Directed Rank-Width and Displit Decomposition*

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Abstract. *Rank-width* is a graph complexity measure that has many structural properties. It is known that the rank-width of an undirected graph is the maximum over all induced prime graphs with respect to *split decomposition* and an undirected graph has rank-width at most 1 if and only if it is a distance-hereditary graph. We are interested in an extension of these results to directed graphs. We give several characterizations of directed graphs of rank-width 1 and we prove that the rank-width of a directed graph is the maximum over all induced prime graphs with respect to *displit decomposition*, a new decomposition on directed graphs.

1 Introduction

Rank-width [18,19] is a graph complexity measure introduced by Oum and Seymour in their investigations on recognition algorithms for undirected graphs of *clique-width* [4] at most k, for fixed k. It is known that a class of graphs has bounded rank-width if and only if it has bounded clique-width [19]. However, rank-width has better algorithmic properties: undirected graphs of rank-width at most k can be recognized by a cubic-time algorithm [13] and are characterized by a finite list of undirected graphs to exclude as *vertex-minors* [18].

Another interesting fact is that rank-width is related to *split decomposition*. The split decomposition, introduced by Cunningham [5], is a generalisation of the well known *modular decomposition* [10,16]. It was defined on graphs (directed or not), but only the undirected case has been widely studied in literature. Split decomposition of undirected graphs can be computed in linear time [7], and can be used in several problems such as: circle graph recognition [9,21], parity graph recognition [3,7], and solving some optimization problems [5,3,11,20]. The rank-width of an undirected graph is the maximum over the rank-width of its induced prime graphs with respect to split decomposition. Moreover, undirected graphs of rank-width at most 1 are exactly *distance hereditary* graphs [18], which are graphs that are *completely decomposable* by the split decomposition.

Despite all these positive results of rank-width on clique-width, clique-width has an undeniable advantage on rank-width: it is defined for undirected as well as directed graphs and its definition can be extended to relational structures. In

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his investigations for an extension of rank-width to relational structures, Kanté defined in [15] a notion of rank-width for directed graphs, called GF(4)-*rank-width*, and that generalized the rank-width of undirected graphs. He, moreover, generalized two results on undirected graphs: directed graphs of GF(4)-rank-width k can be recognized by a cubic-time algorithm and are also characterized by a finite list of directed graphs to exclude as vertex-minors. It is thus natural to ask whether we can generalize all the results known for rank-width of undirected graphs.

In this paper, we are interested in a characterization of directed graphs of GF(4)-rank-width 1, similar to the one for undirected graphs. In the literature, there exist several characterizations of undirected graphs of rank-width 1 that we recall in the following.

Theorem 1 ([1,12,18]). *Let G be a connected undirected graph. Then the following conditions are equivalent:*

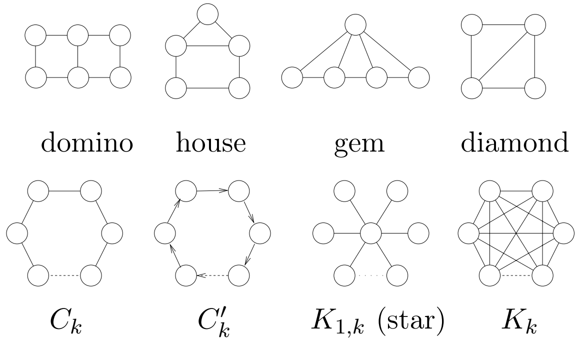
1. *G is completely decomposable by the split decomposition (*i.e.*, every node in the split decomposition tree is degenerated).*
2. *G can be obtained from a single vertex by creating twins or adding pendant vertices.*
3. *G has rank-width* 1*.*
4. *For every $W \subseteq V_G$ with $|W| \geq 4$, $G[W]$ has a non trivial split.*
5. *G is (house, hole, domino, gem)-free.*
6. *G is distance hereditary (*i.e.*, for every $x, y \in V_G$, every chordless path between x and y has the same length).*

The main result of this paper is the extension of Theorem 1 to directed graphs (Theorem 6). We will show in particular that directed graphs of GF(4)-rank-width 1 are obtained by orienting in a certain way distance hereditary graphs and are exactly directed graphs completely decomposable by the *displit decomposition*, a new decomposition that generalizes split decomposition. As a consequence we get that the GF(4)-rank-width of a directed graph is the maximum over the GF(4)-rank-width of its induced prime graphs with respect to displit decomposition.

The paper is organized as follows. We give some notations in Section 2 and recall the notion of GF(4)-rank-width in Section 3. In Section 4 we define the notion of displit decomposition and derive some basic properties. In Section 5 we prove our main result. We conclude by a comparison between the split decomposition of directed graphs introduced by Cunningham [5] and the displit decomposition.

2 Preliminaries

When the context is clear we will write u to denote the set $\{u\}$. We denote by 2^V the power-set of a set V and we let $\mathbb{N}$ be the set of natural integers. A function $f : 2^V \to \mathbb{N}$ is said *symmetric* if for any $X \subseteq V,\ f(X) = f(V \backslash X)$; it is said *sub-modular* if for any $X, Y \subseteq V, f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.



For sets R and C, an (R,C)-*matrix* is a matrix where the rows are indexed by elements in R and columns indexed by elements in C. For an (R,C)-matrix M, if $X \subseteq R$ and $Y \subseteq C$, we let $M[X,Y]$ be the sub-matrix of M where the rows and the columns are indexed by X and Y respectively. If M is an (X,Y)-matrix, M^t denotes the transposed (Y,X)-matrix. A Y-*vector* is an (X,Y)-matrix where $|X| = 1$. The matrix rank function is denoted by rk.

A *directed graph* (or *digraph*) G is a couple (V_G, E_G) where V_G is the set of *vertices* and E_G, the set of *edges*, is a set of ordered pairs (x,y) with $x, y \in V_G$ and $x \neq y$. We consider *undirected graphs* as special cases of directed graphs where $(x,y) \in E_G \Leftrightarrow (y,x) \in E_G$ (edges are denoted xy in this case). Unless otherwise specified, a graph is considered as directed. If G is a digraph and x a vertex of G, we denote by $N_G^+(x)$ the set $\{y \mid (x,y) \in E_G\}$, by $N_G^-(x)$ the set $\{y \mid (y,x) \in E_G\}$ and by $N_G(x)$ the set $N_G^+(x) \cup N_G^-(x)$. The *degree* of x is $|N_G(x)|$.

For a graph G, we denote by $G[X]$ the sub-graph of G induced by $X \subseteq V_G$ and we let $G - X$ be the sub-graph $G[V_G \backslash X]$. If G is a digraph, let $u(G)$ be the undirected graph obtained from G by forgetting the directions of edges, *i.e.*, $u(G) = (V_G, E_G \cup \{(y,x) \mid (x,y) \in E_G\})$. A digraph G is said *strongly connected* if for every pair $x, y \in V_G$, there is a sequence $x_0 = x, x_1, \ldots x_k = y$ such that $(x_i, x_{i+1}) \in E_G$ for every $i \in \{0, \ldots k-1\}$, and it is said *connected* if $u(G)$ is connected.

An undirected graph is *acyclic* if it does not contain simple cycles of length at least 3. A *tree* is an acyclic connected undirected graph. In order to avoid confusions, the vertices of trees will be called *nodes*. The nodes of degree at most 1 in trees are called *leaves* and denoted by L_T. A *sub-cubic* tree is a tree such that the degree of each node is at most 3.

A *layout* of a set V is a pair $(T, \mathcal{L})$ of an undirected tree T and a bijective function $\mathcal{L} : V \to \mathrm{L}_T$. For each edge (u,v) of T, we let X_{uv} be the set of leaves reachable from u by a path going through v. Each edge (u,v) of T induces a bipartition $\{X_{uv}, \mathrm{L}_T \backslash X_{uv}\}$ of L_T, and thus a bipartition $\{X^{uv}, V \backslash X^{uv}\} = \{\mathcal{L}^{-1}(X_{uv}), \mathcal{L}^{-1}(\mathrm{L}_T \backslash X_{uv})\}$ of V.

3 Rank-Width of Digraphs

In [15] Kanté defined a notion of rank-width for digraphs named GF(4)-*rank-width*. This notion is based on a function, called *cut-rank function*, that measures how some bipartitions of sets of vertices are connected. The cut-rank function is based on a representation of digraphs by matrices over the field GF(4). We recall that GF(4) has four elements $\{0, 1, \mathbb{a}, \mathbb{a}^2\}$ with the property that $1 + \mathbb{a} + \mathbb{a}^2 = 0$ and $\mathbb{a}^3 = 1$ and is of characteristic 2.

For a digraph G, we denote by M_G the (V_G, V_G)-matrix over GF(4) where:

$$M_G[x, y] = \begin{cases} 0 & \text{if } (x, y) \notin E_G \text{ and } (y, x) \notin E_G \\ \mathbb{a} & \text{if } (x, y) \in E_G \text{ and } (y, x) \notin E_G \\ \mathbb{a}^2 & \text{if } (y, x) \in E_G \text{ and } (x, y) \notin E_G \\ 1 & \text{if } (x, y) \in E_G \text{ and } (y, x) \in E_G. \end{cases}$$

For every subset X of V_G, we let $\mathrm{cutrk}_G^{(4)}(X)$, called *cut-rank function*, be $\mathrm{rk}\left(M_G[X, V_G \backslash X]\right)$.

Lemma 1 ([15]). *For every digraph G, the function $\mathrm{cutrk}_G^{(4)}$ is symmetric and sub-modular.*

Definition 1 (GF(4)-Rank-Width). *A* sub-cubic layout *of a digraph G is a layout $(T, \mathcal{L})$ of V_G where T is sub-cubic. Let $(T, \mathcal{L})$ be a sub-cubic layout of a digraph G. The* GF(4)-rank-width of an edge (u, v) of T *is* $\mathrm{cutrk}_G^{(4)}(X^{uv})$*. The* GF(4)-rank-width of a sub-cubic layout $(T, \mathcal{L})$ *is the maximum* GF(4)*-rank-width over all edges of T. The* GF(4)-rank-width of G, *denoted by* $\mathrm{rwd}^{(4)}(G)$, *is the minimum* GF(4)*-rank-width over all sub-cubic layouts of G.*

Observation 1. *Since* GF(4) *is an extension of* GF(2), *for every undirected graph G, we have* $\mathrm{rwd}^{(4)}(G) = \mathrm{rwd}(G)$, *where* $\mathrm{rwd}(G)$ *denotes the rank-width of G.*

4 Displit Decomposition

4.1 Bi-Partitive Families

Two bipartitions $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ of a set V *overlap* if $X_i \cap Y_j \neq \emptyset$ for every $i, j \in \{1, 2\}$.

Definition 2 (Bi-Partitive Family). *Let V be a finite set and let $\mathcal{F}$ be a family of bipartitions of V. Then $\mathcal{F}$ is* bi-partitive *if:*

- $\{\emptyset, V\} \notin \mathcal{F}$,
- *for all $v \in V$, $\{\{v\}, V \backslash \{v\}\} \in \mathcal{F}$ and*
- *for all $\{X_1, X_2\} \in \mathcal{F}$ and $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap, then $\{X_i \cap Y_j, V \backslash (X_i \cap Y_j)\} \in \mathcal{F}$, for every $i, j \in \{1, 2\}$.*

A member $\{X_1, X_2\}$ of a bi-partitive family $\mathcal{F}$ is trivial *if $|X_1| \leq 1$ or $|X_2| \leq 1$, and is* strong *if there is no $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap.*

Bi-partitive families have been studied in [6]. They are very close to partitive families [2,16] introduced in order to generalize properties of modular decomposition. An example of a bi-partitive family is the family of splits[1] in a strongly connected digraph [5]. The following proposition gives another example of a bi-partitive family.

Proposition 1 (Folklore). *Let $f : 2^V \to \mathbb{N}$ be a symmetric and sub-modular function and let $m = \min_{\emptyset \subsetneq X \subsetneq V} f(X)$. Then the family $\mathcal{F} = \{\{X, V\backslash X\} \mid f(X) = m\}$ is bi-partitive.*

Proof. Let $\{X, V\backslash X\}$ and $\{Y, V\backslash Y\}$ be in $\mathcal{F}$ such that $\{X, V\backslash X\}$ and $\{Y, V\backslash Y\}$ overlap. Thus $f(X \cap Y) + f(X \cup Y) \leq 2m$. Since $X \cap Y$ and $X \cup Y$ are non-empty, $f(X \cap Y) \geq m$ and $f(X \cup Y) \geq m$. Thus $f(X \cap Y) = f(X \cup Y) = m$ and $\{X \cap Y, V\backslash(X \cap Y)\}$ and $\{X \cup Y, V\backslash(X \cup Y)\}$ are in $\mathcal{F}$. □

A major result on bi-partitive families, that we recall in the following theorem, is that every bi-partitive family can be represented by a unique labeled tree.

Theorem 2. *Let $\mathcal{F}$ be a bi-partitive family on a finite set V. Then there is a unique layout $(T, \mathcal{L})$ of V, called the* representative layout, *such that each internal node of T has at least 3 neighbors, is marked `degenerate`, `linear` or `prime` and:*

- *For every $(u, v) \in E_T$, the bipartition $\{X^{uv}, V\backslash X^{uv}\}$ is a strong bipartition in $\mathcal{F}$ and there is no other strong bipartition in $\mathcal{F}$.*
- *For every internal node u of T:*
  - *If u is `degenerated`, then for every $\emptyset \subsetneq W \subsetneq N_T(u)$, the bipartition $\{\cup_{v\in W} X^{uv}, V\backslash \cup_{v\in W} X^{uv}\}$ is in $\mathcal{F}$.*
  - *If u is `linear`, there is an ordering $v_1, \ldots, v_k$ of $N_T(u)$ such that for every $1 \leq i \leq j < k$, the bipartition $\{\cup_{\ell\in\{i,\ldots,j\}} X^{uv_\ell}, V\backslash \cup_{\ell\in\{i,\ldots,j\}} X^{uv_\ell}\}$ is in $\mathcal{F}$.*
- *There is no other bipartition in $\mathcal{F}$.*

(By convention, an internal node of degree 3 is always degenerated.)

Remark 1. Theorem 2 is proved in [6] using a different formalism. It follows also directly from results on partitive families [2,16] using the simple bijection $f(\mathcal{F}) = \{X \subseteq V\backslash\{v\} \mid \{X, V\backslash X\} \in \mathcal{F}\}$ between bi-partitive families on V and partitive families on $V\backslash\{v\}$, where $v \in V$ is fixed.

Remark 2. If $\mathcal{F}$ is a bi-partitive family with the additional property:

- for all $\{X_1, X_2\} \in \mathcal{F}$ and $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap, $\{X_1 \Delta Y_1, X_1 \Delta Y_2\} \in \mathcal{F}$ [2],

[1] A *split* in a digraph G is a bipartition $\{X, V_G\backslash X\}$ of V_G, where $\emptyset \subsetneq X \subsetneq V_G$, such that for every $u, v \in X$, $(N_G^+(u) \setminus X \neq \emptyset) \wedge (N_G^+(v) \setminus X \neq \emptyset) \Rightarrow (N_G^+(u) \setminus X = N_G^+(v) \setminus X)$, and $(N_G^-(u) \setminus X \neq \emptyset) \wedge (N_G^-(v) \setminus X \neq \emptyset) \Rightarrow (N_G^-(u) \setminus X = N_G^-(v) \setminus X)$.

[2] For two sets X and Y, we let $X \Delta Y$ be the set $X\backslash Y \cup Y\backslash X$.

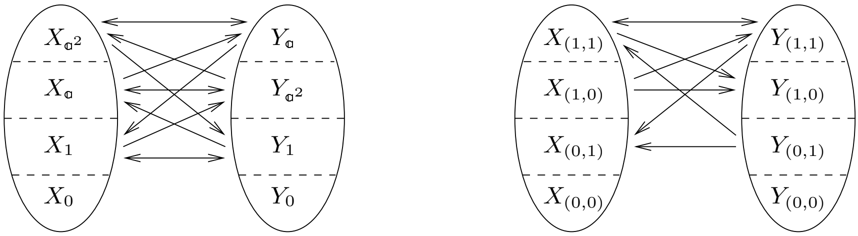


Fig. 1. Schematic view of a displit (left) and a Cunningham's split (right)

then $\mathcal{F}$ is said to be *strongly bi-partitive*. The representative layout of a strongly bi-partitive family has no `linear` node. Cunningham showed that the family of splits in a connected undirected graph is strongly bi-partitive [5]. Another example is the family of bi-joins in an undirected graph [17].

4.2 Displits

Definition 3 (Displit). *Let G be a digraph. A bipartition $\{X_1, X_2\}$ of V_G is a* displit *if $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and* $\text{cutrk}_G^{(4)}(X_1) \leq 1$.

Figure 1 shows a comparison between displits and splits on digraphs. A digraph G is *degenerated* (for the displit decomposition) if every bipartition of V_G is a displit, and G is *prime* if every displit in G is trivial. Finally G is *linear* if there is an ordering $x_1, \ldots, x_n$ of its vertices such that the family of displits in G is $\{\{\{x_i, \ldots, x_j\}, V_G \setminus \{x_i, \ldots, x_j\}\} \mid 1 \leq i \leq j < n\}$. By convention, a graph with at most 3 vertices is only degenerated.

By Proposition 1, the family of displits in a connected digraph is bi-partitive. By Theorem 2, this family can be represented by a unique labeled layout, that we call *displit decomposition*.

Observation 2. *If $\{X_1, X_2\}$ is a displit in G, then $\{X_1, X_2\}$ is a split in $u(G)$. The converse is not necessarily true.*

4.3 Quotient Graphs

Let $(T, \mathcal{L})$ be a displit decomposition of a connected digraph G and let u be an internal node of T. We recall that for every node v in $N_T(u)$, X_{uv} is the set of leaves reachable from u by a path going through v. The set $\{X^{uv} = \mathcal{L}^{-1}(X_{uv}) \mid v \in N_T(u)\}$ is a proper partition of V_G, and for every $v \in N_T(u)$, $\{X^{uv}, V_G \setminus X^{uv}\}$ is a displit.

For every $v \in N_T(u)$, we choose a vertex x_v in X^{uv} such that x_v is adjacent to a vertex in $V_G \setminus X^{uv}$. Such a x_v always exists since G is connected. Let $C(u)$ be the graph of vertex set $N_T(u)$ and of edge set $\{(v, w) \mid (x_v, x_w) \in E_G\}$. It is worth noticing that $C(u)$ is isomorphic to $G[\{x_v \mid v \in N_T(u)\}]$, and that $C(u)$ is not unique for a node u. Then we will consider $C(u)$ as an induced sub-graph of G. We now prove or state some technical lemmas.

Lemma 2. *Let $\{X, Y\}$ be a displit in G, and let $x \in X$ and $y \in Y$ such that x is adjacent to y. Let $\{X', Y'\}$ be a bipartition of V_G with $Y' \subseteq Y$. Then* $\text{cutrk}_G^{(4)}(Y') = \text{cutrk}_{G'}^{(4)}(Y')$, *where $G' = G[Y \cup \{x\}]$.*

Proof. Obviously $\text{cutrk}_{G'}^{(4)}(Y') \leq \text{cutrk}_G^{(4)}(Y')$. By definition of displits, there is an X-vector A and a Y-vector B such that $M_G[X, Y] = A^t \cdot B$. Since x is adjacent to a vertex in Y, $A[x] \neq 0$. Thus $M_G[X, Y'] = A[x]^{-1} \cdot A^t \cdot M_G[\{x\}, Y']$. Therefore, $\text{rk}(M_G[X' \backslash (X \backslash \{x\}), Y']) = \text{rk}(M_G[X', Y'])$ since all rows in $M_G[X, Y']$ are generated by the row $M_G[\{x\}, Y']$. □

Lemma 3. *Let $(T, \mathcal{L})$ be a displit decomposition of a digraph G and let u be a node of T. If u is prime (resp. degenerated, linear), then $C(u)$ is prime (resp. degenerated, linear).*

Proof. Let $\{X, Y\}$ be a bi-partition of $V_{C(u)}$, let $X' = \cup_{v \in X} X^{uv}$ and let $Y' = V_G \backslash X'$. We show that $\{X, Y\}$ is a displit in $C(u)$ if and only if $\{X', Y'\}$ is a displit in G. Trivially, if $\{X', Y'\}$ is a displit in G, then $\{X, Y\}$ is a displit in $C(u)$.

Now suppose that $\{X, Y\}$ is a displit in $C(u)$. $\{X', Y'\}$ does not overlap $\{X^{uv}, V_G \backslash X^{uv}\}$ for every $v \in N_T(u)$. We apply $|N_T(u)|$ times Lemma 2, for all $\{X^{uv}, V_G \backslash X^{uv}\}$. Thus $\{X', Y'\}$ is a displit if and only if $\{X, Y\}$ is a displit. □

The following lemmas give characterization of degenerated and linear digraphs. (Proofs are omitted.)

Lemma 4. *If G is degenerated with at least 4 vertices, then either $u(G)$ is a star, or G is C_3' where each of the 3 vertices is substituted by a complete graph (maybe with 0 vertex).*

Lemma 5. *If G is linear and has at least 4 vertices, then there is an ordering $(x_1, \ldots, x_n)$ of vertices of V_G, and a function $f : V_G \to \{0, 1, 2\}$ such that for all $j > i$:*

- *$(x_i, x_j) \in E_G$ if $f(x_i) \equiv f(x_j) \pmod 3$ or $f(x_i) \equiv f(x_j) + 1 \pmod 3$,*
- *$(x_j, x_i) \in E_G$ if $f(x_i) \equiv f(x_j) - 1 \pmod 3$ or $f(x_i) \equiv f(x_j) + 1 \pmod 3$,*
- *there are no other edges in the graph.*

Theorem 3. *Let G be a connected digraph with at least 3 vertices, and let $(T, \mathcal{L})$ be its displit decomposition. Then* $\text{rwd}^{(4)}(G) = \max\{\text{rwd}^{(4)}(C(u)) \mid u \in V_T \backslash L_T\}$.

Proof. Let $m = \max\{\text{rwd}^{(4)}(C(u)) \mid u \in V_T \backslash L_T\}$. Obviously $m \leq \text{rwd}^{(4)}(G)$ (since $C(u)$ is an induced sub-graph of G). For every $u \in V_T \backslash L_T$, let $(T_u, \mathcal{L}_u)$ be a sub-cubic layout of $C(u)$ of GF(4)-rank-width at most m. We suppose w.l.o.g. that the T_u are pairwise disjoint. We construct a sub-cubic layout $(T', \mathcal{L}')$ of G of GF(4)-rank-width at most m. Let T' be the union of all T_u (for $u \in V_T \backslash L_T$), after the identification of the vertices u in T_v and v in T_u for every $(u, v) \in E_{T - L_T}$, and after contraction of every vertex of degree 2. For all $x \in V_G$, let $\mathcal{L}'(x) = \mathcal{L}_u(\mathcal{L}(x))$ where $\{u\} = N_T(\mathcal{L}(x))$.

It is not hard to see that $(T', \mathcal{L}')$ is a sub-cubic layout of G. Moreover, by Lemma 2, in T' every edge has GF(4)-rank-width at most m. □

4.4 Decomposition Algorithm

It is known that the split decomposition of an undirected graph can be computed in linear time [7], and the split decomposition of a digraph in time $O(m\log(n))$ [14]. We present here a simple $O(nm)$ algorithm to compute the displit decomposition of a digraph. This algorithm is a simple adaptation of [9]. Due to space limitation, we present only the main lines, stated in the following two lemmas without proofs.

Lemma 6. *Let x and y be two vertices of a connected digraph G. We can compute in time $O(n+m)$ a non trivial displit $\{X,Y\}$ such that $x\in X$ and $y\in Y$ (if it exists).*

Lemma 7. *Given a digraph G, we can compute in time $O(nm)$ a family $\mathcal{F}$ of non overlapping displits such that for every displit $\{X,Y\}$ in G, either $\{X,Y\}\in\mathcal{F}$, or there is a bipartition $\{X',Y'\}\in\mathcal{F}$ such that $\{X,Y\}$ and $\{X',Y'\}$ overlap.*

The family constructed in the previous lemma contains obviously all strong displits in G. A final $O(nm)$ procedure finds every non-strong displits in $\mathcal{F}$. This leads to the following theorem.

Theorem 4. *The displit decomposition of every digraph can be computed in time $O(nm)$.*

5 Digraphs of GF(4)-Rank-Width 1

In [15] Kanté defined a notion of *vertex-minor* for digraphs that extended the one for undirected graphs. He also characterized the class of digraphs of GF(4)-rank-width at most k in the following.

Theorem 5 ([15]). *For each k, there is a finite list $\mathcal{C}_k$ of digraphs having at most $(6^{k+1}-1)/5$ vertices such that a digraph G has* GF(4)*-rank-width at most k if and only if no digraph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.*

When $k=1$, the digraphs to exclude as vertex-minors have at most 7 vertices. However, we do not know any polynomial-time algorithm that checks whether a given graph is a vertex-minor of another. We will give in this section several characterizations of digraphs of GF(4)-rank-width 1. As a consequence we get an algorithm for recognizing digraphs of GF(4)-rank-width 1.

A vertex x of a digraph G is a *pendant vertex* of another vertex y if y is the only neighbor of x in G. Two vertices x and y of a digraph G are called *dtwins* if x and y verify one of the following exclusive conditions ($A=N^+_{G-y}(x),\ B=N^-_{G-y}(x)$):

1. $N^+_{G-x}(y)=A,\ N^-_{G-x}(y)=B$ or,
2. $N^+_{G-x}(y)=B,\ N^-_{G-x}(y)=(B\backslash A)\cup(A\backslash B)$ or,
3. $N^+_{G-x}(y)=(A\backslash B)\cup(B\backslash A),\ N^-_{G-x}(y)=A$.

We say that a digraph is *completely decomposable by the displit decomposition* if every node in the displit decomposition is degenerated or linear. The main result of this paper is the following theorem, analogous to Theorem 1.

Theorem 6. *Let G be a connected digraph with at least 2 vertices. Then the following conditions are equivalent:*

1. *G is completely decomposable by the displit decomposition.*
2. *G can be obtained from a single vertex by creating dtwins or adding pendant vertices.*
3. *G has* GF(4)*-rank-width* 1*.*
4. *For every $W \subseteq V$ with $|W| \geq 4$, $G[W]$ has a non-trivial displit.*
5. *$u(G)$ is distance-hereditary and for every $W \subseteq V$ with $|W| \leq 5$, we have* $\mathrm{rwd}^{(4)}(G[W]) \leq 1$*.*

Condition 5 gives a characterization of digraphs of GF(4)-rank-width 1 by forbidden induced sub-graphs: a digraph has GF(4)-rank-width 1 if and only if it is $(\mathcal{H}, \mathcal{C})$-free, where $\mathcal{H}$ is the set of digraphs G such that $u(G)$ is a house, a gem, a domino or a hole ($C_k, k \geq 5$), and $\mathcal{C}$ is the set of connected digraphs G with at most 5 vertices such that $\mathrm{rwd}^{(4)}(G) > 1$ and for every $x \in V_G$, $\mathrm{rwd}^{(4)}(G-x) \leq 1$.

Before proving Theorem 6, let us state and prove two technical propositions. The following is immediate from the definitions.

Proposition 2. *Let x and y be two vertices of a digraph G. Then $\{x, y\}$ is a displit if and only if x and y are dtwins or x is a pendant vertex of y or y is a pendant vertex of x.*

The following proposition is a straightforward adaptation of [18, Proposition 7.1].

Proposition 3. *Let x and y be dtwins of a digraph G such that $G - x$ has at least one edge. Then* $\mathrm{rwd}^{(4)}(G - x) = \mathrm{rwd}^{(4)}(G)$*.*

Proof. By definition of GF(4)-rank-width we have $\mathrm{rwd}^{(4)}(G - x) \leq \mathrm{rwd}^{(4)}(G)$. We will prove that $\mathrm{rwd}^{(4)}(G-x) \geq \mathrm{rwd}^{(4)}(G)$. Let $(T, \mathcal{L})$ be a sub-cubic layout of GF(4)-rank-width $k = \mathrm{rwd}^{(4)}(G-x)$ of $G-x$. By definition, there is a bijection $\mathcal{L}$ between V_{G-x} and L_T. Let $v = \mathcal{L}(y)$ and let $u \in V_T$ such that $uv \in E_T$. Let T' be obtained from T as follows: $V_{T'}$ is the set $V_T \cup \{u', w\}$ (where u' and w are two new nodes) and $E_{T'}$ the set $(E_T \backslash \{uv\}) \cup \{uu', u'v, u'w\}$. We let $\mathcal{L}' : V_G \rightarrow \mathrm{L}_{T'}$ be such that $\mathcal{L}'(x) = w$ and for every $z \in V_G \backslash x,\ \mathcal{L}'(z) = \mathcal{L}(z)$.

It is clear that $(T', \mathcal{L}')$ is a sub-cubic layout of G. We claim that the GF(4)-rank-width of $(T', \mathcal{L}')$ is equal to the GF(4)-rank-width of $(T, \mathcal{L})$.

It is clear that the GF(4)-rank-width of the edges $u'v$ and $u'w$ are at most 1. Since x and y are dtwins, the GF(4)-rank-width of the edge uu' is at most 1 (Proposition 2). Moreover, the other edges of T' are in T, then their GF(4)-rank-width in $(T', \mathcal{L}')$ is equal to their GF(4)-rank-width in $(T, \mathcal{L})$ (Lemma 2). Since $G-x$ has at least one edge we have $\mathrm{rwd}^{(4)}(G-x) \geq 1$. Therefore $\mathrm{rwd}^{(4)}(G-x) \geq \mathrm{rwd}^{(4)}(G)$. □

We can now begin the proof of Theorem 6.

Proof (Proof of Theorem 6). $1 \rightarrow 2$). By induction on $|V_G|$. It is trivial if $|V_G| \leq 2$. Otherwise, let $(T, \mathcal{L})$ be the displit decomposition of G, and let u be a leaf in $T - L_T$. If u is degenerated, let $\{v, w\} \subseteq N_T(u) \cap L_T$. Otherwise, u is linear and has at least 4 neighbors. Let $v_1, \ldots v_k$ be its ordering. If $N_T(u) \backslash L_T \subseteq \{v_2, \ldots, v_{k-1}\}$, take $v = v_1$ and $w = v_k$. Otherwise, take $v = v_2$ and $w = v_3$. In all cases, $\{\mathcal{L}^{-1}(\{v, w\}), V_G \backslash \mathcal{L}^{-1}(\{v, w\})\}$ is a displit. By Proposition 2, either $x = \mathcal{L}^{-1}(v)$ and $y = \mathcal{L}^{-1}(w)$ are dtwins, or one is a pendant vertex of the other. If x and y are dtwins or x is a pendant vertex of y, we let $G' = G - x$, otherwise $G' = G - y$. By induction G' is obtained from a single vertex by creating dtwins or adding pendant vertices.

$2 \rightarrow 3$). By induction on $|V_G|$. It is trivial if $|V_G| \leq 2$. Otherwise, let $x \in V_G$ be the last added vertex. If x is a pendant vertex, let $\{y\} = N_G(x)$, otherwise let y be the dtwin of x. By induction, $\mathrm{rwd}^{(4)}(G - x) = 1$. Using Proposition 3, $\mathrm{rwd}^{(4)}(G) = 1$.

$3 \rightarrow 4$). If $\mathrm{rwd}^{(4)}(G) \leq 1$, then for every $W \subseteq V_G$, $\mathrm{rwd}^{(4)}(G[W]) \leq 1$. When $|W| \geq 4$, a sub-cubic layout of $G[W]$ has an edge (u, v) such that $\{X^{uv}, V \backslash X^{uv}\}$ is non-trivial, and thus $G[W]$ has a non-trivial displit.

$4 \rightarrow 1$). Suppose that G is not completely decomposable. Then the displit decomposition of G has a prime node u. By definition of a representative layout, the degree of u is at least 4. By Lemma 3, the quotient graph $C(u)$ is prime and is an induced sub-graph of G with at least 4 vertices.

$3 \rightarrow 5$). By Observation 2, $\mathrm{rwd}(u(G)) = 1$ since the layout of GF(4)-rank-width 1 for G is a layout of rank-width 1 for $u(G)$. Thus by Theorem 1, $u(G)$ is distance hereditary. Moreover, for every $W \subseteq V$, we have $\mathrm{rwd}^{(4)}(G[W]) \leq 1$.

$5 \rightarrow 3$). Due to space limitation we will give only a sketch of the proof. Suppose that G is a digraph such that $\mathrm{rwd}^{(4)}(G) > 1$ and such that $u(G)$ is distance hereditary. Let W be a minimal subset of V_G such that $\mathrm{rwd}^{(4)}(G[W]) > 1$. Working on the split decomposition of $u(G[W])$, one can show successively that:

- $u(G[W])$ has no pendant vertex,
- if $u(G[W])$ has a false twin, then $G[W]$ has at most 4 vertices,
- if $u(G[W])$ has no false twin and no pendant vertex, then $u(G)$ is complete,
- and if $u(G[W])$ is complete, then $G[W]$ has at most 5 vertices.

Thus there is a $W \subseteq V_G$ of size at most 5 such that $\mathrm{rwd}^{(4)}(G[W]) > 1$. □

As a corollary of Theorems 4 and 6, we get an algorithm for recognizing digraphs of GF(4)-rank-width 1.

Corollary 1. *Digraphs of* GF(4)*-rank-width* 1 *can be recognized in time* $O(nm)$*.*

6 Concluding Remarks

Differences with Cunningham's split decomposition of digraphs. Cunningham showed that the family of splits in a strongly connected digraph is bi-partitive.

He also gave a characterization of degenerated and linear digraphs for the split decomposition: a digraph is degenerated for the split decomposition if and only if it is complete or is a star, and is linear if and only if it is a *circle of transitive tournaments* (CTT) [5].

The displit decomposition and the split decomposition of digraphs are both generalization of the split decomposition of undirected graphs. A first difference is that for the displit decomposition the graph has only to be connected.

The quotient graphs of the displit decomposition are induced sub-graphs of the original graph; this is not necessarily true for the split decomposition of digraphs.

Finally, the split decomposition and the displit decomposition are mutually exclusive. For all $k \geq 3$, the graph C'_k is linear for the split decomposition (and thus completely decomposable) since it is a CTT, but it is prime for the displit decomposition since $u(C'_k)$ is prime for the split decomposition. In the other hand, we can construct an infinite family of graphs linear for the displit decomposition and prime for the split decomposition.

Links between bi-rank-width and Cunningham's split decomposition. Kanté defined another digraph parameter called *bi-rank-with*, and showed relations between GF(4)-rank-width and bi-rank-width [15]. A strongly connected digraph is completely decomposable by Cunningham's split decomposition if and only if it has bi-rank-width 2. It is open to find another characterization for digraphs of bi-rank-width 2.

Generalization to 2-structures. A *2-structure* is a complete digraph with labels on edges. We mention that GF(4)-rank-width and displit decomposition can be generalized to 2-structures over finite fields. For a field $\mathbb{F}$, we obtain a decomposition for 2-structures over $\mathbb{F}$ with a characterization theorem similar to Theorem 6. An interesting case is GF(3), which gives a decomposition theory for oriented graphs (*i.e.*, directed anti-symmetric graph).

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