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" TRANSFORMATIONAL METHODS APPLIED TO SOME ONE-DIMENSIONAL PROBLEMS CONCERNING THE EQUATIONS OF THE NEUTRON TRANSPORT THEORY".

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TRANSFORMATIONAL METHODS APPLIED TO SOME ONE-DIMENSIONAL PROBLEMS CONCERNING THE EQUATIONS OF THE NEUTRON TRANSPORT THEORY

by

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1. Introductory remarks.

There is a very great bibliography concerning the mathematical theory of slowing down and diffusion of the neutrons within the moderating medium of an atomic fission pile. The methods used for such study must allow for the differences in the reactivity that take place in the reactors chiefly during the starting and because of the sinking and the prising of the control bars, so doing a quantitative scheme of the behaviour of the fast neutrons produced in the nuclear fission, to reach, by slowing in the moderating medium, the thermal energy.
The neutrons, endowed with such energy, are "trapped" in turn , by the uranium nuclei producing new nuclear fissions, and so on, by the well known chain reaction.
The methods used for the study of the neutron diffusion in a moderator recall or upon the research, in opportune conditions , of the solution of the Maxwell-Boltzmann integro-differential equation of the transport theory , or on the research of opportune supplementary conditions of solutions of the partial differential equations, of diffusion, derived from the application of these so called "phenomenological theory".
This lecture is concerning the transport theory point of view and, in particular, the application of transformational methods to some one-dimensional problems.
As well known, a neutron is a heavy uncharged elementary particle.
Of the forces which act upon it the nuclear forces are by far the most important and they are only one that need be taken into a account under the conditions in which one is interested in the diffusion of neutrons.

Since these forces have an extremely short range , it follows that:

1) the motion of a neutron can be described in terms of it collisions with atomic nuclei and with other freely moving neutrons;

2) these collisions are well-defined events;

3) between such collisions a neutron moves with a constant velocity, that is in a straight line with a constant speed;

4) the mutual collisions of freely moving neutrons may safely be neglected and only the collision of neutron with atomic nuclei with a surrounding medium need be taken into account;

5) for a neutron travelling at a given speed through a given medium the probability of neutron collision per unit path lenght is a constant;

6) the neutron or neutrons emerging from a collision do so at the point of space where the collision took place.

We will indicate with N the neutron density in a medium (the neutron density N in a medium is a function of the position, denoted by the vector $\vec{r}$, the direction of the neutron $\vec{\Omega}$, its velocity v and the time t) .

Let $\Sigma(v)$ be the total macroscopic cross section for all processes; $\Sigma(v)$ is the inverse mean free path. Let $c(v)$ be the mean number of secondary neutrons produced per collision. The quantity $c(v)\ \Sigma(v)$ is the mean number of secondary of unit path. Let $c(v')\ f\ (v', \vec{\Omega} \mapsto v, \vec{\Omega})\ dv\, d\Omega$ be the mean number of neutrons produced in the velocity range dv and cone $d\Omega$ when a neutron of velocity v' and direction $\vec{\Omega}'$ undergoes a collision with a stationary nucleus. Let $S(\vec{r}\ , v, \vec{\Omega}, t)$ be the source strenght of neutrons in the particular volume element. The rate of change

of $N(\vec{r}, v, \vec{\Omega}, t)$ with time is equal to the number of neutrons scattered to the velocity v and direction $\vec{\Omega}$ from other directions and velocity less the loss due to leakage and scattering out of v and $\vec{\Omega}$.
The transport equation is therefore :

$$1) \quad \frac{\partial N(\vec{r}, v, \vec{\Omega}, t)}{\partial t} = - v \vec{\Omega} \cdot \text{grad } N - v \Sigma(v) N + \iint v' c(v') \Sigma(v') f(v', \vec{\Omega}' \to v, \vec{\Omega}) N(\vec{r}, v', \vec{\Omega}', t) dv' d\Omega' + S(\vec{r}, v, \vec{\Omega}, t) .$$

The physical meaning of this equation is the following . The rate of change $\frac{\partial N}{\partial t}$ is equal to the neutrons scattering to v and $\vec{\Omega}$ plus sources minus the leakage and the scattering out of v and $\vec{\Omega}$.
In the case of the isotropic scattering, the density of the neutrons at $\vec{r}$ is

$$N_o(\vec{r}, v') = \int N(\vec{r}, \vec{\Omega}', v') d\Omega' ,$$

and the following integral equation is obtained

$$(2) \quad v N_o(\vec{r}, v) = \int \frac{\exp(-\rho\Sigma)}{4\pi\rho^2} \left[S(\vec{r}', v, t - \frac{\rho}{v}) + \int v' \Sigma_s (v') f(v' \to v) . N_o(\vec{r}', v', t - \frac{\rho}{v}) dv' \right] dV$$

(dV = volume element) .

If the neutron distribution does not vary with time, then $N(\vec{r}, v, \vec{\Omega})$ satisfies the equation :

$$(3) \quad v\vec{\Omega} . \text{grad } N + v \Sigma(v) N = \iint v'(c(v') \Sigma(v') f(v', \vec{\Omega}' \to v, \vec{\Omega}) N(\vec{r}, v', \vec{\Omega}') dv' d\Omega' + S(\vec{r}, \vec{\Omega}, v) .$$

This equation does not have, generally, a solution and the following modified equation is considered :

$$(4) \qquad v\vec{\Omega}\cdot \text{grad}\ N + (\alpha + v\Sigma)\, N =$$

$$= \iint v'c(v')\ \Sigma(v')\ f(v', \vec{\Omega}' \longrightarrow v, \vec{\Omega})\ N(\vec{r}, v', \vec{\Omega}')\, dv'\, d\Omega',$$

(α is a constant) .

This equation has solutions for certain eigenvalues α_i . The eigenfunctions of the equation satisfy to the following conditions :

a) Continuity at the boundary between two media ;

b) $N_i = 0$ at the free surface for all incoming directions.

A general solution of (1) is :

$$(5) \qquad N(\vec{r}, v, \vec{\Omega}, t) = \sum_i a_i\, N_i(\vec{r}, v, \vec{\Omega})\ \exp\ (\alpha_i t) \ .$$

(We assume that the N_i are a complete set) .
For large t , it is :

$$(6) \qquad N(\vec{r}, \vec{\Omega}, v, t) = N_o(\vec{r}, v, \vec{\Omega})\ \exp\ (\alpha_o t) ,$$

where α_o is the maximum of the α_i' s .

($\alpha_o < 0$) subcritical system, $\alpha_o = 0$ critical system,

$\alpha_o > 0$ supercritical system) .

The coefficients a_i of the series are determined from the boundary conditions which may be specified at some particular time.

2. The one group theory .

Consider the time-dependent equation (1) . If all neutrons have the same velocity v_o, then it is :

$$(7)\quad \begin{cases} N(\vec{r}, \vec{\Omega}, v, t) = \delta(v-v_o)\, N(\vec{r}, \vec{\Omega}, t), \\ f(v', \vec{\Omega}' \rightarrow v, \Omega) = \delta(v'-v_o)\, f(\vec{\Omega}' \rightarrow \vec{\Omega}), \\ S(\vec{r}, \vec{\Omega}, v, t) = \delta(v-v_o)\, S(\vec{r}, \vec{\Omega}, t). \end{cases}$$

Let :

$$(8)\quad \begin{cases} N(\vec{r}, \vec{\Omega}, t) = \int N(\vec{r}, \vec{\Omega}, v, t)\, dv, \\ f(\vec{\Omega}' \rightarrow \vec{\Omega}) = \int f(v', \vec{\Omega}' \rightarrow v, \vec{\Omega})\, dv, \\ S(\vec{r}, \vec{\Omega}, t) = \int S(\vec{r}, \vec{\Omega}, v, t)\, dv. \end{cases}$$

Integrating the transport equation over v, results in the following one velocity group equation :

$$(9)\quad \frac{\partial N(\vec{r}, \vec{\Omega}, t)}{\partial t} = - v_o \vec{\Omega} \cdot \text{grad}\ N - v_o \Sigma N + $$
$$+ v_o c(v_o)\, \Sigma(v_o) \int N(\vec{r}\,', \vec{\Omega}', t)\, f(\vec{\Omega}' \rightarrow \vec{\Omega})\, d\Omega' + S(\vec{r}, \vec{\Omega}, t)$$

[$v_o N = \psi(\vec{r}, \vec{\Omega}, t)$ = angular distribution of the neutron flux.

If we assume :

A) time independence of N ;

B) No variation of Σ and c with neutron velocity ;

C) f $(\vec{\Omega}' \rightarrow \vec{\Omega})$ independent of v; and integrate (1) without the term $\frac{\partial N}{\partial t}$, we obtain :

$$(10) \qquad \vec{\Omega}\,.\,\mathrm{grad}\ \psi(\vec{r},\vec{\Omega}) + \Sigma\,\psi(\vec{r},\vec{\Omega}) = c\,\Sigma\int\psi(\vec{r},\vec{\Omega}')\,f(\vec{\Omega}'\longrightarrow\vec{\Omega})\,d\Omega' + S(\vec{r},\vec{\Omega}),$$

where :

$$(11) \qquad \begin{cases} \psi(\vec{r},\vec{\Omega}) = \int vN(\vec{r},v,\vec{\Omega})\,dv, \\ S(\vec{r},\vec{\Omega}) = \int S(\vec{r},v,\vec{\Omega})\,dv. \end{cases}$$

3. Solution of the one group transport equation in an infinite uniform medium in the case of the time-independence and with a plane source at x = 0.

We consider now the one group time indipendent transport equation for an infinite uniform medium with a plane source at $x = 0$.

That is the equation :

$$(1) \qquad \mu\frac{\partial\psi(x,\mu)}{\partial x} + \Sigma\psi(x,\mu) = \frac{1}{2}\,c\,\Sigma\int_{-1}^{+1}\psi(x,\mu)\,d\mu + \frac{\delta(x)}{4\pi},$$

where $\delta(x)$ is the Dirac "Delta distribution ".

We apply to this equation the method of the Fourier transform. We indicate with $\pi(\tau,\mu)$ the Fourier transformation of $\psi(x,\mu)$, that is we put :

$$\pi(\tau,\mu) = \int_{-\infty}^{+\infty}\psi(x,\mu)\,\exp(-i\,\tau\,x)\,dx\,.$$

We have

$$(2) \quad i\tau\mu\,\pi(\tau,\mu) + \Sigma\,\pi(\tau,\mu) = \frac{c\Sigma}{2}\int_{-1}^{+1}\pi(\tau,\mu)\,d\mu + \frac{1}{4\pi}\,.$$

Now we indicate with $\pi_0(\tau)$ the μ indipendent quantity at the second member of (2) ; and we have ;

(3) $$\pi(\tau,\mu) = \pi_0(\tau) \quad / \, (\Sigma + i\,\tau\,\mu) \, .$$

Substituting back in the equation (2) we have :

$$(\Sigma + i\,\tau\,\mu) \; \frac{\pi_0(\tau)}{\Sigma + i\tau\mu} = \frac{1}{2}\, c\,\Sigma \int_{-1}^{+1} \frac{\pi_0(\tau)}{\Sigma + i\tau\mu}\, d\mu + \frac{1}{4\pi}$$

(4) $$\pi_0(\tau) = \frac{1}{4\,\pi\left[1 - \frac{c\Sigma}{\tau}\ \mathrm{arctg}\ \frac{\tau}{\Sigma}\right.} \, .$$

From the Fourier inversion fourmula we obtain :

(5) $$\psi(x,\mu) = \frac{1}{8\,\pi^2} \int_{-\infty}^{+\infty} e^{i\,\tau\,x} (\Sigma + i\tau\mu)^{-1} \left[1 - \frac{c\Sigma}{2i\tau}\ \lg \left(\frac{\Sigma + i\tau}{\Sigma - i\tau}\right)\right]^{-1} d\tau \, .$$

The total flux is obtained by integration over μ ;

$$\Phi(x) = 2\pi \int_{-1}^{+1} \psi(x,\mu)\, d\mu \, ,$$

that is :

(6) $$\Phi(x) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{i\,\tau\,x} \left[1 - \frac{c\Sigma}{2i\tau}\ \log\ \left(\frac{\Sigma - i\tau}{\Sigma + i\tau}\right)\right]^{-1} .$$

$$(i\,\tau\,)^{-1}\ \log\ \left(\frac{\Sigma + i\tau}{\Sigma - i\tau}\right) d\tau \, .$$

The integral can be evaluated by the method of residues .

The integrand has a simple pole where the term in the denominator vanishes , i.e. where :

(7) $$c\ \Sigma\ \log\ \frac{\Sigma + i\tau}{\Sigma - i\tau} = 2i\,\tau \, .$$

If $c < 1$ the poles are at $\tau = \pm\, i\, K$, where K is given by

(8) $$K/\Sigma = \mathrm{tgh}\ \left(\frac{K}{c\,\Sigma}\right) .$$

(The principal rooth of this equation is real). An examination of the integrand reveals a singularity at $\tau = +i\Sigma$. This is of the form log z for $z \rightarrow 0$. To avoid this singularity the plane is cut and the deformed contour is taken along the imaginary axis to $i\Sigma$ and back again . We have :

I) an asymptotic part of the solution : that is arising from the residue at $\tau = iK$

II) a transient part which is only important near to the source; that arises from the contributions of the integral around the cut (fig. 1) .

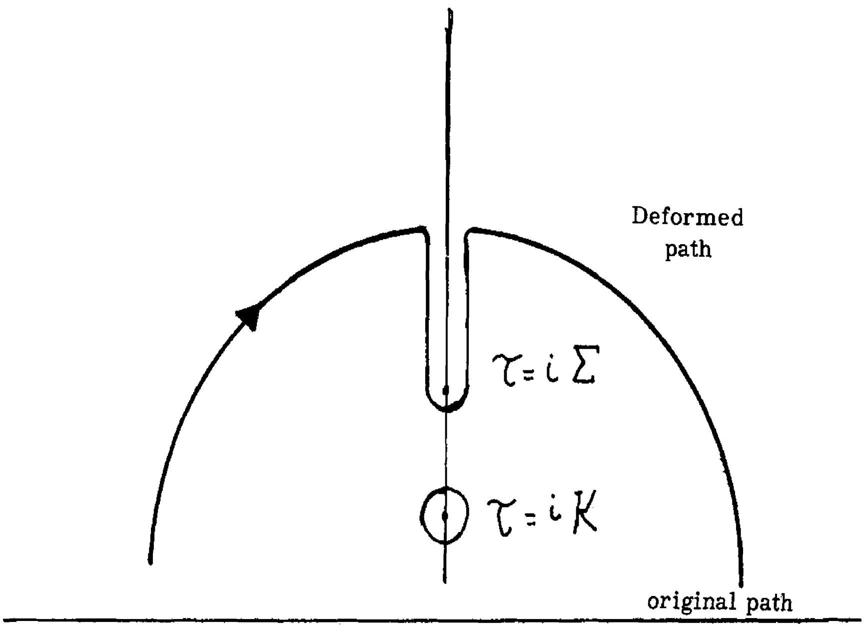

Fig. 1 .

For $x < 0$, the integral can be evaluated by taking a path of integration in the lower half of the complex plane. A contribution then arises from the pole $\tau = -iK$.

We have therefore a first part of the solution, that is :

$$\Phi_1 = \frac{2(1-c)}{c} \cdot \frac{\Sigma^2 - K^2}{K^2 - \Sigma^2(1-c)} \cdot \frac{K}{2(1-c)\Sigma} \exp(-K|x|) . \tag{9}$$

This part dominates in Φ at large values of x.

A second part of the solution corresponds to $c > 1$, that is :

$$\Phi_2 = \int_0^\infty \frac{2\Sigma^2(\eta+1)\exp\left[-(\eta+1)\Sigma|x|\right] d\eta}{\left[2\Sigma(\eta+1) - c\Sigma \log(2\eta^{-1}+1)\right]^2 + \pi^2 c^2 \Sigma^2} . \tag{10}$$

When x is small, the major contribution to the integral comes from large values of η, i.e. the integral is given approximately by :

$$\int_0^\infty \frac{\exp(-(\eta+1)|x|\Sigma)\, d\eta}{2(\eta+1)} = \frac{1}{2} F_1(\Sigma|x|) . \tag{11}$$

This is important near the source $x = 0$ and decreases rapidly as $\exp(-\Sigma x)$ when x tends to ∞

4. Solution of the time-dependent transport equation without sources in a semi-infinite medium.

Now we consider the time-dependent Boltzmann integro-differential equation for the case of the neutron transport in a semi-infinite medium without sources and for the one-group theory with non-isotropic collisions . That is we consider the integro-differential equation :

$$\frac{1}{v} \frac{\partial N(x,\mu,t)}{\partial t} + \mu \frac{\partial N(x,\mu,t)}{\partial x} = \tag{1}$$
$$= \eta\sigma_s \int_{-1}^{+1} N(x,\mu',t) \frac{1+3P\mu\mu'}{2} d\mu' - (\sigma_s + \sigma_c)\eta N(x,\mu,t) ,$$

where η = number per unit of volume of collision centers for the neutron ;

σ_c = cross section of capture of the neutron ;

σ_4 = cross section of scattering.

P = constant

and

$$f(\vec{\Omega}' \longrightarrow \vec{\Omega}) = (1+3P\mu\mu')/2 .$$

Putting

$$\eta\sigma_c = \Lambda^{-1}, \quad v\eta\sigma_c = \tau^{-1},$$

we obtain the equation :

$$\frac{\partial N(x,\mu,t)}{\partial t} + v\mu\frac{\partial N(x,\mu,t)}{\partial x} = \left(-\frac{v}{\Lambda}+\frac{1}{\tau}\right) N(x,\mu,t) + \tag{2}$$

$$+\frac{v}{\Lambda}\int_{-1}^{+1} N(x,\mu',t)\,\frac{1+3P\mu\mu'}{2}\,d\mu', \quad (\Lambda, v, \tau, P \text{ constants})$$

We have the following boundary condition:

$$N(x,\mu,t) = 0 \qquad \text{for} \quad x = 0, \quad \mu < 0 . \tag{I}$$

We put now :

$$\omega = \frac{v}{\Lambda} + \frac{1}{\tau}, \quad \omega_0 = \frac{v}{\Lambda} . \tag{3}$$

Writing

$$N(x,\mu,t) = \exp(-\omega t)\; F(x,\mu,t) , \tag{4}$$

we obtain :

$$\frac{\partial F(x,\mu,t)}{\partial t} + v\mu\frac{\partial F(x,\mu,t)}{\partial x} = \tag{5}$$

$$= \omega_0 \int_{-1}^{+1} F(x,\mu',t)\,\frac{1+3P\mu\mu'}{2}\,d\mu' ,$$

with

$$F(x,\mu,t) = 0 \qquad \text{for } x = 0, \ \mu < 0 .$$

We apply a $\mathcal{L}_{t,p}\,\mathcal{L}_{x,q}$ (double $\mathcal{L}$ -transformation) and we put :

$$\varphi(p,\mu) = \int_{-\infty}^{+\infty} \exp(-pt) \ F(0,\mu,t)\,dt ,$$

$$\Psi(q,\mu) = \int_{-\infty}^{0} \exp(qx)\, N(x,\mu,0)\, dx ,$$

$$\Phi(p,q,\mu) = \mathcal{L}_{t,p}\mathcal{L}_{x,q}\left[N(x,\mu,t)\right] = \int_{-\infty}^{0} e^{qx}\, dx \int_{0}^{+\infty} e^{-pt} N(x,\mu,t)dt \rbrace$$

We obtain the integral equation :

(6) $$-\Psi(q,\mu) + p\Phi(p,q,\mu) + v\mu \left[\varphi(p,q,\mu) - q\Phi(p,q,\mu)\right] = \\ = \omega_0 \int_{-1}^{+1} \Phi(p,q,\mu') \frac{1+3P\mu\mu'}{2}\, d\mu ,$$

with

(7) $$\varphi(p,\mu) = 0 \ \text{ for } \mu < 0.$$

One considers $F(0,\mu,t)$ as a given function and $\varphi(p,\mu)$ is a known function. The function $\psi(q,\mu)$ is dependent from the initial values of $F(x,\mu,t)$ and will be determined when the function φ is given . In order to solve the integral equation (6) we write :

(8) $$\varphi = \sum_{m=1}^{\infty} \frac{\varphi_m(\mu)}{p^m} ,$$ (9) $$\Psi = \sum_{n=1}^{\infty} \frac{\Psi_n(\mu)}{q^n}$$

(10) $$\Phi = \sum_{1}^{\infty}{}_{mn} \frac{\Phi_{mn}(\mu)}{p^n\, q^n}$$

and we observe that we shall have :

(11) $\qquad \varphi_m(\mu) = 0 \qquad$ for $\mu < 0$, $(m=1,2,3,\dots\dots)$.

Substituting in the integral equation (6) , we obtain :

$$(12)\quad \sum_{m,n=1}^{\infty} \frac{\Phi_{mn}(\mu)}{p^{m-1}q^{n}} - \sum_{n-1}^{\infty} \frac{\Psi_n(\mu)}{q^n} + v\mu \left[\sum_{m=1}^{\infty} \frac{\varphi_m(\mu)}{p^m} - \sum_{m,n=1}^{\infty} \frac{\Phi_{mn}(\mu)}{p^m q^{n-1}} \right]$$

$$= \omega_0 \sum_{m,n=1}^{\infty} \frac{1}{p^m q^n} \int_{1}^{+1} \Phi_{m,n}(\mu') \frac{1+3P\mu\mu'}{2} d\mu' .$$

Therefore we have necessarily :

$$(13)\quad \Phi_{m1} = \varphi_m(\mu) , \quad (m = 1,2,3,\dots\dots)$$

$$(14)\quad \Phi_{1n} = \Psi_n(\mu) , \quad (n = 1,2,3, \dots\dots)$$

$$(15)\quad \Phi_{m+1,n}(\mu) - v\mu\, \Phi_{m,n+1}(\mu) =$$

$$= \omega_0 \int_{-1}^{+1} \Phi_{mn}(\mu) \frac{1+3P\mu\mu}{2} d\mu' , \quad (m,n = 1,2,3,\dots\dots) .$$

The equations (13) give the coefficients Φ_{m1} of the double series Φ in terms of the φ_m . The equation (15) is recurrent and gives $\Phi_{m,2}$, $\Phi_{m,3}, \dots\dots, \Phi_{m,n}, \dots\dots$ in terms of the φ_m .

When we have $\Phi_{m,n}$, we have also Φ_m, and the equation (14) gives the functions Ψ_n in terms of the functions φ_m.

Applying the theorems on the $\mathcal{L}$-transformation of the series, we obtain , after the calculations :

$$(16)\quad \Phi(p,q,\mu) = \mathcal{L}_{t,p}\mathcal{L}_{x,q} \left[\sum_{1}^{\infty}{}_{mn} \Phi_{mn}(\mu) \frac{t^{m-1}}{(m-1)!} \cdot \frac{(-x)^{n-1}}{(n-1)!} \right],$$

that is (unicity of the L-transformation):

$$(17) \qquad F(x,\mu,t) = \sum_{1}^{\infty}{}_{mn} \quad \Phi_{mn}(\mu) \frac{t^{m-1}}{(m-1)!} \cdot \frac{(-x)^{n-1}}{(n-1)!} \cdot$$

One demonstrates immediately that it is

$$F(x,\mu,t) = 0 \text{ for } \quad x = 0, \qquad \mu < 0 \ .$$

It is easily possible to demostrate also that for $F(x,\mu,t)$ exists an exponential majorant series

$$H\left(1 + 3P\frac{\omega_o}{\omega}\right) e^{\omega\left(t-\frac{x}{v}\right)}$$

Finally we obtain the density function :

For $\mu > 0$:

$$(18) \quad N(x,\mu,t) = e^{-\omega t} \sum_{m=1}^{\infty} \frac{t^{m-1}}{(m-1)!} \left\{ \sum_{m=0}^{\infty} \frac{\varphi_{m+n}(\mu)}{(v\mu)^n} \frac{(-x)^n}{n!} + \right.$$

$$\left. + \ 3P\omega_o \sum_{m=1}^{\infty} \frac{\alpha_{m+2n-2,\,2n-2}}{v^{2n-1}} \frac{x^{2n-1}}{(2n-1)!} \right\}$$

For $\mu < 0$:

$$(19) \quad N = 3P\,\omega_o\, e^{-\omega t} \sum_{mn=1}^{\infty} \frac{\alpha_{m+2n-2,\,2n-2}}{v^{2n-1}} \frac{t^{m-1}}{(m-1)!} \frac{x^{2n-1}}{(2n-1)!} \ .$$

And it is

$$N(0,\mu,t) = 0 \quad \text{for} \quad \mu < 0$$

So our problem is solved .

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