_2)} \max_{X \in P} D(X) \tag{15}$$

and

$$\sigma(t_1, t_2) := \inf_{P \in Z(\mathcal{P}(t_1, t_2))} \max_{X \in P} D(X) \tag{16}$$

satisfy

$$d(X_1, X_2) = \sigma(t_1, t_2). \tag{17}$$

Proof. From $Z(\mathcal{P}(t_1, t_2)) \subset \mathbb{P}''(X_1, X_2)$ it follows that

$$d(X_1, X_2) \leq \sigma(t_1, t_2).$$

Thus it remains to show

$$\sigma(t_1, t_2) \leq d(X_1, X_2). \tag{18}$$

This is not obvious since the pre-image $Z^{-1}(P)$ of $P \in \mathbb{P}''(X_1, X_2)$ might not contain a path $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$. Instead we prove a weaker result, stated in the next proposition, which suffices to verify (18). $\square$

Proposition 2. *For any* $P \in \mathbb{P}''(Z(t_1), Z(t_2))$ *there exists a* $\boldsymbol{p} \in \mathcal{P}(t_1, t_2)$ *such that*

$$\max_{X \in Z(\boldsymbol{p})} D(X) \leq \max_{X \in P} D(X). \tag{19}$$

Proof. We first note that the pre-image $m := Z^{-1}(P)$ of a given $P \in \mathbb{P}''(Z(t_1), Z(t_2))$ is closed. In fact, if $t_j \in m$ for all $j \in \mathbb{N}$ and $t_j \to t_0$ then $t_0 \in \Omega$ since $t_0 \in \partial\Omega$ would imply $D(Z(t_j)) = \Theta(t_j) \to \infty$ whereas $Z(t_j) \in P$ yields $D(Z(t_j)) \leq \text{const} < \infty$. Since Z is continuous we have $Z(t_j) \to Z(t_0)$ whence $Z(t_0) \in P$. Hence m is closed and therefore compact. If m is connected we set $\boldsymbol{p} := m$ and obtain (19). Thus we now assume that m is disconnected and write m as disjoint union $m = \bigcup_{\alpha \in J} m_\alpha$ of its compact connected components m_α.

Consider two such components m_α and m_β, $\alpha \neq \beta$, for which $Z(m_\alpha) \cap Z(m_\beta)$ is nonvoid. Then there are points $t \in m_\alpha$ and $\bar{t} \in m_\beta$ such that $Z(t) = Z(\bar{t})$. Let $j \in \{1, \ldots, N\}$ be the first index such that $t^j \neq \bar{t}^j$, say, $t^j < \bar{t}^j$. Then it follows that

$$Z(t)(e^{i\varphi}) \equiv A_j \quad \text{for } \varphi \in [t^j, \bar{t}^j].$$

Consider the path $\gamma_1 := \{(t^1, \ldots, t^{j-1}, s, t^{j+1}, \ldots, t^N) \colon t^j \leq s \leq \bar{t}^j\}$ and set $Y(t_1) = Z(t)$ for $t_1 \in \gamma_1$. Then $Y(t_1) \in U(t_1)$ for $t_1 \in \gamma_1$. If $t^{j+1} =$

$\bar{t}^{j+1}, \ldots, t^N = \bar{t}^N$, the path γ_1 connects t and $\bar{t}$ in T, and $Y(t_1) \equiv Z(t) = Z(\bar{t})$ for all $t_1 \in \gamma_1$. Otherwise we proceed in the same way for the next index k with $t^k \neq \bar{t}^k$ and obtain a path γ_2 that connects $(t^1, \ldots, t^{j-1}, \bar{t}^j, t^{j+1}, \ldots, t^k, \ldots, t^N)$ with $(t^1, \ldots, t^{j-1}, \bar{t}^j, t^{j+1}, \ldots, \bar{t}^k, \ldots, t^N)$, and $Y(t_2) \equiv Z(t) = Z(\bar{t})$ for $t_2 \in \gamma_2$. After at most N steps we have constructed a path $\gamma_{\alpha\beta} \in \mathcal{P}(t, \bar{t})$ in T with $D(Z(\tau)) \leq D(Z(t)) = D(Z(\bar{t}))$ for all $\tau \in \gamma_{\alpha\beta}$. Then $m_{\alpha\beta} := m_\alpha \cup m_\beta \cup \gamma_{\alpha\beta} \in \mathcal{P}(t, \bar{t})$, and

(20) $$\max_{X \in Z(m_{\alpha\beta})} D(X) \leq \max_{X \in P} D(X).$$

On the other hand, if $Z(m_\alpha) \cap Z(m_\beta) = \emptyset$ we set $m_{\alpha\beta} := m_\alpha \cup m_\beta$; in this case (20) is clearly satisfied. Set

$$m' := \bigcup_{(\alpha,\beta) \in J \times J} m_{\alpha\beta}.$$

Then

(21) $$\sup_{X \in Z(m')} D(X) \leq \max_{X \in P} D(X) < \infty.$$

Since $\Theta(t) \to \infty$ as $\operatorname{dist}(t, \partial T) \to 0$ for $t \in T$, we conclude that $\boldsymbol{p} := \overline{m'}$ is a compact subset of T. Moreover we infer from the connectedness of P and the above construction that m' is connected, whence $\boldsymbol{p}$ is connected, since the closure of a connected set is connected. Hence $\boldsymbol{p}$ is an element of $\mathcal{P}(t_1, t_2)$, and (21) implies (19) because of the continuity of Z. □

Corollary 3. *If $X_1 = Z(t_1)$, $X_2 = Z(t_2)$ for $t_1, t_2 \in T$, $\boldsymbol{p}^* \in \mathcal{P}(t_1, t_2)$, and $\max_{t \in \boldsymbol{p}^*} D(Z(t)) = \sigma(t_1, t_2)$ then $P^* := Z(\boldsymbol{p}^*)$ satisfies $\max_{X \in P^*} D(X) = d(X_1, X_2)$. This means: The image $P^* = Z(\boldsymbol{p}^*)$ of a minimal path $\boldsymbol{p}^*$ of $\mathcal{P}(t_1, t_2)$ is a minimal path of $P(X_1, X_2)$.*

Proof. The assertion is an immediate consequence of Proposition 1. □

In particular we obtain:

Corollary 4. *The saddle point property*

$$D(X_3) = \inf_{\boldsymbol{p} \in \mathcal{P}(t_1, t_2)} \max_{t \in \boldsymbol{p}} D(Z(t))$$

in the Corollaries 1 and 2 is equivalent to

(22) $$D(X_3) = \inf_{P \in \mathbb{P}''(X_1, X_2)} \max_{X \in P} D(X).$$

6.4 The Douglas Functional. Convergence Theorems for Harmonic Mappings

Let $C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ be the class of continuous mappings $\xi : \mathbb{R} \to \mathbb{R}^3$ that are 2π-periodic, i.e. which satisfy $\xi(\theta + 2\pi) = \xi(\theta)$ for any $\theta \in \mathbb{R}$. Then the **Douglas functional** A_0 is a function $A_0 : C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3) \to \mathbb{R}$ defined by

$$(1) \qquad A_0(\xi) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\xi(\theta) - \xi(\varphi)|^2}{4 \sin^2 \frac{1}{2}(\theta - \varphi)} \, d\theta \, d\varphi \le \infty.$$

Because of

$$|e^{i\theta} - e^{i\varphi}|^2 = 4 \sin^2 \frac{1}{2}(\theta - \varphi)$$

we can write $A_0(\xi)$ as

$$(2) \qquad A_0(\xi) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\xi(\theta) - \xi(\varphi)|^2}{|e^{i\theta} - e^{i\varphi}|^2} \, d\theta \, d\varphi.$$

We recall the following well-known result:

Lemma 1. *Let $\xi \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$. Then the uniquely determined mapping $H \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with*

$$\Delta H = 0 \quad \text{in } B, \quad H(e^{i\theta}) = \xi(\theta) \quad \text{for } \theta \in \mathbb{R},$$

is given by

$$(3) \quad H(\rho e^{i\theta}) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta), \quad 0 \le \rho \le 1, \, \theta \in \mathbb{R},$$

$$a_n := \frac{1}{\pi} \int_0^{2\pi} \xi(\theta) \cos n\theta \, d\theta, \quad b_n := \frac{1}{\pi} \int_0^{2\pi} \xi(\theta) \sin n\theta \, d\theta.$$

We call H the "harmonic extension of ξ".

Theorem 1. *Let $H \in C^0(\overline{B}, \mathbb{R}^3)$ be the harmonic extension of $\xi \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$. Then*

$$(4) \qquad D(H) = A_0(\xi).$$

Proof. H is given by (3). Then

$$|a_n|^2 = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \xi(\theta) \xi(\varphi) \cos n\theta \cos n\varphi \, d\theta \, d\varphi,$$

$$|b_n|^2 = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \xi(\theta) \xi(\varphi) \sin n\theta \sin n\varphi \, d\theta \, d\varphi,$$

whence

$$|a_n|^2 + |b_n|^2 = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} \xi(\theta)\xi(\varphi)\cos n(\theta-\varphi)\,d\theta\,d\varphi \quad \text{for } n \ge 1.$$

Because of

$$\int_0^{2\pi} \cos n(\theta-\varphi)\,d\varphi = \int_0^{2\pi} \cos n(\theta-\varphi)\,d\theta = 0$$

we obtain

$$|a_n|^2 + |b_n|^2 = -\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} |\xi(\theta)-\xi(\varphi)|^2 \cos n(\theta-\varphi)\,d\theta\,d\varphi, \quad n \ge 1.$$

Furthermore,

$$D_{B_r}(H) = \frac{1}{2}\int_{B_r} |\nabla H|^2\,du\,dv \quad \text{with } B_r := \{(u,v)\in\mathbb{R}^2 \colon u^2+v^2<r^2\}$$

is computed as

$$D_{B_r}(H) = \frac{\pi}{2}\sum_{n=1}^{\infty} nr^{2n}(|a_n|^2+|b_n|^2) \quad \text{for } 0<r<1.$$

Setting

$$Q(r,\alpha) := \begin{cases} -\sum_{n=1}^{\infty} nr^{2n}\cos n\alpha & \text{for } 0\le r<1, \\ \frac{1}{4\sin^2\frac{1}{2}\alpha} & \text{for } r=1, \end{cases} \tag{5}$$

we obtain

$$D_{B_r}(H) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} Q(r,\theta-\varphi)|\xi(\theta)-\xi(\varphi)|^2\,d\theta\,d\varphi \quad \text{for } 0<r<1. \tag{6}$$

Furthermore,

$$-2Q(r,\alpha) = \sum_{n=1}^{\infty} nr^{2n}e^{in\alpha} + \sum_{n=1}^{\infty} nr^{2n}e^{-in\alpha}.$$

Setting $z := r^2e^{i\alpha}$, we find that

$$\begin{aligned} -2Q(r,\alpha) &= z\sum_{n=1}^{\infty} nz^{n-1} + \overline{z}\sum_{n=1}^{\infty} n\overline{z}^{n-1} = \frac{z}{(1-z)^2} + \frac{\overline{z}}{(1-\overline{z})^2} \\ &= \frac{z(1-\overline{z})^2+\overline{z}(1-z)^2}{(1-z)^2(1-\overline{z})^2} = \frac{(z+\overline{z})-4|z|^2+(z+\overline{z})|z|^2}{[1-(z+\overline{z})+z\overline{z}]^2}, \end{aligned}$$

and so

$$Q(r,\alpha) = r^2 \frac{a-b}{(a+b)^2}, \quad a := (1+r^2)^2 \sin^2 \frac{\alpha}{2}, \quad b := (1-r^2)^2 \cos^2 \frac{\alpha}{2}.$$

Hence

$$\frac{Q(r,\alpha)}{Q(1,\alpha)} = \frac{4r^2 \sin^2 \frac{\alpha}{2}}{a+b} \, \frac{a-b}{a+b} \quad \text{for } \alpha \not\equiv 0 \bmod 2\pi$$

which implies

$$\begin{aligned} &Q(r,\alpha) \le Q(1,\alpha) \quad \text{for } 0 \le r < 1, \\ &\lim_{r \to 1-0} Q(r,\alpha) = Q(1,\alpha) \quad \text{for } \alpha \not\equiv 0 \bmod 2\pi. \end{aligned} \tag{7}$$

If $A_0(\xi) < \infty$ then

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \to A_0(\xi) \quad \text{as } r \to 1-0$$

on account of Lebesgue's convergence theorem. Since

$$D_{B_r}(H) \to D(H) \quad \text{as } r \to 1-0$$

we infer from (6) that $D(H) = A_0(\xi)$.

Conversely, if $D(H) < \infty$, $0 < \varepsilon < \pi$, and

$$R(\varepsilon) := \{(\theta, \varphi) \in [0, 2\pi] \times [0, 2\pi] \colon |e^{i\theta} - e^{i\varphi}| > \varepsilon\}$$

we have

$$\begin{aligned} &\int_{R(\varepsilon)} Q(1, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad \le \lim_{r \to 1-0} \int_{R(\varepsilon)} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad \le \lim_{r \to 1-0} \int_0^{2\pi} \int_0^{2\pi} Q(r, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi \\ &\quad = \lim_{r \to 1-0} 4\pi D_{B_r}(H) = 4\pi D(H) < \infty. \end{aligned}$$

With $\varepsilon \to +0$ we obtain

$$A_0(\xi) = \lim_{\varepsilon \to +0} \frac{1}{4\pi} \int_{R(\varepsilon)} Q(1, \theta - \varphi) |\xi(\theta) - \xi(\varphi)|^2 \, d\theta \, d\varphi < \infty,$$

and then the reasoning above yields $A_0(\xi) = D(H)$. Thus we have proved (4), since our arguments imply that $D(H) = \infty$ if and only if $A_0(\xi) = \infty$. □

Corollary 1. *Let $\{\xi_j\}$ be a sequence in $C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ with the following properties*:

(i) $\xi_j(\theta) \rightrightarrows 0$ *on* $\mathbb{R}$ *as* $j \to \infty$.
(ii) *There is a mapping* $\eta \in C^0_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ *such that* $A_0(\eta) < \infty$ *and*

$$(8) \qquad |\xi_j(\theta) - \xi_j(\varphi)| \le |\eta(\theta) - \eta(\varphi)| \quad \textit{for all } j \in \mathbb{N} \textit{ and } \theta, \varphi \in \mathbb{R}.$$

Then we have the relation

$$\lim_{j \to \infty} A_0(\xi_j) = 0.$$

Proof. As $Q(1, \theta - \varphi)|\eta(\theta) - \eta(\varphi)|^2$ is an L^1-majorant of the functions $Q(1, \theta - \varphi)|\xi_j(\theta) - \xi_j(\varphi)|^2$ on $[0, 2\pi] \times [0, 2\pi]$, the assertion is an immediate consequence of Lebesgue's convergence theorem. □

Let $\overline{\mathcal{H}}(B, \mathbb{R}^3)$ be the class of mappings $H \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with $\Delta H = 0$ in B. For any $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$ we define the value $D_0(H)$ by

$$(9) \qquad D_0(H) := A_0(\xi) \quad \text{where } \xi(\theta) := H(e^{i\theta}), \ \theta \in \mathbb{R}.$$

The function $D_0 : \overline{\mathcal{H}}(B, \mathbb{R}^3) \to \mathbb{R}$ is also denoted as **Douglas functional**. Because of (2) we can as well write

$$(10) \qquad D_0(H) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|H(e^{i\theta}) - H(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} \, d\theta \, d\varphi.$$

An immediate consequence of Theorem 1 is

Corollary 2. *For any* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3)$ *we have*

$$(11) \qquad D(H) = D_0(H).$$

For $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ with $\xi(\theta) := H(e^{i\theta})$ for $\theta \in \mathbb{R}$ we define the norm

$$(12) \qquad \|H\|_{1,B} := \|\xi\|_{C^0([0,2\pi],\mathbb{R}^3)} + \sqrt{A_0(\xi)};$$

in virtue of (9), (11), and the maximum principle it agrees with the norm

$$(13) \qquad \|H\|_{1,B} := \|H\|_{C^0(\overline{B},\mathbb{R}^3)} + \sqrt{D(H)}$$

introduced in 6.3, (1); hence $\overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ equipped with the norm $\|\cdot\|_{1,B}$ is complete, i.e. a Banach space.

From Corollaries 1 and 2 we infer the following important result:

Theorem 2 (E. Heinz [14]). *Let $\{H_j\}$ be a sequence in $\overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H^1_2(B, \mathbb{R}^3)$ with the boundary values $\{\xi_j\}$, $\xi_j(\theta) = H_j(e^{i\theta})$ for $\theta \in \mathbb{R}$, and assume the following*:

(i) $\xi_j(\theta) \rightrightarrows \xi(\theta)$ *on* $\mathbb{R}$ *for* $j \to \infty$ *with* $A_0(\xi) < \infty$.
(ii) *There is a number* $\kappa > 0$ *such that*

$$|\xi_j(\theta) - \xi_j(\varphi)| \le \kappa |\xi(\theta) - \xi(\varphi)| \quad \textit{for all } j \in \mathbb{N} \textit{ and } \theta, \varphi \in \mathbb{R}.$$

Then we have

$$\|H_j - H\|_{1,B} = \|H_j - H\|_{C^0(\overline{B},\mathbb{R}^3)} + \sqrt{D(H_j - H)} \to 0 \quad \textit{as } j \to \infty,$$

where H *is the harmonic extension of* ξ, *and in particular* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) \cap H_2^1(B, \mathbb{R}^3)$ *and* $H_j \to H$ *in* $H_2^1(B, \mathbb{R}^3)$.

Now we want to prove a second kind of convergence theorem for harmonic mappings. We begin with deriving an *isoperimetric inequality for harmonic surfaces* due to M. Morse and C. Tompkins [3].

Theorem 3. *For any* $H \in \overline{\mathcal{H}}(B, \mathbb{R}^3) := C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \cap \{\Delta H = 0 \textit{ in } B\}$ *we have*

$$A(H) \le \frac{1}{4}\left(\int_{\partial B} |dH|\right)^2. \tag{14}$$

Proof. We may assume that $\int_{\partial B} |dH|$ is finite, because otherwise (14) is certainly true. The area $A(H)$ of H is defined as

$$A(H) = \int_B |H_u \wedge H_v| \, du \, dv.$$

We transform $H(u, v)$ to polar coordinates r, θ around the origin by setting

$$X(r, \theta) := H(r\cos\theta, r\sin\theta), \quad 0 \le r \le 1, \ 0 \le \theta \le 2\pi,$$

and obtain

$$A(H) = \int_0^1 \int_0^{2\pi} |X_r \wedge X_\theta| \, d\theta \, dr.$$

Poisson's integral formula yields

$$X(r, \theta) = \int_0^{2\pi} K(r, \varphi - \theta)\xi(\varphi) \, d\varphi, \quad \xi(\varphi) := X(1, \varphi),$$

where $K(r, \alpha)$ denotes the Poisson kernel

$$K(r, \alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}.$$

As in the proof of 4.7, Proposition 1, we obtain

$$X_\theta(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{\omega(r, \theta, \varphi)} \, d\xi(\varphi)$$

where

$$\omega(r,\theta,\varphi) := 1 - 2r\cos(\theta-\varphi) + r^2.$$

By using the computation of the proof of 4.7, Proposition 2, we find in addition that

$$X_r(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(\varphi-\theta)}{\omega(r,\theta,\varphi)}\, d\xi(\varphi).$$

Therefore,

$$X_r(r,\theta)\wedge X_\theta(r,\theta) = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)\sin(\varphi-\theta)}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, d\xi(\varphi)\wedge\, d\xi(\psi).$$

Interchanging φ and ψ on the right-hand side, the left-hand side remains the same. Adding the two expressions, dividing by 2, and noting the relation $d\xi(\varphi)\wedge d\xi(\psi) = -d\xi(\psi)\wedge d\xi(\varphi)$, we arrive at

$$\begin{aligned} &X_r(r,\theta)\wedge X_\theta(r,\theta) \\ &\quad = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)[\sin(\varphi-\theta)-\sin(\psi-\theta)]}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, d\xi(\varphi)\wedge d\xi(\psi). \end{aligned}$$

Furthermore, the identity

$$\sin\varphi - \sin\psi = 2\cos\frac{\varphi+\psi}{2}\sin\frac{\varphi-\psi}{2}$$

implies

$$\sin(\varphi-\theta)-\sin(\psi-\theta) = 2\cos\big[\tfrac{1}{2}(\varphi+\psi)-\theta\big]\sin\tfrac{1}{2}(\varphi-\psi)$$

whence

$$|\sin(\varphi-\theta)-\sin(\psi-\theta)| \le 2|\sin\tfrac{1}{2}(\varphi-\psi)|,$$

and therefore

$$|X_r(r,\theta)\wedge X_\theta(r,\theta)| \le \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(1-r^2)|\sin\frac{1}{2}(\varphi-\psi)|}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\, |d\xi(\varphi)||d\xi(\psi)|.$$

For $0<\varepsilon<\rho<1$ we set

$$a(\varepsilon,\rho) := \int_\varepsilon^\rho\int_0^{2\pi} |X_r(r,\theta)\wedge X_\theta(r,\theta)|\, d\theta\, dr.$$

Then

$$a(\varepsilon,\rho) \le \int_0^{2\pi}\int_0^{2\pi} \mathfrak{I}(\varphi,\psi)|\sin\tfrac{1}{2}(\varphi-\psi)||d\xi(\varphi)||d\xi(\psi)|$$

with

$$\begin{aligned}\mathcal{I}(\varphi,\psi) &:= \frac{1}{\pi}\int_\varepsilon^\rho \mathcal{I}^*(r,\varphi,\psi)\,dr,\\ \mathcal{I}^*(r,\varphi,\psi) &:= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{\omega(r,\theta,\varphi)\omega(r,\theta,\psi)}\,d\theta.\end{aligned}$$

Fix $\psi \in [0,2\pi]$ and $r \in (\varepsilon,\rho)$, and consider a harmonic function f in the unit disk B with $f \in C^0(\overline{B})$ which has the boundary values

$$f(e^{i\theta}) := \frac{1}{\omega(r,\theta,\psi)} = \frac{1}{1-2r\cos(\psi-\theta)+r^2}.$$

For $0 \le R \le 1$ we write

$$h(R,\varphi) := f(Re^{i\varphi}).$$

Then Poisson's integral formula yields

$$h(R,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-R^2}{\omega(R,\theta,\varphi)\omega(r,\theta,\psi)}\,d\theta,$$

whence

$$\mathcal{I}^*(r,\varphi,\psi) = h(r,\varphi).$$

In order to determine the function h, we recall that for fixed r with $0<r<1$ the Poisson kernel

$$g(R,\theta) := \frac{R^2-r^2}{R^2-2rR\cos(\psi-\theta)+r^2}$$

is a harmonic function (written in polar coordinates) of R,θ in $\{R>1\}$, i.e. in the exterior of $B=\{w\in\mathbb{C}\colon |w|<1\}$, and $g(1,\theta)=(1-r^2)h(1,\theta)$. Hence h is obtained from $(1-r^2)^{-1}g$ by reflection at the unit circle $\partial B=\{R=1\}$, that is, by replacing R by $\frac{1}{R}$:

$$h(R,\varphi) = \frac{1}{1-r^2}\;\frac{R^{-2}-r^2}{R^{-2}-2rR^{-1}\cos(\psi-\varphi)+r^2}.$$

Thus we infer

$$\mathcal{I}^*(r,\varphi,\psi) = \frac{1+r^2}{1-2r^2\cos(\psi-\varphi)+r^4}$$

whence

$$\begin{aligned}\mathcal{I}(\varphi,\psi) &= \frac{1}{\pi}\int_\varepsilon^\rho \frac{1+r^2}{1-2r^2\cos(\psi-\varphi)+r^4}\,dr\\ &= \frac{1}{2\pi}\int_\varepsilon^\rho\left\{\frac{1}{1-2r\cos\frac{1}{2}(\psi-\varphi)+r^2}+\frac{1}{1+2r\cos\frac{1}{2}(\psi-\varphi)+r^2}\right\}dr\\ &= \frac{1}{2\pi}\frac{1}{|\sin\frac{1}{2}(\psi-\varphi)|}\;[S(r,\varphi,\psi)]_\varepsilon^\rho\end{aligned}$$

with

$$S(r,\varphi,\psi) := \operatorname{arctg}\left(\frac{r-\cos\frac{1}{2}(\psi-\varphi)}{|\sin\frac{1}{2}(\psi-\varphi)|}\right) + \operatorname{arctg}\left(\frac{r+\cos\frac{1}{2}(\psi-\varphi)}{|\sin\frac{1}{2}(\psi-\varphi)|}\right).$$

Using the formula

$$\operatorname{arctg} a + \operatorname{arctg} b = \operatorname{arctg}\frac{a+b}{1-ab}$$

we obtain

$$\mathcal{I}(\varphi,\psi) = \left[\frac{1}{2\pi|\sin\frac{1}{2}(\psi-\varphi)|}\operatorname{arctg}\left(\frac{2r|\sin\frac{1}{2}(\psi-\varphi)|}{1-r^2}\right)\right]_{r=\varepsilon}^{r=\rho}.$$

Therefore,

$$a(\varepsilon,\rho) \le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left[\operatorname{arctg}\left(\frac{2r|\sin\frac{1}{2}(\psi-\varphi)|}{1-r^2}\right)\right]_{r=\varepsilon}^{r=\rho}|d\xi(\varphi)||d\xi(\psi)|. \tag{15}$$

Since $[\dots] \to \frac{\pi}{2}$ as $\varepsilon \to +0$ and $\rho \to 1-0$, and $a(\varepsilon,\rho) \to A$, we finally see that

$$A \le \frac{1}{4}L^2 \quad \text{with } L := \int_0^{2\pi}|d\xi(\theta)|. \tag{16}$$

□

Now we prove a convergence theorem for the area of harmonic mappings discovered by M. Morse and C. Tompkins [3].

Theorem 4. *Let $\{H_j\}$ be a sequence of harmonic mappings in $\overline{\mathcal{H}}(B,\mathbb{R}^3)$ with the following two properties*:

(i) $\|H_j - H\|_{C^0(\overline{B},\mathbb{R}^3)} \to 0$ *as* $j\to\infty$ *for some* $H \in \overline{\mathcal{H}}(B,\mathbb{R}^3)$;
(ii) $\int_{\partial B}|dH_j| \to \int_{\partial B}|dH|$ *as* $j \to \infty$.

Then the area of H_j tends to the area of H, i.e.

$$\lim_{j\to\infty} A(H_j) = A(H). \tag{17}$$

Proof. Analogously to the preceding proof we introduce X_j, X and ξ_j, ξ by

$$X_j(r,\theta) := H_j(re^{i\theta}), \quad X(r,\theta) := H(re^{i\theta}),$$
$$\xi_j(\theta) := X_j(1,\theta), \quad \xi(\theta) := X(1,\theta).$$

For $\alpha,\beta\in\mathbb{R}$ with $0<\beta-\alpha<2\pi$ we set

$$L_j(\alpha,\beta) := \int_\alpha^\beta |d\xi_j(\theta)|, \quad L(\alpha,\beta) := \int_\alpha^\beta |d\xi(\theta)|.$$

1° *Claim*: $L_j(\alpha,\beta) \to L(\alpha,\beta)$ *as* $j\to\infty$ *uniformly in* α,β.

Otherwise there would be an $\varepsilon > 0$ and a subsequence of indices $j_p \to \infty$ as $p \to \infty$ and sequences $\alpha_{j_p} \to \alpha$, $\beta_{j_p} \to \beta$ such that

$$|L_j(\alpha_j, \beta_j) - L(\alpha_j, \beta_j)| \geq 2\varepsilon \quad \text{for all } j = j_p,\ p \in \mathbb{N}.$$

Since $L(\alpha_{j_p}, \beta_{j_p}) \to L(\alpha, \beta)$ for $p \to \infty$ we may assume that

$$|L(\alpha_j, \beta_j) - L(\alpha, \beta)| < \varepsilon \quad \text{for all } j = j_p,$$

and so we obtain

$$|L_j(\alpha_j, \beta_j) - L(\alpha, \beta)| > \varepsilon \quad \text{for all } j = j_p.$$

Since the arc length is lower semicontinuous with respect to uniform convergence we obtain

$$\liminf_{p\to\infty} L_{j_p}(\alpha_{j_p}, \beta_{j_p}) \geq L(\alpha, \beta) + \varepsilon.$$

By passing to a suitable subsequence of $\{j_p\}$, which will again be denoted by $\{j_p\}$, we may even assume that

$$\lim_{p\to\infty} L_{j_p}(\alpha_{j_p}, \beta_{j_p}) \geq L(\alpha, \beta) + \varepsilon. \tag{18}$$

On the other hand, if γ_j and γ are the complementary arcs to $\{e^{i\theta}\colon \alpha_j \leq \theta \leq \beta_j\}$ and $\{e^{i\theta}\colon \alpha \leq \theta \leq \beta\}$ respectively in ∂B, and

$$L_j := \int_{\gamma_j} |dH_j|, \quad L := \int_\gamma |dH|,$$

we get

$$\liminf_{p\to\infty} L_{j_p} \geq L. \tag{19}$$

Adding (18) and (19) we would obtain

$$\liminf_{p\to\infty} \int_{\partial B} |dH_{j_p}| \geq \int_{\partial B} |dH| + \varepsilon,$$

which contradicts assumption (ii). Thus the claim 1° is proved.

2° Set

$$l(\sigma) := \sup\left\{ \int_\gamma |dH|, \int_\gamma |dH_j|\colon \gamma \subset \partial B,\ \text{length } \gamma = \sigma, j \in \mathbb{N} \right\}. \tag{20}$$

Because of 1° we obtain

$$l(\sigma) \to 0 \quad \text{as } \sigma \to +0. \tag{21}$$

Furthermore there is a number $\lambda > 0$ such that

$$\int_{\partial B} |dH_j| \leq \lambda \quad \text{for all } j \in \mathbb{N}.$$

Set

$$a_j(R, \rho) := \int_R^\rho \int_0^{2\pi} |X_{j,r}(r,\theta) \wedge X_{j,\theta}(r,\theta)| \, d\theta \, dr$$

for $0 < R < \rho < 1$. By (15) we have

$$a_j(R,\rho) \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} [\chi(r,\psi,\varphi)]_{r=R}^{r=\rho} |d\xi_j(\varphi)||d\xi_j(\psi)|$$

with

$$\chi(r,\psi,\varphi) := \operatorname{arctg}\left(\frac{2r|\sin\frac{1}{2}(\psi - \varphi)|}{1 - r^2} \right).$$

We decompose the domain of integration $\Omega := \{(\varphi, \psi) \colon 0 < \varphi, \psi < 2\pi\}$ into the disjoint sets Ω_1 and Ω_2 defined by

$$\Omega_1 := \{(\varphi,\psi) \in \Omega \colon \|\psi - \varphi\| \leq \sigma\}, \quad \Omega_2 := \{(\varphi,\psi) \in \Omega \colon \|\psi - \varphi\| > \sigma\},$$

where $\|\psi - \varphi\|$ denotes the length of the shorter arc on ∂B with the endpoints $e^{i\varphi}$ and $e^{i\psi}$. Then

$$a_j(R,\rho) \leq I_j^1(R,\rho) + I_j^2(R,\rho)$$

with

$$I_j^k(R,\rho) := \frac{1}{2\pi} \int_{\Omega_k} [\chi(r,\psi,\varphi)]_{r=R}^{r=\rho} |d\xi_j(\varphi)||d\xi_j(\psi)|, \quad k = 1, 2.$$

On Ω_1 we estimate $[\chi(r,\psi,\varphi)]_R^\rho$ from above by $\frac{\pi}{2}$ and obtain

$$I_j^1(R,\rho) \leq \frac{1}{2\pi} \cdot \frac{\pi}{2} \int_{\Omega_1} |d\xi_j(\varphi)||d\xi_j(\psi)| \leq \frac{1}{4} \lambda l(\sigma).$$

On Ω_2 we find

$$[\chi(r,\psi,\varphi)]_R^\rho \leq \frac{\pi}{2} - \operatorname{arctg}\left\{ \frac{2R|\sin(\sigma/2)|}{1 - R^2} \right\}.$$

For $0 < \sigma < 1$ we certainly have $\sin(\sigma/2) \geq \sigma/4$, and so

$$[\chi(r,\psi,\varphi)]_R^\rho \leq \frac{\pi}{2} - \operatorname{arctg} \frac{R\sigma}{2(1 - R^2)}$$

whence

$$I_j^2(R,\rho) \leq \frac{\lambda^2}{2\pi} \left\{ \frac{\pi}{2} - \operatorname{arctg} \frac{R\sigma}{2(1 - R^2)} \right\}.$$

Then we obtain for $a_j(R,1) := \lim_{\rho\to 1-0} a_j(R,\rho)$ the estimate

$$a_j(R,1) \le \frac{\lambda}{4} l(\sigma) + \frac{\lambda^2}{2\pi}\left\{\frac{\pi}{2} - \operatorname{arctg}\frac{R\sigma}{2(1-R^2)}\right\} \quad \text{for } 0<\sigma<1.$$

Now we choose an arbitrary $\varepsilon > 0$; then there is some $\sigma \in (0,1)$ such that $l(\sigma) < \varepsilon/(2\lambda)$. Moreover we can find an $R \in (0,1)$ depending on ϵ and σ such that

$$\frac{\lambda^2}{2\pi}\left\{\frac{\pi}{2} - \operatorname{arctg}\frac{R\sigma}{2(1-R^2)}\right\} < \frac{\varepsilon}{8}.$$

Then we obtain

$$A_{B\setminus B_R}(H_j) = a_j(R,1) < \frac{\varepsilon}{4} \quad \text{for } B_R = \{w\colon |w|<R\},\ B := B_1,$$

and the same reasoning yields

$$A_{B\setminus B_R}(H) < \frac{\varepsilon}{4}.$$

On B_R we have $\nabla H_j \rightrightarrows \nabla H$; therefore there is a number $j_0 \in \mathbb{N}$ such that

$$|A_{B_R}(H) - A_{B_R}(H_j)| < \frac{\varepsilon}{2} \quad \text{for } j > j_0(\varepsilon).$$

It follows that

$$\begin{aligned} |A(H) - A(H_j)| &\le |A_{B_R}(H) - A_{B_R}(H_j)| + A_{B\setminus B_R}(H) + A_{B\setminus B_R}(H_j) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \quad \text{for } j > j_0(\varepsilon), \end{aligned}$$

and so: $A(H_j) \to A(H)$ as $j \to \infty$. □

For minimal surfaces we obtain a stronger convergence result:

Theorem 5. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are continuous on $\overline{B}$ and satisfy*

(i) $\|X_j - X\|_{C^0(\overline{B},\mathbb{R}^3)} \to 0$ *for some* $X \in C^0(\overline{B},\mathbb{R}^3)$;
(ii) $\int_{\partial B} |dX_j| \to \int_{\partial B} |dX|$ *as* $j \to \infty$.

Then X is a minimal surface in B, and

$$\lim_{j\to\infty} D(X_j) = D(X). \tag{22}$$

Moreover, $X_j \to X$ in $H_2^1(B,\mathbb{R}^3)$.

Proof. By assumption we have in B

$$\Delta X_j = 0 \quad \text{and} \quad |D_u X_j|^2 = |D_v X_j|^2, \quad \langle D_u X_j, D_v X_j\rangle = 0 \quad \text{in } B. \tag{23}$$

Furthermore relation (i) implies $\nabla^s X_j \rightrightarrows \nabla^s X$ on every $B' \subset\subset B$ and for any $s \geq 1$. Therefore (23) implies

$$\Delta X = 0 \quad \text{and} \quad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B, \tag{24}$$

i.e. X is a minimal surface in B. From (23) and (24) we infer

$$D(X_j) = A(X_j) \quad \text{and} \quad D(X) = A(X),$$

and (17) of Theorem 4 yields $A(X_j) \to A(X)$. This implies (22). Finally, a standard reasoning shows that $X_j \rightharpoonup X$ in $H_2^1(B, \mathbb{R}^3)$. Then, in conjunction with (22), we obtain $X_j \to X$ in $H_2^1(B, \mathbb{R}^3)$. □

An immediate consequence of this theorem are the next two results:

Corollary 3. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are of class $\overline{\mathcal{C}}^*(\Gamma)$ and satisfy $X_j \rightrightarrows X$ on $\overline{B}$. Then X is a minimal surface in B of class $\overline{\mathcal{C}}^*(\Gamma)$, and*

$$\|X - X_j\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Corollary 4. *Let $\{X_j\}$ be a sequence of minimal surfaces in B which are of class $\overline{\mathcal{C}}(\Gamma_j)$ and satisfy $X_j \rightrightarrows X$ on $\overline{B}$. We also assume that $\Gamma, \Gamma_1, \Gamma_2, \dots$ are closed rectifiable Jordan curves in $\mathbb{R}^3$ such that $\Gamma_j \to \Gamma$ (in the sense of Fréchet), and that the lengths $L(\Gamma_j)$ of Γ_j tend to the length $L(\Gamma)$ of Γ. Then X is a minimal surface in B of class $\overline{\mathcal{C}}(\Gamma)$, and $D(X_j) \to D(X) < \infty$ as well as*

$$\|X - X_j\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Remark 1. If we in Theorem 3 of 4.3 assume in addition that $L(\Gamma_n) \to L(\Gamma)$ then the extracted subsequence $\{X_{n_p}\}$ of the quoted theorem also satisfies $\|X - X_{n_p}\|_{H_2^1(B,\mathbb{R}^3)} \to 0$ as $p \to \infty$.

6.5 When Is the Limes Superior of a Sequence of Paths Again a Path?

In the next section we need a generalization of the reasoning used in the proof of Theorem 1 of 6.2 to prove the existence of a minimizing path $\boldsymbol{p}^*$ joining two minimizers. Since we shall operate in the metric space $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, we shall formulate this generalization in the context of a general metric space (E, d) with a distance function d.

Let $\{M_j\}$ be a sequence of subsets M_j of E. Following the example of Hausdorff we define the **Limes Inferior of $\{M_j\}$** by

$$\liminf_{j\to\infty} M_j := \big\{x \in E\colon \text{there is a sequence of points } x_j \in M_j,\ j \in \mathbb{N},\ \text{with } d(x, x_j) \to 0\big\},$$

and the **Limes Superior of $\{M_j\}$** by

$$\limsup_{j\to\infty} M_j := \big\{x \in E\colon \text{there is an increasing sequence of indices } j_l \to \infty \\ \text{and a sequence of points } x_l \in M_{j_l} \text{ with } d(x, x_l) \to 0\big\}.$$

Proposition 1. *We have*

$$\limsup_{j\to\infty} M_j = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{j\geq k} M_j}. \tag{1}$$

Proof. Set

$$M := \limsup_{j\to\infty} M_j \quad \text{and} \quad M^* := \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{j\geq k} M_j}.$$

(i) Let $x \in M$; then $d(x, x_l) \to 0$ for some sequence of points $x_l \in M_{j_l}$ with increasing $j_l \to \infty$. Given $k \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $j_l \geq k$ for all $l \geq N$ whence $x_l \in M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots$ for $l \geq N$, and therefore $x \in \text{closure}(M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots)$ for any $k \in \mathbb{N}$. Hence $x \in M^*$, and consequently $M \subset M^*$.

(ii) Conversely let $x \in M^*$. Then $x \in \text{closure}(M_k \cup M_{k+1} \cup M_{k+2} \cup \cdots)$ for all $k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ we can find a point $x_k \in M_k \cup M_{k+1} \cup \cdots$ with $d(x, x_k) < 2^{-k}$. By induction we can now extract a subsequence $\{x_{j_l}\}$ of points $x_{j_l} \in M_{j_l}$ where $\{j_l\}$ is an increasing sequence of indices $j_l \to \infty$. Clearly, $d(x, x_{j_l}) \to 0$, and so $x \in M$, whence $M^* \subset M$. □

The following results are well known:

Proposition 2. *If $\{M_j\}_{j\in\mathbb{N}}$ is a family of connected subsets of E with the property $M_j \cap M_k \neq \emptyset$ for all $j, k \in \mathbb{N}$ then $\bigcup_{j\in\mathbb{N}} M_j$ is connected.*

Proposition 3. *If M is a connected subset of E then also $\overline{M}$.*

Proposition 4. *If $\{M_j\}$ is a sequence of compact, connected subsets M_j of E with $M_1 \supset M_2 \supset M_3 \supset \cdots$ then $\bigcap_{j\in\mathbb{N}} M_j$ is connected.*

Proof. See e.g. Alexandroff and Hopf [1], p. 118. □

A straight-forward consequence of Propositions 1–4 is:

Theorem 1. *If $\{M_j\}$ is a sequence of compact, connected subsets of (E, d) such that $M_j \cap M_k$ is nonempty for all $j, k \in \mathbb{N}$, and that $\bigcup_{j\geq k} M_j$ is relatively compact for any $k \in \mathbb{N}$, then $\limsup_{j\to\infty} M_j$ is connected and compact.*

We note that it was this result that we have used in 6.2 to establish the existence of a minimal path $\boldsymbol{p}^*$. Now we prove the following generalization of Theorem 1 that will be employed in 6.6. We use the following notation: A **path in $\boldsymbol{E}$** *is a nonempty compact, connected subset of E.*

Theorem 2. *Let* $\{M_n\}$ *be a sequence of paths in* (E,d) *such that* $\bigcup_{j\in\mathbb{N}} M_j$ *is relatively compact and* $\liminf_{j\to\infty} M_j$ *is nonempty. Then also* $\limsup_{j\to\infty} M_j$ *is a path in* (E,d).

Proof. Set $M := \limsup_{j\to\infty} M_j$. By (1), M is a closed subset of the compact subset closure$(M_1 \cup M_2 \cup M_3 \cup \cdots\}$, and so M is compact and nonempty, as $\liminf_{j\to\infty} M_j \subset M$.

Suppose now that M were not connected. Then there are two open sets Ω' and Ω'' in E such that the sets $M' := M \cap \Omega'$ and $M'' := M \cap \Omega''$ are nonvoid as well as disjoint and satisfy $M = M' \cup M''$. Clearly M' and M'' are compact subsets of E whence $\delta := \operatorname{dist}(M', M'') > 0$. Set $\varepsilon := \delta/4$ and define the sets

$$M'_\varepsilon := \{x \in E\colon \operatorname{dist}(x, M') < \varepsilon\}, \quad M''_\varepsilon := \{x \in E\colon \operatorname{dist}(x, M'') < \varepsilon\}.$$

Moreover let x be an arbitrary point of $\liminf M_j$. Then there is a sequence $\{x_j\}$ of points $x_j \in M_j$ with $\operatorname{dist}(x, x_j) \to 0$. We may assume that x is contained in M', because the case $x \in M''$ can be handled analogously. Then there is a number $N(\varepsilon) \in \mathbb{N}$ such that

$$M_j \cap M'_\varepsilon \neq \emptyset \quad \text{for all } j > N(\varepsilon).$$

Furthermore, since M'' is nonvoid there is a subsequence $\{M_{j_l}\}$ such that $M_{j_l} \cap M''_\varepsilon \neq \emptyset$ for all $l \in \mathbb{N}$; in addition we can assume that $j_l > N(\varepsilon)$ for all $l \in \mathbb{N}$. In this way we obtain

$$M_{j_l} \cap M'_\varepsilon \neq \emptyset \quad \text{and} \quad M_{j_l} \cap M''_\varepsilon \neq \emptyset \quad \text{for all } l \in \mathbb{N}.$$

Thus, for any $l \in \mathbb{N}$, we can choose points $x'_l \in M_{j_l} \cap M'_\varepsilon$ and $x''_l \in M_{j_l} \cap M''_\varepsilon$. Fix l; since M_{j_l} is connected there is a finite set $\{z_1, \ldots, z_m\}$ of points in M_{j_l} with $z_1 = x'_l$, $z_m = x''_l$, and $d(z_k, z_{k+1}) < \varepsilon$ for $k = 1, \ldots, m-1$ (see e.g. Querenburg [1], p. 46). Since $\operatorname{dist}(M'_\varepsilon, M''_\varepsilon) > \delta - 2\varepsilon = 2\varepsilon$, at least one of the points $z_1, \ldots, z_m$ is not contained in $M'_\varepsilon \cup M''_\varepsilon$, we call it y_l. Then $\{y_l\}$ is a sequence of points with $y_l \in M_{j_l}$ and

$$\operatorname{dist}(y_l, M) \geq \varepsilon \quad \text{for } l \in \mathbb{N}. \tag{2}$$

As $\bigcup_{j\in\mathbb{N}} M_j$ is relatively compact, there is a subsequence $\{y_{l_k}\}$ of $\{y_l\}$ that converges to some point y, i.e. $d(y_{l_k}, y) \to 0$ as $k \to \infty$. By definition of M we have $y \in M$, contrary to (2). Thus the compact set is also connected, and so M is a path. □

6.6 Unstable Minimal Surfaces in Rectifiable Boundaries

Now we want to carry over the results of Section 6.3 to rectifiable closed Jordan curves Γ in $\mathbb{R}^3$ that satisfy a (global) *chord-arc condition* of the following kind:

There is a positive constant μ such that for any two points x_1, x_2 of Γ we have

$$\ell(x_1, x_2) \le \mu |x_1 - x_2| \tag{1}$$

where $\ell(x_1, x_2)$ is the length of the shorter one of the two subarcs of Γ bounded by x_1 and x_2.

This assumption will be denoted as **Condition ($\boldsymbol{\mu}$)**.

By $\overline{\mathcal{C}}^*(\Gamma)$ we denote the class of surfaces $X \in \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$ which satisfy some fixed preassigned 3-point condition

$$X(w_k) = Q_k, \quad k = 0, 1, 2, \quad w_k := \exp(i\psi_k), \quad \psi_k = \frac{2\pi k}{3}, \tag{2}$$

where Q_0, Q_1, Q_2 are three fixed points on Γ.

Furthermore let $\mathcal{H}^*(\Gamma)$ be the subset of mappings $X \in \overline{\mathcal{C}}^*(\Gamma)$ which are harmonic in B. As in 6.3 we could equip both $\overline{\mathcal{C}}^*(\Gamma)$ and $\mathcal{H}^*(\Gamma)$ with the distance function

$$d_1(X, Y) := \|X - Y\|_{1,B}, \quad X, Y \in \overline{H}{}_2^1(B, \mathbb{R}^3), \tag{3}$$

which is derived from the norm

$$\|X\|_{1,B} := \|X\|_{C^0(\overline{B}, \mathbb{R}^3)} + \sqrt{D(X)} \tag{4}$$

of the Banach space $\overline{H}{}_2^1(B, \mathbb{R}^3) := H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$. Unfortunately we have to work with the metric space $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_0)$,

$$d_0(X, Y) := \|X - Y\|_{C^0(\overline{B}, \mathbb{R}^3)}, \tag{5}$$

as we are unable to apply Theorem 2 of 6.5 in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_1)$. Instead we can obtain a version of this result in $(\overline{H}{}_2^1(B, \mathbb{R}^3), d_0)$; this will be stated as Lemma 2. We note that this deficiency is the reason why we cannot carry over the results obtained for polygons in full strength to general boundary contours Γ. We begin our discussion of the general case with a suitable approximation device.

Lemma 1. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Then there exists a sequence $\{\Gamma_j\}$ of simple, closed polygons Γ_j in $\mathbb{R}^3$ and a sequence of homeomorphisms $\phi_j : \Gamma \to \Gamma_j$ from Γ onto Γ_j such that the following holds*:

(i) $Q_1, Q_2, Q_3 \in \Gamma_j$ *for all* $j \in \mathbb{N}$.
(ii) Γ_j *has* N_j+3 (≥ 4) *consecutive vertices which lie on* Γ, *given by* $Q_0, A_1(j), \dots, A_{l_j}(j), Q_1, A_{l_j+1}(j), \dots, A_{m_j}(j), Q_2, A_{m_j+1}(j), \dots, A_{N_j}(j), Q_0$.
(iii) $\Delta(\Gamma_j) \to 0$ *where* $\Delta(\Gamma_j)$ *denotes the length of the largest edge of* Γ_j.

(iv) *The length* $L(\Gamma_j)$ *of the polygons* Γ_j *tends to the length* $L(\Gamma)$ *of* Γ.
(v) *We have*

$$\max_{x\in\Gamma} |x - \phi_j(x)| \to 0 \quad \textit{as } j \to \infty$$

and $\phi_j(x) = x$ *if* x *is a vertex of* Γ_j.
(vi) *The subarc on* Γ *bounded by two consecutive vertices of* Γ_j *is the shorter one of the two subarcs of* Γ *bounded by these vertices if it contains no other vertex of* Γ_j.
(vii) *For any* $x', x'' \in \Gamma$ *and any* $j \in \mathbb{N}$ *we have*

$$|\phi_j(x') - \phi_j(x'')| \le l(x', x'')$$

where $l(x', x'')$ *is the length of the shorter arc on* Γ *with endpoints* x' *and* x''.

We call $\{\Gamma_j\}$ *an* **approximating sequence of inscribed polygons for** $\boldsymbol{\Gamma}$ **and** $\boldsymbol{Q_0, Q_1, Q_2}$.

The proof of this lemma is tedious, but elementary, and will therefore be omitted.

Lemma 2. *Let* $\{\Gamma_j\}$ *be an approximating sequence of inscribed polygons for* Γ, *and* $Q_0, Q_1, Q_2 \in \Gamma$, *and* $\{P_j\}$ *be a sequence of paths* (*i.e. compact and connected sets*) P_j *in* $(\mathcal{H}^*(\Gamma_j), d_0)$ *such that*

$$\sup\{D(X)\colon X \in P_j\} \le \kappa \quad \textit{for all } j \in \mathbb{N} \textit{ and some } \kappa > 0. \tag{6}$$

Moreover, suppose that there is a sequence $\{Y_j\}$ *of points* $Y_j \in \mathcal{H}^*(\Gamma_j)$ *with*

$$d_0(Y_j, Y) \to 0 \quad \textit{as } j \to \infty \textit{ for some } Y \in \overline{H}{}^1_2(B, \mathbb{R}^3). \tag{7}$$

Then $P := \limsup_{j\to\infty} P_j$ *is a path in* $(\mathcal{H}^*(\Gamma), d_0)$, *and*

$$\sup\{D(X)\colon X \in P\} \le \kappa. \tag{8}$$

Proof. By Theorem 3 of 4.3 we obtain that $\bigcup_{j\in\mathbb{N}} P_j$ is relatively compact in $(\overline{H}{}^1_2(B, \mathbb{R}^N), d_0)$, and (7) implies that $\liminf_{j\to\infty} P_j$ is nonempty. Then we infer from 6.5, Proposition 1 and Theorem 2, and 4.3, Theorem 3, that $P := \limsup_{j\to\infty} P_j$ is a path in $(\mathcal{H}^*(\Gamma), d_0)$, and (8) follows from (6) since D is sequentially weakly lower semicontinuous on $H^1_2(B, \mathbb{R}^3)$. $\square$

Remark 1. A path in $(\overline{H}{}^1_2(B, \mathbb{R}^3), d_1)$ is also a path in $(\overline{H}{}^1_2(B, \mathbb{R}^3), d_0)$. This ensues from the following two statements:

1° A d_1-compact set in $\overline{H}{}^1_2(B, \mathbb{R}^3)$ is also d_0-compact.
2° A d_1-connected set in $\overline{H}{}^1_2(B, \mathbb{R}^3)$ is as well d_0-connected.

Proof. (a) We first recall that a subset of a metric space is compact if and only if it is sequentially compact.

Let $M \subset \overline{H}_2^1(B, \mathbb{R}^3)$ be d_1-compact, and $\{x_j\}$ be a sequence in M. Then there exists a subsequence $\{x_{j_k}\}$ with $d_1(x_{j_k}, x) \to 0$ for some $x \in M$. It follows that $d_0(x_{j_k}, x) \to 0$; consequently M is d_0-compact.

(b) Suppose now that $M \subset \overline{H}_2^1(B, \mathbb{R}^3)$ is d_1-connected, but not d_0-connected. Then there exist two d_0-closed sets M' and M'' which are nonvoid and satisfy $M = M' \cup M''$ and $M' \cap M'' = \emptyset$. We claim that both M' and M'' are d_1-closed. For instance, if $\{x_j\}$ is a sequence in M' with $d_1(x_j, x) \to 0$ then $d_0(x_j, x) \to 0$, and therefore $x \in M'$ since M' is d_0-closed. Analogously we see that M'' is d_1-closed. Consequently M is d_1-disconnected, contrary to our assumption. □

Remark 2. In virtue of Remark 1 we can carry over the results of 6.3, Theorems 1–3, from $(\overline{\mathcal{C}}^*(\Gamma), d_1)$ to $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, Γ being a closed simple polygon. We merely have to replace expressions of the kind $\max_P D$ for paths P by $\sup_P D$ since D is no longer continuous on a d_0-path. Of course, $\sup_P D$ could be infinite, and it might be infinite for any path P containing two given points $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$. It will be seen later that the latter does not occur; cf. Lemma 6.

Let us now consider an arbitrary boundary contour Γ satisfying Condition (μ), and three points $Q_0, Q_1, Q_2 \in \Gamma$. We choose an approximating sequence $\{\Gamma_j\}$ of inscribed polygons Γ_j for Γ and Q_0, Q_1, Q_2. As in 6.1 we define for each Γ_j the set T_j of points $t = (t^1, \dots, t^{N(j)}) \in \mathbb{R}^{N(j)}$ satisfying

$$\psi_0 < t^1 < \cdots < t^{l_j} < \psi_1 < t^{l_j+1} < \cdots < t^{m_j} < \psi_2 < t^{m_j+1} < \cdots < t^{N_j} < \psi_3,$$

$\psi_k := \frac{2k\pi}{3}$, $k = 0, 1, 2, 3$, and $\overline{\mathcal{C}}^*(\Gamma_j)$ and $\mathcal{H}^*(\Gamma_j)$ are the subsets of mappings X of class $\overline{\mathcal{C}}(\Gamma_j)$ or $\mathcal{H}(\Gamma_j)$ respectively satisfying (2).

With every $t \in T_j$ we associate the set

$$U_j(t) := \{X \in \overline{\mathcal{C}}^*(\Gamma_j) \colon X(\exp(it^k)) = A_k(j),\ k = 1, \dots, N_j\}, \tag{9}$$

and the corresponding Courant function $\Theta_j : T_j \to \mathbb{R}$ is defined by

$$\Theta_j(t) := \inf\{D(X) \colon X \in U_j(t)\} \quad \text{for } t \in T_j. \tag{10}$$

Furthermore the associated Courant mapping $Z_j : T_j \to \overline{\mathcal{C}}^*(\Gamma_j)$ is the mapping $t \mapsto Z_j(t)$ where $Z_j(t)$ is the uniquely determined minimizer of D in $U_j(t)$, i.e.

$$\Theta_j(t) = D(Z_j(t)) \quad \text{for } t \in T_j. \tag{11}$$

The beautiful properties of Z_j and Θ_j are discussed in 6.1 and 6.3. We set

$$\mathcal{W}_j^*(\Gamma_j) := Z_j(T_j), \quad j \in \mathbb{N}. \tag{12}$$

Furthermore Θ_j is of class $C^1(T_j)$, and the minimal surfaces of class $\overline{\mathcal{C}}^*(\Gamma_j)$ are in 1–1 correspondence with the critical points of Θ_j in T_j.

Lemma 3. *Given $H \in \mathcal{H}^*(\Gamma)$ there are points $t_j = (t_j^1, \dots, t_j^{N_j}) \in T_j$ such that*

$$(13) \qquad H(\exp(it_j^k)) = A_k(j) \quad \text{for } 1 \le k \le N_j \text{ and for all } j \in \mathbb{N}.$$

Then

$$(14) \qquad d_1(Z_j(t_j), H) \to 0 \quad \text{as } j \to \infty.$$

Proof. The first statement is obvious since H satisfies the Plateau boundary condition.

Let now H_j be the harmonic extension of $\phi_j(H|_{\partial B})$ to B, and set $Y_j := H_j - H$. Then, by (vii) of Lemma 1 and (1), we obtain for any $\alpha, \beta \in \mathbb{R}$ that

$$\begin{aligned} |Y_j(e^{i\alpha}) - Y_j(e^{i\beta})| &\le |H_j(e^{i\alpha}) - H_j(e^{i\beta})| + |H(e^{i\alpha}) - H(e^{i\beta})| \\ &\le l(H(e^{i\alpha}), H(e^{i\beta})) + |H(e^{i\alpha}) - H(e^{i\beta})| \\ &\le (\mu + 1)|H(e^{i\alpha}) - H(e^{i\beta})|. \end{aligned}$$

Furthermore, $\phi_j(H|_{\partial B}) \to H|_{\partial B}$ in $C^0(\partial B, \mathbb{R}^3)$, i.e. $Y_j \to 0$ in $C^0(\partial B, \mathbb{R}^3)$. Then 6.4, Theorem 2 implies

$$(15) \qquad d_1(H_j, H) \to 0 \quad \text{as } j \to \infty.$$

From (13) it follows that $H_j \in U_j(t_j)$, $j \in \mathbb{N}$, and then

$$(16) \qquad D(Z_j(t_j)) \le D(H_j) \le \text{const} \quad \text{for all } j \in \mathbb{N}$$

because of (10), (11), and (15). Since $Z_j(t_j)$ and H_j lie in $U_j(t_j)$, we infer from Lemma 1(iii) that

$$\|Z_j(t_j) - H_j\|_{C^0(\partial B, \mathbb{R}^3)} \le \Delta(\Gamma_j) \to 0 \quad \text{as } j \to \infty.$$

In conjunction with (15) we arrive at

$$(17) \qquad d_0(Z_j(t_j), H) \to 0 \quad \text{as } j \to \infty.$$

Because of (16) we can extract from any subsequence of $\{Z_j(t_j)\}$ another subsequence $\{Z_{j_k}(t_{j_k})\}$ which converges weakly in $H_2^1(B, \mathbb{R}^3)$ and therefore strongly in $L_2(B, \mathbb{R}^3)$ to some element X, and (17) implies $X = H$. Hence

$$(18) \qquad Z_{j_k}(t_{j_k}) \rightharpoonup H \quad \text{in } H_2^1(B, \mathbb{R}^3) \text{ as } k \to \infty,$$

and consequently

$$D(H) \le \liminf_{k \to \infty} D(Z_{j_k}(t_{j_k})).$$

On the other hand,

$$\limsup_{k \to \infty} D(Z_{j_k}(t_{j_k})) \le D(H)$$

in virtue of (15) and (16), and so we obtain

$$D(Z_{j_k}(t_{j_k})) \to D(H).$$

In conjunction with (18) we arrive at $Z_{j_k}(t_{j_k}) \to H$ in $H_2^1(B, \mathbb{R}^3)$, and then a standard reasoning implies

$$\|Z_j(t_j) - H\|_{H_2^1(B,\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Together with (17) we finally have (14). □

Lemma 4. *Equip $\overline{\mathcal{C}}^*(\Gamma)$ with the metric d_0; then there is a continuous mapping $(X, \lambda) \mapsto R(X, \lambda)$ from $\overline{\mathcal{C}}^*(\Gamma) \times [0,1]$ into $\overline{\mathcal{C}}^*(\Gamma)$ such that $R(X, 0) = X$, $R(X, 1) = H \in \mathcal{H}^*(\Gamma)$ with $X|_{\partial B} = H|_{\partial B}$, and $d(\lambda) := D(R(X, \lambda))$ decreases from $d(0) = D(X)$ to $d(1) = D(H)$.*

Proof. Choose $H \in \mathcal{H}^*(\Gamma)$ with $H|_{\partial B} = X|_{\partial B}$ for some $X \in \overline{\mathcal{C}}^*(\Gamma)$, and set

$$R(X, \lambda) := \lambda H + (1 - \lambda)X = H + (1 - \lambda)(X - H) \quad \text{for } 0 \le \lambda \le 1.$$

By Dirichlet's principle we have $D(H, \phi) = 0$ for all $\phi \in H_2^1(B, \mathbb{R}^3)$ with $\phi|_{\partial B} = 0$ whence

$$D(R(X, \lambda)) = D(H) + (1 - \lambda)^2 D(X - H) \quad \text{for } 0 \le \lambda \le 1. \qquad \square$$

Let $\mathbb{P}(X_1, X_2)$ be the set of all paths P in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$ with $X_1, X_2 \in P$.

Lemma 5. *Let $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$, and $H_1, H_2 \in \mathcal{H}^*(\Gamma)$ be the harmonic mappings with $H_k|_{\partial B} = X_k|_{\partial B}$, $k = 1, 2$. Then we have:*

(i) *$\mathbb{P}(X_1, X_2)$ is nonvoid if and only if $\mathbb{P}(H_1, H_2)$ is nonvoid.*
(ii) *Assume that $\mathbb{P}(H_1, H_2)$ is nonvoid. Then*

$$(19) \qquad \sup_P D > \max\{D(X_1), D(X_2)\} \quad \textit{for all } P \in \mathbb{P}(X_1, X_2)$$

implies

$$(20) \qquad \sup_P D > \max\{D(H_1), D(H_2)\} \quad \textit{for all } P \in \mathbb{P}(H_1, H_2).$$

Proof. If $P \in \mathbb{P}(X_1, X_2)$ then $R(P, 1) \in \mathbb{P}(H_1, H_2)$. Conversely, if $P \in \mathbb{P}(H_1, H_2)$ and $P_1 := \{R(X_1, \lambda) : 0 \le \lambda \le 1\}$, $P_2 := \{R(X_2, \lambda) : 0 \le \lambda \le 1\}$ then $\tilde{P} := P_1 \cup P \cup P_2 \in \mathbb{P}(X_1, X_2)$, and so (i) is proved.

Suppose that there is some $P \in \mathbb{P}(H_1, H_2)$ with $\sup_P D \le \max\{D(H_1), D(H_2)\}$. Then $\sup_{\tilde{P}} D \le \max\{D(X_1), D(X_2)\}$ on account of Lemma 4. Hence (19) implies (20). □

Lemma 6. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Then any two points H_1 and H_2 of $\mathcal{H}^*(\Gamma)$ with $H_1 \neq H_2$ can be joined by a path P^* (i.e. a compact connected subset) in $(\mathcal{H}^*(\Gamma), d_0)$ such that*

$$\sup_{P^*} D \leq \max\left\{D(H_1), D(H_2), \frac{1}{4\pi}L^2(\Gamma)\right\}. \tag{21}$$

Proof. Let $\{\Gamma_j\}$ be an approximating sequence of inscribed polygons Γ_j for Γ and Q_0, Q_1, Q_2 with the associated points $t_{j,1}$ and $t_{j,2}$ in $T_j \subset \mathbb{R}^{N_j}$ for H_1 and H_2 such that $\phi_j(H_1|_{\partial B})$ and $\phi_j(H_2|_{\partial B})$ are the boundary values of harmonic mappings $H_{j,1}$ and $H_{j,2}$ in $U_j(t_{j,1})$ and $U_j(t_{j,2})$ respectively; see Lemma 3. Set

$$Y_{j,1} := Z_j(t_{j,1}), \quad Y_{j,2} := Z_j(t_{j,2}), \quad j \in \mathbb{N}.$$

In virtue of Lemma 3, (14) we have

$$d_1(Y_{j,1}, H_1) \to 0 \quad \text{and} \quad d_1(Y_{j,2}, H_2) \to 0 \quad \text{as } j \to \infty, \tag{22}$$

in particular

$$D(Y_{j,1}) \to D(H_1) \quad \text{and} \quad D(Y_{j,2}) \to D(H_2). \tag{23}$$

Consider the set $\mathcal{P}_j := \mathcal{P}_j(t_{j,1}, t_{j,2})$ of all paths $\boldsymbol{p}$ in T_j joining $t_{j,1}$ and $t_{j,2}$. By (22) and $H_1 \neq H_2$ we may assume that $t_{j,1} \neq t_{j,2}$ for all $j \in \mathbb{N}$. In virtue of 6.3, Theorem 4, there is a minimal path $\boldsymbol{p}_j^*$ in $\mathcal{P}_j$ such that

$$c_j := \max_{X \in Z_j(\boldsymbol{p}_j^*)} D(X) = \inf_{\boldsymbol{p} \in \mathcal{P}_j} \max_{X \in Z_j(\boldsymbol{p})} D(X), \quad j \in \mathbb{N}.$$

We claim that for all $j \in \mathbb{N}$

$$c_j \leq \max\left\{D(Y_{j,1}), D(Y_{j,2}), \frac{1}{4\pi}L^2(\Gamma_j)\right\}. \tag{24}$$

In fact, suppose that

$$c_j > \max\{D(Y_{j,1}), D(Y_{j,2})\}.$$

Then it follows from Theorem 4 of 6.3 that there is an unstable minimal surface $Y_j \in P_j^* := Z(\boldsymbol{p}_j^*)$ with $c_j = D(Y_j)$. The isoperimetric inequality yields

$$c_j \leq \frac{1}{4\pi}L^2(\Gamma_j),$$

and so we have (24). On the other hand, if $c_j \leq \max\{D(Y_{j,1}), D(Y_{j,2})\}$, (24) is clearly fulfilled, and so we have verified (24) for all $j \in \mathbb{N}$.

In conjunction with (23) and $L(\Gamma_j) \to L(\Gamma)$ we conclude that the sequence $\{c_j\}$ is bounded, and so (passing to a subsequence and renaming it) we may assume that $c_j \to \kappa$ for some $\kappa \geq 0$. Then H_1 and H_2 lie in

$\liminf_{j\to\infty} P_j^*$, and Lemma 2 implies that $P^* := \limsup_{j\to\infty} P_j^*$ is a path in $(\mathcal{H}^*(\Gamma), d_0)$ joining H_1 and H_2. The weak lower semicontinuity D with respect to weak convergence in $H_2^1(B, \mathbb{R}^3)$ in conjunction with the definition of P^* yields $\sup_{P^*} D \le \kappa$, and so we arrive at

$$\sup_{P^*} D \le \kappa = \lim_{j\to\infty} c_j \le \max\left\{D(H_1), D(H_2), \frac{1}{4\pi} L^2(\Gamma)\right\}. \tag{25}$$

□

Lemma 7. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ). Suppose also that H_1, H_2 are two different points of $(\mathcal{H}^*(\Gamma), d_0)$ such that*

$$\sup_{P'} D > \max\{D(H_1), D(H_2)\} \quad \textit{for all } P' \in \mathbb{P}'(H_1, H_2) \tag{26}$$

where $\mathbb{P}'(H_1, H_2)$ denotes the set of all paths in $(\mathcal{H}^*(\Gamma), d_0)$ joining H_1 and H_2. Then there exists some path $P^* \in (\mathcal{H}^*(\Gamma), d_0)$ and some minimal surface $H_3 \in P^*$ with

$$D(H_3) = c := \sup_{P^*} D \tag{27}$$

which is d_0-unstable, i.e. in every d_0-neighborhood of H_3 there exists an $X \in \overline{\mathcal{C}}^*(\Gamma)$ such that $D(X) < D(H_3)$.

Proof. Let $P^* \in \mathbb{P}'(H_1, H_2)$ be the path constructed in the proof of Lemma 6. Then

$$c := \sup_{P^*} D > \max\{D(H_1), D(H_2)\}$$

in virtue of (26). By (25) we have

$$c \le \kappa = \lim_{j\to\infty} c_j, \quad c_j := \max_{P_j^*} D.$$

In conjunction with (23) we then obtain

$$\max_{P_j^*} D > \max\{D(Y_{j,1}), D(Y_{j,2})\} \quad \text{for } j \gg 1.$$

Using the proof of Theorem 4 in 6.3 we conclude that for $j \gg 1$ there is a minimal surface $X_j \in P_j^*$ satisfying $D(X_j) = c_j$, and a standard reasoning (cf. 4.3, Theorem 3) shows that there is a subsequence $\{X_{j_k}\}$ and a minimal surface $X_0 \in P^*$ such that $d_0(X_{j_k}, X_0) \to 0$ and $X_{j_k} \rightharpoonup X_0$ in $H_2^1(B, \mathbb{R}^3)$. Furthermore, $L(\Gamma_j) \to L(\Gamma)$. Then we infer $D(X_{j_k}) \to D(X_0)$ on account of 6.4, Theorem 5. Therefore

$$\kappa = \lim_{k\to\infty} c_{j_k} = D(X_0) \le c.$$

Thus the minimal surface $X_0 \in P^*$ satisfies $D(X_0) = c$. In order to obtain an unstable minimal surface $H_3 \in P^*$ with $D(H_3) = c$ we consider the set K_c

of all minimal surfaces $H \in P^*$ with $D(H) = c$ which is a closed subset of (P^*, d_0) on account of 4.3, Theorem 3, and 6.4, Theorem 5. Furthermore K_c is a nonvoid and proper subset of P^* since $X_0 \in K_c$ and $H_1, H_2 \notin K_c$. Hence, on account of the connectedness of P^*, there exists a boundary point H_3 of K_c with $H_3 \in K_c$, therefore

$$\mathcal{N}_\varepsilon := (P^* \setminus K_c) \cap \{X \colon d_0(X, H_3) < \varepsilon\} \neq \emptyset \quad \text{for every } \varepsilon > 0.$$

If for any $\varepsilon > 0$ there is an $X \in \mathcal{N}_\varepsilon$ with $D(X) < c$, we have shown that H_3 is unstable. It remains to consider the case when we have $D(X) = c$ for all $X \in \mathcal{N}_\varepsilon$ and any $\varepsilon \in (0, \varepsilon_0)$ for some positive ε_0. Pick some $X \in \mathcal{N}_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_0)$. Then $D(X) = c$, and X is harmonic as $\mathcal{N}_\varepsilon \subset P^* \subset \mathbb{P}'$. Since $X \notin K_c$ we conclude that X is not conformal; therefore we have

$$\partial D(X, \lambda) \neq 0 \tag{28}$$

for some vector field $\lambda \in C^2(\overline{B}, \mathbb{R}^2)$ that is tangential at ∂B. Then we can find a C^1-family $\sigma(\cdot, t) : \overline{B} \to \overline{B}$ of diffeomorphisms of $\overline{B}$ onto itself such that $\sigma(w, 0) = w$ for all $w \in \overline{B}$ and

$$\frac{d}{dt} D(X \circ \sigma(\cdot, t))\bigg|_{t=0} < 0,$$

whence $D(X \circ \sigma(\cdot, t)) < D(X) = c$ for $0 < t \ll 1$. Thus $D(Y) < D(X)$ for $Y := X \circ \sigma(\cdot, t) \in \overline{\mathcal{C}}(\Gamma)$ as well as $d_0(Y, X) \ll 1$ for $0 < t \ll 1$, and therefore $d_0(Y, H_3) < \varepsilon$ for $0 < t \ll 1$. By the reasoning of 6.1, Proposition 8, one can achieve that (28) holds for some admissible λ with $\lambda(w_k) = 0$ for $w_k = \exp(i\psi_k)$, $k = 0, 1, 2$. Approximating λ in a suitable way one can construct $\sigma(\cdot, t)$ in such a way that also $\sigma(w_k, t) = w_k$ for $|t| \ll 1$ is fulfilled, and therefore Y lies even in $\overline{\mathcal{C}}^*(\Gamma)$. □

Now we can state the main result of this section.

Theorem 1. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let X_1, X_2 be two points of $(\overline{\mathcal{C}}^*(\Gamma), d_0)$ such that $X_1|_{\partial B} \neq X_2|_{\partial B}$ and*

$$\sup_P D > \max\{D(X_1), D(X_2)\} \quad \textit{for all } P \in \mathbb{P}(X_1, X_2). \tag{29}$$

Then there exists a D-unstable (and therefore also A-unstable) minimal surface $X_3 \in (\overline{\mathcal{C}}^*(\Gamma), d_0)$, i.e. for any $\varepsilon > 0$ there is an $X \in \overline{\mathcal{C}}^*(\Gamma)$ with $d_0(X, X_3) < \varepsilon$ and $D(X) < D(X_3)$.

Proof. Let $H_1, H_2 \in \mathcal{H}^*(\Gamma)$ be the harmonic surfaces with $H_k|_{\partial B} = X_k|_{\partial B}$, $k = 1, 2$. By Lemma 6 the set $\mathbb{P}'(H_1, H_2)$ is nonvoid, and so also $\mathbb{P}(X_1, X_2)$ is nonvoid according to Lemma 5(i). Thus assumption (29) makes sense, and Lemma 5(ii) yields

$$\sup_P D > \max\{D(H_1), D(H_2)\} \quad \text{for all } P \in \mathbb{P}(H_1, H_2).$$

Since $\mathbb{P}'(H_1, H_2) \subset \mathbb{P}(H_1, H_2)$ we also have

$$\sup_{P'} D > \max\{D(H_1), D(H_2)\} \quad \text{for all } P' \in \mathbb{P}'(H_1, H_2).$$

Moreover, $X_1|_{\partial B} \neq X_2|_{\partial B}$ implies $H_1 \neq H_2$. Now the assertion follows from Lemma 7 and from 6.3, Remark 2. □

As a corollary of the preceding result we obtain

Theorem 2. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ be two different minimal surfaces, each furnishing a strict local minimum for D in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$, i.e.*

(30) $\quad D(X_k) < D(X) \quad$ *for any $X \in \overline{\mathcal{C}}^*(\Gamma)$ with $d_0(X, X_k) < \varepsilon$,* $\quad k = 1, 2,$

for any positive $\varepsilon \ll 1$. Then there is a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which is both D-unstable and A-unstable in $(\overline{\mathcal{C}}^*(\Gamma), d_0)$.

Another corollary of Theorem 1 is the following result:

Theorem 3. *Let Γ be a closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying Condition (μ), and let $X_1, X_2 \in \overline{\mathcal{C}}^*(\Gamma)$ be two minimal surfaces separated by a wall, i.e. which satisfy (29). Then there exists a third minimal surface $X_3 \in \overline{\mathcal{C}}^*(\Gamma)$ which is both D-unstable and A-unstable in $\overline{\mathcal{C}}^*(\Gamma)$.*

6.7 Scholia

6.7.1 Historical Remarks and References to the Literature

The results of the present chapter are a special part of Morse theory that formerly ran under the headline "Theorem of the Wall". Nowadays one speaks of the "Mountain Pass Theorem", referring to the path-breaking work by A. Ambrosetti and P. Rabinowitz [1]. A presentation of applications of this theorem to various variational problems can be found in the texts of M. Struwe [13] and E. Zeidler [1]. Originally Morse theory worked very well for one-dimensional variational integrals, say, geodesics whereas already minimal surfaces lead to enormous difficulties. In a remarkable competition, the first results were found by M. Shiffman [2–5] and by M. Morse & C. Tompkins [1–5] almost simultaneously. Of particular interest is that work by Morse & Tompkins which uses the "theorem of the wall", while their general Morse-theoretic statements are more or less useless as they are based on topological assumptions which cannot be verified in a concrete situation. A general Morse theory for minimal surfaces in $\mathbb{R}^4$ was developed by M. Struwe [4,8] and J. Jost & M. Struwe [1]. Furthermore A. Tromba [10–12] obtained a Morse-theoretic result for minimal surfaces in three-dimensional space which is presented in the last chapter of Vol. 3.

Somewhat later than Shiffman and Morse & Tompkins, Courant found a new approach to the "theorem of the wall" that works for minimal surfaces bounded by a polygonal contour; cf. R. Courant [13] and [15], and Shiffman [4] showed how Courant's "polygonal theory" can also be used to establish the "theorem of the wall" for rectifiable boundary contours. This work was carried over by E. Heinz [12–14] to surfaces of constant mean curvature H with $|H| < \frac{1}{(2R)}$ which are contained in a ball of radius R. In his remarkable paper [2], Ströhmer was able to establish the "theorem of the wall" for surfaces of prescribed mean curvature $H(x)$ under the most general assumption $|H(x)| \leq \frac{1}{R}$. Previously G. Ströhmer [1] had generalized the Courant–Shiffman theory to minimal surfaces in a Riemannian manifold of nonpositive sectional curvature. Further contributions by Ströhmer concern the semi-free problem [3], and in [4] the Plateau problem for more general integrals. M. Shiffman [8] developed a "mountain pass theorem" for general parametric integrals of the type

$$\mathcal{F}(X) = \int_B F(X_u \wedge X_v)\, du\, dv.$$

Unfortunately, Shiffman's reasoning is not stringent, as has been pointed out by R. Jakob (cf. [2], p. 403). Nevertheless, Shiffman's paper contains quite ingenious ideas which, combined with the technique developed by Courant and Heinz, enabled R. Jakob to establish a modified version of Shiffman's theory (see Jakob [1,2,4,5]).

6.7.2 The Theorem of the Wall for Minimal Surfaces in Textbooks

The first textbook presentation can be found in Courant [15], Chapter VI, Sections 7 and 8. The results of 6.6 can also be derived by using Courant's *pinching lemma* 6.10 (cf. pp. 236–237, 241–243) instead of Lemma 1 in 6.6.

J.C.C. Nitsche gave a detailed and very precise presentation of Shiffman's approach to unstable minimal surfaces in §§419–433 of his treatise [28], with applications to several examples in §§ 434–436.

An interesting and completely new approach to the "mountain pass theorem" for minimal surfaces, based on Douglas's functional, was given by M. Struwe [11], with a correction in Imbusch and Struwe [1]. In this work, an infinite-dimensional version of the mountain-pass lemma is used to prove the existence of unstable minimal surfaces directly for boundary contours Γ of class C^2, without the detour of approximating Γ by polygonal contours Γ_j. This enabled Struwe to work with a metric d_1 instead of d_0, just as we did in 6.3, which leads to somewhat stronger existence results than those in 6.6 for $\Gamma \in C^2$, i.e. to results as presented in 6.3 for polygonal Γ. It is a challenging problem to carry over Struwe's approach to related problems.

J. Jost [17], Corollary 4.4.11, proved the following result: *Let Γ be a closed Jordan curve in a compact Riemannian manifold that contains no minimal spheres* (*e.g. if the sectional curvature of M is nonpositive*). *Suppose that Γ bounds two homotopic minimal surfaces $X_1, X_2 : B \to M$ both of which are strict relative minima of Dirichlet's integral D* (*with respect to the C^0- or H_2^1-topology*). *Then there exists a third minimal surface $X_3 : B \to M$ bounded by Γ and satisfying*

$$D(X_3) > \max\{D(X_1), D(X_2)\}.$$

He also noted that for proving such a result the compactness of M is not really needed; it is sufficient to assume that $X_1(B)$ and $X_2(B)$ lie in a bounded, strictly convex subset of N, without further restrictions on M. Therefore one in particular obtains a corresponding result in $\mathbb{R}^n$. Moreover, Ströhmer's results in his papers [1] and [4] can be obtained in this way.

Another "instability result" of J. Jost [17] is his Theorem 4.6.1 which holds for *Jordan curves Γ of class C^2 in $\mathbb{R}^n$*: *Let $X_1, X_2 : B \to \mathbb{R}^n$ be minimal surfaces of class $\mathcal{C}^*(\Gamma)$ such that $X_1(B) \neq X_2(B)$.* (i) *If both X_1 and X_2 are strict local minimizers, then there is a third minimal surface in $\mathcal{C}^*(\Gamma)$ which is unstable.* (ii) *If both X_1 and X_2 are global minimizers, then one either has a third and unstable minimal surface X_3, or there is a continuous family $X(\cdot, t)$ with $X(\cdot, 0) = X_1$, $X(\cdot, 1) = X_2$, and $D(X(\cdot, t)) \equiv$ const.*

Generalizations of this result are indicated in Jost [17], p. 160, Remark (1).

6.7.3 Sources for This Chapter

In writing this chapter we have extensively used Courant's work in [15], Heinz's papers [13] and [14], as well as a first draft by R. Jakob. In addition, Jakob's papers [1,2] and several lectures that he gave to us were of help; we are very grateful for his support in drawing up the material and for his criticism of our first draft.

6.7.4 Multiply Connected Unstable Minimal Surfaces

In [8] Struwe used his approach from [11] to prove the existence of unstable minimal surfaces of annulus type. J. Hohrein [1] discussed the existence of unstable minimal surfaces of higher genus in Riemannian manifolds of non-positive curvature, employing ideas of Struwe [4]. Unstable minimal surfaces X of annulus type in a Riemannian manifold M were studied by H. Kim [1] assuming that the boundary of X lies in a ball $B_r(p)$ of normal range, which in particular means that the radius r of the ball satisfies $r < \pi/(2\sqrt{\kappa})$ where κ is an upper bound for the sectional curvature of M.

6.7.5 Quasi-Minimal Surfaces

It is not known whether the Courant function Θ associated with a polygon Γ is of class C^2, and so it is impossible to develop a Morse theory for Θ. To overcome this difficulty, Marx and Shiffman have set up a modified variational problem which leads to a modification Θ^* of Θ with a much better behavior; cf. Courant [15], pp. 235–236, and I. Marx [1]. The original work of Shiffman was never published, and the proofs given in Marx's paper are incomplete (see e.g. E. Heinz [20], p. 84, and [25], pp. 200–201). A satisfactory theory of the *variational problem of Marx–Shiffman* was developed only much later by E. Heinz [19–24], with further contributions by F. Sauvigny [1–3,6], and R. Jakob [6–10]. In the sequel we shall present a brief summary of this work.

Let Γ be a simple closed polygon with $N+3$ (≥ 4) consecutive vertices $Q_1, Q_2, \dots, Q_{N+3}$, and set $Q_{N+4} := Q_1$, $Q_0 := Q_{N+3}$. Consider the set of points $t = (t^1, t^2, \dots, t^N)$ with $0 < t^1 < t^2 < \cdots < t^N < \pi$, and set

$$t^{N+\nu} := \frac{1}{2}\pi(1+\nu) \quad \text{for } \nu = 1,2,3, \quad t^0 := 0.$$

We assume that the angles at the corners Q_j are neither 0 nor π, i.e. for $\xi_k := Q_k - Q_{k-1}$, any two vectors ξ_k, ξ_{k+1} are linearly independent. By Γ_k we denote the straight lines

$$\Gamma_k := \{s\xi_k \colon s \in \mathbb{R}\}.$$

For $t \in T$ we define $U^*(t)$ as the set of surfaces

$$X \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$$

which map the circular arcs $\gamma_k := \{e^{i\varphi} \colon t^k < \varphi < t^{k+1}\}$ into the straight lines $\Gamma'_k := Q_k + \Gamma_k$. Then $X(w_k) = Q_k$ for $w_k := e^{it^k}$ and $1 \leq k \leq N+3$. We want to minimize D in the class $U^*(t)$. This will be achieved by minimizing D in the class V, defined by

$$V := \{X \in H_2^1(B, \mathbb{R}^3) \colon X|_{\partial B}(\gamma_j) \subset \Gamma'_j,\ j = 1,2,\dots,N+3\},$$

and then proving that the minimizer in V actually belongs to $U^*(t)$. Since we have no control over the boundary values of elements of V we need a Poincaré inequality for the elements of V; in fact, such an inequality for the elements of $V \cap C^1(B, \mathbb{R}^3)$ will suffice. This will be achieved by formula (2) of the following

Lemma 1. *Let $\tau_0 \in [0, 2\pi]$, $w_0 := e^{i\tau_0}$, $0 < \varepsilon_0 < \pi$, $0 < \varepsilon_1 < 1$, $\gamma^- := \{e^{i\varphi} \colon \tau_0 - \varepsilon_0 < \varphi < \tau_0\}$, $\gamma^+ := \{e^{i\varphi} \colon \tau_0 < \varphi < \tau_0 + \varepsilon_0\}$; $e^-, e^+ \in \mathbb{R}^3$ with $|e^-| = |e^+| = 1$ and $\langle e^-, e^+\rangle \leq 1 - \varepsilon_1$, $\Gamma^+ := \{se^+ \colon s \in \mathbb{R}\}$, $\Gamma^- := \{se^- \colon s \in \mathbb{R}\}$; finally suppose that $Z \in H_2^1(B, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$ and $Z|_{\gamma^+}(w) \in \Gamma^+$ $\mathcal{H}^1$-a.e. on γ^+, $Z|_{\gamma^-}(w) \in \Gamma^-$ $\mathcal{H}^1$-a.e. on γ^-. Then there are numbers $c_1 = c_1(\varepsilon_0, \varepsilon_1)$, $c_2 = c_2(\varepsilon_0, \varepsilon_1)$, and $\delta_0 = \delta_0(\varepsilon_0) \in (0,1)$ with the following properties*:

(i) *For any $\delta \in (0, \delta_0]$ there is a $\delta^* \in (\delta, \sqrt{\delta})$ with*

$$(1) \qquad |Z(w)| \leq c_1 \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \sqrt{D(Z)} \quad \textit{for } w \in B \textit{ with } |w - w_0| = \delta^*.$$

(ii) *We have*

$$\int_B |Z|^2 \, du \, dv \le c_2 D(Z). \tag{2}$$

Proof. (i) By 4.4, Proposition 2, there is a $\delta^* \in (\delta, \sqrt{\delta})$ such that

$$|Z(w) - Z(w')| \le 2\left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \sqrt{D(Z)} \tag{3}$$
$$\text{for } w, w' \in \overline{B} \quad \text{with } |w - w_0| = \delta^* \text{ and } |w' - w_0| = \delta^*.$$

Let w_1, w_2 be the two end points of the circular arc $\{w \in \overline{B}\colon |w - w_0| = \delta^*\}$. For $0 < \delta < \varepsilon_0$ we can assume that $w_1 \in \gamma^-$, $w_2 \in \gamma^+$. Then (3) and $\langle e^-, e^+ \rangle \le 1 - \varepsilon_1$ imply

$$|Z(w_1)| \le c_0(\varepsilon_0, \varepsilon_1)\left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \cdot \sqrt{D(Z)}. \tag{4}$$

Now (1) follows from (3), (4), and $|Z(w)| \le |Z(w_1)| + |Z(w) - Z(w_1)|$.

(ii) Fix some δ_0 with $0 < \delta_0 < \varepsilon$ and choose $\delta := \delta_0$ in (i). Since

$$\int_{1-\delta_0}^{1} \left(\int_0^{2\pi} |Z_\varphi(re^{i\varphi})|^2 \, d\varphi\right) \frac{dr}{r} \le 2D(Z),$$

there are numbers $\delta_1 \in (0, \delta_0)$ and $r_1 := 1 - \delta_1$ with $r_1 > 1 - \delta_0$ and

$$\int_0^{2\pi} |Z_\varphi(r_1 e^{i\varphi})|^2 \, d\varphi \le \left(\int_{1-\delta_0}^{1} \frac{dr}{r}\right)^{-1} \cdot 2D(Z) < \frac{2}{\delta_0} D(Z)$$

whence

$$|Z(r_1 e^{i\varphi}) - Z(r_1 e^{i\psi})| \le [4\pi\delta_0^{-1} D(Z)]^{\frac{1}{2}} \quad \text{for } 0 \le \varphi, \psi \le 2\pi.$$

Since the arcs $\{w \in B\colon |w| = r_1\}$ and $\{w \in B\colon |w - w_0| = \delta^*\}$ intersect we obtain in conjunction with (1) that

$$|Z(w)| \le \left[c_1 \cdot \left(\log \frac{1}{\delta_0}\right)^{-\frac{1}{2}} + \left(\frac{4\pi}{\delta_0}\right)^{\frac{1}{2}}\right] \sqrt{D(Z)} \tag{5}$$
$$\text{for } \{w \in B\colon |w| = r_1\}.$$

Choose some $\varepsilon > 0$ and consider the function

$$f(r) := \varepsilon + \int_0^{2\pi} |Z(re^{i\varphi})|^2 \, d\varphi, \quad r \in (0, 1).$$

From

$$\frac{d}{dr}\sqrt{f(r)} = \frac{f'(r)}{2\sqrt{f(r)}} = \frac{\int_0^{2\pi} 2\langle Z(re^{i\varphi}), Z_r(re^{i\varphi})\rangle\, d\varphi}{2\sqrt{f(r)}}$$

we infer by Schwarz's inequality that

$$\left|\frac{d}{dr}\sqrt{f(r)}\right| \le \left(\int_0^{2\pi} |Z_r(re^{i\varphi})|^2\, d\varphi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\sqrt{f(r)} - \sqrt{f(r_1)} &\le \left|\int_{r_1}^{r} \frac{d}{dr}\sqrt{f(r)}\, dr\right| \le \left|\int_{r_1}^{r}\left|\frac{d}{dr}\sqrt{f(r)}\right| dr\right| \\ &\le \left|\int_{r_1}^{r} \frac{1}{\sqrt{r}} \cdot \left(\int_0^{2\pi} \left|Z_r(r\, e^{i\varphi})\right|^2 r d\varphi\right)^{\frac{1}{2}} dr\right|,\end{aligned}$$

and by Schwarz's inequality,

$$\sqrt{f(r)} \le \sqrt{f(r_1)} + \left(\sqrt{\log\frac{1}{r}} + \sqrt{\log\frac{1}{r_1}}\right)\sqrt{2D(Z)}.$$

Squaring and letting ε tend to zero we obtain the estimate

$$\int_0^{2\pi} |Z(re^{i\varphi})|^2\, d\varphi \le 2\int_0^{2\pi} |Z(r_1 e^{i\varphi})|^2\, d\varphi + 8\left(\log\frac{1}{r} + \log\frac{1}{r_1}\right)D(Z).$$

In virtue of $\frac{1}{r_1} < \frac{1}{1-\delta_0}$ and (5) we arrive at

$$\int_0^{2\pi} |Z(re^{i\varphi})|^2\, d\varphi \le c(\varepsilon_0, \varepsilon_1)\cdot\left(1 + \log\frac{1}{r}\right)D(Z) \quad \text{for } 0 < r < 1.$$

Multiplying by r and integrating with respect to r from 0 to 1 we obtain (2). $\square$

Now we fix some arbitrary $t \in T$. Depending on Γ there are numbers $q > 0$ and $\mu = \mu(t) \in (0,1)$ such that

$$\begin{aligned}&|Q_k| \le q, \quad |t^j - t^k| \ge \mu, \quad |\langle \xi_k, \xi_{k+1}\rangle| \le 1-\mu \\ &\quad \text{for } 1 \le j,k \le N+3,\ j \ne k.\end{aligned} \tag{6}$$

Proposition 1. *There exists a uniquely determined mapping $X \in U^*(t)$ with $D(X) = \inf_{U^*(t)} D$ which is harmonic in B.*

Proof. Set $d := \inf_V D$ and $d^* := \inf_{U^*(t)} D$. By $U^*(t) \subset V$ and (6) it follows that

$$0 \le d \le d^*(q,\mu) < \infty. \tag{7}$$

Choose a sequence of mappings $X_n \in V$ with $D(X_n) \to d$. By Dirichlet's principle we can assume that the X_n are harmonic in B, in particular $X_n \in C^1(B, \mathbb{R}^3)$. Since $Z := X_n - X_l$ satisfies the assumptions of Lemma 1 we have

$$\int_B |X_n - X_l|^2 \, du \, dv \leq c_2 D(X_n - X_l) \quad \text{for any } n, l \in \mathbb{N}.$$

Furthermore, $\frac{1}{2}(X_k + X_l) \in V$ because V is a convex set, whence

$$D(X_n + X_l) = 4D\big(\tfrac{1}{2}(X_n + X_l)\big) \geq 4d,$$

and therefore

$$\begin{aligned} D(X_n - X_l) &= 2D(X_n) + 2D(X_l) - D(X_n + X_l) \\ &\leq 2D(X_n) + 2D(X_l) - 4d \to 0 \quad \text{as } n, l \to \infty. \end{aligned}$$

Thus $\{X_n\}$ is a Cauchy sequence in $H_2^1(B, \mathbb{R}^3)$, and so there is an $X \in H_2^1(B, \mathbb{R}^3)$ with $X_n \to X$ in $H_2^1(B, \mathbb{R}^3)$ as $n \to \infty$. Then we also have $X_n \rightrightarrows X$ in B' for any $B' \subset\subset B$; hence X is harmonic in B. Since V is a closed subset of $H_2^1(B, \mathbb{R}^3)$ we see that $X \in V$, and $D(X_n) \to d$ yields $D(X) = d$. Consequently X is a minimizer of D in V.

Suppose that $Y \in V$ were another minimizer. Then

$$D(X - Y) = 2D(X) + 2D(Y) - D(X + Y) \leq 2d + 2d - 4d \leq 0$$

and consequently $D(X - Y) = 0$. By Lemma 1(ii), follows

$$\int_B |X - Y|^2 \, du \, dv = 0,$$

and therefore $X = Y$. Thus D possesses exactly one minimizer X in V. If we can show that $X \in C^0(\overline{B}, \mathbb{R}^3)$ it follows that X lies in $U^*(t)$, and consequently

$$\inf_V D = \inf_{U^*(t)} D$$

because of $U^*(t) \subset V$, and so it would be shown that X is the unique minimizer of D in $U^*(t)$.

Standard elliptic regularity theory yields that X is real analytic on the set $\overline{B} \setminus \{w_1, \ldots, w_{N+3}\}$, $w_k := e^{it^k}$, and

$$\text{(8)} \qquad \begin{aligned} &\langle X_r(w), Q_{k+1} - Q_k \rangle = 0 \\ &R_k X(w) = X(w) \end{aligned} \qquad \text{for } w \in \gamma_k, \; k = 1, \ldots, N+3.$$

Here $R_k : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the reflection of $\mathbb{R}^3$ in the straight line $\Gamma_k' = Q_k + \Gamma_k$.

It remains to prove the continuity of X at the points $w = w_k$. Set

$$X^+(w) := X(w) \quad \text{for } |w| < 1, \quad X^-(w) := X(\overline{w}^{-1}) \quad \text{for } |w| > 1.$$

Here $\overline{w}^{-1} = w/|w|^2$ is the mirror point of w with respect to the unit circle ∂B. Let P be the exterior of the convex hull of $w_1, \ldots, w_{N+3}$, and Ω_k be the subset of $B \cap P$ bounded by $\overline{\gamma}_k$ and the linear segment σ_k with the endpoints w_k and w_{k+1}. By (8) and Schwarz's reflection principle, X^- can be extended to a harmonic mapping in P, which will again be denoted by X^-, and one has

$$X^-(w) = R_k X^+(w) \quad \text{for } w \in \Omega_k. \tag{9}$$

Then for $0 < \rho^2 \leq \delta_1(\mu) \ll 1$ the function $|X^- - Q_k|^2$ is subharmonic in $P \cap B_\rho(w_k)$, whereas $|X^+ - Q_k|^2$ is subharmonic in $B \cap B_\rho(w_k)$, and by (9) it follows that $|X^-(w) - Q_k|^2 = |X^+(w) - Q_k|^2$ holds both for $w \in \Omega_k$ and for $w \in \Omega_{k-1}$. Hence the function

$$g(w) := \begin{cases} |X^+(w) - Q_k|^2 & \text{for } w \in B_\rho(w_k) \cap B, \ w \neq w_k, \\ |X^-(w) - Q_k|^2 & \text{for } w \in B_\rho(w_k) \cap (\mathbb{C} \setminus B), \ w \neq w_k, \end{cases} \tag{10}$$

is subharmonic in the punctured disk $B'_\rho(w_k) := \{w \in \mathbb{R}^2 \colon 0 < |w - w_k| < \rho\}$. By (7) we have $D(X - Q_k) = D(x) \leq d^*$, and so the Courant–Lebesgue Lemma yields together with $|\langle \xi_k, \xi_{k+1} \rangle| \leq 1 - \mu$ for each k:

For any $\delta \in (0, \delta_1]$ and any k with $1 \leq k \leq N + 3$ there is a number $\rho = \rho(\delta, k) \in (\delta, \sqrt{\delta})$ such that

$$|X(w) - Q_k| \leq c_3(\mu)\sqrt{d^*} \cdot \left(\log \frac{1}{\delta}\right)^{-1/2} =: M_\delta \tag{11}$$
$$\text{for } w \in B \text{ with } |w - w_k| = \rho.$$

Hence for any $k = 1, \ldots, N + 3$ there is a sequence $\{\rho_j\}$ of numbers $\rho_j > 0$ with $\rho_j \to 0$ as $j \to \infty$ such that

$$m_j := \max\{|X(w) - Q_k| \colon w \in \overline{B}, \ |w - w_k| = \rho_j\} \to 0 \quad \text{as } j \to \infty. \tag{12}$$

Applying the maximum principle to the subharmonic function g on the annulus $A(\rho_j, \rho) := \{w \in \mathbb{C} \colon \rho_j \leq |w - w_k| \leq \rho\}$ for $j \gg 1$ we infer from (11) and (12) that

$$\max\{g(w) \colon w \in A(\rho_j, \rho)\} \leq \max\{m_j^2, M_\delta^2\}.$$

Letting j tend to infinity it follows that

$$\max\{g(w) \colon w \in B'_\rho(w_k)\} \leq M_\delta^2.$$

In conjunction with (10) we obtain that X is also continuous at the points $w_1, \ldots, w_{N+3}$, and therefore $X \in C^0(\overline{B}, \mathbb{R}^3)$ and $X \in U^*(t)$. $\square$

Definition 1. *The* **Marx–Shiffman mapping** $Z^* : T \to H_2^1(B, \mathbb{R}^3)$ *is defined by* $Z^*(t) := X$ *for* $t \in T$ *where* X *denotes the uniquely determined minimizer of* D *in* $U^*(t)$, *and the* **Marx–Shiffman function** $\Theta^* : T \to \mathbb{R}$ *is given by*

$$\Theta^*(t) := D(Z^*(t)) = \inf_{U^*(t)} D. \tag{13}$$

Any mapping $Z^*(t) : \overline{B} \to \mathbb{R}^3$ *is called a* **quasi-minimal surface** (*cf. I. Marx* [1]). *If we want to emphasize the dependence of* Z^* *and* Θ^* *on* t *and on the vertices* $Q_1, \dots, Q_{N+3}$ *we write* $Z^*(t, Q_1, \dots, Q_{N+3})$ *and as well as* $\Theta^*(t, Q_1, \dots, Q_{N+3})$.

Remark 1. The three point condition $X(w_k) = Q_k$, $k = N+1, N+2, N+3$ with $w_k = e^{it^k}$ and $t^{N+\nu} = \frac{1}{2}\pi(1+\nu)$ for $\nu = 1, 2, 3$ is only needed if we want to compare Θ^* with the Courant function Θ. Otherwise we can replace t by $t^* = (t^1, t^2, \dots, t^{N+3})$ and T by $T^* := \{t^* \in \mathbb{R}^{N+3} : t^1 < t^2 < \cdots < t^{N+3} < t^1 + 2\pi\}$. Then the statements on Θ^* and Z^* as functions of $t \in T$ also hold (with obvious alterations) if we consider Θ^*, Z^* as functions of $t^* \in T^*$.

Corollary 1. *For* $t \in T$ *there is a number* $c_4 = c_4(q, \mu(t)) > 0$ *such that*

$$\max_{\overline{B}} |Z^*(t)| \le c_4.$$

Furthermore, for any $t \in T$ *and* $\varepsilon > 0$ *there is a number* $\delta_2 = \delta_2(q, \mu(t), \varepsilon) > 0$ *such that*

$$|Z^*(t)(w) - Z^*(t)(w')| < \varepsilon \quad \text{for any } w, w' \in \overline{B} \text{ with } |w - w'| < \delta_2.$$

Remark 2. This corollary together with Proposition 1 implies that the mappings $Z^*(t, Q_1, \dots, Q_{N+3})$ and $Z^*(t^*, Q_1, \dots, Q_{N+3})$ depend continuously on the data $t \in T$ and $t^* \in T^*$ respectively and on $Q_1, \dots, Q_{N+3}$.

Corollary 2. *For any* $t \in T$, *the mapping* $Z^*(t) \in U^*(t)$ *is of class* $C^0(\overline{B}, \mathbb{R}^3) \cap C^\omega(\overline{B} \setminus \{w_1, \dots, w_{N+3}\}, \mathbb{R}^3)$, *harmonic in* $\overline{B}' := \overline{B} \setminus \{w_1, \dots, w_{N+3}\}$, *and satisfies the boundary conditions* (8) *on* $\gamma_1 \cup \cdots \cup \gamma_{N+3}$.

Proposition 2. *Let* Θ *and* Θ^* *be the Courant function and the Marx–Shiffman function associated with a given simple and closed polygon* Γ. *Then we have*:

(i) $\Theta^*(t) \le \Theta(t)$ *for all* $t \in T$;
(ii) $\Theta^*(t) = \Theta(t)$ *if* $t \in T$ *is a critical point of* Θ;
(iii) *There are polygons* Γ *such that* $\Theta^* \ne \Theta$, *i.e.*

$$\Theta^*(t) < \Theta(t) \quad \text{for some } t \in T.$$

Proof. (i) Since $U(t) \subset U^*(t)$ for any $t \in T$, it follows that

$$\Theta^*(t) = \inf_{U^*(t)} D \le \inf_{U(t)} D = \Theta(t).$$

(ii) Let $t \in T$ be a critical point of Θ, and note that

$$\Theta(t) = D(Z(t)), \quad \Theta^*(t) = D(Z^*(t)).$$

By a similar reasoning as in the proof of Proposition 2 in 6.1 we infer that $Z(t) = Z^*(t)$.

(iii) The third assertion follows from the **Lewerenz examples**: *For any $N \in \mathbb{N}$ there is a closed simple polygon Γ with $N+3$ vertices such that the corresponding functions Θ and Θ^* do not coincide.*

It suffices to construct an example for $N = 1$; the other cases will be obtained by a slight modification of this example. So we are looking for a polygon with four vertices Q_1, Q_2, Q_3, Q_4 such that the corresponding functions Θ, Θ^* satisfy $\Theta^*(t) < \Theta(t)$ for some $t \in T$. (Note that T here reduces to an interval.) For the sake of convenience we parametrize all surfaces on the semidisk $B^+ := \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1 \text{ and } v > 0\}$ instead of the disk $B := \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1\}$. Let Γ_ε be the polygon determined by the four successive corners

$$\begin{aligned} &Q_1^\varepsilon := (0,0,-\varepsilon), \quad Q_2^\varepsilon := (0,0,\varepsilon), \\ &Q_3^\varepsilon = Q_3 := (1,1,-1), \quad Q_4^\varepsilon = Q_4 := (-1,1,1), \end{aligned}$$

where $\varepsilon > 0$ is a parameter that will be fixed later on. Set

$$w_1 := (-1,0), \quad w_2 := (1,0), \quad w_3 := (\alpha,\alpha), \quad w_4 := (-\alpha,\alpha), \quad \alpha := \frac{1}{\sqrt{2}}.$$

By $Z_\varepsilon = (Z_\varepsilon^1, Z_\varepsilon^2, Z_\varepsilon^3)$ we denote the uniquely determined minimizer of D_{B^+},

$$D_{B^+}(X) := \frac{1}{2} \int_{B^+} |\nabla X|^2 \, du \, dv,$$

among all $X \in H_2^1(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$ which map ∂B^+ monotonically onto Γ_ε such that $X(w_1) = Q_1^\varepsilon$, $X(w_2) = Q_2^\varepsilon$, $X(w_3) = Q_3$, $X(w_4) = Q_4$, and $\mathcal{C}_\varepsilon$ be the class of all such X. Consider the surface Z'_ε defined by

$$Z'_\varepsilon(u,v) := \big(-Z_\varepsilon^1(-u,v), Z_\varepsilon^2(-u,v), -Z_\varepsilon^3(-u,v)\big).$$

One easily checks that $Z'_\varepsilon \in \mathcal{C}_\varepsilon$ and $D_{B^+}(Z_\varepsilon) = D_{B^+}(Z'_\varepsilon)$, whence we obtain $Z_\varepsilon = Z'_\varepsilon$. Thus, for any $(u,v) \in \overline{B^+}$, $Z_\varepsilon^1(-u,v) = -Z_\varepsilon^1(u,v)$, $Z_\varepsilon^2(-u,v) = Z_\varepsilon^2(u,v)$, $Z_\varepsilon^3(-u,v) = -Z_\varepsilon^3(u,v)$. In particular it follows that

$$Z_\varepsilon^1(0,v) = 0, \quad Z_\varepsilon^3(0,v) = 0 \quad \text{for } 0 \le v \le 1,$$

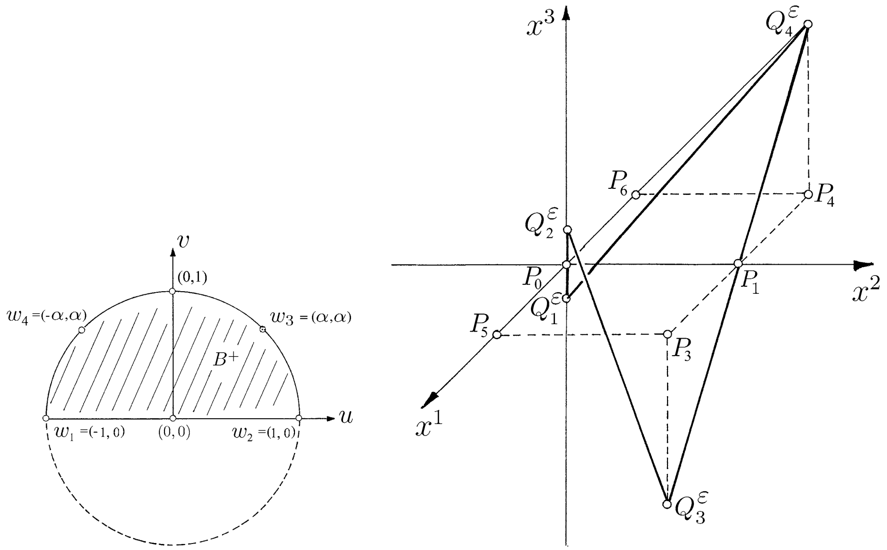

Fig. 1. Lewerenz curve

and $Z_\epsilon \in \mathcal{C}_\epsilon$ immediately yields

$$Z_\varepsilon^2(0,0) = 0, \quad Z_\varepsilon^2(0,1) = 1,$$

that is

$$Z_\varepsilon(0,0) = P_0 := (0,0,0), \quad Z_\varepsilon(0,1) = P_1 := (0,1,0).$$

Fix some ε' with $0 < \varepsilon' \ll 1$. By an elementary construction we can find surfaces $Y_\varepsilon \in \mathcal{C}_\varepsilon$ such that $D_{B^+}(Y_\varepsilon) \leq \text{const}$ for $0 < \varepsilon < \varepsilon'$. Then $D_{B^+}(Z_\varepsilon) \leq \text{const}$ for $0 < \varepsilon < \varepsilon'$, and by the Courant–Lebesgue Lemma there exists a sequence of numbers $\varepsilon_j \in (0,1)$ with $\varepsilon_j \to 0$ such that the harmonic mappings Z_{ε_j} converge uniformly on $\overline{B^+}$ to some $Z_0 \in C^0(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3)$ which maps ∂B^+ monotonically onto the polygon Γ_0 with the three vertices P_0, Q_3, Q_4 such that

$$Z_0(u,0) = P_0 \quad \text{for } -1 \leq u \leq 1, \quad Z_0(w_3) = Q_3, \quad Z_0(w_4) = Q_4$$

and

$$Z(0,1) = P_1.$$

Now we reflect Z_ε^3 and Z_0^3 symmetrically at the u-axis, setting

$$\zeta_\varepsilon(u,v) := \begin{cases} Z_\varepsilon^3(u,v) & \text{for } v \geq 0, \\ Z_\varepsilon^3(u,-v) & \text{for } v \leq 0, \end{cases} \quad \zeta_0(u,v) := \begin{cases} Z_0^3(u,v) & \text{for } v \geq 0, \\ Z_0^3(u,-v) & \text{for } v \leq 0, \end{cases}$$

where $(u,v) \in \overline{B}$ and $B := \{(u,v)\colon u^2 + v^2 < 1\}$.

Then we consider the functions $h_\varepsilon, h_0 \in H_2^1(B) \cap C^0(\overline{B}) \cap C^2(B)$ which are harmonic in B and satisfy

$$h_\varepsilon|_{\partial B} = \zeta_\varepsilon|_{\partial B}, \quad h_0|_{\partial B} = \zeta_0|_{\partial B}.$$

By Dirichlet's principle,

$$D_B(h_\varepsilon) \le D_B(\zeta_\varepsilon) \quad \text{and} \quad D_B(h_0) \le D_B(\zeta_0),$$

and the equality sign holds if and only if $h_\varepsilon = \zeta_\varepsilon$ and $h_0 = \zeta_0$ respectively. The symmetry of ζ_ε and ζ_0 implies

$$D_B(\zeta_\varepsilon) = 2D_{B^+}(\zeta_\varepsilon) \quad \text{and} \quad D_B(\zeta_0) = 2D_{B^+}(\zeta_0).$$

Let $w^* = (u, -v)$ be the mirror point of $w = (u, v)$. Then

$$\zeta_\varepsilon(w) = \zeta_\varepsilon(w^*) \quad \text{and} \quad \zeta_0(w) = \zeta_0(w^*) \quad \text{for any } w \in \partial B.$$

Set

$$h_\varepsilon^*(w) := h_\varepsilon(w^*) \quad \text{and} \quad h_0^*(w) := h_0(w^*) \quad \text{for } w \in \overline{B}.$$

Then h_ε^*, h_0^* are continuous on $\overline{B}$, harmonic in B, and $h_\varepsilon^*|_{\partial B} = h_\varepsilon|_{\partial B}$, $h_0^*|_{\partial B} = h_0|_{\partial B}$. The maximum principle implies $h_\varepsilon^* = h_\varepsilon$ and $h_0^* = h_0$, and so

$$h_\varepsilon(u, -v) = h_\varepsilon(u, v), \quad h_0(u, -v) = h_0(u, v) \quad \text{for } (u, v) \in \overline{B}.$$

This in turn yields

$$D_B(h_\varepsilon) = 2D_{B^+}(h_\varepsilon), \quad D_B(h_0) = 2D_{B^+}(h_0),$$

therefore

$$D_{B^+}(h_\varepsilon) \le D_{B^+}(\zeta_\varepsilon), \quad D_{B^+}(h_0) \le D_{B^+}(\zeta_0),$$

and equality occurs if and only if $h_\varepsilon = \zeta_\varepsilon$ and $h_0 = \zeta_0$ respectively.

Furthermore, $Z_\varepsilon^3(-u, v) = -Z_\varepsilon^3(u, v)$ for $(u, v) \in \overline{B}$ yields

$$h_\varepsilon(-u, v) = -h_\varepsilon(u, v) \quad \text{and} \quad h_0(-u, v) = -h_0(u, v) \quad \text{for } (u, v) \in \partial B.$$

Then an analogous reasoning furnishes

$$h_0(-u, v) = -h_0(u, v) \quad \text{for } (u, v) \in \overline{B},$$

and so

$$h_0(0, v) = 0 \quad \text{for all } v \text{ with } |v| \le 1.$$

Moreover, Z_0 maps the quarter circle $\{(\cos\theta, \sin\theta)\colon 0 \le \theta \le \frac{\pi}{2}\}$ onto the polygonal subarc $P_0Q_3P_1$ of Γ_0, whence $h_0(u, v) \le 0$ on the boundary of the semidisk $S^+ := \{(u, v) \in B\colon u > 0\}$. The maximum principle yields $h_0(u, v) < 0$ in S^+, in particular

$$h_0(u,0) < 0 \quad \text{for } 0 < u < 1,$$

and similarly

$$h_0(u,0) > 0 \quad \text{for } -1 < u < 0.$$

Since $Z_{\varepsilon_j} \rightrightarrows Z_0$ on $\overline{B^+}$, it follows that $\zeta_{\varepsilon_j}|_{\partial B} \rightrightarrows \zeta_0|_{\partial B}$, and therefore $h_{\varepsilon_j} \rightrightarrows h_0$ on $\overline{B}$. Hence there are numbers $\varepsilon_0 > 0$ and $u_1^+, u_2^+, u_1^-, u_1^-$ such that $0 < u_1^+ < u_2^+ < 1$, $-1 < u_1^- < u_2^- < 0$, and

$$h_{\varepsilon_0}(u,0) < 0 \quad \text{for } u_1^+ < u < u_2^+, \quad h_{\varepsilon_0}(u,0) > 0 \quad \text{for } u_1^- < u < u_2^-.$$

Now we define a new harmonic mapping X_{ε_0} by

$$X_{\varepsilon_0} := (Z^1_{\varepsilon_0}, Z^2_{\varepsilon_0}, h_{\varepsilon_0}|_{\overline{B}^+})$$

which satisfies

$$X_{\varepsilon_0}(w_1) = Q_1^{\varepsilon_0}, \quad X_{\varepsilon_0}(w_2) = Q_2^{\varepsilon_0}, \quad X_{\varepsilon_0}(0,0) = P_0,$$

hence

$$h_{\varepsilon_0}(-1,0) = -\varepsilon_0, \quad h_{\varepsilon_0}(1,0) = \varepsilon_0, \quad h_{\varepsilon_0}(0,0) = 0.$$

Therefore X_{ε_0} is not monotonic on $\{(u,0)\colon -1 \le u \le 1\}$. Thus h_{ε_0} does not coincide with ζ_{ε_0}; hence $D_{B^+}(h_{\varepsilon_0}) < D_{B^+}(\zeta_{\varepsilon_0})$ and therefore

$$D_{B^+}(X_{\varepsilon_0}) < D_{B^+}(Z_{\varepsilon_0}).$$

We also note that X_ε is an admissible mapping for Shiffman's variational problem since it maps the subarcs of ∂B^+ between w_j and w_{j+1} into the straight lines through $Q_j^{\varepsilon_0}$ and $Q_{j+1}^{\varepsilon_0}$ (with $w_{j+4} := w_j$, $Q_{j+4}^{\varepsilon_0} := Q_j^{\varepsilon_0}$). Hence, for w_1, w_2, w_3, w_4 and Γ_{ε_0} the "Marx–Shiffman minimizer" furnishes a smaller value for D_{B^+} than the "Courant minimizer", and consequently $\Theta^*(t) < \Theta(t)$ for some $t \in T$ if we return to our original notation.

By a slight modification of the preceding reasoning one can construct "Lewerenz examples" Γ with more than four vertices. □

The elementary results that we so far have proved are taken from Lewerenz [1] and Heinz [19]. The following work is much more profound and rests on classical results by H. Poincaré, L. Schlesinger [1–4], and J. Plemelj [1] about the Riemann–Hilbert problem. Here we can only present the statements of Heinz's principal theorems without any proof.

The main result of [19] is

Theorem 1. *For* $t^* = (t^1, \dots, t^{N+3}) \in T^* := \{t^* \in \mathbb{R}^{N+3}\colon t^1 < \cdots < t^{N+3} < t^1 + 2\pi\}$ *we set* $X(u,v,t^*) := Z^*(t^*)(w)$, $w = (u,v)$, *where* $Z^*(t^*)$ *is the quasi-minimal surface (i.e. the Marx–Shiffman mapping), bounded by the polygon* Γ, *that belongs to* $t^* \in T^*$ *(see Definition 1 and Remark 1). Let* $t_0^* \in (t_0^1, \dots, t_0^{N+3}) \in T^*$, $\mathring{w}_k := \exp(i t_0^k)$ *for* $k = 1, \dots, N+3$ *and* $\mathring{w} = \mathring{u} + i\mathring{v} \mathrel{\hat{=}} (\mathring{u}, \mathring{v}) \in \overline{B}$ *with* $\mathring{w} \ne \mathring{w}_1, \dots, \mathring{w}_{N+3}$. *Then, in a sufficiently small neighborhood of* $(\mathring{u}, \mathring{v}, t_0^*)$, *the mapping* $X(u,v,t^*)$ *possesses a convergent power series expansion.*

In [21] Heinz also allowed the corners $Q_1, \dots, Q_{N+3}$ of Γ to vary under the assumption that none of the angles at $\mathring{Q}_1, \dots, \mathring{Q}_{N+3}$ of $\mathring{\Gamma}$ will be 90°. Then it turns out that $X(u, v, t^*, Q)$ can be expanded in a convergent power series of the variables (u, v, t^*, Q) near $(\mathring{u}, \mathring{v}, t_0^*, \mathring{Q})$ where $Q := (Q_1, \dots, Q_{N+3}), \mathring{Q} := (\mathring{Q}_1, \dots, \mathring{Q}_{N+3}), X(w_k, t^*, Q) = Q_k$ for $w_k = \exp(it^k)$ and $X(\mathring{w}_k, t_0^*, \mathring{Q}_k) = \mathring{Q}_k$ for $k = 1, \dots, N+3$.

In [20] (and with simplified proofs in [23]) Heinz proved analyticity of the Shiffman function $\Theta^*(t)$ in $t \in T$:

Theorem 2. *For $t \in T$ set $X = X(\cdot, t) = Z^*(t)$. Then one has:*

(i) *In $B \times T$ the mapping X satisfies*

$$w^2 X_w(w) \cdot X_w(w) = \frac{i}{8\pi} \sum_{k=1}^{N+3} R_k(t) \frac{w_k + w}{w_k - w}, \quad w_k := \exp(it^k),$$

where the $R_k(t)$ are real analytic in $t \in T$ and satisfy

$$\sum_{k=1}^{N+3} R_k(t) = 0 \quad \textit{and} \quad \sum_{k=1}^{N+3} w_k R_k(t) = 0.$$

(ii) *$\Theta^*(t)$ is real analytic in T, and*

$$\frac{\partial \Theta^*(t)}{\partial t^k} = R_k(t) \quad \textit{for } k = 1, \dots, N.$$

(iii) *$X(\cdot, t)$ is minimal surface (i.e. $\Delta X(\cdot, t) = 0$ and $X_w(\cdot, t) \cdot X_w(\cdot, t) = 0$ in B) if and only if $\nabla \Theta^*(t) = 0$.*

According to Heinz [21], also the function $\Theta^*(t, Q) = D(X(\cdot, t, Q))$ is real analytic in $t \in T$ and $Q \in \mathbb{R}^{3(N+3)}$ if we avoid angles of 90° at the vertices $\mathring{Q}_k$ of the polygon $\mathring{\Gamma}$ that is to be varied; in fact, it suffices that the angle at one of the vertices $\mathring{Q}_k$ is different from 90° (see Heinz [21], pp. 33–34).

In order to formulate further results it will be convenient to use the following notations:

$$\mathcal{M}(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon \Delta X = 0, X_w \cdot X_w = 0\}$$

is the class of disk-type minimal surfaces $X : B \to \mathbb{R}^3$ bounded by Γ, and $\mathcal{M}^*(\Gamma)$ is the subclass

$$\mathcal{M}^*(\Gamma) := \{X \in \overline{\mathcal{C}}(\Gamma) \colon \Delta X = 0, X_w \cdot X_w = 0\}$$

of $X \in \mathcal{M}(\Gamma)$ satisfying the three-point condition

$$(*) \qquad X(w_k) = Q_k \quad \text{for } k = N+1, N+2, N+3$$

and $w_{N+\nu} := \exp(it^{N+\nu})$, $1 \le \nu \le 3$, and $t^{N+\nu} = \frac{\pi}{2}(1+\nu)$. As always in the present context, Γ is a polygon in $\mathbb{R}^3$ with the "true vertices" $Q_1, Q_2, \ldots, Q_{N+3}$. By $\mathcal{S}(Q)$ we denote the class of quasi-minimal surfaces $X = Z^*(t)$ with "$\partial X \subset \Gamma_1' \cup \Gamma_2' \cup \cdots \cup \Gamma_{N+3}'$", precisely speaking:

$$\mathcal{S}(Q) := \{X\colon X = Z^*(t) \text{ for some } t \in T\}, \quad Q := (Q_1, \ldots, Q_{N+3}),$$

where $T = \{t = (t^1, \ldots, t^N)\colon 0 < t^1 < \cdots < t^N < \pi\}$ and $Z^*(t)$ is the Marx–Shiffman mapping for $t \in T$, i.e. the minimizer of D in the class $U^*(t)$ of surfaces $Y \in H_2^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ with $Y(\gamma_k) \subset \Gamma_k' = Q_k + \Gamma_k$, $k = 1, \ldots, N+3$. In particular, the elements $X \in \mathcal{S}(Q)$ satisfy the same 3-point condition ($*$) as the elements $X \in \mathcal{M}^*(\Gamma)$, and $X(w_k) = Q_k$ for $k = 1, \ldots, N+3$.

Finally we denote by $\mathcal{S}_k(Q)$ the class of quasi-minimal surfaces $X \in \mathcal{S}(Q)$ which are minimal surfaces, i.e.:

$$\begin{aligned} \mathcal{S}_{\mathcal{M}}(Q) &= \{X \in \mathcal{S}(Q)\colon X_w \cdot X_w = 0\} \\ &= \{X\colon X = Z^*(t) \text{ for some } t \in T \text{ with } \nabla\Theta^*(t) = 0\}. \end{aligned}$$

Then

$$\mathcal{M}^*(\Gamma) \subset \mathcal{S}_{\mathcal{M}}(Q) \subset \mathcal{S}(Q).$$

Now we want to define the notion of a *branch point* of a quasi-minimal surface and of its *branch point order*.

Proposition 3 (E. Heinz [19], Satz 2; [22], pp. 549–550; [23], pp. 385–386). *Let $X = X(\cdot, t) = Z^*(t) \in \mathcal{S}(Q)$, $t \in T$, $V := \{w_1, \ldots, w_{N+3}\}$. Then for any $\zeta \in \overline{B}$ there exist $A \in \mathbb{C}^3$ with $A \neq 0$, $\nu \in \mathbb{Z}$ with $\nu \ge 0$, and $\alpha \in (-1, 0]$ such that $X_w = \frac{1}{2}(X_u - iX_v)$ has the asymptotic representation*

$$X_w(w,t) = A \cdot (w-\zeta)^{\nu+\alpha} + o(|w-\zeta|^{\nu+\alpha}) \quad \textit{for } w \in \overline{B}, w \to \zeta. \tag{14}$$

Moreover, $\alpha = 0$ if $\zeta \notin V$.

This expansion is uniquely determined.

Definition 2. *One calls $\zeta \in \overline{B}$ a* **branch point** *of $X \in \mathcal{S}(Q)$ if $\nu > 0$, and $\nu = \nu(\zeta)$ is said to be the* **order** *of the branch point ζ. If $\zeta \in \overline{B}$ is not a branch point, we set $\nu(\zeta) = 0$. Clearly the set $\Sigma(X)$ of branch points $\zeta \in \overline{B}$ is finite, and for any $\zeta \in \overline{B}$ we have:*

$$\zeta \in \Sigma(X) \quad \textit{if and only if} \quad |X_w(w,\tau)| \to 0 \quad \textit{as } w \to \zeta,\ w \in \overline{B}.$$

The **total order of branch points of** *X will be called $\kappa(X)$; it is defined as*

$$\kappa(X) := \sum_{\zeta \in B} \nu(\zeta) + \frac{1}{2} \sum_{\zeta \in \partial B} \nu(\zeta). \tag{15}$$

In order to estimate $\kappa(X)$ for $X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$, we need one more definition:

Definition 3. *For $X \in X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one defines the* **Schwarz operator** *$S = S^Y : \mathrm{dom}(S) \to L_2(B)$ by $S := -\Delta + 2KE$ on its domain*

$$\mathrm{dom}\, S = \{\varphi \in \mathring{H}_2^1(B) \cap C^2(B) \colon S\varphi \in L_2(B)\},$$

where $E := |X_u|^2$, and K is the Gauss curvature of X. By $\ker S$ *we denote the kernel of S,*

$$\ker S := \{\varphi \in \mathrm{dom}\, S \colon S\varphi = 0\}.$$

Realizing that for any critical point t of Θ^* the pairing $\langle Y(\cdot,t), \cdot\rangle$ with the unit normal field $Y(\cdot,t) := |X_u \wedge X_v|^{-1}(X_u \wedge X_v)(\cdot,t)$ maps the vector space

$$V^t := \left\{ \sum_{k=1}^{N} c^k X_{t^k}(\cdot,t) \colon c = (c^1,\dots,c^N) \in \ker D^2\Theta^*(t) \right\}$$

onto the kernel of $S^{X(\cdot,t)}$ with

$$\dim\bigl\{\ker(\langle X(\cdot,t),\cdot\rangle)\bigr\} = 2\kappa(X(\cdot,t)) - \#\bigl\{e^{it^\ell} \in \Sigma(X(\cdot,t)) \colon 1 \le \ell \le N\bigr\}.$$

E. Heinz (cf. [22], p. 563, Satz 3) obtained the following fundamental result:

Theorem 3. *For $X = X(\cdot,t) = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$, $t \in T$, with the Schwarz operator $S = S^X$ one has*

$$\dim\ker S^X + \mathrm{rank}\, \nabla^2\Theta^*(t) + 2\kappa(X) = N, \tag{16}$$

where $\nabla^2\Theta^*(t)$ denotes then Hessian matrix of Θ^* at t:

$$\nabla^2\Theta^*(t) = \left(\frac{\partial^2\Theta^*(t)}{\partial t^j \partial t^k}\right)_{j,k=1,\dots,N}.$$

Corollary 3. *For $X = X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one has $\kappa(X) \le \frac{N}{2}$, and $\kappa(X) = \frac{N}{2}$ if and only if $\lambda = 0$ is not an eigenvalue of S^X and $\Theta^*_{t^j t^k}(t) = 0$ for $1 \le j,k \le N$.*

Corollary 4. *For $X = X(\cdot,t) \in \mathcal{S}_{\mathcal{M}}(Q)$, the Hessian matrix $\nabla^2\Theta^*(t)$ is invertible if $\lambda = 0$ is not an eigenvalue of S^X and X has no branch points in $\overline{B}$.*

What can one say about $\kappa(X)$ if $X = X(\cdot,t)$ merely is an element of $\mathcal{S}(Q)$, but not a minimal surface? E. Heinz [24] has found that in this case one still has

$$\kappa(X) \le \frac{N}{2}. \tag{17}$$

Furthermore, for any $N \in \mathbb{N}$ he constructed a closed simple polygon Γ in $\mathbb{R}^3$ with $N+3$ vertices $Q_1, \dots, Q_{N+3}$ and a surface $X = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ such that $\kappa(X) = \frac{N}{2}$. Then $\nabla \Theta^*(t) = 0$ and $\nabla^2 \Theta^*(t) = 0$.

There is a much sharper estimate for the total branch point order $\kappa(X)$ of a quasi-minimal surface $X = X(\cdot, t) \in \mathcal{S}(Q)$, due to F. Sauvigny [6], p. 300:

Let α_k be the angle $\alpha \in (-1, 0]$ appearing in the expansion (14) of $X_w(w,t)$ at the point $\zeta = w_k := \exp(it^k) \in \partial B$, $k = 1, 2, \dots, N+3$.

The Heinz expansion (14) can be written as

$$X_w(w,t) = (w-\zeta)^{\nu+\alpha} g(w) \quad \text{as } w \to \zeta,$$

with $g(\zeta) \neq 0$. This expansion has the companion

$$X_{ww}(w,t) = \frac{\beta}{w-\zeta} X_w(w,t) + (w-\zeta)^\beta g_w(w), \quad \beta := \nu + \alpha > -1.$$

The "normal component" Y^* of any $Y \in \mathbb{C}^3$ is

$$Y^* := Y - |X_w(w,t)|^{-2} \langle Y, X_{\overline{w}}(w,t) \rangle X_w(w,t).$$

Thus

$$X^*_{ww}(w) = (w-\zeta)^\beta g^*_w(w),$$

and consequently the curvature function

$$\Psi(w) := 2|X_w(w,t)|^{-2} \cdot |X^*_{ww}(w,t)| \tag{18}$$

is integrable on B. If $X_w \cdot X_w = 0$ (i.e. $X \in \mathcal{S}_{\mathcal{M}}(Q)$) one finds that $\Psi = -E \cdot K$, hence $\int_B \Psi \, du \, dv$ in this case is the total curvature $\int_X |K| \, dA = -\int_B EK \, du \, dv$ of X. Now we can formulate Sauvigny's result:

Theorem 4. *For any quasi-minimal surface $X = X(\cdot, t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one has*

$$2\pi \cdot [1 + \kappa(X)] = \pi \sum_{k=1}^{N+3} |\alpha_k| + \int_B \Psi \, du \, dv. \tag{19}$$

This is the analog of formula (19) in Section 2.11 of Vol. 2.

For $X = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ one can relate the second variation $\delta^2 A(X, Y)$ of the area A at X in normal direction $Y = \lambda \mathcal{W}^{-1}(X_u \wedge X_v)$, $\lambda \in C^1_0(B)$, $\mathcal{W} = |X_u \wedge X_v|$, to the Hessian matrix $\nabla^2 \Theta(t)$:

Theorem 5 (F. Sauvigny [5], pp. 180–181). *If $X = X(\cdot, t) = Z^*(t) \in \mathcal{S}_{\mathcal{M}}(Q)$ and $\delta^2 A(X, Y) \geq 0$ for all normal directions Y then $\nabla^2 \Theta^*(t)$ is positive semidefinite. If X has no branch points in $\overline{B}$ and X is* **strictly stable**, *that is, if*

$$\int_B (|\nabla \varphi|^2 + 2EK\zeta^2) \, du \, dv > 0$$

for all $\varphi \in \mathring{H}^1_2(B) \cap C^0(\overline{B})$, then $\nabla^2 \Theta^*(t)$ is positive definite.

In his paper [4], Sauvigny was even able to show that *the Morse index* $m(X)$ *of a mapping* $X = X(\cdot, t) \in \mathcal{S}_{\mathcal{M}}(Q)$, i.e. the number of negative eigenvalues of the Schwarz operator, *agrees with the Morse index of* $\nabla^2 \Theta^*(t)$, i.e. with the number of negative eigenvalues of this symmetric $N \times N$-matrix (cf. [2], p. 186, Theorem 3; a weaker version of this result was already formulated by I. Marx [1], without proof). Furthermore Sauvigny in [4] generalized Heinz's identity (16) (which only holds for surfaces in $\mathbb{R}^3$) to an inequality for surfaces and polygons in $\mathbb{R}^p$, namely,

$$\dim \ker \nabla^2 \Theta^*(t) \le \dim \ker S^X + 2\kappa(X). \tag{20}$$

For this purpose we note that most results discussed in this subsection can be carried over from $\mathbb{R}^3$ to $\mathbb{R}^p$ with $p > 3$, except for Theorem 3 and for the addendum to Theorem 1 (cf. Heinz [21]), which are restricted to $\mathbb{R}^3$.

We finally mention several other results for surfaces $X \in \mathcal{M}^*(\Gamma)$ bounded by polygons Γ:

Theorem 6 (F. Sauvigny [3]). *If Γ is an extreme, simple polygon of total curvature $k(\Gamma) < 4\pi$ then Γ bounds exactly one minimal surface, i.e. $\#\mathcal{M}^*(\Gamma) = 1$. This surface has no branch points in $\overline{B}$.*

As usual Γ is called *extreme* if it lies on the boundary of a compact convex set.

The *total curvature* $k(\Gamma)$ is defined as the sum $\eta_1 + \eta_2 + \cdots + \eta_{N+3}$ of the unoriented angles $\eta_k :\equiv \sphericalangle(\xi_{k-1}, \xi_k) \in (0, \pi)$.

The result above can be generalized to $\mathbb{R}^p$ with $p > 3$ if one replaces the assumption $k(\Gamma) < 4\pi$ by the stronger condition $k(\Gamma) < \frac{10\pi}{3}$ (cf. Sauvigny [3]). At last we quote three finiteness results:

Theorem 7 (R. Jakob [6–8]). *Let Γ be a simple, closed, extreme polygon Γ in $\mathbb{R}^3$. Then every immersed, stable minimal surface spanning Γ is an isolated point of $\mathcal{M}^*(\Gamma)$. In particular, $\mathcal{M}^*(\Gamma)$ contains only finitely many stable minimal surfaces without branch points.*

This result can be generalized in the following way:

Theorem 8 (R. Jakob [9]). *Let Γ be a simple, closed, extreme polygon in $\mathbb{R}^3$ whose angles at the corners are different from $\frac{\pi}{2}$. Then there exists a neighborhood $\mathcal{N}(\Gamma)$ of Γ in $\mathbb{R}^3$ and an integer $\mu(\Gamma)$ such that the number of immersed, stable minimal surfaces in $\mathcal{M}^*(\Gamma')$ is bounded by $\mu(\Gamma)$ for any simple closed polygon Γ' which is contained in $\mathcal{N}(\Gamma)$ and has as many vertices as Γ.*

Recently, R. Jakob [10] has also obtained the following generalization of Theorem 7:

Theorem 9 (R. Jakob [10]). *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed polygon having the following two properties: Firstly it has to bound only minimal surfaces* *without boundary branch points, and secondly its total curvature, i.e. the sum of the exterior angles* $\{\eta_k\}$ *at its* $N+3$ *vertices, has to be smaller than* 6π*. Then every immersed minimal surface spanning* Γ *is an isolated point of the space* $\mathcal{M}^*(\Gamma)$ *of all disk-type minimal surfaces spanning* Γ*, and in particular* Γ *can bound only finitely many immersed minimal surfaces of disk-type.*

Sketch of the proof: At first we prove that any immersed $X^* \in \mathcal{M}^*(\Gamma)$ is an isolated point of $\mathcal{M}^*(\Gamma)$ with respect to the $\|\cdot\|_{C^0(\bar{B})}$-norm. Hence we assume the contrary, i.e. the existence of some immersed minimal surface X^* and of some sequence $\{X_j\} \subset \mathcal{M}^*(\Gamma)$ satisfying $\|X_j - X^*\|_{C^0(\bar{B})} \to 0$. Now Heinz's formula (16) in Theorem 3 implies that the Schwarz operator S^X (cf. Definition 3) of any immersed minimal surface X which is a non-isolated point of $\mathcal{M}^*(\Gamma)$ must have a non-trivial kernel. There are two possibilities: Either 0 is the smallest eigenvalue of S^{X^*}, or 0 is the nth eigenvalue of S^{X^*} for some $n > 1$. In the first case X^* is a stable immersed minimal surface and therefore an isolated point of $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$ by Theorem 1.1 in Jakob [10]. Hence, only the second case can hold true here. Now let u_n be some eigenfunction of S^{X^*} corresponding to the nth eigenvalue $\lambda_n = 0$, for some $n > 1$. In this case it is known (cf. Theorem 2.3 in Jakob [10]) that the zero set of u_n is not empty and subdivides B into at least 2 disjoint nodal domains. Sauvigny's Gauss–Bonnet formula (19) in Theorem 4 yields especially under the requirement that the total curvature $\sum_{k=1}^{N+3} \eta_k$ of Γ, as defined below Theorem 6, is smaller than 6π:

$$2\pi - \int_B KE\,du\,dv = \pi \sum_{k=1}^{N+3} |\alpha_k| = \sum_{k=1}^{N+3} \eta_k < 6\pi, \tag{21}$$

thus $\int_B |KE|\,du\,dv < 4\pi$ for every immersed minimal surface $X = X(\cdot, t) \in \mathcal{M}^*(\Gamma)$. Here we have used the fact that the exterior angles η_k at the vertices of Γ coincide with the angles $-\pi\alpha_k$ of X, where α_k appears in the exponent of the leading summand in the asymptotic expansion (14) of $X_w(\cdot, t)$ about each e^{it_k} respectively, on account of the fact that particularly the points $\{e^{it_k}\}_{k=1,\dots,N+3}$ are not branch points of the immersed surface X. Thus there is at least one nodal domain D of u_n such that

$$\int_D |(KE)^*|\,du\,dv < 2\pi. \tag{22}$$

Now again by Theorem 2.3 in Jakob [10] there are two possibilities: (i) ∂D is a finite, disjoint union of piecewise analytic, closed Jordan curves, or (ii) ∂D is piecewise real analytic about each of its points with the exception of at most finitely many points $e^{it_{k_j}}$, for some subcollection $\{k_j\} \subset \{1, \dots, N+3\}$, about each of which ∂D fails to be representable as a graph of a Lipschitz-continuous function. We shall first examine case (i): The stability theorem of Barbosa and do Carmo [4], in its version for minimal surfaces with polygonal boundaries (cf. Theorem 2.4 in Jakob [10]), guarantees that the restriction $X^*|_D$ of X^* has to be even strictly stable, i.e. one has

$$\lambda_{\min}(S^{X^*|_D}) > 0 \tag{23}$$

for the smallest eigenvalue $\lambda_{\min}(S^{X^*|_D})$ of the Schwarz operator assigned to the restricted surface $X^*|_D$. But on the other hand, since there holds $S^{X^*}(u_n) = 0$ on B, the restriction $u_n|_D$ satisfies in particular

$$S^{X^*|_D}(u_n|_D) = 0 \quad \text{on } D$$

and is moreover of class $\mathring{H}^1_2(D) \cap C^\omega(D)$, because $u_n|_D$ vanishes identically on the piecewise real analytic boundary of D and is continuous on $\bar{B}$. Hence u_n is an eigenfunction of $S^{X^*|_D}$ corresponding to the eigenvalue 0, in contradiction to (23). In case (ii) the argument is slightly more involved. Firstly one has to prove that on the considered domain D there still exists some function $\phi^* \in S\mathring{H}^1_2(D)$ which minimizes the quadratic form

$$J^{X^*|_D}(\phi) := \int_D \{|\nabla\phi|^2 + 2(KE)^*\phi^2\}\, du\, dv,$$

assigned to $S^{X^*|_D}$, on the $L^2(D)$-sphere

$$S\mathring{H}^1_2(D) := \{\phi \in \mathring{H}^1_2(D) \colon \|\phi\|_{L^2(D)} = 1\}$$

of $\mathring{H}^1_2(D)$, and which satisfies

$$J^{X^*|_D}(\phi^*) = \lambda_{\min}(S^{X^*|_D}). \tag{24}$$

Moreover we need the following pointwise estimate of $|KE|$ due to Heinz (see (3.3) in Heinz [22] or (26) in Jakob [9]) about each of the points $e^{it_1}, \dots, e^{it_{N+3}}$:

$$|KE|(w) \le \text{const}(X, \Gamma)|w - e^{it_k}|^{-2+\alpha} \quad \text{for all } w \in \bar{B} \cap B_\delta(e^{it_k}) \setminus \{e^{it_k}\} \tag{25}$$

for $\delta < \frac{1}{2}\min_{k=1,\dots,N+3}\{|e^{it_k} - e^{it_{k-1}}|\}$ and for some $\alpha > 0$ depending only on Γ. Now combining (22), (25) and the absolute continuity of the Lebesgue integral we can infer the existence of some sufficiently small $\delta > 0$ such that the enlargement $\tilde{D} := D \cup \bigcup_j (B_\delta(e^{it_{k_j}}) \cap B)$ of D is a finitely connected domain whose boundary is a disjoint union of piecewise real analytic closed Jordan curves and still satisfies

$$\int_{\tilde{D}} |(KE)^*|\, du\, dv < 2\pi. \tag{26}$$

Hence, we can apply the above mentioned stability theorem, i.e. Theorem 2.4 in Jakob [10], to $X^*|_{\tilde{D}}$ and obtain

$$\lambda_{\min}(S^{X^*|_{\tilde{D}}}) > 0. \tag{27}$$

Next we extend ϕ^* onto $\tilde{D}$ by simply setting $\tilde{\phi}^*(w) = 0$ for $w \in \tilde{D} \setminus D$, obtaining $\tilde{\phi}^* \in S\mathring{H}_2^1(\tilde{D})$. Now since $\tilde{D}$ has a piecewise real analytic boundary we know that $\lambda_{\min}(S^{X^*|_{\tilde{D}}}) = \min_{S\mathring{H}_2^1(\tilde{D})} J^{X^*|_{\tilde{D}}}$. Combining this with (24) and (27) we achieve:

$$\begin{aligned} (28) \qquad \lambda_{\min}(S^{X^*|_D}) &= J^{X^*|_D}(\phi^*) = J^{X^*|_{\tilde{D}}}(\tilde{\phi}^*) \geq \min_{S\mathring{H}_2^1(\tilde{D})} J^{X^*|_{\tilde{D}}} \\ &= \lambda_{\min}(S^{X^*|_{\tilde{D}}}) > 0. \end{aligned}$$

But on the other hand we know that $u_n|_D \in H_2^1(D) \cap C^0(\bar{D})$, on account of $u_n \in H_2^1(B) \cap C^0(\bar{B})$, and that $u_n = 0$ on ∂D, from which one can deduce that $u_n|_D \in \mathring{H}_2^1(D)$. Since we also know that $u_n|_D \in C^\omega(D)$ and that $S^{X^*|_D}(u_n|_D) = 0$ on D, we obtain that $u_n|_D$ is an element of the domain of $S^{X^*|_D}$, and thus an eigenfunction of $S^{X^*|_D}$ corresponding to the eigenvalue 0, in contradiction to (28), which proves the first assertion of Theorem 9.

In order to derive from this the "finiteness statement" of Theorem 9, it suffices to show the closedness of the subset $\mathcal{M}_i^*(\Gamma)$ of immersed minimal surfaces within the compact space $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$. Thus let $\{X_j\}$ be a sequence in $\mathcal{M}_i^*(\Gamma)$ converging to some X^* in $\mathcal{M}^*(\Gamma)$. As in (21) we infer the constant value $\int_B |(KE)_j|\, dw \equiv \sum_{k=1}^{N+3} \eta_k - 2\pi < 4\pi$ for every j from Sauvigny's Gauss–Bonnet formula. Now, by Theorem 1 in Sauvigny [10], this is in fact a sufficient condition for the limit minimal surface X^* to be free of interior branch points again. Finally, since X^* is spanned by Γ, it must be free of boundary branch points as well, just by assumption on Γ. Hence X^* is an element of $\mathcal{M}_i^*(\Gamma)$, and consequently $\mathcal{M}_i^*(\Gamma)$ inherits the compactness of $(\mathcal{M}^*(\Gamma), \|\cdot\|_{C^0(\bar{B})})$.

Chapter 7

Graphs with Prescribed Mean Curvature

This chapter is devoted to nonparametric surfaces of prescribed mean curvature H, that is, to ***H*-surfaces** which can be represented as graphs over planar domains. Nonparametric minimal surfaces, i.e. graphs with $H = 0$, were already considered in Section 2.2, and the celebrated two-dimensional Bernstein theorem was described in Section 2.4. Generalizations of this result are presented in Volume 3 of this treatise.

One can find a wealth of theorems on nonparametric minimal surfaces and H-surfaces in the monographs of J.C.C. Nitsche [28], D. Gilbarg and N. Trudinger [1], U. Massari and M. Miranda [1], E. Giusti [4], as well as in the notes [8] of L. Simon, in his survey paper [9], and in his encyclopaedia article [17], IV. Clearly the abundance of this material deserves a thorough and comprehensive presentation which exceeds the scope of the present book. For this reason we merely describe some *existence and uniqueness results* for the *nonparametric Plateau problem* (i.e. the *Dirichlet problem*) for minimal surfaces and, more generally, for H-surfaces, which can be derived from the solution of the *parametric Plateau problem* for minimal surfaces, studied in Chapter 4, and for H-surfaces that will be treated in Vol. 2.

We shall base our investigations on the results of Chapter 5 concerning stable minimal- and H-surfaces, and so we will use the same notations as in Chapter 5. The discussion ends in Section 7.3 with a presentation of some basic estimates for nonparametric H-surfaces, namely Heinz's *maximal radius theorem*, Serrin's *maximal height theorem*, and *Finn's area estimate*. Furthermore a gradient estimate for nonparametric H-surfaces is derived. The section closes with an energy estimate for the difference of two solutions of the H-surface equation, which can be used to prove *unique* solvability of the H-surface equation even in cases when the classical maximum principle fails. An application of this estimate is a theorem about the removability of isolated singularities of nonparametric H-surfaces which generalizes *Bers's celebrated result* that isolated singularities of solutions for the minimal surface equation can be removed.

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The basic feature of this chapter is the Gaussian approach viewing graphs as regular parametric surfaces whose normals $N = (N^1, N^2, N^3)$ point into the upper hemisphere

$$S^2_+ := \{x \in \mathbb{R}^3 \colon \langle x, e\rangle > 0\}$$

where e denotes some unit vector in $\mathbb{R}^3$. Applying a rotation we can assume that $e = e_3 = (0, 0, 1)$, and then $N(B) \subset S^2_+$ means $N^3 > 0$.

7.1 H-Surfaces with a One-to-One Projection onto a Plane, and the Nonparametric Dirichlet Problem

In Section 4.9 Radó's result on minimal surfaces with a 1–1 projection onto a plane was presented, using H. Kneser's lemma. Now we take up these considerations following F. Sauvigny [1,2], and the textbook [16], where in Chapter XII, §9, the Dirichlet problem for the nonparametric H-surface equation is solved by a continuity method.

For the following we assume that $H(x, y, z)$ is a real-valued function on $\mathbb{R}^3$ of class $C^{1,\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$, satisfying

$$\sup_{\mathbb{R}^3} |H| \le h_0 \quad \text{and} \quad H_z(x, y, z) \ge 0 \quad \text{on } \mathbb{R}^3 \tag{1}$$

for some $h_0 \in (0, \infty)$. Set

$$r_0 = \frac{1}{2h_0}. \tag{2}$$

Let $\mathbb{R}^2$ be the x, y-plane with the points $p = (x, y)$. The Euclidean distance of two points $p = (x, y)$ and $p' = (x', y')$ is denoted by

$$|p - p'| := \sqrt{(x - x')^2 + (y - y')^2}.$$

The disk with radius $r > 0$ and center $p_0 = (x_0, y_0)$ is

$$B_r(p_0) := \{p \in \mathbb{R}^2 \colon |p - p_0| < r\}.$$

Specifically we introduce the disk

$$\Omega_0 := B_{r_0}(0) = \{p \in \mathbb{R}^2 \colon |p| < r_0\} \tag{3}$$

of radius r_0 about the origin, and the closed circular cylinder

$$\mathcal{Z} := \overline{\Omega}_0 \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 \colon (x, y) \in \overline{\Omega}_0,\ z \in \mathbb{R}\}. \tag{4}$$

Definition 1. (i) *A bounded open set Ω of $\mathbb{R}^2$ is called a* **Jordan domain** *if it is bounded by a closed Jordan curve.*

(ii) *A Jordan domain Ω in $\mathbb{R}^2$ with $0 \in \Omega \subset \Omega_0$ is said to be* $2h_0$**-convex** *if for every point $p' \in \partial\Omega$ there is a closed disk $S_0 := \overline{B}_{r_0}(p_0)$ such that*

$$\overline{\Omega} \subset S_0 \quad \textit{and} \quad p' \in \partial\Omega \cap \partial S_0. \tag{5}$$

We call S_0 a **support disk** *of Ω at the point $p' \in \partial\Omega$.*

Remark 1. A Jordan domain Ω with $0 \in \Omega \subset \Omega_0$ with $\partial\Omega \in C^{2,\alpha}, 0 < \alpha < 1$, is $2h_0$-convex if and only if the curvature κ of the positive-oriented boundary $\partial\Omega$ satisfies $\kappa(p) \geq 1/r_0 = 2h_0$ at each point $p \in \partial\Omega$.

Let Γ be a rectifiable closed Jordan curve in $\mathbb{R}^3$, and recall that $\mathcal{C}(\Gamma)$ denotes the class of surfaces $X : B \to \mathbb{R}^3$ bounded by Γ. We fix a *three-point condition*

$$X(\zeta_k) = Q_k \quad \text{for } k = 1, 2, 3, \tag{$*$}$$

with $\zeta_k = \exp(\frac{2\pi k}{3} i)$ and three given distinct points $Q_k \in \Gamma$, thereby expressing the orientation of Γ. As usual we denote by $\mathcal{C}^*(\Gamma)$ the class of surfaces $X \in \mathcal{C}(\Gamma)$ satisfying $(*)$.

Now we consider regular curves $\Gamma \in C^{3,\alpha}$ which lie as *graphs above the boundary $\partial\Omega$* of a $2h_0$-convex Jordan domain. This means the following: *There is a function $\gamma \in C^{3,\alpha}(\partial\Omega)$ above the boundary $\partial\Omega \in C^{3,\alpha}$ such that*

$$\Gamma = \{(p, \gamma(p)) \in \mathbb{R}^3 \colon p \in \partial\Omega\}. \tag{6}$$

Then we write:

$$\Gamma = \operatorname{graph} \gamma. \tag{7}$$

Furthermore we assume that $Q_k = (q_k, \gamma(q_k))$, $q_k \in \partial\Omega$, holds where q_1, q_2, q_3 induce a positive orientation of $\partial\Omega$.

Theorem 1. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with $0 \in \Omega \subset \Omega_0$ which is $2h_0$-convex, $\partial\Omega \in C^{3,\alpha}$, and suppose that $\Gamma \in C^{3,\alpha}$ is given as a graph γ for some $\gamma \in C^{3,\alpha}(\partial\Omega)$, whereas $H \in C^{1,\alpha}$ satisfies (1). Then there exists exactly one stable H-surface $X \in \mathcal{C}^*(\Gamma)$. This surface is an immersion and even an embedding of $\overline{\Omega}$ into $\mathbb{R}^3$, and it can be represented nonparametrically as* graph ζ, *where $\zeta \in C^{3,\alpha}(\overline{\Omega})$ is a solution of the boundary value problem*

$$\begin{aligned} &\mathcal{M}\zeta = 2H(\cdot, \zeta)(1 + |\nabla\zeta|^2)^{3/2} \quad \textit{in } \Omega, \\ &\zeta = \gamma \quad \textit{on } \partial\Omega, \end{aligned} \tag{8}$$

and $\mathcal{M}\zeta$ denotes the minimal surface operator

$$\mathcal{M}\zeta := (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1 + \zeta_x^2)\zeta_{yy}. \tag{9}$$

Proof. (i) Consider the vector field

$$Q(x,y,z) := \frac{1}{2}\Big(\int_0^x H(t,y,z)\,dt, \int_0^y H(x,t,z)\,dt, 0\Big),$$

satisfying $\operatorname{div} Q = H$ on $\mathcal{Z}$, and the associated functional

$$E(X) := \int_B \Big(\frac{1}{2}|\nabla X|^2 + 2[Q(X), X_u, X_v]\Big)\,du\,dv$$

defined by formula (4) of Section 5.3.

By minimizing E among all $X \in \mathcal{C}^*(\Gamma)$ with $X(\overline{B}) \subset \mathcal{Z}$ one obtains an H-surface X contained in $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $X(B) \subset \operatorname{int} \mathcal{Z}$ (cf. Gulliver and Spruck [1], Hildebrandt [10]; these results are described in Chapter 4 of Vol. 2). On account of 5.3, Theorem 1, *the H-surface X is stable* since it also minimizes

$$F(X) := \int_B (|X_u \wedge X_v| + 2[Q(X), X_u, X_v])\,du\,dv$$

in the class $\{X \in \mathcal{C}^*(\Gamma) : X(B) \subset \mathcal{Z}\}$ and satisfies $X(B) \subset \operatorname{int} \mathcal{Z}$.

(ii) Now we consider an arbitrary stable H-surface X of class $\mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ with $X(\overline{B}) \subset \mathcal{Z}$. We write

$$X(w) = (X^1(w), X^2(w), X^3(w)) = (f(w), X^3(w))$$

where $f : \overline{B} \to \mathbb{R}^2$ denotes the associated planar mapping

(10) $$f(w) := (X^1(w), X^2(w)), \quad w \in \overline{B}.$$

One realizes that $f|_{\partial B}$ maps ∂B homeomorphically onto $\partial\Omega$. We claim that

(11) $$f(B) \subset \Omega.$$ □

Otherwise we could find a point $\tilde{w} \in B$ with $f(\tilde{w}) \notin \Omega$. Then there is a support disk S_0 of Ω at some $p' \in \partial\Omega \cap \partial S_0$ such that $f(\tilde{w}) \notin \operatorname{int} S_0$, and $\Omega \subset \operatorname{int} S_0$. Let $S_0 = \overline{B}_{r_0}(p_0)$ and consider the family $\Phi(w,\lambda)$, $w \in \overline{B}$, $\lambda \in [0,1]$, of functions

$$\Phi(w,\lambda) := |f(w) - \lambda p_0|^2, \quad w \in \overline{B},$$

which satisfy

$$\Phi(w,\lambda) \le r_0^2 \quad \text{for } w \in \partial B \text{ and } 0 \le \lambda \le 1.$$

On account of $X(B) \subset \operatorname{int} Z$ we have

$$\Phi(w,0) < r_0^2 \quad \text{for } w \in B$$

whereas $\Phi(\tilde{w}, 1) > r_0^2$. Then there is a $\lambda^* \in (0,1)$ and a point $w^* \in B$ with

$$\Phi(w^*, \lambda^*) = r_0^2 \quad \text{and} \quad \Phi(w, \lambda^*) \le r_0^2 \quad \text{on } \overline{B}. \tag{12}$$

The conformality relation $X_w \cdot X_w = 0$ implies $|\nabla X^3|^2 \le |\nabla f|^2$, and so

$$\begin{aligned} \Delta\Phi(\cdot, \lambda^*) &= 2|\nabla f|^2 + 2\langle f - \lambda^* p_0, \Delta f\rangle \\ &\ge 2|\nabla f|^2 - 2|f - \lambda^* p_0||\Delta f| \ge 2|\nabla f|^2 - 2r_0|\Delta X| \\ &\ge 2|\nabla f|^2 - 2r_0 \cdot 2h_0|X_u \wedge X_v| \ge 2|\nabla f|^2 - 2|X_u||X_v| \\ &= 2|\nabla f|^2 - |\nabla X|^2 \ge 2|\nabla f|^2 - |\nabla f|^2 - |\nabla X^3|^2 \ge 0, \end{aligned}$$

that is,

$$\Delta\Phi(\cdot, \lambda^*) \ge 0 \quad \text{in } B. \tag{13}$$

By virtue of the maximum principle we infer from (12) and (13) that $\Phi(w, \lambda^*) \equiv r_0^2$ for $w \in \overline{B}$, which evidently is not true. Thus (11) is valid.

(iii) For each point $w' \in \partial B$ with the image $p' := f(w') \in \partial\Omega$ we consider the support disk $S_0 = \overline{B}_{r_0}(p_0)$ and define the auxiliary function $\Phi : \overline{B} \to \mathbb{R}^2$ defined by

$$\Phi(w) := |f(w) - p_0|^2$$

which satisfies

$$\Phi(w) \le r_0^2 \quad \text{in } \overline{B} \quad \text{and} \quad \Phi(w') = r_0^2.$$

By the same reasoning as before we have

$$\Delta\Phi \ge 0 \quad \text{in } B.$$

Then the boundary point lemma of E. Hopf yields

$$\frac{\partial\Phi}{\partial\nu}(w') = 2\left\langle f(w') - p_0, \frac{\partial f}{\partial\nu}(w')\right\rangle > 0 \tag{14}$$

for the derivative in direction of the exterior normal ν to ∂B at $w' \in \partial B$. This immediately implies

$$\frac{\partial X}{\partial\nu}(w') \ne 0 \quad \text{for all } w' \in \partial B, \tag{15}$$

and consequently the H-surface X has no boundary branch points.

Furthermore, Φ assumes its maximum at $w' \in \partial B$. Therefore

$$\frac{\partial\Phi}{\partial\tau}(w') = 0 \tag{16}$$

holds true where $\frac{\partial}{\partial\tau}$ denotes the tangential derivative to ∂B at w'.

Equation (16) implies

$$\left\langle f(w') - p_0, \frac{\partial f}{\partial \tau}(w') \right\rangle = 0, \quad p_0 = (x_0^1, x_0^2),$$

whence

$$X_\tau^1(w') = -\lambda[X^2(w') - x_0^2], \quad X_\tau^2(w') = \lambda[X^1(w') - x_0^1] \tag{17}$$

for some $\lambda \in \mathbb{R}$.

Because of

$$|X_\tau(w')|^2 = |X_\nu(w')|^2 > 0$$

and

$$|X_\tau^3(w')|^2 \le c|f_\tau(w')|^2$$

for some constant c we arrive at

$$|f_\tau(w')|^2 > 0. \tag{18}$$

Since f is positive-oriented it follows that (17) holds with some $\lambda > 0$, and we infer from (14) that the Jacobian

$$J_f = \det(f_u, f_v) = \det(f_\nu, f_\tau)$$

satisfies

$$J_f(w') = (X_\nu^1 X_\tau^2 - X_\tau^1 X_\nu^2)(w') = \frac{\lambda}{2}\Phi_\nu(w') > 0.$$

Thus we have found

$$J_f(w') > 0 \quad \text{for all } w' \in \partial B \tag{19}$$

which is equivalent to

$$N^3(w') = \langle N(w'), e_3 \rangle > 0 \quad \text{for all } w' \in \partial B. \tag{20}$$

Invoking the fundamental Theorem 2 of Section 5.3 on stable H-surfaces, we arrive at

$$N^3(w) = \langle N(w), e_3 \rangle > 0 \quad \text{for all } w \in \overline{B}. \tag{21}$$

(iv) Now we want to show that X has no branch points in B, using formula (21) and applying an index-sum argument to the mapping $f : \overline{B} \to \mathbb{R}^2$ (see Sauvigny [16], Chapter III).

We use the asymptotic expansion of an H-surface X at an interior branch point $w_0 \in B$ which is obtained by the Hartmann–Wintner technique (cf. Vol. 2, Chapter 3) and has the same form as for minimal surfaces: There is a vector $A \in \mathbb{C}^3$ with $A \neq 0$ and $A \cdot A = 0$, and an integer $n \ge 1$ such that

(22) $$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n) \quad \text{as } w \to w_0.$$

If w_0 is a regular point of X, i.e. if $X_w(w_0) \neq 0$, then the same formula holds with $n = 0$. As explained in Section 5.1, the normal

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v)$$

satisfies

$$\lim_{w \to w_0} N(w) = |a \wedge b|^{-1}(a \wedge b) = |a|^{-2}(a \wedge b),$$

where $A = a - ib$; $a, b \in \mathbb{R}^3 \setminus \{0\}$, $|a| = |b|$, $\langle a, b\rangle = 0$. Since $H \in C^{1,\alpha}$, it follows $X \in C^{3,\alpha}(B, \mathbb{R}^3)$, and by 5.1, Theorem 1, we have: N is of the class $C^{3,\alpha}(B, \mathbb{R}^3)$ and satisfies equation (12) of 5.1. Set

$$a := (a^1, a^2, a^3), \quad b = (b^1, b^2, b^3).$$

Then (21) yields

(23) $$a^1 b^2 - a^2 b^1 > 0.$$

We integrate the first two equations of (22),

$$\left.\begin{aligned} X_w^1(w) &= A^1(w - w_0)^n + o(|w - w_0|^n) \\ X_w^2(w) &= A^2(w - w_0)^n + o(|w - w_0|^n) \end{aligned}\right. \quad \text{as } w \to w_0,$$

$A^1 = a^1 - ib^1$, $A^2 = a^2 - ib^2$. This leads to

$$\begin{aligned} X^1(w) &= X^1(w_0) + \frac{1}{n+1}[A^1(w - w_0)^{n+1} + \overline{A}^1(\overline{w} - \overline{w}_0)^{n+1}] \\ &\quad + o(|w - w_0|^{n+1}), \\ X^2(w) &= X^2(w_0) + \frac{1}{n+1}[A^2(w - w_0)^{n+1} + \overline{A}^2(\overline{w} - \overline{w}_0)^{n+1}] \\ &\quad + o(|w - w_0|^{n+1}) \end{aligned}$$

as $w \to w_0$. Using polar coordinates r, φ with $w = w_0 + re^{i\varphi}$, it follows

$$\begin{aligned} X^1(w_0 + re^{i\varphi}) &= X^1(w_0) + \frac{2}{n+1}[a^1 \cos(n+1)\varphi + b^1 \sin(n+1)\varphi] r^{n+1} \\ &\quad + o(r^{n+1}), \\ X^2(w_0 + re^{i\varphi}) &= X^2(w_0) + \frac{2}{n+1}[a^2 \cos(n+1)\varphi + b^2 \sin(n+1)\varphi] r^{n+1} \\ &\quad + o(r^{n+1}) \end{aligned}$$

as $r \to 0$.

When $l : \mathbb{C} \to \mathbb{C}$ denotes the mapping given by the matrix

$$\frac{2}{n+1}\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix}, \tag{24}$$

we obtain for $f(w) = X^1(w) + iX^2(w)$ the expansion

$$f(w) = f(w_0) + l((w - w_0)^{n+1}) + o(|w - w_0|^{n+1}) \quad \text{as } w \to w_0. \tag{25}$$

From (23)–(25) we infer that

$$f(w) \neq f(w_0) \quad \text{for } 0 < |w - w_0| < \epsilon \ll 1.$$

Furthermore, the "topological index" $i(f, w_0)$ of f at w_0 is given by

$$i(f, w_0) = n + 1. \tag{26}$$

The mapping $f : B \to \mathbb{C}$ is open and satisfies (11): $f(B) \subset \Omega$. Since $f|_{\partial B}$ yields a homeomorphism of ∂B onto $\partial \Omega$ and $f \in C^0(\overline{B}, \mathbb{R}^2)$, $\mathbb{R}^2 \widehat{=} \mathbb{C}$, it follows that $f(B) = \Omega$. Then an arbitrarily chosen point $z^* \in \Omega$ has at least one and at most finitely many pre-images $w_1, \ldots, w_k$ in B, i.e.

$$f(w_\nu) = z^* \quad \text{for } \nu = 1, \ldots, k.$$

As $f|_{\partial B}$ is positive-oriented, the index-sum formula yields

$$\sum_{\nu=1}^{k} i(f, w_\nu) = 1$$

which together with (26) implies $k = 1$ and $i(f, w^*) = 1$ for $w^* := w_1$. Therefore $f|_B$ is a one-to-one mapping of B onto Ω, and (23)–(25) imply that the Jacobian $J_f(w^*)$ of f at $w^* \in B$ satisfies $J_f(w^*) > 0$. Thus $f|_B$ is a diffeomorphism from B onto Ω with $J_f(w) > 0$ for all $w \in B$, i.e. $f|_B$ is orientation preserving.

(v) Now we introduce $\zeta \in C^{3,\alpha}(\overline{\Omega})$ by

$$\zeta := X^3 \circ f^{-1}, \tag{27}$$

which solves the Dirichlet problem (8). Using (1): $H_z \geq 0$, the maximum principle implies that the solution of (8) is uniquely determined; see e.g. F. Sauvigny [16], Chapter VI, pp. 365–370, or Gilbarg–Trudinger [1]. Therefore, any two stable H-surfaces within the class $\{X \in \mathcal{C}^*(\Gamma) \colon X(B) \subset \mathcal{Z}\}$ coincide.

Remark 2. Mutatis mutandis, Theorem 1 remains valid if the bounding contour Γ is allowed to creep vertically along the z-axis finitely many times. The planar map f then possesses finitely many intervals of constancy on ∂B which correspond to the creeping intervals of $X|_{\partial B}$. However, the parametric H-surface X has no branch points on $\overline{B}$ and is uniquely determined within the class of stable H-surfaces $\in \mathcal{C}^*(\Gamma)$. The Dirichlet boundary values of $\zeta := X^3 \circ f^{-1}$ on $\partial \Omega$ jump finitely often. Even in this case one can verify the unique solvability of the Dirichlet problem (8) by an "energy method" due to J.C.C. Nitsche.

Remark 3. S. Hildebrandt and F. Sauvigny [4–7] have studied the phenomenon that minimal surfaces with a free boundary on a surface $\mathcal{S}$ having edges that may creep along such edges. This work is described in Vol. 3. Generalizations of these results to H-surfaces can be found in papers by F. Müller [5–11].

Via a simultaneous approximation of the "projection domain Ω" and the boundary values one can derive the following result from Theorem 1:

Theorem 2 (Nonparametric Dirichlet problem). *Let $\gamma \in C^0(\partial\Omega)$ be prescribed boundary values on a $2h_0$-convex Jordan domain Ω with $0 \in \Omega \subset \Omega_0$. Then the Dirichlet problem (8) possesses exactly one solution $\zeta \in C^0(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$.*

Proof. The uniqueness of a solution of (8) is proved in the same way as before, using the maximum principle. Another way to establish unique solvability of (8) is to apply Corollary 1 of Section 7.3.

Hence we only have to show the existence of a solution. This will be achieved with the aid of a suitable approximation procedure, approximating Ω by smoothly bounded Ω_n and $\gamma : \partial\Omega \to \mathbb{R}$ by smooth functions $\gamma_n : \partial\Omega_n \to \mathbb{R}$, and applying Theorem 1 to the "approximating problems"

$$\begin{aligned} \mathcal{M}\zeta_n &= 2H(\cdot,\zeta_n)(1+|\nabla\zeta_n|^2)^{3/2} \quad \text{in } \Omega_n, \\ \zeta_n &= \gamma_n \quad \text{on } \partial\Omega_n. \end{aligned} \tag{28}$$

Let us sketch this approach.

(i) First we construct a sequence $\{\Omega_n\}$ of $2h_0$-convex domains Ω_n with $\partial\Omega_n \in C^{3,\alpha}$ and $0 \in \Omega_n \subset \Omega$ such that

$$\operatorname{dist}(\partial\Omega_n, \partial\Omega) \to 0 \quad \text{as } n \to \infty \tag{29}$$

and

$$\operatorname{length}(\partial\Omega_n) \nearrow \operatorname{length}(\partial\Omega) \quad \text{as } n \to \infty$$

(see F. Sauvigny [1,2] for details). We can write

$$\partial\Omega = \omega(I), \quad \partial\Omega_n = \omega_n(I), \quad I := [0, 2\pi]$$

where ω and ω_n are 2π-periodic mappings $\mathbb{R} \to \mathbb{R}^2$ which provide monotonic, positive-oriented representations of $\partial\Omega$ and $\partial\Omega_n$ respectively such that $\omega \in \operatorname{Lip}(\mathbb{R}, \mathbb{R}^2)$, $\omega_n \in C^{3,\alpha}(\mathbb{R}, \mathbb{R}^2)$. Using polar coordinates about the origin, we can write ω_n and ω in the form

$$\omega_n(\theta) = (r_n(\theta)\cos\theta, r_n(\theta)\sin\theta), \quad \omega(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \tag{30}$$

where $r_n(\theta)$ and $r(\theta)$ are 2π-periodic. Because of (29) we can assume that

$$\omega_n(\theta) \rightrightarrows \omega(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty; \quad \text{equivalently:} \quad r_n(\theta) \rightrightarrows r(\theta). \tag{31}$$

$\square$

Since the $2h_0$-convex curve ω fulfills a chord-arc condition, we can choose the ω_n in such a way that the ω_n satisfy a uniform chord-arc condition, i.e. there is an $\epsilon > 0$ and an $M_0 > 0$ such that

$$\int_{\theta_1}^{\theta_2} |\dot{\omega}_n(\theta)| d\theta \le M_0 |\omega_n(\theta_2) - \omega_n(\theta_1)| \quad \text{for all } n \in \mathbb{N} \tag{32}$$
$$\text{and all } \theta_1, \theta_2 \in \mathbb{R} \text{ with } \theta_1 \le \theta_2 \text{ and } |\omega_n(\theta_1) - \omega_n(\theta_2)| \le \epsilon.$$

Now we interpret the boundary values $\gamma : \partial\Omega \to \mathbb{R}$ as a continuous, 2π-periodic function $\gamma(\theta)$ of the polar angle θ, and we approximate γ uniformly on $\mathbb{R}$ by 2π-periodic functions $\gamma_n(\theta)$, $\theta \in \mathbb{R}$, which are of class $C^{3,\alpha}(\mathbb{R})$:

$$\gamma_n(\theta) \rightrightarrows \gamma(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty. \tag{33}$$

Set

$$\psi_n(\theta) := (\omega_n(\theta), \gamma_n(\theta)), \quad \psi(\theta) := (\omega(\theta), \gamma(\theta)), \quad \theta \in \mathbb{R}. \tag{34}$$

Then we obtain the Jordan contours

$$\Gamma_n := \psi_n(I) \in C^{3,\alpha}, \quad \Gamma := \psi(I), \quad I = [0, 2\pi], \tag{35}$$

whose representations ψ_n and ψ satisfy

$$\psi_n(\theta) \rightrightarrows \psi(\theta) \quad \text{on } \mathbb{R} \text{ as } n \to \infty. \tag{36}$$

This yields the following *auxiliary statement*: For each $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that

$$|\psi_n(\theta_1) - \psi_n(\theta_2)| \le \epsilon \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R} \text{ with } |\theta_1 - \theta_2| \le \delta(\epsilon), \ n \in \mathbb{N}. \tag{37}$$

(ii) On account of Theorem 1 we obtain: For each $n \in \mathbb{N}$ there is an $X_n \in \mathcal{C}^*(\Gamma) \cap C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, satisfying

$$\Delta X_n = 2H(X_n) X_{n,u} \wedge X_{n,v} \quad \text{and} \quad X_{n,w} \cdot X_{n,w} = 0, \tag{38}$$

which admits an equivalent representation

$$Z_n(x,y) = (x, y, \zeta_n(x,y)), \quad (x,y) \in \Omega_n.$$

Here $\zeta_n \in C^{3,\alpha}(\overline{\Omega}_n)$ is a solution of the equation

$$\mathcal{M}\zeta_n = 2H(\cdot, \zeta_n)(1 + |\nabla\zeta_n|^2)^{3/2} \quad \text{in } \Omega_n, \tag{39}$$

which is obtained by

$$\zeta_n = X_n^3 \circ f_n^{-1}, \tag{40}$$

where $f_n : \overline{B} \to \mathbb{R}^2$ is a diffeomorphism from $\overline{B}$ onto $\overline{\Omega}_n$ with $f_n \in C^{3,\alpha}(\overline{B}, \mathbb{R}^2)$. By (33) we have

$$m_0 := \sup\{|\gamma_n(\theta)| \colon \theta \in \mathbb{R}, n \in \mathbb{N}\} < \infty.$$

Then it follows from Theorem 4 in Section 7.3 that

$$\sup_{\Omega_n} |\zeta_n| \le m_0 + h_0^{-1} \quad \text{for all } n \in \mathbb{N}, \tag{41}$$

and

$$\begin{aligned} D(X_n) &= A(X_n) = A(Z_n) \\ &\le 3 \operatorname{meas} \Omega + m_0[2h_0 \operatorname{meas} \Omega + \operatorname{length}(\partial\Omega)] =: c_0. \end{aligned} \tag{42}$$

(iii) Now we want to prove a result that will be used to prove equicontinuity of the sequence $\{X_n\}$. To this end we consider an arbitrary mapping $X = (X^1, X^2, X^3) \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ satisfying $X(B) \subset \Omega_0 \times \mathbb{R} = \operatorname{int} \mathcal{Z}$, $\mathcal{Z} = \overline{\Omega}_0 \times \mathbb{R}$, $\Omega_0 = B_{r_0}(0)$, $r_0 = (2h_0)^{-1}$, and

$$\Delta X = 2H(X) X_u \wedge X_v \quad \text{and} \quad X_w \cdot X_w = 0 \quad \text{in } B$$

with $\sup_{\mathbb{R}^3} |H| \le h_0$. Let $f := (X^1, X^2)$ be the associated planar mapping; it satisfies

$$f(B) \subset \Omega_0$$

and

$$|\nabla X|^2 \le 2|\nabla f|^2 \quad \text{in } B.$$

Lemma 1. *Let $0 < \epsilon < r_0$, $p^* \in \Omega_0$, $\Omega := \Omega_0 \cap B_\epsilon(p^*)$, G a subdomain of B, and suppose that $f(\partial G) \subset \overline{B}_\epsilon(p^*) = \{p \in \mathbb{R}^2 \colon |p - p^*| \le \epsilon\}$. Then we have*

$$f(G) \subset \Omega.$$

Proof. We essentially apply the same reasoning as in part (ii) of the proof of Theorem 1. Suppose that the assertion is not valid. Then there is a point $\tilde{w} \in G$ with $f(\tilde{w}) \notin \Omega$. Since Ω is $2h_0$-convex, there exists a support disk $S_0 = \overline{B}_{r_0}(p_0)$ at some point $p' \in \partial\Omega \cap \partial S_0$ such that $f(\tilde{w}) \notin B_{r_0}(p_0)$ and $\Omega \subset B_{r_0}(p_0)$. Set

$$\Phi(w, \lambda) := |f(w) - \lambda p_0|^2 \quad \text{for } w \in \overline{G} \text{ and } 0 \le \lambda \le 1.$$

For $w \in \partial G$ it follows that $|f(w) - p_0| \le r_0$ and $|f(w)| \le \epsilon$ whence

$$|f(w) - \lambda p_0| \le \lambda |f(w) - p_0| + (1 - \lambda)|f(w)| \le \lambda r_0 + (1 - \lambda)\epsilon < r_0,$$

and therefore

$$\Phi(w, \lambda) < r_0^2 \quad \text{for all } w \in \partial G \text{ and } \lambda \in [0, 1].$$

Furthermore, $f(G) \subset f(B) \subset \Omega_0 = B_{r_0}(0)$ implies

$$\Phi(w, 0) < r_0^2 \quad \text{for all } w \in G,$$

and $f(\tilde{w}) \notin B_{r_0}(p_0)$ yields

$$\Phi(\tilde{w}, 1) > r_0^2.$$

Then there exists some $\lambda^* \in (0,1)$ and some $w^* \in G$ with

$$\Phi(w^*, \lambda^*) = r_0^2 \quad \text{and} \quad \Phi(w, \lambda^*) \le r_0^2 \quad \text{for all } w \in \overline{G}.$$

By virtue of

$$\begin{aligned} \Delta\Phi(\cdot, \lambda^*) &= 2|\nabla f|^2 + 2\langle f - \lambda^* p_0, \Delta f\rangle \\ &\ge 2\{|\nabla f|^2 - |f - \lambda^* p_0||\Delta f|\} \ge 2\{|\nabla f|^2 - r_0|\Delta X|\} \\ &\ge 2\{|\nabla f|^2 - 2h_0 r_0 |X_u \wedge X_v|\} \\ &\ge 2\{|\nabla f|^2 - |X_u||X_v|\} \ge 2\{|\nabla f|^2 - \tfrac{1}{2}|\nabla X|^2\} \ge 0, \end{aligned}$$

the function $\Phi(\cdot, \lambda^*)$ is subharmonic in G and assumes its maximum at some point $w^* \in G$. This yields $\Phi(w, \lambda^*) \equiv r_0^2$ for all $w \in G$, a contradiction to $\Phi(w, \lambda^*) < r_0^2$ for all $w \in \partial G$. □

The next result is evident:

Lemma 2. *Let G be a subdomain of B such that $\operatorname{osc}_{\partial G} X \le \epsilon$. Then there is a point $P^* = (p^*, z^*) \in \mathcal{Z}$ such that*

$$X(\partial G) \subset K_\epsilon(P^*) := \{P \in \mathbb{R}^3 \colon |P - P^*| \le \epsilon\},$$

and, in particular, $f(\partial G) \subset \overline{B}_\epsilon(p^*)$.

For $P^* = (p^*, z^*) \in \mathbb{R}^2 \times \mathbb{R}$ we introduce the *spherical box* $N_{\epsilon,\mu}(P^*)$ with $0 < \epsilon < h_0^{-1}$ and $\mu > 0$ by

$$N_{\epsilon,\mu}(P^*) := \{P = (p, z) \in \mathbb{R}^2 \times \mathbb{R} \colon |p - p^*| \le \epsilon,\ |z - z^*| \le \mu + \eta(p - p^*, \epsilon)\}$$

with

$$\eta(p - p^*, \epsilon) := \sqrt{h_0^{-2} - |p - p^*|^2} - \sqrt{h_0^{-2} - \epsilon^2} \quad \text{for } h_0 > 0$$

and $\eta := 0$ for $h_0 = 0$.

If $h_0 > 0$, the boundary of $N_{\epsilon,\mu}(P^*)$ consists of the cylinder

$$\{(p, z) \in \mathbb{R}^3 \colon |p - p^*| = \epsilon, |z - z^*| \le \mu\}$$

and the two spherical caps

$$\begin{aligned} F^+_{\epsilon,\mu}(P^*) &:= \{(p, z) \in \mathbb{R}^3 \colon |p - p^*| \le \epsilon, z = z^* + \mu + \eta(p - p^*, \epsilon)\}, \\ F^-_{\epsilon,\mu}(P^*) &:= \{(p, z) \in \mathbb{R}^3 \colon |p - p^*| \le \epsilon, z = z^* - \mu - \eta(p - p^*, \epsilon)\}. \end{aligned}$$

Lemma 3. *We have*

$$K_\epsilon(P^*) \subset N_{\epsilon,\epsilon}(P^*) \subset K_{2\epsilon}(P^*)$$

for $h_0 = 0$ *as well as for* $h_0 > 0$ *provided that* $\epsilon < r_0 = \frac{1}{2}h_0^{-1}$.

Proof. The first inclusion is evident, and the second is evident for $h_0 = 0$, hence we have to verify it for $h_0 > 0$. We may assume that $P^* = 0$.

Suppose now $P = (p, z) \in N_{\epsilon,\epsilon}(0)$, i.e. $|p|^2 \le \epsilon^2$ and $|z| \le \epsilon + \eta(p, \epsilon)$. Then

$$\begin{aligned} |z| &\le \epsilon + \sqrt{h_0^{-2} - |p|^2} - \sqrt{h_0^{-2} - \epsilon^2} \le \epsilon + \sqrt{h_0^{-2}} - \sqrt{h_0^{-2} - \epsilon^2} \\ &\le \epsilon + \frac{h_0^{-2} - (h_0^{-2} - \epsilon^2)}{\sqrt{h_0^{-2}}} = (1 + \epsilon h_0)\epsilon < \frac{3}{2}\epsilon. \end{aligned}$$

Therefore,

$$|p|^2 + z^2 \le \epsilon^2 + \tfrac{9}{4}\epsilon^2 < 4\epsilon^2,$$

and so $P \in K_{2\epsilon}(0)$. $\square$

Lemma 4. *Let* $0 < \epsilon < r_0 = (2h_0)^{-1}$, *and suppose that* $\mathrm{osc}_{\partial G} X \le \epsilon$ *holds true for some subdomain* G *of* B. *Then we have*:

(i) *There is a point* $P^* = (p^*, z^*) \in \mathcal{Z}$ *such that*

$$X(\partial G) \subset K_\epsilon(P^*) \quad \text{and} \quad f(\partial G) \subset \overline{B}_\epsilon(p^*).$$

(ii) *We have* $f(G) \subset B_\epsilon(p^*)$.

(iii) *Finally we obtain*

$$X(G) \subset N_{\epsilon,\epsilon}(P^*) \subset K_{2\epsilon}(P^*).$$

Proof. Assertion (i) follows from Lemma 2, and (ii) is a consequence of Lemma 1. Because of Lemma 3 it suffices to prove $X(G) \subset N_{\epsilon,\epsilon}(P^*)$. If $h_0 = 0$, this is implied by the maximum principle for harmonic mappings. Thus we may assume $h_0 > 0$. By (ii) we have $|f(w) - p^*| < \epsilon$ for $w \in G$; therefore we only have to show

$$|X^3(w) - z^*| \le \epsilon + \eta(f(w) - p^*, \epsilon) \quad \text{for all } w \in \overline{G}.$$

If this were not true, we could find a number $\mu > \epsilon$ and some point $w' \in \overline{G}$ such that

$$|X^3(w') - z^*| = \mu + \eta(f(w') - p^*, \epsilon)$$

and

$$|X^3(w) - z^*| \le \mu + \eta(f(w) - p^*, \epsilon) \quad \text{for all } w \in \overline{G}.$$

Furthermore, we infer from $X(\partial G) \subset K_\epsilon(P^*) \subset N_{\epsilon,\epsilon}(P^*)$ that

$$(43) \qquad |X^3(w) - z^*| \le \epsilon + \eta(f(w) - p^*, \epsilon) \quad \text{for } w \in \partial G.$$

Thus we obtain $w' \in G$. Consequently, $X(\overline{G})$ either lies entirely below $F^+_{\epsilon,\mu}(P^*)$ or above $F^-_{\epsilon,\mu}(P^*)$ and touches the corresponding cap at some point $X(w')$ with $w' \in G$. It suffices to consider the first case. Then we have

$$X^3(w) - z^* \le \mu + \sqrt{h_0^{-2} - |f(w) - p^*|^2} - \sqrt{h_0^{-2} - \epsilon^2} \quad \text{for } w \in \overline{G}$$

and equality for $w = w'$. Setting

$$\Phi(w) := |f(w) - p^*|^2 + \left| X^3(w) - z^* - \mu + \sqrt{h_0^{-2} - \epsilon^2} \right|^2,$$

this means

$$\Phi(w) \le h_0^{-2} \quad \text{for all } w \in \overline{G} \quad \text{and} \quad \Phi(w') = h_0^{-2}, \quad w' \in G.$$

We have

$$|\Delta X| \le 2h_0 |X_u \wedge X_v| \le h_0 |\nabla X|^2$$

and

$$\Delta\Phi = 2|\nabla X|^2 + 2\langle Y, \Delta X\rangle$$

with

$$Y := \Big(f - p^*, X^3 - z^* - \mu + \sqrt{h_0^{-2} - \epsilon^2} \Big).$$

This yields

$$|Y(w)| = \sqrt{\Phi(w)} \le h_0^{-1} \quad \text{for } w \in \overline{G},$$

whence

$$\begin{aligned} \Delta\Phi &\ge 2|\nabla X|^2 - 2|Y||\Delta X| \\ &\ge 2|\nabla X|^2 - 2h_0^{-1} h_0 |\nabla X|^2 = 0 \quad \text{in } G. \end{aligned}$$

Thus Φ is subharmonic in G and satisfies

$$\Phi(w') = h_0^{-2} = \max_{\overline{G}} \Phi \quad \text{for some } w' \in G,$$

whence $\Phi(w) \equiv h_0^{-2}$ holds true for all $w \in \overline{G}$. This, however, is a contradiction to the property (43) which implies $\Phi(w) < h_0^{-2}$ for $w \in \partial G$ on account of $\epsilon < \mu$. □

(iv) Now we use (37), (42), and the Courant–Lebesgue lemma to make the oscillation $\operatorname{osc}_{\partial G} X_n$ of the X_n uniformly small for appropriate subdomains G of B whose boundaries are either circles or two-gons. By Lemma 4(iii), it follows that the X_n, $n \in \mathbb{N}$, are *equicontinuous* on $\overline{B}$. Furthermore, the f_n are *uniformly bounded* on B since $f_n(\overline{B}) \subset \overline{\Omega}_0$, and (40), (41) imply

$$\sup_{\overline{B}} |X_n^3| \le m_0 + h_0^{-1} \quad \text{for all } n \in \mathbb{N}.$$

(This can also be proved by a reasoning similar to (iii).) Thus the X_n are also *uniformly bounded* on $\overline{B}$. By Arzelà–Ascoli's theorem we may then assume that the X_n converge uniformly on $\overline{B}$ to some $X \in C^0(\overline{B}, \mathbb{R}^3)$, and on account of (42) we may also assume that

$$X_n \rightharpoonup X \quad \text{in } H_2^1(B, \mathbb{R}^3).$$

This implies $X \in \overline{\mathcal{C}}(\Gamma)$.

(v) From (38) we infer

$$|\Delta X_n| \le h_0 |\nabla X_n|^2 \quad \text{in } B \text{ for all } n \in \mathbb{N}.$$

In conjunction with $X_n(w) \rightrightarrows X(w)$ on $\overline{B}$, an a priori estimate due to E. Heinz yields:

For any $B' \subset\subset B$ there is a number $c(B') > 0$ such that

$$\sup_{B'} |\nabla X_n| \le c(B') \quad \textit{for all } n \in \mathbb{N} \tag{44}$$

holds true; cf. Vol. II, Section 2.2, Proposition 1.

Then we infer from (38) and (44) by a standard reasoning that

$$\|X_n\|_{C^{3,\alpha}(B', \mathbb{R}^3)} \le c^*(B', \alpha) \quad \text{for all } n \in \mathbb{N}$$

and all $B' \subset\subset B$, and we obtain $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^{3,\alpha}(B, \mathbb{R}^3)$ as well as $X_n \to X$ in $C^{3,\beta}(B', \mathbb{R}^3)$, $0 < \beta < \alpha$, for all $B' \subset\subset B$ and

$$\Delta X = 2H(X) X_u \wedge X_v \quad \text{in } B.$$

Moreover, (38) yields also

$$X_w \cdot X_w = 0 \quad \text{in } B.$$

Thus X is an H-surface of class $\mathcal{C}(\Gamma)$.

Let N and N_n be the normals of X and X_n respectively. From $N_n^3(w) > 0$ on B we infer

$$N^3(w) \ge 0 \quad \text{in } B, \tag{45}$$

and Theorem 1 in Section 5.1 yields

$$\Delta N + 2pN = -2\Lambda \operatorname{grad} H(X).$$

Since $H_z \ge 0$ it follows

$$\Delta N^3 + 2pN^3 \le 0. \tag{46}$$

Invoking a reasoning due to E. Heinz [5], Lemma 6, we infer from (45) and (46) that

$$N^3(w) > 0 \quad \text{in } B. \tag{47}$$

Another possibility to verify (47) is to invoke Moser's inequality (cf. Sauvigny [16], vol. 2, p. 369).

Now we proceed as in the proof of Theorem 1 and conclude that X has no branch points in B and that $f := (X^1, X^2)$ furnishes a homeomorphic mapping from $\overline{B}$ onto $\overline{\Omega}$ which is diffeomorphic from B onto Ω, and $f \in C^{3,\alpha}(B, \mathbb{R}^2)$. Then $\zeta := X^3 \circ f^{-1}$ is of class $C^0(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ and solves the Dirichlet problem (8).

Remark 4. Since $Z_n^3 := X_n^3 \circ f_n^{-1}$ and $f_n \rightrightarrows f$ in B, one can derive the equicontinuity of the X_n^3 from formula (9) in Section 7.3.

7.2 Unique Solvability of Plateau's Problem for Contours with a Nonconvex Projection onto a Plane

In this section we consider closed Jordan curves Γ in $\mathbb{R}^3$ which possess a one-to-one projection onto a closed Jordan curve $\underline{\Gamma}$ lying in a plane Π, which we identify with $\mathbb{R}^2$. The points in $\mathbb{R}^2$ are described by $p = (x, y)$, and $P = (x, y, z)$ denote the points in $\mathbb{R}^3$.

Radó's theorem states: *If $\underline{\Gamma}$ is convex then there exists exactly one minimal surface of class $\mathcal{C}^*(\Gamma)$, and this surface is nonparametric.* The existence follows from Theorem 2 in Section 7.1, and the uniqueness was proved in Section 4.9. Inspecting this proof, we realize that only planes were used as comparison surfaces for a given minimal surface $X \in \mathcal{C}(\Gamma)$ in order to derive a nonparametric representation

$$Z(x, y) = (x, y, \zeta(x, y)), \quad (x, y) \in \overline{\Omega},$$

of X. Now we shall substitute the plane by *Scherk's first surface* from Section 3.5.6, restricted to its fundamental domain (see also Sauvigny [16], pp. 272–273). This comparison surface leads to a new uniqueness theorem for Plateau's problem in the case that $H = 0$, established by F. Sauvigny [12]. To formulate this result we first repeat the definition of Scherk's surface in a form that we will use, and then we define the *Scherkian tongs* which will replace the ordinary half-space in our considerations.

Definition 1. *For each parameter value $a > 0$ we consider the open square $Q(a) := \{(x, y) \in \mathbb{R}^2 \colon |x|, |y| < \pi/(2a)\}$, where* Scherk's surface $\mathcal{S}(a)$ *is defined as the minimal graph*

$$\begin{gathered} \mathcal{S}(a) := \{(x, y, \sigma(x, y)) \colon (x, y) \in Q(a)\} \\ \textit{with } \sigma(x, y) := \frac{1}{a}[\log\cos(ax) - \log\cos(ay)]. \end{gathered} \tag{1}$$

Then $\sigma_x(x,y) = -\tan(ax)$, $\sigma_y(x,y) = \tan(ay)$; the surface element

$$\omega := \sqrt{1+\sigma_x^2+\sigma_y^2} \tag{2}$$

and the upwards pointing unit normal

$$\Sigma := (-\sigma_x/\omega, -\sigma_y/\omega, 1/\omega) \tag{3}$$

are given by

$$\begin{aligned} \omega(x,y) &= \{1+\tan^2(ax)+\tan^2(ay)\}^{1/2} \\ \Sigma(x,y) &= \omega^{-1}(x,y)(\tan(ax), -\tan(ay), 1) \end{aligned} \quad \text{for } (x,y) \in Q(a). \tag{4}$$

The intersection of $\mathcal{S}(a)$ and the x,z-plane is a principal-curvature line

$$\left(x, 0, \frac{1}{a}\log\cos(ax)\right), \quad |x| < \frac{\pi}{2a}, \tag{5}$$

with the oriented curvature

$$\kappa(x) = -a\cos(ax), \quad |x| < \frac{\pi}{2a}. \tag{6}$$

In the limit $a \to +0$ we obtain $\sigma(x,y) = 0$ and $Q(0) = \mathbb{R}^2$, i.e. the Scherkian surface tends to the x,y-plane $\{z = 0\}$.

Definition 2. *For all parameter values $a \geq 0$ we define the* Scherkian halfspace (*or* Scherkian tongs) $\mathcal{S}_+(a)$ *as the set*

$$\mathcal{S}_+(a) := \{(x,y,z) \in \mathbb{R}^3 \colon (x,y) \in Q(a),\ x > \sigma(y,z)\}$$

whose boundary is the Scherk surface

$$\begin{gathered} \partial\mathcal{S}_+(a) = \{(\sigma(y,z), y, z) \colon (y,z) \in Q(a)\} \\ \text{with } \sigma(y,z) = \frac{1}{a}[\log\cos(ay) - \log\cos(az)], \end{gathered}$$

which lies over the y,z-plane.

Rotating $\mathcal{S}_+(a)$ about the z-axis such that the plane vector $e_1 = (1,0,0)$ is transformed into the vector

$$\nu = (\nu_1, \nu_2, 0) \in S^1 \times \{0\},$$

and translating the origin $0 \in \mathbb{R}^3$ into the point $P_0 = (x_0, y_0, z_0)$ of $\mathbb{R}^3$, we obtain the general Scherkian halfspace (*or* tongs)

$$\mathcal{S}_+(a, P_0, \nu) \subset \mathbb{R}^3.$$

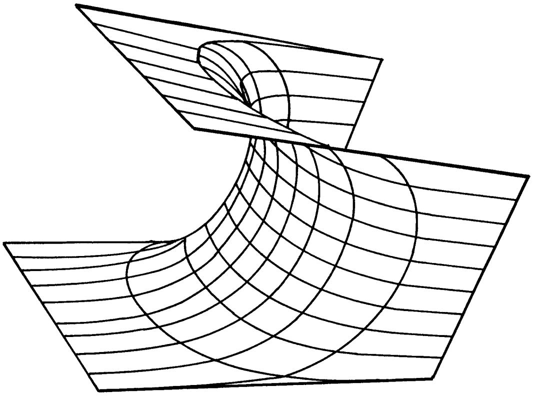

Fig. 1. Scherkian tongs

Note that the open set $\mathcal{S}_+(a, P_0, \nu)$ "emanates" from its boundary point $P_0 \in \partial\mathcal{S}_+(a, P_0, \nu)$ "in the direction ν" and possesses a square of side-length π/a as projection domain perpendicular to ν.

Now we can formulate the main result of this section, Sauvigny's uniqueness theorem.

Theorem 1. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with $\underline{\Gamma} := \partial\Omega \in C^{3,\alpha}$. Furthermore, consider boundary values $\gamma \in C^{3,\alpha}(\underline{\Gamma})$ and define the Jordan contour Γ in $\mathbb{R}^3$ by*

$$(7) \qquad \Gamma := \{(p, \gamma(p)) \in \mathbb{R}^3 \colon p \in \underline{\Gamma}\},$$

which has a 1–1 projection onto $\underline{\Gamma} = \partial\Omega$. Let $\nu : \partial\Omega \to S^1 \times \{0\}$ be the interior unit normal to $\partial\Omega$, and suppose that for each point $p_0 \in \partial\Omega$ there is a parameter value $a_0 = a(p_0)$ such that for $P_0 := (p_0, \gamma(p_0)) \in \Gamma$ and $\nu_0 := \nu(p_0)$ we have

$$(8) \qquad \Gamma \setminus \{P_0\} \subset \mathcal{S}_+(a_0, P_0, \nu_0).$$

As usual we fix a three-point condition $(*)$ on Γ and denote by $\mathcal{C}^*(\Gamma)$ the class of admissible surfaces $X : B \to \mathbb{R}^3$ satisfying $(*)$.

Then there exists exactly one minimal surface $X \in \mathcal{C}^*(\Gamma)$. This surface is a $C^{3,\alpha}$-immersion of $\overline{B}$ into $\mathbb{R}^3$ and possesses a nonparametric representation $(x, y, \zeta(x, y)), (x, y) \in \overline{\Omega}$, as graph of a solution $\zeta \in C^{3,\alpha}(\overline{\Omega})$ of the Dirichlet problem

$$(9) \qquad \mathcal{M}\zeta = 0 \quad \text{in } \Omega, \quad \zeta(p) = \gamma(p) \quad \text{on } \partial\Omega,$$

for the minimal surface equation.

The basic tool to be used in the proof of Theorem 1 is a *comparison principle* that will allow us to compare an arbitrary parametric minimal surface with one of Scherk's minimal graphs $\partial\mathcal{S}_+(a_0, P_0, \nu_0)$ as well as with other minimal graphs.

Theorem 2. *Let $X = (X^1, X^2, X^3) : B \to \mathbb{R}^3$ be a minimal surface with the associate planar mapping $f := (X^1, X^2) : B \to \mathbb{R}^2$, satisfying $f(B) \subset \Omega$, and the surface normal $N : B \to S^2$. Secondly, consider a solution $\eta \in C^2(\Omega)$ of the minimal surface equation $\mathcal{M}\eta = 0$ in some domain Ω of $\mathbb{R}^2$ with the normal $\Xi : \Omega \to S^2_+$ (= open upper hemisphere of S^2) and its pull-back*

$$T = (T^1, T^2, T^3) := \Xi \circ f : B \to S^2_+. \tag{10}$$

Then the auxiliary function

$$\Phi := X^3 - \eta(X^1, X^2) = X^3 - \eta \circ f \tag{11}$$

satisfies the elliptic differential equation

$$\frac{\partial}{\partial u}(T^3 \Phi_u) + \frac{\partial}{\partial v}(T^3 \Phi_v) - [e_3, T_v, N]\Phi_u - [T_u, e_3, N]\Phi_v = 0 \quad \text{in } B, \tag{12}$$

where $e_3 := (0, 0, 1)$, $T^3 = \langle T, e_3 \rangle$, and $[a, b, c]$ denotes the triple product $\langle a, b \wedge c \rangle$.

Proof. (i) We have

$$\Delta X = 0 \quad \text{in } B \tag{13}$$

as well as

$$N \wedge X_u = X_v, \quad N \wedge X_v = -X_u \quad \text{in } B. \tag{14}$$

Now we consider the reparametrization

$$Y := (f, \eta \circ f) = (X^1, X^2, \eta(X^1, X^2)) \tag{15}$$

of the minimal graph $(x, y, \eta(x, y)), (x, y) \in \Omega$. Since the mean curvature of Y is identically zero on B, we obtain the parameter-invariant equation

$$Y_u \wedge T_v + T_u \wedge Y_v = 0 \quad \text{in } B, \tag{16}$$

whatever the sign and the zero set of the Jacobian $J_f = X^1_u X^2_v - X^2_u X^1_v$ might be.

(ii) Set $W := \sqrt{1 + \eta_x^2 \circ f + \eta_y^2 \circ f}$. Because of

$$\Xi = \{1 + \eta_x^2 + \eta_y^2\}^{-1/2}(-\eta_x, -\eta_y, 1)$$

we have

$$T = W^{-1} \cdot (-\eta_x \circ f, -\eta_y \circ f, 1), \quad \text{i.e.} \quad T^3 = 1/W.$$

This leads to

$$
\begin{aligned}
\Phi_u &= -(\eta_x \circ f)X_u^1 - (\eta_y \circ f)X_u^2 + X_u^3 = W\langle T, X_u\rangle,\\
\Phi_v &= -(\eta_x \circ f)X_v^1 - (\eta_y \circ f)X_v^2 + X_v^3 = W\langle T, X_v\rangle.
\end{aligned} \tag{17}
$$

Differentiating $Y = (f, X^3 + [\eta \circ f - X^3])$ we obtain

$$
\begin{aligned}
Y_u &= X_u + [(\eta_x \circ f)X_u^1 + (\eta_y \circ f)X_u^2 - X_u^3]e_3,\\
Y_v &= X_v + [(\eta_x \circ f)X_v^1 + (\eta_y \circ f)X_v^2 - X_v^3]e_3.
\end{aligned}
$$

On account of (17), we then arrive at

$$
\begin{aligned}
Y_u &= X_u - \Phi_u e_3,\\
Y_v &= X_v - \Phi_v e_3.
\end{aligned} \tag{18}
$$

From (17) we infer that the expression

$$
L\Phi := \frac{\partial}{\partial u}(W^{-1}\Phi_u) + \frac{\partial}{\partial v}(W^{-1}\Phi_v) \tag{19}
$$

satisfies

$$
\begin{aligned}
L\Phi &= \frac{\partial}{\partial u}\langle T, X_u\rangle + \frac{\partial}{\partial v}\langle T, X_v\rangle\\
&= \langle T, \Delta X\rangle + \langle T_u, X_u\rangle + \langle T_v, X_v\rangle.
\end{aligned}
$$

In virtue of (13) and (14) we get

$$
L\Phi = [X_u, T_v, N] + [T_u, X_v, N],
$$

and (18) then yields

$$
\begin{aligned}
L\Phi &= [Y_u + \Phi_u e_3, T_v, N] + [T_u, Y_v + \Phi_v e_3, N]\\
&= \langle Y_u \wedge T_v + T_u \wedge Y_v, N\rangle + [e_3, T_v, N]\Phi_u + [T_u, e_3, N]\Phi_v.
\end{aligned}
$$

By (16) it follows that

$$
L\Phi = [e_3, T_v, N]\Phi_u + [T_u, e_3, N]\Phi_v. \tag{20}
$$

From (19), (20), and $T^3 = 1/W$ we finally obtain (12). □

If we apply Theorem 2 to $\eta := \sigma$, defined by (1), we find:

Corollary 1. *If $X = (f, X^3) : B \to \mathbb{R}^3$, is a minimal surface with the normal N satisfying $f(B) \subset Q(a)$, then $\Phi := X^3 - \sigma \circ f$ satisfies*

$$
\frac{\partial}{\partial u}(\Sigma^3\Phi_u) + \frac{\partial}{\partial v}(\Sigma^3\Phi_v) - [e_3, \Sigma_v, N]\Phi_u - [\Sigma_u, e_3, N]\Phi_v = 0 \quad \text{in } B. \tag{21}
$$

Now we turn to the

Proof of Theorem 1. We proceed in four steps. First we show that any minimal surface $X \in \mathcal{C}^*(\Gamma)$ "lies above $\overline{\Omega}$", that means, $f(B) \subset \Omega$. Secondly we prove that X meets the bounding Scherkian graphs transversally. In the third step we show that a minimizer of D in $\mathcal{C}^*(\Gamma)$ possesses a nonparametric representation above Ω. Finally we use the comparison principle of Theorem 2 to identify any minimal surface $X \in \mathcal{C}^*(\Gamma)$ with this minimal graph.

(i) *Step 1* (*Inclusion Principle*). We claim that

$$f(B) \subset \Omega. \tag{22}$$

To verify this assertion we pick an arbitrary point $p_0 = (x_0, y_0) \in \partial\Omega$, set $P_0 = (p_0, \gamma(p_0))$, $a_0 = a(p_0)$, $\nu_0 = \nu(p_0)$, and note that

$$\Gamma \setminus \{P_0\} \subset \mathcal{S}_+(a_0, P_0, \nu_0)$$

an account of assumption (8). We want to show that

$$X(B) \subset \mathcal{S}_+(a_0, P_0, v_0). \tag{23}$$

By a translation in z-direction and a rotation about the z-axis we arrange for $P_0 = 0$ and $\nu_0 = e_1 = (1, 0, 0)$, and so (8) in combination with the boundary condition $X(\partial B) = \Gamma$ takes on the form

$$X(\partial B \setminus \{w_0\}) \subset \mathcal{S}_+(a_0) \quad \text{with } p_0 = f(w_0),\ w_0 \in \partial B. \tag{24}$$

Consider the auxiliary function $\Psi \in C^{3,\alpha}(\overline{B})$ which is defined by

$$\Psi(w) := X^1(w) - \sigma(X^2(w), X^3(w)) \quad \text{for } w \in \overline{B}. \tag{25}$$

This function is built in the same way as the function Φ in Corollary 1, only that the z-direction is interchanged with the x-direction. Therefore it satisfies an elliptic differential equation in B since the Scherk surface $\mathcal{S}_+(a_0)$ lies as a graph over a square $\{(y, z) \colon |y|, |z| < \pi/(2a_0)\}$ in the y, z-plane. This equation is of the same kind as (21), and by (24) we have

$$\Psi(w) > 0 \quad \text{for all } w \in \partial B \setminus \{w_0\}, \quad \text{and} \quad \Psi(w_0) = 0. \tag{26}$$

Then the maximum (or, rather, the minimum) principle yields

$$\Psi(w) > 0 \quad \text{for all } w \in B. \tag{27}$$

Thus the assumption (24) implies

$$X(B) \subset \mathcal{S}_+(a_0).$$

If we return to the original assumption (8), we obtain (23) for all $p_0 \in \partial\Omega$, and so we arrive at (22).

(ii) *Step 2* (*Transversality at the Boundary*). In the situation (26) and (27), the boundary point lemma of E. Hopf implies

$$\frac{\partial}{\partial n_0}\Psi(w_0) < 0 \quad \text{for } w_0 \in \partial B \quad \text{and} \quad n_0 = w_0, \tag{28}$$

and we also have $X(w_0) = P_0 = 0$. Without loss of generality we may assume that $w_0 = (0,1)$. Then (28) states that the function $\Psi(u,v)$ satisfies

$$\Psi_v(0,1) < 0.$$

Furthermore, we have

$$\begin{aligned}\Psi_v(0,1) &= X_v^1(0,1) - \sigma_y(0,0)X_v^2(0,1) - \sigma_z(0,0)X_v^3(0,1)\\ &= X_v^1(0,1),\end{aligned}$$

and therefore

$$X_v^1(0,1) < 0,$$

whence

$$|X_u(0,1)| = |X_v(0,1)| > 0,$$

and a reasoning analogous to that in the proof of Theorem 1 in Section 7.1 yields

$$(X_u^1 X_v^2 - X_v^1 X_u^2)(0,1) > 0.$$

Performing a rotation of B we finally obtain

$$|X_u(w_0)| = |X_v(w_0)| > 0 \quad \text{for all } w_0 \in \partial B \tag{29}$$

and

$$J_f(w_0) = (X_u^1 X_v^2 - X_v^1 X_u^2)(w_0) > 0 \quad \text{for all } w_0 \in \partial B. \tag{30}$$

This implies for the normal $N = (N^1, N^2, N^3)$ of X the inequality

$$N^3(w_0) = \langle N(w_0), e_3\rangle > 0 \quad \text{for all } w_0 \in \partial B. \tag{31}$$

Consequently, X meets the bounding Scherkian graphs transversally.

(iii) *Step 3.* Now we take a minimizer $\tilde{X}$ of D, and therefore also of A, in $\mathcal{C}^*(\Gamma)$. Then its normal $\tilde{N}$ satisfies

$$\langle \tilde{N}(w), e_3\rangle = \tilde{N}^3(w) > 0 \quad \text{on } \overline{B}$$

on account of Section 5.3, Theorem 2. Via the arguments in parts (iv) and (v) of the proof the Theorem 1 in Section 7.1, we see that the plane mapping $\tilde{f} = (\tilde{X}^1, \tilde{X}^2)$ yields a positive-oriented diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and $\tilde{\zeta} := \tilde{X}^3 \circ \tilde{f}^{-1}$ solves the boundary value problem (9).

(iv) *Step 4.* At last, we consider an *arbitrary* minimal surface $X \in \mathcal{C}^*(\Gamma)$, which might be nonstable, and compare it with the minimal graph

$$\{(x, y, \tilde{\zeta}(x, y)) \colon (x, y) \in \overline{\Omega}\}$$

that was obtained in (iii). It satisfies as well the inclusion property (22) and the transversality relations (29)–(31). To identify X with $\tilde{X}$ we consider the auxiliary function

$$\Phi := X^3 - \tilde{\zeta}(X^1, X^2) \in C^2(\overline{B})$$

from Theorem 2, which fulfills the boundary condition

$$\Phi(w) = 0 \quad \text{for all } w \in \partial B.$$

Since Φ satisfies the elliptic equation (12), we conclude that

$$\Phi(w) \equiv 0 \quad \text{on } \overline{B} \quad \Leftrightarrow \quad X^3 = \tilde{\zeta}(X^1, X^2). \tag{32}$$

This implies in particular for $f = (X^1, X^2)$ that

$$X^3_u = \tilde{\zeta}_x(f) X^1_u + \tilde{\zeta}_y(f) X^2_u, \quad X^3_v = \tilde{\zeta}_x(f) X^1_v + \tilde{\zeta}_y(f) X^2_v,$$

from which we infer in virtue of (31) that

$$N = \{(1 + \tilde{\zeta}_x^2 + \tilde{\zeta}_y^2)^{-1/2}(-\tilde{\zeta}_x, -\tilde{\zeta}_y, 1)\} \circ f$$

and therefore $N^3(w) > 0$ on $\overline{B}$. Now we conclude as in Step 4 that $f = (X^1, X^2)$ yields a positive-oriented diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$, and $\zeta := X^3 \circ f^{-1}$ solves (9). On the other hand, the identity (32) is equivalent to $X^3 = \tilde{\zeta} \circ f$ whence $\tilde{\zeta} = X^3 \circ f^{-1} = \zeta$. Consequently X and $\tilde{X}$ can only differ by a conformal mapping φ from $\overline{B}$ onto itself, i.e. $X = \tilde{X} \circ \varphi$, and this implies $X = \tilde{X}$ since both surfaces fulfill the same three-point condition $(*)$. This completes the proof of the theorem. □

Remark 1. In the paper [12] by Sauvigny, boundary values $\gamma : \partial\Omega \to \mathbb{R}$ are explicitly investigated for nonconvex domains with $\partial\Omega \in C^{3,\alpha}$ such that (9) is solvable. These boundary values satisfy a Lipschitz condition with a Lipschitz constant less than one.

We note that, according to a result by Osserman and Finn (see Finn [9]), (9) cannot be solved for all boundary values $\gamma \in C^0(\partial\Omega)$ if Ω is nonconvex; a detailed discussion of the pertinent results can be found in the treatise by J.C.C. Nitsche [28], §§406–411, and also §§648–653. For special classes of boundary values, a solution of the nonparametric problem (9) for nonconvex Ω was also provided by C.P. Lau [1], F. Schulz and G. Williams [1], and G. Williams [1].

Remark 2. H. Wenk [1] improved the results of this section substituting Scherk's surface by the catenoid as comparison surface. This approach is more intricate; however, multiply connected minimal surfaces are then accessible.

7.3 Miscellaneous Estimates for Nonparametric H-Surfaces

In the sequel we assume that Ω is a bounded Jordan domain in $\mathbb{R}^2$, and that $H : \mathbb{R}^3 \to \mathbb{R}$ denotes a mean curvature function of class $C^{1,\alpha}(\mathbb{R}^3)$.

We consider solutions $\zeta \in C^{3,\alpha}$ of the *nonparametric mean curvature equation* where the mean curvature is the prescribed curvature function $H(x, y, z)$, i.e. we consider nonparametric surfaces

$$\mathcal{S} := \operatorname{graph} \zeta = \{(x, y, \zeta(x, y)) \in \mathbb{R}^3 \colon (x, y)) \in \overline{\Omega}\},$$

the height function $z = \zeta(x, y)$ of which satisfies

$$\text{(1)} \qquad \mathcal{M}\zeta(x, y) = 2H(x, y, \zeta(x, y))[1 + |\nabla\zeta(x, y)|^2]^{3/2} \quad \text{in } \Omega,$$

where $\mathcal{M}$ denotes the *minimal surface operator*

$$\text{(2)} \qquad \mathcal{M}\zeta = (1 + \zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1 + \zeta_x^2)\zeta_{yy}.$$

(Sometimes, weaker assumptions on H and X suffice.) We begin with the *Maximal Radius Theorem* due to E. Heinz [26] whose proof is almost elementary.

Theorem 1. *If there is a solution $\zeta \in C^2(\Omega)$ of (1) for a disk $\Omega = B_R(p_0)$ of radius $R > 0$, satisfying*

$$\text{(3)} \qquad \inf_{\Omega} |H(x, y, \zeta(x, y))| \geq \beta > 0$$

then it follows $R \leq 1/\beta$.

Proof. Condition (3) implies that either $H(\cdot, \zeta) > 0$ or $H(\cdot, \zeta) < 0$. The second case can be reduced to the first one by the reflection $(x, y, z) \mapsto (x, y, -z)$, and so we can assume that

$$H(x, y, \zeta(x, y)) \geq \beta > 0 \quad \text{for } (x, y) \in \Omega.$$

Let us write (1) in the form

$$\text{(4)} \qquad \frac{\partial}{\partial x}\left(\frac{\zeta_x}{\mathcal{W}}\right) + \frac{\partial}{\partial y}\left(\frac{\zeta_y}{\mathcal{W}}\right) = 2H(\cdot, \zeta) \quad \text{in } \Omega, \quad \mathcal{W} := \sqrt{1 + \zeta_x^2 + \zeta_y^2}.$$

Integrating both sides over the disk $B_r := B_r(p_0)$, $0 < r < R$, we obtain

$$\begin{aligned} 2\pi r^2\beta &\leq \int_{B_r} 2H(x, y, \zeta(x, y))\, dx\, dy \\ &= \int_{\partial B_r} \left(\frac{\zeta_x}{\mathcal{W}}\, dy - \frac{\zeta_y}{\mathcal{W}}\, dx\right) \\ &\leq \int_{\partial B_r} \mathcal{W}^{-1}|\nabla\zeta|\sqrt{dx^2 + dy^2} \leq \int_{\partial B_r} ds = 2\pi r \end{aligned}$$

whence $r \leq 1/\beta$ for all $r \in (0, R)$. Letting $r \to R - 0$ we arrive at $R \leq 1/\beta$. $\square$

One can estimate the supremum of $|\zeta|$ for solutions ζ of (1) by their boundary values on sufficiently small disks; cf. F. Sauvigny [16], Vol. 2, Chap. XII, §9, Proposition 1. This is achieved by comparing the solution with a spherical cap, a technique proposed by S. Bernstein. With the aid of *Bonnet's parallel surface* from Section 5.2 we now estimate the height of solutions of (1), even on arbitrary domains, by their boundary values assuming that $H = \text{const}$. This device was used earlier by H. Liebmann to show that ovaloids of constant mean curvature are necessarily spheres. J. Serrin rediscovered Bonnet's surface in his investigation of the so-called large solutions to Plateau's problem with constant $H > 0$. We now derive Serrin's *Maximal Height Theorem* (cf. J. Serrin [5]).

Theorem 2. *Let $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of* (1) *for $H = \text{const} > 0$ which satisfies*

$$|\zeta(x,y)| \le m \quad \textit{for all } (x,y) \in \partial\Omega \tag{5}$$

with a constant $m > 0$. Then ζ is estimated by

$$-m - \frac{1}{H} \le \zeta(x,y) \le m \quad \textit{for all } (x,y) \in \Omega. \tag{6}$$

Proof. (i) we introduce conformal parameters into $Z : \overline{\Omega} \to \mathbb{R}^3$, given by $Z(x,y) := (x, y, \zeta(x,y))$, $(x,y) \in \overline{\Omega}$, using a positive-oriented uniformization map $f : \overline{B} \to \overline{\Omega}$ which is a homeomorphism from $\overline{B}$ onto $\overline{\Omega}$ and furnishes a conformal mapping from B onto Ω; see Section 4.11, or Sauvigny [16], Chapter VII, §§7–8. Set $X = (X^1, X^2, X^3) := Z \circ f$, and let N be the unit normal of X and Λ its surface element. Then $N^3 \ge 0$, and so the equation $\Delta X = 2H X_u \wedge X_v$ implies

$$\Delta X^3 = 2H(X_u^1 X_v^2 - X_v^1 X_u^2) \ge 0 \quad \text{in } B.$$

Thus X^3 is subharmonic, and therefore $X^3 = \zeta \circ f \le m$ on ∂B satisfies $X^3 \le m$ on B whence $\zeta = X^3 \circ f^{-1} \le m$ on Ω.

(ii) By Theorem 2 of Section 5.2, the parallel surface $Y := X + \frac{1}{H} N$ is again an H-surface satisfying

$$\Delta Y = 2H Y_u \wedge Y_v = -2H\Lambda(H^2 - K)N \quad \text{in } B.$$

The auxiliary function $\Phi := Y^3 = \langle Y, e_3 \rangle$ satisfies

$$\Delta\Phi = -2H\Lambda(H^2 - K)N^3 \le 0 \quad \text{in } B,$$

and so it is superharmonic in B. Since

$$\Phi(w) \ge -m + \frac{1}{H} N^3(w) \ge -m \quad \text{for } w \in \partial B,$$

we obtain $\Phi(w) \ge -m$ for $w \in B$, which means that

$$X^3(w) \geq -m - \frac{1}{H} N^3(w) \geq -m - \frac{1}{H} \quad \text{for all } w \in B$$

holds true. □

Next we derive an *Area Estimate* for nonparametric H-surfaces that repeatedly appears in the work of R. Finn.

Theorem 3. *The area $A(Z) := \int_\Omega \sqrt{1 + |\nabla \zeta|^2}\, dx\, dy$ of an H-surface $Z(x, y) = (x, y, \zeta(x, y))$, corresponding to a solution $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of (1) with $\sup_{\overline{\Omega}} |H(x, y, \zeta(x, y))| \leq h_0$, is bounded by*

(7) $$A(Z) \leq \sup_{\partial\Omega} |\zeta| \cdot \operatorname{length}(\partial\Omega) + [1 + 2h_0 \sup_\Omega |\zeta|] \cdot \operatorname{meas} \Omega.$$

Proof. We shall verify (7) for domains Ω with a smooth boundary. Then the general result follows by approximation. Let us multiply (4) by ζ and integrate over Ω. Then

$$\begin{aligned} 2 \int_\Omega \zeta H(\cdot, \zeta)\, dx\, dy &= \int_\Omega \zeta [(\mathcal{W}^{-1} \zeta_x)_x + (\mathcal{W}^{-1} \zeta_y)_y]\, dx\, dy \\ &= \int_\Omega [(\mathcal{W}^{-1} \zeta \zeta_x)_x + (\mathcal{W}^{-1} \zeta \zeta_y)_y]\, dx\, dy - \int_\Omega \mathcal{W}^{-1} |\nabla \zeta|^2\, dx\, dy \\ &= \int_{\partial\Omega} \mathcal{W}^{-1} \zeta (\zeta_x\, dy - \zeta_y\, dx) - \int_\Omega \mathcal{W}\, dx\, dy + \int_\Omega \mathcal{W}^{-1}\, dx\, dy. \end{aligned}$$

This leads to

(8) $$\int_\Omega \mathcal{W}\, dx\, dy \leq \int_{\partial\Omega} |\zeta|\, ds + \operatorname{meas} \Omega + 2h_0 \int_\Omega |\zeta|\, dx\, dy$$

whence

$$\int_\Omega \mathcal{W}\, dx\, dy \leq \sup_{\partial\Omega} |\zeta| \cdot \operatorname{length}(\partial\Omega) + \left[1 + 2h_0 \sup_\Omega |\zeta|\right] \operatorname{meas} \Omega. \qquad \square$$

Let $\mu := \sup_{\partial\Omega} |\zeta|$ and

$$\eta^\pm := \pm\mu \pm \sqrt{h_0^{-2} - (x^2 + y^2)} \pm \sqrt{h_0^{-2} - r_0^2} \quad \text{with } 0 < r_0 \leq h_0^{-1}.$$

The functions η^+ and η^- are spherical caps over $\Omega_0 := B_{r_0}(0)$. If $0 \in \Omega \subset B_{r_0}(0)$ then

$$\left.\begin{aligned} \mathcal{M}\eta^+ &= -2h_0(1 + |\nabla\eta^+|^2)^{3/2} \leq 2H(\cdot, \eta^+)(1 + |\nabla\eta^+|^2)^{3/2} \\ \mathcal{M}\eta^- &= 2h_0(1 + |\nabla\eta^-|^2)^{3/2} \geq 2H(\cdot, \eta^-)(1 + |\nabla\eta^-|^2)^{3/2} \end{aligned}\right\} \text{ in } \Omega.$$

Assuming $H_z \geq 0$ we can deduce differential inequalities in Ω for $\phi^+ := \zeta - \eta^+$ and $\phi^- := \zeta - \eta^-$ (see e.g. Sauvigny [16], Vol. 1, Chapter VI, §2). Then the maximum principle yields

$$\eta^-(w) \le \zeta(w) \le \eta^+(w) \quad \text{for } w \in \Omega.$$

Therefore

$$|\zeta(w)| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} + \sqrt{h_0^{-2} - r_0^2} \quad \text{for all } w \in \Omega. \tag{9}$$

In the maximal situation $r_0 = h_0^{-1}$ we attain

$$|\zeta(w)| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} \quad \text{in } \Omega.$$

In conjunction with Theorem 3 we obtain:

Theorem 4. *Suppose that* $0 \in \Omega \subset \Omega_0 = B_{r_0}(0)$, $r_0 = 1/h_0$, $H_z(x, y, z) \ge 0$, $|H| \le h_0$, *and let* $\zeta \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *be a solution of* (1). *Then we have*

$$\sup_{\Omega} |\zeta| \le \sup_{\partial\Omega} |\zeta| + h_0^{-1} \tag{10}$$

and

$$\int_\Omega \sqrt{1 + |\nabla\zeta|^2}\, dx\, dy \le 3 \operatorname{meas} \Omega + [2h_0 \operatorname{meas} \Omega + \operatorname{length}(\partial\Omega)] \sup_{\partial\Omega} |\zeta|. \tag{11}$$

Theorem 5 (Gradient estimates for H-graphs). *Suppose that* $H \in C^{1,\alpha}(\mathbb{R}^3)$ *satisfies*

$$H_z(x, y, z) \ge 0 \quad \text{in } \mathbb{R}^3, \quad \sup_{\mathbb{R}^3} |H| \le h_0, \quad \sup_{\mathbb{R}^3} |\nabla H| \le h_1, \tag{12}$$

with positive constants h_0 *and* h_1. *Furthermore let* ζ *be a solution of* (1) *with* $\sup_\Omega |\zeta| \le M$ *for constant* $M > 0$. *Then there is a constant* $M_1 = M_1(h_0 R, h_1 R^2, M R^{-1}) > 0$, *depending only on the quantities* $h_0 R$, $h_1 R^2$, $M R^{-1}$, *such that*

$$|\nabla\zeta(p_0)| \le M_1 \tag{13}$$

holds true for any $p_0 \in \Omega$ *with* $B_R(p_0) \subset\subset \Omega$.

Proof. (i) Let $B_R := B_R(p_0) \subset\subset \Omega$; then by (7)

$$\int_{B_R} \sqrt{1 + |\nabla\zeta|}\, dx\, dy \le 2\pi R M + \pi R^2 + 2h_0 M \pi R^2. \tag{14}$$

Introduce conformal parameters for $Y(x, y) := (x, y, \zeta(x, y))$, $p = (x, y) \in B_R(p_0)$, via a uniformizing mapping $f \in C^{3,\alpha}(\overline{B}, B_R)$ with $f(0) = p_0 = (x_0, y_0)$, and set $X := Y \circ f \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$. Then the resulting H-surface X satisfies

$$D(X) = A(X) = A(Y) = \int_{B_R} \sqrt{1 + |\nabla\zeta|^2}\, dx\, dy,$$

and by (14) we obtain

$$2D(X) \le 4\pi RM + 2\pi R^2 + 4\pi h_0 MR^2. \tag{15}$$

(ii) Now we consider the *normalized plane mapping*

$$F(w) := R^{-1}(X^1(w) - x_0, X^2(w) - y_0), \quad w \in \overline{B}, \tag{16}$$

corresponding to $X(w) = (X^1(w), X^2(w), X^3(w))$. Clearly, $F \in C^{3,\alpha}(\overline{B}, \overline{B})$, $F(0) = 0$, and F is a diffeomorphism of $\overline{B}$ onto itself, which by (16) satisfies

$$\begin{aligned} 2D(F) &\le 2R^{-2}D(X) \\ &\le 4\pi MR^{-1} + 2\pi + 4\pi(MR^{-1})(h_0 R) =: \tau(h_0 R, MR^{-1}). \end{aligned} \tag{17}$$

Furthermore,

$$|\Delta F| \le R^{-1}|\Delta X| \le h_0 R^{-1}|\nabla X|^2 = h_0 R^{-1}(|\nabla X^1|^2 + |\nabla X^2|^2 + |\nabla X^3|^2),$$

and $X_w \cdot X_w = 0$ implies

$$|\nabla X^3|^2 \le |\nabla X^1|^2 + |\nabla X^2|^2 = R^2|\nabla F|^2.$$

This leads to

$$|\Delta F| \le 2h_0 R|\nabla F|^2 \quad \text{in } B. \tag{18}$$

Now we can apply a *distortion estimate* due to E. Heinz in the form derived in F. Sauvigny [16], Chap. XII, §5, formulae (29) and (28), using (17) and (18). This yields a number $\delta = \delta(h_0 R, MR^{-1}) \in (0,1)$ such that

$$|F(w)| \ge 1/2 \quad \text{for all } w \in \partial B_{1-\delta}(0), \tag{19}$$

and numbers $\vartheta(h_0 R, MR^{-1})$ and $\lambda(h_0 R, MR^{-1})$ such that

$$0 < \vartheta \le |\nabla F(w)| \le \lambda \quad \text{for all } w \in B_{1-\delta}(0). \tag{20}$$

(iii) Next we consider the auxiliary function $\Phi \in C^{2,\alpha}(\overline{B})$ defined by

$$\Phi := N^3 = \langle N, e_3 \rangle = \Lambda^{-1}(X_u^1 X_v^2 - X_u^2 X_v^1). \tag{21}$$

Since

$$\Lambda = 2^{-1}|\nabla X|^2 \le R^2|\nabla F|^2 \quad \text{and} \quad X_u^1 X_v^2 - X_u^2 X_v^1 = R^2 J_F,$$

we obtain

$$\Phi \ge |\nabla F|^{-2} J_F > 0 \quad \text{in } B, \tag{22}$$

where J_F is the Jacobian of F. On account of (19) it follows that $F(B_{1-\delta}(0)) \supset B_{1/2}(0)$; therefore

$$\int_{B_{1-\delta}(0)} J_F \, du \, dv = \operatorname{meas} F(B_{1-\delta}(0)) \geq \frac{1}{4}\pi.$$

Furthermore, (20) yields $|\nabla F(w)|^{-2} \geq \lambda^{-2}$ on $B_{1-\delta}(0)$, and so (22) implies

$$\int_{B_{1-\delta}(0)} \Phi \, du \, dv \geq \frac{\pi}{4}\lambda^{-2}(h_0 R, MR^{-1}). \tag{23}$$

By Theorem 1 of Section 5.1 we have

$$\Delta\Phi = -2p\Phi - 2\Lambda H_z(X)$$

with

$$p = 2\Lambda H^2(X) - \Lambda K - \Lambda\langle \operatorname{grad} H(X), N\rangle.$$

Then,

$$-2p = -2\Lambda[2H^2(X) - K] + 2\Lambda\langle \operatorname{grad} H(X), N\rangle \leq 0 + 2\Lambda h_1$$

and

$$-2\Lambda H_z(X) \leq 0.$$

Consequently we have

$$\begin{aligned} \Delta\Phi &\leq 2(R^{-2}\Lambda)(h_1 R^2)\Phi \leq 2|\nabla F|^2(h_1 R^2)\Phi \\ &\overset{(20)}{\leq} 2\lambda^2(h_0 R, MR^{-1})(h_1 R^2)\Phi \quad \text{on } B_{1-\delta}(0). \end{aligned}$$

Setting

$$\sigma(h_0 R, MR^{-1}, h_1 R^2) := 2\lambda^2(h_0 R, MR^{-1})(h_1 R^2),$$

we arrive at

$$\Delta\Phi \leq \sigma\Phi \quad \text{in } B_{1-\delta}(0).$$

(iv) Now we apply a quantitative version of Moser's inequality that in two dimensions had already been proved by E. Heinz [5], Lemma 6′ on p. 216; see also F. Sauvigny [16], Chap. X, §5, Theorem 1. This yields

$$\Phi(0) \geq \exp\left(-\frac{1}{4}(1-\delta)^2\sigma\right)[\pi(1-\delta)^2]^{-1}\int_{B_{1-\delta}(0)} \Phi \, du \, dv.$$

If we use (23) and define $M_1(h_0 R, h_1 R^2, MR^{-1})$ by

$$M_1^{-1} := \exp\left(-\frac{1}{4}(1-\delta)^2\sigma\right)[\pi(1-\delta)^2]^{-1}\frac{\pi}{4}\lambda^{-2}(h_0 R, MR^{-1}),$$

it follows that

$$\Phi(0) \geq 1/M_1.$$

On account of

$$\Phi(0) = N^3(0) = (1 + |\nabla\zeta(p_0)|^2)^{-1/2}$$

we obtain

$$|\nabla\zeta(p_0)| \leq \sqrt{1 + |\nabla\zeta(p_0)|^2} = 1/\Phi(0) \leq M_1,$$

which gives the desired gradient estimate (13). □

Remark 1. This proof of the gradient estimate is due to F. Sauvigny [11]. We note that only the estimate of $|\nabla F|$ from above by λ in (20) was used to derive a bound for $|\nabla\zeta(p_0)|$. If one wants to obtain curvature estimates then the lower bound by ϑ in (20) is needed as well. In Sauvigny [7,8], curvature estimates are derived for solutions ζ of (1), without assuming the monotonicity condition $H_z \ge 0$.

Remark 2. If $H(x,y,z) \equiv \text{const}$, then the graph of a solution ζ of (1) represents a stable *cmc*-surface, and Section 5.5 yields an estimate for the principal curvatures in this class.

Now we prove an estimate for the difference of two solutions of (1), using a similar idea as in the proof of Theorem 2. For $H = 0$, the estimate was derived by J.C.C. Nitsche (see [28], §585). It can be applied to prove uniqueness of solutions to the Dirichlet problem for (1) with discontinuous boundary values.

Theorem 6. *Let Ω be a Jordan domain in $\mathbb{R}^2$ with a rectifiable boundary $\partial\Omega$, and suppose that $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ are two solutions of (1). Then, for any compact subset Q of Ω and with*

$$\mu(Q) := \max\Big\{\max_Q \sqrt{1+|\nabla\zeta_1|^2}, \max_Q \sqrt{1+|\nabla\zeta_2|^2}\Big\} \tag{24}$$

we have

$$\int_Q |\nabla\zeta_1 - \nabla\zeta_2|^2\,dx\,dy \le 2\mu^3(Q)\int_{\partial\Omega} |\zeta_1 - \zeta_2|\,ds, \tag{25}$$

provided that $H_z(x,y,z) \ge 0$ on $\mathbb{R}^3$.

Proof. (i) Let $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be two solutions of (1), and set

$$p_j := \frac{\partial\zeta_j}{\partial x}, \quad q_j := \frac{\partial\zeta_j}{\partial y}, \quad W_j = \sqrt{1+p_j^2+q_j^2}, \quad j = 1,2.$$

By (4) we have

$$\frac{\partial}{\partial x}\left(\frac{p_j}{W_j}\right) + \frac{\partial}{\partial y}\left(\frac{q_j}{W_j}\right) = 2H(\cdot,\zeta_j) \quad \text{in } \Omega, \quad j = 1,2.$$

This leads to

$$\frac{\partial}{\partial x}\left(\frac{p_2}{W_2} - \frac{p_1}{W_1}\right) + \frac{\partial}{\partial y}\left(\frac{q_2}{W_2} - \frac{q_1}{W_1}\right) = 2H(\cdot,\zeta_2) - 2H(\cdot,\zeta_1).$$

If we multiply this equation by $\zeta_2 - \zeta_1$, integrate over $\Omega' \subset\subset \Omega$, and perform an integration by parts, we obtain

$$
\begin{aligned}
&-\int_{\Omega'}\left[(p_2-p_1)\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)+(q_2-q_1)\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)\right]dx\,dy\\
&\quad+\int_{\partial\Omega'}(\zeta_2-\zeta_1)\left[\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)dy-\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)dx\right]\\
&=\int_{\Omega'}2(\zeta_2-\zeta_1)[H(\cdot,\zeta_2)-H(\cdot,\zeta_1)]dx\,dy,
\end{aligned}
$$

provided that $\partial\Omega'$ is piecewise smooth.

(ii) The boundary integral is estimated by

$$\left|\int_{\partial\Omega'}(\zeta_2-\zeta_1)[\ldots]\right|\le 2\int_{\partial\Omega'}|\zeta_2-\zeta_1|\,ds,$$

and we observe

$$H(x,y,z_2)-H(x,y,z_1)=H_z(x,y,\tilde z)(z_2-z_1)$$

with an intermediate value $\tilde z$. Since $H_z\ge 0$, we obtain

$$(z_2-z_1)\cdot[H(x,y,z_2)-H(x,y,z_1)]=H_z(x,y,\tilde z)(z_2-z_1)^2\ge 0,$$

and therefore

$$\int_{\Omega'}2(\zeta_2-\zeta_1)[H(\cdot,\zeta_2)-H(\cdot,\zeta_1)]\,dx\,dy\ge 0.$$

Thus we arrive at

$$\int_{\Omega'}\left[(p_2-p_1)\left(\frac{p_2}{W_2}-\frac{p_1}{W_1}\right)+(q_2-q_1)\left(\frac{q_2}{W_2}-\frac{q_1}{W_1}\right)\right]dx\,dy\le 2\int_{\partial\Omega'}|\zeta_2-\zeta_1|\,ds.$$

(iii) For $0\le t\le 1$ we set

$$p(t):=p_1+t(p_2-p_1),\quad q(t):=q_1+t(q_2-q_1),\quad W(t):=\{1+p(t)^2+q(t)^2\}^{1/2},$$

$$f(t):=(p_2-p_1)\left[\frac{p(t)}{W(t)}-\frac{p_1}{W_1}\right]+(q_2-q_1)\left[\frac{q(t)}{W(t)}-\frac{q_1}{W_1}\right].$$

Note that $f(0)=0$. By the mean value theorem there is a value $t=t(x,y)\in(0,1)$ with $f(1)=f'(t)$, and a brief calculation yields

$$W''(t)=W^{-3}(t)\{|p'(t)|^2+|q'(t)|^2+[p(t)q'(t)-q(t)p'(t)]^2\}=f'(t),$$

whence

$$f'(t)\ge W^{-3}(t)[(p_2-p_1)^2+(q_2-q_1)^2]$$

and

$$W''(t)\ge 0.$$

Therefore

$$W(t) \le \max\{W_1, W_2\} \quad \text{for } 0 \le t \le 1,$$

and consequently

$$f(1) = f'(t) \ge (\max\{W_1, W_2\})^{-3}[(p_2 - p_1)^2 + (q_2 - q_1)^2].$$

(iv) Now we choose an arbitrary compact set Q in Ω and then an open set Ω' with $\partial\Omega' \in C^1$ and $Q \subset \Omega' \subset\subset \Omega$; set

$$\mu(Q) := \max\{W_1(x,y), W_2(x,y) \colon (x,y) \in Q\}.$$

Then $D_Q(\zeta_2 - \zeta_1) := \frac{1}{2}\int_Q |\nabla\zeta_2 - \nabla\zeta_1|^2\, dx\, dy$ is estimated by

$$D_Q(\zeta_2 - \zeta_1) \le \frac{1}{2}\mu^3(Q) \int_Q f(1)\, dx\, dy \le \mu^3(Q) \int_{\partial\Omega'} |\zeta_2 - \zeta_1|\, ds.$$

Approximating Ω from the interior by domains $\Omega' \subset\subset \Omega$ such that $\Omega' \nearrow \Omega$ and length $(\partial\Omega') \to$ length$(\partial\Omega)$, we find

$$D_Q(\zeta_2 - \zeta_1) \le \mu^3(Q) \int_{\partial\Omega} |\zeta_2 - \zeta_1|\, ds. \qquad \square$$

Corollary 1. *If $\zeta_1, \zeta_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ are two solutions of (1) in a Jordan domain Ω with a rectifiable boundary which satisfy $\zeta_1 = \zeta_2$ on $\partial\Omega$, then we have $\zeta_1 = \zeta_2$.*

Proof. The estimate (25) implies $\nabla\zeta_1|_Q = \nabla\zeta_2|_Q$ for any compact Q in Ω, whence $\nabla\zeta_1(p) = \nabla\zeta_2(p)$ for all $p \in \Omega$, and therefore $\zeta_1 - \zeta_2 = \text{const}$. Since $\zeta_1(p) = \zeta_2(p)$ for $p \in \partial\Omega$, we obtain $\zeta_1 = \zeta_2$. $\square$

Remark 3. J.C.C. Nitsche (see [28], §586) has used the technique of the proof for Theorem 6 to establish a

General Maximum Principle. *Let $\zeta_1, \zeta_2 \in C^2(\Omega \setminus A), \Omega \subset \mathbb{R}^2$, be two solutions of $\mathcal{M}\zeta = 0$ in $\Omega \setminus A$ where A is a compact set in $\mathbb{R}^2$ with $\mathcal{H}^1(A) = 0$, $\mathcal{H}^1$ = one-dimensional Hausdorff measure. Furthermore, suppose that*

$$\lim_{p \to p_0} [\zeta_1(p) - \zeta_2(p)] \le M \quad \textit{for all } p_0 \in \partial\Omega \setminus A.$$

Then we obtain $\zeta_1 - \zeta_2 \le M$ in $\Omega \setminus A$. Furthermore, if $\zeta_1(p') - \zeta_2(p') = M$ for a single point $p' \in \Omega \setminus A$, it follows $\zeta_1(p) - \zeta_2(p) \equiv M$.

Independently and at the same time, an n-dimensional version of the maximum principle was proved by De Giorgi and Stampacchia [1] in 1965. These authors as well as Nitsche also established the following result.

General Removability Theorem. *Let Ω be a domain in $\mathbb{R}^2$, A a compact subset of Ω with $\mathcal{H}^1(A) = 0$, and $\zeta \in C^2(\Omega \setminus A)$ be a solution of $\mathcal{M}\zeta = 0$ in $\Omega \setminus A$. Then there is exactly one extension $\zeta^* \in C^2(\Omega)$ of ζ such that $\mathcal{M}\zeta^* = 0$ in Ω.*

For a proof, see J.C.C. Nitsche [28], §§591–593. This result is a powerful generalization of a celebrated theorem by L. Bers [2], published in 1951: *An isolated singularity of a solution of the minimal surface equation* $\mathcal{M}\zeta = 0$ *is removable.*

We shall now generalize this to nonparametric H-surfaces. We will remove sets of exemption points which are specified in

Definition 1. *A subset A of a domain Ω in $\mathbb{R}^2$ is called* admissible singular subset of Ω, *if it is compact and has the following covering property: For each $\epsilon > 0$ there exist $N = N(\epsilon)$ open disks $B_k := \{p \in \mathbb{R}^2 : |p - p_k| < r_k\}$ with $0 < r_k < \epsilon$, $B_k \subset\subset \Omega$,*

$$A \subset \bigcup_{k=1}^{N} B_k, \quad \overline{B}_k \cap \overline{B}_\ell = \emptyset \quad \textit{for } k \neq \ell \tag{26}$$

and

$$\sum_{k=1}^{N} \text{length}(\partial B_k) \leq 2\pi\epsilon. \tag{27}$$

We call $\{B_k\}_{1 \leq k \leq N}$ an ϵ-covering of A.

Remark 4. Obviously, an admissible singular A in Ω is a two-dimensional null set in $\mathbb{R}^2$ which even satisfies $\mathcal{H}^1(A) = 0$; but in addition we require $\overline{B}_k \cap \overline{B}_\ell = \emptyset$. We note that, the *regular part* $\Omega' := \Omega \setminus A$ is connected, and thus Ω' is a domain. For example, any finite subset A of Ω is admissible. Also, any compact, denumerable subset A of Ω with at most finitely many accumulation points is admissible.

We can generalize Theorem 6 in the following way:

Theorem 7. *Let A be an admissible singular subset of a Jordan domain Ω with a rectifiable boundary, $H_z(x, y, z) \geq 0$ on $\mathbb{R}^3$, and suppose that $\zeta_1, \zeta_2 \in C^0(\overline{\Omega} \setminus A) \cap C^2(\Omega \setminus A)$ are solutions of*

$$\mathcal{M}\zeta_j = 2H(\cdot, \zeta_j)W_j^3 \quad \textit{in } \Omega \setminus A, \quad W_j := \sqrt{1 + |\nabla\zeta_j|^2}. \tag{28}$$

Then we have the weighted energy estimate

$$\int_{\Omega \setminus A} \mu |\nabla\zeta_1 - \nabla\zeta_2|^2 \, dx \, dy \leq 2 \int_{\partial\Omega} |\zeta_1 - \zeta_2| \, ds \tag{29}$$

with the positive, continuous weight function $\mu : \Omega \setminus A \to \mathbb{R}$ defined by

$$\mu(x, y) := [\max\{W_1(x, y), W_2(x, y)]^{-3} \quad \textit{for } (x, y) \in \Omega \setminus A. \tag{30}$$

Corollary 2. *Let the assumptions of Theorem 7 be satisfied and suppose also that $\zeta_1 = \zeta_2$ on $\partial\Omega$. Then we have*

$$\zeta_1 = \zeta_2 \quad \text{on } \overline{\Omega} \setminus A. \tag{31}$$

Proof. The weighted estimate (29) together with (30) imply that $\nabla\zeta_1 = \nabla\zeta_2$ in $\Omega \setminus A$. Since $\Omega \setminus A$ is connected we infer $\zeta_1 - \zeta_2 = \text{const}$ on $\overline{\Omega} \setminus A$, and the boundary condition $\zeta_1|_{\partial\Omega} = \zeta_2|_{\partial\Omega}$ finally yields (31). □

As an immediate application of Corollary 2 and of Theorem 1 in Section 7.1 we obtain the following **Theorem on Removable Singularities for *H*-Graphs**.

Theorem 8. *Let A be an admissible singular subset of the domain Ω in $\mathbb{R}^2$, and $\zeta \in C^2(\Omega \setminus A)$ be a solution of*

$$\mathcal{M}\zeta = 2H(\cdot, \zeta)\{1 + |\nabla\zeta|^2\}^{3/2} \quad \text{in } \Omega \setminus A, \tag{32}$$

where H satisfies $\sup_{\mathbb{R}^3} |H| \le h_0$ and $H_z(x, y, z) \ge 0$ on $\mathbb{R}^3$.

Then ζ can be extended to a function of class $C^2(\Omega)$ which satisfies (1).

Proof. Choose $0 < \epsilon < h_0$, and let $\{B_k\}_{1 \le k \le N}$ be an ϵ-covering of A. With the aid of Theorem 2 in Section 7.1 we obtain solutions $\zeta_k \in C^0(\overline{B}_k) \cap C^2(B_k)$ of

$$\begin{aligned} \mathcal{M}\zeta_k &= 2H(\cdot, \zeta_k)\{1 + |\nabla\zeta_k|^2\}^{3/2} \quad \text{in } B_k, \\ \zeta_k &= \zeta \quad \text{on } \partial B_k. \end{aligned}$$

Corollary 2 can be applied to the pair $\{\zeta|_{\overline{B}_k}, \zeta_k\}$, and we obtain $\zeta = \zeta_k$ on $\overline{B}_k \setminus A$, $k = 1, \dots, N(\epsilon)$. Thus it follows $\zeta \in C^2(\Omega)$, and (1) is now an immediate consequence of (32). □

It remains to establish Theorem 7.

Proof of Theorem 7. (i) We first assume that $\partial\Omega \in C^1$. Then we write (28) in the form

$$\operatorname{div}(W_j^{-1}\nabla\zeta_j) = 2H(\cdot, \zeta_j) \quad \text{in } \Omega' := \Omega \setminus A, \quad j = 1, 2.$$

Subtracting the two equations from each other we obtain

$$\begin{aligned} \frac{1}{2}\operatorname{div}[(W_1^{-1}\nabla\zeta_1) - (W_2^{-1}\nabla\zeta_2)] &= H(\cdot, \zeta_1) - H(\cdot, \zeta_2) \\ &= (\zeta_1 - \zeta_2)\int_0^1 H_z(\cdot, \zeta_2 + t(\zeta_1 - \zeta_2))\, dt \quad \text{on } \Omega'. \end{aligned} \tag{33}$$

For any $\zeta \in C^2(\overline{\Omega} \setminus A)$ we define the truncated function $[\zeta]_M$, $0 < M < \infty$, by

$$[\zeta]_M(x,y) := \begin{cases} M & \text{for } \zeta(x,y) \geq M, \\ \zeta(x,y) & \text{for } |\zeta(x,y)| < M, \\ -M & \text{for } \zeta(x,y) \leq -M, \end{cases} \qquad (x,y) \in \overline{\Omega} \setminus A.$$

Clearly, $[\zeta_1 - \zeta_2]_M \in H^1_{2,\text{loc}}(\Omega') \cap L_\infty(\Omega')$. Moreover we infer from (33) that

$$\begin{aligned} (34) \qquad 0 &\leq 2[\zeta_1 - \zeta_2]_M[\zeta_1 - \zeta_2] \int_0^1 H_z(\cdot, \zeta_2 + t(\zeta_1 - \zeta_2))\, dt \\ &= [\zeta_1 - \zeta_2]_M \operatorname{div}[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2] \\ &= \operatorname{div}\{[\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2] \\ &\quad - \langle\nabla[\zeta_1 - \zeta_2]_M, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle \quad \text{on } \Omega'. \end{aligned}$$

We note that the open sets

$$\Omega_M := \{(x,y) \in \Omega' \colon |\zeta_1(x,y) - \zeta_2(x,y)| < M\}, \quad M > 0,$$

exhaust Ω' monotonically, i.e. $\Omega_M \nearrow \Omega'$ as $M \to \infty$, in the sense that $\Omega_M \subset \Omega_{\tilde{M}}$ for $M < \tilde{M}$ and $\Omega' = \bigcup_{M=1}^\infty \Omega_M$.

(ii) Let $\epsilon > 0$, and choose an ϵ-covering $\{B_k\}_{1 \leq k \leq N}$ of A. Define the subdomain Ω_ϵ of Ω' by

$$\Omega_\epsilon := \Omega \setminus \{\overline{B}_1 \cup \cdots \cup \overline{B}_N\}.$$

The Ω_ϵ exhaust the regular domain $\Omega' = \Omega \setminus A$, i.e. $\Omega_\epsilon \to \Omega'$ for $\epsilon \to +0$, but the exhaustion need not be monotonic.

For any $\epsilon > 0$, the vector field $\eta := [\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2]$ belongs to the class $H_2^1(\Omega_\epsilon, \mathbb{R}^2) \cap C^0(\overline{\Omega}_\epsilon, \mathbb{R}^2)$. Thus we may apply an integration by parts to (34) integrated over Ω_ϵ, thereby obtaining

$$\begin{aligned} (35) \qquad &\int_{\Omega_\epsilon \cap \Omega_M} \langle\nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle\, dx\, dy \\ &\qquad \leq \int_{\partial\Omega_\epsilon} \langle[\zeta_1 - \zeta_2]_M[W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2], \nu\rangle\, ds, \end{aligned}$$

where ν denotes the exterior unit normal to the domain Ω_ϵ, which is of class C^1. Since $\eta \in L^\infty(\Omega')$ and

$$\sum_{k=1}^{N(\epsilon)} \text{length}(\partial B_k) \leq 2\pi\epsilon,$$

we infer from (35) for $\epsilon \to 0$ that

$$\begin{aligned} (36) \qquad &\int_{\Omega_M} \langle\nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle\, dx\, dy \\ &\qquad \leq 2\int_{\partial\Omega} |\zeta_1 - \zeta_2|\, ds \quad \text{for all } M > 0. \end{aligned}$$

Now we could use the reasoning from part (iii) in the proof of Theorem 6 to derive the following estimate from (36):

$$(37) \qquad \int_{\Omega_M} \mu |\nabla \zeta_1 - \nabla \zeta_2| \, dx \, dy \le 2 \int_{\partial\Omega} |\zeta_1 - \zeta_2| \, ds \quad \text{for all } M > 0.$$

Instead it might be welcome to the reader if we present the following detailed computation, because it gives some geometric insight. Consider the function $F(p) = \sqrt{1+|p|^2}$ on $\mathbb{R}^2$. Setting $p = (\alpha, \beta)$, we have $F(p) = \sqrt{1+\alpha^2+\beta^2}$, and the Hessian $F_{pp}(p)$ of F is given by

$$F_{pp}(p) = F^{-3}(p)C(p) \quad \text{with } C(p) := \begin{pmatrix} 1+\beta^2 & -\alpha\beta \\ -\alpha\beta & 1+\alpha^2 \end{pmatrix}.$$

With $\gamma = (\xi, \eta) \in \mathbb{R}^2$ we obtain for the quadratic form associated with $C(p)$ that

$$\begin{aligned} \langle \gamma, C(p)\gamma \rangle &= \xi^2 + \beta^2\xi^2 - 2\alpha\beta\xi\eta + \eta^2 + \alpha^2\eta^2 \\ &= \xi^2 + \eta^2 + (\alpha\eta - \beta\xi)^2 \ge \xi^2 + \eta^2 = |\gamma|^2. \end{aligned}$$

Therefore,

$$(38) \qquad \langle \gamma, F_{pp}(p)\gamma \rangle \ge F^{-3}(p)|\gamma|^2.$$

For $p_1, p_2 \in \mathbb{R}^2$ we obtain

$$F_p(p_1) - F_p(p_2) = \int_0^1 F_{pp}(p_2 + t(p_1 - p_2))(p_1 - p_2) \, dt,$$

whence by (38),

$$\begin{aligned} &\langle p_1 - p_2, F_p(p_1) - F_p(p_2) \rangle \\ &\quad = \int_0^1 \langle p_1 - p_2, F_{pp}(p_2 + t(p_1 - p_2))(p_1 - p_2) \rangle \, dt \\ &\quad \ge \left(\int_0^1 F^{-3}(p_2 + t(p_1 - p_2)) \, dt \right) |p_1 - p_2|^2. \end{aligned}$$

By (38), the function $F(p)$ is convex; hence

$$F(p_2 + t(p_1 - p_2)) \le \max\{F(p_1), F(p_2)\} \quad \text{for } 0 \le t \le 1.$$

Then it follows

$$(39) \qquad \langle p_1 - p_2, F_p(p_1) - F_p(p_2) \rangle \ge [\max\{F(p_1), F(p_2)\}]^{-3} |p_1 - p_2|^2.$$

With $p_1 := \nabla\zeta_1(x,y)$ and $p_2 := \nabla\zeta_2(x,y)$ we obtain

$$F_p(p_1) = W_1^{-1}(x,y)\nabla\zeta_1(x,y), \quad F_p(p_2) = W_2^{-1}(x,y)\nabla\zeta_2(x,y),$$

and then

$$\langle \nabla\zeta_1 - \nabla\zeta_2, W_1^{-1}\nabla\zeta_1 - W_2^{-1}\nabla\zeta_2\rangle \geq \mu|\nabla\zeta_1 - \nabla\zeta_2|^2.$$

In conjunction with (36), this implies (37).

Letting M tend to infinity and recalling $\Omega_M \nearrow \Omega'$, we infer with the aid of B. Levi's theorem on monotone convergence the desired inequality (29) with μ given by (30), provided that $\partial\Omega \in C^1$.

(ii) If $\partial\Omega$ is merely a rectifiable Jordan curve, we exhaust Ω by domains Ω_j with $A \subset \Omega_j \subset\subset \Omega$, $\partial\Omega_j \in C^1$, and length $(\partial\Omega_j) \to$ length $(\partial\Omega)$ as $j \to \infty$. Then the desired estimate is obtained from the estimate for Ω_j in the limit $j \to \infty$. $\square$

7.4 Scholia

1. In this chapter we presented an approach to the Dirichlet problem for the minimal surface equation $\mathcal{M}\zeta = 0$ and, more generally, for the nonparametric H-surface equation

$$\mathcal{M}\zeta = 2H(\cdot,\zeta)[1 + |\nabla\zeta|^2]^{3/2} \quad \text{in } \Omega \tag{1}$$

in two dimensions. The special feature of our method is to start with a solution X of the Plateau problem for the parametric equation

$$\Delta X = 2H(X)X_u \wedge X_v \quad \text{in } B \tag{2}$$

and then to show that X possesses an equivalent nonparametric representation $Y(x,y) = (x, y, \zeta(x,y))$ with ζ solving (1), provided that Γ is a graph above the boundary of a $2h_0$-convex domain Ω in $\mathbb{R}^2$ and that $|H| \leq h_0$ as well as $H_z \geq 0$. The transition from the parametric problem to the nonparametric one is based on the *projection theorem* by F. Sauvigny [1,2]. For the minimal surface equation this idea was invented by T. Radó [21] in the proof of his uniqueness theorem for Plateau's problem, see Section 4.9. For the general case, the uniqueness is restricted to stable H-surfaces. Sauvigny's ideas were generalized to the study of free boundary value problems for minimal surfaces (cf. S. Hildebrandt and F. Sauvigny [1–7]; see Vol. 3) and also for H-surfaces (F. Müller [5–11]).

Besides the treatises of Nitsche [28], Gilbarg and Trudinger [1], and Sauvigny [16] we also refer to the monograph on capillarity problems by R. Finn [11] as well as to later work by this author.

2. An independent proof of the removability theorem of Bers was given by R. Finn [1]. Finn's result extends to isolated singularities of solutions to equations of the minimal surface type. L. Bers [5] gave another proof of Finn's theorem using the uniformization theorem, and eventually Finn [6] strengthened Bers's method, thereby obtaining a removability for a more general type of nonlinear elliptic equations. Nitsche's removability theorem appeared first in his paper [12]. De Giorgi and Stampacchia [1] proved: *If* $\zeta \in C^2(\Omega \setminus K)$ *is a solution of the* n*-dimensional minimal surface equation in* $\Omega \setminus K$ *where* Ω *is an open set in* $\mathbb{R}^n$ *and* K *a compact subset of* Ω *with* $\mathcal{H}^{n-1}(K) = 0$, *then* u *extends to a* C^2*-solution on the whole of* Ω. L. Simon [3] showed that it is in fact only necessary for K to be a *locally compact subset* of Ω, and therefore K can extend to the boundary of Ω. Furthermore, Simon's method carries over to equations of the form

$$\sum_{j=1}^{n} D_j F_{p_j}(x, -D\zeta, 1) = H(x),$$

where $F(x,p)$ is a positive definite, elliptic Lagrangian satisfying $\lambda F(x,p) = F(x, \lambda p)$ for $\lambda > 0$. We also refer to work of M. Miranda [1], G. Anzellotti [1], and Hildebrandt and Sauvigny [8].

Chapter 8

Introduction to the Douglas Problem

In this chapter we present an introduction to the *general problem of Plateau* that, justifiedly, is often called the *Douglas problem*. This is the question whether a configuration $\Gamma := \langle \Gamma_1, \dots, \Gamma_k \rangle$ of k nonintersecting closed Jordan curves Γ_j in $\mathbb{R}^3$ may bound multiply connected minimal surfaces of prescribed Euler characteristic and prescribed character of orientability. Here we treat only the simplest form of the Douglas problem, to find a minimal surface $X : \overline{\Omega} \to \mathbb{R}^3$ whose parameter domain Ω is a k-fold connected, bounded, open set in $\mathbb{R}^2$ whose boundary consists of k closed, nonintersecting Jordan curves. Since any such domain can be mapped conformally onto a domain B bounded by k circles, we may choose such k-circle domains as parameter domains for the desired minimal surfaces. However, different from the case $k = 1$ where all parameter domains are conformally equivalent, two admissible parameter domains will in general be of different conformal type if $k \geq 2$. Therefore we are no longer allowed to fix a k-circle domain B a priori as the parameter domain of any solution of the Douglas problem; instead, the determination of B is part of the problem since X has to fulfill the conformality relations.

After discussing some examples in Section 8.1, we state the main result. In Section 8.2 we show that from $\partial D(X, \eta) = 0$ for all C^1-vector fields $\eta : \overline{B} \to \mathbb{R}^2$ on the domain B of X one can derive the conformality relation $\langle X_w, X_w \rangle = 0$. The proof of this fact is a quite nontrivial generalization of the method used in Section 4.5.

Different from the Plateau problem ($k = 1$), the Douglas problem ($k \geq 2$) has in general no "connected" solution. For example, two parallel circles Γ_1 and Γ_2 contained in distinct planes do not bound a connected minimal surface if they are "too far apart". This phenomenon is discussed in Chapter 4 of Vol. 2. Douglas has exhibited a sufficient condition ensuring the existence of connected minimal surfaces bounded by $\Gamma_1, \dots, \Gamma_k$. However, this condition is somewhat difficult to deal with, while Courant's *condition of cohesion* is much easier to handle. This condition is described in Section 8.3, and it is

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shown that it leads to sequences of parameter domains which converge towards nondegenerate domains.

In Section 8.4 we solve the Douglas problem for k-fold connected minimal surfaces, assuming that the condition of cohesion is satisfied. Then, in Section 8.5, we prepare two useful tools which later on will be used to modify surfaces in a suitable way. These modifications were invented by Courant.

The main result is contained in Section 8.6 where we solve the Douglas problem, assuming the so-called Douglas condition. The solution is seen to be a simultaneous minimizer of the area A and the energy D in the class $\mathcal{C}(\Gamma)$ of admissible surfaces, which implies

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D.$$

The "necessary Douglas condition"

$$a(\Gamma) \le a^+(\Gamma)$$

and the "sufficient Douglas condition"

$$a(\Gamma) < a^+(\Gamma)$$

are studied in some detail in Sections 8.7 and 8.8; in particular, we present several examples. As a generalization of Riemann's mapping theorem to multiply connected planar domains we obtain *Koebe's mapping theorem.*

The Scholia (Section 8.9) contain some historical remarks and references to the literature.

8.1 The Douglas Problem. Examples and Main Result

In Chapter 4 we discussed the classical problem of Plateau as it was solved by Douglas and Radó, and we presented the solution found by Courant and, independently, by Tonelli. In the restricted sense formulated in Definition 1 of Section 4.2, Plateau's problem consists in finding a "disk-type" minimal surface spanning a prescribed closed Jordan curve Γ. This is to say, given Γ, we have to find a mapping $X : \overline{B} \to \mathbb{R}^3$ of the closure of the disk $B := \{w \in \mathbb{R}^2 \colon |w| < 1\}$ into $\mathbb{R}^3$ which is harmonic and conformal in B, continuous on $\overline{B}$, and maps ∂B topologically (i.e. homeomorphically) onto Γ.

As mentioned before, this is neither the most general nor the most natural way to formulate Plateau's problem, but merely the simplest and most convenient one, as we do not run into the difficulty that parameter domains of the same topological type may be of different conformal type. However, there is no need to restrict ourselves to minimal surfaces bounded by a single closed curve since boundary configurations consisting of several closed curves may bound multiply connected minimal surfaces. A classical example is the

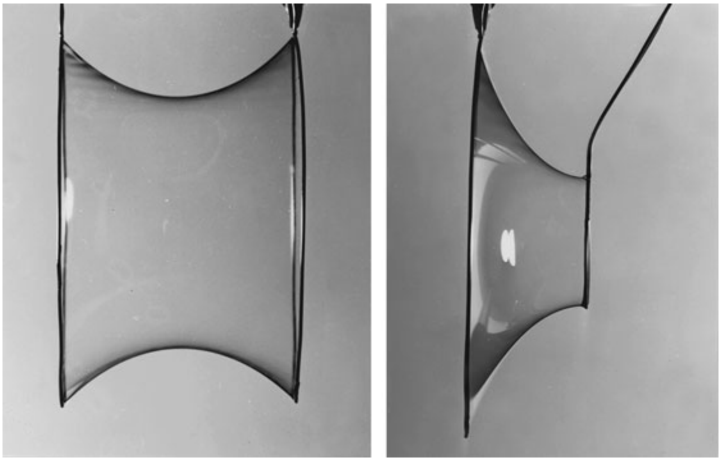

Fig. 1. A soap film experiment: Catenoids held by two coaxial circles

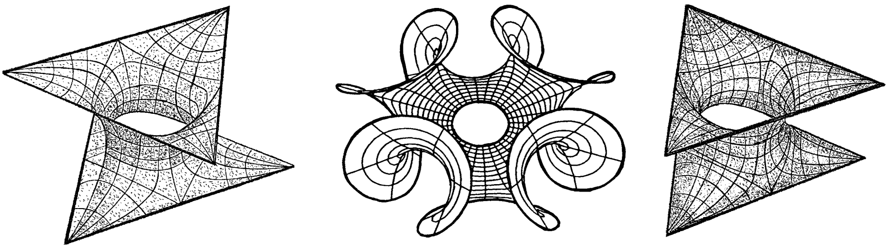

Fig. 2. Minimal surfaces bounded by two closed curves

catenoid, the minimal surface of revolution, which is bounded by two coaxial circles in parallel planes. Moreover, soap film experiments show that certain configurations may bound minimal surfaces of higher topological structure, even nonorientable ones such as surfaces of the type of the Möbius strip. Figures 1–5 depict several such contours as well as minimal surfaces spanning them. In certain cases it is not difficult to see that a topologically more complicated minimal surface may have a smaller area than any disk-type surface bounded by the same contour.

The first to state Plateau's problem in a general form was Jesse Douglas who attacked this question in a series of profound and pioneering papers. Hence many authors speak of the *Douglas problem* instead of what Douglas himself called the

General problem of Plateau. *Given a configuration* $\Gamma = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$ *in* $\mathbb{R}^3$ *consisting of* k *mutually disjoint Jordan curves* Γ_j*, find a minimal surface of prescribed topological type that spans* Γ.

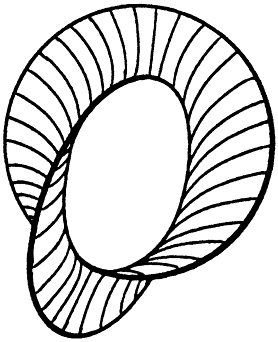

Fig. 3. An annulus-type minimal surface spanned by two interlocking curves

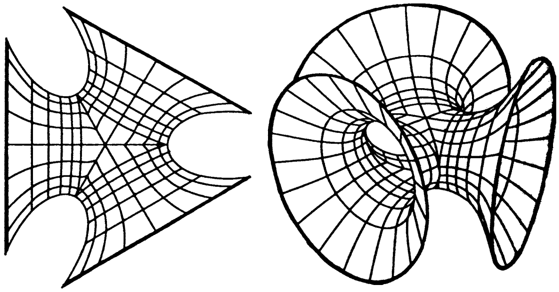

Fig. 4. Two views of a minimal surface of genus zero bounded by three closed curves. Courtesy of K. Polthier

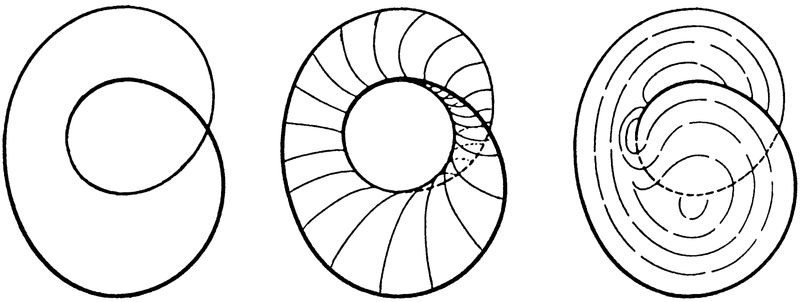

Fig. 5. A closed curve bounding a one-sided minimal surface. This curve also spans a disk-type minimal surface

What is a *surface*, and what is its *topological type*? One might think of a surface as a two-dimensional submanifold of $\mathbb{R}^3$, or better as an embedding $X : M \to \mathbb{R}^3$ of a two-dimensional manifold M with (or without) boundary into $\mathbb{R}^3$. This definition is too restrictive as we want to consider "surfaces" $S = X(M)$ with selfintersections; thus we might think of local embeddings such as immersions $X : M \to \mathbb{R}^3$. But even this class is too narrow as we want to study minimal surfaces with branch points. Moreover, in order to be able to use functional analytic arguments, we would like to operate with mappings X contained in a Sobolev space, which are in fact only equivalence classes of mappings $X : M \to \mathbb{R}^3$, and every representative of X is only determined up to a set of two-dimensional measure zero.

In such general cases we cannot define the topological type of the surface $S = X(M)$ in the usual way. Instead we use the following preliminary definition: *A surface in* $\mathbb{R}^3$ *is a mapping* $X : M \to \mathbb{R}^3$ *of a two-dimensional manifold* M *(with or without boundary), and the topological type of* X *is defined as the topological type of the "parameter manifold"* M. The image set $S := X(M)$ is called the *trace of* X *in* $\mathbb{R}^3$; occasionally one calls S instead of X a surface in $\mathbb{R}^3$, and S is said to be an *embedded* or *immersed surface* respectively if X is an embedding or an immersion. We note that X might be defined only up to a null set in M.

Suppose that M is a compact two-dimensional C^1-manifold whose boundary ∂M consists of k closed Jordan curves, and which is oriented ($\epsilon(M) := 1$) or nonoriented ($\epsilon(M) := -1$). Let $\chi(M) := \alpha_0 - \alpha_1 + \alpha_2$ be the Euler characteristic of M, with $\alpha_0, \alpha_1, \alpha_2$ the number of edges, wedges, and faces of any regular triangulation of M. Then the *topological type of* M, denoted by $[M]$, is defined as

$$[M] := \{\epsilon(M), r(M), \chi(M)\} \quad \text{with } r(M) := k,$$

and the *genus of* M, denoted by $g(M)$, is defined by

$$\chi(M) + r(M) =: \begin{cases} 2 - 2g(M) & \text{if } \epsilon(M) = 1, \\ 2 - g(M) & \text{if } \epsilon(M) = -1. \end{cases}$$

For instance, if M is a k-fold connected, compact region in $\mathbb{R}^2$, then $\epsilon(M) = 1$, $r(M) = k$, $\chi(M) = 2 - k$, and $[M] = \{1, k, 2 - k\}$, $g(M) = 0$. In the present chapter we want to consider surfaces $X : M \to \mathbb{R}^3$, whose parameter sets M have this topological type. The more general case $[M] = \{1, r(M), \chi(M)\}$ is treated in Chapter 4 of Vol. 3, while $[M] = \{-1, r(M), \chi(M)\}$ can be handled by passing to the double cover of M.

The *General Douglas Problem* then reads as follows:

Given a configuration $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *of* k *mutually disjoint Jordan curves* $\Gamma_1, \dots, \Gamma_k$, *find a minimal surface* $X : M \to \mathbb{R}^3$ *of prescribed topological type* $[M] = \{\epsilon(M), k, \chi(M)\}$ *that spans* Γ.

The idea to solve this task is to minimize Dirichlet's integral $D(X)$ among all surfaces $X : M \to \mathbb{R}^3$ bounded by Γ, and of given topological type $[M]$. For technical reasons it is inconvenient to allow all parameter sets M of fixed topological type for competition. Since D is invariant with respect to conformal mappings $\tau : \operatorname{int} M^* \to \operatorname{int} M$, it is sufficient to minimize D in a class of mappings $X : M \to \mathbb{R}^3$ with $M \in \mathcal{N}$ where $\mathcal{N}$ denotes a set of parameter manifolds M containing all conformal types with the fixed topological type $[M]$. In fact, it is not necessary to know a priori what all conformal representations for a given type $[M]$ are; it suffices to make a good guess and to verify that the method works. However, choosing a sequence $\{X_j\}$ of surfaces $X_j : M_j \to \mathbb{R}^3$, bounded by Γ, with $M_j \in \mathcal{N}$ and

$$D(X_j) \to \inf\{D(X) \colon X : M \to \mathbb{R}^3,\ \partial X = \Gamma,\ M \in \mathcal{N}\}$$

as $j \to \infty$ such that X_j converges in some sense to a mapping $X : M \to \mathbb{R}^3$, it is by no means clear that the limit set $M = \lim_{j\to\infty} M_j$ will belong to $\mathcal{N}$; in fact, M might very well jump out of the class $\mathcal{N}$. To prevent this, one has to take suitable precautions such as assuming the *condition of cohesion* or the *Douglas condition.*

Here we shall solve the simple kind of Douglas problem, namely: *Determine a minimal surface* $X : M \to \mathbb{R}^3$, *spanning* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, *defined on a schlicht parameter region* $M \subset \mathbb{R}^2$ *of type* $[M] = \{1, k, 2-k\}$. To obtain a solution, we minimize D in a suitable class $\mathcal{C}(\Gamma)$ of mappings $X : B \to \mathbb{R}^3$ with $B \in \mathcal{N}(k)$ where $\mathcal{N}(k)$ is the class of k-*circle domains* B in $\mathbb{R}^2$. Let us give a precise definition of this kind of domains.

As usual we identify the point $w = (u, v) \in \mathbb{R}^2$ with $w = u + iv \in \mathbb{C}$, and correspondingly $\mathbb{R}^2$ is identified with $\mathbb{C}$. For $q \in \mathbb{C}$ and $r > 0$ we define the disk $B_r(q)$ as

$$B_r(q) := \{w \in \mathbb{C} \colon |w - q| < r\};$$

it is a 1-*circle domain.* If $q = 0$ and $r = 1$, we call the unit disk $B_1(0)$ the *normed* 1-*circle domain.* For $k > 1$, a k-*circle domain* $B(q, r)$ with $q = (q_1, \dots, q_k) \in \mathbb{C}^k$ and $r = (r_1, \dots, r_k) \in \mathbb{R}^k$, $r_1 > 0, \dots, r_k > 0$, is a disk $B_{r_1}(q_1)$, from which $k-1$ closed disks $\overline{B}_{r_2}(q_2), \dots, \overline{B}_{r_k}(q_k)$ are removed which are contained in $B_{r_1}(q_1)$ and which do not intersect. That is,

$$B(q, r) = B_{r_1}(q_1) \setminus \{\overline{B}_{r_2}(q_2) \,\dot\cup \cdots \dot\cup\, \overline{B}_{r_k}(q_k)\},$$

and $|q_1 - q_j| + r_j > r_1$ for $1 < j \le k$ as well as

$$r_j + r_\ell < |q_j - q_\ell| \quad \text{for } j \ne \ell \text{ with } 2 \le j, \ell \le r.$$

If, in addition $q_1 = q_2 = 0$ and $r_1 = 1$, then $B(q, r)$ is called a *normed k-circle domain.* We set $C_j := \partial B_{r_j}(q_j)$.

Let $\mathcal{N}(k)$ be the *class of k-circle domains*, and $\mathcal{N}_1(k)$ be the class of normed k-circle domains.

For $X \in H_2^1(B, \mathbb{R}^3)$ with $B = \operatorname{dom}(X) \in \mathcal{N}(k)$ we define the *area functional* $A(X)$ and the *Dirichlet integral* $D(X)$ as

$$A(X) := \int_B |X_u \wedge X_v| \, du\, dv = \int_B \sqrt{|X_u|^2 |X_v|^2 - \langle X_u, X_v\rangle^2}\, du\, dv,$$
$$D(X) := \frac{1}{2} \int_B [|X_u|^2 + |X_v|^2] \, du\, dv.$$

Note that these integrals are extended over the *domain* B of X which may vary with X. If B' is a subdomain of $\mathrm{dom}(X) = B$ we write

$$A_{B'}(X) := \int_{B'} |X_u \wedge X_v| \, du \, dv, \quad D_{B'}(X) := \frac{1}{2} \int_{B'} |\nabla X|^2 \, du \, dv.$$

Recall that

$$A(X) \le D(X) \quad \text{for any } X \in H_2^1(B, \mathbb{R}^3)$$

and

$$A(X) = D(X) \quad \text{if and only if} \quad \langle X_w, X_w \rangle = 0$$

where

$$X_w := \frac{1}{2}(X_u - iX_v), \quad X_{\overline{w}} := \frac{1}{2}(X_u + iX_v).$$

The real form of the *conformality relation* $\langle X_w, X_w \rangle = 0$ is

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{1}$$

For a boundary contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ of k mutually disjoint, closed Jordan curves $\Gamma_1, \dots, \Gamma_k$ we define the *Douglas class* $\mathcal{C}(\Gamma)$ of admissible mappings $X : B \to \mathbb{R}^3$ for the variational procedure that we are going to set up:

Definition 1. *A mapping* $X \in H_2^1(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ *with* $B = \mathrm{dom}(X) \in \mathcal{N}(k)$ *belongs to* $\mathcal{C}(\Gamma)$ *if the Sobolev trace* $X|_{\partial B}$ *maps* ∂B *in a weakly monotonic way onto* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$. *By this we mean the following: There is an enumeration* $C_1, \dots, C_k$ *of the boundary circles of* B *such that* $X|_{C_j}$ *maps* C_j *in a weakly monotonic way onto* Γ_j, $j = 1, \dots, k$.

If in the sequel we consider a mapping $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X)$ and $\partial B = C_1 \dot\cup \cdots \dot\cup C_k$, we tacitly assume the boundary circles C_j to be enumerated in such a way that

$$\Gamma_1 = X(C_1), \quad \dots, \quad \Gamma_k = X(C_k).$$

We note that $\mathcal{C}(\Gamma)$ is nonempty if $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ is rectifiable, which means that each of the curves $\Gamma_1, \dots, \Gamma_k$ is rectifiable.

Now we can formulate the principal result of this chapter.

Theorem 1. *Let* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *be a boundary contour consisting of* k *mutually disjoint, closed, rectifiable Jordan curves in* $\mathbb{R}^3$, *and suppose that* Γ *satisfies either Courant's condition of cohesion or the Douglas condition. Then the following holds true:*

(i) *There is a minimizer* $X \in \mathcal{C}(\Gamma)$ *of Dirichlet's integral in* $\mathcal{C}(\Gamma)$, *that is,*

$$D(X) = \inf_{\mathcal{C}(\Gamma)} D. \tag{2}$$

Every such minimizer X *is a minimal surface, i.e.* X *is harmonic in* B *and satisfies the conformality relations* (1); *moreover,* X *is continuous on* $\overline{B}$ *and yields a topological mapping from* ∂B *onto* Γ.

(ii) *In addition, we have*

$$\inf_{\mathcal{C}(\Gamma)} D = \inf_{\mathcal{C}(\Gamma)} A. \tag{3}$$

This implies that every minimizer of D *in* $\mathcal{C}(\Gamma)$ *is also a minimizer of* A *in* $\mathcal{C}(\Gamma)$. *Conversely, every conformally parametrized minimizer of* A *in* $\mathcal{C}(\Gamma)$ *is also a minimizer of* D *in* $\mathcal{C}(\Gamma)$.

(iii) *Set* $\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$. *Then we even have*

$$\inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} A. \tag{4}$$

Courant's condition of cohesion and the Douglas condition will be stated in Sections 8.3 and 8.6 respectively.

Without proof we mention the following result that will be derived in Vol. 2, Section 2.3 (for $k \geq 2$):

Theorem 2. *Suppose that* $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ *is of class* $C^{k,\alpha}$, $k \geq 1$, $\alpha \in (0,1)$. *Then each minimal surface* $X : B \to \mathbb{R}^3$ *of class* $\overline{\mathcal{C}}(\Gamma)$ *is also of class* $C^{k,\alpha}(\overline{B}, \mathbb{R}^3)$.

8.2 Conformality of Minimizers of D in $\mathcal{C}(\Gamma)$

Following ideas of R. Courant and H. Lewy we shall prove:

Theorem 1. *If* $X \in \mathcal{C}(\Gamma)$ *is a minimizer of* D *in* $\mathcal{C}(\Gamma)$ *then it satisfies*

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0. \tag{1}$$

We first recall the following result that was proved in Section 4.5:

Lemma 1. *If* $X \in \mathcal{C}(\Gamma)$ *minimizes* D *in* $\mathcal{C}(\Gamma)$ *then its inner variation* $\partial D(X, \eta)$ *vanishes for all vector fields* $\eta \in C^1(\overline{B}, \mathbb{R}^2)$ *with* $B = \mathrm{dom}(X)$.

Therefore Theorem 1 follows from

Theorem 2. *If* $X \in \mathcal{C}(\Gamma)$ *with* $B = \mathrm{dom}(X)$ *satisfies*

$$\partial D(X, \eta) = 0 \quad \textit{for all } \eta \in C^1(\overline{B}, \mathbb{R}^2), \tag{2}$$

then the conformality relations (1) *hold true.*

Before we begin with the proof of this theorem, we will derive some auxiliary results. The first one was proved in Section 4.5:

Lemma 2. *If $X \in \mathcal{C}(\Gamma)$ with $B = \operatorname{dom}(X)$ then*

$$a := |X_u|^2 - |X_v|^2, \quad b := 2\langle X_u, X_v \rangle \tag{3}$$

are of class $L_1(B)$, and for any $\eta = (\eta^1, \eta^2) \in C^1(B, \mathbb{R}^2)$ we have

$$\partial D(X, \eta) = \frac{1}{2} \int_B [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, du\, dv. \tag{4}$$

Let X be of class $H^1_2(B, \mathbb{R}^3)$, and consider a conformal mapping $\nu : B^* \to B$ from $B^* \subset \mathbb{C}$ onto B. Then $X^* := X \circ \nu$ satisfies $D(X) = D(X^*)$, i.e.

$$\int_B |\nabla X|^2\, du\, dv = \int_{B^*} |\nabla X^*|^2\, du\, dv.$$

Since

$$\partial D(X, \eta) = \frac{d}{d\epsilon} D(X \circ \tau_\epsilon)\bigg|_{\epsilon=0}$$

where τ_ϵ denotes an "inner variation" of the form

$$\tau_\epsilon(w) = w - \epsilon\lambda(w) + o(\epsilon), \quad |\epsilon| \ll 1$$

we obtain

Lemma 3. *Let ν be a conformal mapping from $\overline{B}^*$ onto $\overline{B}$ and $X \in H^1_2(B, \mathbb{R}^2)$, $X^* = X \circ \tau$. Then*

$$\partial D(X, \eta) = 0 \quad \textit{for all } \eta \in C^1(\overline{B}, \mathbb{R}^2)$$

is equivalent to

$$\partial D(X, \zeta) = 0 \quad \textit{for all } \zeta \in C^1(\overline{B}^*, \mathbb{R}^2).$$

Lemma 4. *For any $B \in \mathcal{N}(k)$ there is a Möbius transformation f such that $f(B) \in \mathcal{N}_1(k)$.*

Proof. For $k = 1$, f is given by $f(w) := \frac{1}{r_1}(w - q_1)$ if $B = B_{r_1}(q_1)$. If $k \geq 2$ and $B = B_{r_1}(q_1) \setminus \{\overline{B}_{r_2}(q_2) \cup \cdots \cup \overline{B}_{r_k}(q_k)\}$ then $f := \varphi \circ \psi$ with

$$\varphi(w) := \frac{w_1 - q_1}{r_1}, \quad \psi(z) := \frac{z - p_2}{\overline{p}_2 z - 1} \quad \text{with } p_2 := \varphi(q_2)$$

solves the task. □

Lemma 5. *If $X \in \mathcal{C}(\Gamma)$ with $B = \operatorname{dom}(X)$ and*

$$\phi := 4\langle X_w, X_w \rangle = a - ib, \quad a = |X_u|^2 - |X_v|^2, \quad b = 2\langle X_u, X_v \rangle,$$

then (2) is equivalent to

$$(5) \qquad \int_B \eta_{\overline{w}} \phi \, du \, dv = 0 \quad \textit{for all } \eta \in C^1(\overline{B}, \mathbb{C}).$$

Furthermore, if ν *is a Möbius transformation and* $B = \nu(B^*)$, $B, B^* \in \mathcal{N}(k)$ *as well as* $X^* = X \circ \nu$, $\phi^* := \langle X^*_w, X^*_w \rangle$, *then* $X^* \in \mathcal{C}(\Gamma)$ *with* $B^* = \mathrm{dom}(X^*)$ *satisfies*

$$(5') \qquad \int_{B^*} \zeta_{\overline{w}} \phi^* \, du \, dv = 0 \quad \textit{for all } \zeta \in C^1(\overline{B}^*, \mathbb{C}).$$

(*Here and in the sequel,* C^1 *means continuously differentiable in the "real" sense, i.e.* $C^1(\overline{B}, \mathbb{C})$ *is identified with* $C^1(\overline{B}, \mathbb{R}^2)$*, etc.*)

Proof. The equivalence of (2) and (5) follows from (4) and the identity

$$\mathrm{Re}(\eta_{\overline{w}} \phi) = \tfrac{1}{2}[(\eta^1_u - \eta^2_v)a + (\eta^2_u + \eta^1_v)b].$$

Furthermore, equation (5) implies (5′) on account of Lemmas 2 and 3, using the first assertion of Lemma 5. □

Lemma 6. *If* $X \in \mathcal{C}(\Gamma)$ *with* $B = \mathrm{dom}(X)$ *satisfies*

$$(6) \qquad \int_B [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)] \, du \, dv = 0 \quad \textit{for all } \eta \in C^\infty_c(B, \mathbb{R}^2),$$

then $\phi := 4\langle X_w, X_w \rangle = a - ib$ *with* a, b *given by* (3) *is holomorphic in* B*, i.e.* $\phi_{\overline{w}}(w) = 0$ *for all* $w \in B$*.*

Proof. Let $\mu = (\mu_1, \mu_2) \in C^\infty_c(B, \mathbb{R}^2)$ and set $\eta := \mathcal{S}_\delta \mu = k_\delta * \mu$ where $\mathcal{S}_\delta$ is a mollifier with a symmetric kernel k_δ. Then $\eta \in C^\infty_c(B, \mathbb{R}^2)$ if $0 < \delta \ll 1$, and $a^\delta := \mathcal{S}_\delta a$, $b^\delta := \mathcal{S}_\delta b$ are of class $C^\infty(B)$, and we infer from (6) the relation

$$\int_B [a^\delta(\mu^1_u - \mu^2_v) + b^\delta(\mu^2_u + \mu^1_v)] \, du \, dv = 0.$$

An integration by parts yields

$$\int_B [-(a^\delta_u + b^\delta_v)\mu^1 + (a^\delta_v - b^\delta_u)\mu^2] \, du \, dv = 0$$

for all $\mu \in C^\infty_c(B', \mathbb{R}^2)$ with $B' \subset\subset B$ and $0 < \delta < \delta_0(B') \le \mathrm{dist}(B', \partial B)$. Hence $a^\delta, -b^\delta$ satisfy

$$a^\delta_u = (-b^\delta)_v, \quad a^\delta_v = -(-b^\delta)_u \quad \text{in } B'$$

and so $\phi^\delta := a^\delta - ib^\delta$ is holomorphic in $B' \subset\subset B$ for $0 < \delta < \delta_0(B')$. Since $\phi^\delta \to \phi$ in $L_1(B', \mathbb{C})$ as $\delta \to 0$ for $B' \subset\subset B$, we infer that ϕ is holomorphic in any $B' \subset\subset B$ and therefore also in B. □

Proof of Theorem 2. We have to show that the holomorphic function $\phi(w)$ vanishes identically in B. We shall proceed in five steps. First we prove:

(i) *Let α be a closed C^1-Jordan curve in B which partitions $B \setminus \alpha$ into two disjoint open sets B_1 and B_2, i.e. $B = B_1 \dot{\cup} \alpha \dot{\cup} B_2$. Suppose also that $\eta = (\eta^1, \eta^2) \in C^1(\overline{B}, \mathbb{R}^2)$, written in the complex form $\eta = \eta^1 + i\eta^2$, is holomorphic in B_1 and satisfies $\eta(w) = 0$ for any $w \in \partial B_2 \setminus \alpha$. Then we have*

$$\operatorname{Im} \int_\beta \eta(w)\phi(w)\, dw = 0 \tag{7}$$

for any closed C^1-curve β in B_1 that is homologous to α (where $\int_\beta \dots dw$ is the complex line integral along β).

In fact, $\eta_{\overline{w}} = 0$ on B_1 and (6) imply

$$\operatorname{Re} \int_{B_2} \eta_{\overline{w}} \phi\, du\, dv = 0,$$

whence

$$\int_{B_2} [a(\eta^1_u - \eta^2_v) + b(\eta^2_u + \eta^1_v)]\, du\, dv = 0.$$

Since $\eta = 0$ on $\partial B_2 \setminus \alpha$, an integration by parts yields

$$\begin{aligned} 0 = & \int_\alpha (a\eta^2 - b\eta^1) du + (a\eta^1 + b\eta^2)\, dv \\ & - \int_{B_2} [(a_u\eta^1 + b_u\eta^2) + (b_v\eta^1 - a_v\eta^2)]\, du\, dv. \end{aligned}$$

Furthermore,

$$2 \operatorname{Re}(\eta\phi_{\overline{w}}) = (a_u\eta^1 + b_u\eta^2) + (b_v\eta^1 - a_v\eta^2),$$

and

$$\operatorname{Im}(\phi\eta\, dw) = (a\eta^2 - b\eta^1)\, du + (a\eta^1 + b\eta^2)\, dv.$$

Since $\phi_{\overline{w}} = 0$ in B it follows

$$\operatorname{Im} \int_\alpha \phi\eta\, dw = 0.$$

As $\phi\eta$ is holomorphic in B_1 we also have

$$\int_\alpha \phi\eta\, dw = \int_\beta \phi\eta\, dw$$

and so we obtain (7). Thus assertion (i) is proved.

For any M in $\mathbb{C}$ we define the "thickening" $B_\delta(M)$ by

$$B_\delta(M) := \{w \in \mathbb{C}\colon \operatorname{dist}(w, M) < \delta\},$$

and then the annuli $A_j(\delta)$ of width $\delta > 0$ about the circles $C_j = \partial B_{r_j}(q_j)$ which bound the domain $B \in \mathcal{N}(k)$ given by

$$B = B_{r_1}(q_1) \Big\backslash \bigcup_{j=2}^{k} \overline{B}_{r_j}(q_j)$$

with $\overline{B}_{r_j}(q_j) \subset B_{r_1}(q_1)$ and $\overline{B}_{r_j}(q_j) \cap \overline{B}_{r_\ell}(q_\ell) = \emptyset$ for $2 \le j, \ell \le k$, $j \ne \ell$:

$$A_j(\delta) := B \cap B_\delta(C_j), \quad j = 1, \dots, k.$$

We have

$$A_j(\delta) \cap A_\ell(\delta) = \emptyset \quad \text{for } j \ne \ell,\ 1 \le j, \ell \le k,$$

provided that

$$\delta < \delta_0 := \tfrac{1}{2} \min\{\operatorname{dist}(C_j, C_\ell)\colon j \ne \ell,\ 1 \le j, \ell \le k\}.$$

Now we turn to the second step in the proof of Theorem 2, which consists in proving the following result:

(ii) *For any closed C^1-curve in $A_j(\delta)$, $0 < \delta < \delta_0$, which is homologous to C_j, we have*

$$\int_{\beta_j} \phi(w)\, dw = 0 \tag{8}$$

and

$$\int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \tag{9}$$

for $j = 1, \dots, k$.

To prove this result, we fix some $j \in \{1, \dots, k\}$ and consider three vector fields $\eta_1, \eta_2, \eta_3 \in C_c^\infty(B \cup C_j, \mathbb{C})$ with

$$\frac{\partial}{\partial \overline{w}} \eta_\ell(w) = 0 \quad \text{in } A_j(\delta), \quad \ell = 1, 2, 3,$$

satisfying

$$\eta_1(w) := \begin{cases} \zeta & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta), \end{cases}$$

where ζ is an arbitrary complex number,

$$\eta_2(w) := \begin{cases} w - q_j & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta), \end{cases}$$

$$\eta_3(w) := \begin{cases} -i(w - q_j) & \text{for } w \in \overline{A}_j(\delta), \\ 0 & \text{for } w \in \overline{B} \setminus \overline{A}_j(2\delta). \end{cases}$$

Let C'_j be the circle $\partial A_j(\delta) \setminus C_j$ and apply step (i) to $\alpha := C'_j$ and $\eta := \eta_1$. Then, for any closed curve β_j in $A_j(\delta)$ homologous to α and therefore homologous to C_j, it follows that

$$\operatorname{Im}\left[\zeta \int_{\beta_j} \phi(w)\, dw\right] = 0 \quad \text{for all } \zeta \in \mathbb{C}.$$

This yields formula (8).

Applying the same reasoning to $\eta := \eta_2$ and $\eta := \eta_3$ respectively, we obtain

$$\operatorname{Im} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0 \quad \text{and} \quad \operatorname{Re} \int_{\beta_j} (w - q_j)\phi(w)\, dw = 0,$$

which proves formula (9).

Remark. One can as well choose

$$\eta_2(w) := (w - q_j)^n \quad \text{and} \quad \eta_3(w) := -i(w - q_j)^n \quad \text{on } \overline{A}_j(\delta)$$

with $n \in \mathbb{Z} \setminus \{0\}$ and

$$\eta_2(w) := 0 \quad \text{and} \quad \eta_3(w) := 0 \quad \text{on } \overline{B} \setminus \overline{A}_j(2\delta).$$

Then one obtains

$$\int_{\beta_j} (w - q_j)^n \phi(w)\, dw = 0 \quad \text{for all } n \in \mathbb{Z} \tag{10}$$

and $\beta_j \subset A_j(\delta)$, $0 < \delta < \delta_0$. If $k = 2$ and

$$B = \{w \in \mathbb{C}\colon 0 < r < |w| < 1\} \in \mathcal{N}_1(2),$$

then $\phi(w)$ is holomorphic in B, and thus it can be expanded into a convergent Laurent series:

$$\phi(w) = \sum_{n=-\infty}^{\infty} a_n w^n \quad \text{for } w \in B.$$

Formula (10) then becomes

$$\int_{\beta} w^n \phi(w)\, dw = 0 \quad \text{for all } n \in \mathbb{Z}$$

and $\beta = \{w \in \mathbb{C}\colon |w| = \rho\}$ with $r < \rho < 1$, and we obtain $a_n = 0$ for all $n \in \mathbb{Z}$, i.e. $\phi(w) \equiv 0$. Since every $\tilde{B} \in \mathcal{N}(2)$ is equivalent to some $B \in \mathcal{N}_1(2)$, the assertion of Theorem 2 is proved in case that $k = 2$, and for $k = 1$ the proof follows in the same way. Thus the proof becomes really interesting for $k \geq 3$.

On account of Lemmas 4 and 5, it suffices to prove Theorem 2 under the *additional assumption*

$$B \in \mathcal{N}_1(k) \tag{11}$$

which from now on will be required. In other words, we assume that

$$r_1 = 1, \quad q_1 = q_2 = 0. \tag{11$'$}$$

Now we turn to the third step of the proof. We are going to show

(iii) *One has*: $(w - q_j)^2\phi(w)$ *is continuous on* $B \cup C_j$, *and*

$$\operatorname{Im}[(w - q_j)^2\phi(w)] = 0 \quad \textit{for } w \in C_j, \quad 1 \leq j \leq k. \tag{12}$$

We will first verify (12) for the case $j = 1$ where $q_1 = 0$ and $r_1 = 1$; by a suitable Möbius transformation any of the cases $j = 2, \ldots, k$ will be reduced to $j = 1$.

Fix some $\delta \in (0, \delta_0)$, and let ψ be an arbitrary real valued function with $\psi \in C^1(\overline{B})$ and

$$\psi(w) = 0 \quad \text{for } w \in \overline{B} \text{ with } |w| \leq 1 - 2\delta.$$

Set

$$\eta(w) := -i[w\psi(w)] \quad \text{for } w \in \overline{B}.$$

By (6) we have

$$0 = \operatorname{Re} \int_B \eta_{\overline{w}}\phi \, du \, dv = \lim_{R \to 1-0} \operatorname{Re} \int_{B \cap B_R(0)} \eta_{\overline{w}}\phi \, du \, dv.$$

As in the proof of step (i) it follows that

$$0 = -\lim_{R \to 1-0} \operatorname{Im} \int_{\partial B_R(0)} iw\psi(w)\phi(w) \, dw.$$

With $w = \mathrm{Re}^{i\theta}$ and $dw = iw \, d\theta$ we obtain

$$0 = \lim_{R \to 1-0} \int_0^{2\pi} \psi(\mathrm{Re}^{i\theta}) h(\mathrm{Re}^{i\theta}) \, d\theta \tag{13}$$

if we denote by $h : B \to \mathbb{R}$ the harmonic function

$$h(w) := \operatorname{Im}[w^2\phi(w)], \quad w \in B.$$

Suppose now that ψ depends also on a further parameter $z \in \overline{B}_\rho(0)$ such that $\psi(w,z)$ is of class C^1 for (w,z) satisfying $1-\delta \le |w| \le 1$, $|z| \le \rho \le 1-\sigma$ for $\sigma \in (0, 2\delta)$. Then we obtain for $f := \mathrm{Re}[\eta_{\overline{w}}(\cdot, z)\phi]$ that

$$\begin{aligned}\left|\int_{B\cap B_R(0)} f\,du\,dv\right| &= \left|\int_B f\,du\,dv - \int_{B\setminus B_R(0)} f\,du\,dv\right| \\ &= \left|\int_{B\setminus B_R(0)} f\,du\,dv\right| \\ &\le M\cdot \int_{B\setminus B_R(0)} |\phi|\,du\,dv \quad \text{for } R > 1-\sigma\end{aligned}$$

where

$$M := \sup\{|\eta_{\overline{w}}(w,z)| \colon 1-\delta \le |w| \le 1,\ |z| \le \rho\} < \infty.$$

Thus we achieve the uniform convergence of $\int_{B\cap B_R(0)} f(w,z)\,du\,dv$ to zero as $R \to 1-0$ for $z \in \overline{B}_\rho(0)$, i.e.

$$\mathrm{Re}\int_{B\cap B_R(0)} \eta_{\overline{w}}(w,z)\phi(w)\,du\,dv \to 0 \quad \text{uniformly in } z \in \overline{B}_\rho(0) \text{ as } R \to 1-0,$$

since $|\phi| \in L_1(B)$. This implies that the convergence in (13) is uniform with respect to $z \in \overline{B}_\rho(0)$, i.e.

$$(14) \quad \int_0^{2\pi} \psi(\mathrm{Re}^{i\theta}, z) h(\mathrm{Re}^{i\theta})\,d\theta \to 0 \quad \text{uniformly in } z \in \overline{B}_\rho(0) \text{ as } R \to 1-0.$$

For $0 \le r \le \rho < 1-\sigma < R < 1$ and $w = \mathrm{Re}^{i\theta}$, $z = re^{i\vartheta}$ we introduce the Poisson kernel $K(w,z)$ of the ball $B_R(0)$ with respect to $w \in \partial B_R(0)$ and $z \in \overline{B}_\rho(0)$,

$$K(w,z) := \frac{R^2 - r^2}{2\pi[R^2 - 2rR\cos(\theta - \vartheta) + r^2]}.$$

Furthermore let ξ be a radial cut-off function of class $C^\infty(\mathbb{R})$ with $\xi(r) = 1$ for $r \ge 1-\sigma/2$ and $\xi(r) = 0$ for $r \le 1-\sigma$, $0 < \sigma < 2\delta$, and set

$$\psi(w,z) := \xi(|w|)K(w,z)$$

for $z \in \overline{B}_\rho(0)$, $0 < \rho < 1-\sigma$, and $1-2\delta < 1-\sigma \le |w| \le 1$. Then $\psi(w,z)$ has the properties required above, and for $R = |w| \ge 1-\sigma/2$ one has $\xi(|w|) = 1$. Consequently it follows from (14) that

$$H_R(z) := \int_0^{2\pi} K(\mathrm{Re}^{i\theta}, z)h(\mathrm{Re}^{i\theta})d\theta, \quad z \in B_R(0),$$

satisfies

$$(15) \quad \|H_R\|_{C^0(\overline{B}_\rho(0))} \to 0 \quad \text{as } R \to 1-0 \text{ for any } \rho < 1-\sigma,\ 0 < \sigma < 2\delta.$$

By Poisson's formula and Schwarz's theorem it follows that H_R is harmonic in the disk $B_R(0)$ and can be extended to a continuous function on $\overline{B}_R(0)$ satisfying

$$H_R(w) = h(w) \quad \text{for } w \in \partial B_R(0). \tag{16}$$

In the sequel, $A(r, r')$ denotes the annulus

$$A(r, r') := \{w \in \mathbb{C} \colon r < |w| < r'\} \quad \text{for } 0 < r < r' < \infty.$$

For $R_0 := 1 - 2\delta < R < 1$ we now consider the excess function $E_R : \overline{A}(R_0, R) \to \mathbb{R}$ defined by

$$E_R(w) := h(w) - H_R(w) \quad \text{for } w \in \overline{A}(R_0, R),$$

which is continuous on $\overline{A}(R_0, R)$, harmonic in $A(R_0, R)$, and vanishes on the circle $\partial B_R(0)$ according to (16). By reflection in this circle we can extend E_R to a continuous function on $\overline{A}(R_0, R')$ with $R' := R^2/R_0$ which is harmonic in $A(R_0, R')$ and satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| = \max_{\partial B_{R'}(0)} |E_R|. \tag{17}$$

Set

$$C = C(R_0) := 2 \max_{\partial B_{R_0}(0)} |h|, \quad R_0 = 1 - 2\delta,$$

and for arbitrarily chosen $\epsilon > 0$ we pick a number σ with

$$0 < \sigma < \min\left\{\frac{\delta}{2}, \frac{\epsilon\delta}{2C}\right\}. \tag{18}$$

Because of (15) there is a number $R_1 \in (1 - (\sigma/2), 1)$ such that

$$\max_{\partial B_{R_0}(0)} |H_R| < C/2 \quad \text{for all } R \in (R_1, 1),$$

and so $E_R = h - H_R$ satisfies

$$\max_{\partial B_{R_0}(0)} |E_R| < C \quad \text{for all } R \in (R_1, 1).$$

In conjunction with (17) the maximum principle then implies

$$\max_{\overline{A}(R_0, R')} |E_R| < C \quad \text{for all } R \in (R_1, 1) \tag{19}$$

where $R_0 = 1 - 2\delta$ and $R' = R^2/R_0$.

For $R \in (R_1, 1)$ we have $1 - \sigma/2 < R < 1$ and therefore $R - (1 - \sigma) > \sigma/2 > 0$. For any $w \in A(1 - \sigma, R)$ it follows that

$$\operatorname{dist}(w, \partial A(R_0, R')) > (1-\sigma) - R_0 = 2\delta - \sigma > \delta.$$

Applying Cauchy's estimate to ∇E_R on $A(1-\sigma, R)$ we then infer from (19) that

$$\max_{\overline{A}(1-\sigma,R)} |\nabla E_R| \le \frac{C(R_0)}{\delta} \quad \text{for } R \in (R_1, 1).$$

Since $E_R(w) = 0$ for $|w| = R$, we can write

$$|E_R((1-\sigma)e^{i\theta})| \le \int_{1-\sigma}^{R} |\partial_r E_R(re^{i\theta})|\, dr \le \sigma \frac{C}{\delta} < \frac{\epsilon\delta}{2C} \cdot \frac{C}{\delta}$$

whence

$$|E_R(w)| < \frac{\epsilon}{2} \quad \text{for } |w| = 1-\sigma \text{ and } R_1 < R < 1,$$

where $R_1 \in (1-\sigma/2, 1)$ was chosen above and σ is a fixed number satisfying (18).

Applying once more (15) it follows that for the chosen σ there is a number $R_2 \in [R_1, 1)$ such that

$$\max_{\overline{B}_{1-\sigma}(0)} |H_R| < \frac{\epsilon}{2} \quad \text{for all } R \in (R_2, 1).$$

Because of

$$h(w) = E_R(w) + H_R(w) \quad \text{for } w \in \overline{A}(R_0, R)$$

and $R_0 = 1 - 2\delta < 1 - \sigma < 1 - \sigma/2 < R_1 \le R_2 < R < 1$ we arrive at

$$|h(w)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for } |w| = 1 - \sigma.$$

This implies for the harmonic function $h(w) = \operatorname{Im}[w^2\phi(w)]$ that

$$\lim_{\sigma \to +0} \max_{\partial B_{1-\sigma}(0)} |h| = 0,$$

and so we can extend h continuously to $B \cup C_1$ with $C_1 = \partial B_1(0)$ by setting

$$h(w) = 0 \quad \text{for } w \in C_1,$$

which completes the proof of (12) for $j = 1$.

Note that for the proof of (12) in the case $j = 1$ we only have used $q_1 = 0$, $r_1 = 1$ and the fact that $C_1 = \partial B_1(0)$ contains the other boundary circles $C_2, \dots, C_k$ in its interior domain $B_1(0)$. Therefore we can reduce the cases $j = 2, \dots, k$ to $j = 1$ by applying the Möbius transformation $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, defined by

$$z = \mu(w) := \frac{r_j}{w - q_j}$$

where $C_j = \partial B_{r_j}(q_j) = \{w \in \mathbb{C}\colon |w - q_j| = r_j\}$. The mapping μ maps B into another k-circle domain B^* whose exterior circle is $C_1 = \partial B_1(0)$, and $C_1 = \mu(C_j)$. Let $\nu := \mu^{-1}$ be the inverse of μ, and set

$$X^* := X \circ \nu \quad \text{with } \operatorname{dom}(X^*) = B^*.$$

Then by Lemma 5 we have

$$\int_{B^*} \zeta_{\overline{w}} \phi^* \, du \, dv = 0 \quad \text{for all } \zeta \in C^1(\overline{B}^*, \mathbb{R}^2),$$

and the above reasoning yields

$$\operatorname{Im}[z^2 \phi^*(z)] = 0 \quad \text{for } z \in C_1$$

and $\phi^*(z) = 4\langle X^*_z, X^*_z \rangle = a^*(z) - ib^*(z)$. A straight-forward computation yields

$$(w - q_j)^2 \phi(w) = z^2 \phi^*(z) \quad \text{for } z \in C_1 \text{ and } w = \nu(z) \in C_j.$$

Thus we have shown that $(w - q_j)^2 \phi(w)$ is continuous on $B \cup C_j$ and

$$\operatorname{Im}[(w - q_j)^2 \phi(w)] = 0 \quad \text{for } w \in C_j,\ 2 \le j \le k,$$

and so the proof of assertion (iii) is complete. □

Let us review the assertion of (iii). We have shown that each of the holomorphic functions

$$F_j(w) := (w - q_j)^2 \phi(w), \quad w \in B,$$

$1 \le j \le k$, has a harmonic imaginary part $h_j := \operatorname{Im} F_j$ which can continuously be extended to $B \cup C_j$ by setting $h_j = 0$ on C_j. Then the reflection principle for harmonic functions yields that h_j can be extended as a harmonic function beyond C_j. Inspecting the Cauchy–Riemann equations, it follows that F_j can be extended holomorphically across C_j, and therefore ϕ can be extended holomorphically to some domain G with $\overline{B} \subset G \subset \mathbb{C}$. This implies that either $\phi(w) \equiv 0$ in $\overline{B}$, or else ϕ has finitely many zeros in $\overline{B}$. Employing a method due to Hans Lewy we will show that the second case is impossible, thus verifying the assertion of Theorem 2. To this end we turn to the next step of the proof:

(iv) *If $\phi(w) \not\equiv 0$ in $\overline{B}$ then ϕ has at least four zeros on each boundary circle C_j of B.*

To prove this, let r, θ be polar coordinates around q_j defined by $w = q_j + re^{i\theta}$, and introduce the 2π-periodic functions

$$f_j(\theta) := r_j^2 e^{i2\theta} \phi(q_j + r_j e^{i\theta}), \quad j = 1, \ldots, k,$$

that are real analytic in θ and satisfy $f_j(\theta) \in \mathbb{R}$ for $\theta \in \mathbb{R}$ on account of (12). By step (ii) applied to $\beta_j := C_j$ it follows that

$$i \int_0^{2\pi} f_j(\theta)\, d\theta = 0 \quad \text{and} \quad i r_j^{-1} \int_0^{2\pi} e^{-i\theta} f_j(\theta)\, d\theta = 0,$$

whence

$$(20) \quad \int_0^{2\pi} f_j(\theta)\, d\theta = 0, \quad \int_0^{2\pi} f_j(\theta) \cos\theta\, d\theta = 0, \quad \int_0^{2\pi} f_j(\theta) \sin\theta\, d\theta = 0.$$

Then $f_j(\theta) \not\equiv \text{const}$, because the first equation would imply $f_j(\theta) \equiv 0$ and therefore $\phi(w) \equiv 0$ on $\partial B_{r_j}(q_j)$ which is impossible since $\phi(w)$ has only finitely many zeros in $\overline{B}$. Moreover $\int_0^{2\pi} f_j(\theta)\, d\theta = 0$ shows that $f_j(\theta)$ must change its sign in $[0, 2\pi)$ at least once, and so it has a positive maximum and a negative minimum. Correspondingly $f_j(\theta)$ possesses two zeros $\theta_0, \theta_1 \in [0, 2\pi)$, i.e. $|\theta_0 - \theta_1| < 2\pi$ since f_j is periodic. By choosing the polar angle θ suitably we can assume that $f_j(\theta)$ has the two zeros θ_0 and $-\theta_0$ with some $\theta_0 \in (0, \pi)$, while the three equations (20) remain valid. This yields

$$(21) \qquad \int_{-\pi}^{\pi} f_j(\theta)[\cos\theta - \cos\theta_0]\, d\theta = 0,$$

and so the function $f_j(\theta)[\cos\theta - \cos\theta_0]$ changes its sign in $(-\pi, \pi)$. Since $g(\theta) := \cos\theta - \cos\theta_0$ with $g'(\theta) = -\sin\theta$ satisfies $g'(\theta) > 0$ for $-\pi < \theta < 0$, $g'(\theta) < 0$ for $0 < \theta < \pi$, it follows that

$$g(\theta) < 0 \quad \text{on } (-\pi, -\theta_0) \cup (\theta_0, \pi), \quad g(\theta) > 0 \quad \text{on } (-\theta_0, \theta_0).$$

If $f_j(\theta)$ would have no other zero than θ_0 and $-\theta_0$ then $f_j(\theta)g(\theta)$ did not change its sign in $(-\pi, \pi)$, but this contradicts (21). Thus there is a third zero θ_3 of $f_j(\theta)$ in $(-\pi, \pi)$. We claim that there is even a fourth zero θ_4 of f_j in $(-\pi, \pi)$. In fact suppose that $f_j(\theta) \neq 0$ for $\theta \in (-\pi, \pi)$ with $\theta \neq \pm\theta_0, \theta_3$. If $\theta_3 \in (-\theta_0, \theta_0)$ then again $f_j(\theta)g(\theta)$ would not change its sign, a contradiction to (21). The other two cases $\theta_3 < -\theta_0$ and $\theta_0 < \theta_3$ can be transformed to the case $-\theta_0 < \theta_3 < \theta_0$ by a shift of θ which keeps (21) fixed because of (20). This completes the proof of assertion (iv). □

Now we turn to the final step in the proof of Theorem 2:

(v) *We have $\phi(w) \equiv 0$ in B.*

Suppose that this were false. Then $\phi(w)$ had only finitely many zeros in $\overline{B}$ as we have observed before. Let $w_m \in B$ be the interior zeros of ϕ with the multiplicities μ_m, $m = 1, \dots, M$, and $\zeta_\ell \in \partial B$ be the boundary zeros of ϕ with the multiplicities ν_ℓ, $\ell = 1, \dots, L$. Set $N := \mu_1 + \cdots + \mu_M$, and choose $\rho > 0$ sufficiently small. Then, by Rouché's formula, the number $N \geq 0$ is given by

$$N = \frac{1}{2\pi i} \int_{\partial G_\rho} \frac{\phi'(w)}{\phi(w)}\, dw, \quad G_\rho := B \setminus \bigcup_{\ell=1}^{L} \overline{B}_\rho(\zeta_\ell).$$

The boundary ∂G_ρ consists of $\beta_j(\rho) := C_j \cap \partial G_\rho$, $j = 1, \dots, k$, and of the circular arcs $\gamma_\ell(\rho) := \partial B_\rho(\zeta_\ell) \cap B$, $\ell = 1, \dots, L$. Recall also that $F_j(w) = (w - q_j)^2 \phi(w)$ is holomorphic in $B \cup C_j$ and *real valued on* C_j. Then we have

$$d \log F_j(w) = d \log (w - q_j)^2 + d \log \phi(w) \quad \text{on } \beta_j,$$

whence

$$\frac{\phi'(w)}{\phi(w)}\, dw = \frac{F_j'(w)}{F_j(w)}\, dw - \frac{2}{w - q_j}\, dw \quad \text{for } w \in \beta_j.$$

This implies

$$\frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{\phi'(w)}{\phi(w)}\, dw = I_j(\rho) \setminus K_j(\rho)$$

with

$$I_j(\rho) := \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{F_j'(w)}{F_j(w)}\, dw$$

and

$$K_j(\rho) := 2 \frac{1}{2\pi i} \int_{\beta_j(\rho)} \frac{dw}{w - q_j}.$$

We have

$$\lim_{\rho \to +0} K_j(\rho) = \begin{cases} 2 & \text{for } j = 1, \\ -2 & \text{for } j = 2, \dots, k, \end{cases}$$

and it will be proved below that

$$\lim_{\rho \to +0} I_j(\rho) = 0. \tag{22}$$

Thus

$$N = \lim_{\rho \to +0} \sum_{j=1}^{k} [I_j(\rho) - K_j(\rho)] + \lim_{\rho \to +0} \sum_{\ell=1}^{L} P_\ell(\rho)$$

with

$$P_\ell(\rho) := \frac{1}{2\pi i} \int_{\gamma_\ell(\rho)} \frac{\phi'(w)}{\phi(w)}\, dw.$$

Since ϕ is mirror symmetric with respect to the inversion at C_j it follows that (for $\gamma_\ell^*(\rho)$ as reflection of $\gamma_\ell(\rho)$ at C_j)

$$\begin{aligned} \lim_{\rho \to +0} P_\ell(\rho) &= \frac{1}{4\pi i} \lim_{\rho \to +0} \int_{\gamma_\ell(\rho) \cup \gamma_\ell^*(\rho)} \frac{\phi'(w)}{\phi(w)}\, dw \\ &= \frac{1}{4\pi i} \lim_{\rho \to +0} \int_{-\partial B_\rho(\zeta_\ell)} \frac{\phi'(w)}{\phi(w)}\, dw = -\frac{\nu_\ell}{2} \end{aligned}$$

since the positive orientation of G_ρ implies that the circles $\partial B_\rho(\zeta_\ell)$ are to be taken as negatively oriented. Since $L \geq 4k$ and $\nu_\ell \geq 1$ it follows that

$$N = -2 + 2(k-1) - \frac{1}{2}\sum_{\ell=1}^{L} \nu_\ell \leq -4 + 2k - \frac{1}{2} \cdot 4k = -4,$$

a contradiction to $N \geq 0$. Therefore we obtain $\phi(w) \equiv 0$ on $\overline{B}$.

It remains to prove (22). Since

$$2\pi i I_j(\rho) = \int_{\beta_j(\rho)} d\log|F_j(w)| = \int_{\beta_j'(\rho)} d\log|\psi(\theta)|$$

with $\psi(\theta) := F_j(q_j + r_j e^{i\theta})$ and

$$\beta_j'(\rho) = [0, \theta_1 - \epsilon(\rho)] \cup \bigcup_{s=1}^{p-1} [\theta_s + \epsilon(\rho), \theta_{s+1} - \epsilon(\rho)] \cup [\theta_p + \epsilon(\rho), 2\pi],$$

where $\epsilon = \epsilon(\rho) \to +0$ as $\rho \to +0$, and $\zeta_s := e^{i\theta_s}$ are the zeros of F_j on C_j, we obtain

$$\int_{\beta_j'(\rho)} d\log|\psi(\theta)| = \sum_{s=1}^{p+1} [\log|\psi(\theta)|]_{a_s(\rho)}^{b_s(\rho)}$$

with

$$\begin{aligned}
&a_1(\rho) = 0, \quad a_2(\rho) = \theta_1 + \epsilon(\rho), \quad \ldots, \quad a_p(\rho) = \theta_{p-1} + \epsilon(\rho),\\
&a_{p+1}(\rho) = \theta_p + \epsilon(\rho),\\
&b_1(\rho) = \theta_1 - \epsilon(\rho), \quad b_2(\rho) = \theta_2 - \epsilon(\rho), \quad \ldots, \quad b_p(\rho) = \theta_p - \epsilon(\rho),\\
&b_{p+1}(\rho) = 2\pi.
\end{aligned}$$

Thus we infer from $\psi(0) = \psi(2\pi)$

$$\begin{aligned}
\int_{\beta_j'(\rho)} d\log|\psi(\theta)| &= \sum_{s=1}^{p} \big[\log|\psi(b_s(\rho))| - \log|\psi(a_{s+1}(\rho))|\big]\\
&= \sum_{s=1}^{p} \log\left|\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))}\right| \to 0 \quad \text{for } \rho \to +0
\end{aligned}$$

since

$$\frac{\psi(\theta_s - \epsilon(\rho))}{\psi(\theta_s + \epsilon(\rho))} \to 1 \quad \text{as } \rho \to +0.$$

Thus we conclude $I_j(\rho) \to 0$ as $\rho \to +0$, and we have verified (22).

This completes the proof of Theorem 2. □

8.3 Cohesive Sequences of Mappings

Let $\{B_m\}$ be a sequence of k-circle domains

$$B_m = B(q^{(m)}, r^{(m)}) \in \mathcal{N}(k)$$

with $q^{(m)} = (q_1^{(m)}, \dots, q_k^{(m)}) \in \mathbb{C}^k$, $r^{(m)} = (r_1^{(m)}, \dots, r_k^{(m)}) \in \mathbb{R}^k$, $r_j^{(m)} > 0$.

We say that $\{B_m\}$ converges to the domain $B := B_{r_1}(q_1) \setminus \bigcup_{j=2}^k \overline{B_{r_j}(q_j)}$, symbol:

$$B_m \to B \quad \text{as } m \to \infty, \quad \text{or} \quad \lim_{m\to\infty} B_m = B,$$

if $q^{(m)} \to q$ in $\mathbb{C}^k$ and $r^{(m)} \to r$ in $\mathbb{R}^k$.

By $\overline{\mathcal{N}}(k)$ and $\overline{\mathcal{N}}_1(k)$ we denote the set of domains B in $\mathbb{C}$ that are limits of converging sequences $\{B_m\}$ in $\mathcal{N}(k)$ and $\mathcal{N}_1(k)$ respectively.

Clearly the limit B of a sequence $\{B_m\} \subset \mathcal{N}(k)$ need not be a k-circle domain again, i.e. B might be "degenerate" in the sense that $B \in \overline{\mathcal{N}}(k) \setminus \mathcal{N}(k)$. Let us investigate how the boundary circles

$$C_j^{(m)} := \partial B_{r_j^{(m)}}(q_j^{(m)}) \quad \text{of } B_m = B_{r_1^{(m)}}(q_1^{(m)}) \Big\backslash \bigcup_{j=2}^k \overline{B_{r_j^{(m)}}(q_j^{(m)})}$$

behave if the B_m converge to a degenerate domain with the "boundary circles" $C_j = \partial B_{r_j}(q_j)$. Here r_j might be zero; then C_j is just the point q_j, i.e. $C_j^{(m)} \to q_j$. Another form of degeneration is that two limit circles C_j and C_ℓ, $j \neq \ell$, are true circles which "touch" each other (this includes the possibility $C_j = C_\ell$).

We distinguish three kinds of degeneration:

Type 1. Two limits C_j and C_ℓ, $j \neq \ell$, are true circles which touch each other, i.e. either $C_j = C_\ell$ or $C_j \cap C_\ell = \{w_0\}$ for some $w_0 \in \overline{B}$.

Type 2. One limit C_ℓ is a point p which lies on a true limit circle C_j.

Type 3. One limit C_ℓ is a point p which does not lie on any true limit circle.

For our purposes it suffices to consider degenerate limits B of domains $B_m \in \mathcal{N}_1(k)$. Here we have for all $m \in \mathbb{N}$ that

$$C_1^{(m)} = C := \partial B_1(0), \quad C_2^{(m)} = \partial B_{r_2^{(m)}}(0), \quad 0 < r_2^{(m)} < 1.$$

Case (a): $k = 2$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to 0$, i.e. $C_1 = C_2 = C$ (type 1) or $C_2 = \{0\}$ (type 3), whereas type 2 cannot occur for a degenerate limit B.

Case (b): $k \geq 3$. Then either $r_2^{(m)} \to 1$ or $r_2^{(m)} \to r \in [0, 1)$.

(b1) If $r_2^{(m)} \to 1$ then $C_1 = C_2 = C$ and $C_j = \{q_j\}$ with $j = 3, \dots, k$. Thus B is both of type 1 and 2.

(b2) If $r_2^{(m)} \to r_2$ with $0 \le r_2 < 1$, then $C_1 = C$ and either $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$ or $C_2 = \{0\}$. Here we have at least one of the following possibilities:

(i) B is of type 1 with $C_j \cap C_\ell = \{w_0\}$ for some $w_0 \in \overline{B}$, and possibly also of type 2 or type 3 or both.
(ii) B is not of type 1, but of type 2, or of type 3, or both of type 2 and 3.

The following result is obvious:

Lemma 1. *From any sequence of domains $B_m \in \mathcal{N}_1(k)$ we can extract a subsequence $\{B_{m_j}\}$ with $B_{m_j} \to B \in \overline{\mathcal{N}}_1(k)$ as $j \to \infty$.*

We now want to state conditions ensuring that the limit B of domains $B_m \in \mathcal{N}_1(k)$ is nondegenerate, that is $B \in \mathcal{N}_1(k)$. A first result in this direction is

Proposition 1. *Let $\{X_m\}$ be a sequence of mappings $X_m \in \mathcal{C}(\Gamma)$ with $B_m =$ $\mathrm{dom}(X_m) \in \mathcal{N}_1(k)$, $k \ge 2$, where $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ is a contour consisting of k rectifiable, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$ and suppose that $B_m \to B$ for $m \to \infty$ as well as*

$$D(X_m) \le M \quad \textit{for all } m \in \mathbb{N}$$

and some constant $M > 0$. Then $B \in \overline{\mathcal{N}}_1(k)$ cannot be degenerate of type 1.

Proof. Let $\mu(\Gamma)$ be the minimal distance of the curves $\Gamma_1, \dots, \Gamma_k$ from each other, i.e.

$$\mu(\Gamma) := \min\{\mathrm{dist}(\Gamma_j, \Gamma_\ell) \colon 1 \le j, \ell \le k,\ j \ne \ell\} > 0. \tag{1}$$

If B were of type 1, there would be $j, \ell \in \{1, \dots, k\}$ with $j \ne \ell$ such that $C_j^{(m)} \to C_j$ and $C_\ell^{(m)} \to C_\ell$ as $m \to \infty$, where C_j and C_ℓ are true circles with $C_j \cap C_\ell \ne \emptyset$. Let $w_0 \in C_j \cap C_\ell$, and introduce polar coordinates ρ, θ about $w_0 \colon w = w_0 + \rho e^{i\theta}$. There is a representative

$$Z_m(\rho, \theta) := X_m(w_0 + \rho e^{i\theta})$$

of X_m which, for almost all $\rho \in (0,1)$, is absolutely continuous in $\theta \in [\theta_1, \theta_2]$ along each arc $\gamma(\rho) := \{w_0 + \rho e^{i\theta} \colon \theta_1 \le \theta \le \theta_2\}$ contained in $\overline{B}_m$; we call $\gamma(\rho)$ X_m-admissible. The Courant–Lebesgue lemma yields: *For each $m \in \mathbb{N}$ and each $\delta \in (0,1)$ there is an X_m-admissible* arc $\gamma_m(\rho) = \{w_0 + \rho e^{i\theta} \colon \theta_1^m \le \theta \le \theta_2^m\}$ *in $\overline{B}_m$ with $\delta < \rho < \sqrt{\delta}$ such that*

$$\mathrm{osc}(Z_m, \gamma_m(\rho)) \le 2\left\{2\pi M \left(\log \frac{1}{\delta}\right)^{-1}\right\}^{1/2}. \tag{2}$$

Furthermore, there is an $R > 0$ such that $\partial B_r(w_0)$ intersects C_j and C_ℓ for $0 < r < 2R$. Let δ be an arbitrary number with $0 < \sqrt{\delta} < R$. Since $C_j^{(m)} \to C_j$ and $C_\ell^{(m)} \to C_\ell$ as $m \to \infty$, there is a number $N(\delta, R) \in \mathbb{N}$ such that the following holds:

For $m > N(\delta, R)$ and $\delta < \rho < R$ the circle $\partial B_\rho(w_0)$ intersects $C_j^{(m)}$ and $C_\ell^{(m)}$.

Then there is an X_m-admissible subarc $\gamma_m(\rho)$ of $\partial B_\rho(w_0) \cap B_m$ satisfying $\delta < \rho < \sqrt{\delta}$ which has its endpoints on two circles $C_{j'}^{(m)}$ and $C_{\ell'}^{(m)}$ (which might be different from $C_j^{(m)}$ and $C_\ell^{(m)}$), and, moreover, which satisfies (2).

It follows that

$$\mu(\Gamma) \le \operatorname{dist}(\Gamma_{j'}, \Gamma_{\ell'}) \le 2\sqrt{\frac{2\pi M}{\log \frac{1}{\delta}}} \quad \text{for } 0 < \delta \ll 1$$

whence we obtain $\mu(\Gamma) = 0$ letting $\delta \to +0$, a contradiction to (1). $\square$

Corollary 1. *Under the assumptions of Proposition 1, the limit $B \in \overline{\mathcal{N}}_1(k)$ of the domains $B_m \in \mathcal{N}_1(k)$ can only be degenerate of type 3 if $k = 2$. Moreover, if $k \ge 3$ then B can only be degenerate of type 2, or of type 3, or both.*

These two types of degeneration may indeed occur if we do not impose a further condition, namely a *condition of cohesion*.

If we operate with sequences in the class $\overline{\mathcal{C}}(\Gamma)$, defined by

$$\overline{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^3)$$

one can conveniently use Courant's condition of cohesion. In this way one can solve the minimum problem

$$\text{“}D \to \min \quad \text{in } \overline{\mathcal{C}}(\Gamma)\text{”}.$$

In the same way one could also solve the problem

$$\text{“}A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma)\text{”}$$

where $A^\epsilon = (1-\epsilon)A + \epsilon D$, $0 < \epsilon \le 1$, but this would require a strong regularity theorem, which we here want to avoid in order to make the minimization procedure as transparent as possible. The prize for this is that we have to work with another condition of cohesion which is a bit more cumbersome to formulate than Courant's condition. This second condition is a simplified version of a stipulation introduced by M. Kurzke [1]; cf. also Kurzke and von der Mosel [1].

Definition 1. *A sequence $\{X_m\}$ of mappings $X_m \in \overline{H}_2^1(B_m, \mathbb{R}^3)$ with $B_m \in \mathcal{N}(k)$, $k \ge 2$, is said to be C-**cohesive** if there is an $\epsilon > 0$ such that, for each $m \in \mathbb{N}$, any closed continuous curve $c : S^1 \to \mathbb{R}^2$ with $\gamma := c(S^1) \subset \overline{B}_m$ and $\operatorname{diam} X_m|_\gamma < \epsilon$ is homotopic to zero in $\overline{B}_m$.*

For $X \in H_2^1(B, \mathbb{R}^3)$, the composition $X \circ c$ of X with a closed curve $c \in C^0(S^1, \overline{B})$ is not defined in the usual sense. In order to give it a well-defined meaning we restrict ourselves to special curves c. Suppose that γ is a closed Jordan curve in $\overline{B}$, i.e. the image $\gamma = c(S^1)$ of S^1 under a homeomorphism $c : S^1 \to \gamma \subset \overline{B}$. If the inner domain G of γ is strong Lipschitz (i.e. $G \in C^{0,1}$) then X has a well-defined trace $Z = $ "$X|_\gamma$" on $\gamma = \partial G$, which is of class $L_2(\gamma, \mathbb{R}^3)$. If Z has a continuous representative $\gamma \to \mathbb{R}^3$ we denote it again by Z and call it the *continuous representative of* X *on* γ. Then $Z \circ c : S^1 \to \mathbb{R}^3$ is a well-defined, closed, continuous curve in $\mathbb{R}^3$. (Note that G need not be a subdomain of B.)

In applications G will be either (i) a disk, or (ii) a two-gon bounded by two circular arcs γ_1 and γ_2. In case (i), X is represented by a mapping $X^*(r, \theta)$ with respect to polar coordinates r, θ about the origin of the disk G of radius $R \in (0, 1)$ such that $X^*(r, \theta)$ is absolutely continuous with respect to $\theta \in \mathbb{R}$ for all $r \in (0, 1) \setminus N_1$ where N_1 is a 1-dimensional null set and $R \notin N_1$, and similarly $X^*(r, \theta)$ is absolutely continuous with respect to $r \in (\epsilon, 1 - \epsilon)$, $0 < \epsilon \ll 1$, for almost all $\theta \in \mathbb{R}$. Then the continuous representative $Z = $ "$X|_\gamma$" of X on the circle $\gamma = \partial G$ is given by $Z = X^*(R, \cdot)$. In case (ii), γ_1 is a subarc of ∂B, $B = \operatorname{dom}(X)$, and γ_2 is a circular subarc in B with the same endpoints as γ_1. Here the continuous representative $Z = $ "$X|_\gamma$" is the continuous trace of X on γ_1 (recall that for $X \in \mathcal{C}(\Gamma)$ we have "$X|_{\partial B}$" $\in C^0(\partial B, \mathbb{R}^3)$), while on γ_2 the trace $Z = $ "$X|_\gamma$" is given as in (i) by

$$Z(w_0 + Re^{i\theta}) = X^*(R, \theta) \quad \text{for } \theta_1 \le \theta \le \theta_2,$$

where $X^*(r, \theta)$ is a representation of X in polar coordinates around a point w_0 such that $X^*(R, \theta)$ is absolutely continuous in $\theta \in [\theta_1, \theta_2]$.

Definition 2. *A sequence* $\{X_m\}$ *of mappings* $X_m \in H_2^1(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$ *is called* **separating** *if the following holds: For any* $\epsilon > 0$ *there is an* $m_0(\epsilon) \in \mathbb{N}$ *such that for any* $m > m_0(\epsilon)$ *there exists a closed Jordan curve* γ_m *in* $\overline{B}_m$ *bounding a strong Lipschitz interior* B_m^* *such that:*

(i) X_m *possesses a well-defined continuous trace* $Z_m :=$ "$X_m|_{\gamma_m}$" *on* $\gamma_m = \partial B_m^*$;
(ii) $\operatorname{diam} Z_m(\gamma_m) < \epsilon$;
(iii) *A homeomorphic representation* $c_m : S^1 \to \gamma_m$ *of* γ_m *is not homotopic to zero in* $\overline{B}_m$.

Definition 3. *A sequence* $\{X_m\}$ *of mappings* $X_m \in H_2^1(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$ *is said to be* **cohesive** *if none of its subsequences is separating.*

An immediate consequence of these two definitions is

Proposition 2. *A sequence* $\{X_m\}$ *of mappings* $X_m \in H_2^1(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$ *is cohesive if and only if the following holds: For* *every subsequence* $\{X_{m_j}\}$ *of* $\{X_m\}$ *there is an* $\epsilon > 0$ *and a further subsequence* $\{X_{m_{j_\ell}}\}$ *such that for each closed Jordan curve* γ *in* $\overline{B}_{m_{j_\ell}}$ *with a strong Lipschitz interior* G *the continuous trace* $Z_\ell :=$ "$X_{m_{j_\ell}}|_\gamma$" *satisfies* $\operatorname{diam} Z_\ell < \epsilon$, *but a homeomorphic representation* $c : S^1 \to \gamma$ *of* γ *is homotopic to zero in* $B_{m_{j_\ell}}$.

Comparing Proposition 2 with Definition 1 we obtain

Proposition 3. *Any* C*-cohesive sequence* $\{X_m\}$ *of mappings* X_m *of class* $\overline{H}{}^1_2(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$ *is also cohesive.*

Because of this, we in the sequel investigate only cohesive sequences.

Proposition 4. *Let* $\{X_m\}$ *be a sequence of mappings* $X_m \in H^1_2(B_m, \mathbb{R}^3)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}(k)$, *and* $\{\sigma_m\}$ *be a sequence of Möbius transformations from* $\overline{B}{}^*_m$ *onto* $\overline{B}_m$, $B^*_m \in \mathcal{N}(k)$. *Then we have:*

(i) *If* $\{X_m\}$ *is separating, then also* $\{X_m \circ \sigma_m\}$.
(ii) *If* $\{X_m\}$ *is cohesive, then also* $\{X_m \circ \sigma_m\}$.

Proof. We only have to observe that every σ_m is a diffeomorphism from $\overline{B}{}^*_m$ onto $\overline{B}_m$; hence $X_m \circ \sigma_m \in H^1_2(\overline{B}{}^*_m, \mathbb{R}^3)$; furthermore, if γ is a Jordan curve in $\overline{B}_m$ bounding a strong Lipschitz domain, then $\sigma_m^{-1}(\gamma)$ is a Jordan curve in $\overline{B}{}^*_m$ bounding a strong Lipschitz domain. (We also note: If γ consists of circular arcs, then the same holds for $\sigma_m^{-1}(\gamma)$.) □

Theorem 1. *Let* $\{X_m\}$ *be a cohesive sequence of mappings* $X_m \in \mathcal{C}(\Gamma)$ *with* $B_m = \operatorname{dom}(X_m) \in \mathcal{N}_1(k)$, $k \geq 2$, *whose contour* $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ *consists of* k *rectifiable, mutually disjoint Jordan curves* $\Gamma_1, \dots, \Gamma_k$. *Suppose also that there is a constant* $M > 0$ *such that*

$$D(X_m) \leq M \quad \textit{for all } m \in \mathbb{N},$$

and that $B_m \to B$. *Then* B *is of class* $\mathcal{N}_1(k)$.

Proof. Clearly, $B \in \overline{\mathcal{N}}_1(k)$. If B were degenerate, it could not be of type 1 on account of Proposition 1; so we have to show that B can neither be of type 2 nor of type 3. □

Suppose first that B were of type 3, that is: *One or several circles shrink to a point* $p \in \overline{B}$ *which stays away from other limit points or limit circles.* Since $C_1^{(m)} \equiv C := \partial B_1(0)$ for all $m \in \mathbb{N}$, we have $C_1 = C$, and therefore $p \notin C$, i.e. $p \in \overline{B} \setminus C$. Thus the index set $I := \{\ell \in \mathbb{N} \colon 2 \leq \ell \leq k\}$ consists of two disjoint, nonempty sets I_1 and I_2 such that

$$C_j^{(m)} \to \{p\} \quad \text{as } m \to \infty \quad \text{for } j \in I_1,$$

$$C_\ell^{(m)} \to C_\ell \text{ (= point or circle)} \quad \text{as } m \to \infty \text{ with } p \notin C_\ell \quad \text{for } \ell \in I_2.$$

Then we can find a number $\rho_0 \in (0,1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ the following holds true:

$$\begin{aligned} C_j^{(m)} &\subset B_{\rho_0}(p) \quad \text{for } j \in I_1, \\ C_\ell^{(m)} &\cap \overline{B}_{\rho_0}(p) = \emptyset \quad \text{for } \ell \in I_2. \end{aligned} \tag{3}$$

Secondly, for any $\rho_1 \in (0, \rho_0)$ there is an $m_1 = m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \geq m_0$ such that

$$C_j^{(m)} \subset B_{\rho_1}(p) \quad \text{for } j \in I_1 \text{ and } m > m_1(\rho_1).$$

We clearly have

$$\{w \in \mathbb{C} \colon \rho_1 \leq |w - p| \leq \rho_0\} \subset B_m \quad \text{for } m > m_1(\rho_1).$$

Furthermore, by virtue of a well-known extension theorem, there are Sobolev functions $Y_m \in H_2^1(B_1(0), \mathbb{R}^3)$ on the unit disk $B_1(0)$ satisfying

$$Y_m|_{B_m} = X_m \quad \text{for all } m \in \mathbb{N}.$$

We introduce polar coordinates r, θ about p, and choose representations $\tilde{Z}_m(r,\theta)$ of X_m, restricted to $B_{\rho_0}(p) \setminus B_{\rho_1}(p)$, for $m > m_1(\rho_1)$ which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. $\theta \in \mathbb{R}$. By the Courant–Lebesgue lemma we have:

(4) *For any $\epsilon > 0$ there is a number $\delta^*(\epsilon, M, \rho_0) \in (0,1)$, depending only on ϵ, M, ρ_0, which has the following properties*:
(i) $\delta^* < \sqrt{\delta^*} \leq \rho_0$;
(ii) *for any $\rho_1 \in (0, \delta^*)$, any δ with $\rho_1 < \delta < \delta^*$, and all $m > m_1(\rho_1)$, there is a subset $J_m(\delta)$ of $(\delta, \sqrt{\delta})$ with* $\operatorname{meas} J_m(\delta) > 0$ *and* $\operatorname{osc} \tilde{Z}_m(r, \cdot) < \epsilon$ *for all $r \in J_m(\delta)$*;
(iii) *$\tilde{Z}_m(r, \cdot)$ is the trace of X_m on $\partial B_r(p)$ for any $r \in (\rho_1, \rho_0) \setminus \mathcal{S}_m$ where $\mathcal{S}_m$ is a one-dimensional null set, and so we can assume that $J_m(\delta) \subset (\rho_1, \rho_0) \setminus \mathcal{S}_m$.*

Let us now fix some $\epsilon > 0$ and then some $\rho_1 > 0$ with $\rho_1 < \delta^*(\epsilon, M, \rho_0)$. Furthermore we choose some $\delta > 0$ satisfying

$$\rho_1 < \delta < \delta^*(\epsilon, M, \rho_0).$$

Then

$$\{w \in \mathbb{C}\colon \delta < |w-p| < \sqrt{\delta}\} \subset B_{\rho_0}(p) \setminus B_{\rho_1}(p) \subset B_m \quad \text{for all } m > m_1(\rho_1).$$

For any $m > m_1(\rho_1)$ we choose some $r_m \in J_m(\delta)$ and set $\gamma_m := \partial B_{r_m}(p)$. Then γ_m is a Jordan curve in B_m which bounds the strong Lipschitz domain $B_m^* := B_{r_m}(p)$. By construction, Y_m is defined on B_m^*, and $X_m(w) = Y_m(w)$ for $w \in B_{\rho_0}(p) \setminus B_{\rho_1}(p)$. Thus X_m possesses an absolutely continuous representation $Z_m := \tilde{Z}_m(r_m, \cdot) =$ "$X_m|_{\gamma_m}$" with diam $Z_m(\gamma_m) < \epsilon$. Furthermore we have $C_j^{(m)} \subset B_m^*$ for $j \in I_1$. Therefore no homeomorphic representation $c_m : S^1 \to \gamma_m$ of γ_m is homotopic to zero in $\overline{B}_m$.

Since $\epsilon > 0$ can be chosen arbitrarily, we see that $\{X_m\}$ contains a separating subsequence, a contradiction, since $\{X_m\}$ was assumed to be cohesive.

Now we turn to the last possibility: Suppose that $B := \lim_{m\to\infty} B_m$ is of type 2. Then we have $k \geq 3$, see Corollary 1. Here we again have $C_1^{(m)} \equiv C = \partial B_1(0)$ for all $m \in \mathbb{N}$, whence $C_1 = C$, and either $C_2 = \{0\}$ or $C_2 = \partial B_{r_2}(0)$ with $0 < r_2 < 1$. Furthermore, type 2 means that *one sequence of circles, say* $\{C_j^{(m)}\}$, *converges to a true circle* C_j, $1 \leq j \leq k$, *while one or several other sequences* $\{C_\ell^{(m)}\}$ *shrink to a point* p *of* C_j. Here we can decompose $I' := \{\ell \in \mathbb{N}\colon 1 \leq \ell \leq k,\ \ell \neq j\}$ into $I_1' := \{\ell \in I'\colon C_\ell^{(m)} \to \{p\} \text{ as } m \to \infty\}$ and $I_2' := I' \setminus I_1'$; then the limits C_ℓ of $C_\ell^{(m)}$ for $m \to \infty$ and $\ell \in I_2'$ are either points or circles which stay away from p.

We can find a number $\rho_0 \in (0,1)$ and an index $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ the following holds true:

$$(*) \qquad \begin{cases} \partial B_{\rho_0}(p) \text{ intersects } C_j^{(m)} \text{ in exactly two points;} \\ C_\ell^{(m)} \subset B_{\rho_0}(p) \cap B_m := S_{\rho_0}^m(p) \quad \text{for } \ell \in I_1'; \\ C_\ell^{(m)} \cap \overline{B}_{\rho_0}(p) = \emptyset \quad \text{for } \ell \in I_2'. \end{cases}$$

Checking the three cases $j = 1$, $j = 2$, and $3 \leq j \leq k$, one realizes that both I_1' and I_2' are nonempty.

For any $\rho_1 \in (0, \rho_0)$ there is an $m_1 = m_1(\rho_1) \in \mathbb{N}$ with $m_1(\rho_1) \geq m_0$ such that

$$C_\ell^{(m)} \subset B_{\rho_1}(p) \cap B_m =: S_{\rho_1}^m(p) \quad \text{for } \ell \in I_1' \text{ and } m > m_1(\rho_1).$$

As in the preceding discussion we choose extensions $Y_m \in H_2^1(B_1(0), \mathbb{R}^3)$ of X_m from B_m to $B_1(0)$. Then we introduce polar coordinates r, θ about p, and choose representations $\tilde{Z}_m(r,\theta)$ of X_m, restricted to $S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$ for $m > m_1(\rho_1)$ which are absolutely continuous in θ for a.a. $r \in (\rho_1, \rho_0)$, and absolutely continuous in $r \in (\rho_1, \rho_0)$ for a.a. θ such that $w = p + re^{i\theta} \in S_{\rho_0}^m(p) \setminus S_{\rho_1}^m(p)$.

Applying the Courant–Lebesgue lemma, we obtain analogously to (4):

$$(4') \quad \begin{cases} \textit{For any } \epsilon > 0 \textit{ there is a number } \delta^*(\epsilon, M, \rho_0) \in (0,1)\textit{, depending only} \\ \textit{on } \epsilon, M, \rho_0\textit{, which has the following properties:} \\ \text{(i) } \delta^* < \sqrt{\delta^*} \le \rho_0; \\ \text{(ii) } \textit{for any } \rho_1 \in (0, \delta^*)\textit{, any } \delta \textit{ with } \rho_1 < \delta < \delta^*\textit{, and all } m > \\ m_1(\rho_1)\textit{, there is a subset } J_m(\delta) \textit{ of } (\delta, \sqrt{\delta}) \textit{ with } \operatorname{meas} J_m(\delta) > 0\textit{, and} \\ \operatorname{osc} \tilde{Z}_m(r, \cdot) < \epsilon/2 \textit{ for all } r \in J_m(\delta); \\ \text{(iii) } \tilde{Z}_m(r, \cdot) \textit{ is the trace of } X_m \textit{ on } \partial B_r(p) \cap \overline{B}_m \textit{ for any } r \in \\ (\rho_1, \rho_0) \setminus \mathcal{S}_m \textit{ where } 1\text{-meas}\, \mathcal{S}_m = 0\textit{, and so we can assume that} \\ J_m(\delta) \subset (\rho_1, \rho_0) \setminus \mathcal{S}_m. \end{cases}$$

Let us now fix some $\epsilon > 0$ and then $\rho_1 > 0$ with $\rho_1 < \delta^*(\epsilon, M, \rho_0)$. Furthermore, choose some $\delta > 0$ satisfying

$$\rho_1 < \delta < \delta^*(\epsilon, M, \rho_0).$$

Then it follows that, for $\rho \in (\delta, \sqrt{\delta})$ and $m > m_1(\rho_1)$, the circle $\partial B_\rho(p)$ meets $C_j^{(m)}$ in exactly two points $w'_m(\rho)$ and $w''_m(\rho)$, and that $\gamma'_m(\rho) := \partial B_\rho(p) \cap \overline{B}_m$ is a connected circular arc in $\overline{B}_m$ with the endpoints $w'_m(\rho)$ and $w''_m(\rho)$. Their image points $Q'_m(\rho)$ and $Q''_m(\rho)$ under $\tilde{Z}_m(\rho, \cdot)$ lie on Γ_j and decompose this curve into two arcs; denote the "smaller one" by $\Gamma^*(m, \rho)$. Then there is a function $\eta : (0, \infty) \to (0, \infty)$ with $\eta(t) \to +0$ as $t \to +0$ such that

$$\operatorname{diam} \Gamma^*(m, \rho) < \eta(\rho) \quad \text{for } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta).$$

We can arrange for

$$\operatorname{diam} \Gamma^*(m, \rho) < \epsilon/2 \quad \text{for } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta)$$

by choosing the number $\delta^*(\epsilon, M, \rho_0) > 0$ even smaller if necessary (see the application of the Courant–Lebesgue lemma in Section 4.3).

Instead of $(*)$, we even have

$$(**) \qquad \begin{cases} C_\ell^{(m)} \subset B_\rho(p) \cap B_m =: S_\rho^m(p) \quad \text{for } \ell \in I'_1, \\ C_\ell^{(m)} \cap \overline{B}_\rho(p) = \emptyset \quad \text{for } \ell \in I'_2, \\ \text{provided that } m > m_1(\rho_1) \text{ and } \rho \in J_m(\delta). \end{cases}$$

Choose some $r_m \in J_m(\delta) \subset (\delta, \sqrt{\delta})$ and set

$$\Gamma'_m := \text{image of } \gamma'_m(r_m) \text{ under the mapping } \tilde{Z}_m;$$
$$\Gamma''_m := \Gamma^*(m, r_m) = \text{image of } \gamma''_m(r_m) \text{ under } X_m;$$

here $\gamma''_m(r_m)$ is the connected arc on $C_j^{(m)}$, bounded by $w'_m(r_m), w''_m(r_m)$, which is mapped by the Sobolev trace $X_m|_{C_j^{(m)}}$ in a continuous way onto Γ''_m. Then we have

$$\operatorname{diam} \Gamma'_m + \operatorname{diam} \Gamma''_m < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } m > m_1(\rho_1).$$

Consider the closed Jordan curve $\gamma_m := \gamma'_m(r_m) \cup \gamma''_m(r_m)$ in $\overline{B}_m$, which bounds a two-gon B^*_m in $B_1(0)$; B^*_m is a strong Lipschitz domain. Because of (**), one realizes that no homeomorphic representation $c_m : S^1 \to \gamma_m$ of γ_m is homotopic to zero in $\overline{B}_m$. There is a continuous representation Z_m of X_m on γ_m given by

$$Z_m := X_m(r_m, \cdot) \quad \text{on } \gamma'_m$$

and

$$Z_m := \text{trace of } X_m \quad \text{on } \gamma''_m.$$

Then it follows

$$\operatorname{diam} Z_m(\gamma_m) \le \operatorname{diam} \Gamma'_m + \operatorname{diam} \Gamma''_m < \epsilon \quad \text{for } m > m_1(\rho_1).$$

Since $\epsilon > 0$ is arbitrary, we obtain that $\{X_m\}$ contains a separating subsequence, and so it cannot be cohesive, a contradiction to the assumption.

Thus we have shown that B cannot be degenerate, i.e. $B \in \mathcal{N}_1(k)$. $\square$

Proposition 5. *Let $\{X_m\}$ be a sequence of mappings $X_m \in H^1_2(B_m, \mathbb{R}^3)$ with $B_m = \text{dom } (X_m) \in \mathcal{N}_1(k)$, and suppose that $B_m \to B \in \mathcal{N}_1(k)$ and $D(X_m) \to L$ as $m \to \infty$. Then there is a sequence of diffeomorphisms σ_m from $\overline{B}$ onto $\overline{B}_m$ such that the following holds true:*

(i) *$X^*_m := X_m \circ \sigma_m \in H^1_2(B, \mathbb{R}^3)$ for all $m \in \mathbb{N}$, and if $X_m \in \mathcal{C}(\Gamma)$ then $X^*_m \in \mathcal{C}(\Gamma)$;*
(ii) *$D(X^*_m) \to L$ as $m \to \infty$;*
(iii) *$\{X^*_m\}$ is cohesive if and only if $\{X_m\}$ is cohesive;*
(iv) *If $X_m \in \overline{\mathcal{C}}(\Gamma)$ then $X^*_m \in \overline{\mathcal{C}}(\Gamma)$, and $\{X^*_m\}$ is C-cohesive if and only if $\{X_m\}$ is C-cohesive.*

Proof. Since the limit domain is nondegenerate, it is not difficult to prove that there is a sequence $\{\sigma_m\}$ of diffeomorphisms from $\overline{B}$ onto $\overline{B}_m$ which converges to the identity $id_{\overline{B}}$ on $\overline{B}$ with respect to the $C^1(\overline{B}, \mathbb{R}^2)$-norm. (This would not be true if $B \in \overline{\mathcal{N}}_1(k) \setminus \mathcal{N}_1(k)$). Setting $X^*_m := X_m \circ \sigma_m$, the assertions follow at once. $\square$

Theorem 2. *Let $\{X_m\}$ be a cohesive sequence of mappings $X_m \in \mathcal{C}(\Gamma)$ with $\text{dom}(X_m) \equiv B \in \mathcal{N}_1(k)$ for all $m \in \mathbb{N}$, $k \ge 2$, whose boundary contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consists of k rectifiable, closed, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$. Suppose also that there is a constant $M > 0$ such that*

$$D(X_m) \le M \quad \textit{for all } m \in \mathbb{N}.$$

Then the boundary traces $X_m|_{\partial B}$ are equicontinuous on ∂B, and there is a subsequence $\{X_{m_\ell}\}$ of $\{X_m\}$ such that the traces $X_{m_\ell}|_{\partial B}$ converge uniformly on ∂B.

Proof. We can essentially proceed as in the proof of Theorem 1 of Section 4.3 noting that $X_m|_{C_j}$ maps C_j continuously and in a weakly monotonic way onto Γ_j. One only has to ensure that small arcs on C_j are mapped onto small subarcs of Γ_j. In the case $k = 1$ this was achieved by imposing a three-point condition upon $\{X_m\}$; for $k \geq 2$ the same will be attained by the cohesivity condition. In fact, mapping small arcs on C_j onto large arcs on Γ_j corresponds to mapping large arcs on C_j onto small arcs of Γ_j, and by the Courant-Lebesgue Lemma one would obtain Jordan curves γ_m in B bounding strong Lipschitz domains B_m^* such that the continuous trace $Z_m :=$ "$X_m|_{\gamma_m}$" of X_m on γ_m satisfies "$\operatorname{diam} Z_m(\gamma_m) = \text{small}$", but γ_m cannot be contracted continuously in $\overline{B}$ to some point of $\overline{B}$ since $\overline{B} \cap \overline{B}_m^*$ possesses at least one hole. □

Corresponding to Theorem 3 of Section 4.3 we obtain the following generalizations of Theorems 1 and 2 above:

Theorem 3. *The assertions of Theorems 1 and 2 remain valid if we replace the assumption "$X_m \in \mathcal{C}(\Gamma)$" by "$X_m \in \mathcal{C}(\Gamma^m)$" where $\Gamma^m = \langle \Gamma_1^m, \dots, \Gamma_k^m \rangle$ are boundary contours converging in the sense of Fréchet ("$\Gamma^m \to \Gamma$ as $m \to \infty$") to some contour $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consisting of k rectifiable, closed, mutually disjoint Jordan curves.*

8.4 Solution of the Douglas Problem

Using the results obtained in Sections 8.2 and 8.3 we can now solve the Douglas problem under the assumption that $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \geq 2$, bounds a cohesive minimizing sequence in $\mathcal{C}(\Gamma)$ for the Dirichlet integral.

Theorem 1. *Let $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \geq 2$, be a boundary configuration consisting of rectifiable, closed, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$ in $\mathbb{R}^3$, and suppose that Γ fulfills the following* **condition of cohesion**: *There is a cohesive sequence $\{X_m\}$ of surfaces $X_m \in \mathcal{C}(\Gamma)$ with*

$$D(X_m) \to d(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D.$$

Then there exists a minimizer X of the energy D in $\mathcal{C}(\Gamma)$ which is of class $C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and satisfies

(1) $$\Delta X = 0 \quad \text{in } B$$

as well as

(2) $$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B.$$

Proof. Consider a cohesive sequence of $X_m \in \mathcal{C}(\Gamma)$ with $D(X_m) \to d(\Gamma)$ and $B_m = \text{dom}(X_m) \in \mathcal{N}(k)$. By Lemma 4 of Section 8.2 there are Möbius transformations f_m mapping $\overline{B}^*_m \in \mathcal{N}_1(k)$ onto $\overline{B}_m$. Set $X^*_m := X_m \circ f_m \in \mathcal{C}(\Gamma)$; then $B^*_m = \text{dom}(X^*_m) \in \mathcal{N}_1(k), D(X^*_m) \to d(\Gamma)$, and $\{X^*_m\}$ is cohesive too on account of Proposition 4 in Section 8.3. Furthermore, there is a constant $M_0 > 0$ such that $D(X^*_m) \le M_0$ for all $m \in \mathbb{N}$. By Lemma 1 of Section 8.3 we can extract a subsequence $\{B^*_{m_j}\}$ of $\{B^*_m\}$ such that $B^*_{m_j} \to B \in \overline{\mathcal{N}}_1(k)$. Applying Theorem 1 of Section 8.3 we infer that B is nondegenerate, i.e. $B \in \mathcal{N}_1(k)$, and by Proposition 5 of the same section we find diffeomorphisms σ_{m_j} from $\overline{B}$ onto $\overline{B}^*_{m_j}$ such that $X^{**}_{m_j} := X^*_{m_j} \circ \sigma_{m_j}$, $j \in \mathbb{N}$, defines a cohesive sequence of mappings $X^{**}_{m_j} \in \mathcal{C}(\Gamma)$ with $D(X^{**}_{m_j}) \to d(\Gamma)$. In virtue of Theorem 2 of Section 8.3, the boundary traces $X^{**}_{m_j}|_{\partial B}$ are compact in $C^0(\partial B, \mathbb{R}^3)$, and so we can assume without loss of generality that the cohesive minimizing sequence we have started with, satisfies also

(i) $\text{dom}(X_m) \equiv B$ for all $m \in \mathbb{N}$;
(ii) $X_m|_{\partial B} \to \phi$ in $C^0(\partial B, \mathbb{R}^3)$.

If we replace X_m by the solution $H_m \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^2)$ of the Dirichlet problem

$$\Delta H_m = 0 \quad \text{in } B, \quad H_m|_{\partial B} = X_m|_{\partial B}$$

the sequence $\{H_m\}$ possesses all properties of $\{X_m\}$. Renaming H_m as X_m, we therefore obtain also

(iii) $X_m \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and $\Delta X_m = 0$ in B.

Because of (ii) and (iii) there is a mapping $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ which is harmonic in B and satisfies

$$X_m \to X \quad \text{in } C^0(\overline{B}, \mathbb{R}^3). \tag{3}$$

Because of $D(X_m) \le M_0$ and $X_m \in \mathcal{C}(\Gamma)$ as well as (i) we can also assume that X_m converges weakly in $H^1_2(B, \mathbb{R}^3)$ to X, whence $X \in \mathcal{C}(\Gamma)$, and therefore

$$d(\Gamma) \le D(X).$$

Furthermore, the weak lower semicontinuity of D in $H^1_2(B, \mathbb{R}^3)$ yields

$$D(X) \le \lim_{n \to \infty} D(X_n) = d(\Gamma),$$

and so we obtain

$$D(X) = d(\Gamma).$$

That is, X minimizes D in $\mathcal{C}(\Gamma)$ and satisfies (1). Finally, Theorem 1 of Section 8.2 leads to the conformality relations (2), and so the proof is complete. $\square$

Remark 1. If we assume the existence of a C-cohesive sequence of surfaces $X_m \in \overline{\mathcal{C}}(\Gamma)$ with

$$D(X_m) \to \overline{d}(\Gamma) := \inf_{\overline{\mathcal{C}}(\Gamma)} D,$$

a similar reasoning as above leads to a minimal surface $X \in \overline{\mathcal{C}}(\Gamma)$ minimizing D in $\overline{\mathcal{C}}(\Gamma)$. This is the original approach of Courant [15].

Remark 2. As we have noted earlier, C-cohesiveness implies cohesiveness. Using Theorem 1 one can also show the converse. Thus the two conditions actually are equivalent, and so they lead to the same result. Hence it seems superfluous to work with cohesiveness instead of C-cohesiveness, as it is more troublesome to work with. Its usefulness will become apparent when we will minimize

$$A^\epsilon = (1 - \epsilon)A + \epsilon D$$

for some $\epsilon \in [0, 1]$, in order to prove

$$\inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = \inf_{\overline{\mathcal{C}}(\Gamma)} D. \tag{4}$$

Then it seems impossible, or at least much more cumbersome, to operate in $\overline{\mathcal{C}}(\Gamma)$, and it appears to be more natural to work in $\mathcal{C}(\Gamma)$.

The same holds true if one wants to minimize a Cartan functional under Plateau boundary conditions.

In the sequel we want to solve the Douglas problem assuming the ("sufficient") condition of Douglas, thereby verifying also (4). For this purpose we need two technical results that will be provided in the next section.

8.5 Useful Modifications of Surfaces

First we will show that we can replace small parts of a surface by the constant surface $X_0(w) \equiv 0$ without gaining much energy. This argument works for general functionals

$$\mathcal{F}(X) := \int_B F(X, \nabla X)\, du\, dv$$

and surfaces $X \in H_2^1(B, \mathbb{R}^3)$, $B = \mathrm{dom}(X) \in \mathcal{N}(k)$, with a Lagrangian $F(x, p) \in C^0(\mathbb{R}^3 \times \mathbb{R}^6)$ satisfying

$$0 \le F(x, p) \le \tfrac{1}{2}\mu|p|^2$$

for some constant $\mu > 0$. For $\Omega \subset B$ we set

$$\mathcal{F}_\Omega(X) := \int_\Omega F(X, \nabla X)\, du\, dv.$$

Proposition 1. *Suppose that $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}(k)$. Then, for any $\delta > 0$ and any point $p \in B$, there exists a number $r_0 \in (0, \mathrm{dist}(p, \partial B))$, depending on X, δ, p, and μ, such that for any $r \in (0, r_0)$ there is a surface $Z^r \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(Z^r) = B$ and*

$$\mathcal{F}(Z^r) < \mathcal{F}(X) + \delta \quad \textit{as well as} \quad Z^r(w) \equiv 0 \quad \textit{on } B_r(p).$$

Proof. Choose any $\delta > 0$ and $p \in B$; then there is some $R \in (0,1)$ with $R < \mathrm{dist}(p, \partial B)$ such that

$$\int_{B_\rho(p)} |\nabla X|^2 \, du \, dv < \delta_0 := \frac{\delta}{2\mu} \quad \text{for all } \rho \in (0, R). \tag{1}$$

Then we take some $\rho \in (0, R)$ such that the trace $X|_{\partial B_\rho(p)}$ is absolutely continuous on $\partial B_\rho(p)$. Set

$$M := \sup_{\partial B_\rho(p)} |X|,$$

and choose some $H \in H_2^1(B_\rho(p), \mathbb{R}^3)$ with

$$\Delta H = 0 \quad \text{in } B_\rho(p), \quad H = X \quad \text{on } \partial B_\rho(p).$$

Then $H - X \in \mathring{H}_2^1(B_\rho(p), \mathbb{R}^3)$, and the maximum principle implies

$$\sup_{B_\rho(p)} |H| = M. \tag{2}$$

Furthermore, Dirichlet's principle yields

$$\int_{B_\rho(p)} |\nabla H|^2 \, du \, dv \le \int_{B_\rho(p)} |\nabla X|^2 \, du \, dv < \delta_0. \tag{3}$$

For some constant $\epsilon \in (0, \rho)$ to be fixed later we set

$$\varphi(s, \epsilon^2) := \begin{cases} 1 & \text{for } \epsilon < s, \\ 0 & \text{for } 0 \le s \le \epsilon^2 \end{cases} \tag{4}$$

and

$$\varphi(s, \epsilon^2) := 1 + \frac{\log \epsilon - \log s}{\log \epsilon} \quad \text{for } \epsilon^2 \le s \le \epsilon. \tag{5}$$

By means of $\varphi(\cdot, \epsilon^2) \in \mathrm{Lip}([0, \infty))$ we define $Y(\cdot, \epsilon^2)$ as

$$Y(w, \epsilon^2) := \begin{cases} X(w) & \text{for } |w - p| \ge \rho, \\ \varphi(|w - p|, \epsilon^2) H(w) & \text{for } |w - p| < \rho. \end{cases}$$

Writing

$$\phi(w) := \varphi(|w-p|, \epsilon^2),$$

we obtain

$$\begin{aligned}\int_{B_\rho(p)} |\nabla\phi|^2\,du\,dv &= |\log\epsilon|^{-2}\int_0^{2\pi}\int_{\epsilon^2}^{\epsilon} r^{-2} r\,dr\,d\theta\\ &= -\frac{2\pi}{\log\epsilon} =: \delta_1(\epsilon) > 0\end{aligned}$$

and then

$$\begin{aligned}\int_{B_\rho(p)} |\nabla Y(\cdot,\epsilon^2)|^2\,du\,dv &= \int_{B_\rho(p)} \{|\phi_u H + \phi H_u|^2 + |\phi_v H + \phi H_v|^2\}\,du\,dv\\ &\le 2M^2\int_{B_\rho(p)} |\nabla\phi|^2\,du\,dv + 2\int_{B_\rho(p)} |\nabla H|^2\,du\,dv\\ &\le 2M^2\delta_1(\epsilon) + 2\delta_0 < 4\delta_0 \quad \text{for } 0<\epsilon<\epsilon_0\end{aligned}$$

if we choose $\epsilon_0 \in (0,\rho)$ so small that $M^2\delta_1(\epsilon) < \delta_0$ for $0<\epsilon<\epsilon_0$. Set $r := \epsilon^2$ with $0<\epsilon<\epsilon_0$ and $Z^r := Y(\cdot,\epsilon^2)$; then

$$\begin{aligned}\mathcal{F}(Z^r) &= \mathcal{F}_{B\setminus B_\rho(p)}(X) + \mathcal{F}_{B_\rho(p)}(Z^r)\\ &\le \mathcal{F}(X) + \frac{\mu}{2}\int_{B_\rho(p)} |\nabla Z^r|^2\,du\,dv\\ &< \mathcal{F}(X) + 2\delta_0\mu = \mathcal{F}(X) + \delta \quad \text{for } r\in(0,\epsilon_0^2),\end{aligned}$$

and similarly

$$\int_B |\nabla Z^r|^2\,du\,dv \le \int_B |\nabla X|^2\,du\,dv + 4\delta_0.$$

Since $|Z^r| \le |X|$, it follows $Z^r \in H_2^1(B,\mathbb{R}^3)$. Furthermore, $B_{\sqrt{r}}(p) \subset\subset B$, and

$$Z^r(w) \equiv 0 \quad \text{on } B_r(p), \qquad Z^r(w) \equiv X(w) \quad \text{on } B\setminus B_{\sqrt{r}}(p).$$

This implies $Z^r \in \mathcal{C}(\Gamma)$ since $X \in \mathcal{C}(\Gamma)$. Setting $r_0 := \epsilon_0^2$, the assertion is proved. □

Proposition 2 (Pinching method). *Let $\tilde{\Gamma}$ be a boundary configuration consisting of k rectifiable, closed, mutually disjoint Jordan curves in $\mathbb{R}^3$. Then, for given $K>0$ and $\delta>0$, there is a constant $\eta_0 \in (0,1)$ depending only on $\tilde{\Gamma}, K, \delta$, such that for every point $Q \in \mathbb{R}^3$ and any $\eta \in (0,\eta_0)$ there is a Lipschitz mapping $\Phi = \Phi_{\eta,Q}$ from $\mathbb{R}^3$ onto $\mathbb{R}^3$ with the following properties: If X is an arbitrary mapping of class $\mathcal{C}(\tilde{\Gamma})$ with $\operatorname{dom}(X) = B$ and $D(X) \le K$, then we have*

(i) *$\Gamma^* := \Phi(\tilde{\Gamma})$ consists of k rectifiable, closed, mutually disjoint Jordan curves such that the Fréchet distance $\Delta(\tilde{\Gamma},\Gamma^*)$ of $\tilde{\Gamma}$ and Γ^* satisfies $\Delta(\tilde{\Gamma},\Gamma^*) < \delta$;*

(ii) $\Phi \circ X \in \mathcal{C}(\Gamma^*)$, *and* $\mathrm{dom}(\Phi \circ X) = B$;
(iii) $\Phi(x) \equiv x$ *for* $x \in \mathbb{R}^3$ *with* $|x - Q| \geq \eta$;
(iv) $\Phi(x) \equiv Q$ *for* $x \in \mathbb{R}^3$ *with* $|x - Q| \leq \eta^2$;
(v) *For* $A^\epsilon := (1-\epsilon)A + \epsilon D$, $0 \leq \epsilon \leq 1$, *we have*

$$A^\epsilon(\Phi \circ X) \leq A^\epsilon(X) + \delta.$$

Proof. Choose $\eta_0 \in (0, 1/3)$ so small that

$$3|\log \eta_0|^{-1} < \delta/K \tag{6}$$

and

$$\eta_0 < \tfrac{1}{2} \min\{\mathrm{dist}(\tilde{\Gamma}_j, \tilde{\Gamma}_\ell) \colon j \neq \ell, j, \ell = 1, \ldots, k\}$$

where $\tilde{\Gamma} = \langle \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_k \rangle$. For $\eta \in (0, \eta_0)$ we define the Lipschitz function $\varphi_\eta : [0, \infty) \to \mathbb{R}$ by

$$\varphi_\eta(s) := \begin{cases} 1 & \text{for } \eta < s, \\ 2 - \frac{\log s}{\log \eta} & \text{for } \eta^2 \leq s \leq \eta, \\ 0 & \text{for } 0 \leq s < \eta^2. \end{cases}$$

Then, fixing an arbitrary point $Q \in \mathbb{R}^3$, we define the mapping $\Phi_{\eta,Q} \equiv \Phi_\eta : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Phi_\eta(x) := Q + \varphi_\eta(|x - Q|)\{x - Q\} \quad \text{for } x \in \mathbb{R}^3.$$

Clearly, Φ_η is a Lipschitz map from $\mathbb{R}^3$ onto itself which "pinches" the ball $K_{\eta^2}(Q) := \{x \in \mathbb{R}^3 \colon |x - Q| \leq \eta^2\}$ to the point Q and maps $\mathbb{R}^3 \setminus K_{\eta^2}(Q)$ in a $1-1$ way onto $\mathbb{R}^3 \setminus \{Q\}$. This immediately implies the properties (i)–(iv) of $X^* := \Phi_\eta(X)$. It remains to show (v). We first note that

$$\begin{aligned} &X^*(w) = Q \quad \text{and} \quad \nabla X^*(w) = 0 \quad \text{a.e. on } B' := \{w \in B \colon |X(w) - Q| \leq \eta^2\}, \\ &X^*(w) = X(w) \quad \text{and} \quad \nabla X^*(w) = \nabla X(w) \\ &\quad \text{a.e. on } B'' := \{w \in B \colon |X(w) - Q| \geq \eta\}. \end{aligned}$$

Thus we have to compute ∇X^* on $R := \{w \in B \colon \eta^2 < |X(w) - Q| < \eta\}$. Set

$$e(w) := |X(w) - Q|^{-1}\{X(w) - Q\} \quad \text{for } w \in R;$$

then $|e(w)| = 1$ on R. Furthermore, we have on R:

$$\begin{aligned} &X^* = Q + \varphi_\eta(|X - Q|)\{X - Q\}, \quad \varphi_\eta(|X - Q|) = 2 - \frac{\log|X - Q|}{\log \eta}, \\ &\frac{\partial}{\partial u}\varphi_\eta(|X - Q|) = \frac{-e \cdot X_u}{(\log \eta)|X - Q|}, \quad \frac{\partial}{\partial v}\varphi_\eta(|X - Q|) = \frac{-e \cdot X_v}{(\log \eta)|X - Q|}. \end{aligned}$$

Then,

$$X_u^* = \varphi_\eta(|X-Q|)X_u - \frac{1}{\log\eta}(e\cdot X_u)e, \tag{7}$$
$$X_v^* = \varphi_\eta(|X-Q|)X_v - \frac{1}{\log\eta}(e\cdot X_v)e \quad \text{on } R,$$

whence by $0 \le \varphi_\eta(|X-Q|) \le 1$, $|e|=1$, $-\log\eta = |\log\eta| > 1$ we obtain on R:

$$\begin{aligned} |X_u^*|^2 &\le |X_u|^2 - 2(\log\eta)^{-1}|X_u|^2 + |\log\eta|^2|X_u|^2 \\ &\le (1+3|\log\eta|^{-1})|X_u|^2, \\ |X_v^*|^2 &\le (1+3|\log\eta|^{-1})|X_v|^2. \end{aligned}$$

On account of (6), this leads to

$$D_R(X^*) \le D_R(X) + (\delta/K)D(X) \le D_R(X) + \delta. \tag{8}$$

Now we are going to estimate $A_R(X^*)$. From (7) we infer by setting $\psi := \varphi_\eta(|X-Q|)$ that

$$X_u^* \wedge X_v^* = \psi^2 X_u \wedge X_v + \psi|\log\eta|^{-1}\{(e\cdot X_v)(e\wedge X_u) + (e\cdot X_u)(e\wedge X_v)\}$$

whence

$$\begin{aligned} |X_u^* \wedge X_v^*| &\le |X_u \wedge X_v| + |\log\eta|^{-1}2|X_u||X_v| \\ &\le |X_u \wedge X_v| + |\log\eta|^{-1}|\nabla X|^2 \quad \text{on } R. \end{aligned}$$

This implies

$$\begin{aligned} A_R(X^*) &\le A_R(X) + |\log\eta|^{-1}2D_R(X) \\ &\le A_R(X) + (\delta/K)D(X) \le A_R(X) + \delta. \end{aligned} \tag{9}$$

From (8), (9), and $A_R^\epsilon = (1-\epsilon)A_R + \epsilon D_R$ we infer

$$A_R^\epsilon(X^*) \le A_R^\epsilon(X) + \delta \quad \text{for any } \epsilon \in [0,1].$$

Furthermore,

$$A_{B'}^\epsilon(X^*) = 0, \quad A_{B''}^\epsilon(X^*) = A_{B''}^\epsilon(X).$$

Since $B = B' \,\dot\cup\, R \,\dot\cup\, B''$, we arrive at

$$\begin{aligned} A^\epsilon(X^*) &= A_{B'}^\epsilon(X^*) + A_R^\epsilon(X^*) + A_{B''}^\epsilon(X^*) \\ &\le 0 + A_R^\epsilon(X) + \delta + A_{B''}^\epsilon(X) \le A^\epsilon(X) \end{aligned}$$

for any $\epsilon \in [0,1]$. This completes the proof. □

8.6 Douglas Condition and Douglas Problem

For $\epsilon \in [0,1]$ we consider the conformally invariant functionals

$$A^\epsilon(X) := (1-\epsilon)A(X) + \epsilon D(X)$$

which satisfy

$$A^0(X) = A(X), \quad A^1(X) = D(X)$$

and

$$A(X) \le A^\epsilon(X) \le D(X) \quad \text{for any } \epsilon \in [0,1]. \tag{1}$$

Furthermore, for $0 < \epsilon \le 1$ we have

$$A(X) = A^\epsilon(X) = D(X) \quad \text{if and only if} \quad \langle X_w, X_w \rangle = 0, \tag{2}$$

and

$$\langle X_w, X_w \rangle = 0 \quad \Leftrightarrow \quad |X_u|^2 = |X_v|^2 \quad \text{and} \quad \langle X_u, X_v \rangle = 0.$$

As a first result we shall prove that the problem

$$A^\epsilon \to \min \quad \text{in } \mathcal{C}(\Gamma)$$

has a solution $X^\epsilon \in \mathcal{C}(\Gamma)$ for any $\epsilon \in (0, \epsilon_0]$ with $0 < \epsilon_0 \ll 1$ provided that $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$, $k \ge 2$, satisfies the Douglas condition. The proof follows essentially the same lines as in Section 8.4, but it is somewhat more involved.

In order to define the Douglas condition for $k > 1$ we have to consider the class of mappings $X : B \to \mathbb{R}^3$ whose domains B are disconnected. Precisely speaking we assume that B is a set $\{B^1, \dots, B^s\}$, $s > 1$, of k_ν-circle domains $B^\nu \in \mathcal{N}(k_\nu)$ with

$$k = k_1 + k_2 + \cdots + k_s,$$

and X is a collection $\{X^{(1)}, \dots, X^{(s)}\}$ of mappings

$$X^{(\nu)} \in H^{1,2}(B^\nu, \mathbb{R}^3) \cap C^0(\partial B^\nu, \mathbb{R}^3)$$

such that $X^{(\nu)}|_{\partial B^\nu}$ is a weakly monotonic mapping of ∂B^ν onto a configuration of k_ν disjoint closed, rectifiable Jordan curves $\Gamma_1, \dots, \Gamma_{k_\nu}$. The set $\mathcal{C}^+(\Gamma)$ of such maps X is called the **class of splitting mappings bounded by** Γ.

Now we define $A^\epsilon(X)$ for $X = \{X^{(1)}, \dots, X^{(s)}\} \in \mathcal{C}^+(\Gamma)$ by

$$A^\epsilon(X) := A^\epsilon(X^{(1)}) + \cdots + A^\epsilon(X^{(s)}),$$

and then

$$d(\Gamma, \epsilon) := \inf_{\mathcal{C}(\Gamma)} A^\epsilon, \quad d^+(\Gamma, \epsilon) := \inf_{\mathcal{C}^+(\Gamma)} A^\epsilon,$$

in particular

$$a(\Gamma) := \inf_{\mathcal{C}(\Gamma)} A, \quad a^+(\Gamma) := \inf_{\mathcal{C}^+(\Gamma)} A,$$

that is, $a(\Gamma) = d(\Gamma, 0)$ and $a^+(\Gamma) = d^+(\Gamma, 0)$.

Definition 1. *The* **Douglas condition** *is the hypothesis*

$$a(\Gamma) < a^+(\Gamma).$$

In the following discussion we need a third function of ϵ *besides* $d(\Gamma, \epsilon)$ *and* $d^+(\Gamma, \epsilon)$, *namely*

$$d^*(\Gamma, \epsilon) := \inf\Big\{ \liminf_{m\to\infty} A^\epsilon(X_m) \colon \{X_m\} = \textit{separating sequence}$$
$$\textit{of } X_m \in \mathcal{C}(\Gamma) \Big\}.$$

Lemma 1. *The infima* $d(\Gamma, \epsilon), d^+(\Gamma, \epsilon), d^*(\Gamma, \epsilon)$ *are nondecreasing functions of* $\epsilon \in [0, 1]$, *and*

$$d(\Gamma, 0) = \lim_{\epsilon\to+0} d(\Gamma, \epsilon), \quad d^+(\Gamma, 0) = \lim_{\epsilon\to+0} d^+(\Gamma, \epsilon). \tag{3}$$

Proof. Since $A \le D$ we obtain for $0 < \epsilon \le \epsilon'$ that

$$\begin{aligned} A^\epsilon(X) &= A(X) + \epsilon[D(X) - A(X)] \\ &\le A(X) + \epsilon'[D(X) - A(X)] = A^{\epsilon'}(X), \end{aligned}$$

which shows that $d(\Gamma, \cdot), d^+(\Gamma, \cdot), d^*(\Gamma, \cdot)$ are nondecreasing, whence in particular

$$d(\Gamma, 0) \le \lim_{\epsilon\to+0} d(\Gamma, \epsilon).$$

Suppose that

$$\delta := \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - d(\Gamma, 0) > 0.$$

Then there is a mapping $X \in \mathcal{C}(\Gamma)$ such that

$$A(X) \le d(\Gamma, 0) + \frac{\delta}{2} = \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2}.$$

Choosing $\epsilon^* \in (0, 1)$ so small that

$$0 \le \epsilon^*[D(X) - A(X)] \le \frac{\delta}{4},$$

it follows

$$\begin{aligned} A^{\epsilon^*}(X) &= A(X) + \epsilon^*[D(X) - A(X)] \le A(X) + \frac{\delta}{4} \\ &\le \lim_{\epsilon\to+0} d(\Gamma, \epsilon) - \frac{\delta}{2} + \frac{\delta}{4} \\ &\le d(\Gamma, \epsilon^*) - \frac{\delta}{4} \le A^{\epsilon^*}(X) - \frac{\delta}{4}, \end{aligned}$$

a contradiction. Thus we have $\delta = 0$ and therefore $d(\Gamma, \epsilon) \to d(\Gamma, 0)$ as $\epsilon \to 0$. Analogously, the second relation in (3) is proved. $\square$

Lemma 2. *Let $\epsilon \in (0,1]$ and $M \geq 0$, and consider a sequence $\{\Gamma^m\}$ of boundary configurations converging to the configuration Γ in the sense of Fréchet ($\Gamma^m \to \Gamma$) as $m \to \infty$, where Γ^m and Γ consist of k closed, disjoint, rectifiable Jordan curves. Then for any cohesive sequence $\{X_m\}$ of mappings $X_m \in \mathcal{C}(\Gamma^m)$ with*

$$D(X_m) \leq M \quad \text{for all } m \in \mathbb{N} \tag{4}$$

there exists a mapping $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}_1(k)$ such that

$$d(\Gamma, \epsilon) \leq A^\epsilon(X) \leq \liminf_{m\to\infty} A^\epsilon(X_m). \tag{5}$$

Proof. For $k = 1$ each sequence of mappings $X_m \in \mathcal{C}(\Gamma^m)$ is cohesive, and the assertion follows using the results of Chapter 4. Thus we now suppose $k \geq 2$. There is a subsequence $\{X_{m_j}\}$ such that $\{A^\epsilon(X_{m_j})\}$ converges and

$$\lim_{j\to\infty} A^\epsilon(X_{m_j}) = \liminf_{m\to\infty} A^\epsilon(X_m). \tag{6}$$

Because of (4) we can also achieve that

$$D(X_{m_j}) \to L \in [0, M_0] \quad \text{as } j \to \infty. \tag{6$'$}$$

Applying the results of Section 8.3 (and using Theorem 3 of that section instead of Theorems 1 and 2), we obtain by the reasoning used in the proof of Theorem 1 of Section 8.4 that we can also assume that $\{X_{m_j}\}$ is a cohesive sequence with $\mathrm{dom}(X_{m_j}) = B \in \mathcal{N}_1(k)$ for all $j \in \mathbb{N}$, while (6) and (6′) remain unaltered.

Using (4) and a suitable variant of Poincaré's theorem we can in addition assume that

$$X_{m_j} \rightharpoonup X \quad \text{in } H_2^1(B, \mathbb{R}^3)$$

and

$$X_{m_j}|_{\partial B} \to X|_{\partial B} \quad \text{in } L_2(B, \mathbb{R}^3)$$

as $j \to \infty$, and by Theorem 3 of Section 8.3 also

$$X_{m_j}|_{\partial B} \to X|_{\partial B} \quad \text{in } C^0(\partial B, \mathbb{R}^3).$$

Since $X_m \in \mathcal{C}(\Gamma^m)$ and $\Gamma^m \to \Gamma$ it follows $X \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(X) = B \in \mathcal{N}_1(k)$, and the lower semicontinuity of A^ϵ with respect to weak convergence of sequences in $H_2^1(B, \mathbb{R}^3)$ yields

$$A^\epsilon(X) \leq \liminf_{j\to\infty} A^\epsilon(X_{m_j}). \tag{7}$$

Then we infer (5) from (6), (7), and the fact that $X \in \mathcal{C}(\Gamma)$ implies $d(\Gamma, \epsilon) \leq A^\epsilon(X)$. $\square$

Lemma 3. *For all $\epsilon \in [0,1]$ we have*

$$d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon).$$

Proof. For any separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ we have

$$d(\Gamma,\epsilon) \le A^\epsilon(X_m) \quad \text{for all } m \in \mathbb{N},$$

which implies

$$d(\Gamma,\epsilon) \le d^*(\Gamma,\epsilon).$$

Thus we have to prove

$$d^*(\Gamma,\epsilon) \le d^+(\Gamma,\epsilon). \tag{8}$$

For $k=1$, nothing is to be proved since then $d^+(\Gamma,\epsilon)=\infty$ as $\mathcal{C}^+(\Gamma)=\emptyset$. Thus we assume $k \ge 2$. We have to show: For any partition $\{\Gamma^1,\dots,\Gamma^s\}$ of $\Gamma = \langle \Gamma_1,\dots,\Gamma_k\rangle$ with $s \ge 2$ one has

$$d^*(\Gamma,\epsilon) \le \sum_{j=1}^s d(\Gamma^j,\epsilon).$$

This is equivalent to the following assertion:

For every number $\eta > 0$ there is a separating sequence $\{X_m\}$ of mappings $X_m \in \mathcal{C}(\Gamma)$ such that

$$\liminf_{m\to\infty} A^\epsilon(X_m) \le \sum_{j=1}^s d(\Gamma^j,\epsilon) + \eta. \tag{9}$$

We begin with $s=2$ and an arbitrary partition $\{\Gamma^1,\Gamma^2\}$ of Γ. For an arbitrary chosen $\delta > 0$ there are $X^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B_\nu = \mathrm{dom}(X^\nu) \in \mathcal{N}(k_\nu), \nu = 1,2$, $k_1 + k_2 = k$, such that

$$A^\epsilon(X^{(\nu)}) \le d(\Gamma^\nu,\epsilon) + \delta \quad \text{for } \nu = 1,2.$$

Applying Proposition 1 of Section 8.5 to $\mathcal{F} := A^\epsilon$ we construct new mappings $Z_\nu \in \mathcal{C}(\Gamma^\nu)$ with $\mathrm{dom}(Z_\nu) = B^\nu \in \mathcal{N}(k_\nu)$ and

$$Z_\nu|_{B_{2r}(p_\nu)} = 0 \quad \text{for some disks } B_{2r}(p_\nu) \subset\subset B^\nu$$

such that

$$A^\epsilon(Z_\nu) \le A^\epsilon(X^{(\nu)}) + \delta \quad \text{for } \nu = 1,2.$$

Shifting B^2 suitably we may assume that $p_1 = p_2$; set

$$p := p_1 = p_2.$$

Let ρ be the inversion with respect to the circle $\partial B_{2r}(p)$ and set

$$B_2^* := \rho(B^2 \setminus B_{2r}(p)).$$

Furthermore, let C^* be the "outer" boundary circle of B_2^*, and B^* be the disk bounded by C^*. Set

$$B_1^* := B^1 \setminus B^*$$

and

$$Z_1^* := Z_1|_{B_1^*}, \quad Z_2^* := Z_2 \circ \rho^{-1}|_{B_2^*}.$$

Then

$$X^* := \begin{cases} Z_1^* & \text{on } B_1^*, \\ Z_2^* & \text{on } B_2^* \end{cases}$$

defines a mapping $X^* \in \mathcal{C}(\Gamma)$ with

$$\mathrm{dom}(X^*) = B_1^* \cup B_2^* \in \mathcal{N}(k).$$

Since A^ϵ is conformally invariant, it follows that

$$\begin{aligned} A^\epsilon(X^*) &= A^\epsilon(Z_1^*) + A^\epsilon(Z_2^*) \\ &= A^\epsilon(Z_1|_{B_1^*}) + A^\epsilon(Z_2|_{B^2 \setminus B_{2r}(p)}) \\ &= A^\epsilon(Z_1) + A^\epsilon(Z_2) \\ &= A^\epsilon(X^{(1)}) + \delta + A^\epsilon(X^{(2)}) + \delta \\ &\le d(\Gamma^1, \epsilon) + d(\Gamma^2, \epsilon) + 4\delta. \end{aligned}$$

Given $\eta > 0$ we choose $\delta := \eta/4$ and $X_m := X^*$ for all $m \in \mathbb{N}$. Then $\{X_m\}$ is a separating sequence satisfying (9) for a partition $\{\Gamma^1, \Gamma^2\}$ of Γ.

Similarly, if Γ is partitioned as $\{\Gamma^1, \dots, \Gamma^s\}$, we fix $\delta > 0$ and choose $X^{(\nu)} \in \mathcal{C}(\Gamma^\nu)$ with $B^\nu = \mathrm{dom}(X^{(\nu)}) \in \mathcal{N}(k_\nu)$, $k_1 + \dots + k_s = k$, such that

$$A^\epsilon(X^{(\nu)}) \le d(\Gamma^\nu, \epsilon) + \delta, \quad \nu = 1, \dots, s.$$

By the above procedure, carried out $(s-1)$ times, we find a mapping $X^* \in \mathcal{C}(\Gamma)$ with $\mathrm{dom}(X^*) \in \mathcal{N}(k)$ satisfying

$$A^\epsilon(X^*) \le \sum_{\nu=1}^{s} A^\epsilon(X^\nu) + 2^s \delta$$

whence

$$A^\epsilon(X^*) \le \sum_{\nu=1}^{s} d(\Gamma^\nu, \epsilon) + (s + 2^s)\delta.$$

Choosing $\delta := (s+2^s)^{-1}\eta$ and considering the separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with $X_m := X^*$ for all $m \in \mathbb{N}$, we obtain (9), and the proof of (8) is complete. $\square$

Lemma 4. (a) *Let $\Gamma^m \to \Gamma$ as $m \to \infty$ in the Fréchet sense, and $\{X_m\}$ be a sequence of mappings $X_m \in \mathcal{C}(\Gamma^m)$ with*

$$\Gamma^m = \langle \Gamma_1^{(m)}, \dots, \Gamma_k^{(m)} \rangle, \quad \Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$$

consisting of k rectifiable, closed, mutually disjoint Jordan curves. Then

$$d(\Gamma, \epsilon) \le \liminf_{m\to\infty} A^\epsilon(X_m) \quad \textit{for any } \epsilon \in (0,1]. \tag{10}$$

(b) *For any $\epsilon \in (0,1]$ we have*

$$d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon). \tag{11}$$

Proof. We fix ϵ with $0 < \epsilon \le 1$.

(a) Inequality (10) is trivially satisfied if the right-hand side is $= \infty$. Thus we may assume that $\{A^\epsilon(X_m)\}$ converges as $m \to \infty$, i.e.

$$\liminf_{m\to\infty} A^\epsilon(X_m) = \lim_{m\to\infty} A^\epsilon(X_m) < \infty. \tag{12}$$

Since $D(X_m) \le \epsilon^{-1} A^\epsilon(X_m)$ we have

$$D(X_m) \le M_0 \quad \text{for all } m \in \mathbb{N} \tag{13}$$

and some constant $M_0 = M_0(\epsilon) < \infty$. Then (10) follows from Lemma 2 if $\{X_m\}$ is cohesive, $k \ge 2$, and for $k = 1$ one infers (10) for any sequence on account of Chapter 4.

Now we are going to prove (10) by induction over k where we can restrict ourselves to noncohesive sequences $\{X_m\}$.

Induction hypothesis. Suppose that (10) is satisfied for boundary configurations consisting of at most $k-1$ closed curves.

Consider now a noncohesive sequence $\{X_m\}$ of $X_m \in \mathcal{C}(\Gamma^m)$ with $\operatorname{dom}(X_m) \in \mathcal{N}(k)$ satisfying (12) and therefore also (13). As $\{X_m\}$ is noncohesive, it possesses a separating subsequence which we may again call $\{X_m\}$. By Lemma 4 of Section 8.2 and Proposition 4 of Section 8.3 we can also assume that $B_m \in \mathcal{N}_1(k)$. Then there exist points $Q_m \in \mathbb{R}^3$, numbers $\eta_m > 0$ with $\eta_m \to 0$, and closed rectifiable Jordan curves γ_m in $\overline{B}_m$ bounding a strong Lipschitz interior B_m^* in $\mathbb{R}^2$ such that X_m possesses a well-defined continuous trace $Z_m =$ "$X_m|_{\gamma_m}$" on $\gamma_m = \partial B_m^*$ with

$$\sup_{\gamma_m} |Z_m - Q_m| \le \eta_m^2,$$

and any topological representation $c_m : S^1 \to \gamma_m$ of γ_m is not homotopic to zero in $\overline{B}_m$.

Then we choose a sequence of numbers $\delta_j > 0$ with $\delta_j \to 0$ and apply Proposition 2 of Section 8.5 with $\delta := \delta_j$ and $K := M_0(\epsilon)$. Let $\eta_{0,j}$ be the corresponding numbers $\eta_0 \in (0,1)$. For a suitable sequence $\{m_j\}$ of $m_j \in \mathbb{N}$ with $m_1 < m_2 < m_3 < \cdots$ we have $\eta_{m_j} < \eta_{0,j}$ for all $j \in \mathbb{N}$. Renaming $X_{m_j}, Q_{m_j}, Z_{m_j}, \eta_{m_j}$ as X_j, Q_j, Z_j, η_j respectively, it follows

$$\eta_j < \eta_{0,j} \quad \text{for all } j \in \mathbb{N},$$

and there are mappings

$$\Phi_j := \Phi_{\eta_j, Q_j} = \Phi_{\eta_j} : \mathbb{R}^3 \to \mathbb{R}^3$$

with the following properties:

(i) $\Gamma^{j*} := \Phi_j(\Gamma^j)$ is a configuration of k closed, disjoint Jordan curves such that the Fréchet distance $\Delta(\Gamma^j, \Gamma^{j*})$ of Γ^j and Γ^{j*} satisfies

$$\Delta(\Gamma^j, \Gamma^{j*}) < \delta_j \quad \text{for all } j \in \mathbb{N};$$

(ii) $\Phi_j \circ X_j \in \mathcal{C}(\Gamma^{j*})$ and $\mathrm{dom}(\Phi_j \circ X_j) = B_j$;
(iii) $\Phi_j(x) \equiv x$ for $x \in \mathbb{R}^3$ with $|x - Q_j| \geq \eta_j$;
(iv) $\Phi_j(x) \equiv Q_j$ for $x \in \mathbb{R}^3$ with $|x - Q_j| \leq \eta_j^2$;
(v) $A^\epsilon(\Phi_j \circ X_j) \leq A^\epsilon(X_j) + \delta_j$.

In particular we have

$$\Phi_j \circ Z_j = Q_j \quad \text{for all } j \in \mathbb{N}.$$

Then we define

$$B_j^1 := B_j \cap B_j^*, \quad B_j^2 := B_j \setminus \overline{B}_j^1$$

where B_j^* is the "inner domain" of γ_j. This means: Cutting along γ_j we decompose B_j into

$$B_j = B_j^1 \,\dot{\cup}\, \gamma_j \,\dot{\cup}\, B_j^2,$$

where B_j^1, B_j^2 are disjoint subdomains of B_j. Since γ_j cannot be contracted in $\overline{B}_j$ to a point, both B_j^1 and B_j^2 contain at least one of the boundary circles of B_j. Thus there is a circle β_j in B_j^1 whose center does not lie in $\overline{B}_j$. Let ρ_j be the inversion with respect to β_j, and set

$$\begin{aligned} E_j^1 &:= \overline{B}_j^{**} \cup \rho_j(B_j^1) \quad \text{with } B_j^{**} := \text{“inner domain” of } \rho_j(\gamma_j), \\ E_j^2 &:= \overline{B}_j^* \cup B_j^2. \end{aligned}$$

We note that $E_j^1 \in \mathcal{N}(k')$, $E_j^2 \in \mathcal{N}(k'')$ with $1 \leq k', k'' < k$ and $k = k' + k''$. Now we define new mappings $X_j^1 \in H_2^1(E_j^1, \mathbb{R}^3)$, $X_j^2 \in H_2^1(E_j^2, \mathbb{R}^3)$ by

$$X_j^1 := \begin{cases} \Phi_j \circ X_j \circ \rho_j^{-1} & \text{on } \rho_j(B_j^1), \\ Q_j & \text{on } \overline{B}_j^{**}, \end{cases}$$

$$X_j^2 := \begin{cases} \Phi_j \circ X_j & \text{on } B_j^2, \\ Q_j & \text{on } \overline{B}_j^{*}. \end{cases}$$

Roughly speaking, this process amounts to "pinching" X_j to a point in the neighborhood of the closed curve γ_j and to decomposing the resulting surface into two surfaces of "lower topological type" by cutting through γ_j.

Then there is a decomposition $\Gamma = \{\tilde{\Gamma}^1, \tilde{\Gamma}^2\}$ of Γ and correspondingly a decomposition $\Gamma^j = \{\tilde{\Gamma}^{j,1}, \tilde{\Gamma}^{j,2}\}$ of Γ^j such that

$$X_j^1 \in \mathcal{C}(\Phi_j(\tilde{\Gamma}^{j,1})), \quad X_j^2 \in \mathcal{C}(\Phi_j(\tilde{\Gamma}^{j,2}))$$

and

$$\Phi_j(\tilde{\Gamma}^{j,1}) \to \tilde{\Gamma}^1, \quad \Phi_j(\tilde{\Gamma}^{j,2}) \to \tilde{\Gamma}^2 \quad \text{in the sense of Fréchet.}$$

Furthermore, the construction yields

$$\begin{aligned} A^\epsilon(X_j^1) + A^\epsilon(X_j^2) &= A^\epsilon(\Phi_j \circ X_j|_{B_j^1}) + A^\epsilon(\Phi_j \circ X_j|_{B_j^2}) \\ &= A^\epsilon(\Phi_j \circ X_j), \end{aligned}$$

and the induction hypothesis implies

$$d(\Gamma^\ell, \epsilon) \le \liminf_{j\to\infty} A^\epsilon(X_j^\ell) \quad \text{for } \ell = 1, 2.$$

The partition $\Gamma = \{\tilde{\Gamma}^1, \tilde{\Gamma}^2\}$ leads to

$$d^+(\Gamma, \epsilon) \le d(\tilde{\Gamma}^1, \epsilon) + d(\tilde{\Gamma}^2, \epsilon).$$

Therefore

$$\begin{aligned} d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) &\le d(\tilde{\Gamma}^1, \epsilon) + d(\tilde{\Gamma}^2, \epsilon) \\ &\le \liminf_{j\to\infty} A^\epsilon(X_j^1) + \liminf_{j\to\infty} A^\epsilon(X_j^2) \\ &\le \liminf_{j\to\infty} [A^\epsilon(X_j^1) + A^\epsilon(X_j^2)] \\ &= \liminf_{j\to\infty} A^\epsilon(\Phi_j \circ X_j) \\ &\le \liminf_{j\to\infty} [A^\epsilon(X_j) + \delta_j]. \end{aligned}$$

Since $\delta_j \to 0$, we arrive at

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \le \liminf_{m\to\infty} A^\epsilon(X_m), \tag{14}$$

which completes the proof by induction, and so we have verified assertion (a).

(b) For $k = 1$ we have $d^*(\Gamma, \epsilon) = d^+(\Gamma, \epsilon) = \infty$, and so (11) holds true.

If $k \ge 2$ then Lemma 3 yields $d^*(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) < \infty$. Thus it suffices to show $d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon)$. In fact, for given $\delta > 0$ there is a separating sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with

$$\liminf_{m\to\infty} A^\epsilon(X_m) \le d^*(\Gamma, \epsilon) + \delta.$$

By the same proof as in (a) we obtain (14) for this sequence. Thus,

$$d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon) + \delta \quad \text{for any } \delta > 0,$$

whence

$$d^+(\Gamma, \epsilon) \le d^*(\Gamma, \epsilon),$$

which finishes the proof of (b). □

Theorem 1. *If the Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied, $k \ge 2$, then there is an $\epsilon_0 \in (0,1]$ such that for each $\epsilon \in (0, \epsilon_0]$ there exists a mapping $X^\epsilon \in \mathcal{C}(\Gamma)$ with*

$$A^\epsilon(X^\epsilon) = d(\Gamma, \epsilon) \tag{15}$$

and

$$|X_u^\epsilon|^2 = |X_v^\epsilon|^2, \quad \langle X_u^\epsilon, X_v^\epsilon \rangle = 0. \tag{16}$$

Proof. Since

$$\lim_{\epsilon \to +0} d(\Gamma, \epsilon) = d(\Gamma, 0) = a(\Gamma) < a^+(\Gamma) = d^+(\Gamma, 0) = \lim_{\epsilon \to +0} d^+(\Gamma, \epsilon),$$

there is an ϵ_0 with $0 < \epsilon_0 \le 1$ such that

$$d(\Gamma, \epsilon) < d^+(\Gamma, \epsilon) \quad \text{for } 0 < \epsilon \le \epsilon_0. \tag{17}$$

Fix some $\epsilon \in (0, \epsilon_0]$ and choose a sequence $\{X_m\}$ in $\mathcal{C}(\Gamma)$ with

$$A^\epsilon(X_m) \to d(\Gamma, \epsilon) \quad \text{as } m \to \infty.$$

If $\{X_m\}$ were not cohesive, there would exist a separating subsequence $\{X_{m_j}\}$, whence

$$d^*(\Gamma, \epsilon) \le \lim_{j \to \infty} A^\epsilon(X_{m_j}) = d(\Gamma, \epsilon),$$

and by (11) we would have

$$d^+(\Gamma, \epsilon) = d^*(\Gamma, \epsilon) \le d(\Gamma, \epsilon),$$

a contradiction to (17). Thus $\{X_m\}$ has to be cohesive, and $D(X_m) \le M_0(\epsilon)$ for all $m \in \mathbb{N}$ since $A^\epsilon(X_m) \le \text{const}$ and $\epsilon D(X_m) \le A^\epsilon(X_m)$. Hence we can apply Lemma 2 to $\Gamma^m \equiv \Gamma$ for all $m \in \mathbb{N}$, and consequently there is a $X^\epsilon \in \mathcal{C}(\Gamma)$ such that

$$d(\Gamma, \epsilon) \le A^\epsilon(X^\epsilon) \le \liminf_{m \to \infty} A^\epsilon(X_m) = d(\Gamma, \epsilon),$$

which yields (15), i.e.

$$A^\epsilon(X^\epsilon) \le A^\epsilon(X) \quad \text{for all } X \in \mathcal{C}(\Gamma).$$

By Theorem 1 of Section 8.2 this implies (16). □

Now we can prove the main result (cf. Theorem 1 in Section 8.2):

Theorem 2. *Suppose that the Douglas condition $a(\Gamma) < a^+(\Gamma)$ holds. Then there is a mapping $X \in \overline{\mathcal{C}}(\Gamma)$ with*

$$A(X) = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D = D(X) \tag{18}$$

satisfying the conformality relations

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \quad \text{in } B$$

as well as $X \in C^2(B, \mathbb{R}^3)$ and

$$\Delta X = 0 \quad \text{in } B. \tag{19}$$

Furthermore, X maps ∂B homeomorphically onto Γ.

Proof. Let $\epsilon_0 > 0$ be as in Theorem 1 and consider the mapping $X^\epsilon \in \mathcal{C}(\Gamma)$, $0 < \epsilon \le \epsilon_0$, satisfying (15) and (16). Then $A(X^\epsilon) = D(X^\epsilon)$, and consequently

$$d(\Gamma, \epsilon) = A^\epsilon(X^\epsilon) = A(X^\epsilon) = D(X^\epsilon) \quad \text{for } 0 < \epsilon \le \epsilon_0.$$

For an arbitrary $Y \in \mathcal{C}(\Gamma)$ we have

$$A^\epsilon(X^\epsilon) \le A^\epsilon(Y) \le D(Y),$$

whence

$$d(\Gamma) \le D(X^\epsilon) = A^\epsilon(X^\epsilon) \le A^\epsilon(Y) \le D(Y) \quad \text{for all } Y \in \mathcal{C}(\Gamma).$$

This yields

$$d(\Gamma) \le D(X^\epsilon) \le d(\Gamma)$$

and therefore

$$d(\Gamma) = D(X^\epsilon) \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

Then it follows for all $Y \in \mathcal{C}(\Gamma)$ and any $\epsilon, \epsilon' \in (0, \epsilon_0]$:

$$a(\Gamma) \le A(X^\epsilon) = A^\epsilon(X^\epsilon) = A^{\epsilon'}(X^{\epsilon'}) \le A^{\epsilon'}(Y).$$

Since $A^{\epsilon'}(Y) \to A(Y)$ as $\epsilon' \to +0$, we arrive at

$$a(\Gamma) \le A(X^\epsilon) \le a(\Gamma),$$

which implies

$$a(\Gamma) = A(X^\epsilon).$$

Thus we have

$$A(X^\epsilon) = D(X^\epsilon) = a(\Gamma) = d(\Gamma) \quad \text{for } 0 < \epsilon \le \epsilon_0,$$

that is,

$$A(X^\epsilon) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D = D(X^\epsilon). \tag{20}$$

Fix some $\epsilon \in (0, \epsilon_0]$ and set $X = X^\epsilon$. From

$$D(X) = \inf_{\mathcal{C}(\Gamma)} D$$

it follows that X is harmonic in B, and by virtue of $X \in H_2^1(B, \mathbb{R}^3)$ and $X|_{\partial B} \in C^0(\partial B, \mathbb{R}^3)$ we conclude that $X \in C^0(\overline{B}, \mathbb{R}^3)$, i.e. $X \in \overline{\mathcal{C}}(\Gamma)$. On account of (20) we obtain

$$A(X) = \inf_{\mathcal{C}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} A = \inf_{\overline{\mathcal{C}}(\Gamma)} D = \inf_{\mathcal{C}(\Gamma)} D = D(X).$$

Finally one proves in the same way as for Theorem 3 in Section 4.5 that X maps ∂B homeomorphically onto Γ. This completes the proof of the theorem. □

8.7 Further Discussion of the Douglas Condition

We had formulated the Douglas condition as the assumption that

$$a(\Gamma) < a^+(\Gamma) \tag{1}$$

holds true. Jesse Douglas [28] noted that (1) is equivalent to the assumption

$$d(\Gamma) < d^+(\Gamma) \tag{2}$$

where $d(\Gamma)$ and $d^+(\Gamma)$ are defined by

$$d(\Gamma) := \inf_{\mathcal{C}(\Gamma)} D, \quad d^+(\Gamma) := \inf_{\mathcal{C}^+(\Gamma)} D.$$

Using the notation of the previous section, this means

$$d(\Gamma) = d(\Gamma, 1), \quad d^+(\Gamma) = d^+(\Gamma, 1).$$

In fact, Douglas pointed out that the gist of his method to find a minimal surface X bounded by $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consisted in using exclusively Dirichlet's integral D instead of the area, replacing condition (1) by (2), cf. [28], p. 232, and all later authors proceeded in the same way. In order to prove that his solution is area minimizing, Douglas showed

$$a(\Gamma) = d(\Gamma), \tag{3}$$

and the proof of this identity he based on a theorem by P. Koebe, according to which every polyhedral surface possesses an a.e.-conformal representation of the same topological type. Our proof of (3) in Theorem 2 of Section 8.6 required no such tool, but was based on the assumption (1). Now we want to show that $a(\Gamma) = d(\Gamma)$ and $a^+(\Gamma) = d^+(\Gamma)$ holds for any contour Γ, without using any conformal mapping theorem. This in turn will yield the equivalence of the conditions (1) and (2) which are often called the *sufficient condition of Douglas*.

First, however, we note that for any contour $\Gamma = \langle \Gamma_1, \ldots, \Gamma_k \rangle$ one has the two inequalities

$$a(\Gamma) \le a^+(\Gamma) \quad \text{and} \quad d(\Gamma) \le d^+(\Gamma), \tag{4}$$

which are sometimes denoted as *necessary condition of Douglas*. Clearly, (4) follows from the inequality

$$d(\Gamma, \epsilon) \le d^+(\Gamma, \epsilon) \quad \text{for } \epsilon \in [0, 1], \tag{5}$$

which was established in Lemma 3 of Section 8.6.

Furthermore we recall (cf. Theorem 2 of Section 8.6):

$$\textit{Inequality (1) implies } a(\Gamma) = d(\Gamma). \tag{6}$$

Theorem 1. *We have*

$$a(\Gamma) = d(\Gamma) \quad \textit{for } k \ge 1 \tag{7}$$

and

$$a^+(\Gamma) = d^+(\Gamma) \quad \textit{for } k \ge 2 \tag{8}$$

Proof. (i) For $k = 1$, the identity (7) was proved in Chapter 4.

(ii) Let $k = 2$ and $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$.

(α) $a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2) \overset{\text{(i)}}{=} d(\Gamma_1) + d(\Gamma_2) = d^+(\Gamma)$.

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$ by (6).

(γ) If $a(\Gamma) = a^+(\Gamma)$ then $a(\Gamma) \overset{(\alpha)}{=} d^+(\Gamma) \overset{(4)}{\ge} d(\Gamma)$, and trivially we have

$$a(\Gamma) \le d(\Gamma) \quad \text{for any } k \ge 1 \tag{9}$$

because of $A \le D$. Thus $a(\Gamma) = d(\Gamma)$ also in case (γ), and by (4) it follows $a(\Gamma) = d(\Gamma)$ in any case if $k = 2$.

(iii) Let $k = 3$ and $\Gamma = \langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$.

$$(\alpha) \qquad \begin{aligned} a^+(\Gamma) &= \min\{a(\Gamma_\mu) + a(\Gamma_\nu) + a(\Gamma_\rho), a(\Gamma_\mu, \Gamma_\nu) + a(\Gamma_\rho) \colon \\ &\qquad (\mu, \nu, \rho) \sim (1, 2, 3)\} \\ &= \min\{d(\Gamma_\mu) + d(\Gamma_\nu) + d(\Gamma_\rho), d(\Gamma_\mu, \Gamma_\nu) + d(\Gamma_\rho) \colon \\ &\qquad (\mu, \nu, \rho) \sim (1, 2, 3)\} \\ &= d^+(\Gamma). \end{aligned}$$

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$ by (6).

(γ) If $a(\Gamma) = a^+(\Gamma)$, then by (α), (4) and (9) it follows

$$a(\Gamma) = d^+(\Gamma) \geq d(\Gamma) \geq a(\Gamma),$$

whence $a(\Gamma) = d(\Gamma)$ in any case on account of (4), if $k = 3$.

(iv) The general case is proved by induction: Suppose that (7) is verified for $k \leq N$. Then we obtain for $k = N + 1$:

(α) $a^+(\Gamma) = d^+(\Gamma)$. In fact,

$$\begin{aligned} a^+(\Gamma) &= \min\{a(\Gamma^1) + \cdots + a(\Gamma^s): \\ &\qquad \{\Gamma^1, \ldots, \Gamma^s\} = \text{partition of } \Gamma \text{ with } s \geq 2\}, \\ d^+(\Gamma) &= \min\{d(\Gamma^1) + \cdots + d(\Gamma^s): \\ &\qquad \{\Gamma^1, \ldots, \Gamma^s\} = \text{partition of } \Gamma \text{ with } s \geq 2\}, \end{aligned}$$

and Γ^ℓ consists of k_ℓ closed curves, $k_1 + \cdots + k_s = N + 1$, whence $k_\ell \leq N$ for $\ell = 1, \ldots, s$, and since (7) holds for $k \leq N$, we obtain $a(\Gamma^1) = d(\Gamma^1), \ldots, a(\Gamma^s) = d(\Gamma^s)$; therefore we have (8) for $k = N + 1$.

(β) If $a(\Gamma) < a^+(\Gamma)$ then $a(\Gamma) = d(\Gamma)$.

(γ) If $a(\Gamma) = a^+(\Gamma)$, then by (α), (4), and (9):

$$a(\Gamma) = d^+(\Gamma) \geq d(\Gamma) \geq a(\Gamma)$$

whence $a(\Gamma) = d(\Gamma)$ in any case on account of $a(\Gamma) \leq a^+(\Gamma)$. □

Similarly one proves

Theorem 2. *For any $\epsilon \in [0, 1]$ we have*

$$a(\Gamma, \epsilon) = d(\Gamma, \epsilon) = a(\Gamma) = d(\Gamma) \quad \text{if } k \geq 1 \tag{10}$$

and

$$a^+(\Gamma, \epsilon) = d^+(\Gamma, \epsilon) = a^+(\Gamma) = d^+(\Gamma) \quad \text{if } k \geq 2; \tag{11}$$

therefore also

$$a^+(\Gamma) - a(\Gamma) = d^+(\Gamma) - d(\Gamma) = d^+(\Gamma, \epsilon) - d(\Gamma, \epsilon) \quad \text{if } k \geq 2. \tag{12}$$

Corollary 1. *The conditions* (1) *and* (2) *are equivalent.*

Corollary 2. *In Theorem* 1 *of Section* 8.6 *we can choose $\epsilon_0 = 1$.*

8.8 Examples

We now exhibit some examples when the sufficient Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied.

$\boxed{1}$ Let $k = 2$, and consider two closed, rectifiable, disjoint Jordan curves Γ_1 and Γ_2 that lie in planes Π_1 and Π_2 which intersect in a straight line L. By S_1 and S_2 we denote the two bounded planar domains in Π_1 and Π_2 with the boundary contours Γ_1 and Γ_2 respectively. Then

$$a(\Gamma_1) = \text{area}(S_1), \quad a(\Gamma_2) = \text{area}(S_2).$$

Suppose that $S_1 \cap S_2$ is nonempty. Then S_1 and S_2 intersect in a closed interval I contained in L. The line L decomposes S_1 and S_2 into the pieces S_1^+, S_1^- and S_2^+, S_2^- respectively with $S_1^+ \cap S_1^- := I_1 \subset L$ and $S_2^+ \cap S_2^- =: I_2 \subset L$. Take an interior point $P \in L$, a bisectrix L' of one of the angles between Π_1 and Π_2 meeting L at P perpendicularly, and consider a sufficiently small circular cylinder Z with the axis L'. Then Z intersects $S_1^+, S_1^-, S_2^+, S_2^-$ in closed curves $\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-$ consisting of semi-ellipses $\epsilon_1^+, \epsilon_1^-, \epsilon_2^+, \epsilon_2^-$ and an interval $j \subset I$. Let $E_1^+, E_1^-, E_2^+, E_2^-$ be the "full" semi-ellipses bounded by $\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-$ respectively. Then $\gamma_1 := \gamma_1^+ \cup \gamma_2^+$ spans a nonparametric minimal surface M_1 with

$$\text{area}(M_1) < \text{area}(E_1^+ \cup E_2^+),$$

and $\gamma_2 := \gamma_1^- \cup \gamma_2^-$ spans a nonparametric minimal surface M_2 with

$$\text{area}(M_2) < \text{area}(E_1^- \cup E_2^-).$$

Then the set

$$\Sigma := (S_1 \cup S_2 \cup M_1 \cup M_2) \setminus (E_1^+ \cup E_2^+ \cup E_1^- \cup E_2^-)$$

has an area less than that of $S_1 \cup S_2$, i.e.

$$\text{area}(\Sigma) < \text{area}(S_1 \cup S_2) = \text{area}(S_1) + \text{area}(S_2).$$

We can construct a mapping $X \in \mathcal{C}(\Gamma)$, $\Gamma := \langle \Gamma_1, \Gamma_2 \rangle$, such that

$$\Sigma = X(\overline{B}), \quad B = \text{dom}(X) \in \mathcal{N}(2),$$

and thus we have

$$a(\Gamma) \le A(X) = \text{area}(\Sigma) < a(\Gamma_1) + a(\Gamma_2) = a^+(\Gamma).$$

Hence we have

Proposition 1. $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ *satisfies the Douglas condition if* Γ_1 *and* Γ_2 *fulfill the assumptions stated above. In particular, we have* $a(\Gamma) < a^+(\Gamma)$ *for* $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ *if* Γ_1 *and* Γ_2 *are closed, rectifiable, disjoint planar Jordan curves in* $\mathbb{R}^3$ *which are linked.*

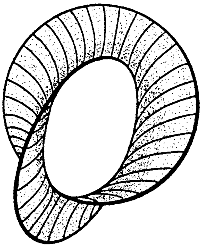

Fig. 1. An annulus-type minimal surface bounded by two interlocking closed curves

Actually, it is irrelevant that Γ_1 and Γ_2 are planar, and a similar reasoning as above yields

Theorem 1. *Suppose that the contour $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ consists of two closed, rectifiable, disjoint Jordan curves in $\mathbb{R}^3$ which are linked. Then $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$, and so there is a minimal surface $X \in \mathcal{C}(\Gamma)$ with $B = \mathrm{dom}(X) \in \mathcal{N}(2)$ and $A(X) = a(\Gamma)$, i.e. X is an area-minimizing minimal surface of annulus type bounded by two linked closed curves Γ_1 and Γ_2.*

J. Douglas (cf. [13], p. 351) obtained Theorem 1 as a corollary of the following

Theorem 2. *Let Γ_1 and Γ_2 be two nonintersecting, closed, rectifiable Jordan curves in $\mathbb{R}^3$, and suppose that there are minimal surfaces $X_1 \in \mathcal{C}(\Gamma_1), X_2 \in \mathcal{C}(\Gamma_2)$ with $A(X_1) = a(\Gamma_1)$, $A(X_2) = a(\Gamma_2)$ such that $X_1(w_1) = X_2(w_2)$ for some $w_1, w_2 \in B = B_1(0) = \mathrm{dom}(X_1) = \mathrm{dom}(X_2)$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ satisfies the Douglas condition $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$, and so there is an annulus-type minimal surface $X \in \mathcal{C}(\Gamma)$ with $A(X) = a(\Gamma)$.*

Remark 1. Instead of giving a geometric proof for $a(\Gamma) < a^+(\Gamma)$, Douglas derived the inequality $d(\Gamma) < d^+(\Gamma)$ in an analytic way working with the Dirichlet integral and arranging for $w_1 = w_2 = 0$. Using the harmonic mapping $H : \{r < |w| < 1\} \to \mathbb{R}^3$, $0 < r < 1$, with the boundary values $H(w) = X_1(w)$ for $|w| = 1, H(w) = X_2(w)$ for $|w| = r$. Then it can be shown that

$$D(H) < D(X_1) + D(X_2) = d(\Gamma_1) + d(\Gamma_2) \quad \text{for } 0 < r \ll 1,$$

which implies $d(\Gamma) < d^+(\Gamma)$ for $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$, and we know that this inequality is equivalent to $a(\Gamma) < a^+(\Gamma)$.

Essentially the same proof can be found in J.C.C. Nitsche [28], pp. 531–533.

Remark 2. Both Douglas and Nitsche assumed in addition that w_1 is not a branch point of X_1 and w_2 is not a branch point of X_2. These requirements are now superfluous because of the Osserman–Alt–Gulliver result.

Remark 3. Note that for planar Γ_1 and Γ_2 the result of Theorem 2 is essentially contained in Proposition 1. Furthermore, the proof of this proposition can be modified to yield Theorem 2.

$\boxed{2}$ Obviously the Douglas condition $a(\Gamma) < a^+(\Gamma) = a(\Gamma_1) + a(\Gamma_2)$ for $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is satisfied if Γ_1 and Γ_2 bound a doubly connected surface S with

$$\text{area}(S) < a(\Gamma_1) + a(\Gamma_2),$$

say, the lateral surfaces of a conical frustum, or a cylindrical surface. This simple observation was used in the construction of a one-parameter family of triply-connected minimal surfaces bounded by three coaxial circles $\Gamma_1, \Gamma_2, \Gamma_3$; see Section 4.15.

$\boxed{3}$ Finally we note that the Douglas condition $a(\Gamma) < a^+(\Gamma)$ is satisfied for $\Gamma = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_k \rangle$, $k \geq 2$, if the distinct, closed, rectifiable Jordan curves $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ form the boundary of a bounded, k-fold connected domain Ω in $\mathbb{R}^2$:

$$\partial\Omega = \Gamma_1 \dot\cup \Gamma_2 \dot\cup \cdots \dot\cup \Gamma_k.$$

In fact, each contour Γ_j bounds a simply connected, bounded domain Ω_j in $\mathbb{R}^2$, and we may assume that

$$\Omega = \Omega_1 \setminus \{\Omega_2 \cup \cdots \cup \Omega_k\},$$

i.e. Γ_1 is the "exterior" boundary curve of Ω. Then

$$\text{(1)} \qquad \text{area}(\Omega) = \text{area}(\Omega_1) - \left\{ \sum_{j=2}^{k} \text{area}(\Omega_j) \right\}.$$

Let $\Gamma = \{\Gamma^1, \dots, \Gamma^s\}$ be an arbitrary partition of the boundary curves $\Gamma_1, \dots, \Gamma_k$, $s \geq 2$. We may assume that Γ_1 belongs to Γ^1, i.e. $\Gamma^1 = \langle \Gamma_1, \Gamma_{j_2}, \dots, \Gamma_{j_\ell} \rangle$ with $1 < j_2 < \cdots < j_\ell$ and $1 \leq \ell < k$. Then

$$a^+(\Gamma) := \inf\{a(\Gamma^1) + \cdots + a(\Gamma^s) \colon \{\Gamma^1, \dots, \Gamma^s\} = \text{partition of } \Gamma\}$$

whence

$$\text{(2)} \qquad a^+(\Gamma) \geq a(\Gamma^1) = \text{area}(\Omega_1) - \sum_{\nu=2}^{\ell} \text{area}(\Omega_{j_\nu})$$

($\sum_{\nu=2}^{\ell} = 0$ if $\ell = 1$). From (1) and (2) we infer

$$a^+(\Gamma) > \text{area}(\Omega) = a(\Gamma).$$

Thus we can apply Theorem 2 of Section 8.6. Combining this with the reasoning that was used in Section 4.11 to prove Riemann's mapping theorem (cf. Theorem 1 in Section 4.11), we obtain **Koebe's mapping theorem**:

Theorem 3. *Let Ω be a k-fold connected domain in $\mathbb{C}$ whose boundary consists of k closed, mutually disjoint Jordan curves $\Gamma_1, \ldots, \Gamma_k$. Then there exists a homeomorphism f from $\overline{B}$ onto $\overline{\Omega}$, $B \in \mathcal{N}(k)$, which is holomorphic in B and satisfies $f'(w) \neq 0$ for all $w \in B$.*

P. Koebe also proved that f is uniquely determined up to a Möbius transformation, i.e. if f^* is another mapping like f from $\overline{B}^*$ onto $\overline{\Omega}$, $B^* \in \mathcal{N}(k)$, then there is a Möbius transformation τ from $\overline{B}^*$ onto $\overline{B}$ with $f^* = f \circ \tau$. An elegant proof of this fact can be found in Courant and Hurwitz [1], pp. 517–519. In another form, a uniqueness result is stated and proved in R. Courant [15], pp. 187–191: $f^* = f$ if $f, f^* \in \mathcal{N}_1(k)$ and $f(\zeta) = f^*(\zeta)$ for a fixed point $\zeta \in \partial B_1(0)$.

8.9 Scholia

1. The first to study general Plateau problems for minimal surfaces of higher topological type was Jesse Douglas; his work was truly pioneering, and his ideas and insights are as exciting and important nowadays as at the time when they were published, more than half a century ago. It seems that Douglas was the first to grasp the idea that a minimizing sequence could be degenerating in topological type, and he interpreted such a conceivable degeneration as a change in the conformal structure. He based his notion of degeneration on the representation of Riemann surfaces as branched coverings of the sphere. Then degeneration meant "disappearance of branch cuts". The intuitive meaning of degeneration is the shrinking of handles and the tendency to separate the Riemann surface into several components. Since degeneration is unavoidable in general, Douglas had the idea of minimizing not over surfaces of a *fixed* topological type but also over all possible reductions of the given type. In this set of Riemann surfaces of varying topological type, Douglas introduced a notion of convergence as convergence of branch points in the representation of the surfaces as branched coverings of the sphere. The compactness of this set of Riemann surfaces seemed to be a trivial matter to him since his whole argument reads: *"This is because the set can be referred to a finite number of parameters, e.g., the position of the branch points ..."*. This reasoning is, however, rather inaccurate since the position of branch points alone does not determine the structure of the surface. Douglas also argued on a rather intuitive level when it came to the lower semicontinuity of Dirichlet's integral with respect to the convergence of surfaces. Taking the compactness of the above set of Riemann surfaces and the lower semicontinuity of Dirichlet's integral for granted, it is then obvious that an absolute minimum of Dirichlet's integral in the class of surfaces considered by Douglas must be achieved, either in a surface of desired (highest) topological type or in one of reduced type. In this way Douglas was led to his celebrated solution of the **general Plateau problem**:

Given a boundary configuration $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ consisting of $k \geq 1$ closed, rectifiable, mutually disjoint Jordan curves $\Gamma_1, \dots, \Gamma_k$ in $\mathbb{R}^3$, there is a connected minimal surface X of prescribed Euler characteristic and prescribed character of orientability, bounded by Γ, provided that the infimum $a(\Gamma)$ of area for all admissible surfaces is less than the infimum $a^+(\Gamma)$ of Dirichlet's integral or of the sum of Dirichlet integrals for surfaces of lower type bounded by Γ.

Here a possibly disconnected surface Y bounded by Γ is called of *lower type* if at least one of the following degenerations occurs:

(i) Y has a smaller Euler characteristic than prescribed;
(ii) Y is disconnected and consists of several connected pieces of total characteristic (= sum of the characteristics of the connected pieces) not greater than prescribed, and each piece is bounded by complementary subsets of $\{\Gamma_1, \dots, \Gamma_k\}$ which together make up Γ.

J. Douglas published this most general result in his 1939 paper [28]. Already in 1931 he had treated the case $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ for annulus-type minimal surfaces (cf. [18]), and one-sided minimal surfaces in a given contour he had discussed 1932 in his paper [15]. Further work dealing with the general Plateau problem are his papers [27,29] and [31].

2. R. Courant [9,11], and M. Shiffman [3,5] put the pioneering work of Douglas on a solid basis by solving the variational problem "$D \to \min$" within a class of surfaces of fixed topological type. In this context we also mention H. Lewy's lecture notes [3] from 1939.

Courant gave a very clear exposition of his method in his treatise [15] from 1950 for minimal surfaces $X : B \to \mathbb{R}^3$ with $B \in N$, where the class N of parameter domains comprises either (a) schlicht k-circle domains, or (b) slit domains, or (c) Riemann domains over the w-plane bounded by k unit circles and having branch points of total multiplicity $2k - 2$ (cf. Courant [15], pp. 144–145, 149); other types are briefly discussed in [15], pp. 164–166.

3. Douglas has based his investigations on the use of symmetric Riemann surfaces without boundary which are obtained as doubles of Riemann surfaces of genus g with k boundary curves. This idea is also employed in the study of the general Plateau problem by F. Tomi and A. Tromba [5], which will be presented in Chapter 4 of Vol. 3.

An exposition of how to solve the Douglas problem for surfaces of higher topological type or for nonorientable surfaces is presented in Courant [15], pp. 160–164. In particular, the existence proof for surfaces of the topological type of the Möbius strip with one boundary contour is worked out in detail. For the general case, Courant refers to Shiffman [3].

Another presentation of the work of Douglas is given in the treatise [6] of J. Jost. In Nitsche's *Vorlesungen* [28], the Douglas problem for annulus-type minimal surfaces with two boundary curves is treated. C.B. Morrey [8], Chapter 9, described a solution of the Douglas problem for k-fold connected minimal surfaces.

The Douglas problem for H-surfaces was studied by H. Werner [1] for $H = \text{const}$, and for variable H by S. Luckhaus [1].

Beautiful soap film experiments with minimal surfaces are described in papers by Courant [10] and by Almgren and Taylor [1].

4. Douglas also treated the case of configurations $\Gamma = \langle \Gamma_1, \dots, \Gamma_k \rangle$ with *non-rectifiable* curves. In this regard we refer to Section 17 of his paper [28], pp. 279–287.

5. The idea to prove Koebe's mapping theorem via the solution of the general Plateau problem was also conceived by Douglas in [11] and [28]. Courant presented an elaboration of this approach in Chapter 5, pp. 167–198, of his treatise [15].

A generalization of Lichtenstein's mapping theorem to Riemannian metrics on multiply connected domains is due to J. Jost [6] and [17]; the original approach by Morrey [8] is incorrect. A new proof in the spirit of Section 4.11 was given in the paper [8] by Hildebrandt and von der Mosel. Jost [6] treated the Douglas problem for orientable minimal surfaces in a Riemannian manifold; see also Morrey [3] and [8]. The nonorientable case was worked out by F. Bernatzki [1].

6. The presentation of this chapter is based on the work of Courant [15] and on the papers of Kurzke [1], Kurzke and von der Mosel [1], and Hildebrandt and von der Mosel [6,8].

Problems

Here we formulate some major open problems for minimal surfaces, mostly in the context of Plateau's or Douglas's problem. Many of them are unsolved since a long time, see e.g. J.C.C. Nitsche [28,37].

Let Γ be a rectifiable curve in $\mathbb{R}^3$, and $\mathcal{C}(\Gamma)$ be the class of admissible curves for Plateau's problem defined in Section 4.2. Furthermore, for any $X \in \mathcal{C}(\Gamma)$ let $A(X)$ be the area functional and $D(X)$ the Dirichlet integral of X. One has

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D.$$

Hence an absolute minimizer of D is one of A, and conversely, a conformally parametrized absolute minimizer of A is an absolute minimizer of D.

1. Is it true that a relative minimizer of D in $\mathcal{C}(\Gamma)$ is also a relative minimizer of A in $\mathcal{C}(\Gamma)$ with respect to some suitable norm on $\mathcal{C}(\Gamma)$, say, $\|\cdot\|_{C^0}$, $\|\cdot\|_{C^0} + \sqrt{D(\cdot)}$, $\|\cdot\|_{C^1}$, $\|\cdot\|_{C^{1,\alpha}}, \dots$? The converse holds for conformally parametrized relative minimizers of A. The problem might be easier to solve if one assumes $\Gamma \in C^{m,\alpha}$. (See also Appendix 1.)
   A similar question can be raised for the Douglas problem, for H-surfaces, or for minimal surfaces in a Riemann surface.

Let $\mathcal{C}^*(\Gamma)$ be the class of surfaces in $\mathcal{C}(\Gamma)$ which satisfy a preassigned three-point condition.

2. How many minimal surfaces of class $\mathcal{C}^*(\Gamma)$ are bounded by a given "well-behaved" closed Jordan curve Γ? Here well-behaved might be interpreted as *regular and real analytic*, or as *regular and of class* C^k, or as *piecewise linear* (i.e. Γ be a *polygon*), or in another suitable way.
   An analogous question can be asked for the Douglas problem, or for cmc-surfaces.
3. Is the number of immersed, stable minimal surfaces in $\mathcal{C}^*(\Gamma)$ finite?
4. Can one give upper or lower bounds on the number $N(\Gamma)$ of minimal surfaces in $\mathcal{C}^*(\Gamma)$ in terms of bounds on the geometric data of X? Can

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one find such bounds for the number $N_s(\Gamma)$ of stable minimal surfaces in $\mathcal{C}^*(\Gamma)$? Similarly for the number $N_i(\Gamma)$ of immersed minimal surfaces in $\mathcal{C}^*(\Gamma)$.

A closed, connected component $\mathcal{K}_c$ of a level set $\mathcal{M}_c(\Gamma) := \{X$ is minimal surface of class $\mathcal{C}^*(\Gamma)$ with $D(X) = A(X) = c\}$, $c \in \mathbb{R}$, is called a *block of minimal surfaces*.

5. Do blocks of minimal surfaces in $\mathbb{R}^3$ always consist of single elements? If not, what is their topological and analytic structure?
6. Can one solve Plateau's problem for minimal surfaces in a Riemann manifold M, if Γ is an arbitrary Jordan curve in M? The same question can be posed for the Douglas problem, or for H-surfaces (cf. Section 4.12, Remark 1)?
7. Is it possible to solve Plateau's problem within the class of immersed minimal surfaces, either by a continuity method or by a variational method?
8. Can minimizers of D in $\mathcal{C}(\Gamma)$ possess boundary branch points?
9. Can one show the existence of immersed, but unstable minimal surfaces in $\mathcal{C}^*(\Gamma)$ via a mountain pass theorem?
10. Is it possible to derive general uniqueness theorems which include and combine those of Radó and Nitsche as well as that for small perturbations of certain planar curves?
11. Let $\mathcal{M}(\Gamma)$ be the class of minimal surfaces in $\mathcal{C}(\Gamma)$. What is the interrelation between the classes $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ if Γ is a regular C^k-curve and Γ' is a polygon or another C^k-curve close to Γ? In particular, how does $N(\Gamma)$ change under perturbations of Γ? What happens to $N(\Gamma)$, $N_i(\Gamma)$ and $N_s(\Gamma)$ if the total curvature of Γ changes beyond 6π?
12. Is there an index formula for polygons similar to the formula established by Böhme/Tromba and Tomi/Tromba for smooth contours?
13. Given a closed polygon Γ, can one give a classification of minimal surfaces in $\mathcal{C}^*(\Gamma)$ by Courant's and Shiffman's functions, using ideas of Heinz and Sauvigny? Is Courant's function of class C^2? Can one prove an index-sum formula in the nondegenerate situation?
14. Is it possible to estimate the modulus of continuity of the normal of a minimal surface up to and including the boundary, at least for special classes of minimal surfaces?
15. Can one derive estimates of the Gaussian curvature for stable minimal immersions, possibly up to and including the boundary?

Appendix 1

On Relative Minimizers of Area and Energy

Let Γ be a closed, rectifiable Jordan curve in $\mathbb{R}^3$, and $\mathcal{C}(\Gamma)$ be the class of disk-type surfaces $X : B \to \mathbb{R}^3$ bounded by Γ. Since

$$\inf_{\mathcal{C}(\Gamma)} A = \inf_{\mathcal{C}(\Gamma)} D,$$

it follows:

Any minimizer of Dirichlet's integral D in $\mathcal{C}(\Gamma)$ is a minimal surface that minimizes the area A in $\mathcal{C}(\Gamma)$. Conversely, any conformally parametrized minimizer of A in $\mathcal{C}(\Gamma)$ is a minimal surface which minimizes D in $\mathcal{C}(\Gamma)$.

In Problem 1 we have raised the question whether a similar result holds for relative minimizers of A and D. Clearly, *a minimal surface $X \in \mathcal{C}(\Gamma)$ is a relative minimizer of D in $\mathcal{C}(\Gamma)$ with respect to some suitable norm $\|\cdot\|$ if X is a relative minimizer of A in $\mathcal{C}(\Gamma)$ with respect to $\|\cdot\|$*. In fact, from

$$A(X) \le A(Y) \quad \text{for all } Y \in \mathcal{C}(\Gamma) \text{ with } \|X - Y\| < \epsilon$$

for some $\epsilon > 0$ it follows that

$$D(X) = A(X) \le A(Y) \le D(Y).$$

Here we prove a weak converse of this result giving a partial solution to Problem 1.

Let $\mathcal{M}(\Gamma)$ be the set of minimal surfaces $X \in \mathcal{C}(\Gamma)$, and denote by $\mathcal{M}_{\mathrm{im}}(\Gamma)$ the class of immersed minimal surfaces in $\mathcal{C}(\Gamma) \cap C^1(\overline{B}, \mathbb{R}^3)$. For the following we assume that Γ is a regular contour of class $C^{1,\mu}, 0 < \mu < 1$, which implies $\mathcal{M}(\Gamma) \subset C^{1,\mu}(\overline{B}, \mathbb{R}^3)$.

For the sake of brevity, we write $C^0, C^1, C^{1,\nu}$ instead of $C^0(\overline{B}, \mathbb{R}^3)$, $C^1(\overline{B}, \mathbb{R}^3)$, $C^{1,\nu}(\overline{B}, \mathbb{R}^3)$, and correspondingly $\|\cdot\|_{C^0}$ for $\|\cdot\|_{C^0(\overline{B}, \mathbb{R}^3)}$, etc.

We have the following results, proved in Hildebrandt and Sauvigny [9]:

Theorem 1. *Let $X \in \mathcal{M}_{\mathrm{im}}(\Gamma)$ be a relative minimizer of D in the following sense: There is an $\epsilon > 0$ such that*

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$$D(X) \le D(Z) \quad \textit{for all } Z \in \mathcal{C}(\Gamma) \cap C^1(\overline{B}, \mathbb{R}^3) \textit{ with } \|Z - X\|_{C^1} < \epsilon.$$

Then there exists a $\delta(\epsilon) > 0$ *such that*

$$\begin{aligned} A(X) \le A(Y) \quad & \textit{for all } Y \in \mathcal{C}(\Gamma) \cap C^{1,\mu}(\overline{B}, \mathbb{R}^3) \\ & \textit{with } \|Y - X\|_{C^{1,\mu}} < \delta(\epsilon), \end{aligned}$$

i.e. X is a relative minimizer of A.

Remark. We can rephrase this result as follows: *A relative minimizer X of D with respect to the C^1-norm is a relative minimizer of A with respect to the $C^{1,\mu}$-norm.* Note that the C^1-minimum property is stronger than the $C^{1,\mu}$-minimum property since the C^1-norm is weaker than the $C^{1,\mu}$-norm.

Similarly one can prove

Theorem 2. *Suppose that the immersed minimal surface $X \in \mathcal{C}(\Gamma)$ is of the class $C^2(\overline{B}, \mathbb{R}^2)$, and assume that X is a relative minimizer of D with respect to the $C^{1,\mu}$-norm for some $\mu \in (0,1)$. Then it is also a relative minimizer of A with respect to the C^2-norm.*

Before we come to the proofs, we have to collect some results.

First, fix some $X \in \mathcal{M}_{\mathrm{im}}(\Gamma)$. Then $X \in C^{1,\mu}(\overline{B}, \mathbb{R}^3)$, and the line element $d\sigma$ on X, given by

$$d\sigma^2 := \langle dX, dX \rangle = \Lambda(X)\{du^2 + dv^2\},$$

satisfies

$$\Lambda(X) = |X_u|^2 \ge \delta_0(X) \tag{1}$$

for some positive constant $\delta_0(X)$.

Consider a "perturbation" Y of X satisfying $Y \in C^{1,\mu}(\overline{B}, \mathbb{R}^3)$ and

$$\|Y - X\|_{C^{1,\mu}} < \eta \tag{2}$$

for some $\eta \in (0,1)$, and let ds be the line element on Y. We write

$$ds^2 := \langle dY, dY \rangle = a\, du^2 + 2b\, du\, dv + c\, dv^2$$

and denote by

$$\mathcal{D} := ac - b^2$$

the associate discriminant. Then $a, b, c \in C^{0,\mu}(\overline{B})$, and there is a constant $\kappa(X) > 0$ such that the matrices

$$(\gamma_{jk}) := \begin{pmatrix} \Lambda(X) & 0 \\ 0 & \Lambda(X) \end{pmatrix}, \quad (g_{jk}) := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

satisfy

$$\|g_{jk} - \gamma_{jk}\|_{C^{0,\mu}(\overline{B})} \le \kappa(X)\eta, \tag{3}$$

and for

$$\eta < \tfrac{1}{6}\Lambda(X)\kappa^{-1}(X) \tag{4}$$

we obtain

$$\sqrt{\mathcal{D}} \ge \lambda_0 \quad \text{with } \lambda_0 := \sqrt{5/18}\,\Lambda(X) > 0. \tag{5}$$

This implies $\sqrt{\mathcal{D}} \in C^{0,\mu}(\overline{B})$ and

$$\|\sqrt{\mathcal{D}} - \Lambda(X)\|_{C^{0,\mu}(\overline{B})} \le \kappa'(X)\eta \tag{6}$$

for some constant $\kappa'(X) > 0$, and we note that $\sqrt{\mathcal{D}} = \Lambda$ if $Y = X$.

Now we want to construct a diffeomorphism $f : \overline{B} \to \overline{\Omega}$ of class $C^{1,\mu}(\overline{B}, \mathbb{R}^2)$ such that the pull-back $(f^{-1})^*\, ds^2$ of ds^2 under the inverse f^{-1} of f is transferred into the form

$$(f^{-1})^*\, ds^2 = \lambda\{dx^2 + dy^2\}$$

and that $f : (u, v) \mapsto (x, y) = f(u, v)$ is close to the identical map $id_{\overline{B}}$ if η is sufficiently small. For this purpose we follow I.N. Vekua [2], §§1–4, and F. Sauvigny [16], Chapter XII, especially §8. Interpreting B and Ω as domains in the complex plane $\mathbb{C}$, which is identified with $\mathbb{R}^2$, the mapping $w \mapsto z = f(w)$ with $w = u + iv \in \overline{B}$, $z = x + iy \in \mathbb{C}$, will be constructed as a solution of the *complex Beltrami equation*

$$f_{\overline{w}} - q(w) f_w = 0 \quad \text{in } \overline{B} \tag{7}$$

where the complex potential $q(w)$ is defined by

$$q := \frac{a - \sqrt{\mathcal{D}} + ib}{a + \sqrt{\mathcal{D}} - ib} = \frac{a - c + 2ib}{a + c + 2\sqrt{\mathcal{D}}}. \tag{8}$$

Using the estimates (3)–(6) we easily verify the subsequent

Proposition 1. *For each $\epsilon_1 > 0$ there is a number $\eta_1(X, \epsilon_1) > 0$ such that $q \in C^{0,\mu}(\overline{B}, \mathbb{C})$ and that the $C^{0,\mu}(\overline{B}, \mathbb{C})$-norm $\|q\|_\mu := \|q\|_{C^{0,\mu}(\overline{B},\mathbb{C})}$ of the mapping $w \mapsto q(w)$ satisfies*

$$\|q\|_\mu < \epsilon_1 \quad \textit{provided that} \quad \|Y - X\|_{C^{1,\mu}} < \eta_1.$$

In order to construct a solution f of (7), we use the following ansatz which is described in Vekua [2], pp. 359–360:

$$f(w) = w + T_B[\psi](w), \quad w \in \overline{B},\ \psi \in C^{0,\mu}(\overline{B}, \mathbb{C}), \tag{9}$$

where T_B is the *Cauchy operator* $\psi \mapsto T_B[\psi]$ given by

$$T_B[\psi](w) := -\frac{1}{\pi}\int_B \frac{\psi(z)}{z-w}\,dx\,dy, \quad w = u + iv \in \overline{B},\ z = x + iy \in \mathbb{C}. \tag{10}$$

When we insert (9) into (7), we are led to a *Tricomi integral equation* for $\psi \in C^{0,\mu}(\overline{B}, \mathbb{C})$,

$$\psi - q\Pi_B[\psi] = q, \tag{11}$$

containing *Vekua's integral operator* Π_B, which is defined by a Cauchy principal value:

$$\Pi_B[\psi](w) := \lim_{\rho \to +0}\left\{-\frac{1}{\pi}\int_{B\setminus B_\rho(w)} \frac{\psi(z)}{(z-w)^2}\,dx\,dy\right\}, \quad w \in \overline{B}. \tag{12}$$

On account of Vekua [2], Chapter I, §8, Satz 1.33, the mapping $\psi \mapsto \Pi_B[\psi]$ defines a bounded linear operator on $C^{0,\mu}(\overline{B}, \mathbb{C})$; consequently, there is a constant $M(\mu) > 1$ such that

$$\|\Pi_B[\psi]\|_\mu \le M(\mu)\|\psi\|_\mu \quad \text{for all } \psi \in C^{0,\mu}(\overline{B}, \mathbb{C}) \tag{13}$$

where $\|\cdot\|_\mu$ is the Hölder norm on $E := C^{0,\mu}(\overline{B}, \mathbb{C})$.

Suppose now that q satisfies the smallness condition

$$M(\mu)\|q\|_\mu < \tfrac{1}{4}. \tag{14}$$

Then the affine mapping $L_q : E \to E$, defined by

$$L_q[\psi] := q + q\Pi_B[\psi] \tag{15}$$

yields a contraction on the Banach space $(E, \|\cdot\|_\mu)$, and so L_q has a uniquely determined fixed point ψ, which satisfies

$$\|\psi\|_\mu \le \frac{\|q\|_\mu}{1 - M(\mu)\|q\|_\mu} \tag{16}$$

and (15) implies (11).

By Vekua's theorem from [2], quoted before, T_B is a bounded linear mapping from E into $C^{1,\mu}(\overline{B}, \mathbb{C})$, and one has

$$(T_B[\psi])_w = \Pi_B[\psi], \quad (T_B[\psi])_{\overline{w}} = \psi \tag{17}$$

(cf. Vekua [2], p. 52, (8.18), or Sauvigny [16], vol. 2, p. 359, (21)). Then the equation $L_q[\psi] = \psi$ is equivalent to

$$(0 + \psi) - q(1 + \Pi_B[\psi]) = 0,$$

and therefore $f(w) = w + T_B[\psi](w), w \in \overline{B}$, is of class $C^{1,\mu}(\overline{B}, \mathbb{C})$ and satisfies (7).

The Jacobian J_f of f is estimated from below by

$$
\begin{aligned}
(18) \qquad J_f &= |f_w|^2 - |f_{\overline{w}}|^2 = |1 + \Pi_B[\psi]|^2 - |\psi|^2 \\
&\geq |1 - M(\mu)\|\psi\|_\mu|^2 - \|\psi\|_\mu^2 > \left|1 - \frac{1}{3}\right|^2 - \frac{1}{3^2} = \frac{1}{3} \quad \text{on } \overline{B},
\end{aligned}
$$

and therefore f is a local diffeomorphism.

Now we want to show that f furnishes a 1–1 mapping from $\overline{B}$ onto its image. From (13), (14), (16), (17) we infer: There is a constant $c_0(\mu)$ with

$$0 < c_0(\mu) < \frac{1}{2M(\mu)}$$

such that for

$$\|q\|_\mu \leq c_0(\mu)$$

the uniquely determined fixed point of L_q satisfies

$$|T_B[\psi](w) - T_B[\psi](w')| \leq \tfrac{1}{2}|w - w'| \quad \text{for } w, w' \in \overline{B}.$$

From

$$|f(w) - f(w')| \geq |w - w'| - |T_B[\psi](w) - T_B[\psi](w')|$$

we then infer

$$|f(w) - f(w')| \geq \tfrac{1}{2}|w - w'| \quad \text{for all } w, w' \in \overline{B}.$$

Thus the mapping $f = \mathrm{id}_{\overline{B}} + T_B[\psi]$ yields a diffeomorphism of class $C^{1,\mu}(\overline{B}, \mathbb{C})$ from $\overline{B}$ onto $\overline{\Omega}$ with $\Omega := f(B)$. Since there is a constant $c_1(\mu)$ such that

$$\|T_B[\psi]\|_{C^{1,\mu}(\overline{B},\mathbb{C})} \leq \tfrac{1}{2}c_1(\mu)\|\psi\|_\mu,$$

we have by (14) and (15)

$$\|\psi\|_\mu \leq 2\|q\|_\mu \quad \text{if } \|q\|_\mu \leq c_0(\mu).$$

Thus,

$$\|f - \mathrm{id}_{\overline{B}}\|_{C^{1,\mu}(\overline{B},\mathbb{C})} \leq c_1(\mu)\|q\|_\mu \quad \text{if } \|q\|_\mu \leq c_0(\mu).$$

Summarizing the preceding results we obtain

Proposition 2. *There are constants $c_0(\mu) > 0$ and $c_1(\mu) > 0$ such that for any $q \in C^{0,\mu}(\overline{B}, \mathbb{C})$ with $\|q\|_\mu \leq c_0(\mu)$ the Tricomi equation (11) has exactly one solution $\psi \in C^{0,\mu}(\overline{B}, \mathbb{C})$ satisfying $\|\psi\|_\mu \leq 2\|q\|_\mu$, and the associate mapping $f := id_{\overline{B}} + T_B[\psi]$ is of class $C^{1,\mu}(\overline{B}, \mathbb{C})$ and satisfies the Beltrami equation (7). Furthermore, f is a diffeomorphism from $\overline{B}$ onto $\overline{\Omega}$ where Ω is* *a simply connected domain whose boundary is a regular closed Jordan curve* $\gamma \in C^{1,\mu}$, *and we have*

$$\|f - id_{\overline{B}}\|_{C^{1,\mu}(\overline{B},\mathbb{C})} \le c_1(\mu)\|q\|_\mu.$$

The inverse $g := f^{-1}$ *transforms* ds^2 *into the isothermal form*

$$g^* ds^2 = \lambda(x,y)\{dx^2 + dy^2\}, \quad z = x + iy \hat{=} (x,y) \in \overline{\Omega},$$

and we have $g \in C^{1,\mu}(\overline{\Omega},\mathbb{C})$ *and* $\lambda \in C^{0,\mu}(\overline{\Omega})$, $\lambda(x,y) > 0$.

Given $Y \in \mathcal{C}(\Gamma)$, we have $a = |Y_u|^2, b = \langle Y_u, Y_v\rangle$, $c = |Y_v|^2, \mathcal{D} = ac - b^2$, and so the potential q in (8) depends on Y. Consequently, also $\psi, f, g, \Omega, \gamma$ and λ depend on Y. We express this dependence by writing $f(Y) = f(Y,\cdot)$, $g(Y) = g(Y,\cdot), \Omega(Y), \gamma(Y) = \partial\Omega(Y), \lambda(Y) = \lambda(Y,\cdot)$ for f, g, etc.

A standard estimation procedure shows that $g(Y)$ is close to $\mathrm{id}_{\overline{\Omega}}$ with respect to the $C^{1,\mu}$-norm if $f(Y)$ is close to $\mathrm{id}_{\overline{B}}$ with respect to the $C^{1,\mu}$-norm. Combining this observation with the Propositions 1 and 2, we obtain

Proposition 3. *There is an* $\epsilon_2^* \in (0,1)$ *with the following property*: *For any* ϵ_2 *with* $0 < \epsilon_2 \le \epsilon_2^*$ *there exists a number* $\eta_2(X,\epsilon_2) > 0$ *such that*

$$\begin{aligned}
&\|f(Y) - id_{\overline{B}}\|_{C^{1,\mu}(\overline{B},\mathbb{C})} < \epsilon_2,\\
&\|g(Y) - id_{\overline{\Omega}(Y)}\|_{C^{1,\mu}(\overline{\Omega}(Y),\mathbb{C})} < \epsilon_2,\\
&|\lambda(x,y) - 1| < \epsilon_2 \quad \text{for all } z = x + iy \in \overline{\Omega}(Y),
\end{aligned}$$

provided that

$$\|Y - X\|_{C^{1,\mu}} < \eta_2(X,\epsilon_2).$$

Now we state a reparametrization result for perturbations Y:

Proposition 4. *There is a number* $\epsilon_3^* \in (0,1)$ *with the following property*: *For any* ϵ_3 *with* $0 < \epsilon_3 \le \epsilon_3^*$ *and any* $\nu \in (0,\mu)$ *there exists a number* $\eta_3(X,\epsilon_3,\nu) > 0$ *such that the following holds*:

For any Y *with* $\|Y - X\|_{C^{1,\mu}} < \eta_3(X,\epsilon_3,\nu)$ *there is a diffeomorphism* $\varphi(Y) = \varphi(Y,\cdot) \in C^{1,\mu}(\overline{B},\overline{\Omega}(Y))$ *from* $\overline{B}$ *onto* $\overline{\Omega}(Y)$ *which maps* B *conformally onto* $\Omega(Y)$ *and satisfies*

$$\|\varphi(Y) - \mathrm{id}_{\overline{B}}\|_{C^{1,\nu}(\overline{B},\mathbb{C})} < \epsilon_3.$$

In complex notation this means: *The mapping* $w \mapsto z = \varphi(Y,w)$, $w \in \overline{B}$, *is holomorphic in* B, *univalent on* $\overline{B}$, *and satisfies* $\varphi(Y,\overline{B}) = \overline{\Omega}(Y)$ *and* $\varphi_w(Y,w) \ne 0$ *on* B.

Proof. Fix three different points $w_1, w_2, w_3 \in \partial B$ and set $p_j(Y) := f(Y,w_j) \in \gamma(Y)$, $j = 1,2,3$. Then $f(X,w_j) = w_j$ and $p_j(Y) \to w_j$ if $Y \to X$ in $C^{1,\mu}$. By the Riemann mapping theorem there is a conformal mapping $\varphi(Y)$ from

B onto $\Omega(Y)$ which extends to a homeomorphism from $\overline{B}$ onto $\overline{\Omega}(Y)$ and satisfies $\varphi(Y, w_j) = p_j(Y)$ for $j = 1, 2, 3$. On account of the Proposition 2 we may assume that each $\gamma(Y)$ has an arc length representation $\xi(Y, s)$ with $\|\xi(Y, \cdot)\|_{C^{1,\mu}} \leq H_0(\mu)$ where the constant $H_0(\mu)$ is independent of Y provided that $\|Y - X\|_{C^{1,\mu}} < \eta_3 \ll 1$.

Choose a sequence $\{Y_k\}$ with $\|Y_k - X\|_{C^{1,\mu}} \to 0$ as $k \to \infty$. Then the Jordan curves $\gamma(Y_k)$ tend to $\gamma(X) = \partial B$ in the sense of Fréchet, the points $p_j(Y_k) \in \gamma(Y_k)$ tend to $w_j \in \partial B$ as $k \to \infty$, $j = 1, 2, 3$, and the Dirichlet integrals of $\varphi(Y_k)$ are uniformly bounded. Applying Theorem 3 of Section 4.3, it follows that $\varphi(Y_k)$ tends uniformly on $\overline{B}$ to the conformal mapping of B onto itself keeping the points w_1, w_2, w_3 fixed. Hence,

$$\varphi(Y_k) \rightrightarrows \mathrm{id}_{\overline{B}} = \varphi(X) \quad \text{on } \overline{B} \text{ as } k \to \infty. \tag{19}$$

On the other hand, we infer from a slight variation of the Kellogg–Warschawski theorem (cf. Kellogg [1], Warschawski [1–5], as well as Sauvigny [16], Chapter IV, §8) that

$$\|\varphi(Y_k)\|_{C^{1,\mu}(\overline{B},\mathbb{C})} \leq H(\mu) \quad \text{for all } k \in \mathbb{N}$$

with some constant $H(\mu)$. Hence, from any subsequence of $\{Y_k\}$ we can extract another subsequence $\{Y_{k_l}\}$ with

$$\varphi(Y_{k_l}) \to \varphi_0 \quad \text{in } C^{1,\nu}(\overline{B}, \mathbb{C}) \text{ as } l \to \infty$$

for some $\varphi_0 \in C^{1,\mu}(\overline{B}, \mathbb{C})$, and by (19) it follows that $\varphi_0 = \mathrm{id}_{\overline{B}}$. Then a standard reasoning yields

$$\varphi(Y_k) \to \mathrm{id}_{\overline{B}} \quad \text{in } C^{1,\nu}(\overline{B}, \mathbb{C}) \text{ as } k \to \infty.$$

Finally, a well-known argument yields the assertion of the Proposition. □

Now we combine the results of the Propositions 3 and 4. Introducing the mapping $\tau(Y) : \overline{B} \to \overline{B}$ by

$$\tau(Y) := g(Y) \circ \varphi(Y), \tag{20}$$

we obtain

Proposition 5. *There is a number $\epsilon_4^* \in (0, 1)$ with the following property: For any ϵ_4 with $0 < \epsilon_4 < \epsilon_4^*$ and any $\nu \in (0, \mu)$ there exists a further number $\eta_4(X, \epsilon_4, \nu) > 0$ such that the following holds:*

For any Y with $\|Y - X\|_{C^{1,\mu}} < \eta_4(X, \epsilon_4, \nu)$ we can find a diffeomorphism $\tau(Y) \in C^{1,\nu}(\overline{B}, \mathbb{C})$ from $\overline{B}$ onto itself such that

$$\|\tau(Y) - \mathrm{id}_{\overline{B}}\|_{C^{1,\nu}(\overline{B},\mathbb{C})} < \epsilon_4 \tag{21}$$

and $Z := Y \circ \tau(Y)$ is conformal in the sense

$$|Z_u|^2 = |Z_v|^2 > 0, \quad \langle Z_u, Z_v \rangle = 0 \quad \textit{on } \overline{B}. \tag{22}$$

Now we turn to the

Proof of Theorem 1. Given $Y \in \mathcal{C}(\Gamma)$ with $\|Y - X\|_{C^{1,\mu}} < \eta(X, \epsilon_4, \nu)$, we form $Z := Y \circ \tau(Y) \in \mathcal{C}(\Gamma)$ as above and consider the number $\epsilon > 0$ as in the assertion of the theorem. Then a straight-forward estimation shows the following: Choosing $\epsilon_4 > 0$ sufficiently small, we can find a number $\delta(\epsilon)$ with $0 < \delta(\epsilon) < \epsilon/2$ such that

$$\|Z - Y\|_{C^1} < \frac{\epsilon}{2} \quad \text{for } \|Y - X\|_{C^{1,\mu}} < \delta(\epsilon).$$

Then

$$\|Z - X\|_{C^1} \le \|Z - Y\|_{C^1} + \|Y - X\|_{C^1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } \|Y - X\|_{C^{1,\mu}} < \delta(\epsilon).$$

By assumption we then have

$$D(X) \le D(Z).$$

Moreover, (2) and (22) yield

$$A(X) = D(X) \quad \text{and} \quad A(Z) = D(Z),$$

and the parameter invariance of A yields

$$A(Z) = A(Y).$$

Finally, we also have $A(Y) \le D(Y)$, and so we arrive at

$$A(X) = D(X) \le D(Z) = A(Z) = A(Y) \quad \text{for } \|Y - X\|_{C^{1,\mu}} < \delta(\epsilon).$$

This completes the proof of the theorem. $\square$

The proof of Theorem 2 goes along the same lines.

Appendix 2

Minimal Surfaces in Heisenberg Groups

Recently problems concerning minimal surfaces in so-called Carnot–Carathéodory structures, particularly in Heisenberg groups, have found much attention. This interesting work is beyond the scope of our treatise, but we want at least to point out that problems have been treated which are in many ways similar to questions and phenomena studied for classical minimal surfaces in Euclidean and Riemannian spaces. Besides the construction of special examples, several authors have studied existence, uniqueness, and regularity questions, isoperimetric inequalities, Bernstein problems and calibrations. The literature on this work is already quite extensive, and therefore we only quote a few recent publications whose bibliographies will give further references: Garofalo and Nhieu [1], Garofalo and Pauls [1], Danielli, Garofalo, and Nhieu [1–3], Ambrosio, Serra Cassano, and Vittone [1], Adesi, Serra Cassano, and Vittone [1], Cheng and Hwang [1], Cheng, Hwang, Malchiodi, and Yang [1], Cheng, Hwang, and Yang [1,2], Pauls [1,2], Ritoré [1].

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The following references are not complete with respect to the early literature but cover only some of the essential papers. A very detailed and essentially complete bibliography of the literature on two-dimensional minimal surfaces until 1970 is given in Nitsche's treatise [28] (cf. also Nitsche [37]). Nitsche's bibliography is particularly helpful for the historically interested reader as each of its more than 1200 items is discussed or at least briefly mentioned in the right context, and the page numbers attached to each bibliographic reference make it very easy to locate the corresponding discussion. We have tried to collect as much as possible of the more recent literature and to include some cross-references to adjacent areas; completeness in the latter direction has neither been aspired not attained.

We particularly refer the reader to the following Lecture notes:

MSG: Minimal submanifolds and geodesics. Proceedings of the Japan–United States Seminar on Minimal Submanifolds, including Geodesics, Tokyo, 1977. Kagai Publications, Tokyo, 1978

SDG: Seminar on differential geometry, edited by S.T. Yau, Ann. Math. Studies **102**, Princeton, 1982

SMS: Seminar on minimal submanifolds, edited by Enrico Bombieri. Ann. Math. Studies **103**, Princeton, 1983

TVMA: Théorie des variétés minimales et applications. Séminaire Palaiseau, Oct. 1983–June 1984. Astérisque **154–155** (1987)

GACG: Geometric analysis and computer graphics. Proceedings of the Conference on Differential Geometry, Calculus of Variations and Computer Graphics, edited by P. Concus, R. Finn, D.A. Hoffman. Math. Sci. Res. Inst. **17**. Springer, New York, 1991

GTMS: Global theory of minimal surfaces, edited by D. Hoffman. Proceedings of the Clay Mathematics Institute 2001 Summer School, MSRI, Berkeley, June 25–27, 2001. Clay Math. Proceedings **2**, Am. Math. Soc., Providence, 2005

We also mention the following report by H. Rosenberg which appeared in May 1992:

Some recent developments in the theory of properly embedded minimal surfaces in $\mathbb{R}^3$. Séminaire Bourbaki **34**, Exp. No. 759 (1992), 73 pp.

Furthermore, we refer to:

EMS: Encyclopaedia of Math. Sciences **90**, Geometry V, Minimal surfaces (ed. R. Osserman), Springer, 1997. This volume contains the following reports:

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I. D. Hoffman, H. Karcher, Complete embedded minimal surfaces of finite total curvature.
II. H. Fujimoto, Nevanlinna theory and minimal surfaces.
III. S. Hildebrandt, Boundary value problems for minimal surfaces.
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