Connected Feedback Vertex Set in Planar Graphs

Alexander Grigoriev[1] and René Sitters[2]

[1] Department of Quantitative Economics, Maastricht University, The Netherlands
a.grigoriev@ke.unimaas.nl
[2] Department of Econometrics and Operations Research, VU University, Amsterdam, The Netherlands
rsitters@feweb.vu.nl

Abstract. We study the problem of finding a minimum tree spanning the faces of a given planar graph. We show that a constant factor approximation follows from the unconnected version if the minimum degree is 3. Moreover, we present a polynomial time approximation scheme for both the connected and unconnected version.

1 Introduction

Given a planar graph, what is the smallest subgraph connecting all the faces? The simplicity and naturalness of this question is the main motivation for the study in this paper. Bodlaender et al. [5] call this the *face cover tree problem* and to the best of our knowledge they were the first to study it. They show that the problem can be solved efficiently for edge-weighted graphs of bounded treewidth. In this paper we consider unweighted planar graphs with the minimum degree at least three. This is a natural restriction since allowing vertices of degree two makes its complexity polynomially equivalent to the problem with polynomially bounded edge weights.

Interestingly, the problem does not depend on the embedding since any tree hitting all faces will, in fact, hit all cycles of the graph.

Lemma 1. *Let G be a connected planar graph and $T \subseteq G$ be a tree such that, for a given embedding of G, every face has at least one vertex in T. Then, every cycle of G has a vertex in T.*

Proof. Every cycle separates the embedded graph in an inner and outer part. Each of the two parts contains at least one face. Therefore, the cycle and the tree must have at least one vertex in common. □

The problem of finding the smallest set of vertices hitting all cycles is well-studied and known as the *feedback vertex set problem.* A natural variant for planar graphs is the problem of hitting all faces with a minimum number of vertices. By Lemma 1, the connected versions of these two problems are equivalent and independent of the embedding. This is the problem we study here and call it the *connected feedback vertex set problem.*

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Planar Feedback Vertex Set (Planar FVS): Given an unweighted planar graph, find the smallest set S of vertices such that every *cycle* of the graph has at least one vertex in S.

Face Hitting Set (FHS): Given an unweighted planar graph with an embedding, find the smallest set S of vertices such that every *face* of the graph has at least one vertex in S.

Connected Planar Feedback Vertex Set (Connected Planar FVS): Given an unweighted planar graph, find the smallest tree T such that every *cycle* (or equivalently, every face in an embedding) of the graph has a vertex in T.

1.1 Related Results

The feedback vertex set problem is extensively studied. It is APX-hard in general graphs and can be approximated efficiently within a factor 2; see Becker and Geiger [4] and Bafna et al [1]. For planar graphs the problem is NP-hard [12] and a PTAS was given by Demaine and Hajiaghayi [8]. Goemans and Williamson apply the primal-dual method to obtain a (9/4)-approximation [13], which was later reduced to 2 by Chudak et al [7].

Regarding the connectivity constraint, two obvious related problems, are the problem of spanning all vertices and the problem of spanning all the edges of the graph. The latter is known as *connected vertex cover* and was introduced in 1977 by Garey and Johnson [11], who showed it to be NP-hard even when restricted to planar graphs with maximum degree 4. The 2-approximation algorithm for vertex cover in general graphs by Savage [14] transfers directly to the connected problem. Recently, Escoffier et al. [10] have shown that connected vertex cover admits a PTAS for planar graphs. A PTAS for connected dominating set in planar graphs was given in [8] as well.

1.2 Our Results

We give an overview on structural properties, complexity and approximability results for the connected feedback vertex set problem in planar graphs. We show that if the minimum vertex degree is three, then the ratio between the connected and unconnected problem is bounded by a constant. This provides a polynomial time constant approximation algorithm for connected planar FVS. Another interesting consequence of this structural result is that the diameter of a 3-polytope is in the order of the smallest set of vertices hitting all facets. Further, we show that FHS and connected planar FVS are strongly NP-hard and give polynomial time approximation schemes for both problems. Along the text we pose several interesting open questions.

2 Structural Results

2.1 Insightful Observations

We start with some simple lemmas to get an insight in the relation between FVS and FHS and the dependence on the embedding. Then we give the main result of this section on the relation between connected and unconnected FVS in planar graphs. We end with a small discussion on the application to diameters of polytopes.

Lemma 2. *For any planar graph G and embedding Γ_G with at least two faces we have* $\mathrm{FHS}(\Gamma_G) \leq \mathrm{FVS}(G)$. *Otherwise,* $\mathrm{FHS}(\Gamma_G) = 1$ *and* $\mathrm{FVS}(G) = 0$.

Lemma 3. *For any planar graph G with faces F we have* $\mathrm{FVS}(G) \leq |F| - 1$ *and this bound is tight.*

The proofs of Lemmas 2 and 3 are not complicated and we leave those for a reader.

Lemma 4. *For any planar graph G and embedding Γ_G we have* $\mathrm{FVS}(G) \leq 2\mathrm{FHS}(\Gamma_G) - 1$ *and this bound is tight for* $0, 1$ *or* 2*-connected graphs. If G is 3-connected then* $\mathrm{FVS}(G) \leq 2\mathrm{FHS}(\Gamma_G) - 2$.

Proof. Let S_1 be a minimum FHS in Γ_G. Now consider the graph H containing all uncovered cycles and let F_H be its faces. Each face f in H must contain a point from S_1 in its interior. Hence, $|S_1| \geq |F_H|$. Let S_2 be a minimum FVS in H. Then by Lemma 3 $|S_2| \leq |F_H| - 1 \leq |S_1| - 1$. Note that $S_1 \cup S_2$ is a FVS in G. Hence, $\mathrm{FVS}(G) \leq |S_1| + |S_2| \leq 2|S_1| - 1 = 2\mathrm{FHS}(\Gamma_G) - 1$. □

For 3-connected graphs the embedding is unique and so is the minimum value of FHS. In general the optimal value differs by at most a factor two for different embeddings and this bound is tight; see Figure 1(A).

Lemma 5. *Let Γ_1, Γ_2 be two embeddings of planar graph G. Then* $\mathrm{FHS}(\Gamma_1) \leq 2\mathrm{FHS}(\Gamma_2) - 1$.

Proof. If G contains only one face (i.e., it is a forest), then $\mathrm{FHS}(\Gamma_1) = \mathrm{FHS}(\Gamma_2) = 1$. In the other case Lemma 2 says $\mathrm{FHS}(\Gamma_1) \leq \mathrm{FVS}(G)$. By Lemma 4 we have $\mathrm{FVS}(G) \leq 2\mathrm{FHS}(\Gamma_2) - 1$. Combining these inequalities the lemma follows. □

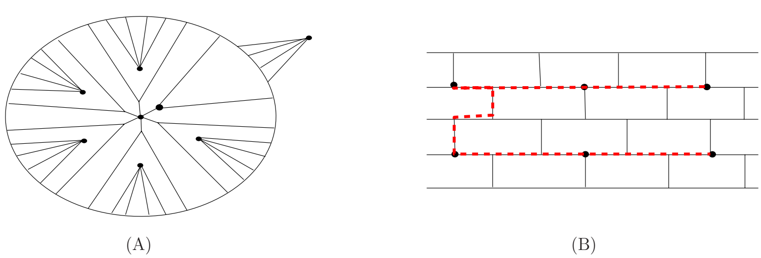

Fig. 1. (A) Tight example for the dependence on the embedding. If all k triangles are directed inwards, the optimal FHS has size $k + 1$. If all are directed outwards the optimal value is $2k + 1$. (B) In an infinite honeycomb graph the ratio between FHS and the connected FVS is 3.

2.2 FVS and FHS in Planar Graphs with Minimum Degree 3

The minimum FHS may be arbitrarily much smaller than the connected FVS if we allow vertices of degree 2. However, restricting to a minimum degree of 3 ensures a constantly bounded ratio between the connected and unconnected problem.

Theorem 1. *Let G be a connected planar graph with minimum degree* 3 *and let* OPT^C *be the optimal value for connected planar FVS. Further, let* OPT *be the optimal value for* FHS, *for some given embedding of* G*. Then,* $\text{OPT}^C = 0$ *if* $\text{OPT} = 1$ *and* $\text{OPT}^C \leq 11\text{OPT} - 14$ *otherwise.*

Proof. The case $\text{OPT} = 1$ is trivial. Now assume $\text{OPT} \geq 2$. The outline of the proof is as follows. Given a planar graph $G = (V, E)$ plus embedding, let S be a face hitting set and T be a minimum Steiner tree on S. We will construct curves in the embedding that go from edges in T to vertices in S such that no two curves intersect. On one hand, the number of curves will be $\Omega(|T|)$. On the other hand, we will see that non-intersection of curves implies that their number is $O(|S|)$. Combined we get $|T| = O(|S|)$.

To simplify the construction of curves we define a graph G' that follows from G after contractions of edges. We can partition T in a collection $\mathcal{P}$ of at most $2|S| - 2$ paths such that any two paths may only have an endpoint in common. By minimality of T, any path is a shortest path between its endpoints. We leave any path $P_i \in \mathcal{P}$ of length $1, 2$ or 3 unchanged, where the length is the number of edges. If the length P_i is two, we denote its inner vertex by q_i. If the length is three we denote one of its inner vertices by q_i. Any path P_i of length at least four is reduced to length exactly four by contracting all points that are at distance at least two from the two endpoints of P_i in a single point q_i. Let the resulting (multi) graph be $G' = (V', E')$. The following properties are easy to verify. A short justification is given below.

(i) All points $q_1, q_2, \ldots,$ are different.
(ii) If $\text{length}(P_i) \leq 3$ then $\text{degree}(q_i) \geq \text{length}(P_i)$.
(iii) If $\text{length}(P_i) \geq 4$ then $\text{degree}(q_i) \geq \text{length}(P_i) - 1$.
(iv) Solution S is a FVS for G' as well.
(v) For any $v \in V'$, $\text{degree}(v) \geq 3$.

Explanation: (i) Paths only share endpoints and these are not contracted, i.e., no edge was contracted to any of those points. (ii) Obvious. (iii) Every point on P_i has at least one edge not in P_i. (iv) A contraction creates no extra faces. Vertices in S are not contracted. (v) The contraction of two adjacent points with degrees $d_1, d_2 \geq 3$ results in a point with degree $d_1 + d_2 - 2 \geq 4$.

Claim. There are no multiple edges in G'.

Proof. Since G is a simple graph, a multiple edge can only appear if at least one of the two endpoints is a contracted point. More precisely, either (i) there are edges (u_1, v) and (u_2, v) in E such that u_1 and u_2 are contracted in a single points q_i, or (ii) there are edges (u_1, v_1) and (u_2, v_2) in E such that u_1 and u_2 are contracted in q_i and v_1 and v_2 are contracted in q_j.

In case (i), the point v cannot be on the part of P_i between u_1 and u_2 since then all three points would be contracted in q_i. Therefore, the edges (u_1, v) and (u_2, v) plus the part of P_i between u_1 and u_2 form a simple cycle C in G. By Lemma 1, C must have a vertex from T and this can only be v. But then, we can strictly reduce the length of the T as follows. Remove from T the path from u_1 to u_2. Assume, w.l.o.g., that u_1 and v are in the same component. Now add edge (u_2, v) and remove the remaining redundant path to u_1. The argument for (ii) is similar. The edges edges $(u_1, v_1), (u_2, v_2)$ plus the part of P_i between u_1 and u_2 and the part of P_j between v_1 and v_2 form a simple cycle in G and must therefore contain a point from T. But there is no such point on these parts by definition. □

Note that a face in G' may have more than one vertex from S. For each face f we fix one vertex $s(f) \in S$. Now, consider a point q_i and let N_i be the set of neighboring edges whose endpoints are not in S. For each i with $|N_i| \geq 2$ we do the following. Consider two edges from N_i that are consecutive in the embedding, i.e., they appear consecutively among edges from N_i when we walk around q_i. Let f be a face that touches q_i between these edges. We draw a curve inside f from q_i to $s(f)$. Call this a *face-curve*. We do this for all $|N_i|$ pairs of consecutive edges of q_i. If, on the other hand, $|N_i| = 0$ or $|N_i| = 1$ we add a curve from q_i to each neighbor that is an element from S. Call these *edge curves*. Finally, for each path P_i of length one in G we define the point in the plane on the middle of edge P_i as r_i and draw one curve from r_i to $s(f)$, where f is a face adjacent to r_i. These curves are also called face-curves. Note that for each path $P_i \in \mathcal{P}$, we either defined a vertex q_i in G' or defined a point r_i in the embedding of G'.

We define the bipartite (multi-)graph $H = (\Pi \cup \Sigma, \mathcal{A})$ as follows. Let $\Pi = \{\pi_1, \ldots, \pi_{|\mathcal{P}|}\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_|S|\}$. For each curve defined in the process above there is an edge in H, i.e., for each curve from q_i or r_i to s_j there is an edge (π_i, σ_j).

Claim. The graph H is planar and degree$(\pi_i) \geq$ length$(P_i) - 1$ for each $P_i \in \mathcal{P}$.

Proof. The first follows directly from the following observations. All points q_i and r_i are different and none coincides with points from S. Each curve either lies inside a single face or corresponds to a single edge. All curves inside a face have a common endpoint.

If length$(P_i) = 1$ then degree$(\pi_i) = 1$. If length$(P_i) \in \{2, 3\}$ then degree$(q_i) \geq 3$. Now, either $|N_i| \geq 2$ in which case we added $|N_i|$ face-curves, or $|N_i| \leq 1$ in which case we added degree$(q_i) - |N_i| \geq 3 - 1 = 2$ edge-curves. Now assume length$(P_i) \geq 4$. Note that in that case no neighbor of q_i can be a vertex $s \in S$, since in that case we can reduce the length of $T \subseteq G$ by adding edge (q_i, s) and removing the path in T from q_i to one of the two endpoints of P_i. Hence, $|N_i| \geq$ degree$(q_i) \geq$ length$(P_i) - 1$. □

We will show that H has few edges. Consider the embedding of H defined naturally by the curves in the embedding of G'. In general, H may have faces of length two and may be disconnected. To facilitate the analysis we add edges to H until it is connected. Note that we can always do this without creating new faces of length two. Denote the new graph by H'. We prove through Claim 2.2 that H' does not have many edges by showing that it has no faces of length two. In the proof we use the next general statement on planar graphs.

Claim. Let $G = (V, E)$ be a simple planar graph with $|V| \geq 3$ and $s, w \in V$ such that degree$(v) \geq 3$ for all $v \in V \setminus \{s, w\}$ and degree$(w) \geq 1$. Then, there is at least one face that does not contain s on its boundary.

Proof. Remove s from G and consider a component C containing some $v \in V \setminus \{s, w\}$. Since $|V| \geq 3$ this component exists. If $w \in C$ then its degree is at least one. Any other vertex has degree at least two. The sum of the degrees in C is then at least $2n' - 1$, with n' the number of vertices in C. But then C is not a forest and must therefore have a cycle. Since s is not on the cycle there is a face not connected to s. □

Claim. There are no faces of length two in H'.

Proof. Suppose there is a face f of length two in H'. Since H' is connected, there cannot be a point $\sigma_j \in \Sigma$ inside f since it has to be connected to at least one of the two points of f, in which case f has length larger than two. Given that f has no points from Σ in its interior, the two curves in G' that correspond to the two edges of f do not enclose a point from S. We will show that this leads to a contradiction.

For each r_i we defined exactly one curve. So the two curves do not start from a point r_i. For each q_i we either defined edge-curves (at most one to each neighbor) or defined face-curves. Therefore the two curves must both be face-curves. Assume they go from q_i to s_j. Let $J \subseteq G'$ be the graph induced by all vertices enclosed by the two curves and including q_i and s_j. We know that s_j is the only vertex from S in J and, by construction, there is at least one edge $(q_i, w) \in N_i$ in J. Since $w \notin S$ we have $w \neq s_j$ and J has at least three vertices: q_i, w and s_j. Further, any vertex $v \notin \{q_i, s_j\}$ in J has degree at least 3 in J. By Claim 2.2, graph $J \subseteq G'$ has no multiple edges. Now, it follows from Claim 2.2 that there must be a face of J that is not connected to s_j. A contradiction. □

The proof of Theorem 1 now easily follows from an upper and lower bound on the number of edges $|\mathcal{A}|$ in H. By Claim 2.2 we have

$$|\mathcal{A}| = \sum_{\pi_i \in \Pi} \text{degree}(\pi_i) \geq \sum_{P_i \in \mathcal{P}} (\text{length}(P_i) - 1) = |T| - |\mathcal{P}|. \qquad (1)$$

Since we assumed $|S| \geq 2$, the number of vertices in H is $|\Pi| + |\Sigma| \geq 1 + 2 = 3$. Let n, m, f be, respectively, the number of vertices, the number of edges, and the number of faces in H'. Each face in H' is bounded by at least three edges so

$2m \geq 3f$. Since H' is connected we know from Euler's formula that $n+f = m+2$. Hence, $2m \geq 3f = 3m + 6 - 3n$ implying $m \leq 3n - 6$. We obtain,

$$|\mathcal{A}| \leq m \leq 3n - 6 = 3(|\Pi| + |\Sigma|) - 6.$$

Combined with (1) we get that

$$|T| - |\mathcal{P}| \leq |\mathcal{A}| \leq 3(|\Pi| + |\Sigma|) - 6 = 3(|\mathcal{P}| + |S|) - 6. \tag{2}$$

In the definition of $\mathcal{P}$ we remarked that $|\mathcal{P}| \leq 2|S| - 2$. This combined with (2) gives

$$|T| \leq 4|\mathcal{P}| + 3|S| - 6 \leq 4(2|S| - 2) + 3|S| - 6 = 11|S| - 14.$$

If S is an *optimal* solution for the unconnected problem, then

$$\text{Opt}^C \leq |T| \leq 11|S| - 14 = 11\text{Opt} - 14.$$ □

Question 1. What is the right ratio for Theorem 1? We conjecture it is 3. See Figure 1(B).

2.3 Diameter of Polytopes

The 1-skeleton of a 3d-polytope is a 3-connected planar graph and vice versa. We proved that the smallest tree spanning all facets is in the order of the number of points hitting all facets. An easy corollary is that the diameter is not much larger. The famous Hirsch conjecture states that the diameter of any d-polytope is at most $n - d$, with n the number of facets. It is known to be true for $d = 3$. Note that the face hitting set may be much smaller than the number of faces. We believe the next easy corollary is of its own interest.

Corollary 1. *The diameter of a 3-dimensional polytope $\mathcal{P}$ is $O(\text{FHS}(\mathcal{P}))$.*

Proof. Let s_1, s_2 be vertices of the polytope $\mathcal{P}$ and S a smallest set covering all the facets. If $s_i \notin S$ we add a hyperplane that just cuts off s_i. The new polytope $\mathcal{P}'$ has at most two extra facets and we can cover all facets by at most $|S| + 2$ vertices. These vertices are spanned by tree of size at most $11(|S| + 2) - 14 = 11|S| + 4$. Clearly, the tree in $\mathcal{P}'$ induces a tree in $\mathcal{P}$ of at most the same size and which connects s_1 and s_2. □

A similar statement for higher dimensions should depend on the dimension d. For example, the facets of a d-dimensional cube are covered by two opposite vertices while the diameter is d. Hence, the diameter of a d-cube $\mathcal{P}$ is $d/2 \cdot \text{FHS}(\mathcal{P})$, where FHS is the minimum *facet* hitting set. Barnette [3] proved that the diameter of a d-polytope is $O(2^d(n - d))$. Can we replace the n by the minimum FHS?

Question 2. Is there a function $f(d)$ such that for any d-dimensional polytope $\mathcal{P}$, the diameter is bounded by $f(d)\text{FHS}(\mathcal{P})$?

3 Complexity and Approximation

3.1 NP-Hardness of FHS and Connected FVS

Bodlaender et al. [5] show that connected FVS with maximum degree 4 is NP-hard even if every edge has either unit length or an input dependent length K. They reduce from the connected vertex cover problem in planar graphs. Garey and Johnson[11] show that the latter problem is already NP-hard if the maximum degree is 4. To prove NP-hardness for unit lengths we modify the original proof from [11]. NP-hardness of FHS follows easily from the reduction we use for the connected version.

Theorem 2. FHS *is NP-hard in planar graphs with maximum degree 6 and connected* FVS *is NP-hard in planar graphs with maximum degree 9.*

Proof. We concentrate on the proof for the connected FVS problem and we leave the proof for FHS to the extended journal version of this paper. We reduce from the vertex cover problem in planar graphs with maximum degree 3, which is known to be NP-hard [11]. Given a planar graph $G = (V, E)$ with maximum degree 3, we fix some embedding. Let F be the set of faces. We replace each edge by a graph on 10 vertices as in Figure 2. Call this a *bridge*. Let the size of a face be the length of a closed walk along the edges of the face. In each face f of size k we add two rings: an outside ring on $5k$ vertices and an inside ring on $15k$ vertices. Connections between the rings and bridges are illustrated in Figure 2. To enhance the counting we do this for the outer face as well. (Not shown in the figure.) The newly constructed graph G' has maximum vertex degree 9. We claim that G has a vertex cover of size s if and only if G' has a connected feedback vertex set of size $s + 12|E| + |F|$. We omit this technical proof and present it in the full length journal version of this paper. □

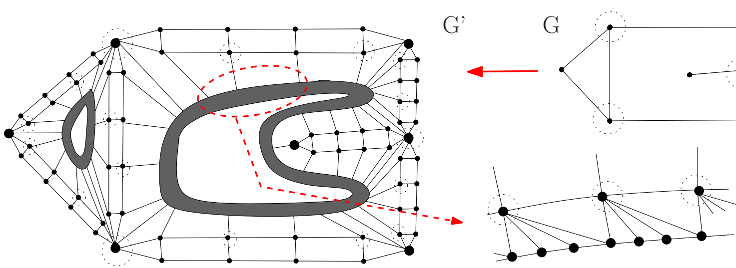

Fig. 2. The reduction. Construction for the outer face is not shown. The encircled vertices indicate a vertex cover in G and a connected feedback vertex set in G'.

3.2 Approximation Schemes for FHS and Connected FVS

First, we consider the connected FVS. We assume that the minimum vertex degree in the graph is at least 3. The polynomial time approximation scheme is a Baker's type algorithm; see Baker [2]. First, we define levels of the planar embedding following the recursive procedure: define level 1 of the embedding as the set of vertices incident to the exterior face of the embedding; assume we constructed level j, then level $j+1$ is defined as the set of vertices incident to the exterior face of the embedding after removal of the first j levels.

Given a desired approximation precision $\varepsilon > 0$, let $k = 2\lceil(\log n)/\varepsilon\rceil$. Let T be a minimum tree hitting all faces in G and let $OPT = |V(T)| - 1$. For any $0 \le j \le k-1$ define set V_j as the union of levels i such that $i \equiv j \mod k$. Since sets $V_j, 0 \le j \le k-1$, define a partition of $V(G)$ into k subsets, there is a subset V_ℓ containing at most OPT/k vertices from T. Denote $q = |V_\ell \cap V(T)|$. Notice that we do not know in advance values ℓ and q. So, the algorithm enumerates all possibilities for those values and chooses the values providing the shortest tree hitting all faces. As $1 \le \ell \le k$ and $1 \le q \le OPT/k$, this enumeration adds a factor $O(n)$ in the running time. From now, we assume that the algorithm picked correct values ℓ and q.

Consider $k+1$ consecutive levels with the first and the last levels from V_ℓ. We call a subgraph induced by such set of levels a *slice*. Clearly, any slice is a $(k+1)$-outerplanar graph. By Bodlaender et al [5], the minimum FHS in k-outerplanar graphs can be found in time $O(n^3 + 2^{9.5539k})$. Thus, by definition of k, we can solve the problem on any slice in polynomial time. Using the same algorithm as in [5], we can solve in polynomial time even more general problem: given a slice and an arbitrary number $1 \le r \le n$, we have to find a minimum forest of at most r components that hits all faces of the slice. We omit the proof of this simple adjustment.

Notice that T can be seen as a collection of at most $q+1$ trees such that each of these trees is located in exactly one slice. Given the minimum forests for each slice and each number of components, by straightforward dynamic program we find the minimum forest hitting all faces of G with at most $q+1$ components, each located in exactly one slice. Let T' be such forest. Notice, $E(T') \le OPT$.

Now, we have a forest T' of at most $q+1$ components that hits all faces in G. Moreover, this forest is shorter than tree T. The only question remains: how to connect the components of T' at small cost? For any two components S and S' of T', let distance $d(S, S')$ be defined as the length of the shortest path connecting S and S'. On this metric, take a minimum spanning tree M. If $(S, S') \in E(M)$, we connect S and S' with the corresponding shortest path. In this way we obtain a connected graph that hits all faces. Hence, we can find a tree of length at most $OPT + \sum_{(S,S') \in E(M)} d(S, S')$.

Lemma 6. $\sum_{(S,S') \in E(M)} d(S, S') \le \varepsilon OPT$.

Proof. As T' has at most $q+1$ components, M contains at most q edges. Assume there is an edge (S, S') in M of length greater than $\varepsilon OPT/q$. Since $OPT \ge qk \ge 2\log n/\varepsilon$, we have that $d(S, S') > 2\log n$. Consider the corresponding shortest path between S and S'. Take a vertex v in the middle of this path. Let the set of faces of G which are not incident to v be referred as $\mathcal{F}$. Consider the distance from v to $f \in \mathcal{F}$ by mean the length of the shortest path from v to the furthest vertex incident to f. By choice of v and the assumption that $d(S, S') > 2\log n$, the distance from v to any face from $\mathcal{F}$ is greater than $\log n$. Therefore, the subgraph of G induced by all vertices on distance at most $1 + \log n$ from v is a tree. Since minimum degree in G is at least 3, the number of vertices in such tree is more than n. A contradiction.

Now, we summarize the main results of this section in the following theorem and corollary.

Theorem 3. *Given a planar graph G of minimum degree 3 and $\varepsilon > 0$, the algorithm above constructs in polynomial time a tree hitting all faces of G with length at most $(1 + \varepsilon)OPT$.*

Applying literally the same modifications to the Baker's algorithm as in Eppstein [9] and Bodlaender and Grigoriev [6] we derive the following corollary.

Corollary 2. *The connected* feedback vertex set *face hitting set problem on graphs embeddable on a surface of bounded genus and having minimum vertex degree 3 admits a polynomial time approximation scheme.*

Without the connectivity constraint the problem becomes much easier. A PTAS for FHS follows directly from the discussion above and we leave the proof to the reader.

Theorem 4. *The face hitting set problem on graphs embeddable on a surface of bounded genus admits a polynomial time approximation scheme.*

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