

Improved Induced Matchings in Sparse Graphs

Rok Erman¹, Łukasz Kowalik², Matjaž Krnc¹, and Tomasz Waleń²

¹ Department of Mathematics, University of Ljubljana, Slovenia

² Institute of Informatics, University of Warsaw, Poland

Abstract. An induced matching in graph G is a matching which is an induced subgraph of G . Clearly, among two vertices with the same neighborhood (called *twins*) at most one is matched in any induced matching, and if one of them is matched then there is another matching of the same size that matches the other vertex. Motivated by this, Kanj, Pelsmajer, Schaefer and Xia [10] studied induced matchings in twinless graphs. They showed that any twinless planar graph contains an induced matching of size at least $\frac{n}{40}$ and that there are twinless planar graphs that do not contain an induced matching of size greater than $\frac{n}{27} + O(1)$. We improve both these bounds to $\frac{n}{28} + O(1)$, which is tight up to an additive constant. This implies that the problem of deciding an whether a planar graph has an induced matching of size k has a kernel of size at most $28k$. We also show for the first time that this problem is FPT for graphs of bounded arboricity.

Kanj et al. presented also an algorithm which decides in $O(2^{159\sqrt{k}} + n)$ -time whether an n -vertex planar graph contains an induced matching of size k . Our results improve the time complexity analysis of their algorithm. However, we show also a more efficient, $O(2^{25.5\sqrt{k}} + n)$ -time algorithm. Its main ingredient is a new, $O^*(4^l)$ -time algorithm for finding a maximum induced matching in a graph of branch-width at most l .

1 Introduction

An induced matching in graph G is a matching which is an induced subgraph of G . It was introduced by Stockmeyer and Vazirani [17] and motivated as the “risk-free” marriage problem (decide whether there exist at least k pairs such that each married person is compatible with no married person except the one he or she is married to). In this paper we study induced matchings in planar, or more generally bounded arboricity graphs, both from combinatorial and computational perspective.

1.1 Combinatorial Perspective

It is a natural and heavily researched area in extremal graph theory to establish lower bounds on the size of various structures in selected graph classes. For example, Nishizeki and Baybars [15] and later Biedl et al. [3] showed tight lower bounds on the size of matching in subclasses of planar graphs, Alon, Mubayi and Thomas [1] show a lower bound on the size of induced forest in sparse graphs.

Kanj, Pelsmajer, Schaefer and Xia [10] were first to consider the size of induced matchings in planar graphs. Graphs like $K_{1,n}$, $K_{2,n}$ show that general planar graphs have no nontrivial lower bound on the induced matching size. Kanj et al. observed that among two vertices with the same neighborhood (called *twins*) at most one is matched in any induced matching, and if one of them is matched then there is another matching of the same size that matches the other vertex. In particular, after removing one of two twins from a graph, the size of maximum induced matching does not change. Motivated by this, Kanj et al. studied induced matchings in twinless graphs. They showed that any twinless planar graph contains an induced matching of size at least $\frac{n}{40}$ and that there are twinless planar graphs that do not contain an induced matching of size greater than $\frac{n}{27} + O(1)$.

In this paper we improve both these bounds to $\frac{n}{28} + O(1)$, which is tight up to an additive constant. The lower bound is also generalized to bounded genus graphs, i.e. we show that any twinless graph of genus g contains an induced matching of size at least $\frac{2(n-10g+9)}{7(7+\sqrt{1+48g})}$. This improves an earlier bound $\frac{2(n-10g+10)}{13(7+\sqrt{1+48g})}$ of Kanj et al. [10].

Kanj et al. showed also that any planar graph of minimum degree 3 contains an induced matching of size $(n+8)/20$. We note that results of Nishizeki and Baybars [15] imply a better bound of $(n+2)/12$ for these graphs, as well as some better bounds for planar graphs of minimum degree 4 and 5.

Finally, we consider graphs of bounded arboricity, i.e. graphs whose edges set can be partitioned into $O(1)$ forests. For example, planar graphs have arboricity 3. Intuitively, graphs of bounded arboricity are uniformly sparse, since this class is equal to the class of graphs of bounded maximum density, where maximum density of a graph G is defined as $d^* = \max_{J \subseteq V, J \neq \emptyset} \frac{|E(G[J])|}{|J|}$ (see e.g. [11] for some relations between classes of sparse graphs). We show that any n -vertex twinless graph of arboricity c contains an induced matching of size $\Omega(\frac{1}{c}n^{1/c})$.

1.2 Computational Perspective

It was shown by Yannakakis [18] that deciding whether a planar graph contains an induced matching is NP-complete. Although the optimization problem is APX-complete in general [8], for planar graphs, and more generally for graphs that do not contain K_5 or $K_{3,3}$ as a minor, there is a PTAS working in $2^{O(1/\epsilon)}n$ time due to Baker [2] and Chen [4]. The PTAS (though with a worse running-time bound) can be generalized to H -minor-free graphs due to Demaine et al. [7].

In the area of parameterized complexity, one asks whether there is an algorithm for the induced matching problem which verifies whether an n -vertex graph contains an induced matching of size k in time $n^{O(1)}f(k)$. If so, then the problem is *fixed parameter tractable* (FPT in short). It is known that the problem is $W[1]$ -hard in general [13], which means that most likely the induced matching problem is not FPT. However, there is a $2^{O(\sqrt{k})}n^{O(1)}$ -time parameterized algorithm for H -minor-free graphs [6] due to Demaine, Fomin, Hajiaghayi and Thilikos. For the (smaller) class of planar graphs, Moser and Sikdar [12] showed

that the problem has a linear *kernel*, which means that one can reduce the problem in polynomial time to the same problem but on instance of size $O(k)$. The result of Kanj et al. mentioned in Section 1.1 implies that the size of the kernel is bounded by $40k$. Our results improve the bound further to $28k$.

We show, using the concept of eliminating twins, that the induced matching problem has a polynomial kernel for graphs of bounded arboricity. This implies that for such graphs there is an FPT algorithm with time complexity of the form $O(n + f(k))$. Since H -minor-free graphs have bounded arboricity, this generalizes the result of Demaine et al. [6] for the special case of the induced matching problem (the results in [6] are stated for all so-called bidimensional problems). This is also particularly interesting because there are classes of bounded arboricity graphs (like 4-regular graphs) for which the problem is APX-hard.

By using the linear kernel and planar separator technique, Kanj et al. showed an $O(2^{159\sqrt{k}} + n)$ -time parameterized algorithm. Our lower bound of the size of induced matching in twinless planar graphs improves the time complexity analysis of their algorithm to $O(2^{133\sqrt{k}} + n)$. However, we show also a more efficient, $O(2^{25.5\sqrt{k}} + n)$ -time algorithm based on the branch-width decomposition.

We also note that the proof of the lower bound gives a very practical and easy-to-implement algorithm for finding large induced forests in planar graphs (see Section 2.4).

2 Lower Bounds

In this section we present some lower bounds on the size of induced matching in subclasses of planar graphs and some classes of twinless sparse graphs. Our general approach is the same as that of Kanj et al., who used the following lemma. (We give a simple proof for completeness).

Lemma 1 (Kanj et al. [10]). *Let \mathcal{G} be a minor-closed family of graphs and let c be a constant such that any graph in \mathcal{G} is c -colorable. Moreover, let G be a graph from \mathcal{G} and let M be a matching in G . Then G contains an induced matching of size at least $|M|/c$.*

Proof. Let M be a matching in G . We obtain graph G' by removing all unmatched vertices and contracting all edges of matching M . Then $G' \in \mathcal{G}$. Color the vertices of G' in c colors. The largest color class in $V(G')$ is an independent set of size at least $|M|/c$. It corresponds to an induced matching in G of size at least $|M|/c$.

It follows that a lower bound on the size of a matching in a subclass \mathcal{H} of a minor-closed graph family implies a bound on the size of an induced matching in \mathcal{H} .

2.1 Planar Graphs of Large Minimum Degree

Kanj et al. showed that a planar graph of minimum degree 3 contains a matching of size at least $(n + 8)/5$. Using this with Lemma 1 and the Four Color Theorem

they obtained that any planar graph of minimum degree 3 contains an induced matching of size at least $(n + 8)/20$. However, this bound can be easily improved by using the following tight bounds for the size of matchings due to Nishizeki and Baybars.

Theorem 1 (Nishizeki and Baybars [15]). *Let G be an n -vertex planar graph of minimum degree δ and let M be a maximum cardinality matching in G . Then,*

- (i) if $\delta = 3$ and $n \geq 10$, then $|M| \geq \frac{n+2}{3}$,
- (ii) if $\delta = 4$ and $n \geq 16$, then $|M| \geq \frac{2n+3}{5}$,
- (iii) if $\delta = 5$ and $n \geq 34$, then $|M| \geq \frac{5n+6}{11}$.

Corollary 1. *Let G be an n -vertex planar graph of minimum degree δ and let M be a maximum cardinality induced matching in G . Then,*

- (i) if $\delta = 3$ and $n \geq 10$, then $|M| \geq \frac{n+2}{12}$,
- (ii) if $\delta = 4$ and $n \geq 16$, then $|M| \geq \frac{2n+3}{20}$,
- (iii) if $\delta = 5$ and $n \geq 34$, then $|M| \geq \frac{5n+6}{44}$. □

Let us note that the above bound $|M| \geq \frac{n+2}{12}$ is tight (up to an additive constant), as we show in Section 3. Let us also note that the paper of Nishizeki and Baybars contains also tight lower bounds on the matching size in graphs of minimum degree 3, 4 and 5 and *vertex connectivity* 1, 2, 3 and 4 and the corresponding bounds for induced matchings can be obtained.

2.2 Twinless Graphs of Bounded Genus

In this section we present an improved lower bound for the size of induced matchings in twinless graph of bounded genus. To this end, we are going to establish a lower bound on the size of a maximum cardinality matching in such graphs, and apply Lemma 1.

We begin with two simple observations.

Lemma 2. *Let uv be an edge in a maximum cardinality matching M in graph G and let I the set of unmatched vertices. If $N(u) \cap I \neq \emptyset$ and $N(v) \cap I \neq \emptyset$, then there is a vertex $x \in I$ such that $N(u) \cap I = N(v) \cap I = \{x\}$. In particular u, v and x form a triangle.*

Proof. Follows from the maximality of M .

Lemma 3. *For any twinless graph G there exists a maximum cardinality matching such that all 1-vertices of G are matched.*

Proof. Let M be a maximum cardinality matching in G and let I be the set of unmatched vertices. Suppose I contains a 1-vertex v . Let y be the sole neighbor of v . Then y is matched for otherwise M is not maximal. Let x be the vertex matched with y by M . Since G is twinless, x has degree at least two. We can now replace the edge xy with the edge vy in matching M and hereby decrease the number of vertices of degree 1 in I , without changing the size of M . After applying the above procedure to all 1-vertices we get the desired matching.

Now we are ready to show a lower bound on the size of a matching in twinless graphs of bounded genus.

Theorem 2. *Every n -vertex twinless graph G of genus g contains a matching of size $\frac{n+10(1-g)-1}{7}$.*

Proof. We will show that if G has no isolated vertices then G contains a matching of size $\frac{n+10(1-g)}{7}$. Since a twinless graph contains at most one isolated vertex then the claimed bound will follow. In what follows, M denotes the matching described in Lemma 3 and $I = V \setminus V(M)$. Note that I is an independent set, by the maximality of M . In what follows, we show a lower bound on $|M|$.

Let $M^\Delta \subset M$ be the set of edges in matching M that form triangles with vertices in I . Similarly, let $I^\Delta \subset I$ be the set of vertices in I that form triangles with edges of M . Let $I_2 \subset I$ denote the vertices of degree two in I and I_{3+} the vertices with degree three or more.

First note that Lemma 2 implies that

$$|M^\Delta| \geq |I^\Delta|. \tag{1}$$

Hence it suffices to bound $|M \setminus M^\Delta|$ from below. Let R be the set of vertices in $M \setminus M^\Delta$ that are adjacent to I . Note that by Lemma 2 each edge of $M \setminus M^\Delta$ has at most one endpoint in R , so

$$|M \setminus M^\Delta| \geq |R|. \tag{2}$$

Now we bound R from below in terms of $|I_2 \setminus I^\Delta|$. Let G_2 be a graph on the vertices R such that G_2 contains an edge uv when there is a vertex $x \in I_2 \setminus I^\Delta$ adjacent to both u and v . Observe that G_2 has genus at most g , because after subdividing its edges we get a subgraph of G . Hence, by Euler's Formula, $|E(G_2)| \leq 3|V(G_2)| - 6 + 6g$. Since $|E(G_2)| = |I_2 \setminus I^\Delta|$ and $V(G_2) = R$, we get $|R| \geq \frac{|I_2 \setminus I^\Delta| + 6 - 6g}{3}$. By (2),

$$|M \setminus M^\Delta| \geq \frac{|I_2 \setminus I^\Delta| + 6 - 6g}{3}. \tag{3}$$

Now we bound $|R|$ from below in terms of $|I_{3+} \setminus I^\Delta|$.

Let G_3 be the bipartite subgraph of G , on the vertices $R \cup (I_{3+} \setminus I^\Delta)$ and with edges incident with $I_{3+} \setminus I^\Delta$. Since G_3 is bipartite its embedding on an orientable surface of genus g has no triangles and we get the following bound on the number of its edges by Euler's Formula :

$$|E(G_3)| \leq 2|V(G_3)| - 4 + 4g.$$

By combining it with the fact that vertices in I_{3+} have degree at least 3, we can bound $|E(G_3)|$ as follows.

$$3 \cdot |I_{3+} \setminus I^\Delta| \leq |E(G_3)| \leq 2 \cdot (|R| + |I_{3+} \setminus I^\Delta|) - 4 + 4g.$$

It gives us $|R| \geq \frac{|I_{3+} \setminus I^\Delta| + 4 - 4g}{2}$ so with (2) we get

$$|M \setminus M^\Delta| \geq \frac{|I_{3+} \setminus I^\Delta| + 4 - 4g}{2}. \tag{4}$$

Now we are going to merge the bounds (3) and (4) into the following bound:

$$|M \setminus M^\Delta| \geq \frac{|I \setminus I^\Delta|}{5} + 2(1 - g). \tag{5}$$

When $|I_2 \setminus I^\Delta| \geq \frac{3}{5}|I \setminus I^\Delta|$, we get (5) from (3). Similarly, when $|I_2 \setminus I^\Delta| \leq \frac{3}{5}|I \setminus I^\Delta|$ we get (5) from (4) by replacing $|I_{3+} \setminus I^\Delta|$ by $|I \setminus I^\Delta| - |I_2 \setminus I^\Delta|$.

By combining (1) and (5) we get

$$\begin{aligned} |M| &= |M \setminus M^\Delta| + |M^\Delta| \geq \frac{|I \setminus I^\Delta|}{5} + 2(1 - g) + |I^\Delta| = \\ &= \frac{|I|}{5} + 2(1 - g) + \frac{4}{5}|I^\Delta| \geq \frac{|I|}{5} + 2(1 - g). \end{aligned}$$

Since $I = n - 2|M|$, we get $|M| \geq \frac{n+10(1-g)}{7}$, as desired.

By using Lemma 1, Four Color Theorem, and Heawood’s Theorem (which states that any graph of genus $g > 0$ is $\lfloor (7 + \sqrt{1 + 48g})/2 \rfloor$ -colorable), we get the following corollaries.

Corollary 2. *Every n -vertex twinless graph of genus g contains an induced matching of size $(2n + 20(1 - g) - 2)/(49 + 7\sqrt{1 + 48g})$.*

Corollary 3. *Every n -vertex twinless planar graph contains an induced matching of size $\frac{n+9}{28}$.*

2.3 Twinless Sparse Graphs

In this section we focus on graphs of bounded arboricity. Let $\text{arb}(G)$ and $d^*(G)$ denote arboricity and the maximum density of graph G , respectively.

Theorem 3. *Any n -vertex twinless graph of maximum density d^* contains a matching of size $\Omega(n^{1/\lceil d^* \rceil})$.*

Proof. Let G be an n -vertex twinless graph of maximum density d^* and let M be a maximum cardinality matching in G . Denote $d = \lceil d^* \rceil$. Let I denote the independent set $V(G) \setminus V(M)$. Let us partition I into vertices of degree at least $d + 1$ and vertices of degree at most d , denoted by $I_{d+1\uparrow}$ and $I_{d\downarrow}$ respectively.

Let $E(V(M), I_{d+1\uparrow})$ denote set the edges between $V(M)$ and $I_{d+1\uparrow}$. Then

$$(d + 1)|I_{d+1\uparrow}| \leq |E(V(M), I_{d+1\uparrow})| \leq d(2|M| + |I_{d+1\uparrow}|),$$

where the second inequality follows from the fact that $E(V(M), I_{d+1\uparrow})$ induces a graph of maximum density at most d^* . By rearranging we get

$$|M| = \Omega(|I_{d+1\uparrow}|/d). \tag{6}$$

On the other hand, since G is twinless,

$$|I_{d\downarrow}| \leq \sum_{i=0}^d \binom{2|M|}{i} = O((2|M|)^d).$$

Hence, $|M| = \Omega(|I_{d\downarrow}|^{1/d})$. Together with (6) we get the claimed bound.

Theorem 4. *Any n -vertex twinless graph G of maximum density d^* contains an induced matching of size $\Omega(\frac{1}{d^*} \cdot n^{1/\lceil d^* \rceil})$.*

Proof. Let M be a maximum cardinality matching in G . Similarly as in Lemma 1 we consider graph G' which is obtained from G by removing all unmatched vertices and contracting all edges of matching M . Consider any set of vertices $S' \subseteq V(G')$. Then S' corresponds to a set $S \subseteq V(G)$, i.e. S' is obtained from S by identifying endpoints of edges of M . Then $|E(G'[S'])| \leq |E(G[S])|$ and $|S'| = |S|/2$, hence $\frac{|E(G'[S'])|}{|S'|} \leq 2 \frac{|E(G[S])|}{|S|} \leq 2d^*(G)$. It follows that $G'[S']$ contains a vertex of degree at most $4d^*$. Since S' was chosen arbitrarily we infer that G' is $\lfloor 4d^* \rfloor$ -degenerate and hence $(\lfloor 4d^* \rfloor + 1)$ -vertex-colorable (by a simple algorithm which chooses a vertex v with the smallest degree, removes it from the graph, colors the resulting graph recursively and assigns to v the smallest color which is unused by v 's neighbors). By choosing the subset of M corresponding to the biggest color class in G' we obtain an induced matching of size $|M|/(\lfloor 4d^* \rfloor + 1) = \Omega(|M|/d^*)$. Since $|M| = \Omega(n^{1/\lceil d^* \rceil})$ by Theorem 3, the claim follows¹.

Although it is more convenient to prove the above result referring to maximum density, we feel that arboricity is more often used as a measure of graph sparsity. However, we can easily reformulate Theorem 4 using the following lemma, which follows from the Nash-Williams Theorem [14].

Lemma 4. *For any graph G with at least one edge, $\lceil d^*(G) \rceil < \text{arb}(G)$. □*

Corollary 4. *Any n -vertex twinless graph G of arboricity c contains an induced matching of size $\Omega(\frac{1}{c}n^{1/c})$.*

Now, if we want to decide whether an n -vertex graph of arboricity bounded by a constant c contains an induced matching of size k , we begin by eliminating twins in linear time (see [10]). Let H be the resulting graph. From Theorem 4 we know that H contains an induced matching of size $\alpha \cdot |V(H)|^{1/c}$, for some constant α . Hence if $k \leq \alpha \cdot |V(H)|^{1/c}$ we answer “yes”, and otherwise we know that $|V(H)| = O(k^c)$ and hence $|E(H)| = O(ck^c) = O(k^c)$. Since we can find a maximum induced matching in H by the exhaustive search, the overall algorithm runs in time $O(n + \exp(k^c))$. We note that one can also *find* induced matchings of size k within this time bound (see Section 4). We summarize it with the following corollary.

¹ Independently, Kanj et al. [10] in the journal version observed that any matching M in a graph of maximum density d^* contains an induced matching of size at least $|M|/(4d^* - 1)$.

Corollary 5. *The induced matching problem for graphs of arboricity bounded by $c = O(1)$ has kernel of size $O(k^c)$. In particular, this problem is fixed parameter tractable for these graphs. \square*

2.4 A Practical Algorithm for Planar Graphs

The discussion in Section 1.2 shows that if we want to find large induced matching in a planar graph *in practice*, then most likely we should use PTAS of Baker or Chen, since they are linear-time (for any fixed approximation ratio) and their time complexities do not hide large constants. However, these algorithms are still very complicated and hard to implement.

Here we want to note that the proof technique of Corollary 3 (introduced by Kanj et al. [10]) can be turned into the following algorithm. Given an input graph G , remove twins, find a maximum matching M , remove the unmatched vertices, contract the edges from M , color the resulting graph and choose the subset of M which corresponds to the biggest color class.

Eliminating twins can be easily done in linear time (see [10]). Finding a maximum matching using Hopcroft-Karp algorithm works in $O(n^{3/2})$ -time for planar graphs and is implemented in many libraries. Since so far there is no fast and simple algorithm for 4-coloring planar graphs, we use 5 colors instead and then the coloring can be found by a simple linear-time algorithm (see e.g. [5]). Because of using 5 colors instead of 4 the constant 28 increases to 35. Then we get a $O(n^{3/2})$ -time algorithm which always finds an induced matching of size at least $n'/35$, where n' is the number of pairwise different vertex neighborhoods in G . If one insists on linear-time, a maximal matching can be used instead of maximum matching M . (Then the constant 35 doubles because any maximal matching has size at least $|M|/2$.)

3 An Upper Bound

In this section we show that the bound in Corollary 3 is tight, up to an additive constant. Namely, we show the following.

Theorem 5. *For any $n_0 \in \mathbb{N}$ there is an n -vertex twinless planar graph G such that $n > n_0$ and any induced matching in G is of size at most $\frac{n}{28} + O(1)$.*

Proof. In what follows we describe an n -vertex planar graph with maximum induced matching of size at most $\frac{n}{28} + O(1)$. It will be clear from our construction that the number of vertices can be made arbitrarily large.

We begin with a graph T_k , which consists of k copies of K_4 and some additional edges. We obtain T_k from the graph drawn in Fig 1 by identifying vertex v_1 with w_1 , v_2 with w_2 and so on. It is easy to see that the resulting graph is still planar, since the cylinder is homeomorphic to a subset of the plane. Also, T_k is twinless.

Note that T_k has $4k$ vertices, $8k + O(1)$ triangular faces and $12k + O(1)$ edges. Now, we build a new graph G_k by extending T_k , as follows:

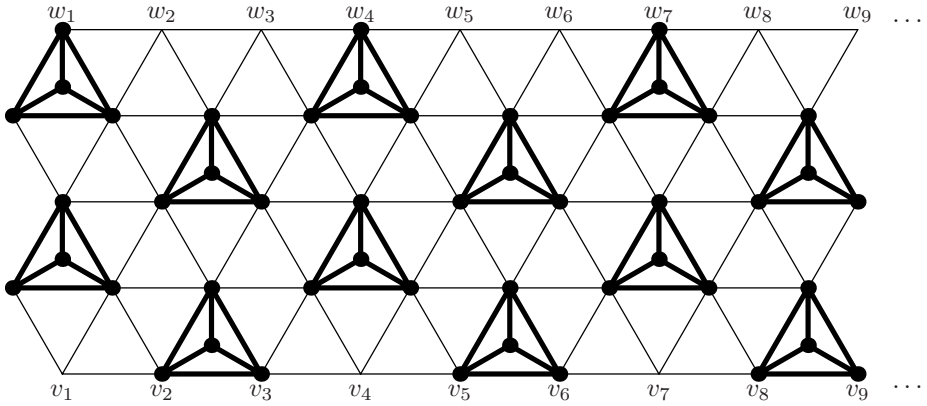


Fig. 1. Building T_k : arranging k copies of K_4 in 4 layers of triangles

- (i) For each 3-face xyz of T_k add a 3-vertex v adjacent to x, y and z ,
- (ii) For each edge xy of T_k add a 2-vertex v adjacent to x and y .
- (iii) For each vertex x of T_k add a 1-vertex v adjacent to x .

Note that by adding vertices like this we do not introduce twins and the graph stays planar. It is clear that G_k has $4k + 8k + 12k + 4k + O(1) = 28k + O(1)$ vertices. Moreover, every edge of G_k is incident with a vertex of one of the k copies of K_4 . On the other hand, if M is an induced matching in G_k , vertices of each copy of K_4 are incident with at most one edge of M . It follows that $|M| \leq k$, so $|M| \leq |V(G_k)|/28 + O(1)$.

In a very similar way, we get that Corollary 1 (i) is also tight.

Corollary 6. *For any $n_0 \in \mathbb{N}$ there is an n -vertex planar graph G of minimum degree 3 such that $n > n_0$ and any induced matching in G is of size at most $\frac{n}{12} + O(1)$.*

Proof. Just remove the 1- and 2-vertices from the graphs constructed in the proof of Theorem 5.

4 An Algorithm Based on Branch-Width

In this section we discuss an algorithm that, given a planar graph G on n vertices and an integer k , either computes a induced matching of size $\geq k$, or concludes that there is no such induced matching. The algorithm requires $O(n + 2^{25.5\sqrt{k}})$ time.

4.1 Preliminaries

A *branch decomposition* of a graph G is a pair (T, r) , where T is a tree with vertices of degree 1 or 3 and r is a bijection from $E(G)$ to the set of leaves of

T . The *order function* $\omega : E(T) \rightarrow 2^{V(G)}$ of a branch decomposition maps every edge e of T to a subset of vertices $\omega(e) \subseteq V(G)$ as follows. The set $\omega(e)$ consists of all vertices of $V(G)$ such that for every vertex $v \in \omega(e)$ there exist two edges $f_1, f_2 \in E(G)$ that are incident with v and the leaves $\tau(f_1), \tau(f_2)$ are in different components of $T - \{e\}$. The *width* of (T, τ) is equal to $\max_{e \in E(T)} |\omega(e)|$ and the branch-width of G , $bw(G)$, is the minimum width over all branch decompositions of G . A set $D \subseteq V(G)$ is a *dominating set* in a graph G if every vertex in $V(G) - D$ is adjacent to a vertex in D .

Now we will introduce a few lemmas, that will connect induced matching problem with branch-width decomposition.

Lemma 5. *In any graph without isolated vertices if D is a minimum dominating set and M is a maximum cardinality matching, then $|D| \leq |M|$.*

Proof. Let $V(M)$ be the set of the vertices of edges from M . We will describe a dominating set of size $|M|$. Let us observe that every vertex is adjacent to some vertex from the matching M , for otherwise the matching M is not maximal. Hence $V(M)$ is dominating. However, it is sufficient to choose just one endpoint for each edge of M . Then clearly all vertices of $V(M)$ are dominated, but we need to be careful about which endpoint we choose to dominate the unmatched vertices. Namely, for each edge of $uv \in M$ we choose its endpoint which has unmatched neighbors (or any endpoint if both have only matched neighbors). It may happen that both endpoints have unmatched neighbors but then by Lemma 2, $N(u) - V(M) = N(v) - V(M) = \{x\}$ for some x , so it does not matter whether we choose u or v .

Lemma 6 ([9]). *For any planar graph G with dominating set D ,*

$$bw(G) \leq 3\sqrt{4.5 \cdot |D|}.$$

Lemma 7. *For any planar graph G with maximum induced matching I ,*

$$bw(G) \leq 3\sqrt{18 \cdot |I|} \cong 12.7 \cdot \sqrt{|I|}.$$

Proof. From Lemma 1 we know that the maximum cardinality matching M of G has size $|M| \leq 4|I|$. Combining lemmas 5 and 6 we get

$$bw(G) \leq 3\sqrt{18 \cdot |I|} \cong 12.7 \cdot \sqrt{|I|}.$$

4.2 Algorithm Outline

Let G be the input planar graph on n vertices and let k be the size of induced matching we look for. As long as there is a pair of twins in G we remove one of them. This can be implemented in $O(n)$ time (see [10]). Let n' be the number of vertices of the resulting graph H . Now we describe an algorithm which *decides* whether H has an induced matching of size k .

- Step 1. If $n' > 28k$ we can answer *True*, since the induced matching of size at least k exists as a consequence of Corollary 3. Otherwise we proceed with the next step and we can assume our graph has $O(k)$ vertices.
- Step 2. Compute the optimal branch-decomposition of graph H . Using algorithm of Seymour and Thomas [16] this step requires $O(k^4)$ time. If $bw(G) \geq 12.7\sqrt{k}$ then as a consequence of Lemma 7 we can return answer *True*. Otherwise we proceed with the next step.
- Step 3. Use the dynamic programming approach for finding a maximum cardinality induced matching in graph G . In Section 4.3 we present an algorithm that solves this problem on graphs with branch-decomposition of width $\leq l$ in $O(m \cdot 4^l)$ time where m is the number of edges in a graph. This step requires $O(k \cdot 4^{12.7\sqrt{k}}) = O(2^{25.5\sqrt{k}})$ time, since $l \leq 12.7\sqrt{k}$.

If we want to *find* the matching, in Step 1, we check whether $n' > 70k$ and if so we find an induced matching of size k by the linear-time algorithm from Section 2.4. Otherwise, we know that our graph has $O(k)$ vertices. Then we find the matching using the self-reducibility approach. Let $T(n')$ denote the time complexity of the decision algorithm described above. First, using the decision problem we determine the size s of the maximum induced matching in H . Then we can test in time $O(k + T(n'))$ whether a chosen edge e belongs to some induced matching of size s : just remove e and the adjacent vertices and test whether there is an induced matching of size $s - 1$. If that is the case, find the induced matching of size $s - 1$ recursively, and otherwise we put back the removed vertices (and their incident edges) and we test another edge, which has not been excluded so far. Clearly this procedure takes overall $O(|E(H)|(k + T(n')))$ time, which is $O(k^2 \cdot 4^{12.7\sqrt{k}}) = O(2^{25.5\sqrt{k}})$.

Theorem 6. *For any planar graph G on n vertices and an integer k , there is an $O(n + 2^{25.5\sqrt{k}})$ -time algorithm which finds in G an induced matching of size k if and only if such a matching exists.*

4.3 Dynamic Programming on Graphs of Bounded Branch-Width

Our approach here is based on the algorithm for dominating set proposed by Fomin and Thilikos in [9]. We closely follow the notation and presentation from their paper.

Let (T', τ) be a branch decomposition of a graph G with m edges, let $\omega' : E(T') \rightarrow 2^{V(G)}$ be the order function of (T', τ) . The tree T' is unrooted, so we build its rooted version T , by choosing an edge xy in T' , putting new vertex v of degree 2 on this edge and making v adjacent to new vertex r , which is the new root of tree T . For every edge $f \in E(T) \cap E(T')$ we define $\omega(f) = \omega'(f)$, and for edges $\omega(xv) = \omega(vy) = \omega'(xy)$ and $\omega(rv) = \emptyset$.

For an edge f of T we define E_f (V_f) as the set of edges (vertices) of tree T that are “below” f , i.e. the set of all edges (vertices) g such that every path containing g and vr in T contains f . Every edge f of T that is not incident to a leaf has two children that are edges of E_f incident to f .

For every edge f of T we color the vertices of $\omega(f)$ in three colors $\{0, 1, 2\}$. We say that an induced matching M is *valid* for a coloring $c : \omega(f) \rightarrow \{0, 1, 2\}$ when for every $x \in \omega(f)$:

- if $c(x) = 2$, then $x \in V(M)$,
- if $c(x) = 1$, then $x \notin V(M)$, but it can be adjacent to some vertex of the matching,
- if $c(x) = 0$, then $x \notin V(M)$, and x is not adjacent to a vertex of the matching (for all $y \in N(x)$, $y \notin V(M)$).

For every edge f of T we use a mapping:

$$A_f : \{0, 1, 2\}^{\omega(f)} \rightarrow \mathbb{N} \cup \{-\infty\}$$

For a coloring $c \in \{0, 1, 2\}^{\omega(f)}$, the value $A_f(c)$ denotes the largest cardinality of an induced matching in the subgraph G_f of G that is defined by inducing the edge set:

$$\{\tau^{-1}(x) : x \in V_f \text{ and } x \text{ is a leaf of } T'\}$$

subject to the condition that the matching is valid with coloring c .

We define $A_f(c) = -\infty$ if there is no valid induced matching in G_f with coloring c .

Let f be a non-leaf edge of T and let f_1, f_2 be the children of f . Define $X_1 = \omega(f) - \omega(f_2)$, $X_2 = \omega(f) - \omega(f_1)$, $X_3 = \omega(f) \cap (\omega(f_1) \cap \omega(f_2))$, $X_4 = (\omega(f_1) \cap \omega(f_2)) - \omega(f)$.

Let us note that for $1 \leq i \neq j \leq 4$ we have $X_i \cap X_j = \emptyset$. Moreover, $\omega(f) = X_1 \cup X_2 \cup X_3$, $\omega(f_1) = X_1 \cup X_3 \cup X_4$, and $\omega(f_2) = X_2 \cup X_3 \cup X_4$.

We say that a coloring c of $\omega(f)$ is *formed* from coloring c_1 of $\omega(f_1)$ and coloring c_2 of $\omega(f_2)$ if

- (F1) For every $x \in X_1$, $c(x) = \max(\{c_1(x)\} \cup \{c_2(y) - 1 : y \in \omega(f_2) \cap N(x)\})$,
- (F2) For every $x \in X_2$, $c(x) = \max(\{c_2(x)\} \cup \{c_1(y) - 1 : y \in \omega(f_1) \cap N(x)\})$,
- (F3) For every $x \in X_3$, $c(x) = \max\{c_1(x), c_2(x)\}$,
- (F4) For every $x \in X_3 \cup X_4$, $c_1(x) + c_2(x) \leq 2$.

If coloring c of $\omega(f)$ is formed from colorings c_1 of $\omega(f_1)$ and coloring c_2 of $\omega(f_2)$, then $A_f(c) \geq A_{f_1}(c_1) + A_{f_2}(c_2)$.

We compute functions A_f from leaves of T by bottom-up fashion.

For a leaf edge $f \in E(T)$, and its leaf node $v \in V(T)$ corresponding to an edge $xy \in E(G)$ we define function A_f as follows:

- if $c(x) \leq 1$ and $c(y) \leq 1$, then $A_f(c) = 0$,
- if $c(x) = c(y) = 2$ then $A_f(c) = 1$,
- otherwise $A_f(c) = -\infty$.

For non-leaf edges f of T we can compute function A_f as follows (f_1, f_2 denote the children of f):

$$A_f(c) = \max\{A_{f_1}(c_1) + A_{f_2}(c_2) \mid c_1, c_2 \text{ forms } c\}$$

If coloring c can not be formed from colorings c_1, c_2 of f_1, f_2 , then we define $A_f(c) = -\infty$.

Let x_i denote $|X_i|$. The number of pairs (c_1, c_2) of colorings that can form a coloring c , can be bounded by

$$3^{x_1+x_2} \cdot 6^{x_3+x_4}$$

since there are three possible colorings of vertices $u \in X_1 \cup X_2$, and six pairs of colorings of vertices $u \in X_3 \cup X_4$, that is:

$$(c_1(u), c_2(u)) \in \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (0, 2)\}.$$

We can observe that if for some coloring c , $A_f(c) \neq -\infty$, we change coloring c into c' by replacing the color of a vertex x with $c(x) = 0$, to a new color $c'(x) = 1$, then $A_f(c') \neq -\infty$, and $A_f(c') \geq A_f(c)$. This leads us to an observation, that during computation of function A_f , instead of investigating pairs of colorings (from sets X_3, X_4) $\{(0, 1), (1, 0), (1, 1)\}$, it is sufficient to check only one pair, namely $(1, 1)$.

We can compute function A_f using a slightly modified formula:

$$A_f(c) = \max\{A_{f_1}(c_1) + A_{f_2}(c_2) \mid c_1, c_2 \text{ satisfies (F1), (F2), (F3) and (F4')}\}$$

where condition (F4') is defined as follows:

$$(F4') \text{ For every } x \in X_3 \cup X_4, (c_1(x), c_2(x)) \in \{(0, 0), (1, 1), (2, 0), (0, 2)\}.$$

The complexity of computing A_f , with this optimization, can be bounded by:

$$3^{x_1+x_2} \cdot 4^{x_3+x_4}$$

Let $l = bw(G)$, and $x_i = |X_i|$, the values x_i are bounded by following inequalities:

$$\begin{aligned} x_1 + x_2 + x_3 &= |\omega(f)| \leq l \\ x_1 + x_3 + x_4 &= |\omega(f_1)| \leq l \\ x_2 + x_3 + x_4 &= |\omega(f_2)| \leq l \end{aligned}$$

The maximum value of linear functions $\log_4 3(x_1 + x_2) + x_3 + x_4$ subject to constraints on x_i is l (which is achieved for $x_1 = x_2 = 0, x_3 = x_4 = 0.5l$). Hence the cost of computing function A_f for a single edge f can be bounded by $O(4^l)$. Since we have to compute function A_f for each edge of tree T , the total time complexity is $O(m \cdot 4^l)$. The size of the maximum induced matching is stored in $A_{vr}(\epsilon)$, (where ϵ is the coloring of the empty set). The matching itself can be easily retrieved using standard methods. This gives us the following theorem.

Theorem 7. *For a graph G on m edges and with given a branch-decomposition of width l , the maximum induced matching of G can be computed in $O(m \cdot 4^l)$ time.*

We finish this section by noting that there is also an $O^*(4^t)$ -time algorithm by Moser and Sikdar [12], where t denotes the tree-width of the input graph. It follows that our algorithm improves on this result, since for any graph G of at least 3 edges, $bw(G) \leq tw(G) + 1 \leq \frac{3}{2}bw(G)$ and the existing algorithms for finding optimal branch-decomposition are regarded as more practical than those for finding optimal tree-decomposition.

Acknowledgments

We are grateful to Fedor Fomin for helpful hints. We are indebted to anonymous IWPEC referees for careful reading and numerous helpful comments.

The work was partially supported by Polish-Slovenian project ‘‘Graph colorings and their applications’’. L. Kowalik and T. Waleń were also supported by a grant from the Polish Ministry of Science and Higher Education, projects N N206 355636 and N206 004 32/0806.

References

1. Alon, N., Mubayi, D., Thomas, R.: Large induced forests in sparse graphs. *J. Graph Theory* 38(3), 113–123 (2001)
2. Baker, B.S.: Approximation algorithms for NP-complete problems on planar graphs. *J. ACM* 41(1), 153–180 (1994)
3. Biedl, T.C., Demaine, E.D., Duncan, C.A., Fleischer, R., Kobourov, S.G.: Tight bounds on maximal and maximum matchings. *Discrete Mathematics* 285(1-3), 7–15 (2004)
4. Chen, Z.-Z.: Efficient approximation schemes for maximization problems on $K_{3,3}$ -free graphs. *J. Algorithms* 26(1), 166–187 (1998)
5. Chiba, N., Nishizeki, T., Saito, N.: A linear algorithm for five-coloring a planar graph. *J. Algorithms* 2, 317–327 (1981)
6. Demaine, E.D., Fomin, F.V., Hajiaghayi, M.T., Thilikos, D.M.: Subexponential parameterized algorithms on bounded-genus graphs and H -minor-free graphs. *J. ACM* 52(6), 866–893 (2005)
7. Demaine, E.D., Hajiaghayi, M.T., Kawarabayashi, K.: Algorithmic graph minor theory: Decomposition, approximation, and coloring. In: *Proc. FOCS 2005*, pp. 637–646. IEEE Computer Society Press, Los Alamitos (2005)
8. Duckworth, W., Manlove, D., Zito, M.: On the approximability of the maximum induced matching problem. *J. Discrete Algorithms* 3(1), 79–91 (2005)
9. Fomin, F.V., Thilikos, D.M.: Dominating sets in planar graphs: Branch-width and exponential speed-up. *SIAM J. Comput.* 36(2), 281–309 (2006)
10. Kanj, I.A., Pelsmajer, M.J., Xia, G., Schaefer, M.: On the induced matching problem. In: *Proc. STACS 2008*, pp. 397–408 (2008); Journal version to appear in *J. Comput. Sys. Sci.*
11. Kowalik, L.: Approximation scheme for lowest outdegree orientation and graph density measures. In: Asano, T. (ed.) *ISAAC 2006*. LNCS, vol. 4288, pp. 557–566. Springer, Heidelberg (2006)
12. Moser, H., Sikdar, S.: The parameterized complexity of the induced matching problem in planar graphs. *Discrete Applied Mathematics* 157, 715–727 (2009)

13. Moser, H., Thilikos, D.M.: Parameterized complexity of finding regular induced subgraphs. In: Broersma, H., Dantchev, S.S., Johnson, M., Szeider, S. (eds.) Proc. ACiD 2006. Texts in Algorithmics, vol. 7, pp. 107–118. King’s College, London (2006)
14. Nash-Williams, C.S.J.A.: Decomposition of finite graphs into forests. *Journal of the London Mathematical Society* 39, 12 (1964)
15. Nishizeki, T., Baybars, I.: Lower bounds on the cardinality of the maximum matchings of planar graphs. *Discrete Mathematics* 28(3), 255–267 (1979)
16. Seymour, P.D., Thomas, R.: Call routing and the ratcatcher. *Combinatorica* 14(2), 217–241 (1994)
17. Stockmeyer, L.J., Vazirani, V.V.: NP-completeness of some generalizations of the maximum matching problem. *Inf. Process. Lett.* 15(1), 14–19 (1982)
18. Yannakakis, M.: Node- and edge-deletion NP-complete problems. In: Proc. STOC 1978, pp. 253–264. ACM Press, New York (1978)