Acyclically 3-Colorable Planar Graphs

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Abstract. In this paper we study the planar graphs that admit an acyclic 3-coloring. We show that testing acyclic 3-colorability is $\mathcal{NP}$-hard for planar graphs of maximum degree 4 and we show that there exist infinite classes of cubic planar graphs that are not acyclically 3-colorable. Further, we show that every planar graph has a subdivision with one vertex per edge that is acyclically 3-colorable. Finally, we characterize the series-parallel graphs such that every 3-coloring is acyclic and we provide a linear-time recognition algorithm for such graphs.

1 Introduction

A *coloring* of a graph is an assignment of *colors* to vertices such that no two adjacent vertices have the same color. A *k-coloring* is a coloring using k colors. Planar graph colorings have been widely studied from both a combinatorial and an algorithmic point of view. The existence of a 4-coloring for every planar graph, proved by Appel and Haken [4,5], is one of the most famous results in Graph Theory. A quadratic-time algorithm is known to compute a 4-coloring of any planar graph [15].

An *acyclic coloring* is a coloring with *no bichromatic cycle*. An *acyclic k-coloring* is an acyclic coloring using k colors. Acyclic colorings have been deeply investigated in the literature. From an algorithmic point of view, Kostochka proved in [12] that deciding whether a graph admits an acyclic 3-coloring is $\mathcal{NP}$-hard. From a combinatorial point of view, the most interesting result is perhaps the one proved by Alon *et al.* in [2], namely that every graph with degree Δ can be acyclically colored with $O(\Delta^{4/3})$ colors, while there exist graphs requiring $\Omega(\Delta^{4/3}/\sqrt[3]{\log \Delta})$ colors in any acyclic coloring.

Acyclic colorings of planar graphs have been first considered in 1973 by Grünbaum, who proved in [10] that there exist planar graphs requiring 5 colors in any acyclic coloring. The same lower bound holds even for bipartite planar graphs [13]. Grünbaum conjectured that such a bound is tight and proved that 9 colors suffice for constructing such a coloring. The Grünbaum upper bound was improved to 8 [14], to 7 [1], to 6 [11], and finally to 5 by Borodin [6].

Since there exist planar graphs requiring 5 colors in any acyclic coloring, it is natural to study which planar graphs can be acyclically 3- or 4-colored. In this paper we study the acyclically 3-colorable planar graphs, from both an algorithmic and a combinatorial perspective. We show the following results.

– In Sect. 3 we prove that deciding whether a planar graph of maximum degree 4 has an acyclic 3-coloring is an $\mathcal{NP}$-complete problem. An $\mathcal{NP}$-hardness proof for deciding acyclic 3-colorability was known for bipartite planar graphs of degeneracy

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2 [12]. The $\mathcal{NP}$-hardness result is not surprising, since an analogous result is known for deciding (possibly non-acyclic) 3-colorability of planar graphs of degree 4 [9]. However, we show an interesting difference between the class of 3-colorable planar graphs and the class of acyclically 3-colorable planar graphs, by exhibiting an infinite number of cubic planar graphs not admitting any acyclic 3-coloring (while K_4 is the only cubic graph that can not be 3-colored [8]). We remark that it is known how to construct acyclic 4-colorings of every cubic (even non-planar) graph [16].
- In Sect. 4 we prove that every planar graph has a subdivision with one vertex per edge that is acyclically 3-colorable. Acyclic colorings of graph subdivisions have been already considered by Wood in [18], where the author observed that every graph has a subdivision with two vertices per edge that is acyclically 3-colorable.
- In Sect. 5 we consider the problem of determining the planar graphs such that every 3-coloring is acyclic. Such a problem has been introduced by Grünbaum [10], who showed that every 3-coloring of a maximal outerplanar graph is acyclic. We improve his result by characterizing the series-parallel graphs such that every 3-coloring is acyclic and by providing a linear-time recognition algorithm. As a side result, we show a simple algorithm for obtaining an acyclic 3-coloring of any series-parallel graph.

In Sect. 6 we conclude and we present some open problems. Some proofs are omitted because of space limitations and can be found in the full version of the paper [3].

2 Preliminaries

A graph G is *k-connected* if removing any k-1 vertices leaves G connected; 3-connected and 2-connected graphs are called *triconnected* and *biconnected* graphs, respectively. The *degree of a vertex* is the number of incident edges. The *degree of a graph* is the maximum degree of the vertices of the graph. In a *cubic* graph (resp. a *subcubic* graph) each vertex has degree exactly 3 (resp. at most 3). A *subdivision* of a graph G is obtained by replacing each edge of G with a path. A *k-subdivision* of G is such that any path replacing an edge of G has at most k internal vertices. The internal (extremal) vertices of the paths replacing the edges of G are called *subdivision vertices* (resp. *main vertices*).

A *planar graph* is a graph with no K_5-minor and no $K_{3,3}$-minor. A planar graph is *maximal* if all its faces are delimited by 3-cycles. An *outerplanar graph* is a graph admitting a planar drawing with all the vertices on the outer face. Combinatorially, an outerplanar graph is a graph with no K_4-minor and no $K_{2,3}$-minor. An outerplanar graph is *maximal* if all its internal faces are delimited by 3-cycles. A *series-parallel graph (SP-graph)* is a graph with no K_4-minor. SP-graphs are inductively defined as follows. An edge (u, v) is an SP-graph with *poles* u and v. Denote by u_i and v_i the poles of an SP-graph graph G_i. A *series composition* of SP-graphs $G_0, \dots, G_k$, with $k \geq 1$, is an SP-graph with poles u=u_0 and v=v_k, containing graphs G_i as subgraphs, and such that v_i=u_{i+1}, for each i=$0, 1, \dots, k-1$. A *parallel composition* of SP-graphs $G_0, \dots, G_k$, with $k \geq 1$, is an SP-graph with poles u=u_0=u_1=. . .=u_k and v=v_0=v_1=. . .=v_k and containing graphs G_i as subgraphs. The SPQ-tree $\mathcal{T}$ of an SP-graph G is the tree, rooted at any node, representing the series and parallel compositions of G.

3 Deciding the Acyclic 3-Colorability of Planar Graphs

In this section we study the problem of deciding whether a given planar graph admits an acyclic 3-coloring. First, we present a very simple proof that Planar Graph Acyclic 3-Colorability is $\mathcal{NP}$-hard. We remark that a proof of $\mathcal{NP}$-hardness for Planar Graph Acyclic 3-Colorability has been already presented by Kostochka in [12]. We later prove an analogous complexity result for planar graphs of maximum degree 4.

Theorem 1. *Planar Graph Acyclic 3-Colorability is $\mathcal{NP}$-complete.*

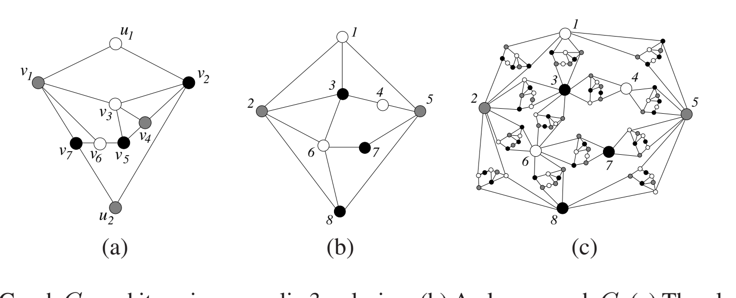
The membership in $\mathcal{NP}$ is trivial. To show the $\mathcal{NP}$-hardness, we sketch a simple reduction from Planar Graph 3-Colorability that uses the graph G_9 shown in Fig. 1.a as a gadget. It is easy to see that G_9 has only one acyclic 3-coloring (up to a switch of the color classes), which satisfies the following properties: (P1) u_1 and u_2 have different colors; (P2) every path connecting u_1 and u_2 contains vertices of all the three colors.

The reduction works as follows. Let G be an instance of Planar Graph 3-Colorability (see Fig. 1.b). Replace each edge (u, v) of G with a copy of G_9 by identifying vertices u and v with u_1 and u_2, respectively (see Fig. 1.c). Let G' be the resulting planar graph. We argue that G admits a 3-coloring if and only if G' admits an acyclic 3-coloring.

First, suppose that G admits a 3-coloring. For each edge (u, v) of G, color the corresponding graph G_9 in G' by assigning the color of u to u_1, the color of v to u_2, and by then completing the unique acyclic 3-coloring of G_9. The resulting coloring of G' is acyclic. Namely, assume, for a contradiction, that G' contains a bichromatic cycle $\mathcal{C}$. Such a cycle is not entirely contained inside a graph G_9 replacing an edge of G in G' (in fact, the 3-coloring of each graph G_9 is acyclic). Hence, $\mathcal{C}$ contains vertices of more than one graph G_9. This implies that $\mathcal{C}$ contains as a subgraph a simple path p connecting vertices u_1 and u_2 of a graph G_9. However, by property P2 of the G_9's coloring, p contains vertices of all the three colors, a contradiction.

Second, if G' admits an acyclic 3-coloring, a coloring of G is obtained from the acyclic 3-coloring of G' by assigning to each vertex of G the color of the corresponding vertex of G'. By property P1, each edge of G connects vertices of distinct colors.

Next, we show that testing whether a planar graph has an acyclic 3-coloring remains an $\mathcal{NP}$-hard problem even when restricted to planar graphs of degree 4.



(a) (b) (c)

Fig. 1. (a) Graph G_9 and its unique acyclic 3-coloring. (b) A planar graph G. (c) The planar graph G' obtained by replacing each edge of G with a copy of G_9.

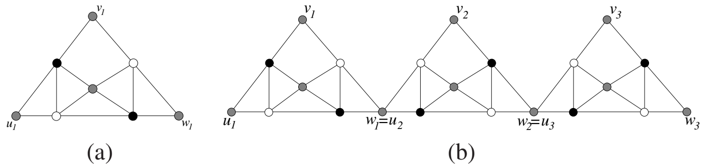


Fig. 2. (a) Graph H_1. (b) Graph H_3.

Theorem 2. *Degree-4 Planar Graph Acyclic 3-Colorability is $\mathcal{NP}$-complete.*

The membership in $\mathcal{NP}$ is trivial. To show the $\mathcal{NP}$-hardness, we sketch a simple reduction from Planar Graph Acyclic 3-Colorability. Consider the family of graphs H_i defined as follows. H_1 is shown in Fig. 2.a. H_i is obtained from a copy of H_{i-1} and a copy of H_1 by renaming vertices u_1, v_1, and w_1 of H_1 with labels u_i, v_i, and w_i, respectively, and by identifying vertex w_{i-1} of H_{i-1} and vertex u_i of H_1. H_3 is shown in Fig. 2.b. Vertices u_j, v_j, and w_j of H_i, for $1 \leq j \leq i$, are the *outlets* of H_i. The family of graphs H_i has been defined in [9] to perform a reduction from *Planar Graph Colorability* to *Degree-4 Planar Graph Colorability*. Here we use the same graph class to reduce Planar Graph Acyclic 3-Colorability to Degree-4 Planar Graph Acyclic 3-Colorability. It is easy to see that H_i satisfies the following properties: (P0) H_i admits an acyclic 3-coloring; (P1) in any acyclic 3-coloring of H_i, the outlets have the same color c_0; (P2) in any acyclic 3-coloring of H_i, for any two outlets x_j and y_k of H_i, there exist two bichromatic paths with colors c_0 and c_1, and with colors c_0 and c_2, respectively, where $x, y \in \{u, v, w\}$ and $j, k \in \{1, 2, \ldots, i\}$.

We reduce Planar Graph Acyclic 3-Colorability to Degree-4 Planar Graph Acyclic 3-Colorability. Let G be any instance of Planar Graph Acyclic 3-Colorability (Fig. 3.a). For each vertex z of G with d neighbors $z_1, z_2, \ldots, z_d$, delete z and its incident edges from G, introduce a copy $H(z)$ of H_d, and add an edge between outlet v_j of $H(z)$ and z_j, for each j=1, 2, ..., d (Fig. 3.b). We argue that the resulting planar graph G' of degree 4 admits an acyclic 3-coloring if and only if G admits an acyclic 3-coloring.

Suppose that G admits an acyclic 3-coloring. Color the outlets z_j corresponding to each vertex z of G with the color of z. By properties P0 and P1, the coloring of each $H(z)$ can be completed to an acyclic 3-coloring. Any cycle $\mathcal{C}'$ of G' either is entirely

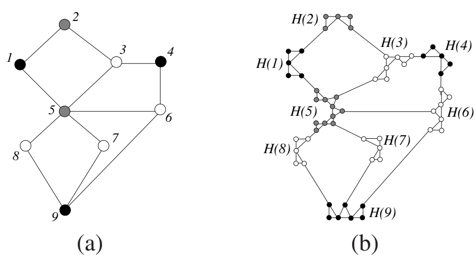


Fig. 3. (a) A planar graph G. (b) Graph G' obtained by replacing each degree-d vertex z of G with a copy $H(z)$ of H_d. For each graph $H(z)$, only its outlets are shown.

contained in a graph $H(z)$ (hence $\mathcal{C}'$ is not bichromatic), or contains vertices of several graphs $H(z)$. In the latter case suppose, for a contradiction, that $\mathcal{C}'$ is bichromatic. Consider the (possibly non-simple) cycle $\mathcal{C}$ of G containing a vertex z if $\mathcal{C}'$ passes through vertices of $H(z)$ and containing an edge (z_1, z_2) if $\mathcal{C}'$ contains an edge between a vertex of $H(z_1)$ and a vertex of $H(z_2)$. Since the outlets of $H(z)$ have the same color of z, the colors of the vertices of $\mathcal{C}$ are a subset of the colors of the vertices of $\mathcal{C}'$; since $\mathcal{C}'$ is bichromatic, $\mathcal{C}$ is bichromatic, as well, contradicting the assumption that the coloring of G is acyclic.

Suppose that G' admits an acyclic 3-coloring. Color G by assigning to each vertex z the color of the outlets of $H(z)$ (by Property P1, all such outlets have the same color). Suppose that G contains a bichromatic cycle $\mathcal{C}$ with colors c_0 and c_1. A bichromatic cycle $\mathcal{C}'$ in G' is found by replacing each vertex z_1 of $\mathcal{C}$ with a path with colors c_0 and c_1 connecting the outlets of $H(z_1)$ adjacent to the outlets of $H(z_2)$ and $H(z_3)$, where z_2 and z_3 are the neighbors of z_1 in $\mathcal{C}$. Such a path exists by Property P2. Then, $\mathcal{C}'$ is a bichromatic cycle in G', contradicting the assumption that the coloring of G' is acyclic.

Now we show infinite classes of cubic planar graphs not admitting any acyclic 3-coloring. Such a result is based on the following lemmata. Denote by $K_{2,3}$ the complete bipartite graph whose vertex sets $V^A_{2,3}$ and $V^B_{2,3}$ have two and three vertices, respectively. Denote by $K_{1,1,2}$ the complete tripartite graph whose vertex sets $V^A_{1,1,2}$, $V^B_{1,1,2}$, and $V^C_{1,1,2}$ have one, one, and two vertices, respectively.

Lemma 1. *Let G be a graph having a vertex z of degree 2 adjacent to two vertices u and v. Let G' be the graph obtained by substituting z with a copy of $K_{2,3}$, where a vertex $u^B_{2,3}$ of $V^B_{2,3}$ is connected to u and a vertex $v^B_{2,3} \neq u^B_{2,3}$ of $V^B_{2,3}$ is connected to v (see Fig. 4.a and Fig. 4.b). Then, G' has an acyclic 3-coloring if and only if G has an acyclic 3-coloring.*

Proof: Suppose that G has an acyclic 3-coloring. Color each vertex of G' not in $K_{2,3}$ as in G, the vertices in $V^B_{2,3}$ with the color c_z of z, and the vertices in $V^A_{2,3}$ with the two colors different from c_z. Every cycle $\mathcal{C}'$ in G' either does not pass through vertices of $K_{2,3}$ (hence it is also a cycle in G and it is not bichromatic), or it is a subgraph of $K_{2,3}$ (hence it is not bichromatic), or it passes through vertices of $K_{2,3}$ and contains a path $\mathcal{P}'$ from $u^B_{2,3}$ to $v^B_{2,3}$ whose vertices do not belong to $K_{2,3}$ (except for $u^B_{2,3}$ and $v^B_{2,3}$). However, $\mathcal{P}'$ is a cycle in G (where $u^B_{2,3}$ and $v^B_{2,3}$ are identified to be the same vertex z), hence it is not bichromatic.

Suppose that G' has an acyclic 3-coloring. In any acyclic coloring of $K_{2,3}$, the vertices in $V^B_{2,3}$ have the same color c_z. Color each vertex of G different from z as in G' and color z with c_z. Every cycle $\mathcal{C}$ in G either does not pass through z (hence it is also a cycle in G' and it is not bichromatic), or passes through z. In the latter case, if $\mathcal{C}$ is bichromatic then each of its vertices has either the color of z or the one of u. However, one vertex in $V^A_{2,3}$, say $x^A_{2,3}$, has the color of u, hence the cycle $\mathcal{C}'$ of G' obtained from $\mathcal{C}$ by replacing (u, z, v) with $(u, u^B_{2,3}, x^A_{2,3}, v^B_{2,3}, v)$ is bichromatic, a contradiction. □

Lemma 2. *Let G be a graph having a vertex z of degree 2 adjacent to two vertices u and v. Let G' be the graph obtained by substituting z with a copy of $K_{1,1,2}$, where a vertex $u^C_{1,1,2}$ of $V^C_{1,1,2}$ is connected to u and a vertex $v^C_{1,1,2} \neq u^C_{1,1,2}$ of $V^C_{1,1,2}$ is connected*

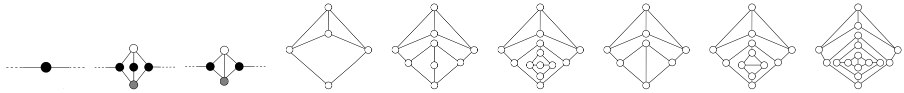


Fig. 4. (a) and (b) Replacement of a degree-2 vertex with a $K_{2,3}$. (a) and (c) Replacement of a degree-2 vertex with a $K_{1,1,2}$. (d) G_5. (e) G_9. (f) G_{13}. (g) G_5^+. (h) G_9^+. (i) G_{13}^+.

to v (see Fig. 4.a and Fig. 4.c). Then, G' has an acyclic 3-coloring if and only if G has an acyclic 3-coloring.

Graph G_5 (Fig. 4.d) has no acyclic 3-coloring and has a degree-2 vertex. For $i > 0$, replace the degree-2 vertex of G_{4i+1} with a copy of $K_{2,3}$, obtaining a graph G_{4i+5} that has a degree-2 vertex and, by Lemma 1, is not acyclically 3-colorable. Figs. 4.e–f show G_9 and G_{13}. Replacing the degree-2 vertex of G_{4i+1} with a copy of $K_{1,1,2}$ yields a graph G^+_{4i+1} that, by Lemma 2, is not acyclically 3-colorable. Figs. 4.g–i show G^+_5, G^+_9, G^+_{13}. Graphs G^+_{4i+1} are cubic, for every $i > 0$.

4 Acyclic 3-Colorings of Planar Graph Subdivisions

In this section we prove the following theorem.

Theorem 3. *Every planar graph has a 1-subdivision that admits an acyclic 3-coloring.*

Proof: It suffices to prove the statement for maximal planar graphs. In fact, suppose that the statement holds for maximal planar graphs. Let G be a planar graph. Augment G to a maximal planar graph G' by adding dummy edges. Then G' has a 1-subdivision G'_s that has an acyclic 3-coloring c. Remove the edges of G'_s corresponding to subdivided dummy edges of G', obtaining a planar graph G_s that is a subdivision of G. Since every cycle of G_s is also a cycle of G'_s, c is an acyclic 3-coloring of G_s.

Consider a planar drawing of any maximal planar graph G. Let G_s be the planar graph obtained by subdividing each edge of G with one subdivision vertex. Partition the vertices of G into disjoint sets $V^0, V^1, \ldots, V^k$ as follows. Let G^0=G; while there are vertices in G^i, denote by V^i the main vertices incident to the outer face of G^i; remove the vertices in V^i and their incident edges from G^i obtaining a graph G^{i+1}. Each edge of G is either incident to two vertices in the same set V^i or to two vertices in sets V^i and V^{i+1}, for some $i \in \{0, 1, \ldots, k-1\}$.

Color the main vertices in V^i with color $c_{j(i)}$, where $j(i) \in \{0, 1, 2\}$ and $j(i) \equiv i \bmod 3$. Color each subdivision vertex adjacent to a vertex in V^i and to a vertex in V^{i+1} with color $c_{j(i+2)}$. See Fig. 5.a. It remains to color each subdivision vertex adjacent to two vertices belonging to the same V^i. Consider the outerplanar subgraph O^i of G induced by the vertices in V^i. Augment O^i to maximal by adding dummy edges. See Fig. 5.b. Let O^i_s be the graph obtained by subdividing each edge of O^i with one subdivision vertex. Each subdivision vertex of G_s adjacent to two vertices belonging to the same V^i, for some $i \in \{1, 2, \ldots, k\}$, is also a subdivision vertex of O^i_s. Hence, a coloring of the subdivision vertices of O^i_s determines a coloring of the subdivision

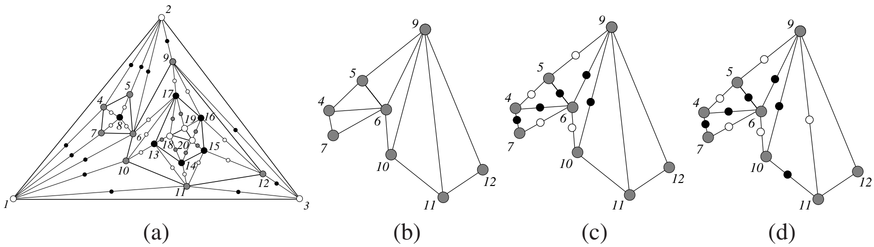


Fig. 5. (a) Coloring the main vertices and the subdivision vertices of G_s adjacent to a vertex in V^i and to a vertex in V^{i+1}. Thick edges connect vertices of G in the same V^i. (b) Subgraph O^2 of G augmented to maximal. (c)–(d) Coloring O_s^2 at steps x and $x+1$ of the algorithm. Not yet colored subdivision vertices of O_s^2 are not shown.

vertices of G_s adjacent to two vertices in the same V^i. We show how to color the subdivision vertices of O_s^i. The algorithm already chose to color all the main vertices of O_s^i with color $c_{j(i)}$. Since O^i is maximal, every internal face of O_s^i has three subdivision vertices. The coloring algorithm consists of several steps. At the first step, consider any internal face f^* of O_s^i. Color two of its subdivision vertices with $c_{j(i+1)}$ and the third one with $c_{j(i+2)}$. At the x-th step, with $x \geq 2$, suppose that the subgraph $O_s^{i,x}$ of O_s^i induced by the colored subdivision vertices and by their neighbors is biconnected. See Fig. 5.c. Consider any internal face of O_s^i of which one subdivision vertex has already been colored. Color the other two subdivision vertices incident to the face, one with $c_{j(i+1)}$ and the other one with $c_{j(i+2)}$. See Fig. 5.d.

We show that the resulting coloring of G_s is acyclic. Consider any simple cycle $\mathcal{C}$. If $\mathcal{C}$ contains main vertices in V^i and V^{i+1}, then $\mathcal{C}$ contains two edges (v_p, v_s) and (v_s, v_q), where v_p and v_q are main vertices in V^i and V^{i+1}, respectively, and v_s is a subdivision vertex. However, v_p, v_q, and v_s have color $c_{j(i)}$, $c_{j(i+1)}$, and $c_{j(i+2)}$, respectively, hence $\mathcal{C}$ is not bichromatic. Otherwise, $\mathcal{C}$ only contains main vertices in the same V^i. Then, $\mathcal{C}$ is also a cycle of O_s^i. We show by induction that the described coloring of O_s^i is acyclic. The coloring of f^* is acyclic. Suppose that, after a certain step of the coloring algorithm for the vertices of O_s^i, the subgraph $O_s^{i,x}$ of O_s^i induced by the colored subdivision vertices and by their neighbors is acyclic. When a new face is considered and two subdivision vertices v_1 and v_2 are colored with colors $c_{j(i+1)}$ and $c_{j(i+2)}$, respectively, every cycle either entirely belongs to $O_s^{i,x}$, hence by induction it is not bichromatic, or passes through v_1, v_2, and their common neighbor, hence it is not bichromatic. □

5 Acyclic 3-Colorings of Series-Parallel Graphs

In this section we consider the problem of determining which are the SP-graphs such that every 3-coloring is acyclic. First, we show a simple algorithm to construct an acyclic 3-coloring of any SP-graph. Let $c(x)$ denote the color assigned to vertex x.

Theorem 4. *Every SP-graph G with poles u and v admits an acyclic 3-coloring such that $c(u){\neq}c(v)$ and every path connecting u and v, except for edge (u,v), contains a vertex w with $c(w){\neq}c(u), c(v)$.*

Proof: We prove the statement by induction on the number n of vertices. Case n=2 is trivial. If $n > 2$, distinguish two cases: (Case 1) G is a series composition of SP-graphs $G_0, \cdots, G_k$, such that G_i has poles u_i and v_i, with u_0=u, v_i=u_{i+1}, and v_k=v; (Case 2) G is a parallel composition of SP-graphs $G_0, \cdots, G_k$ with poles u and v.

In Case 1, apply induction to construct an acyclic 3-coloring of G_i with colors c_0, c_1, and c_2 such that $c(u_i)$=$c_{j(i)}$ and $c(v_i)$=$c_{j(i+1)}$, for each i=$0, 1, \dots, k-1$, where $j(i) \in \{0, 1, 2\}$ and $j(i) \equiv i \bmod 3$. Apply induction to construct an acyclic 3-coloring of G_k with colors c_0, c_1, and c_2 such that $c(u_k)$=$c_{j(k)}$, and such that $c(v_k)$=c_1, if $c(u_k)$=c_0 or $c(u_k)$=c_2, and $c(v_k)$=c_2, if $c(u_k)$=c_1. By construction, $c(u_0$=$u)$=c_0, $c(u_1)$=c_1, $c(u_2)$=c_2. Every path connecting u and v passes through u_0, u_1, and u_2, hence it is not bichromatic. Further, any simple cycle in G is also a cycle in a component G_i. Hence, by induction, the coloring of G is acyclic.

In Case 2, apply induction to construct an acyclic 3-coloring of G_i, for i=$0, 1, \cdots, k$, with colors c_0, c_1, and c_2 such that $c(u)$=c_0, $c(v)$=c_1, and every path connecting u and v in G_i, except for edge (u, v), contains a vertex w with $c(w)$=c_2. By construction, $c(u)$=c_0 and $c(v)$=c_1. Further, every path connecting u and v is also a path in a component G_i which, by induction, contains a vertex with color c_2, unless it is edge (u, v). Let $\mathcal{C}$ be any simple cycle in G. If all the vertices of $\mathcal{C}$ belong to a graph G_i, then $\mathcal{C}$ is not bichromatic by induction. Otherwise, $\mathcal{C}$ contains vertices u and v, hence it consists of two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ connecting u and v and belonging to two distinct components G_i and G_j. At most one of $\mathcal{P}_1$ and $\mathcal{P}_2$, say $\mathcal{P}_1$, coincides with edge (u, v). By induction, $\mathcal{P}_2$ contains a vertex of color c_2. □

Second, we characterize the SP-graphs that have a 3-coloring in which the poles have distinct colors and the SP-graphs that have a 3-coloring in which the poles have the same color.

Corollary 1. *Every SP-graph with poles u and v admits a 3-coloring with $c(u){\neq}c(v)$.*

Lemma 3. *Every SP-graph G with poles u and v admits a 3-coloring with $c(u)$=$c(v)$ if and only if G does not contain edge (u, v).*

Proof: The necessity is trivial. We inductively prove the sufficiency. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \dots, G_k$ and that G does not contain edge (u, v). Then, no component G_i contains (u, v), hence it admits a 3-coloring in which $c(u)$=$c(v)$ by induction. Suppose that G is a series composition of graphs $G_0, G_1, \dots, G_k$. Color G_0 so that $c(u)$=c_0 and the other pole of G_0 has color c_1. Such a coloring exists by Corollary 1. For $1 \leq i \leq k-1$, assume that the color of the pole that G_i shares with G_{i-1} has been already determined to be either c_1 or c_2. Color the pole that G_i shares with G_{i+1} with color c_2 or c_1, respectively, and color G_i so that its poles have colors c_1 and c_2 (such a coloring exists by Corollary 1). Complete the coloring of G by setting $c(v)$=c_0 and by coloring G_k so that its poles have colors c_0 and either c_1 or c_2. Again, such a coloring exists by Corollary 1. □

Third, we characterize the SP-graphs that have a 3-coloring in which there exists a bichromatic path between the poles.

Lemma 4. *Let G be an SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, G admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v if and only if there exists a component that admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v.*

Proof: The necessity comes from the observation that every bichromatic path between u and v in G is internal to a component G_i. We prove the sufficiency. There exists a G_i admitting a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v. By Corollary 1, all other components can be colored with $c(u){\neq}c(v)$, thus completing a 3-coloring of G with the required properties. □

Lemma 5. *Let G be an SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, G admits a 3-coloring with $c(u){\neq}c(v)$ and with a bichromatic path between u and v if and only if the following two conditions are satisfied: (1) Each component admits a 3-coloring with a bichromatic path between its poles and (2) there exists a component G_i with poles u_i and v_i that admits a 3-coloring with $c(u_i)$=$c(v_i)$ and with a bichromatic path between u_i and v_i, and a 3-coloring with $c(u_i){\neq}c(v_i)$ and with a bichromatic path between u_i and v_i, or there exists an odd number of components that admit a 3-coloring in which the poles have different colors and are connected by a bichromatic path.*

Proof: We prove the necessity of (1). Suppose that there exists a G_i that admits no 3-coloring with a bichromatic path between its poles. Every path connecting u and v contains a path between G_i's poles, hence it is not bichromatic. We prove the necessity of (2). Suppose, for a contradiction, that (2) does not hold. Then, in every 3-coloring of G with a bichromatic path between u and v, there is an even number of components G_i such that $c(u_i){\neq}c(v_i)$, hence $c(u)$=$c(v)$. We prove the sufficiency. Suppose that each component G_i admits a 3-coloring with a bichromatic path between its poles. First, suppose that there exists a component G_i with poles u_i and v_i that admits a 3-coloring with $c(u_i)$=$c(v_i)$ and with a bichromatic path between u_i and v_i, and a 3-coloring with $c(u_i){\neq}c(v_i)$ and with a bichromatic path between u_i and v_i. Set $c(u_0)$=c_0. For $0 \leq j \leq i-1$, assume that $c(u_j)$ has already been determined to be either c_0 or c_1; color G_j so that there exists a bichromatic path between u_j and v_j and so that $c(v_j)$ is either c_0 or c_1. Analogously, set $c(v_k)$=c_1. For $k \geq j \geq i+1$, assume that $c(v_j)$ has been determined to be either c_0 or c_1; color G_j so that there exists a bichromatic path between u_j and v_j and so that $c(u_j)$ is either c_0 or c_1. Color G_i so that there exists a bichromatic path between u_i and v_i; this can be done both if $c(u_i)$=$c(v_i)$ and if $c(u_i){\neq}c(v_i)$. Second, suppose that there exists an odd number of components that admit a 3-coloring in which the poles have different colors and are connected by a bichromatic path. Each component has either a 3-coloring with a bichromatic path between its poles and the poles have the same color, or a 3-coloring with a bichromatic path between its poles and the poles have distinct colors. Color each component with such a coloring, so that its poles have colors in $\{c_0, c_1\}$. Since an odd number of components have poles with different colors, $c(u){\neq}c(v)$. □

Lemma 6. *Let G be an SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, G admits a 3-coloring with $c(u)$=$c(v)$ and with a bichromatic path between u and v if and only if G does not contain edge (u, v) and there exists a component admitting a 3-coloring with $c(u)$=$c(v)$ and with a bichromatic path between u and v.*

Lemma 7. *Let G be an SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, G admits a 3-coloring with $c(u)$=$c(v)$ and with a bichromatic path between u and v if and only if the following two conditions are satisfied: (1) Each component admits a 3-coloring with a bichromatic path between its poles and (2) there exists a component G_i with poles u_i and v_i admitting a 3-coloring with $c(u_i)$=$c(v_i)$ and with a bichromatic path between u_i and v_i, and a 3-coloring with $c(u_i) \neq c(v_i)$ and with a bichromatic path between u_i and v_i, or there exists an even number of components admitting a 3-coloring in which the poles have different colors and are connected by a bichromatic path.*

Fourth, we characterize the SP-graphs such that every 3-coloring in which the poles have distinct colors is acyclic and the SP-graphs such that every 3-coloring in which the poles have the same color is acyclic.

Lemma 8. *Let G be an SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, every 3-coloring of G with $c(u) \neq c(v)$ is acyclic if and only if the following two conditions are satisfied: (1) For each component G_i, every 3-coloring with $c(u) \neq c(v)$ is acyclic; (2) there exist no two components admitting a 3-coloring with $c(u) \neq c(v)$ and with a bichromatic path between u and v.*

Lemma 9. *Let G be an SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, every 3-coloring of G with $c(u) \neq c(v)$ is acyclic if and only if the following two conditions are satisfied: (1) For each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i) \neq c(v_i)$ is acyclic; (2) for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i)$=$c(v_i)$ is acyclic.*

Lemma 10. *Let G be an SP-graph with poles u and v. Suppose that G is a parallel composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, every 3-coloring of G with $c(u)$=$c(v)$ is acyclic if and only if one of the following two conditions is satisfied: (1) There exists a component G_i not admitting any 3-coloring with $c(u_i)$=$c(v_i)$; or (2) for each component G_i, every 3-coloring with $c(u)$=$c(v)$ is acyclic and no two components exist admitting a 3-coloring with $c(u)$=$c(v)$ and with a bichromatic path between u and v.*

Lemma 11. *Let G be an SP-graph with poles u and v. Suppose that G is a series composition of SP-graphs $G_0, G_1, \ldots, G_k$. Then, every 3-coloring of G with $c(u)$=$c(v)$ is acyclic if and only if the following three conditions are satisfied: (1) For each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i) \neq c(v_i)$ is acyclic; (2) if $k > 2$, for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i)$=$c(v_i)$ is acyclic; (3) if k=2, for each component G_i with poles u_i and v_i, every 3-coloring with $c(u_i)$=$c(v_i)$ is acyclic, or there exists a component not admitting any 3-coloring in which $c(u_i)$=$c(v_i)$.*

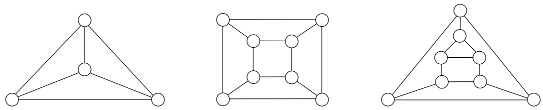


Fig. 6. Triconnected cubic planar graphs with no acyclic 3-coloring

Finally, we conclude by observing that an SP-graph with poles u and v is such that every 3-coloring is acyclic if and only if every 3-coloring in which $c(u){\neq}c(v)$ is acyclic and every 3-coloring in which $c(u){=}c(v)$ is acyclic. The above characterization gives rise to a linear-time recognition algorithm.

Theorem 5. *There exists a linear-time algorithm for deciding whether an SP-graph is such that every 3-coloring is acyclic.*

Proof: The SPQ-tree $\mathcal{T}$ of an SP-graph G can be computed in linear-time (see, e.g., [17]). Then, each node μ of $\mathcal{T}$ with poles u_μ and v_μ can be equipped with values indicating whether: (i) $G(\mu)$ admits a 3-coloring with $c(u_\mu){=}c(v_\mu)$; (ii) $G(\mu)$ admits a 3-coloring with $c(u_\mu){\neq}c(v_\mu)$ and with a bichromatic path between u_μ and v_μ, $G(\mu)$ admits a 3-coloring with $c(u_\mu){=}c(v_\mu)$ and with a bichromatic path between u_μ and v_μ, and $G(\mu)$ admits a 3-coloring with a bichromatic path between u_μ and v_μ; and (iii) every 3-coloring of $G(\mu)$ in which $c(u_\mu){\neq}c(v_\mu)$ is acyclic, every 3-coloring of $G(\mu)$ in which $c(u_\mu){=}c(v_\mu)$ is acyclic, and every 3-coloring of $G(\mu)$ is acyclic. Due to Lemmata 3–11, the computation of such values for μ only requires simple checks on analogous values for the children of μ in $\mathcal{T}$. $\square$

6 Conclusions

In this paper we have shown several results on the acyclic 3-colorability of planar graphs. We proved that recognizing acyclic 3-colorable planar graphs of degree 4 is $\mathcal{NP}$-hard. Further, we exhibited infinite classes of subcubic and cubic planar graphs with no acyclic 3-coloring, result contrasting with the fact that all cubic planar graphs have a 3-coloring, except for K_4 [8]. However, the following problem is still open.

What is the time complexity of testing whether a sub-cubic graph (resp. a cubic graph) admits an acyclic 3-coloring?

The problem is interesting even when restricted to *triconnected* cubic planar graphs. Moreover, we are aware of only three graphs that are cubic, triconnected, and not acyclic 3-colorable (see Fig. 6). The graphs depicted in Figs. 6.a and 6.b were already known to have no acyclic 3-coloring. On the other hand, the graph depicted in Fig. 6.c seems to have gone unnoticed in the literature.

Does an infinite number of triconnected, cubic, and not acyclic 3-colorable planar graphs exist? What is the time complexity of testing whether a triconnected cubic planar graph admits an acyclic 3-coloring?

We have shown that it is possible to test in linear time whether every 3-coloring of an SP-graph is acyclic. Testing and characterizing the same property for general planar graphs seems to be interesting and non-trivial.

Is it possible to test in polynomial time whether every 3-coloring of a given planar graph is acyclic?

Finally, we would like to remind a problem that has been already studied in the literature but that has not been tackled in this paper.

Which is the smallest k such that all planar graphs with girth at least k are acyclic 3-colorable?

The best known lower bound for k is 5 (the second graph of Fig. 6, proposed by Grünbaum, has girth 4 and is not acyclic 3-colorable [10]), while the best known upper bound for k is 7, as proved by Borodin, Kostochka, and Woodall [7].

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