Chapter 3
Credibilistic Portfolio Selection

Credibilistic portfolio selection deals with fuzzy portfolio selection by means of credibility theory. Fuzzy portfolio selection problem was researched from 1990s. Early researchers employed possibility as the basic measure of the occurrence of a fuzzy event and most of them devoted themselves to extending Markowitz's mean-variance selection idea. However, possibility measure is not self-dual. By using possibility, when the investors know the possibility level of a portfolio reaching a target return, they cannot know the possibility level of the opposite event, i.e., the event of this portfolio not being able to achieve the target return! This will confuse and worry the decision maker. Therefore, Huang proposed that we should use the self-dual credibility as the basic measure of the occurrence of a fuzzy event and study the fuzzy portfolio selection problems. To provide an instinct and observable information about loss amount and to accurately evaluate the loss degree, Huang [38] proposed that we should evaluate each likely loss level and the loss occurrence chance instead of just focusing on the average information of loss. Looking at loss from a panoramic perspective, Huang provided a general definition of risk, i.e., the risk curve, and proposed a mean-risk model based on this new definition. In addition, Huang proposed a spectrum of simplified versions of the risk and proposed a system of credibilistic portfolio selection models [41] including mean-risk model [38], β-return-risk model [27], credibility minimization model, mean-variance model [33], mean-semivariance model [37], and entropy optimization model [39].

This chapter will first introduce some necessary knowledge about credibility theory. The reason for adopting credibility measure rather than possibility measure is given. Then we will introduce the definitions of risk and the credibilistic portfolio selection models. Crisp equivalents of the fuzzy models in some special cases will also be presented. After that, we will provide a general solution algorithm for solving the credibilistic portfolio selection models.

X. Huang: Portfolio Analysis: From Probab. to Credibilistic, STUDFUZZ 250, pp. 61–115.
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3.1 Fundamentals of Credibility Theory

In reality, besides randomness, there are many fuzzy phenomena. For example, a "beautiful" girl in many situation is not a very clear concept. In order to describe fuzziness, the concept of fuzzy set was first proposed by Zadeh [95] in 1965 via membership function. Furthermore, to measure a fuzzy event, Zadeh [96] proposed possibility measure. Although possibility is a widely used measure, it is not self-dual. However, for a measure, self-duality property is extremely important. In order to define a self-dual measure, Liu and Liu [55] proposed credibility measure. An axiomatic credibility theory was founded by Liu [58] in 2004 and refined by Liu [60] in 2007. Credibility theory has been fairly well applied in many application areas.

Credibility and Credibility Space

Definition 3.1 *(Liu [60]). Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ, i.e., the largest σ-algebra over Θ. Each element in $\mathcal{P}(\Theta)$ is called an event. The set function* Cr *is called a credibility measure if*
(Axiom 1) (Normality) $\mathrm{Cr}\{\Theta\} = 1$;
(Axiom 2) (Monotonicity) $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$ *whenever* $A \subset B$;
(Axiom 3) (Self-duality) $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ *for any event* A.
(Axiom 4) (Maximality) $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ *for any events* $\{A_i\}$ *with* $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

The value of $\mathrm{Cr}\{A\}$ indicates the level that the event A will occur.

For example, let $\Theta = \{\theta_1, \theta_2\}$. There are only four events: $\emptyset, \{\theta_1\}, \{\theta_2\}, \Theta$. Define $\mathrm{Cr}\{\emptyset\} = 0, \mathrm{Cr}\{\theta_1\} = 0.4, \mathrm{Cr}\{\theta_2\} = 0.6$, and $\mathrm{Cr}\{\Theta\} = 1$. Then the set function Cr is a credibility measure because it satisfies the four axioms.

Let Θ be a nonempty set, $\mathcal{P}$ the power set of Θ, and Cr the credibility measure. From Axioms 1 and 3 we know $\mathrm{Cr}\{\emptyset\} = 0$. From Axiom 2 we know $0 \le \mathrm{Cr}\{A\} \le 1$ for any $A \in \mathcal{P}$ because $\emptyset \subset A \subset \Theta$. That is, the credibility value of a fuzzy event is in the interval $[0, 1]$.

Definition 3.2 *(Liu [60]). Let Θ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ, and* Cr *a credibility measure. Then the triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called a credibility space.*

Fuzzy Variable

Definition 3.3. *A fuzzy variable is defined as a function from a credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to the set of real numbers.*

Remark 3.1. Since $\mathcal{P}(\Theta)$ is the power set of Θ (i.e., the collection of all the subsets of Θ), and a fuzzy variable ξ is a function on a credibility space, for any set B of real numbers, the set

$$\{\xi \in B\} = \{\theta \in \Theta | \xi(\theta) \in B\}$$

is always an element in $\mathcal{P}$. That is, the fuzzy variable is a measurable function and $\{\xi \in B\}$ is an event.

Definition 3.4. *Let ξ_1 and ξ_2 be two fuzzy variables defined on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. We say $\xi_1 = \xi_2$ if $\xi_1(\theta) = \xi_2(\theta)$ for almost all $\theta \in \Theta$.*

Membership Function and Credibility Inversion Theorem

The membership function was first introduced by Zadeh [95] in 1965. In the credibility theory, membership function was defined via credibility.

Definition 3.5 *(Liu [60]). Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then its membership function is derived from the credibility measure by*

$$\mu(t) = (2\mathrm{Cr}\{\xi = t\}) \wedge 1, \quad t \in \Re.$$

If we have got the membership function of a fuzzy variable ξ first, how can we know the credibility degree of a fuzzy event? The following inversion theorem gives the answer.

Theorem 3.1 *(Credibility Inversion Theorem, Liu and Liu [55]). Let ξ be a fuzzy variable with membership function μ. Then for any set A of real numbers, we have*

$$\mathrm{Cr}\{\xi \in A\} = \frac{1}{2}\left(\sup_{t \in A} \mu(t) + 1 - \sup_{t \in A^c} \mu(t)\right). \tag{3.1}$$

Proof: If $\mathrm{Cr}\{\xi \in A\} \le 0.5$, we know from Axiom 2 that $\mathrm{Cr}\{\xi = t\} \le 0.5$ for each $t \in A$. According to Axiom 4 we have

$$\mathrm{Cr}\{\xi \in A\} = \frac{1}{2}(\sup_{t \in A}(2\mathrm{Cr}\{\xi = t\} \wedge 1)) = \frac{1}{2}\sup_{t \in A} \mu(t). \tag{3.2}$$

Since the credibility measure is self-dual, we have $\mathrm{Cr}\{\xi \in A^c\} \ge 0.5$, and $\sup_{t \in A^c} \mathrm{Cr}\{\xi = t\} \ge 0.5$. Therefore

$$\sup_{t \in A^c} \mu(t) = \sup_{t \in A^c}(2\mathrm{Cr}\{\xi = t\} \wedge 1) = 1. \tag{3.3}$$

It follows from (3.2) and (3.3) that (3.1) holds.

If $\text{Cr}\{\xi \in A\} \geq 0.5$, we have $\text{Cr}\{\xi \in A^c\} \leq 0.5$ because the credibility measure is self-dual. From the result of the first case we have

$$\text{Cr}\{\xi \in A\} = 1 - \text{Cr}\{\xi \in A^c\} = 1 - \frac{1}{2}\left(\sup_{t \in A^c} \mu(t) + 1 - \sup_{t \in A} \mu(t)\right)$$
$$= \frac{1}{2}\left(\sup_{t \in A} \mu(t) + 1 - \sup_{t \in A^c} \mu(t)\right).$$

The theorem is proven.

Example 3.1. Let ξ be a fuzzy variable with membership function μ. Then it follows from Theorem 3.1 that the following equations hold:

$$\text{Cr}\{\xi = t\} = \frac{1}{2}\Big(\mu(t) + 1 - \sup_{y \neq t} \mu(y)\Big), \quad \forall t \in \Re; \tag{3.4}$$

$$\text{Cr}\{\xi \leq t\} = \frac{1}{2}\left(\sup_{y \leq t} \mu(y) + 1 - \sup_{y > t} \mu(y)\right), \quad \forall t \in \Re; \tag{3.5}$$

$$\text{Cr}\{\xi \geq t\} = \frac{1}{2}\left(\sup_{y \geq t} \mu(y) + 1 - \sup_{y < t} \mu(y)\right), \quad \forall t \in \Re. \tag{3.6}$$

Especially, if μ is a continuous function, we have

$$\text{Cr}\{\xi = t\} = \frac{\mu(t)}{2}, \quad \forall t \in \Re. \tag{3.7}$$

Remark 3.2. A fuzzy variable has a unique membership function, but a membership function may produce multiple fuzzy variables. For example, let $\Theta = \{\theta_1, \theta_2\}$ and $\mu(\theta_1) = \mu(\theta_2) = 1$. It can be easily proven that $(\Theta, \mathcal{P}, \text{Cr})$ is a credibility space. Define

$$\xi_1(\theta) = \begin{cases} 0, & \text{if } \theta = \theta_1 \\ 1, & \text{if } \theta = \theta_2, \end{cases} \qquad \xi_2(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_1 \\ 0, & \text{if } \theta = \theta_2. \end{cases}$$

We can see that though the fuzzy variables ξ_1 and ξ_2 have the same membership function, i.e., $\mu(t) \equiv 1$ on $t = 0$ or 1, they are two different fuzzy variables in the sense of Definition 3.4. Since one membership function may produce multiple fuzzy variables, we can not define fuzzy variable via membership function. An axiomatic system is needed to define a fuzzy variable and discuss the properties concerning the fuzzy variable to ensure precision and consistency of the researches. That explains why the membership function is defined via credibility in credibility theory. However, for application purposes, we only need to construct the membership function of a fuzzy variable and then use the credibility inversion theorem to derive the credibility and use credibility theory to help solve the application problems. The mathematical requirement for the membership function is simple. It has been proven [58] that a function $\mu :\rightarrow [0, 1]$ is a membership function if and only if

$\sup \mu(t) = 1$. For construction method of membership functions, readers can refer to Triantaphyllou and Mann [90], Chen and Otto [8], Kumar and Ganesh [47], Hong and Chen [26], and Medaglia, Fang, Nuttle and Wilson [70].

Remark 3.3. Membership function indicates the degree that the fuzzy variable ξ takes some prescribed values. If t is an impossible point, the membership degree $\mu(t) = 0$; and if t is the most possible point that the fuzzy variable ξ takes, the membership degree $\mu(t) = 1$. However, the inverse statement is not true. It is the credibility degree rather than membership degree that gives the occurrence chance of the prescribed values. From credibility inversion theorem we know that the credibility degree of a prescribed value depends not only on its membership degree but also on the membership degree of its complementary set.

Why Adopt Credibility?

Possibility measure is an early proposed measure to measure a fuzzy event. Let ξ be a fuzzy variable with membership function μ. Then $\text{Pos}\{A\} = \sup\{\mu(\xi(\theta))|\theta \in A\}$ for any fuzzy event $A \in \mathcal{P}$. Though possibility measure is an important measure and is widely used in fuzzy set theory, it is not self-dual. Yet, self-duality property is absolutely needed in both theory and application research. Without self-duality, confusion will appear. Let us see below what will happen if we adopt possibility to measure the occurrence chance of a fuzzy event.

Example 3.2. A fuzzy variable is called a *triangular fuzzy variable* if it has a triangular membership function (see Fig. 3.1)

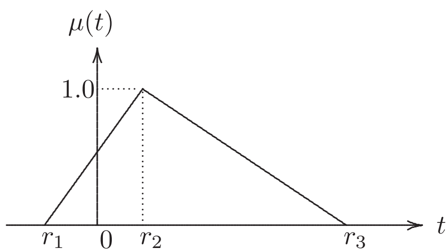

Fig. 3.1 Triangular membership function.

$$\mu(t) = \begin{cases} \dfrac{t - r_1}{r_2 - r_1}, & \text{if } r_1 \le t \le r_2 \\ \dfrac{t - r_3}{r_2 - r_3}, & \text{if } r_2 \le t \le r_3 \\ 0, & \text{otherwise.} \end{cases}$$

We denote it by $\xi = (r_1, r_2, r_3)$ with $r_1 < r_2 < r_3$.

Question 1: Suppose a traveler is going to visit a city. The expenditure is predicted to be a triangular fuzzy variable $\xi = (200, 300, 400)$ dollars (see Fig. 3.2). To ensure that the traveler will have enough money in his traveling, at least how much money should he bring with him?

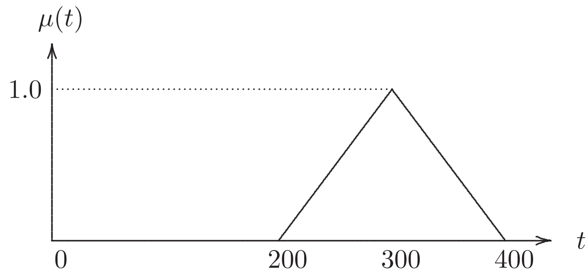

Fig. 3.2 Triangular expenditure $\xi = (200, 300, 400)$.

Judging from common sense, we will say that the traveler should bring 400 dollars with him so that he will have enough money for traveling. However, when using possibility measure, we find

$$\min\left\{t|\text{Pos}\{\xi \le t\} = 1\right\} = 300,$$

which tells us that the traveler just needs to bring with him 300 dollars to ensure that he will have enough money for traveling. This result obviously is contradictory to our judgement and common sense.

Question 2: Suppose we now have a portfolio whose return can be described by a triangular fuzzy variable $\xi = (0, 1.5, 3)$ (see Fig. 3.3). Then which event will be more likely to happen, the event of portfolio return not less than 1.5 or the event of portfolio return less than 1.5?

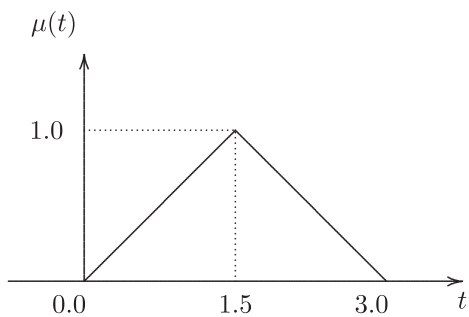

Fig. 3.3 Triangular portfolio return $\xi = (0, 1.5, 3)$.

By using possibility measure, we can calculate that $\text{Pos}\{\xi \geq 1.5\} = 1$, which seems to imply that the portfolio return not less than the value 1.5 will surely happen. However, by using possibility measure, we can also calculate that $\text{Pos}\{\xi < 1.5\} = 1$, which seems to imply that the portfolio return less than the value 1.5 will also surely happen. Is not it strange that two opposite events will both surely happen at the same time? The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle says that a proposition should be either true or false. It is obviously that the judgement made based on possibility is in contradiction with both the law of contradiction and the law of excluded middle.

Now, let us use credibility measure to calculate a fuzzy event. It follows from credibility inversion theorem that for a triangular fuzzy variable $\xi = (r_1, r_2, r_3)$ (Fig. 3.1), we have

$$\text{Cr}\{\xi \leq t\} = \begin{cases} 1, & r_3 \leq t \\ \dfrac{r_3 - 2r_2 + t}{2(r_3 - r_2)}, & r_2 \leq t \leq r_3 \\ \dfrac{t - r_1}{2(r_2 - r_1)}, & r_1 \leq t \leq r_2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Then, for a triangular expenditure $\xi = (200, 300, 400)$, we have

$$\min\left\{t|\text{Cr}\{\xi \leq t\} = 1\right\} = 400,$$

which means that the traveler should bring with him 400 dollars to ensure that he will have enough money for traveling. The result is consistent with our judgement and common sense.

For a triangular portfolio return $\xi = (0, 1.5, 3)$, according to Equation (3.8), we have $\text{Cr}\{\xi \geq 1.5\} = 0.5$, which means that there is only half the chance that the portfolio return will not be less than 1.5. According to Equation (3.8), we know $\text{Cr}\{\xi < 1.5\} = 0.5$, which means that there is only half the chance that the portfolio return will be less than 1.5. It is seen that the result is consistent with our judgement and the confusion disappears.

Some Special Fuzzy Variables

Example 3.3. A fuzzy variable is called a *trapezoidal fuzzy variable* if it has a trapezoidal membership function (see Fig. 3.4)

$$\mu(t) = \begin{cases} \dfrac{t - r_1}{r_2 - r_1}, & \text{if } r_1 \le t \le r_2 \\ 1, & \text{if } r_2 \le t \le r_3 \\ \dfrac{t - r_4}{r_3 - r_4}, & \text{if } r_3 \le t \le r_4 \\ 0, & \text{otherwise.} \end{cases}$$

We denote it by $\xi = (r_1, r_2, r_3, r_4)$ with $r_1 < r_2 < r_3 < r_4$.

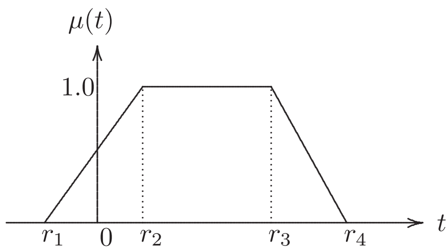

Fig. 3.4 Trapezoidal membership function.

According to credibility inversion theorem, if $r_4 \le t$, we have

$$\text{Cr}\{\xi \le t\} = \frac{1}{2}(1 + 1 - 0) = 1.$$

If $r_3 \le t \le r_4$, then

$$\text{Cr}\{\xi \le t\} = \frac{1}{2}(1 + 1 - \frac{r_4 - t}{r_4 - r_3}) = \frac{r_4 - 2r_3 + t}{2(r_4 - r_3)}.$$

If $r_2 \le t \le r_3$, then

$$\text{Cr}\{\xi \le t\} = \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}.$$

If $r_1 \le t \le r_2$, then

$$\text{Cr}\{\xi \le t\} = \frac{1}{2}(\frac{t - r_1}{r_2 - r_1} + 1 - 1) = \frac{t - r_1}{2(r_2 - r_1)}.$$

If $t < r_1$, then

$$\text{Cr}\{\xi \le t\} = \frac{1}{2}(0 + 1 - 1) = 0.$$

That is,

$$\text{Cr}\{\xi \le t\} = \begin{cases} 1, & \text{if } r_4 \le t \\ \dfrac{r_4 - 2r_3 + t}{2(r_4 - r_3)}, & \text{if } r_3 \le t \le r_4 \\ \dfrac{1}{2}, & \text{if } r_2 \le t \le r_3 \\ \dfrac{t - r_1}{2(r_2 - r_1)}, & \text{if } x_1 \le t \le r_2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Example 3.4. A fuzzy variable ξ is called a *normal fuzzy variable* if it has a normal membership function

$$\mu(t) = 2\left(1 + \exp\left(\frac{\pi|t-e|}{\sqrt{6}\sigma}\right)\right)^{-1}, \quad t \in R, \quad \sigma > 0.$$

We denote it by $\xi \sim \mathcal{N}(e, \sigma)$. It can be calculated that

$$\mu(e+\sigma) = \mu(e-\sigma) = 0.4324, \quad \text{and} \quad \mu(e+2\sigma) = \mu(e-2\sigma) = 0.1428.$$

Two normal membership functions with same σ but different e's are drawn in Fig. 3.5, and two normal membership functions with same e but different σ's are drawn in Fig. 3.6.

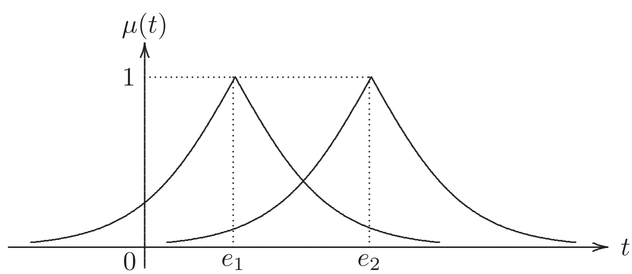

Fig. 3.5 Normal membership functions with same σ but different e's.

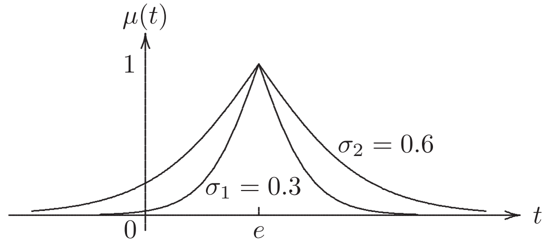

Fig. 3.6 Normal membership functions with same e but different σ's.

Example 3.5. A fuzzy variable ξ is called an *equipossible fuzzy variable* on $[a, b]$ if it has the following membership function (see Fig. 3.7)

$$\mu(t) = \begin{cases} 1, & \text{if } a_1 \le t \le a_2 \\ 0, & \text{otherwise.} \end{cases}$$

We denote it by $\xi = (a, b)$.

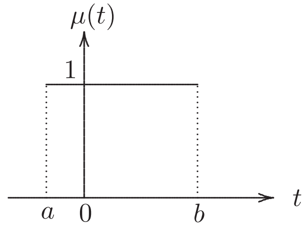

Fig. 3.7 Membership function of an equipossible fuzzy variable.

Credibility Distribution

Definition 3.6 *(Liu [56]). The credibility distribution $\Phi : \Re \to [0, 1]$ of a fuzzy variable ξ is defined by*

$$\Phi(t) = \mathrm{Cr}\{\xi \le t\}. \tag{3.10}$$

Example 3.6. Let ξ be a fuzzy variable with credibility distribution Φ. Then for any number $k > 0$, the credibility distribution of $k\xi$ is

$$\Psi(t) = \Phi\left(\frac{t}{k}\right). \tag{3.11}$$

Theorem 3.2. *Let ξ be a fuzzy variable with membership function μ. Then the credibility distribution of ξ is*

$$\Phi(t) = \frac{1}{2}\left(\sup_{z \le t} \mu(z) + 1 - \sup_{z > t} \mu(z)\right), \quad \forall t \in \Re. \tag{3.12}$$

Independence

Definition 3.7 *(Liu and Gao [61]). The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any sets $B_1, B_2, \cdots, B_n$ of $\Re$, we have*

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{n} \{\xi_i \in B_i\}\right\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \in B_i\}. \tag{3.13}$$

Theorem 3.3 *(Liu [60]). The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if*

$$\mathrm{Cr}\Big\{\bigcup_{i=1}^{n}\{\xi_i \in B_i\}\Big\} = \max_{1\le i\le n} \mathrm{Cr}\{\xi_i \in B_i\}. \tag{3.14}$$

Proof: Since credibility measure is self-dual, the fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if

$$\begin{aligned}\mathrm{Cr}\Big\{\bigcup_{i=1}^{n}\{\xi_i \in B_i\}\Big\} &= 1 - \mathrm{Cr}\Big\{\bigcap_{i=1}^{n}\{\xi_i \in B_i^c\}\Big\}\\ = 1 - \min_{1\le i\le n}\mathrm{Cr}\{\xi_i \in B_i^c\} &= \max_{1\le i\le n}\mathrm{Cr}\{\xi_i \in B_i\}.\end{aligned}$$

Thus, the theorem is proven.

Fuzzy Arithmetic

Definition 3.8 *(Liu [60]). Let $\Re^n \to \Re$ be a function, and $\xi_1, \xi_2, \cdots, \xi_n$ fuzzy variables defined on the credibility space $(\Theta_i, \mathcal{P}(\Theta_i), \mathrm{Cr}_i), i = 1, 2, \cdots, n$, respectively. Then $\xi = f(\xi_1, \xi_2, \cdots, x_n)$ is a fuzzy variable defined as*

$$\xi(\theta) = f\Big(\xi_1(\theta), \xi_2(\theta_2), \cdots, \xi_n(\theta)\Big)$$

for any $\theta \in \Theta$.

Theorem 3.4 *(Extension Principle of Zadeh). Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively, and $\Re^n \to \Re$ a continuous function. Then the membership function μ of $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is derived from the membership functions $\mu_1, \mu_2, \cdots, \mu_n$ for any $t \in \Re$ by*

$$\mu(t) = \sup_{t=f(t_1,t_2,\cdots,t_n)} \min_{1\le i\le n} \mu_i(t_i) \tag{3.15}$$

which is consistent with the expression

$$\mu(t) = \sup_{t_1,t_2,\cdots,t_n\in\Re} \left\{\min_{1\le i\le n} \mu_i(t_i) \;\Big|\; t = f(t_1, t_2, \cdots, t_n)\right\}. \tag{3.16}$$

Here we set $\mu(t) = 0$ if there are not real numbers $t_1, t_2, \cdots, t_n$ such that $t = f(t_1, t_2, \cdots, t_n)$.

Let us now give some examples to show the operations on fuzzy variables.

Example 3.7. Let ξ be a fuzzy variable with membership function ν. Then the membership function μ of $\xi + 2$ is

$$\begin{aligned}\mu(t) &= \{\nu(t_1) | t = t_1 + 2\}\\ &= \nu(t - 2).\end{aligned}$$

That is, the membership value that the fuzzy variable $\xi + 2$ achieves value $t \in \Re$ is the membership value that the fuzzy variable ξ achieves value $t - 2$.

Example 3.8. Let ξ_1 be a fuzzy variable with membership function μ_1, and ξ_2 another fuzzy variable with membership function μ_2. Then the membership function μ of $\xi_1 \cdot \xi_2$ is

$$\mu(t) = \sup_{t_1, t_2 \in R} \{\mu_1(t_1) \wedge \mu_2(t_2) | t = t_1 \cdot t_2\}.$$

Example 3.9. Let ξ_1 be a fuzzy variable with membership function μ_1, and ξ_2 another fuzzy variable with membership function μ_2. Then the membership function μ of $\xi_1 + \xi_2$ is

$$\begin{aligned} \mu(t) &= \sup_{t_1, t_2 \in R} \{\mu_1(t_1) \wedge \mu_2(t_2) | t = t_1 + t_2\} \\ &= \sup_{t_1 \in R} \{\mu_1(t_1) \wedge \mu_2(t - t_1)\}. \end{aligned}$$

Example 3.10. Let $\xi_1 = (a_1, a_2, a_3, a_4)$ and $\xi_2 = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy variables with membership functions μ_1 and μ_2 respectively. Then the membership function μ of $\xi_1 + \xi_2$ is

$$\begin{aligned} \mu(t) &= \sup_{t_1, t_2 \in R} \{\mu_1(t_1) \wedge \mu_2(t_2) | t = t_1 + t_2\} \\ &= \begin{cases} \dfrac{t - (a_1 + b_1)}{(a_2 + b_2) - (a_1 + b_1)}, & \text{if } a_1 + b_1 \le t \le a_2 + b_2 \\ 1, & \text{if } a_2 + b_2 \le t \le a_3 + b_3 \\ \dfrac{t - (a_4 + b_4)}{(a_3 + b_3) - (a_4 + b_4)}, & \text{if } a_3 + b_3 \le t \le a_4 + b_4 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

which shows that the sum of two trapezoidal fuzzy variables $\xi_1 = (a_1, a_2, a_3, a_4)$ and $\xi_2 = (b_1, b_2, b_3, b_4)$ is also a trapezoidal fuzzy variable, and $\xi_1 + \xi_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$.

Similarly, according to Theorem 3.4, we can calculate that the membership function μ of the product of a trapezoidal fuzzy variable $\xi = (a_1, a_2, a_3, a_4)$ and a scalar number ω is

$$\mu_{\omega\xi}(t) = \sup\{\mu_\xi(t_1) | t = \omega t_1\}$$

which produces that

$$\omega \cdot \xi = \begin{cases} (\omega a_1, \omega a_2, \omega a_3, \omega a_4), & \text{if } \omega \ge 0 \\ (\omega a_4, \omega a_3, \omega a_2, \omega a_1), & \text{if } \omega < 0. \end{cases}$$

That is, the product of a scalar number ω and a trapezoidal fuzzy variable $\xi = (a_1, a_2, a_3, a_4)$ is also a trapezoidal fuzzy variable.

Note that the *triangular fuzzy variable* $\xi = (r_1, r_2, r_4)$ is in fact a special *trapezoidal fuzzy variable* $\xi = (r_1, r_2, r_3, r_4)$ when $r_2 = r_3$. Therefore, we can easily get that the sum of two triangular fuzzy variables $\xi_1 = (a_1, a_2, a_3)$ and $\xi_2 = (b_1, b_2, b_3)$ is also a triangular fuzzy variable, and $\xi_1 + \xi_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3$. The product of a scalar number ω and a triangular fuzzy variable $\xi = (a_1, a_2, a_3)$ is also a triangular fuzzy variable, and

$$\omega \cdot \xi = \begin{cases} (\omega a_1, \omega a_2, \omega a_3), & \text{if } \omega \geq 0 \\ (\omega a_3, \omega a_2, \omega a_1), & \text{if } \omega < 0. \end{cases}$$

Example 3.11. Let $\xi_1 \sim \mathcal{N}(e_1, \sigma_1)$ and $\xi_2 \sim \mathcal{N}(e_2, \sigma_2)$ be two normal fuzzy variables. It can be proven that for any real numbers ω_1 and ω_2, the fuzzy variable $\omega_1\xi_1 + \omega_2\xi_2$ is also a normal fuzzy variable whose membership function is

$$\mu(t) = 2\left(1 + \exp\left(\frac{\pi|t - (\omega_1 e_1 + \omega_2 e_2)|}{\sqrt{6}(|\omega_1|\sigma_1 + |\omega_2|\sigma_2)}\right)\right)^{-1}, \quad t \in R.$$

Example 3.12. Let $\xi_1 = (a_1, a_2)$ and $\xi_2 = (b_1, b_2)$ be two equipossible fuzzy variables. It can be proven that the fuzzy variable $\xi_1 + \xi_2$ is also an equipossible fuzzy variable, and

$$\xi_1 + \xi_2 = (a_1 + b_1, a_2 + b_2).$$

Their product $\xi_1 \cdot \xi_2$ is also an equipossible fuzzy variable, and

$$\xi_1 \cdot \xi_2 = \left(\min_{a_1 \leq y \leq a_2, b_1 \leq z \leq b_2} yz, \max_{a_1 \leq y \leq a_2, b_1 \leq z \leq b_2} yz\right).$$

Example 3.13. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively, and $\Re^n \to \Re$ be a function. Then for any set B of real numbers, the credibility $\text{Cr}\{f(t_1, t_2, \cdots, t_n) \in B\}$ is

$$\frac{1}{2}\left(\sup_{f(t_1,t_2,\cdots,t_n)\in B} \min_{1\leq i\leq n} \mu_i(t_i) + 1 - \sup_{f(t_1,t_2,\cdots,t_n)\in B^c} \min_{1\leq i\leq n} \mu_i(t_i)\right).$$

Expected Value

Expected value operator calculates the average value of a fuzzy variable.

Definition 3.9. *(Liu and Liu [55]) Let ξ be a fuzzy variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq t\}\text{d}t - \int_{-\infty}^0 \text{Cr}\{\xi \leq t\}\text{d}t \tag{3.17}$$

provided that at least one of the two integrals is finite.

Example 3.14. Let $\xi = (r_1, r_2, r_3)$ be the triangular fuzzy variable. We know from the credibility inversion theorem that

$$\mathrm{Cr}\{\xi \le t\} = \begin{cases} 1, & r_3 \le t \\ \dfrac{r_3 - 2r_2 + t}{2(r_3 - r_2)}, & r_2 \le t \le r_3 \\ \dfrac{t - r_1}{2(r_2 - r_1)}, & r_1 \le t \le r_2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Cr}\{\xi \ge t\} = \begin{cases} 0, & r_3 \le t \\ \dfrac{r_3 - t}{2(r_3 - r_2)}, & r_2 \le t \le r_3 \\ \dfrac{2r_2 - r_1 - t}{2(r_2 - r_1)}, & r_1 \le t \le r_2 \\ 1, & \text{otherwise.} \end{cases}$$

Thus, if $0 \le r_1 < r_2 < r_3$, we have $\mathrm{Cr}\{\xi \le t\} \equiv 0$ when $t < 0$. Then

$$\begin{aligned} E[\xi] = & \left(\int_0^{r_1} 1\mathrm{d}t + \int_{r_1}^{r_2} \frac{2r_2 - r_1 - t}{2(r_2 - r_1)}\mathrm{d}t + \int_{r_2}^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}\mathrm{d}t \right. \\ & \left. + \int_{r_3}^{+\infty} 0\mathrm{d}t \right) - \int_{-\infty}^{0} 0\mathrm{d}t = \frac{1}{4}(r_1 + 2r_2 + r_3). \end{aligned}$$

If $r_1 < 0 \le r_2$, then

$$\begin{aligned} E[\xi] = & \left(\int_0^{r_2} \frac{2r_2 - r_1 - t}{2(r_2 - r_1)}\mathrm{d}t + \int_{r_2}^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}\mathrm{d}t + \int_{r_3}^{+\infty} 0\mathrm{d}t \right) \\ & - \left(\int_{-\infty}^{r_1} 0\mathrm{d}t + \int_{r_1}^{0} \frac{t - r_1}{2(r_2 - r_1)}\mathrm{d}t \right) = \frac{1}{4}(r_1 + 2r_2 + r_3). \end{aligned}$$

If $r_1 < r_2 < 0 < r_3$, then

$$\begin{aligned} E[\xi] = & \left(\int_0^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}\mathrm{d}t + \int_{r_3}^{+\infty} 0\mathrm{d}t \right) - \left(\int_{-\infty}^{r_1} 0\mathrm{d}t + \right. \\ & \left. \int_{r_1}^{r_2} \frac{t - r_1}{2(r_2 - r_1)}\mathrm{d}t + \int_{r_2}^{0} \frac{r_3 - 2r_2 + t}{2(r_3 - r_2)}\mathrm{d}t \right) = \frac{1}{4}(r_1 + 2r_2 + r_3). \end{aligned}$$

If $r_1 < r_2 < r_3 \le 0$, then

$$E[\xi] = \int_0^{+\infty} 0\mathrm{d}t - \left(\int_{-\infty}^{r_1} 0\mathrm{d}t + \int_{r_1}^{r_2} \frac{t-r_1}{2(r_2-r_1)}\mathrm{d}t + \int_{r_2}^{r_3} \frac{r_3-2r_2+t}{2(r_3-r_2)}\mathrm{d}t + \int_{r_3}^{0} 1\mathrm{d}t\right) = \frac{1}{4}(r_1+2r_2+r_3).$$

Therefore, the expected value of the triangular fuzzy variable $\xi = (r_1, r_2, r_3)$ is always

$$E[\xi] = \frac{1}{4}(r_1+2r_2+r_3). \tag{3.18}$$

Example 3.15. The expected value of a trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$ is

$$E[\xi] = \frac{1}{4}(r_1+r_2+r_3+r_4).$$

Example 3.16. The expected value of a normal fuzzy variable $\xi \sim \mathcal{N}(e, \sigma)$ is

$$E[\xi] = e.$$

Example 3.17. The expected value of an equipossible fuzzy variable $\xi = (r_1, r_2)$ is

$$E[\xi] = (r_2+r_1)/2.$$

Theorem 3.5 *(Liu and Liu [57]) Let ξ_1 and ξ_2 be independent fuzzy variables with finite expected values. Then for any numbers a_1 and a_2, we have*

$$E[a_1\xi_1 + a_2\xi_2] = a_1E[\xi_1] + a_2E[\xi_2]. \tag{3.19}$$

Variance

Definition 3.10 *(Liu and Liu [55]) Let ξ be a fuzzy variable with finite expected value e. Then the variance of ξ is defined by*

$$V[\xi] = E[(\xi - e)^2].$$

Example 3.18. Let ξ be an equipossible fuzzy variable (a, b). Remember that $E[\xi] = e = (a+b)/2$. Then for any positive number t, we have

$$\mathrm{Cr}\{(\xi-e)^2 \ge t\} = \begin{cases} 1/2, & \text{if } t \le (b-a)^2/4 \\ 0, & \text{if } t > (b-a)^2/4. \end{cases}$$

Thus the variance is

$$V[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi-e)^2 \ge t\}\mathrm{d}t = \int_0^{(b-a)^2/4} \frac{1}{2}\mathrm{d}t = \frac{(b-a)^2}{8}.$$

Example 3.19. Let $\xi = (r_1, r_2, r_3)$ be a triangular fuzzy variable. Then its variance is

$$V[\xi] = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha},$$

where $\alpha = \max\{r_2 - r_1, r_3 - r_2\}$ and $\beta = \min\{r_2 - r_1, r_3 - r_2\}$. Especially, when $\xi = (r_1, r_2, r_3)$ is a symmetric triangular fuzzy variable, i.e., $r_3 - r_2 = r_2 - r_1$, its variance is

$$V[\xi] = (r_3 - r_1)^2/24.$$

Example 3.20. Let $\xi = (r_1, r_2, r_3, r_4)$ be a symmetric trapezoidal fuzzy variable, i.e., $r_4 - r_3 = r_2 - r_1$. Then its variance is

$$V[\xi] = ((r_4 - r_1)^2 + (r_4 - r_1)(r_3 - r_2) + (r_3 - r_2)^2)/24.$$

Example 3.21. Let $\xi \sim \mathcal{N}(e, \sigma)$ be a normal fuzzy variable. Then its variance is

$$V[\xi] = \sigma^2.$$

Theorem 3.6 *(Liu [58]) Let a and b be real numbers and ξ a fuzzy variable whose variance exists. Then*

$$V[a\xi + b] = a^2 V[\xi]. \tag{3.20}$$

Example 3.22. Let $\xi_1 \sim \mathcal{N}(e_1, \sigma_1)$ and $\xi_2 \sim \mathcal{N}(e_2, \sigma_2)$ be two normal fuzzy variables, and a_1 and a_2 any real numbers. Then

$$E[a_1\xi_1 + a_2\xi_2] = a_1 e_1 + a_2 e_2 \quad \text{and}$$

$$V[a_1\xi_1 + a_2\xi_2] = (|a_1|\sigma_1 + |a_2|\sigma_2)^2.$$

Semivariance

Definition 3.11 *(Huang [37]) Let ξ be a fuzzy variable with finite expected value e. Then the semivariance of ξ is defined by*

$$SV[\xi] = E[[(\xi - e)^-]^2],$$

where

$$(\xi - e)^- = \begin{cases} \xi - e, & \text{if } \xi \le e \\ 0, & \text{if } \xi > e. \end{cases} \tag{3.21}$$

Example 3.23. Let $\xi = (a, b, c)$ be a triangular fuzzy variable with $b - a > c - b$. Then the semivariance of ξ is

$$SV[\xi] = \frac{(e+a)(e^2 - a^2)}{2(b-a)} - \frac{e^3 - a^3}{3(b-a)} - \frac{ae(e-a)}{b-a}$$

where $e = (a + 2b + c)/4$.

Example 3.24. Let $\xi = (a, b, c)$ be a triangular fuzzy variable with $b - a < c - b$. Then the semivariance of ξ, i.e., $SV[\xi]$ is

$$\frac{3eb + ab - 3ae + a^2 - 2b^2}{6} + \frac{e^3 - 4b^3 + 3ce^2 - 6be^2 + 9b^2e + 3cb^2 - 6cbe}{6(c-b)}$$

where $e = (a + 2b + c)/4$.

Theorem 3.7 *(Huang [37]) Let ξ be a fuzzy variable, $SV[\xi]$ and $V[\xi]$ the semivariance and variance of ξ, respectively. Then $0 \le SV[\xi] \le V[\xi]$.*

Proof: Let e be the expected value of a fuzzy variable ξ. The nonnegativity of variance and semivariance is clear. For any real number t, we have

$$\{\theta \mid (\xi(\theta) - e)^2 \ge t\} \supset \{\theta \mid [(\xi(\theta) - e)^-]^2 \ge t\},$$

which implies that

$$\mathrm{Cr}\{(\xi - e)^2 \ge t\} \ge \mathrm{Cr}\{[(\xi - e)^-]^2 \ge t\}, \quad \forall t$$

because credibility is monotonous.

It follows from the definition of variance and semivariance that

$$V[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \ge t\}\mathrm{d}t \ge \int_0^{+\infty} \mathrm{Cr}\{[(\xi - e)^-]^2 \ge t\}\mathrm{d}t = SV[\xi].$$

Theorem 3.8 *(Huang [37]) Let ξ be a fuzzy variable with symmetric membership function. Then $SV[\xi] = V[\xi]$.*

Proof: Let ξ be a fuzzy variable with symmetric membership function about its expected value e. From the definition of variance, we have

$$V[\xi] = E[(\xi - e)^2] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \ge t\}\mathrm{d}t.$$

Since the membership function of ξ is symmetric about e, we have

$$\mathrm{Cr}\{(\xi - e)^2 \ge t\} = \mathrm{Cr}\{[(\xi - e)^-]^2 \ge t\}, \quad \forall t.$$

Therefore,

$$V[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \ge t\}\mathrm{d}t = \int_0^{+\infty} \mathrm{Cr}\{[(\xi - e)^-]^2 \ge t\}\mathrm{d}t = SV[\xi].$$

Remark 3.4. Theorem 3.8 tells us that when a membership function of portfolio return is symmetrical, the variance value and the semivariance value of the fuzzy portfolio return will be the same. However, when a membership function of a fuzzy portfolio return is asymmetrical, Theorem 3.7 tells us that the variance value and the semivariance value of the fuzzy portfolio return will be different.

β-Value

Definition 3.12 *(Liu [56]) Let ξ be a fuzzy variable, and $\beta \in (0,1]$. Then*

$$\xi_{\sup}(\beta) = \sup\left\{r \;\middle|\; \mathrm{Cr}\{\xi \geq r\} \geq \beta\right\} \tag{3.22}$$

is called the β-value of ξ.

Example 3.25. Let $\xi = [a,b]$ be an equipossible fuzzy variable. Then its β-value is

$$\xi_{\sup}(\beta) = \begin{cases} b, & \text{if } \beta \leq 0.5 \\ a, & \text{if } \beta > 0.5. \end{cases}$$

Example 3.26. Let $\xi = (r_1, r_2, r_3)$ be a triangular fuzzy variable. Then its β-value is

$$\xi_{\sup}(\beta) = \begin{cases} 2\beta r_2 + (1-2\beta)r_3, & \text{if } \beta \leq 0.5 \\ (2\beta-1)r_1 + (2-2\beta)r_2, & \text{if } \beta > 0.5. \end{cases}$$

Example 3.27. Let $\xi = (r_1, r_2, r_3, r_4)$ be a trapeziodal fuzzy variable. Then its β-value is

$$\xi_{\sup}(\beta) = \begin{cases} 2\beta r_3 + (1-2\beta)r_4, & \text{if } \beta \leq 0.5 \\ (2\beta-1)r_1 + (2-2\beta)r_2, & \text{if } \beta > 0.5. \end{cases}$$

Example 3.28. Let $\xi \sim \mathcal{N}(e,\sigma)$ be a normal fuzzy variable. Then its β-value is

$$\xi_{\sup}(\beta) = e - \frac{\sqrt{6}\sigma}{\pi} \ln \frac{\beta}{1-\beta}.$$

Theorem 3.9 *(Liu [58]). Let $\xi_{\sup}(\beta)$ be the β-value of the fuzzy variable ξ. Then $\xi_{\sup}(\beta)$ is a decreasing and left-continuous function of β.*

Theorem 3.10. *Let $\xi_{\sup}(\beta)$ be the β-value of the fuzzy variable ξ. Then if $\lambda \geq 0$, we have $(\lambda\xi)_{\sup}(\beta) = \lambda\xi_{\sup}(\beta)$.*

Theorem 3.11 *(Li and Liu [49]). Let ξ and η be two independent fuzzy variables. Then for any $\beta \in (0,1]$, we have*

$$(\xi + \eta)_{\sup}(\beta) = \xi_{\sup}(\beta) + \eta_{\sup}(\beta) \tag{3.23}$$

Proof: According to monotonicity property of credibility measure, for any $\epsilon > 0$, we have

$$\begin{aligned} &\mathrm{Cr}\{\xi + \eta \geq \xi_{\sup}(\beta) + \eta_{\sup}(\beta) - \epsilon\} \\ &\geq \mathrm{Cr}\Big\{\{\xi \geq \xi_{\sup}(\beta) - \epsilon/2\} \cap \{\eta \geq \eta_{\sup}(\beta) - \epsilon/2\}\Big\}. \end{aligned}$$

Since ξ and η are independent fuzzy variables, according to Definition 3.7, we have

$$\begin{aligned}&\mathrm{Cr}\{\xi+\eta \geq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)-\epsilon\}\\ &\geq \mathrm{Cr}\Big\{\{\xi \geq \xi_{\sup}(\beta)-\epsilon/2\} \cap \{\eta \geq \eta_{\sup}(\beta)-\epsilon/2\}\Big\}\\ &= \mathrm{Cr}\{\xi \geq \xi_{\sup}(\beta)-\epsilon/2\} \wedge \mathrm{Cr}\{\eta \geq \eta_{\sup}(\beta)-\epsilon/2\} \geq \beta\end{aligned}$$

which implies that

$$(\xi+\eta)_{\sup}(\beta) \geq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)-\epsilon. \tag{3.24}$$

According to monotonicity property of credibility measure, for any $\epsilon > 0$, we have

$$\begin{aligned}&\mathrm{Cr}\{\xi+\eta \geq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)+\epsilon\}\\ &\leq \mathrm{Cr}\Big\{\{\xi \geq \xi_{\sup}(\beta)+\epsilon/2\} \cup \{\eta \geq \eta_{\sup}(\beta)+\epsilon/2\}\Big\}.\end{aligned}$$

Since ξ and η are independent fuzzy variables, according to Theorem 3.3, we have

$$\begin{aligned}&\mathrm{Cr}\{\xi+\eta \geq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)+\epsilon\}\\ &\leq \mathrm{Cr}\Big\{\{\xi \geq \xi_{\sup}(\beta)+\epsilon/2\} \cup \{\eta \geq \eta_{\sup}(\beta)+\epsilon/2\}\Big\}\\ &= \mathrm{Cr}\{\xi \geq \xi_{\sup}(\beta)+\epsilon/2\} \vee \mathrm{Cr}\{\eta \geq \eta_{\sup}(\beta)+\epsilon/2\} < \beta\end{aligned}$$

which implies that

$$(\xi+\eta)_{\sup}(\beta) \leq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)+\epsilon. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\xi_{\sup}(\beta)+\eta_{\sup}(\beta)+\epsilon \geq (\xi+\eta)_{\sup}(\beta) \geq \xi_{\sup}(\beta)+\eta_{\sup}(\beta)-\epsilon.$$

Letting $\epsilon \to 0$, we have

$$(\xi+\eta)_{\sup}(\beta) = \xi_{\sup}(\beta)+\eta_{\sup}(\beta).$$

Entropy

Fuzzy entropy is a measure of fuzzy uncertainty. It measures the difficulty degree of predicting the specific value that a fuzzy variable will take.

Definition 3.13 *(Li and Liu [51]). Let ξ be a fuzzy variable with continuous membership function. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{\infty} S(\mathrm{Cr}\{\xi = t\})\mathrm{d}t \tag{3.26}$$

where $S(y) = -y\ln y - (1-y)\ln(1-y)$.

Since for any fuzzy variable ξ with continuous membership function μ, we have $\text{Cr}\{\xi = r\} = \dfrac{\mu(r)}{2}$ for each $r \in R$. Thus, the entropy can be expressed by

$$H[\xi] = -\int_{-\infty}^{\infty}\left(\frac{\mu(r)}{2}\ln\frac{\mu(r)}{2} + \left(1-\frac{\mu(r)}{2}\right)\ln\left(1-\frac{\mu(r)}{2}\right)\right)\mathrm{d}r. \tag{3.27}$$

Example 3.29. Let ξ be a triangular fuzzy variable $\xi = (r_1, r_2, r_3)$. Then its entropy is $H[\xi] = (r_3 - r_1)/2$.

Example 3.30. Let ξ be a trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$. Then its entropy is $H[\xi] = (r_4 - r_1)/2 + (\ln 2 - 0.5)(r_3 - r_2)$.

Example 3.31. Let $\xi \sim \mathcal{N}(e, \sigma)$ be a normal fuzzy variable. Then its entropy is $H[\xi] = \sqrt{6}\pi\sigma/3$.

Remark 3.5. Let ξ be a fuzzy variable with continuous membership function and taking continuous values in the interval $[a, b]$. Then we can find that $H[\xi] \le (b-a)\ln 2$ and that the equality holds if and only if ξ is an equipossible fuzzy variable in the interval $[a, b]$. Since a fuzzy variable with maximum entropy distributes most dispersively and it will be most difficult to predict whether this fuzzy variable will take the specific value, for the safety reason of decision making, if the investors can only give the interval that a security return may lie in and nothing else, they can use the equipossible fuzzy variable to describe this security return.

Remark 3.6. Let ξ be a fuzzy variable with a continuous membership function and having finite expected value e and variance value σ^2. It has been proven [50] that $H[\xi] \le \sqrt{6}\pi\sigma/3$ and that the equality holds if ξ is a normal fuzzy variable with expected value e and variance σ^2. Since a fuzzy variable with maximum entropy distributes most dispersively and it will be most difficult to predict whether this fuzzy variable will take the specific value, for the safety reason of decision making, if the investors can predict only the expected value and variance value of a security return and nothing else, for the safety reason of decision making, they can use the normal fuzzy variable to describe the security return.

3.2 Mean-Risk Model

In reality, some people do not like taking plane because when the plane crashes, it is almost sure that people in the plane will lose their lives though the chance of crashing event is very low. This phenomenon implies that when judging if an event is risky or not, people will consider both the occurrence chance and the severity level of the bad event. This is also true in portfolio investment. To give an instinct information about each likely loss and the corresponding occurrence chance of the loss for portfolio investment with fuzzy returns, Huang [38] defined the concept of risk curve.

3.2.1 Risk Curve

Definition 3.14 *(Huang [38]) Let ξ denote the fuzzy return of a portfolio, and r_f the risk-free interest rate. Then the curve*

$$R(r) = \mathrm{Cr}\{r_f - \xi \geq r\}, \quad \forall r \geq 0 \tag{3.28}$$

is called the risk curve of the portfolio, and r the loss severity indicator.

It is easy to see that $r_f - \xi$ is the deviation of the portfolio return from the risk-free interest rate when $r_f - \xi \geq 0$. Then the value $r_f - \xi$ can easily be understood as a loss. Since the portfolio return is variable, the loss value $r_f - \xi$ may be any non-negative values which can be expressed by

$$r_f - \xi \geq r, \quad r \geq 0.$$

Please note that r is not one specific number but any non-negative numbers, so $r_f - \xi \geq r$ describes all the likely losses of the portfolio, and the curve $R(r)$ gives corresponding occurrence credibility levels of all these losses.

Example 3.32. Let $\xi(a_1, a_1, a_3)$ denote a triangular fuzzy portfolio return. Then risk curve of ξ is as follows,

$$R(r) = \mathrm{Cr}\{r_f - \xi \geq r\} = \begin{cases} 1, & \text{if } r_f - a_3 > r \geq 0 \\ \dfrac{a_3 - 2a_2 + r_f - r}{2(a_3 - a_2)}, & \text{if } r_f - a_2 > r \geq r_f - a_3 \\ \dfrac{r_f - a_1 - r}{2(a_2 - a_1)}, & \text{if } r_f - a_1 > r \geq r_f - a_2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.29}$$

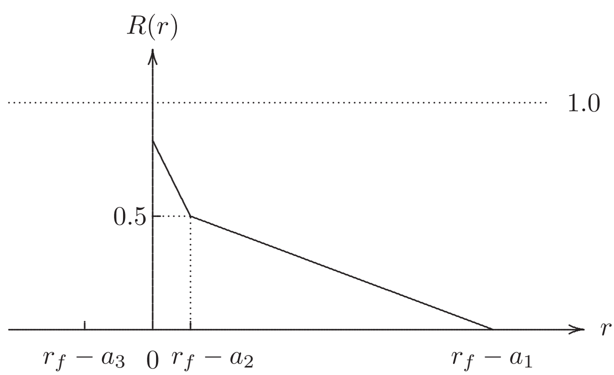

Fig. 3.8 Risk curve of a portfolio with triangular fuzzy return.

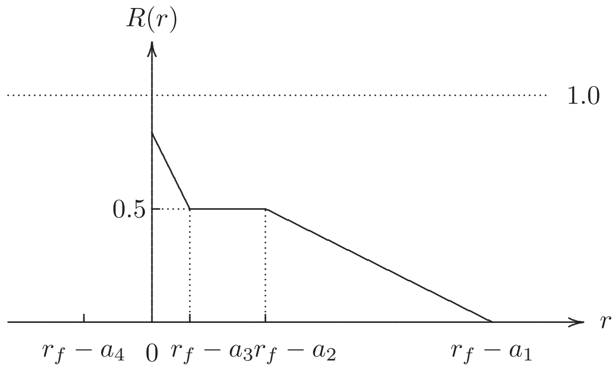

Fig. 3.9 Risk curve of a portfolio with trapezoidal fuzzy return.

Example 3.33. Let $\xi(a_1, a_2, a_3, a_4)$ denote a trapezoidal fuzzy portfolio return. Then risk curve of ξ is as follows,

$$R(r) = \text{Cr}\{r_f - \xi \geq r\} = \begin{cases} 1, & \text{if } r_f - a_4 > r \geq 0 \\ \dfrac{a_4 - 2a_3 + r_f - r}{2(a_4 - a_3)}, & \text{if } r_f - a_3 > r \geq r_f - a_4 \\ 0.5, & \text{if } r_f - a_2 > r \geq r_f - a_3 \\ \dfrac{r_f - a_1 - r}{2(a_2 - a_1)}, & \text{if } r_f - a_1 > r \geq r_f - a_2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

Example 3.34. Let $\xi \sim \mathcal{N}(e, \sigma)$ denote a normal fuzzy portfolio return. Then risk curve of ξ is as follows,

$$R(r) = \text{Cr}\{r_f - \xi \geq r\} = \left(1 + \exp\left(\frac{\pi(e - r_f + r)}{\sqrt{6}\sigma}\right)\right)^{-1}, \quad r \geq 0. \tag{3.31}$$

Example 3.35. Let $\xi = (a, b)$ denote an equipossible fuzzy portfolio return. Then risk curve of ξ is as follows,

$$R(r) = \text{Cr}\{r_f - \xi \geq r\} = \begin{cases} 1, & \text{if } r \leq r_f - b \\ 0.5, & \text{if } r_f - b \leq r \leq r_f - a \\ 0, & \text{otherwise.} \end{cases} \tag{3.32}$$

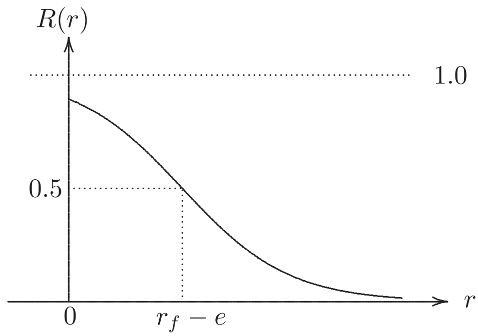

Fig. 3.10 Risk curve of a portfolio with normal fuzzy return.

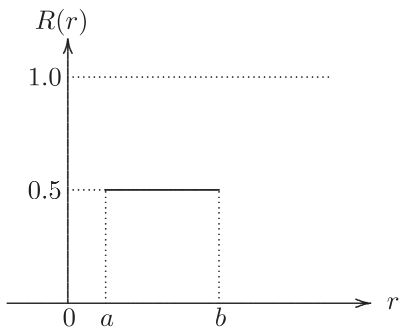

Fig. 3.11 Risk curve of a portfolio with equipossible fuzzy return.

3.2.2 Confidence Curve and Safe Portfolio

To determine a level of a risk, according to Definition 3.14, three inputs should be given. First input is the value r, the loss severity level. Second input is the occurrence chance of the loss event, i.e., $\mathrm{Cr}\{r_f - \xi \geq r\}$. Third input is the investors' subjective judgement to the above two inputs. Different investors have different judgements. Given any value r, an investor should be able to give his/her maximal tolerance towards the occurrence chance of the loss being equal to or greater than r by answering what-if questions in Table 2.5. In fuzzy portfolio selection, occurrence chance of a fuzzy event is measured by credibility value. We call the curve the confidence curve $\alpha(r)$ that gives the investor's maximal tolerance towards the occurrence chances of all the potential losses. Though different investors have different confidence curves, the common trend of the curve is that the severer the loss, the lower the tolerance of occurrence chance of the loss. The general trend of the confidence curve is given in Fig.3.12. Three examples of confidence curve are presented in Subsection 2.2.2 in Chapter 2.

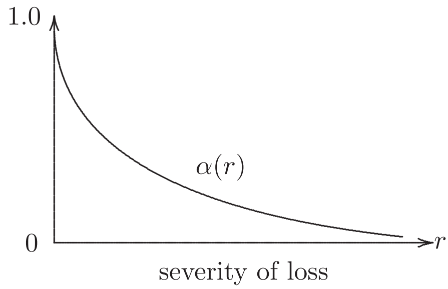

Fig. 3.12 General trend of a confidence curve: The higher the loss value, the lower the tolerance of the occurrence chance of the loss.

It is easy to understand that a portfolio is safe if its risk curve is totally below the investor's confidence curve. A portfolio is regarded to be risky if any part of its risk curve is above the investor's confidence curve (see Fig. 3.12). The mathematical expression of a safe portfolio is as follows:

Let ξ be the fuzzy return of a portfolio A, and $\alpha(r)$ the investor's confidence curve. We say A is a safe portfolio if

$$R(r) = \mathrm{Cr}\{(r_f - \xi) \geq r\} \leq \alpha(r), \quad \forall r \geq 0,$$

where r_f is the risk-free interest rate.

3.2.3 Mean-Risk Model

Let x_i denote the investment proportions in securities i, and ξ_i the i-th security returns which are fuzzy. According to Definition 3.14, the risk curve of a portfolio $(x_1, x_2, \cdots, x_n)$ is

$$R(x_1, x_2, \cdots, x_n; r) = \mathrm{Cr}\left\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq r\right\}.$$

Let $\alpha(r)$ be an investor's confidence curve. The philosophy of mean-risk model is to pursue maximum expected return among the safe portfolios whose risk curves are below the investor's confidence curve. To express it in mathematical way, we have the model as follows:

$$\begin{cases} \max E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{subject to:} \\ \qquad R(x_1, x_2, \cdots, x_n; r) \leq \alpha(r), \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.33}$$

The constraint $R(x_1, x_2, \cdots, x_n; r) \leq \alpha(r)$ requires that the credibility value of each likely loss of a selected portfolio must be lower than the investor's tolerance level. The constraint $x_i \geq 0$ implies that short sales are not allowed in the investment.

3.2.4 Crisp Equivalent

One way of solving the mean-risk model is to convert the expected value and risk curve of the portfolio into their crisp equivalents and use traditional methods to solve the mean-risk model. Luckily, for independent fuzzy security returns, we have the transformation theorem as follows:

Theorem 3.12 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent fuzzy variables with continuous credibility distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If*

$$\lim_{t\to-\infty} \Phi_i(t) = 0, \quad \lim_{t\to\infty} \Phi_i(t) = 1, \quad \textit{for} \quad i = 1, 2, \cdots, n,$$

and $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0,1)$, then for any $\alpha \in (0,1)$, we have

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha), \quad 0 < \alpha < 1 \tag{3.34}$$

where Ψ is the distribution function of fuzzy variable $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$.

Proof: According to monotonicity property of credibility measure, for any given $\alpha \in (0,1)$, we have

$$\mathrm{Cr}\left\{\sum_{i=1}^n \xi_i \leq \sum_{i=1}^n \Phi_i^{-1}(\alpha)\right\} \geq \mathrm{Cr}\left\{\bigcap_{i=1}^n \left(\xi_i \leq \Phi_i^{-1}(\alpha)\right)\right\}.$$

Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent fuzzy variables, according to Equation (3.13), we have

$$\begin{aligned}
&\mathrm{Cr}\left\{\sum_{i=1}^n \xi_i \leq \sum_{i=1}^n \Phi_i^{-1}(\alpha)\right\} \geq \mathrm{Cr}\left\{\bigcap_{i=1}^n \left(\xi_i \leq \Phi_i^{-1}(\alpha)\right)\right\} \\
&= \min_{1\leq i\leq n} \mathrm{Cr}\{\xi_i \leq \Phi_i^{-1}(\alpha)\} = \min_{1\leq i\leq n} \alpha = \alpha.
\end{aligned}$$

On the other hand, for any number $\epsilon > 0$, we have

$$\mathrm{Cr}\left\{\sum_{i=1}^n \xi_i \leq \sum_{i=1}^n \Phi_i^{-1}(\alpha) - \epsilon\right\} \leq \mathrm{Cr}\left\{\bigcup_{i=1}^n \left(\xi_i \leq \Phi_i^{-1}(\alpha) - \frac{\epsilon}{n}\right)\right\}$$

because credibility measure is monotonous. Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent fuzzy variables, according to Equation (3.14), we have

$$\mathrm{Cr}\left\{\sum_{i=1}^{n}\xi_i \le \sum_{i=1}^{n}\Phi_i^{-1}(\alpha)-\epsilon\right\} \le \mathrm{Cr}\left\{\bigcup_{i=1}^{n}\left(\xi_i \le \Phi_i^{-1}(\alpha)-\frac{\epsilon}{n}\right)\right\}$$
$$=\max_{1\le i\le n}\mathrm{Cr}\left\{\xi_i \le \Phi_i^{-1}(\alpha)-\frac{\epsilon}{n}\right\} < \max_{1\le i\le n}\alpha=\alpha.$$

It follows from the continuity of credibility distributions that

$$\mathrm{Cr}\{\xi_1+\xi_2+\cdots+\xi_n \le \Phi_1^{-1}(\alpha)+\Phi_2^{-1}(\alpha)+\cdots+\Phi_n^{-1}(\alpha)\}=\alpha$$

which implies that

$$\Psi^{-1}(\alpha)=\Phi_1^{-1}(\alpha)+\Phi_2^{-1}(\alpha)+\cdots+\Phi_n^{-1}(\alpha).$$

The theorem is proven.

Theorem 3.13. *Let Φ_i denote the credibility distributions of the i-th fuzzy security return rates $\xi_i, i=1,2,\cdots,n$, respectively. Then the mean-risk model (3.33) can be transformed into the following linear model:*

$$\begin{cases}\max x_1E[\xi_1]+x_2E[\xi_2]+\cdots+x_nE[\xi_n]\\ \textit{subject to:}\\ \qquad x_1\Phi_1^{-1}\Big(\alpha(r)\Big)+x_2\Phi_2^{-1}\Big(\alpha(r)\Big)+\cdots+x_n\Phi_n^{-1}\Big(\alpha(r)\Big)\ge r_f-r,\ \forall r\ge 0\\ \qquad x_1+x_2+\cdots+x_n=1\\ \qquad x_i\ge 0,\quad i=1,2,\cdots,n.\end{cases} \tag{3.35}$$

Proof: It follows from linearity property of expected value that the objective function of Model (3.33) can be transformed into the objective function of Model (3.35).

It follows from Theorem 3.12 that the risk curve in Model (3.33) can be transformed into the following linear form

$$R^{-1}(x_1,x_2,\cdots,x_n;r)=x_1\Phi_1^{-1}\Big(\alpha(r)\Big)+x_2\Phi_2^{-1}\Big(\alpha(r)\Big)+\cdots+x_n\Phi_n^{-1}\Big(\alpha(r)\Big).$$

It follows from monotonicity of credibility measure that

$$x_1\Phi_1^{-1}\Big(\alpha(r)\Big)+x_2\Phi_2^{-1}\Big(\alpha(r)\Big)+\cdots+x_n\Phi_n^{-1}\Big(\alpha(r)\Big)\ge r_f-r.$$

Example 3.36. Suppose the return rates of the i-th securities are all triangular fuzzy variables $\xi_i=(a_i,b_i,c_i), i=1,2,\cdots,n$, respectively. Then the fuzzy mean-risk model can be transformed into the following form:

$$\begin{cases} \max \sum_{i=1}^{n} (a_i x_i + 2b_i x_i + c_i x_i) \\ \text{subject to:} \\ \quad \Big(2\alpha(r) - 1\Big) \sum_{i=1}^{n} c_i x_i + \Big(2 - 2\alpha(r)\Big) \sum_{i=1}^{n} b_i x_i \geq r_f - r, \text{ if } \alpha(r) > 0.5 \\ \quad 2\alpha(r) \sum_{i=1}^{n} b_i x_i - \Big(2\alpha(r) - 1\Big) \sum_{i=1}^{n} a_i x_i \geq r_f - r, \text{ if } \alpha(r) \leq 0.5 \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.36}$$

Since all security return rates are triangular fuzzy variables, the portfolio return rate is still a triangular fuzzy variable, i.e.,

$$\xi = \sum_{i=1}^{n} \xi_i x_i = \left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i \right).$$

Thus, we can get Model (3.36) easily.

Example 3.37. Suppose the return rates of the i-th securities are all normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i), i = 1, 2, \cdots, n$, respectively. Then the fuzzy mean-risk model can be transformed into the following form:

$$\begin{cases} \max e_1 x_1 + e_2 x_2 + \cdots + e_n x_n \\ \text{subject to:} \\ \quad \sum_{i=1}^{n} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1 - \alpha(r)}{\alpha(r)} \right) x_i \geq r_f - r, \quad \forall r \geq 0 \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.37}$$

Since all security return rates are normal fuzzy variables, the portfolio return rate is still a normal fuzzy variable, i.e.,

$$\xi = \sum_{i=1}^{n} \xi_i x_i = \left(\sum_{i=1}^{n} e_i x_i, \sum_{i=1}^{n} \sigma_i x_i \right).$$

Thus, we can get Model (3.37) easily.

Example 3.38. Suppose the return rates of the i-th securities are normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i), i = 1, 2, \cdots, m$, and the return rates of the j-th securities are triangular fuzzy variables $\xi_j = (a_j, b_j, c_j), j = m + 1, m +$

$2, \cdots, n$, respectively. Then the fuzzy mean-risk model can be transformed into the following form:

$$\begin{cases} \max \sum_{i=1}^{m} e_i x_i + \sum_{i=m+1}^{n} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \\ \text{subject to:} \\ \qquad \sum_{i=m+1}^{m} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i + \Big(2\alpha(r) - 1\Big) \sum_{i=m+1}^{n} c_i x_i + \\ \qquad \Big(2 - 2\alpha(r)\Big) \sum_{i=m+1}^{n} b_i x_i \geq r_f - r, \text{ if } \alpha(r) > 0.5 \\ \qquad \sum_{i=1}^{m} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i + 2\alpha(r) \sum_{i=m+1}^{n} b_i x_i - \\ \qquad \Big(2\alpha(r) - 1\Big) \sum_{i=m+1}^{n} a_i x_i \geq r_f - r, \text{ if } \alpha(r) \leq 0.5 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.38}$$

3.2.5 An Example

Suppose an investor wants to choose an optimal portfolio from ten securities of which five security return rates are normal fuzzy variables and the rest five the triangular fuzzy variables. The prediction of the return rates of the ten securities is given in Table 3.1. Suppose the monthly risk-free interest rate is 0.01, and the investor gives his/her confidence curve as follows:

$$\alpha(r) = \begin{cases} -2.75r + 0.43, & 0 \leq r \leq 0.12, \\ -0.5r + 0.16, & 0.12 \leq r \leq 0.3, \\ 0.01, & r \geq 0.3. \end{cases}$$

According to the mean-risk selection idea, we build the model as follows:

$$\begin{cases} \max E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \\ \text{subject to:} \\ \qquad R(x_1, x_2, \cdots, x_{10}; r) \leq \alpha(r), \, \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10 \end{cases} \tag{3.39}$$

Table 3.1 Fuzzy Return Rates of 10 Securities

Security i	$\xi_i \sim \mathcal{N}(e_i, \sigma_i)$	Security i	$\xi_i = (a_i, b_i, c_i)$
1	$\mathcal{N}(0.034, 0.12)$	6	$(-0.06, 0.020, 0.15)$
2	$\mathcal{N}(0.033, 0.10)$	7	$(-0.10, 0.030, 0.20)$
3	$\mathcal{N}(0.039, 0.12)$	8	$(-0.12, 0.032, 0.2)$
4	$\mathcal{N}(0.028, 0.08)$	9	$(-0.20, 0.04, 0.28)$
5	$\mathcal{N}(0.025, 0.08)$	10	$(-0.16, 0.03, 0.30)$

where $R(x_1, x_2, \cdots, x_{10}; r)$ is the risk curve of the portfolio defined as

$$R(x_1, x_2, \cdots, x_{10}; r) = \mathrm{Cr}\{0.01 - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}) \geq r\}.$$

According to Model (3.38), we can change Model (3.39) into the following linear programming model. Please note that $\alpha(r) < 0.5$ in the example.

$$\begin{cases} \max \displaystyle\sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \\ \text{subject to:} \\ \qquad \displaystyle\sum_{i=1}^{5} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i + 2\alpha(r) \sum_{i=6}^{10} b_i x_i - \\ \qquad \Big(2\alpha(r) - 1\Big) \displaystyle\sum_{i=6}^{10} a_i x_i \geq 0.01 - r \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.40}$$

Though theoretically, when solving the mean-risk model, r should be any nonnegative numbers, in reality, r can be limited to a certain interval by analyzing the problem. In the example, since the confidence curve is a horizontal line when $r \geq 0.3$ and the risk curve is a decreasing function of r, risk curve will be below the confidence curve if $R(x_1, x_2, \cdots, x_{10}; r) \leq \alpha(r)$ holds for any $r \in [0, 0.3]$. Since risk curve is a continuous function of r, it is enough for us to check if the points on the risk curve are all lower than the points on the confidence curve for $(r = 0, \alpha = 0.43)$, $(r = 0.02, \alpha = 0.375)$, $(r = 0.04, \alpha = 0.32)$, $(r = 0.06, \alpha = 0.265), \cdots, (r = 0.3, \alpha = 0.01)$. That is, we just need to solve Model (3.41) given below. By using "Solver" in "Excel", we get the optimal portfolio shown in Table 3.2. The maximum expected return is 0.042. As shown in Fig. 3.13, risk curve of the optimal portfolio is totally below the investor's confidence curve. Given any loss value r, the loss occurrence credibility is not greater than the investor's tolerable credibility.

Table 3.2 Allocation of Money to Ten Securities

Security i	1	2	3	4	5
Allocation of money	0.00%	0.00 %	0.00%	0.00%	0.00%
Security i	6	7	8	9	10
Allocation of money	0.00%	78.57 %	0.00 %	0.00%	21.43%

Or given any occurrence credibility $\alpha(r)$, the loss level is not greater than the investor's tolerable loss level.

$$\left\{\begin{array}{l}
\max \sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \\
\text{subject to:} \\
\qquad \sum_{i=1}^{5}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{1-0.43}{0.43}\right) x_i + 2\times 0.43 \sum_{i=6}^{10} b_i x_i - \\
\qquad \left(2\times 0.43 - 1\right)\sum_{i=6}^{10} a_i x_i \geq 0.01 \\
\qquad \sum_{i=1}^{5}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{1-0.375}{0.375}\right) x_i + 2\times 0.375 \sum_{i=6}^{10} b_i x_i - \\
\qquad \left(2\times 0.375 - 1\right)\sum_{i=6}^{10} a_i x_i \geq 0.01 - 0.02 \\
\qquad \sum_{i=1}^{5}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{1-0.32}{0.32}\right) x_i + 2\times 0.32 \sum_{i=6}^{10} b_i x_i - \\
\qquad \left(2\times 0.32 - 1\right)\sum_{i=6}^{10} a_i x_i \geq 0.01 - 0.04 \\
\qquad \cdots \\
\qquad \sum_{i=1}^{5}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{1-0.01}{0.01}\right) x_i + 2\times 0.01 \sum_{i=6}^{10} b_i x_i - \\
\qquad \left(2\times 0.01 - 1\right)\sum_{i=6}^{10} a_i x_i \geq 0.01 - 0.3 \\
\qquad x_1 + x_2 + \cdots + x_{10} = 1 \\
\qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10.
\end{array}\right. \tag{3.41}$$

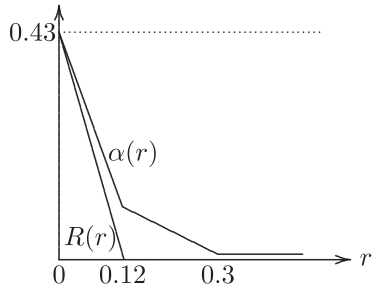

Fig. 3.13 Risk curve $R(r)$ and confidence curve $\alpha(r)$ of Model (3.39).

3.3 β-Return-Risk Model

3.3.1 β-Return-Risk Model

In the mean-risk model, loss is instinct. However, target return is not instinct enough because it is represented by the expected value. Sometimes the investors would like to directly pursue a specific target return rather than an average value. Since the optimal target return may not be obtained in some bad situations, it is natural that people would accept the inability to reach the objective to some extent. However, at a given confidence level which is considered as the safety margin, the objective must be achieved. Based on this idea, Huang [27] proposed the β-return optimization model pursuing the maximal target return at the credibility not less than a predetermined safety level. Replacing expected value by a specific β-return, we get the β-return-risk Model. To understand the β-return-risk selection idea, let us give the definition of β-return and see an example first.

Definition 3.15 *Let x_i be the investment proportions in the i-th securities, $i = 1, 2, \cdots, n$, ξ_i the returns of the i-th securities and β the preset confidence level. The β-return is defined as*

$$\max\{\bar{f} \mid \mathrm{Cr}\left\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geq \bar{f}\right\} \geq \beta\} \tag{3.42}$$

which means the maximal investment return the investor can obtain at confidence level β.

Example 3.39. Suppose we have three securities $\xi_1 = (-0.01, 0.05, 0.1), \xi_2 = (-0.01, 0.06, 0.08)$ and $\xi_3 = (-0.02, 0.08, 0.12)$. There are four money allocation plans. In plan 1, the investor allocates all the money to security 1. In plan 2, the investor allocates all the money to security 2. In plan 3, the investor allocates all the money to security 3. In plan 4, the investor allocates 20% of the money to security 1 and 80% of the money to security 2. The investor sets the confidence level at $\beta = 0.9$. It can be calculated that for plans

Table 3.3 Four Money Allocation Plans

Security i	1	2	3	90%-Return
Money Allocation (plan 1)	100%	0%	0%	0.2%
Money Allocation (plan 2)	0%	100%	0%	0.4%
Money Allocation (plan 3)	0%	0%	100%	0%
Money Allocation (plan 4)	20%	80%	0%	0.36%

1, 2, 3 and 4, the 0.9-return values are $0.2\%, 0.4\%, 0\%, 0.36\%$, respectively. The result is shown in Table 3.3. It can be seen that different money allocation will result in different 0.9-returns. The investor's objective is to find an optimal portfolio which can bring the investor a maximum specific return at a given confidence level, i.e., a maximum β-return. However, a portfolio with maximum β-return may be a risky portfolio. Therefore, before pursuing maximum β-return, the investor has to make sure that the selected portfolio is a safe portfolio. That is to say, the risk curve of the portfolio is first required to be totally below the investor's confidence curve. Then, among the safe portfolios, β-return should be maximized. The mathematical expression of the β-return-risk selection idea is as follows:

$$\begin{cases} \max \bar{f} \\ \text{subject to:} \\ \qquad \mathrm{Cr}\left\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geq \bar{f}\right\} \geq \beta \\ \qquad R(x_1, x_2, \cdots, x_n; r) \leq \alpha(r), \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.43}$$

where $R(x_1, x_2, \cdots, x_n; r)$ is the risk curve of the portfolio, $\alpha(r)$ the investor's confidence curve, and $\bar{f}$ the β-return.

3.3.2 Crisp Equivalent

When security returns are independent fuzzy variables, we can change the β-return-risk model into its equivalent and solve the model in traditional ways.

Theorem 3.14 *Let Φ_i denote the credibility distributions of the i-th fuzzy security return rates $\xi_i, i = 1, 2, \cdots, n$, respectively. Then the β-return-risk model (3.43) can be transformed into the following linear model:*

$$\begin{cases} \max x_1\xi_1(\beta) + x_2\xi_2(\beta) + \cdots + x_n\xi_n(\beta) \\ \textit{subject to:} \\ \qquad x_1\Phi_1^{-1}\Big(\alpha(r)\Big) + x_2\Phi_2^{-1}\Big(\alpha(r)\Big) + \cdots + x_n\Phi_n^{-1}\Big(\alpha(r)\Big) \geq r_f - r, \ \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.44}$$

where $\xi_i(\beta)$ is the β-return value of the i-th security.

Proof: The objective function follows directly from Theorem 3.11, and the constraint follows from Theorem 3.12 and monotonicity property of credibility measure.

Example 3.40. When all the security returns are regarded to be triangular fuzzy variables $\xi_i = (a_i, b_i, c_i)$, the β-return-risk model (3.43) becomes

$$\begin{cases} \max(2\beta - 1)\sum_{i=1}^{n} a_i x_i + 2(1-\beta)\sum_{i=1}^{n} b_i x_i \\ \text{subject to:} \\ \qquad \Big(2\alpha(r) - 1\Big)\sum_{i=1}^{n} c_i x_i + \Big(2 - 2\alpha(r)\Big)\sum_{i=1}^{n} b_i x_i \geq r_f - r, \text{ if } \alpha(r) > 0.5 \\ \qquad 2\alpha(r)\sum_{i=1}^{n} b_i x_i - \Big(2\alpha(r) - 1\Big)\sum_{i=1}^{n} a_i x_i \geq r_f - r, \text{ if } \alpha(r) \leq 0.5 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.45}$$

Please note that the objective function is $(2\beta - 1)\sum_{i=1}^{n} a_i x_i + 2(1-\beta)\sum_{i=1}^{n} b_i x_i$ because the confidence level β should be high enough to be greater than 0.5.

Example 3.41. When all the security returns are regarded to be normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, the β-return-risk model (3.43) becomes

$$\begin{cases} \max \sum_{i=1}^{n} e_i x_i - \frac{\sqrt{6}}{\pi} \ln \frac{\beta}{1-\beta} \sum_{i=1}^{n} \sigma_i x_i \\ \text{subject to:} \\ \qquad \sum_{i=1}^{n} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1 - \alpha(r)}{\alpha(r)} \right) x_i \geq r_f - r, \quad \forall r \geq 0 \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.46}$$

The objective function is $\sum_{i=1}^{n} e_i x_i - \frac{\sqrt{6}}{\pi} \ln \frac{\beta}{1-\beta} \sum_{i=1}^{n} \sigma_i x_i$ because the confidence level β should be high enough to be greater than 0.5.

Example 3.42. Suppose the return rates of the i-th securities are normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i), i = 1, 2, \cdots, m$, and the return rates of the j-th securities are triangular fuzzy variables $\xi_j = (a_j, b_j, c_j), j = m+1, m+2, \cdots, n$, respectively. Then the fuzzy β-return-risk model can be transformed into the following form:

$$
\begin{cases}
\max \sum_{i=1}^{m} e_i x_i - \frac{\sqrt{6}}{\pi} \ln \frac{\beta}{1-\beta} \sum_{i=1}^{m} \sigma_i x_i + \\
\quad \sum_{i=m+1}^{n} (2\beta - 1) a_i x_i + 2(1-\beta) \sum_{i=m+1}^{n} b_i x_i \\
\text{subject to:} \\
\quad \sum_{i=1}^{m} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i + \Big(2\alpha(r) - 1\Big) \sum_{i=m+1}^{n} c_i x_i + \\
\quad \Big(2 - 2\alpha(r)\Big) \sum_{i=m+1}^{n} b_i x_i \ge r_f - r, \text{ if } \alpha(r) > 0.5 \\
\quad \sum_{i=1}^{m} \left(e_i - \frac{\sqrt{6}\sigma_i}{\pi} \ln \frac{1-\alpha(r)}{\alpha(r)} \right) x_i + 2\alpha(r) \sum_{i=m+1}^{n} b_i x_i - \\
\quad \Big(2\alpha(r) - 1\Big) \sum_{i=m+1}^{n} a_i x_i \ge r_f - r, \text{ if } \alpha(r) \le 0.5 \\
\quad x_1 + x_2 + \cdots + x_n = 1 \\
\quad x_i \ge 0, \quad i = 1, 2, \cdots, n.
\end{cases}
\tag{3.47}
$$

Since the sum weighted normal fuzzy variable is still a normal fuzzy variable, $\sum_{i=1}^{m} \xi_i x_i$ is a normal fuzzy variable $\mathcal{N}\Big(\sum_{i=1}^{m} e_i x_i, \sum_{i=1}^{m} \sigma_i x_i\Big)$. Since the sum weighted triangular fuzzy variable is still a triangular fuzzy variable, $\sum_{i=m+1}^{n} \xi_i x_i$ is a triangular fuzzy variable $\Big(\sum_{i=m+1}^{n} a_i x_i, \sum_{i=m+1}^{n} b_i x_i, \sum_{i=m+1}^{n} c_i x_i\Big)$. Since the confidence level β should be high enough to be greater than 0.5, we can get Model (3.47) from Models (3.45) and (3.46) directly.

3.3.3 An Example

Suppose an investor wants to choose an optimal portfolio from ten securities whose return rates are given in Table 3.1 in Subsection 3.2.5. The monthly risk-free interest rate is still 0.01, and the investor gives his/her confidence curve is the same as follows:

$$\alpha(r)=\begin{cases}-2.75r+0.43, & 0\le r\le 0.12,\\ -0.5r+0.16, & 0.12\le r\le 0.3,\\ 0.01, & r\ge 0.3.\end{cases}$$

Suppose this time, the investor wants to pursue a maximum return value at confidence 0.95 among the safe portfolios. According to the β-return-risk selection idea, we build the model as follows:

$$\begin{cases}\max \bar{f}\\ \text{subject to:}\\ \qquad \mathrm{Cr}\{\xi_1x_1+\xi_2x_2+\cdots+\xi_{10}x_{10}\ge \bar{f}\}\ge 0.95\\ \qquad R(x_1,x_2,\cdots,x_{10};r)\le\alpha(r),\ \forall r\ge 0\\ \qquad x_1+x_2+\cdots+x_{10}=1\\ \qquad x_i\ge 0,\quad i=1,2,\cdots,10\end{cases}\tag{3.48}$$

where $R(x_1,x_2,\cdots,x_{10};r)$ is the risk curve of the portfolio defined as

$$R(x_1,x_2,\cdots,x_{10};r)=\mathrm{Cr}\{0.01-(\xi_1x_1+\xi_2x_2+\cdots+\xi_{10}x_{10})\ge r\}.$$

According to Model (3.47), we can change Model (3.48) into the following linear programming form. Note that $\alpha(r)<0.5$ in the example.

$$\begin{cases}\max \sum_{i=1}^{5}e_ix_i-\frac{\sqrt{6}}{\pi}\sum_{i=1}^{5}\sigma_ix_i\ln 19+0.9\sum_{i=6}^{10}a_ix_i+0.1\sum_{i=6}^{10}b_ix_i\\ \text{subject to:}\\ \qquad \sum_{i=1}^{5}\left(e_i-\frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{1-\alpha(r)}{\alpha(r)}\right)x_i+2\alpha(r)\sum_{i=6}^{10}b_ix_i-\\ \qquad \Big(2\alpha(r)-1\Big)\sum_{i=6}^{10}a_ix_i\ge r_f-r\\ \qquad x_1+x_2+\cdots+x_{10}=1\\ \qquad x_i\ge 0,\quad i=1,2,\cdots,10.\end{cases}\tag{3.49}$$

Since the investor's confidence curve is a horizontal line when $r\ge 0.3$ and risk curve of the portfolio is a decreasing function of r, when checking if risk curve of the portfolio is totally below the investor's confidence curve, it is enough to check if the points on the risk curve are all lower than the points on the confidence curve for $(r=0,\alpha=0.43)$, $(r=0.02,\alpha=0.375)$, $(r=$

Table 3.4 Allocation of Money to Ten Securities

Security i	1	2	3	4	5
Allocation of money	0.00%	0.00 %	0.00%	0.00%	0.00%
Security i	6	7	8	9	10
Allocation of money	60%	40 %	0.00%	0.00%	0.00%

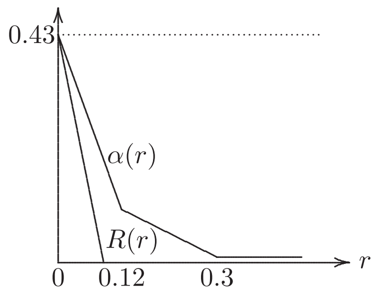

Fig. 3.14 Risk curve $R(r)$ and confidence curve $\alpha(r)$ of Model (3.49).

$0.04, \alpha = 0.32)$, $(r = 0.06, \alpha = 0.265), \cdots, (r = 0.3, \alpha = 0.01)$. By using "Solver" in "Excel", we get the optimal portfolio shown in Table 3.4. The maximum return the investor can obtain at credibility 0.95 is -0.066. As shown in Fig. 3.14, risk curve of the optimal portfolio is totally below the investor's confidence curve. Given every loss value r, the loss occurrence credibility is not greater than the investor's tolerable credibility. Or given every occurrence

Table 3.5 Optimal Portfolios Produced by Different Selection Criteria

Optimal Portfolio	Mean-Risk Criterion	β-Return-Risk Criterion
ξ_1	0.00%	0.00%
ξ_2	0.00 %	0.00%
ξ_3	0.00%	0.00%
ξ_4	0.00%	86.02%
ξ_5	0.00 %	13.98%
ξ_6	0.00 %	0.00%
ξ_7	78.57%	0.00%
ξ_8	0.00%	0.00%
ξ_9	0.00 %	0.00%
ξ_{10}	21.43 %	0.00%
Expected Return	4.2%	3.55%
0.95-Return	−9.9%	−6.6%

credibility $\alpha(r)$, the loss level is not greater than the investor's tolerable loss level.

Remark 3.7. We put the results of examples of mean-risk model in subsection 3.2.5 and β-return-risk model in this subsection together in Table 3.5. It is seen that even when risk-free interest rate, the alternative individual securities and the investor's confidence curve are same, adopting different selection criteria produces different results.

3.4 Credibility Minimization Model

3.4.1 Credibility Minimization Model

When investors are sensitive to only one disastrous low return level, risk curve will degenerate to the credibility of a portfolio return below the specific disaster level, which is proposed to be an alternative definition of risk by Huang [27] in fuzzy portfolio selection. If the investors adopt this definition of risk, they will select the portfolio with minimum occurrence credibility of the specific disastrous return level. Let ξ_i be the i-th security returns and x_i the investment proportions, $i = 1, 2, \cdots, n$, respectively. Taking investment return into account, the investors should select the portfolio whose expected return is not less than a preset expected value and in the meantime whose occurrence credibility of the sensitive bad event is minimum. The selection idea of minimizing the occurrence credibility of the sensitive bad event is expressed as follows:

$$\begin{cases} \min \mathrm{Cr}\left\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \le d\right\} \\ \text{subject to:} \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \ge a \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.50}$$

where d is the concerned disastrous low return level and a the preset minimum expected return that the investors can accept.

Let us recall the definition of risk curve. The curve

$$R(x_1, x_2, \cdots, x_n; r) = \mathrm{Cr}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \ge r\}, \forall r \ge 0$$

is called the risk curve of the portfolio, where r_f is the risk-free interest rate. Let r degenerate to one specific number r_0, then the risk curve becomes

$$\begin{aligned} R(x_1, x_2, \cdots, x_n; r_0) &= \mathrm{Cr}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \ge r_0\} \\ &= \mathrm{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \le r_f - r_0\} \end{aligned}$$

which is just the risk definition of occurrence credibility of a sensitive low return event. It is clear that $r_f - r_0 = d$.

If the investors pre-give a confidence level α, what will be the maximum potential loss for the given confidence level? We can use Value-at-Risk-in-Fuzziness (VaRF) to answer the question.

Definition 3.16 *Let ξ denote a fuzzy return of a portfolio, and r_f the risk-free interest rate. Then Value-at-Risk-in-Fuzziness* (VaRF) *is defined as*

$$\mathrm{VaRF}(\alpha) = \sup\{\bar{r} | \mathrm{Cr}\{r_f - \xi \geq \bar{r}\} \leq 1 - \alpha\}. \tag{3.51}$$

where α is the preset confidence level.

For example, if VaRF(95%) = 8%, it means that there is only a 5% credibility that the portfolio return rate will drop more than 8% below the risk-free interest rate. It is easy to see that VaRF is in fact an inverse version of the risk definition of the credibility of a portfolio return below a specific disaster level.

If the investors adopt VaRF as the investment risk, they will select the portfolio with minimum VaRF value. Taking investment return into account, the investors should select the portfolio whose expected return is not less than a preset level and in the meantime whose VaRF value is the minimum. The selection idea of minimizing VaRF can be expressed by the following model:

$$\begin{cases} \min \bar{r} \\ \text{subject to:} \\ \qquad \mathrm{Cr}\left\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq \bar{r}\right\} \leq 1 - \alpha \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \geq a \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.52}$$

where a is the pre-set tolerable minimum expected return, α the pre-determined confidence level, and $\bar{r}$ the VaRF defined as

$$\mathrm{VaRF}(\alpha) = \sup\{\bar{r} | \mathrm{Cr}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq \bar{r}\} \leq 1 - \alpha\}.$$

It is seen that the VaRF minimization model (3.52) can be regarded as another version of credibility minimization model (3.50).

Mathematically, Model (3.52) is a minmax model because it is equivalent to

$$\begin{cases} \min\limits_{x_1,x_2,\cdots,x_n} \max\limits_{\bar{r}} \bar{r} \\ \text{subject to:} \\ \qquad \mathrm{Cr}\{r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq \bar{r}\} \leq 1 - \alpha \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \geq a \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.53}$$

where $\max \bar{r}$ is the VaRF.

3.4.2 Crisp Equivalent

In some special cases, we can convert the credibility minimization model (3.50) into its crisp equivalent.

Example 3.43. When people invest, the credibility of portfolio return equal to or lower than a sensitive disaster level d should always be required to be less than 0.5. Thus, when security returns are regarded to be all triangular fuzzy variables $\xi_i = (a_i, b_i, c_i)$, the credibility minimization model (3.50) becomes

$$\begin{cases} \min \left(d - \sum\limits_{i=1}^{n} a_i x_i\right) \Big/ \left(\sum\limits_{i=1}^{n} b_i x_i - \sum\limits_{i=1}^{n} a_i x_i\right) \\ \text{subject to:} \\ \qquad \sum\limits_{i=1}^{n} a_i x_i \leq d \leq \sum\limits_{i=1}^{n} b_i x_i \\ \qquad \sum\limits_{i=1}^{n} (a_i x_i + 2b_i x_i + c_i x_i) \geq 4a \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.54}$$

Please note that the constraint $\sum\limits_{i=1}^{n} a_i x_i \leq d \leq \sum\limits_{i=1}^{n} b_i x_i$ is added because the credibility of portfolio return equal to or lower than the concerned disaster level d should be less than 0.5.

Example 3.44. When security returns are regarded to be all normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, the credibility minimization model (3.50) becomes

$$\left\{\begin{array}{l}\max\left(\sum_{i=1}^{n}e_ix_i-d\right)\Big/\sum_{i=1}^{n}\sigma_ix_i\\ \text{subject to:}\\ \qquad d\le e_1x_1+e_2x_2+\cdots+e_nx_n\\ \qquad e_1x_1+e_2x_2+\cdots+e_nx_n\ge a\\ \qquad x_1+x_2+\cdots+x_n=1\\ \qquad x_i\ge 0,\quad i=1,2,\cdots,n.\end{array}\right. \tag{3.55}$$

Model (3.55) can easily be obtained because the sum of weighted normal fuzzy variables is still a normal fuzzy variable. Please note that to minimize the credibility value

$$\left(1+\exp\left(\pi\Big(\sum_{i=1}^{n}e_ix_i-d\Big)\Big/\sqrt{6}\sum_{i=1}^{n}\sigma_ix_i\right)\right)^{-1}$$

we just need to maximize

$$\Big(\sum_{i=1}^{n}e_ix_i-d\Big)\Big/\sum_{i=1}^{n}\sigma_ix_i.$$

A constraint $d\le e_1x_1+e_2x_2+\cdots+e_nx_n$ is added because the chance of portfolio return equal to or less than the concerned disaster level d should be less than 0.5.

Theorem 3.15 *Let Φ_i denote the credibility distributions of the i-th fuzzy security return rates $\xi_i, i=1,2,\cdots,n$, respectively. Then the* VaRF *minimization model (3.52) can be transformed into the following linear model:*

$$\left\{\begin{array}{l}\min r_f-x_1\Phi_1^{-1}(1-\alpha)-x_2\Phi_2^{-1}(1-\alpha)+\cdots-x_n\Phi_n^{-1}(1-\alpha)\\ \textit{subject to:}\\ \qquad x_1E[\xi_1]+x_2E[\xi_2]+\cdots+x_nE[\xi_n]\ge a\\ \qquad x_1+x_2+\cdots+x_n=1\\ \qquad x_i\ge 0,\quad i=1,2,\cdots,n.\end{array}\right. \tag{3.56}$$

Proof: It follows directly from Theorem 3.12 and the monotonicity property of credibility measure.

Example 3.45. Suppose the return rates of the i-th securities are normal fuzzy variables $\xi_i\sim\mathcal{N}(e_i,\sigma_i), i=1,2,\cdots,m$, and the return rates of the j-th securities are triangular fuzzy variables $\xi_j=(a_j,b_j,c_j), j=m+1,m+2,\cdots,n$, respectively. Since confidence level $\alpha>0.5$ and $1-\alpha<0.5$, the VaRF minimization model (3.52) can be transformed into the following form:

$$\begin{cases} \min r_f - \sum_{i=1}^{m}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln\frac{\alpha}{1-\alpha}\right)x_i - \\ \qquad 2(1-\alpha)\sum_{i=m+1}^{n} b_i x_i + \left(1-2\alpha\right)\sum_{i=m+1}^{n} a_i x_i \\ \text{subject to:} \\ \qquad \sum_{i=1}^{m} e_i x_i + \sum_{i=m+1}^{n}\frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \geq a \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.57}$$

3.4.3 An Example

Suppose an investor wants to choose an optimal portfolio from ten securities of which five security return rates are normal fuzzy variables and the rest five the triangular fuzzy variables. The prediction of the return rates of the ten securities is given in Table 3.6. The risk-free interest rate is assumed to be 0.01. Suppose the minimum expected return the investor can accept is 0.03, and the investor wants to minimize the specific potential loss at confidence level 0.95. Then according to the VaRF minimization selection idea , we build the model as follows:

$$\begin{cases} \min \bar{r} \\ \text{subject to:} \\ \qquad \mathrm{Cr}\{0.01 - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}) \geq \bar{r}\} \leq 0.05 \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geq 0.03 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.58}$$

According to model (3.57), we change the model (3.58) into the following form:

$$\begin{cases} \min 0.01 - \sum_{i=1}^{5}\left(e_i - \frac{\sqrt{6}\sigma_i}{\pi}\ln 19\right)x_i - \sum_{i=6}^{10}(0.1 b_i x_i + 0.9 a_i x_i) \\ \text{subject to:} \\ \qquad \sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10}\frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \geq 0.03 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.59}$$

Table 3.6 Fuzzy Return Rates of Ten Securities

Security i	$\xi_i \sim \mathcal{N}(e_i, \sigma_i)$	Security i	$\xi_i = (a_i, b_i, c_i)$
1	$\mathcal{N}(0.033,\ 0.44)$	6	$(-0.008, 0.026, 0.06)$
2	$\mathcal{N}(0.032,\ 0.40)$	7	$(-0.02, 0.030, 0.08)$
3	$\mathcal{N}(0.039,\ 0.45)$	8	$(-0.01, 0.032, 0.08)$
4	$\mathcal{N}(0.031,\ 0.39)$	9	$(-0.05, 0.04, 0.10)$
5	$\mathcal{N}(0.025,\ 0.32)$	10	$(-0.03, 0.03, 0.09)$

Table 3.7 Allocation of Money to Ten Securities

Security i	1	2	3	4	5
Allocation of money	0.00%	0.00 %	0.00%	0.00%	0.00%
Security i	6	7	8	9	10
Allocation of money	46.67%	0.00%	53.33%	0.00%	0.00%

Using "Solver" in "Excel", we obtain the optimal portfolio shown in Table 3.7. The minimum objective value is 0.015, which means that if the investor invests 46.67% of his/her money in security 6 and 53.33% in security 8, the expected return will not be lower than 0.03, and in the meantime there is only a 5% credibility that the portfolio return rate will drop more than 1.5% below the risk-free interest rate.

3.5 Mean-Variance Model

3.5.1 Mean-Variance Model

Risk curve takes a panoramic view of the whole likely loss events. Sometimes people wish to use average information to evaluate the risk. As a counterpart of Markowitz's mean-variance model, Huang [33] proposed credibilistic mean-variance model for portfolio selection with fuzzy returns.

Let ξ_i represent the fuzzy returns of the i-th securities and x_i the investment proportions in the i-th securities, $i = 1, 2, \cdots, n$, respectively. The philosophy of the mean-variance model is to pursue the maximum expected return with the variance not greater than the preset level. Let γ be the maximum variance level the investors can tolerate. The credibilistic mean-variance selection model is expressed as follows:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.60}$$

where E denotes the expected value operator, and V the variance operator of the fuzzy variables. The constraint $V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \le \gamma$ ensures that the optimal portfolio will be selected only from the portfolios whose average square deviations from the expected return are not greater than the tolerable level.

Sometimes, the investors may preset a level of expected return. Then the philosophy of mean-variance model becomes to minimize variance value of the portfolio with the expected value of the portfolio not less than this preset expected return level. Thus the credibilistic mean-variance model is expressed in the following way:

$$\begin{cases} \min V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \ge \lambda \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.61}$$

where λ represents the minimum expected return the investors feel satisfactory. The constraint $E[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \ge \lambda$ ensures that the optimal portfolio is selected only among those satisfactory portfolios, i.e., the portfolios whose expected return will not be less than the preset expected return level.

From Models (3.60) and (3.61), we can see that if we change the preset variance value or expected value, we will get different optimal solution. A portfolio is efficient if it is impossible to obtain higher expected return with no greater variance value, or it is impossible to obtain less variance value with no less expected return. All efficient portfolios make up the efficient frontier. An efficient portfolio is in fact an solution of the following optimization model with two objectives:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \min V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.62}$$

Different investors will find different optimal portfolios from the efficient frontier according to their own preferences to risk aversion, i.e., tradeoff of variance and expected return.

3.5.2 Crisp Equivalent

According to the properties of triangular fuzzy variable, trapezoidal fuzzy variable and normal fuzzy variable, we give the crisp equivalents of credibilistic mean-variance model in some special cases.

When all the security returns are described by symmetrical triangular fuzzy variables $\xi_i = (a_i, b_i, c_i)$, the mean-variance model (3.60) becomes

$$\begin{cases} \max \sum_{i=1}^{n} a_i x_i + 2\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i \\ \text{subject to:} \\ \qquad \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} a_i x_i \le \sqrt{24\gamma} \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.63}$$

When all the security returns are described by symmetrical trapezoidal fuzzy variables $\xi_i = (a_i, b_i, c_i, d_i)$, the mean-variance model (3.60) becomes

$$\begin{cases} \max \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} d_i x_i \\ \text{subject to:} \\ \qquad \left(\sum_{i=1}^{n} d_i x_i - \sum_{i=1}^{n} a_i x_i\right)^2 + \left(\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} b_i x_i\right)^2 \\ \qquad +\left(\sum_{i=1}^{n} d_i x_i - \sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} b_i x_i\right) \le 24\gamma \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.64}$$

When all the security returns are described by normal fuzzy variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, the mean-variance model (3.60) becomes

$$\begin{cases} \max e_1 x_1 + e_2 x_2 + \cdots + e_n x_n \\ \text{subject to:} \\ \qquad \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n \le \sqrt{\gamma} \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.65}$$

3.5.3 An Example

Suppose an investor wants to select his/her portfolio from the ten securities whose returns are given in Table 3.8. If the investor adopts the mean-variance selection idea, and set the minimum expected return at 0.07. Then according to the credibilistic mean-variance model, the investor should select the portfolio according to the following model:

$$\begin{cases} \min V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \\ \text{subject to:} \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geq 0.07 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.66}$$

According to the properties of triangular fuzzy variable, we change Model (3.66) into the following crisp form:

$$\begin{cases} \min \sum_{i=1}^{10} c_i x_i - \sum_{i=1}^{10} a_i x_i \\ \text{subject to:} \\ \qquad \sum_{i=1}^{10} (a_i x_i + 2b_i x_i + c_i x_i) \geq 0.28 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.67}$$

Table 3.8 Fuzzy Return Rates of Ten Securities

Security i	$\xi_i = (a_i, b_i, c_i)$	Security i	$\xi_i = (a_i, b_i, c_i)$
1	(-0.08, 0.02, 0.12)	6	$(-0.09, 0.06, 0.15)$
2	(-0.1, 0.04, 0.14)	7	$(-0.16, 0.09, 0.25)$
3	(-0.12, 0.05, 0.17)	8	$(-0.18, 0.06, 0.24)$
4	(-0.11, 0.06, 0.17)	9	$(-0.15, 0.08, 0.23)$
5	(-0.12, 0.06, 0.18)	10	$(-0.22, 0.1, 0.32)$

Table 3.9 Allocation of Money to Ten Securities

Security i	1	2	3	4	5
Allocation of money	0.00%	0.00 %	0.00%	0.00%	0.00%
Security i	6	7	8	9	10
Allocation of money	0.00 %	66.67 %	0.00 %	0.00%	33.33%

By using "Solver" in "Excel", in order to minimize the variance with the expected return not less than 0.07, the investor should allocate his/her money according to Table 3.9. The objective value of Model (3.67) is 0.433, which means the minimum variance is $0.433^2/24 = 0.0086$.

3.5.4 Mean-Semivariance Model

When the membership functions of the security returns are asymmetrical, variance becomes a deficient measure of risk because when eliminating variance both lower and higher deviations from the expected value are eliminated, yet higher deviations are what we want. Empirical evidences [3, 12, 16, 86] show that there do exist cases that security returns are not symmetrically distributed. Therefore, Huang [37] defined semivariance of fuzzy variable that only measures the lower deviation from the expected value and proposed credibilistic mean-semivariance model in which semivariance replaces variance as the measure of risk.

Let x_i be the investment proportions in securities i, and ξ_i the i-th fuzzy security returns, $i = 1, 2, \cdots, n$, respectively. Similar to mean-variance model, the philosophy of the mean-semivariance model is to maximize the expected return at the given level of risk. Substituting variance with semivariance, we have the credibilistic mean-semivariance model as follows:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad SV[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.68}$$

where γ denotes the maximum semivariance level the investors can tolerate, E the expected value operator, and SV the semivariance of the fuzzy variables.

Sometimes the investors may preset a minimum acceptable expected return level, then the mean-semivariance model is expressed as follows:

$$\begin{cases} \min SV[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \ge \lambda \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases}$$

where λ denotes the minimum expected return that the investors can accept.

From Theorem 3.8 we know that when the membership functions of the security returns are symmetrical, optimal portfolio can be obtained no matter whether we take the variance or the semivariance as the measurement of risk.

However, when membership functions of security returns are asymmetrical, Theorem 3.7 indicates that taking semivariance or variance as the measurement of risk will yield different results. So semivariance can be regarded as an improvement of variance as the measure of risk because semivariance is free from the reliance on symmetrical security returns.

3.6 Entropy Optimization Model

Given a fuzzy portfolio return, investors will usually regard the portfolio to be risky if it is difficult to predict the specific value that the portfolio return may take. Fuzzy entropy measures the difficulty degree of the prediction. When a portfolio return distributes dispersively, the entropy of the return is great, which implies that the return contains much uncertainty and the prediction is difficult; when the portfolio return distributes concentratively, the entropy of the return is small, which implies that the return contains little uncertainty and the prediction is easy. In addition, entropy can well reflect the dispersive degree of the portfolio return no matter if the membership function of the portfolio return is symmetrical or not. Therefore, Huang [39] suggested entropy being another alternative measure of risk and proposed entropy optimization model in which the philosophy is to pursue the maximum expected return among the portfolios whose return distribution is concentrative enough to the required level.

Let ξ_i denote the i-th fuzzy security returns, x_i the investment proportions, $i = 1, 2, \cdots, n$, respectively, and γ the preset entropy level. The mean-entropy model is as follows:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad H[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n \end{cases} \tag{3.69}$$

where E is the expected value operator and H the entropy. The constraint $H[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma$ means that the optimal portfolio must be selected from the portfolios whose returns are concentrative enough to be less than a preset tolerable level. Compared with the mean-variance model (3.60), entropy is more general than variance as a measure of risk because entropy is free from reliance on symmetrical distribution of the security returns, i.e., entropy remains an effective measure of risk when the membership functions of the security returns are asymmetrical. However, when security returns are symmetrical triangular fuzzy variables or the normally distributed fuzzy variables, the optimal solution of the mean-entropy model (3.69) is also the optimal solution of of the mean-variance model (3.60) and vice versa because in these two special cases, entropies can be expressed by the product of some constant and the corresponding variances.

3.7 Hybrid Intelligent Algorithm

We have given the crisp equivalents of the fuzzy portfolio selection models in some special cases so that we can find the optimal portfolios by traditional methods. However, in many cases it is difficult to convert the fuzzy selection models into their equivalents. To produce a general solution algorithm, we integrate fuzzy simulation and genetic algorithm (GA) to produce the hybrid intelligent algorithm. In our algorithm, generally speaking, fuzzy simulation is used to calculate the objective and constraint values, and GA is employed to find the optimal solution. A scheme of the algorithm (Fig. 3.15) is given as follows:

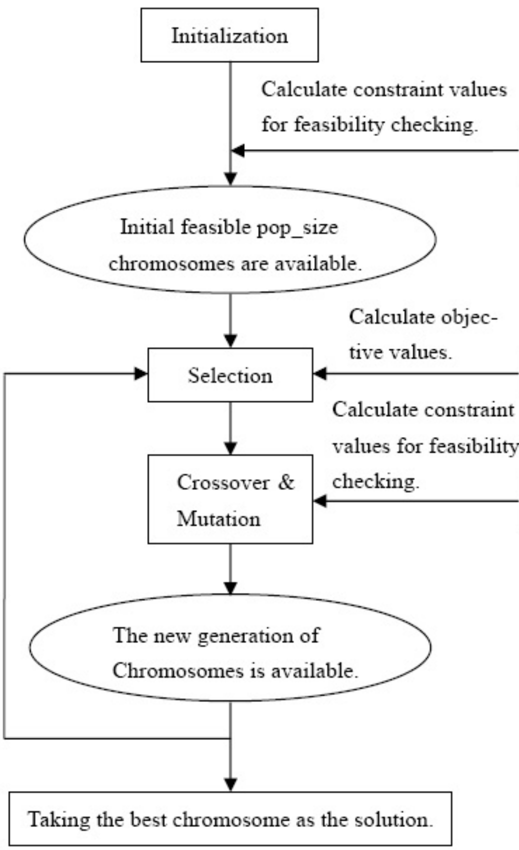

Fig. 3.15 Hybrid intelligent algorithm.

3.7.1 Fuzzy Simulation

Fuzzy simulation has been studied by many scholars. In particular, Liu [56] introduced in detail the technique based on the concept of credibility. Here, we will introduce the simulation procedure for calculating the objective and constraint values appeared in our optimization models, i.e., the expected value, the variance value, the semivariance value, the β-return value, the credibility value, and the entropy value of the fuzzy portfolio return.

Let ξ_i be fuzzy returns with membership functions μ_i, and x_i the investment proportions, $i = 1, 2, \cdots, n$, respectively, where n is the number of securities. For convenience, let $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$, and denote the membership function vector of $\boldsymbol{\xi}$. Since the variance value and the semivariance value is a kind of expected value, we can know the simulation procedure for them if we know the simulation procedure for calculating the expected value. In addition, if we know how to calculate $\mathrm{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r\}$, we can know how to calculate $\mathrm{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq r\}$ because $\mathrm{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq r\} = \mathrm{Cr}\{-f(\boldsymbol{x}, \boldsymbol{\xi}) \geq -r\} = \mathrm{Cr}\{f^{'}(\boldsymbol{x}, \boldsymbol{\xi}) \geq r^{'}\}$. Thus, we in fact only need to calculate the values of the following four types of uncertain functions:

$$\begin{aligned}
U_1: &\quad \boldsymbol{x} \rightarrow \mathrm{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r\}, \\
U_2: &\quad \boldsymbol{x} \rightarrow E\left[f(\boldsymbol{x}, \boldsymbol{\xi})\right], \\
U_3: &\quad \boldsymbol{x} \rightarrow \max\{\bar{f} | \mathrm{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \bar{f}\} \geq \beta\}, \\
U_4: &\quad \boldsymbol{x} \rightarrow H[f(\boldsymbol{x}, \boldsymbol{\xi})].
\end{aligned}$$

Simulation for Credibility Value

According to Theorem 3.4 and the credibility inversion theorem, we know that

$$\begin{aligned}
\mathrm{Cr}\left\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r\right\} = \quad & \frac{1}{2}\left(\sup_{x_1, x_2, \cdots, x_n \in \Re} \left\{ \min_{1 \leq i \leq n} \mu_i(x_i) \;\middle|\; f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r \right\} \right. \\
& \left. +1 - \sup_{x_1, x_2, \cdots, x_n \in \Re} \left\{ \min_{1 \leq i \leq n} \mu_i(x_i) \;\middle|\; f(\boldsymbol{x}, \boldsymbol{\xi}) < r \right\} \right).
\end{aligned}$$

Thus we j times randomly generate real numbers u_{ij} such that $\mu_i(u_{ij}) \geq \varepsilon, i = 1, 2, \cdots, n, j = 1, 2, \cdots, N$ respectively, where ε is a sufficiently small number, and N is a sufficiently large number. Let $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$, and $\mu(\boldsymbol{u}_j) = \mu_{1j}(u_{1j}) \wedge \mu_{2j}(u_{2j}) \wedge \cdots \wedge \mu_{nj}(u_{nj})$. Then the credibility $\mathrm{Cr}\left\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r\right\}$ can be obtained approximately by the following formula

$$L = \frac{1}{2}\left(\max_{1 \leq j \leq N} \left\{ \mu(\boldsymbol{u}_j) \;\middle|\; f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r \right\} + 1 - \max_{1 \leq j \leq N} \left\{ \mu(\boldsymbol{u}_j) \;\middle|\; f(\boldsymbol{x}, \boldsymbol{\xi}) < r \right\} \right),$$

where N is a sufficiently large number.

The fuzzy simulation process for computing $\text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq r\}$ is summarized as follows:

Step 1. Let $j = 1$.
Step 2. Randomly generate real numbers u_{ij} such that $\mu_i(u_{ij}) \geq \varepsilon, i = 1, 2, \cdots, n, j = 1, 2, \cdots, N$ respectively, where ε is a sufficiently small number, and N is a sufficiently large number.
Step 3. Set $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$, and $\mu(\boldsymbol{u}_j) = \mu_{1j}(u_{1j}) \wedge \mu_{2j}(u_{2j}) \wedge \cdots \wedge \mu_{nj}(u_{nj})$.
Step 4. $j \leftarrow j + 1$. Turn back to Step 2 if $j \leq N$, where N is a sufficiently large number. Otherwise, turn to Step 5.
Step 5. Return L.

Simulation for Expected Value

According to the definition of expected value of fuzzy variable, we have

$$E[f(\boldsymbol{x}, \boldsymbol{\xi})] = \int_0^{+\infty} \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq t\}\text{d}t - \int_{-\infty}^0 \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq t\}\text{d}t.$$

Thus we design the procedure as follows:

Step 1. Set $E = 0$.
Step 2. Randomly generate real numbers u_{ik} such that $\mu_i(u_{ij}) \geq \varepsilon, i = 1, 2, \cdots, n, j = 1, 2, \cdots, N$ respectively, where ε is a sufficiently small number, and N is a sufficiently large number. Denote $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$.
Step 3. Set $a = f(\boldsymbol{x}, \boldsymbol{u}_1) \wedge f(\boldsymbol{x}, \boldsymbol{u}_2) \wedge \cdots \wedge f(\boldsymbol{x}, \boldsymbol{u}_N)$, $b = f(\boldsymbol{x}, \boldsymbol{u}_1) \vee f(\boldsymbol{x}, \boldsymbol{u}_2) \vee \cdots \vee f(\boldsymbol{x}, \boldsymbol{u}_N)$.
Step 4. Randomly generate t from $[a, b]$.
Step 5. If $t \geq 0$, then $E \leftarrow E + \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq t\}$.
Step 6. If $t < 0$, then $E \leftarrow E - \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq t\}$.
Step 7. Repeat the fourth to sixth steps for N times, where N is a sufficiently large number.
Step 8. $E\left[f(\boldsymbol{x}, \boldsymbol{\xi})\right] = a \vee 0 + b \wedge 0 + E \cdot (b - a)/N$.

Simulation for *β*-Return

In order to compute the uncertain function U_3, we randomly generate real numbers u_{ij} such that $\mu_i(u_{ij}) \geq \varepsilon, i = 1, 2, \cdots, n, j = 1, 2, \cdots, N$ respectively, where ε is a sufficiently small number, and N is a sufficiently large number. Denote $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$. For any real numbers r, we set

$$\begin{aligned} D(r) = \quad & \frac{1}{2}\left(\max_{1\leq j\leq N}\{\min_{1\leq i\leq n} \boldsymbol{\mu}_i(\boldsymbol{u}_j)|f(\boldsymbol{x}, \boldsymbol{u}_j) \geq r\} + \right. \\ & \left. 1 - \max_{1\leq j\leq N}\{\min_{1\leq i\leq n} \boldsymbol{\mu}_i(\boldsymbol{u}_j)|f(\boldsymbol{x}, \boldsymbol{u}_j) < r\}\right). \end{aligned}$$

Since $D(r)$ is a monotonous function of r, we may employ a bisection search to find the maximal value r such that $D(r) \geq \beta$. This value is an estimation of U_3, i.e., the β-return value. The fuzzy simulation process for computing U_3 is summarized as follows:

Step 1. Randomly generate real numbers u_{ij} such that $\mu_i(u_{ij}) \geq \varepsilon, i=1,2,\cdots,n, j = 1,2,\cdots,N$, respectively, where ε is a sufficiently small positive number, and N a sufficiently large number. Denote $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$.
Step 2. Find the maximal value r such that $D(r) \geq \beta$ by the bisection search.
Step 3. Return r.

Simulation for Entropy Value

According to the entropy definition, we know that

$$H[f(\boldsymbol{x}, \boldsymbol{\xi})] = \int_{-\infty}^{+\infty} S(\text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) = t\})\text{d}t$$

where $S(y) = -y \ln y - (1-y)\ln(1-y)$.

Thus we design the fuzzy simulation procedure for calculating the entropy $H[f(\boldsymbol{x}, \boldsymbol{\xi})]$ as follows:

Step 1. Set $H = 0$.
Step 2. Randomly generate real numbers u_{ij} such that $\mu_i(u_{ij}) \geq \varepsilon, i = 1,2,\cdots,n, j = 1,2,\cdots,N$, respectively, where ε is a sufficiently small positive number, and N a sufficiently large number. Denote $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \cdots, u_{nj})$.
Step 3. Set $a = \min_{1\leq j\leq N} f(\boldsymbol{x}, \boldsymbol{u}_j)$, and $b = \max_{1\leq j\leq N} f(\boldsymbol{x}, \boldsymbol{u}_j)$.
Step 4. Randomly generate t from $[a, b]$.
Step 5. $H \leftarrow H - (y \ln y + (1-y)\ln(1-y))$, where $y = \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) = t\}$.
Step 6. Repeat the fourth and fifth steps for N times, where N is a sufficiently large number.
Step 7. $H[f(\boldsymbol{x}, \boldsymbol{\xi})] = H \cdot (b-a)/N$.

Example 3.46. Let ξ_1 be a triangular fuzzy security return $(-0.1, 0.1, 0.3)$, and ξ_2 the normal fuzzy security return $\mathcal{N}(0.1, 0.1)$. Portfolio A is composed of 40% of ξ_1 and the rest 60% of ξ_2. A run of the simulation with 4000 cycles shows that the credibility value of the portfolio return not greater than 0 is 0.2490, i.e.,

$$\text{Cr}\{0.4\xi_1 + 0.6\xi_2 \leq 0\} = 0.2490.$$

The simulation procedures are as follows:

Step 1. Let $j = 1$.
Step 2. Randomly generate real numbers a from $(-0.1, 0.3)$ and b from $(-0.4, 0.6)$ (we generate b from $(-0.4, 0.6)$ because $\mu(t) \approx 0$ when $t < e - 5\sigma$, and $\mu(t) \approx 0$ when $t > e + 5\sigma$, where e is the expected value and σ the positive square root of variance of the normal fuzzy variable).

Step 3. Calculate $\mu(a) = \dfrac{a+0.1}{0.1+0.1}$ if $a \le 0.1$, and $\mu(a) = \dfrac{0.3-a}{0.3-0.1}$ if $a > 0.1$, and $\mu(b) = 2\left(1+\exp\left(\dfrac{\pi|b-0.1|}{\sqrt{6}\times 0.1}\right)\right)^{-1}$.
Step 4. Set $\nu_j = \mu(a) \wedge \mu(b)$.
Step 5. $j \leftarrow j+1$. Turn back to Step 2 if $j \le 4000$. Otherwise, turn to Step 6.
Step 6. Return

$$L = \frac{1}{2}\left(\max_{1\le j\le N}\left\{\nu_j \mid 0.4a+0.6b \le 0\right\} + 1 - \max_{1\le j\le N}\left\{\nu_j \mid 0.4a+0.6b > 0\right\}\right).$$

Example 3.47. For the above mentioned portfolio A, assume the investors set the confidence level $\beta = 0.9$. A run of the simulation with 3000 cycles shows that the 0.9-return value of the portfolio is -0.0591, i.e.,

$$\xi(0.9) = \sup\{\bar{f}|\text{Cr}\{0.4\xi_1 + 0.6\xi_2 \ge \bar{f}\} \ge 0.9\} = -0.0591.$$

The simulation procedures are as follows:

Step 1. Randomly generate real numbers a_j from $(-0.1, 0.3)$ and b_j from $(-0.4, 0.6)$ for $j = 1, 2, \cdots, 3000$.
Step 2. Calculate $\mu(a_j) = \dfrac{a+0.1}{0.1+0.1}$ if $a_j \le 0.1$, and $\mu(a_j) = \dfrac{0.3-a}{0.3-0.1}$ if $a_j > 0.1$, and $\mu(b_j) = 2\left(1+\exp\left(\dfrac{\pi|b_j-0.1|}{\sqrt{6}\times 0.1}\right)\right)^{-1}$.
Step 3. Set $\nu_j = \mu(a_j) \wedge \mu(b_j)$.
Step 4. Let

$$D(r) = \frac{1}{2}\left(\max_{1\le j\le N}\left\{\nu_j \mid 0.4a+0.6b \le r\right\} + 1 - \max_{1\le j\le N}\left\{\nu_j \mid 0.4a+0.6b > r\right\}\right).$$

Find the maximal value r such that $D(r) \ge 0.9$ by the bisection search.
Step 5. Return r.

Example 3.48. Let ξ_1 be a triangular fuzzy security return $(-0.2, 0, 0.4)$, and ξ_2 the normal fuzzy security return $\mathcal{N}(0.1, 0.1)$. Portfolio A is composed of 40% of ξ_1 and the rest 60% of ξ_2. A run of the simulation with 8000 cycles shows that the semivariance value of the portfolio, i.e.,

$$SV[0.4\xi_1 + 0.6\xi_2] = 0.0432.$$

The simulation procedures are as follows:

Step 1. Randomly 8000 times generate real numbers a_i from $(-0.2, 0.4)$ and b_i from $(-0.4, 0.6), i = 1, 2, \cdots, 8000$.
Step 2. If $0.4a_i + 0.6b_i - 0.08 \le 0$ (the expected value of Portfolio A is 0.08), set $r_i = (0.4a_i + 0.6b_i - 0.08)^2$; otherwise, set $r_i = 0$.

Step 3. Set $a = r_1 \wedge r_2 \wedge \cdots \wedge r_{8000}$ and $b = r_1 \vee r_2 \vee \cdots \vee r_{8000}$.
Step 4. Set $V = 0$.
Step 5. Randomly generate t from $[a, b]$.
Step 6. If $t \geq 0$, then $V \leftarrow V + \text{Cr}\{(0.4\xi_1 + 0.6\xi_2 - 0.08)^2 \geq t\}$; if $t < 0$, then $V \leftarrow V + 0$.
Step 7. Repeat the fifth to sixth steps 8000 times.
Step 6. $V[0.4\xi_1 + 0.6\xi_2] = a \vee 0 + b \wedge 0 + V \cdot (b - a)/8000$.

Example 3.49. For the above mentioned portfolio A, a run of the simulation with 8000 cycles shows that the variance value of the portfolio, i.e.,

$$V[0.4\xi_1 + 0.6\xi_2] = -0.0710.$$

3.7.2 Hybrid Intelligent Algorithm

When the objective and constraint values have been calculated by fuzzy simulation, simulation results are integrated into the GA introduced in Subsection 2.6.3 to produce a hybrid intelligent algorithm. After selection, crossover and mutation, the new population is ready for its next evaluation. The hybrid intelligent algorithm will continue until a given number of cyclic repetitions of the above steps is met. We summarize the algorithm as follows:

Step 1. Initialize *pop_size* chromosomes.
Step 2. Calculate the objective values for all chromosomes by fuzzy simulation.
Step 3. Give the rank order of the chromosomes according to the objective values, and compute the values of the rank-based evaluation function of the chromosomes.
Step 4. Compute the fitness of each chromosome according to the rank-based-evaluation function.
Step 5. Select the chromosomes by spinning the roulette wheel.
Step 6. Update the chromosomes by crossover and mutation operations.
Step 7. Repeat the second to the sixth steps for a given number of cycles.
Step 8. Take the best chromosome as the solution of portfolio selection.

3.7.3 Numerical Example

Suppose an investor adopts credibility minimization selection idea and wants to choose an optimal portfolio from ten securities of which five security return rates are normal fuzzy variables and the rest five the triangular fuzzy variables. The prediction of the return rates of the ten securities is given in Table 3.10. Suppose the minimum expected return the investor can accept is 0.031, and the investor wants to minimize the occurrence credibility of portfolio return below a disaster level -0.08. According to the credibility minimization selection idea introduced in Subsection 3.4.1, we build the model as follows:

$$\begin{cases} \min \mathrm{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10} \le -0.08\} \\ \text{subject to:} \\ \qquad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \ge 0.031 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.70}$$

The constraint $E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}]$ can be calculated via

$$\sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i).$$

The objective (i.e., credibility value) is calculated by fuzzy simulation. Then the simulation result is integrated into the GA introduced in Subsection 2.6.3 to produce the hybrid intelligent algorithm. A run of the algorithm with 10000 generations shows that in order to minimize the credibility of portfolio return not greater than -0.08 with the constraint that the expected return of the portfolio should not be less than 0.031, the investor should allocate his/her money according to Table 3.11. The minimum credibility level of portfolio

Table 3.10 Fuzzy Return Rates of Ten Securities

Security i	$\xi_i \sim \mathcal{N}(e_i, \sigma_i)$	Security i	$\xi_i = (a_i, b_i, c_i)$
1	$\mathcal{N}(0.033,\ 0.12)$	6	$(-0.08, 0.026, 0.09)$
2	$\mathcal{N}(0.032,\ 0.11)$	7	$(-0.09, 0.030, 0.10)$
3	$\mathcal{N}(0.033,\ 0.14)$	8	$(-0.15, 0.032, 0.16)$
4	$\mathcal{N}(0.031,\ 0.11)$	9	$(-0.12, 0.04, 0.10)$
5	$\mathcal{N}(0.025,\ 0.07)$	10	$(-0.12, 0.05, 0.12)$

Table 3.11 Allocation of Money to Ten Securities

Security i	1	2	3	4	5
Allocation of money	0.00%	85.90%	0.00%	0.00%	0.00%
Security i	6	7	8	9	10
Allocation of money	0.00%	0.00 %	0.00%	0.00%	14.10%

return not greater than -0.08 is 0.1805. The hybrid intelligent algorithm is summarized below.

Hybrid Intelligent Algorithm:

Step 1. Determine representation structure of solutions by chromosomes. In the example, the genes $c_1, c_2, \cdots, c_{10}$ in a chromosome $C = (c_1, c_2, \cdots, c_n)$ are restricted in the interval $[0, 1]$. A solution $\boldsymbol{x} = (x_1, x_2, \cdots, x_{10})$ is matched with a chromosome in the following way,

$$x_i = \frac{c_i}{c_1 + c_2 + \cdots + c_{10}}, \quad i = 1, 2, \cdots, n$$

which ensures that $x_1 + x_2 + \cdots + x_{10} = 1$ always holds.

Step 2. Set parameters $P_c = 0.3, P_m = 0.2, pop\_size = 30$ in the GA.

Step 3. Generate the chromosomes $C = (c_1, c_2, \cdots, c_{10})$ from $[0, 1]^{10}$.

Step 4. Calculate the expected return for each chromosome according to the formula

$$\sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i).$$

Then check the feasibility of the chromosome as follows:

If $\sum_{i=1}^{5} e_i x_i + \sum_{i=6}^{10} \frac{1}{4}(a_i x_i + 2b_i x_i + c_i x_i) \geq 0.031$

return 1;

return 0;

in which 1 means feasible, and 0 non-feasible.

Step 5. Repeat the third and fourth steps until feasible *pop_size* numbers of chromosomes are produced.

Step 6. Calculate the objective values (i.e., credibility values) via fuzzy simulation and give the rank order of the chromosomes according to the objective values to make the better chromosomes take the smaller ordinal numbers.

Step 7. Compute the values of the rank-based evaluation function for all the chromosomes.

Step 8. Calculate the fitness of each chromosome according to the rank-based-evaluation function.

Step 9. Select the chromosomes by spinning the roulette wheel.

Step 10. Update the chromosomes by crossover and mutation operations.

Step 11. Repeat the sixth to tenth steps for 10000 cycles.

Step 12. Take the best chromosome as the solution of the portfolio selection problem.