CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E)

P.C. KENDALL

"ON THE DIFFUSION IN THE ATMOSPHERE AND IONOSPHERE"

Corso tenuto a Varenna dal 19-al 27 settembre 1966

$$+\gamma_1\gamma_2\frac{\partial^2 \nu_2}{\partial \eta_1^2} + \gamma_1 \gamma_3 \frac{\partial^2 \nu_2}{\partial \eta_1 \partial \eta_2} + \gamma_2^2 \frac{\partial^2 \nu_3}{\partial \eta_1 \partial \eta_2} +$$

(3.6.24) $$+\gamma_2\gamma_4 \frac{\partial^2 \nu_3}{\partial \eta_1 \partial \eta_3}) = 0 ,$$

" B "

(3.6.25) $$\text{(ii)}\ \nu_5 (\gamma_3 \frac{\partial \nu_2}{\partial \eta_3} + \gamma_2 \nu_1 \frac{\partial \nu_2}{\partial \eta_1} + \cdots) + \cdots = 0 ,$$

$$\text{(iii)}\ \nu_5 (\gamma_3 \frac{\partial \nu_3}{\partial \eta_3} + \gamma_2 \gamma_1 \frac{\partial \nu_3}{\partial \eta_3} + \cdots) + \cdots = 0 ,$$

(3.6.27) $$\text{(iv)}\ \nu_4 = R \nu_5 \nu_6 ,$$

$$\text{(v)}\ \gamma_3 \frac{\partial \nu_5}{\partial \eta_3} + \gamma_2 \frac{\partial (\nu_5 \nu_1)}{\partial \eta_1} + \gamma_1 \frac{\partial (\nu_5 \nu_2)}{\partial \eta_1} + \gamma_3 \frac{\partial (\nu_5 \nu_2)}{\partial \eta_2} +$$

(3.6.28) $$+\gamma_2 \frac{\partial (\nu_5 \nu_3)}{\partial \eta_2} + \gamma_4 \frac{\partial (\nu_5 \nu_3)}{\partial \eta_3} = 0 ,$$

(3.6.29) $$\text{(vi)}\quad C_v \nu_5 (\gamma_3 \frac{\partial \nu_6}{\partial \eta_3} + \gamma_2 \gamma_1 \frac{\partial \nu_6}{\partial \eta_1} + \cdots) + \cdots = 0 .$$

The system " B " is absolutely invariant under the group :

$$P^{(2)} : \bar{\eta}_1 = \eta_1 + \gamma_5 a , \quad \bar{\eta}_2 = \eta_2 - \gamma_6 a , \quad \bar{\eta}_3 = \eta_3 + \gamma_7 a ,$$

(3.6.30) $$\bar{\nu}_j = \nu_j \qquad (j = 1, \cdots, 6) ,$$

whose invariants , as in the previous case, may be chosen as follows:

C.I.M.E. lectures
by
P.C. KENDALL
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Introduction

This set of lectures is not intended to be a comprehensive survey of the ionosphere. It is intended to lead the student with great speed to certain specialized research areas of possible interest to applied mathematicians . I apologise to the many ionospheric physicists (both experimental and theoretical) whose work is not mentioned here; the short list of references included at the end reflects current theory. Present day achievements have only been made possible through the patient experimental work and international collaboration of many scientists over decades.

P. C. Kendall

Lecture 1

In his lectures Professor Ferraro will show how diffusion of a plasma (a gas composed of equal numbers of ions and electrons) takes place through a neutral atmosphere. The electrons are often very mobile and would in absence of electrostatic forces disperse to infinity in a very short time. The ions are bigger and heavier and so cannot move as quickly through the neutral atmosphere. Thus, the tendency of the electrons to "boil off" is prevented by the electrostatic forces between ions and electrons, and an electric field is set up. This has the effect of increasing by a factor of two the pressure gradient which causes diffusion; thus causing faster diffusion, known as ambipolar diffusion The purpose of this lecture is to establish the equations of the problem including the effects of a uniform magnetic field $\underline{H}_0$. In general these equations have not yet been fully solved. I will describe what can be done, and also mention the difficulties. Practical application of the equations may be made in the upper ionosphere, in diffusion of meteor trains and in the laboratory, wherever the plasma is a minot constituent.

Solution of the problem (ion-neutral collisions only)

The main problem is how to work out the electric field arising from attempted charge separation. In the case when the electrons are perfectly mobile (i.e. we can ignore collisions of electrons with both ions and neutral atoms) this can be done in full (provided that certain other minor simplifications are made) . In particular , it will be seen that the external electric field gives rise to complicated drift motions of the joint ion-electron gas.

The equation of motion of the electrons is

(1) $$N\, m_e \frac{d\underline{V}_e}{dt} = - \underline{\nabla} p_e - e\, N\, (\underline{E} + \underline{V}_e \times \underline{B})$$

Here,

m_e = mass of electron

p_e = electron pressure

N = electron density = ion density

e = electronic charge

$\underline{V}_e$ = electron velocity

d/dt = mobile operator $\partial/\partial t + (\underline{V}_e . \underline{\nabla})$

$\underline{E}$ = total electric field in fixed axes

The magnetic field is assumed constant and uniform (i.e. the number of ions and electrons is so small that their associated electric currents may be neglected). Thus the electric field is a potential field, i.e.

(2) $$\text{curl } \underline{E} = 0 \qquad \therefore \quad \underline{E} = - \underline{\nabla}\, \Omega$$

Neglecting terms which involve m_e (which is small)

$$0 = - \underline{\nabla}\, p_e - e\, N\, (\underline{E} + \underline{V}_e \times \underline{B}),$$

(3) i.e.

$$\underline{\nabla}\, p_e - e\, N \underline{\nabla}\, \Omega = - N\, e\, \underline{V}_e \times \underline{B}$$

So, in an atmosphere at <u>uniform temperature</u> T, if k is Boltzmann constant such that

$$p_e = N\, k\, T,$$

equation (3) becomes

(4) $$\underline{\nabla}\left(\frac{kT}{e}\log N - \Omega\right) = -\underline{V}_e \times \underline{B}$$

Write

(5) $$\Omega = \frac{kT}{e}\log N + \Omega_H$$

Then

(6) $$\underline{B}\cdot\underline{\nabla}\,\Omega_H = 0,$$

showing that the potential Ω_H is constant along a magnetic field line. Further, if

$$\underline{E}_H = -\underline{\nabla}\,\Omega_H$$

we obtain

(7) $$\underline{V}_e = \underline{E}_H \times \underline{B}\,/B^2$$

This is known as the Hall drift or Hall motion of electrons. The electric field $\underline{E}_H$ is at right angles to the magnetic field and if present must be regarded as an applied electric field, external in origin. The electric potential kT (log N) / e is internal in origin. One might well ask what happens when $N \to 0$; apparently the electric field becomes infinitely large. The answer lies in our assumption of electrical neutrality, which becomes invalid for small values of N.

The equation of motion of the ions is

$$N m_i \frac{d\underline{V}_i}{dt} + N m_i \nu_{ia}(\underline{V}_i - \underline{V}_a) = -\underline{\nabla}\, p_i + e N (\underline{E} + \underline{V}_i \times \underline{B}) .$$

Here,

m_i = mass of ion

p_i = ion pressure

N = ion density = electron density

$\underline{V}_i$ = ion velocity

$\underline{V}_a$ = neutral air velocity

ν_{ia} = collision frequency ions - neutral atoms

In the diffusion approximation it is usual to neglect the acceleration. (The student should find out why) . Then

(8) $$N m_i \nu_{ia}(\underline{V}_i - \underline{V}_a) = - \underline{\nabla} p_i + e N (\underline{E} + \underline{V}_i \times \underline{B})$$

Adding (3) and (8) to eliminate $\underline{E}$.

(9) $$N m_i \nu_{ia} (\underline{V}_i - \underline{V}_a) = - \underline{\nabla} (p_i + p_e) + e N \underline{V}_i \times \underline{B} - e N \underline{V}_e \times \underline{B}$$

Writing

(10) $$p_i = p_e = N k T .$$

and

(11) $$\underline{F} = \underline{V}_a - \frac{2 k T}{N m_i \nu_{ia}} \underline{\nabla} N - \frac{e}{m_i \nu_{ia}} \underline{V}_e \times \underline{B}$$

we see that

(12) $$\underline{V}_i + \frac{e}{m_i \nu_{ia}} \underline{B} \times \underline{V}_i = \underline{F}$$

This must be solved for $\underline{V}_i$. Then the diffusion equation is formed by using the continuity equation, which in absence of production and loss is

(13) $$\frac{\partial N}{\partial t} = - \operatorname{div} (N \underline{V}_i) = - \underline{\nabla}.(N \underline{V}_i)$$

Denote by $\parallel$ and $\perp$ components parallel to and perpendicular to $\underline{B}$. Then

(14) $$\underline{V}_i^{\parallel} = \underline{F}^{\parallel} = \underline{V}_a^{\parallel} - \frac{2\,k\,T}{N\,m_i\,\nu_{ia}}\,\underline{\nabla}^{\parallel} N$$

and

(15) $$\underline{V}_i^{\perp} + \frac{e}{m_i\,\nu_{ia}}\,\underline{B} \times \underline{V}_i^{\perp} = \underline{F}^{\perp}$$

There are many ways of solving this equation. A quick convenient method is to introduce coordinates x, y , z with the z-axis parallel to $\underline{B}$ (Fig. 1).

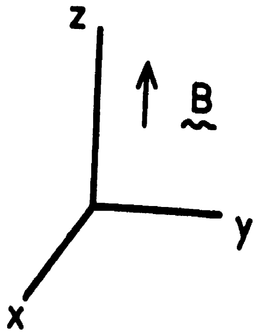

Fig. 1

Then if

$$\underline{V}_i^{\perp} = (V_1,\ V_2,\ 0)$$

we have

$$\underline{B} \times \underline{V}_i^{\perp} = (\,-BV_2,\ BV_1,\ 0)\ .$$

It follows that if we introduce

$$j = \sqrt{(-1)}$$

and write

$$\underline{V}_i^{\perp} = V_1 + jV_2$$

we obtain

$$\underline{B} \times \underline{V}_i^{\perp} = B\, j\, \underline{V}_i^{\perp}.$$

So equation (15) becomes

$$\underline{V}_i^{\perp} = \underline{F}^{\perp} \Big/ \left(1 + \frac{e B}{m_i \nu_{ia}}\, j\right), \tag{16}$$

where

$$\underline{F}^{\perp} = \underline{V}_a^{\perp} - \frac{2 k T}{N m_i \nu_{ia}} \underline{\nabla}^{\perp} N + \frac{e B}{m_i \nu_{ia}}\, j\, \underline{V}_e^{\perp} \tag{17}$$

Equations (14) (16) and (17) describe the motion completely and with (13) give an equation governing N. First we discuss two special cases. These will help to clarify the general problem.

<u>Special case I B = 0 (no magnetic field)</u>

Equation (12) gives at once

$$\underline{V}_i = \underline{V}_a - \frac{2 k T}{N m_i \nu_{ia}} \underline{\nabla} N$$

This is the case of ordinary ambipolar diffusion. We define the coefficient of ambipolar diffusion <u>in this case</u> to be

$$D_a = \frac{2 k T}{m_i \nu_{ia}} \text{ cm}^2/\text{sec}$$

so from (13) if T and ν_{ia} are constants,

(18) $$\frac{\partial N}{\partial t} = - \operatorname{div} (N \underline{V}_a) + D_a \nabla^2 N$$

This is the isotropic diffusion equation. The term $\operatorname{div} (N \underline{V}_a)$ is called a transport term. Its presence indicates that all diffusion takes place relative to the wind $\underline{V}_a$ in the neutral air. Note the factor 2 in D_a.

Special case II $eB/m_i \nu_{ia} \to \infty$

Physically

$$\nu_{ia} = \text{collision frequency}$$

$$\frac{eB}{\pi m_i} = \text{Larmor frequency}^{(*)}$$

(*) In absence of electric fields the motion of a single ion is given by

$$m_i \frac{d\underline{V}}{dt} = e \underline{V} \times \underline{B}$$

i.e.

$$\frac{d\underline{V}}{dt} = \underline{\omega} \times \underline{V}$$

where

$$\underline{\omega} = - e\underline{B}/m_i .$$

Thus in absence of other forces the velocity vector rotates with angular velocity $\omega = eB/m_i$ in a certain sense. The motion is one of spiralling about the magnetic field lines.

Thus case II corresponds to many spirals between collisions and magnetic effects dominate.

Clearly $\underline{V}_i^{\parallel}$ is unchanged from (14) ; but from (16) and (17)

$$\underline{V}_i^{\perp} = \underline{V}_e^{\perp} = \underline{E}_H \times \underline{B}/B^2 \quad .$$

The diffusion equation then becomes

$$\frac{\partial N}{\partial t} = - \operatorname{div} \left(N \, \underline{V}_a^{\parallel}\right) - \operatorname{div} \left(N \, \underline{V}_e^{\perp}\right) + D_a \frac{\partial^2 N}{\partial z^2} \quad , \tag{19}$$

where z is the coordinate along the magnetic field. Note that (a) the only effective neutral wind component is <u>along</u> a field line (b) the Hall drift, caused by electric fields is at right angles to a field line.

<u>In general</u>

Rearranging equation (17) ,

$$\underline{F}^{\perp} = \underline{V}_a^{\perp} - \underline{V}_e^{\perp} - \frac{D_a}{N} \underline{\nabla}^{\perp} N + \left(1 + \frac{e\,B}{m_i \nu_{ia}} j\right) \underline{V}_e^{\perp}$$

Thus from (16)

$$\underline{V}_i^{\perp} = \underline{V}_e^{\perp} + \left(1 + \frac{e\,B}{m_i \nu_{ia}} j\right)^{-1} \left(\underline{V}_a^{\perp} - \underline{V}_e^{\perp} - \frac{D_a}{N} \underline{\nabla}^{\perp} N\right)$$

Also from (14)

$$\underline{V}_i^{\parallel} = \underline{V}_a^{\parallel} - \frac{D_a}{N} \underline{\nabla}^{\parallel} N$$

So, letting $s = e\,B/m_i \nu_{ia}$ we have

$$\frac{\partial N}{\partial t} = - \operatorname{div}(N \underline{V}_a^{\parallel}) - \frac{1}{1+s^2} \operatorname{div}(N \underline{V}_a^{\perp}) + \frac{s}{1+s^2} \operatorname{div}(N \underline{k} \times \underline{V}_a^{\perp})$$

$$- \operatorname{div}(N \underline{V}_e^{\perp}) + \frac{1}{1+s^2} \operatorname{div}(N \underline{V}_e^{\perp}) - \frac{s}{1+s^2} \operatorname{div}(N \underline{k} \times \underline{V}_e^{\perp})$$

$$+ D_a \frac{\partial^2 N}{\partial z^2}$$

$$+ \underline{\nabla}^{\perp} \cdot \left\{ \frac{D_a}{1+s^2} \underline{\nabla}^{\perp} N - \frac{s\, D_a}{1+s^2} \underline{k} \times \underline{\nabla}^{\perp} N \right\},$$

where $\underline{k} = (0, 0, 1)$.

Thus as $\underline{\nabla}^{\perp} \cdot (\underline{k} \times \underline{\nabla}^{\perp} N) = \underline{\nabla} \cdot (\underline{k} \times \underline{\nabla} N) \equiv 0$, we have

$$\frac{\partial N}{\partial t} = - \operatorname{div}(N \underline{V}_a^{\parallel}) - \frac{1}{1+s^2} \operatorname{div}(N \underline{V}_a^{\perp}) + \frac{s}{1+s^2} \operatorname{div}(N \underline{k} \times \underline{V}_a^{\perp})$$

$$- \frac{s^2}{1+s^2} \operatorname{div}(N \underline{V}_e^{\perp}) - \frac{s}{1+s^2} \operatorname{div}(N \underline{k} \times \underline{V}_e^{\perp})$$

$$(20) \qquad + \frac{D_a}{1+s^2} \left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} \right) + D_a \frac{\partial^2 N}{\partial z^2}$$

Note how on putting $s = 0$ this equation reduces to (18) and how it reduces to (19) on putting $s = \infty$. The terms in $\underline{V}_e^{\perp}$ disappear when $s = 0$ and the first one remains when $s \to \infty$. They might be called "Hall terms" . Note also how complicated the effects of electric fields $\underline{E}_H$ (such that $\underline{V}_e = \underline{E}_H \times \underline{B}/B^2$) and winds $\underline{V}_a$ become when $s \sim 1$. The <u>effective</u> wind is not $\underline{V}_a$ but is

$$\underline{V}_{\text{effective}} = \underline{V}_a^{\parallel} + \frac{\underline{V}_a^{\perp} - s\, \underline{k} \times \underline{V}_a^{\perp}}{1+s^2}$$

Asymptotic solutions of the ambipolar diffusion problem

Consider equation (20) in the case when $\underline{E}_H = 0$ and $\underline{V}_a = 0$ i.e. diffusion in a stationary atmosphere with no electric fields present other than internal ones. Then

$$(21) \qquad \frac{\partial N}{\partial t} = \frac{D_a}{1+s^2}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) + D_a \frac{\partial^2 N}{\partial z^2}$$

Write

$$x' = x\sqrt{[(1+s^2)/D_a]} \quad y' = y\sqrt{[(1+s^2)/D_a]} \quad z' = z/\sqrt{D_a}$$

Then (21) becomes

$$(22) \qquad \frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial x'^2} + \frac{\partial^2 N}{\partial y'^2} + \frac{\partial^2 N}{\partial z'^2}$$

Suppose that diffusion takes place over a region of infinite extent in all directions, a given cloud of plasma being released at time $t = 0$. If the plasma is released as a point source such that if $r = \sqrt{(x'^2 + y'^2 + z'^2)}$,

$$N(r, 0) = \delta(r) \quad (*)$$

then it may be shown that

$$N(r, t) = \frac{1}{(4\pi t)^{3/2}} \exp - \frac{x'^2 + y'^2 + z'^2}{4t}.$$

It follows that for a general cloud of plasma such that

$$N = N(x', y', z', t)$$

(*) $\delta(r) \equiv 0$ if $r \neq 0$ and is so defined that $\iiint \delta(r)\, dx'\, dy'\, dz' = 1$

we have

$$N(x', y', z', t) \iiint \frac{N(x_o, y_o, z_o, 0)}{(4\pi t)^{3/2}} \exp\left[- \frac{(x'-x_o)^2+(y'-y_o)^2+(z'-z_o)^2}{4t} \right] dx_o\, dy_o\, dz_o \,.$$

So the asymptotic shape of any plasma cloud as $t \to \infty$ is

$$N = \frac{N_o'}{(4\pi t)^{3/2}} \exp - \frac{x'^2 + y'^2 + z'^2}{4t} \,,$$

where N'_o = total electron content $= \iiint N\, dx'\, dy'\, dz'$. Converting back we see that if

$$N_o = \iiint N\, dx\, dy\, dz \,,$$

as $t \to \infty$ the asymptotic form is obtained, namely,

$$N = \frac{(1+s^2)\, N_o}{(4\pi D_a t)^{3/2}} \exp\left[- \frac{(1+s^2)(x^2+y^2)+z^2}{4 D_a t} \right] \tag{23}$$

This shows that in a uniform magnetic field a discrete plasma cloud will take up the shape of an ellipsoid of revolution, elongated in the direction of the field.

Dispersion of "meteor trails"

A "meteor trail" is an infinitely long cloud of plasma left behind as the meteor passes through the ionosphere. (see Fig. 2 for notation) . At time $t = 0$ one might suppose the trail to be of small cross-section.

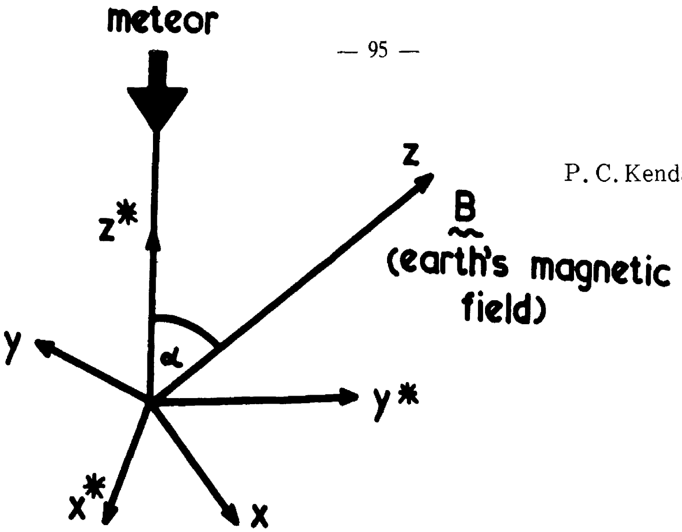

Fig. 2

Clearly the problem is in two dimensions, at right angles to the meteor train . Assuming that an equation of type (21) holds we may put

$$x^* = x \quad y^* = y \cos\alpha + z \sin\alpha \quad z^* = - y \sin\alpha + z \cos\alpha$$

and

$$N = N(x^*, y^*, t)$$

giving

$$\frac{\partial N}{\partial t} = \frac{D_a}{1+s^2}\frac{\partial^2 N}{\partial x^{*2}} + D_a \left(\sin^2\alpha + \frac{\cos^2\alpha}{1+s^2} \right) \frac{\partial^2 N}{\partial y^{*2}} \tag{24}$$

Unfortunately we have omitted electron collisions. Nevertheless, this serves as useful illustrative problem.

Exercise Show that a solution of the equation

$$\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial x'^2} + \frac{\partial^2 N}{\partial y'^2}$$

is

$$N = \frac{n_o a^2}{(a^2 + 4t)} \exp - \frac{(x'^2 + y'^2)}{(a^2 + 4t)} \ .$$

Hence obtain a solution of equation (24) and discuss its nature.

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Note . When electron-ion collisions and electron -neutral atom collisions are included the ambipolar diffusion equation can not be solved. A recent paper by Holway (1965) J.G.R. 70 3635 is known to be incorrect as curl $\underline{E} \neq 0$.

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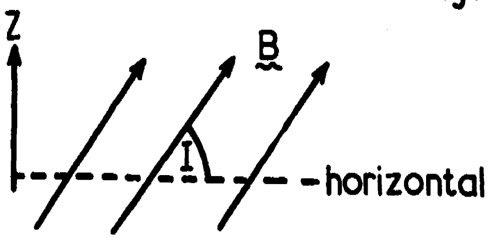
Lecture 2

Diffusion in a plane stratified atmosphere

The diffusion equation

Figure 3



I = angle of dip

= angle between $\underset{\sim}{B}$ and horizontal

All variables are assumed to be functions of z only, where z = height above some fixed reference level (Fig. 3) . Roughly speaking we choose

$$z = 0 \text{ at base of ionospheric F layer}$$

The preceding results on ambipolar diffusion may be used to simplify the derivation, for in the F2 layer, 300 km above the ground

$$\nu_{ia} \sim 1 \sec^{-1}$$

Therefore

$$\frac{eB}{m_i \nu_{ia}} = \frac{1.6 \times 10^{-20} \times 0.3}{26 \times 10^{-24} \times 1} \approx 200$$

giving

$$s \gg 1$$

It follows that

(i) The only relevant equation of motion of the ions is that along a field line i.e. parallel to $\underline{B}$

(ii) The motion $\underline{V}_i^{\perp}$ at right angles to the field lines is determined by the Hall electric field. Thus

$$\underline{V}_i^{\perp} = \underline{E}_H \times \underline{B}/B^2$$

For a self consistent model we must take $\underline{V}_i^{\perp}$ to be uniform.

The physical meaning of (i) & (ii) is that the ions can spiral freely between collisions with neutral atoms, and do so many times. Their random motions across field lines are therefore considerably reduced (Fig. 4).

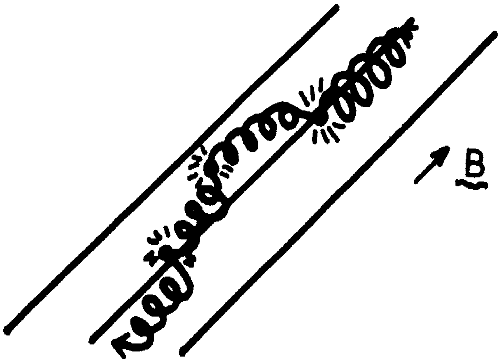


Figure 4

Schematic view of motion of ion

However, the Hall drift is as though the field lines were themselves moving with velocity $\underline{V}_i^{\perp}$, giving a transverse velocity $\underline{V}_i^{\perp}$ to the motion of the ions.

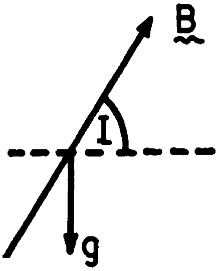


Figure 5

With $\partial/\partial t = 0$ the equation of motion of the plasma along a field line is (Fig. 5)

$$\frac{2NkT}{D}\left(\underline{V}_i^{\parallel} - \underline{V}_a^{\parallel}\right) = -\,\underline{\nabla}^{\parallel}(2NkT) + Nm_i\,\underline{g}^{\parallel}$$

(The partial pressure of the plasma is $p_i + p_e = NkT + NkT$).
We assume that the temperature of all constituents is the same and is uniform and constant. Here

$$D = 2\,D_{ia} = 2 \times \text{coeff. diff. ions} \rightarrow \text{neutrals}$$

and from gas theory

$$D \propto 1/n_a \quad (n_a = \text{neutral air density})$$

The neutral atmosphere is in equilibrium. Thus the neutral air density is

$$n_a = n_o \exp(-z/H)\,,$$

where

$$H = \text{scale height} = kT/m_a g$$

We shall assume that $m_a = m_i$.

Thus if $\hat{\underline{T}}$ denotes a unit tangent along a field line

$$\underline{\nabla}^{\parallel} = \underline{\hat{T}}\,(\underline{\hat{T}}.\underline{\nabla})$$

and

$$\underline{g}^{\parallel} = -\,g \sin I\, \underline{\hat{T}}$$

giving

$$\underline{V}_i^{\parallel} = \underline{V}_a^{\parallel} - \frac{D}{N}\left(\underline{\hat{T}}.\underline{\nabla} N + \frac{N}{2H} \sin I\right) \underline{\hat{T}}$$

The diffusion equation is formed from the continuity equation

$$\frac{\partial N}{\partial t} = Q - L - \underline{\nabla}.(N\underline{V}_i)$$

Here

Q = rate of electron production (by sun)

L = loss rate (by recombination)

The equation could be written

$$\underline{\frac{\partial N}{\partial t} = Q - L - W_H \cos I \frac{\partial N}{\partial z} - V_a \cos I \sin I \frac{\partial N}{\partial z}}$$

(1) $$\underline{+\, D\mathcal{D}\, N}\,,$$

where

W_H = Hall drift $\perp^r$ to field lines (assumed uniform)

V_a = Horizontal velocity of neutral air (assumed uniform)

$\mathcal{D}$ = diffusion operator defined by :

$$D\mathcal{D}N = \underline{\nabla}.\left\{D\,(\underline{\hat{T}}.\underline{\nabla} N + \frac{N}{2H} \sin I)\, \underline{\hat{T}}\right\}$$

$$= (\underline{\hat{T}}.\underline{\nabla})\left\{D\,(\underline{\hat{T}}.\underline{\nabla} N + \frac{N}{2H} \sin I)\right\}$$

But in this case

$$\hat{\underline{T}} . \underline{\nabla} \approx \sin I \frac{\partial}{\partial z}$$

Thus $$D \mathfrak{D} N = \sin^2 I \frac{\partial}{\partial z} D \left(\frac{\partial N}{\partial z} + \frac{N}{2H} \right),$$

giving , as $D \propto \exp(z / H)$

(2) $$\mathfrak{D} N = \left(\frac{\partial^2 N}{\partial z^2} + \frac{3}{2H} \frac{\partial N}{\partial z} + \frac{N}{2H^2} \right) \sin^2 I$$

This equation is due to Professor V.C.A. Ferraro.

We have to solve equation (1) , where $\mathfrak{D}$ is given by (2) . The rate of production may be taken to be the Chapman function

(3) $$Q = q_0 \exp \left(1 - \frac{z}{H} - e^{-z/H} \sec \chi \right) ,$$

where χ is the zenith angle of the sun, during daylight hours. At night $Q = 0$. The constant q_0 is the rate of maximum production.

Let δ = Northern declination of sun, ϕ' = local time in radians measured from noon ($\phi' = 0 \rightarrow$ noon) . Then spherical trigonometry gives (Fig. 6)

$$\cos \chi = \cos \left(\frac{\pi}{2} - \delta\right) \cos \theta + \sin \left(\frac{\pi}{2} - \delta\right) \sin \theta \cos \phi'$$

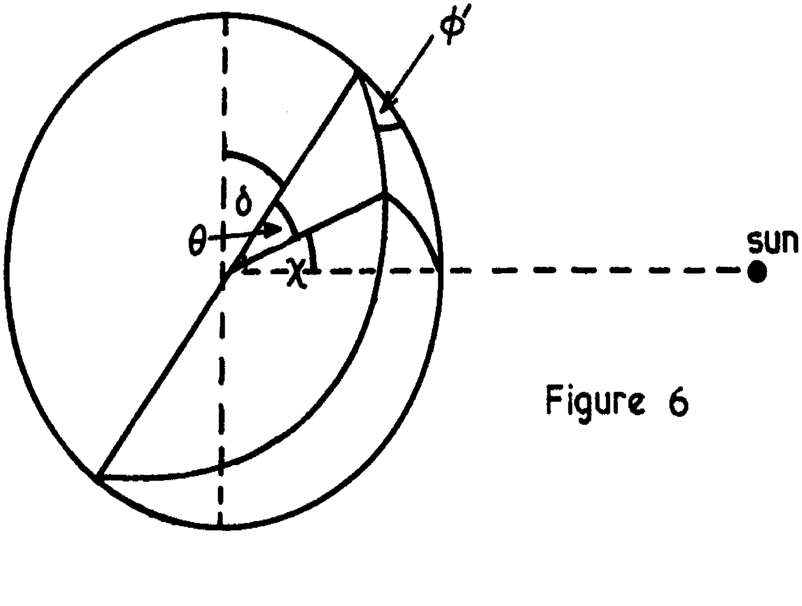


Figure 6

$$= \sin \delta \cos \theta + \cos \delta \sin \theta \cos \phi'$$

The length of day, ψ, is given by the equation

$$0 = \sin \delta \cos \theta + \cos \delta \sin \theta \cos \frac{1}{2}\psi$$

Thus time can be measured in radians from sunrise by writing

$$\phi = \frac{1}{2}\psi + \phi'$$

Then we have

sunrise at $\phi = 0$

sunset at $\phi = \psi$

and

$$\cos \chi = \sin \delta \cos \theta + \cos \delta \sin \theta \cos (\phi - \frac{1}{2}\psi) \tag{4}$$

$$= \text{function of time } \phi$$

The loss rate is proportional to N and is given by

$$L = \beta_o N \exp (-\lambda z/H) \tag{5}$$

The exact value of λ for the F2 layer is believed to be

$$\lambda = 1.75 ,$$

and numerical calculations are possible in this case. However, a wide range of analytic solutions is available in the case $\lambda = 1$. The constant β_o has dimensions sec^{-1} .

Illustrative solutions

The problem may be formulated in dimensionless variables as follows. Note that for simplicity's sake we consider only the case

$$W_H = 0 \qquad V_a = 0$$

Write

(6) $$t = \Upsilon \phi$$

where

$$\Upsilon = \text{number of secs in a radian} = 1.37 \times 10^{4}$$

(7) $$\beta = \Upsilon \beta_o$$

(8) $$\gamma = \frac{H^2}{\Upsilon D_o \sin^2 I} \quad (D = D_o e^{z/H})$$

Then

$$\frac{\partial N}{\partial \phi} = \Upsilon q_o \exp\left(1 - \frac{z}{H} - \frac{e^{-z/H}}{\cos\chi}\right) - \beta N e^{-\lambda z/H} + \frac{e^{z/H}}{\gamma} H^2 \mathfrak{D} N$$

Also put

$$\zeta = e^{-z/2H} \quad \text{and} \quad N = \zeta \nu$$

Then

(9) $$\frac{\partial \nu}{\partial \phi} = \Upsilon q_o e \zeta e^{-\zeta^2 \sec\chi} - \beta \zeta^{2\lambda} + \frac{1}{4\gamma} \frac{\partial^2 \nu}{\partial \zeta^2}$$

The solution of equation (9) is required under the conditions

(i) Solution is periodic $\phi = 0, 2\pi$

(ii) There are no sources at $z = \infty$ ($\zeta = 0$)

(iii) $\nu \to 0$ as $z \to -\infty$ ($\zeta \to \infty$)

Gliddon (1958 a, b) has provided solutions of this equation <u>in full</u> for the case $\lambda = 1$ following earlier work by Ferraro and Özdogan for the case $\lambda = 0$. There are possible generalizations in the case $\lambda = 1$ which appear in the Journal of Atmospheric and Terrestrial Physic under the authorship of Gliddon or Gliddon and Kendall. The analysis and elegant mathematics are due to Gliddon.

Exercise Show by direct substitution that a periodic solution of equation (9) with $\lambda = 1$ and $\cos\chi = \sin\phi$ is

$$v(x,\phi) = \frac{1}{2}\tau q_0 e(\beta\gamma)^{-\frac{1}{4}} \int_0^{\phi} x\{\Delta(\phi,\phi_0)\}^{-\frac{3}{2}} \exp\left\{-\frac{x^2\Gamma(\phi,\phi_0)}{4\Delta(\phi,\phi_0)}\right\} d\phi_0$$

$$+\frac{1}{2}\tau q_0 e(\beta\gamma)^{-\frac{1}{4}} \sum_{n=1}^{\infty} \int_0^{\pi} x\{\Delta(\phi,\phi_0 - 2n\pi)\}^{-\frac{3}{2}} \exp -\frac{x^2\Gamma(\phi,\phi_0-2n\pi)}{4\Delta(\phi,\phi_0-2n\pi)} d\phi_0$$

where

$$\Gamma(\phi_0,\phi) = (\beta\gamma)^{-\frac{1}{2}} \operatorname{cosec}\phi \cosh\left\{\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}(\phi_0-\phi)\right\} + \sinh\left\{\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}(\phi_0-\phi)\right\}$$

$$\Delta(\phi_0,\phi) = (\beta\gamma)^{-\frac{1}{2}} \operatorname{cosec}\phi \sinh\left\{\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}(\phi_0-\phi)\right\} + \cosh\left\{\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}(\phi_0-\phi)\right\}$$

and

$$x^2 = 4(\beta\gamma)^{\frac{1}{2}}\zeta^2$$

Note This solution does not satisfy condition (ii) ; however, Gliddon corrected this in his later papers.

There is scope for more analytical work on equation (9) for the case $\lambda = 1.75$ (or possibly $\lambda = 2$) .

The equilibrium problem

At noon and in the early afternoon the sun is nearly overhead for about 3 hours, so that near-equilibrium may be reached. Then

$$\partial N/\partial\phi \approx 0$$

and, choosing an overhead sun for convenience,

$$\cos\chi \approx 1 \;.$$

Thus

$$(10) \qquad \frac{\partial^2 v}{\partial \zeta^2} - 4\beta\gamma\zeta^{2\lambda} v = -4\gamma\tau q_o e \zeta e^{-\zeta^2}$$

One boundary condition may be found from the expression for $\underline{V}_i^{\parallel}$.
If $\underline{V}_a = 0$

$$\underline{V}_i^{\parallel} \propto D\left(\frac{\partial N}{\partial z} + \frac{N}{2H}\right) \sin I$$

Thus

$$\underline{V}_i^{\parallel} \propto \partial v / \partial \zeta$$

and we require

$$\text{(i)} \quad \partial v/\partial\zeta = 0 \quad \text{at } \zeta = 0$$

Also

$$\text{(ii)} \quad v \to 0 \quad \text{as } \zeta \to \infty \, .$$

These are called <u>two point boundary conditions.</u> There are two methods of solution (a) the method of binary splitting and (b) the use of lattice techniques. In the next lecture I shall give <u>recommended formulae</u> and describe the numerical methods used in detail, and briefly, before describing how to integrate numerically equation (9) with $\partial/\partial\phi \neq 0$.

Note that $\tau q_o e$ is a scaling factor which may be omitted from the calculations and inserted afterwards.

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Numerical and analytical methods

The binary splitting method

We present two methods of integrating the equation

(1) $$\frac{\partial^2 v}{\partial \zeta^2} - 4\beta\gamma\zeta^{2\lambda} v = -4\gamma\zeta e^{-\zeta^2}$$

in preparation for the equation

(2) $$\frac{\partial v}{\partial \phi} = \zeta e^{-\zeta^2 \sec\chi} - \beta\zeta^{2\lambda} v + \frac{1}{4\gamma}\frac{\partial^2 v}{\partial \zeta^2}$$

The first is the binary splitting method used by Rishbeth and Barron.

Briefly, we observe that equation (1) may be integrated under the initial conditions

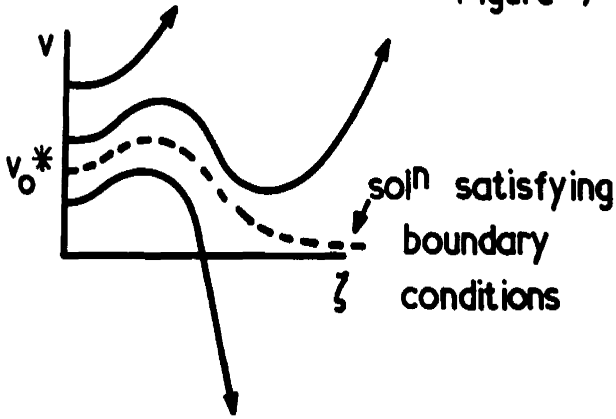
$$v = v_0 , \quad \partial v/\partial\zeta = 0 \quad \text{at} \quad \zeta = 0 ,$$

where v_0 is a constant,. However, in general we see that the solution will not satisfy the condition

$$v \to 0 \quad \text{as} \quad \zeta \to \infty$$

except for a particular value of v_0, say , $v_0 = v_0^*$. In fact solutions fall into two types: either $v \to \infty$ or $v \to -\infty$. The different types of solution are shown in Fig. 7 .

Figure 7



In practice as $N > 0$ always we know that a solution may be discarded as soon as v becomes -ve i.e. when $v < 0$. We also may assume that a solution is such that $v \to \infty$ when v reaches 10. We start with two solutions corresponding to $v_o = v_1$ and $v_o = v_2$ such that

for v_1 $\quad v \to -\infty$

for v_2 $\quad v \to +\infty$

Then test the solution for

$$v_3 = \frac{1}{2}(v_1 + v_2)$$

If it is such that $v \to -\infty$ we replace v_1 by v_3 .
If it is such that $v \to +\infty$ we replace v_2 by v_3 .
In this way we refine the value of v_o as far as the accuracy of the computer permits. The two final solutions give an approximation to the exact solution, and it is obvious where they fail to give a good approximation (Fig. 8);

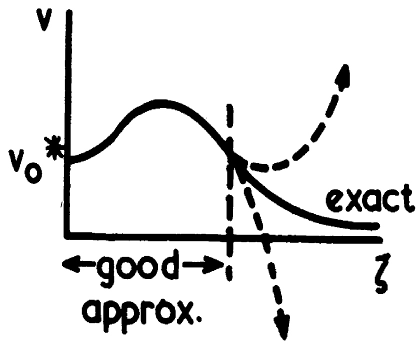


Figure 8

A suitable starting pair is $v_0 = 0$ and $v_0 = 9$ (say). The methods recommended are the Runge-Kutta method for starting the integration and Hamming's method for continuing it. These two integration formulae are equally accurate (of order h^5) but the Runge-Kutta method is unstable when v starts getting large. This can cause convergence to a false value of v_0^*. The reader should note that there are many other integration formulae, including Runge-Kutta formulae of different orders.

Runge-Kutta formula

To integrate the system of d.e.'s

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n) \qquad i = 1, 2, \ldots, n$$

we have for $n = 2$,

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2)$$

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The Runge-Kutta formula for the increments in y_1 and y_2 correspon-ding to an increment h in x is

$$\Delta y_i = \frac{1}{6} (K_{oi} + 2K_{1i} + 2K_{2i} + K_{3i}) \quad (i = 1, 2)$$

where

$$K_{oi} = h\, f_i(x^o, y_1^o, y_2^o)$$

$$K_{1i} = h\, f_i(x^o + \tfrac{1}{2}h,\ y_1^o + \tfrac{1}{2} K_{o1},\ y_2^o + \tfrac{1}{2} K_{o2})$$

$$K_{2i} = h\, f_i(x^o + \tfrac{1}{2} h,\ y_1^o + \tfrac{1}{2} K_{11},\ y_2^o + \tfrac{1}{2} K_{12})$$

$$K_{3i} = h\, f_i(x^o + h,\ y_1^o + K_{21},\ y_2^o + K_{22}) \qquad (i = 1, 2)$$

To use this in integrating (1) we have

$$n = 2 \qquad \zeta = x \qquad y_1 = v \qquad y_2 = \partial v / \partial \zeta$$

Then

$$\frac{dy_1}{dx} = y_2 \qquad = f_1(x, y_1, y_2)$$

$$\frac{dy_2}{dx} = 4\beta\gamma x^{2\lambda}\, y_1 - 4\gamma x e^{-x^2} \qquad = f_2(x, y_1, y_2)$$

The procedure is quite straightforward, but <u>we only make 5 steps,</u> then start using Hamming's method (with same steplength h).

<u>Hamming's method (see Ralston and Wilf</u>

To integrate numerically the d.e.

$$\frac{dy}{dx} = f(x, y)$$

let x_i $(i = 1, 2, \ldots)$ be a sequence of values of x; spaced at equal intervals, h, and such that the values $y(x_i)$ $(o \leq i \leq n)$ are known. We require to find $y(x_{n+1})$.

To simplify the notation, write

$$y_i = y(x_i) \qquad , \left(\frac{dy}{dx}\right)_{x=x_i} = f_i$$

then y_{n+1} is evaluated by the following method :

(1) Predictor : $p_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2})$

(2) Modifier : $m_{n+1} = p_{n+1} - \frac{112}{121}(p_n - C_n)$; $m'_{n+1} = f(x_{n+1}, m_{n+1})$

(3) Corrector: $C_{n+1} = \frac{1}{8}\left[9y_n - y_{n-2} + 3h(m'_{n+1} + 2f_n - f_{n-1})\right]$

(4) Final Value : $y_{n+1} = C_{n+1} + \frac{9}{121}(p_{n+1} - C_{n+1})$

Lattice techniques

Assume that the solution at $\zeta = 0$ is v_o, at $\zeta = h$ is v_1, at $\zeta = 2h$ is v_2 and so on. Then

$$\frac{\partial^2 v}{\partial \zeta^2} \approx \frac{v_{n-1} - 2v_n + v_{n+1}}{h^2}$$

Suppose $\zeta = \zeta_n$ at $v = v_n$. The finite difference approximation to equation (1) is then

$$v_{n-1} - (2 + 4\beta\gamma h^2 \zeta_n^{2\lambda}) v_n + v_{n+1}$$

(3)
$$= -4\gamma h^2 \zeta_n e^{-\zeta_n^2}$$

At $\zeta = 0$ we have $\partial v/\partial\zeta = 0$. The correct procedure here is to introduce a spurious value v_{-1} outside the boundary. Then on the boundary $\zeta = 0$,

$$\frac{\partial v}{\partial \zeta} \approx \frac{v_1 - v_{-1}}{2h} = 0$$

(4)
$$\therefore \quad v_1 = v_{-1}$$

The d.e. is satisfied on the boundary, so we obtain from (4) with $n = 0$ in (3) ($\zeta_o = 0$)

(5)
$$-v_o + v_1 = 0$$

There is no easy way of exactly ensuring that $v \to 0$ as $\zeta \to \infty$. From (1) we observe that for large values of ζ ($\zeta \gg 1$)

$$v \approx \zeta^{1-2\lambda} e^{-\zeta^2} / \beta$$

Thus we choose for n large enough (= N, say)

(6)
$$v_N = \zeta_N^{1-2\lambda} e^{-\zeta_N^2} / \beta$$

and impose this as a boundary condition. In practice $\beta\gamma \approx 30$ and we could choose $\zeta_N = 1$ as a remarkably good approximation to ∞.

Equations (3) (5) and (6) give us the set of N-1 equations

$$
\begin{aligned}
-v_o + v_1 &= 0\\
v_0 - A_1 v_1 + v_2 &= F_1\\
v_1 - A_2 v_2 + v_3 &= F_2\\
&\;\;\dots \qquad\qquad (7)\\
v_{N-3} - A_{N-2} v_{N-2} + v_{N-1} &= F_{N-2}\\
v_{N-2} - A_{N-1} v_{N-1} &= F_{N-1} - \gamma \zeta_N^{1-2\lambda} e^{-\zeta_N^2} / \beta
\end{aligned}
$$

where

$$A_n = 2 + 4\beta\gamma h^2 \zeta_n^{2\lambda} \qquad\qquad F_n = -4\gamma h^2 \zeta_n e^{-\zeta_n^2}$$

This system of equations is called a <u>tridiagonal</u> system. It is straightforward enough to solve for $v_o \dots v_{N-1}$ by using an algorithm given by Richtmyer.

<u>Algorithm for solving tridiagonal system</u> (see Richtmyer)

Suppose the general form is

(8) $$C_i v_{i-1} + B_i v_i + A_i v_{i+1} = K_i \qquad i = 0, 1, \dots, N-1$$

Set (9) $$v_i = r_i v_{i+1} + H_i$$

(10) $$v_{i-1} = r_{i-1} v_i + H_{i-1}$$

Then

$$C_i r_{i-1} v_i + C_i H_{i-1} + B_i v_i + A_i v_{i+1} = K_i$$

(11) or $$v_i \; C_i r_{i-1} + B_i = K_i - C_i H_{i-1} - A_i v_{i+1}$$

Then comparing (9) and (11)

(12) $$r_i = \frac{-A_i}{C_i r_{i-1} + B_i} \quad ; \quad (13) \quad H_i = \frac{K_i - C_i H_{i-1}}{C_i r_{i-1} + B_i}$$

Now from 1^{st} equation ; i.e. 1^{st} boundary condition

$$B_0 v_0 + A_0 v_1 = K_0$$

or $$v_0 = \frac{K_0}{B_0} - \frac{A_0}{B_0} \; v_1$$

so (14) $$r_0 = -\frac{A_0}{B_0} \quad ; \quad (15) \quad H_0 = \frac{K_0}{B_0}$$

Method of Solution (1) Calculate H_i, r_i for $i = 1, ..., N-1$ using (12) - (1

(2) **Using (10) repeatedly calculate v_i**

Numerical solution of $\partial N/\partial t = \partial^2 N/\partial x^2$ (Note that the methodes given here are only two of many possibilities)

Well formulated problem (Fig. 9)

h_t = distance between t lattice

h_x = distance between x lattice

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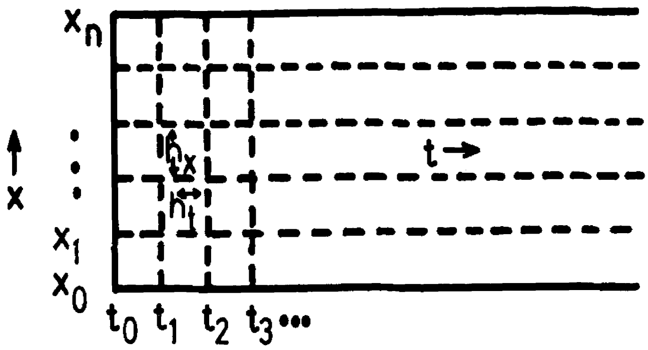


Fig. 9

In a well formulated problem N will be given at time $t = t_o$ as a function of x and it is required to find $N = N(x, t)$ at all later times under given boundary conditions: one on $x = x_o$ and one on $x = x_n$.

<u>Wrong numerical method</u> (i.e. wrong for slow computers)

Denote by brackets $\{ - \}_t$ the value of any function at time t. Then

$$\frac{\partial N}{\partial t} \approx \frac{\{N_o\}_{t+h_t} - \{N_o\}_t}{h_t}$$

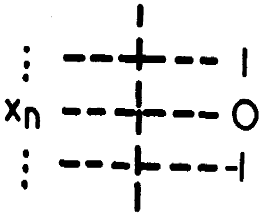
and

$$\frac{\partial^2 N}{\partial x^2} \approx \frac{\{N_{-1} + N_1 - 2N_o\}_t}{h_x^2}$$

giving on substituting into $\partial N/\partial t = \partial^2 N/\partial x^2$

$$\{N\}_{t+h_t} = \{N\}_t + \frac{h_t}{h_x^2}\{N_{-1} + N_1 - 2N_o\}_t$$

<u>Note</u> It is common practice to refer suffixes to the central point as origin, thus at x_n we have the situation shown in Fig. 10

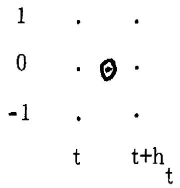


i.e. N_{-1} is N_{n-1}, N_0 is N_n and N_1 is N_{n+1}

Fig. 10

This gives quick and unstable integration from time t to time $t + h_t$. The method is <u>not recommended</u> (except for very fast computers where the steplength can be made small enough to achieve stability.

<u>Crank-Nicolson method</u> (See Smith)



Evaluate all derivatives at a point ⊙ midway between the two t lattice points at $x = x_n$.

$$\frac{\partial N}{\partial t} \approx \frac{\{N_o\}_{t+h_t} - \{N_o\}_t}{h_t}$$

$$\frac{\partial^2 N}{\partial x^2} \approx \left\{\frac{N_{-1} + N_1 - 2N_o}{h_x^2}\right\}_{t+\frac{1}{2}h_t}$$

But we have no known values of N at $t + \frac{1}{2} h_t$, so we use the approximation

$$f_{t+\frac{1}{2}h_t} \approx \frac{1}{2}\left[\{f\}_t + \{f\}_{t+h_t}\right]$$

Then, substituting into $\partial N/\partial t = \partial^2 N/\partial x^2$ using the abbreviation

$$s = h_t / 2h_x^2$$

gives

$$\{N_o\}_{t+h_t} - \{N_o\}_t =$$

$$s\left[\{N_{-1} + N_1 - 2N_o\}_{t+h_t} + \{N_{-1} + N_1 - 2N_o\}_t\right]$$

that is

$$-s\{N_{-1}\}_{t+h_t} + (1+2s)\{N_o\}_{t+h_t} - s\{N_1\}_{t+h_t}$$

$$= s\{N_{-1}\}_t + (1-2s)\{N_o\}_t + s\{N_1\}_t$$

Thus the integration proceeds from time t to $t+h_t$ by solving a system of tridiagonal equations like (7) . The first and last equations of the set are formed by introducing spurious points outside the boundary, if necessary, and by assuming that the differential equation is satisfied there.

Consider the operator $$\sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\, \partial / \partial x_i \qquad (*)$$

This may be reduced to the form $\partial f_1 / \partial x_1'$ by suitable transformations given below, thus making soluble the generalised diffusion equation

$$\sum_{i=1}^{n} f_i\, \partial N / \partial x_i = q - L + \mathfrak{D} N \qquad (**)$$

which can then be subjected to automatic numerical procedures.

In full (*) becomes

$$f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \ldots\ldots + f_n \frac{\partial}{\partial x_n}$$

Consider the system of ordinary differential equations

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$$

$$\frac{dx_3}{dx_1} = \frac{f_3}{f_1}$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$\frac{dx_n}{dx_1} = \frac{f_n}{f_1}$$

with $x_2 = x_{2o}$, $x_3 = x_{3o}$, ..., $x_n = x_{no}$ at $x_1 = 0$.

This defines a series of functions

$$x_2 = x_2(x_1, x_{2o}, x_{3o}, \ldots, x_{no})$$

$$x_3 = x_3(x_1, x_{2o}, x_{3o}, \ldots, x_{no})$$

$$\begin{matrix} \cdot \\ \cdot \end{matrix}$$

$$x_n = x_n(x_1, x_{2o}, x_{3o}, \ldots, x_{no})$$

Consider the transformation from the variables $x_1, x_2, \ldots x_n$ to variables $x_1', x_2' \ldots x_n'$ defined by

$$x_1' = x_1$$

$$x_2' = x_{2o}(x_1, x_2, \ldots x_n)$$

$$x_3' = x_{3o}(x_1, x_2, \ldots x_n)$$

$$\cdots\cdots$$

$$x_n' = x_{no}(x_1, x_2, \ldots x_n)$$

Then we know that

$$\frac{\partial}{\partial x_1'} = \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial x_1}\frac{\partial}{\partial x_3} + \ldots + \frac{\partial x_n}{\partial x_1}\frac{\partial}{\partial x_n}$$

$$= \frac{\partial}{\partial x_1} + \frac{f_2}{f_1}\frac{\partial}{\partial x_2} + \frac{f_3}{f_1}\frac{\partial}{\partial x_3} + \ldots + \frac{f_n}{f_1}\frac{\partial}{\partial x_n}$$

and so ($*$) becomes

$$f_1 \frac{\partial}{\partial x_1'} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$$

Note that all the transformations of variables can be carried out numerically, even if exact integrals cannet be found. The variables $x_{2o}, x_{3o}, \ldots x_{no}$ retain the values of $x_2, x_3 \ldots x_n$ at the start of the integration. At any subsequent point we obtain from ($**$)

$$N = N(x_1', x_{2o}, x_{3o}, \ldots x_{no} + \text{other coordinates})$$

The values of $x_1, \ldots x_n$ at which this is the solution are obtained by integrating the system of equations. Note that in any case these would be needed in the computer at the correct stage of the calculation.

Exercise . Use the above analysis to derive the transformation which reduces the operator

$$\frac{\partial}{\partial t} + W_o \sin \overline{\omega t + \delta} \frac{\partial}{\partial a}$$

to the form $\frac{\partial}{\partial t'}$, where ω, δ and W_o are constants.

Answer $t' = t, \quad a' = a + \frac{W_o}{\omega} \cos \overline{\omega t + \delta}$

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Lecture 4

The diffusion operator with curved magnetic field lines (neutral air at rest)

We choose to write the diffusion equation in the form

$$\frac{\partial N}{\partial t} = Q - L + W\mathfrak{D}_1 N + D\mathfrak{D}_2 N \tag{1}$$

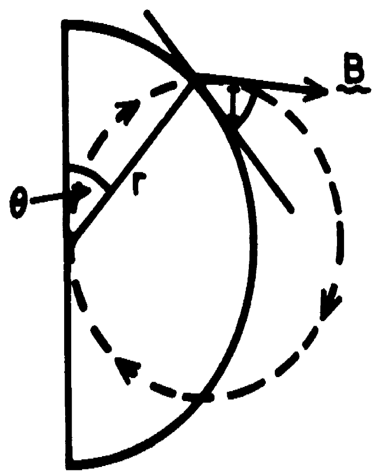
where W is a constant with dimensions of velocity, $\mathfrak{D}_1$ is the Hall drift operator defined by

$$W\mathfrak{D}_1 N = -\underline{\nabla}.(N\underline{V}_i^{\perp}) \tag{2}$$

and $\mathfrak{D}_2$ is the diffusion operator defined by

$$D\mathfrak{D}_2 N = -\underline{\nabla}.(N\underline{V}_i^{\parallel}) \tag{3}$$

Geometry of geomagnetic dipole field lines (Fig. 11)



Field lines are family of curves

(4) $$r = a \sin^2 \theta$$

Thus

(5) $$\sin I = 2 \cos \theta / \Delta \qquad \cos I = \sin \theta / \Delta$$

where

(6) $$\Delta = \sqrt{(1 + 3 \cos^2 \theta)}$$

Also

(7) $$\tan I = 2 \cot \theta$$

Thus the tangent $\hat{\underline{T}}$ to a field line is

(8) $$\hat{\underline{T}} = (\sin I, \cos I, 0)$$

$$\underline{\nabla}^{\parallel} = \hat{\underline{T}} (\hat{\underline{T}} . \underline{\nabla})$$

$$= \hat{\underline{T}} \left(\sin I \frac{\partial}{\partial r} + \frac{\cos I}{r} \frac{\partial}{\partial \theta} \right)$$

<u>Form of $\mathfrak{D}_2$</u>

Thus as

(10) $$\frac{2NkT}{D} \underline{V}_i^{\parallel} = - \underline{\nabla}^{\parallel}(2NkT) + N m_i \underline{g}^{\parallel}$$

we have

(11) $$N\underline{V}_i^{\parallel} = - D\hat{\underline{T}} \left(\sin I \frac{\partial N}{\partial r} + \frac{\cos I}{r} \frac{\partial N}{\partial \theta} + \frac{N \sin I}{2H} \right)$$

Hence $$\underline{\nabla} . N\underline{V}_i^{\parallel} = -(\hat{\underline{T}} . \underline{\nabla}) D \left(\sin I \frac{\partial N}{\partial r} + \frac{\cos I}{r^2} \frac{\partial N}{\partial \theta} + \frac{N \sin I}{2 H} \right)$$

(12) $$- D(\underline{\nabla} . \hat{\underline{T}}) \left(\sin I \frac{\partial N}{\partial r} + \frac{\cos I}{r} \frac{\partial N}{\partial \theta} + \frac{N \sin I}{2H} \right)$$

Using the fact that $D \propto \exp(z/H)$, where $z = r - r_0$ and $r = r_0$ is the base of the F2 layer, this gives

$$\mathfrak{D}_2 N = \left(\sin I \frac{\partial}{\partial r} + \frac{\cos I}{r} \frac{\partial}{\partial \theta}\right)\left(\sin I \frac{\partial N}{\partial r} + \frac{\cos I}{r} \frac{\partial N}{\partial \theta} + \frac{N \sin I}{2H}\right)$$

$$+ \left\{\frac{\sin I}{H} + (\underline{\nabla} . \underline{\hat{T}})\right\}\left(\sin I \frac{\partial N}{\partial r} + \frac{\cos I}{r} \frac{\partial N}{\partial \theta} + \frac{N \sin I}{2 H}\right) \tag{13}$$

It may be verified that

$$\underline{\nabla} . \underline{\hat{T}} = (9 \cos \theta + 15 \cos^3 \theta) / r \Delta^3 \tag{14}$$

Form of $\mathfrak{D}_1$

We have

$$\underline{V}_{\perp} = \underline{E}_H \times \underline{B}/B^2 \tag{15}$$

Also, the electrostatic potential is constant along a field line so that

$$\Omega_H = - F(\phi, a), \tag{16}$$

where ϕ is the longtitude ($t = \tau\phi$, where τ = number of seconds in a radian $= 1.37 \times 10^4$). Then

$$\underline{E}_H = - \text{grad } \Omega_H, \tag{17}$$

and so in spherical polar coordinates (r, θ, ϕ)

$$\underline{E}_H = \left(\frac{1}{\sin^2 \theta}, -\frac{2 \cos \theta}{\sin^3 \theta}, 0\right) \frac{\partial F}{\partial a}$$

$$+ \left(0, 0, \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi}\right) \tag{18}$$

That is

$$\underline{E}_H = \frac{\Delta}{\sin^3\theta}(\cos I, \; -\sin I, \; 0)\frac{\partial F}{\partial a} \tag{19}$$
$$+ \left(0, \; 0, \; \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\right)$$

Also if M = dipole moment, $B = M\,\Delta/r^3$ and

$$\underline{B} = \frac{M\Delta}{r^3}(\sin I, \; \cos I, \; 0) \tag{20}$$

Thus

$$\underline{E}_H \times \underline{B} = \frac{M\Delta^2}{r^3 \sin^3\theta}\frac{\partial F}{\partial a}(0, 0, 1) \tag{21}$$
$$- \frac{M\Delta}{r^4\sin\theta}\frac{\partial F}{\partial \phi}(\cos I, \; -\sin I, \; 0)$$

But

$$\underline{\nabla}.(\underline{E}_H \times \underline{B}) = \underline{B}.(\underline{\nabla}\times\underline{E}_H) - \underline{E}_H.(\underline{\nabla}\times\underline{B}) \tag{22}$$
$$\doteq 0$$

So

$$W\mathfrak{D}_1 N \doteq -(\underline{E}_H \times \underline{B}).\underline{\nabla}\frac{N}{B^2} \tag{23}$$
$$= \frac{-M\Delta^2}{r^3\sin^3\theta}\frac{\partial F}{\partial a}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{r^6}{M^2\Delta^2}N\right)$$
$$+\frac{M\Delta}{r^4\sin\theta}\frac{\partial F}{\partial\phi}\left(\cos I\frac{\partial}{\partial r} - \frac{\sin I}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{r^6}{M^2\Delta^2}N\right)$$

And as $r = a\sin^2\theta$

$$W\mathfrak{D}_1 N = -\frac{a^2}{M}\frac{\partial F}{\partial a}\frac{\partial N}{\partial\phi} + \frac{r^2\,\partial F/\partial\phi}{M\Delta\sin\theta}\left(\cos I\frac{\partial N}{\partial r} - \frac{\sin I}{r}\frac{\partial N}{\partial\theta}\right) \tag{24}$$
$$+\frac{6r(1+\cos^2\theta)}{M(1+3\cos^2\theta)^2}\frac{\partial F}{\partial\phi}N$$

Finally to make this dimensionless we may substitute

$$F = \frac{W\,M}{r_o^2}\; f\,(a, \phi), \tag{25}$$

where f is dimensionless giving

$$\begin{aligned} \mathcal{D}_1 N = &- \frac{a^2}{r_o^2}\,\frac{\partial f}{\partial a}\,\frac{\partial N}{\partial \phi} + \frac{r^2\,\partial f/\partial\phi}{r_o^2\,\Delta\,\sin\theta}\left(\cos I\,\frac{\partial N}{\partial r} - \frac{\sin I}{r}\,\frac{\partial N}{\partial \theta}\right) \\ &+ \frac{6r(1+\cos^2\theta)}{r_o^2\,\Delta^4}\,\frac{\partial f}{\partial \phi}\,N \end{aligned} \tag{26}$$

Finally, we make the substitution

$$N = e^{-z/2H}\,V \tag{27}$$

to obtain the dimensionless form of (1), namely,

$$\begin{aligned} \frac{\partial V}{\partial \phi} = &\ \gamma\, q_o\ e\ \exp\left(-\frac{z}{2H} - \frac{e^{-z/H}}{\cos\chi}\right) - \beta\, e^{-\lambda z/H}\, V \\ &+ W'\,\mathcal{D}_1'\,V + \frac{e^{z/H}}{\gamma}\,\mathcal{D}_2'\,V \end{aligned} \tag{28}$$

$$\begin{aligned} H^{-1}\mathcal{D}_1'\,V = \Big\{ &- \frac{a^2}{r_o^2}\,\frac{\partial f}{\partial a}\,\frac{\partial V}{\partial \phi} + \frac{r^2\,\partial f/\partial\phi}{r_o^2\,\Delta\,\sin\theta}\left(\cos I\,\frac{\partial V}{\partial r} - \frac{\sin I}{r}\,\frac{\partial V}{\partial \theta}\right) \\ &+ \left[-\frac{r^2\,\cos I}{2Hr_o^2\,\Delta \sin\theta} + \frac{6r\,(1+\cos^2\theta)}{r_o^2\,\Delta^4 H}\right]\frac{\partial f}{\partial \phi}\,V \Big\} \end{aligned} \tag{29}$$

$$H^{-2}\mathfrak{D}_2' \, V = \left(\sin I \frac{\partial}{\partial r} + \frac{\cos I}{r} \frac{\partial}{\partial \theta}\right)^2 V$$

(30) $$+ \left(\frac{\sin I}{2H} + \underline{\nabla}.\hat{\underline{T}}\right)\left(\sin I \frac{\partial}{\partial r} + \frac{\cos I}{r}\frac{\partial}{\partial \theta}\right) V$$

where

(31) $$W' = \uparrow W/H \; ,$$

and β and γ are as before[1], except that a factor $\sin^2 I$ is omitted fro γ. We wish to integrate equation (28), which is a partial differential equation in 3 independent variables. This is now possible, and I first outline the difficulties.

<u>Difficulties</u>

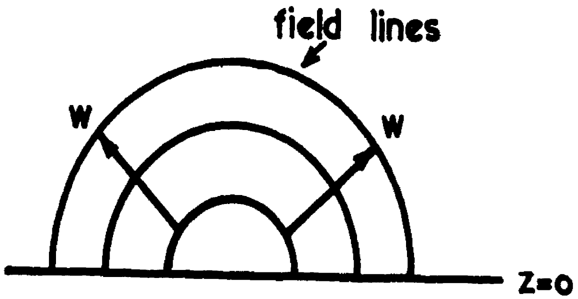


Fig. 12

Roughly speaking we are integrating outwards along lines $\perp^r$ to $\underline{B}$ (Fig. 12). The line $z = 0$ is not a natural boundary in our coordinate system (Fig. 13).

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1. See Lecture 2, equations 7 and 8.

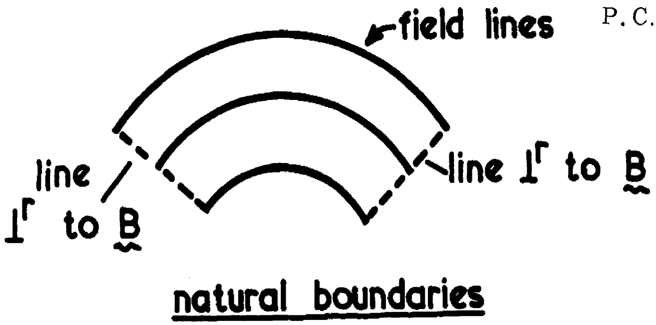


Fig. 13

Also as $r \to \infty$ the total length involved along a field line becomes infinite and the region of integration becomes infinite. We therefore need a coordinate transformation which

(a) transforms $z = 0$ into a natural boundary

(b) transforms $z = \infty$ into a finite point.

A suitable transformation is

(32) $$x^2 = \frac{e^{-hr/H} - e^{-ha/H}}{e^{-hr_o/H} - e^{-ha/H}}$$

(33) $$= 1 + \frac{e^{-hr/H} - e^{-hr_o/H}}{e^{-hr_o/H} - e^{-ha/H}}$$

where

h = dimensionless constant > 0

and

$x > 0$ in Northern hemisphere

$x < 0$ in Southern hemisphere

If $a \gg r$

$$x \approx e^{-hz/2H}$$

On equator $r = a$,

$$x = 0 .$$

Also at $r = r_o$,

$$x = 1 \text{ (Northern) or } -1 \text{ (Southern)}$$

We thus obtain the mapping shown in Fig. 14

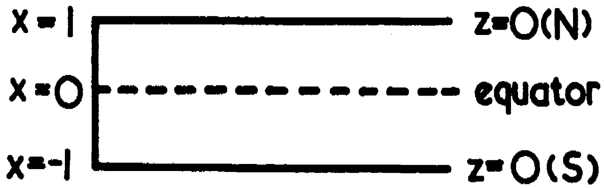


Fig. 14

Choose as independent variables

$$x^2 = 1 + \frac{e^{-hr/H} - e^{-hr_o/H}}{e^{-hr_o/H} - e^{-ha/H}} = 1 + \frac{\Lambda_1}{\Lambda_2} \tag{34}$$

and

$$a = \frac{r}{\sin^2\theta}$$

Then

$$\frac{\partial}{\partial r} = \frac{1}{\sin^2\theta}\frac{\partial}{\partial a} + \left(-\frac{h}{H}\frac{e^{-hr/H}}{\Lambda_2} - \frac{h}{H}\frac{\Lambda_1 e^{-ha/H}}{\Lambda_2^{\,2}}\frac{\partial a}{\partial r}\right)\frac{\partial}{\partial (x^2} \tag{35}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta} = -\frac{2\cos\theta}{\sin^3\theta}\frac{\partial}{\partial a} - \left(\frac{h}{H}\frac{\Lambda_1 e^{-ha/H}}{\Lambda_2^{\,2}}\frac{\partial a}{r\partial\theta}\right)\frac{\partial}{\partial(x^2)} \tag{36}$$

Thus

$$\text{(37)}\qquad 2\cos\theta\frac{\partial}{\partial r}+\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\equiv-\frac{2h}{H}\frac{e^{-hr/H}}{\Lambda_2}\cos\theta\frac{\partial}{\partial(x^2)}$$

giving

$$\text{(38)}\qquad \sin I\frac{\partial}{\partial r}+\frac{\cos I}{r}\frac{\partial}{\partial\theta}\equiv\frac{-h\sin I\, e^{-hr/H}}{H(e^{-hr_o/H}-e^{-ha/H})}\frac{\partial}{\partial(x^2)}$$

Similarly

$$\sin\theta\frac{\partial}{\partial r}-\frac{2\cos\theta}{r}\frac{\partial}{\partial\theta}\equiv\frac{\Delta^2}{\sin^3\theta}\left(\frac{\partial}{\partial a}-\frac{h}{H}\frac{\Lambda_1 e^{-ha/H}}{\Lambda_2^2}\frac{\partial}{\partial(x^2)}\right)$$

$$-\frac{h\sin\theta\, e^{-hr/H}}{H\Lambda_2}\frac{\partial}{\partial(x^2)}$$

giving

$$\cos I\frac{\partial}{\partial r}-\frac{\sin I}{r}\frac{\partial}{\partial\theta}\approx\frac{\Delta}{\sin^3\theta}\frac{\partial}{\partial a}$$

$$\text{(39)}\qquad -\frac{h}{H}\left\{\frac{(x^2-1)\Delta e^{-ha/H}}{\Lambda_2\sin^3\theta}+\frac{\sin\theta\, e^{-hr/H}}{\Delta\ \Lambda_2}\frac{\partial}{\partial(x^2)}\right\}$$

Therefore

$$\mathcal{D}_1{}'V=\frac{a^2H}{r_o^2}\frac{\partial f}{\partial\phi}\frac{\partial V}{\partial a}-\frac{a^2H}{r_o^2}\frac{\partial f}{\partial a}\frac{\partial V}{\partial\phi}$$

$$\text{(40)}\qquad -\frac{h r^2\,\partial f/\partial\phi}{r_o^2\Delta\sin\theta}\left\{\frac{(x^2-1)\Delta e^{-ha/H}}{\Lambda_2\sin^3\theta}+\frac{\sin\theta\, e^{-hr/H}}{\Delta(e^{-hr_o/H}-e^{-ha/H})}\frac{\partial V}{\partial(x^2)}\right.$$

$$(40) \qquad + \left[\frac{6rH(1+\cos^2\theta)}{r_o^2 \Delta^4} - \frac{r^2 \cos I}{2r_o^2 \Delta \sin\theta} \right] \frac{\partial f}{\partial \phi} V$$

and

$$\mathfrak{D}_2{}' V = \left(\frac{\frac{1}{2}h \sin I \, e^{-hr/H}}{e^{-hr_o/H} - e^{-ha/H}} \right)^2 \left(\frac{1}{x^2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{x^3}\frac{\partial V}{\partial x} \right)$$

$$(41) \qquad + \frac{[h^2 \sin^2 I + 2(hH/a\Delta^4)]}{2x(e^{-hr_o/H} - e^{-ha/H})} \left(e^{-hr/H} \frac{\partial V}{\partial x} \right)$$

$$- \left(\frac{\sin I}{2} + H \, \underline{V}.\hat{\underline{I}} \right) \left(\frac{\frac{1}{2}h \sin I \, e^{-hr/H}}{x(e^{-hr_o/H} - e^{-ha/H})} \right) \frac{\partial V}{\partial x}$$

which simplifies to (see (14))

$$(42) \qquad \mathfrak{D}_2{}' V = \left\{ \frac{h \sin I \, e^{-hr/H}}{2x \, \Lambda_2} \right\}^2 \frac{\partial^2 V}{\partial x^2}$$

$$- \frac{\frac{1}{4}h \sin^2 I \, e^{-hr/H}}{x\Lambda_2} \left\{ \frac{(1+h)e^{-hr/H} - e^{-ha/H}}{e^{-hr/H} - e^{-ha/H}} + \frac{15\mu^4 + 10\mu^2 - 1}{r\,\mu^2\Delta^2/H} - 2h \right\} \frac{\partial V}{\partial x}$$

Check : If $a \gg r_o$ and $h = 1$, $x \approx e^{-z/2H}$, $I = 90°$,

then $\mathfrak{D}_2{}' V \approx \frac{1}{4} e^{-z/H} \partial^2 V/\partial x^2$ in agreement with Lecture 2, equation 9 ($\zeta = x$) .

We shall take

$$(43) \qquad f = (r_o^2/a^2) \quad \cos \phi$$

giving a Hall drift which is independent of a at the equator and $\propto \sin \phi$.

The full diffusion equation is thus

$$\left(1 - \frac{2\,W'\,H}{a}\cos\phi\right)\frac{\partial V}{\partial \phi} + W'H\sin\phi\,\frac{\partial V}{\partial a}$$

$$= \frac{e^{z/H}}{\gamma}\left\{\frac{h\sin I\ e^{-hr/H}}{2x\,\Lambda_2}\right\}^2\frac{\partial^2 V}{\partial x^2}$$

$$+\left[-\frac{\frac{1}{4}h\sin^2 I\ e^{(z-hr)/H}}{x\Lambda_2\gamma}\left\{\frac{(1+h)e^{-hr/H} - e^{-ha/H}}{e^{-hr/H} - e^{-ha/H}} + \frac{15\mu^4 + 10\mu^2 - 1}{r\mu^2\Delta^2/H} - 2h\right\}\right.$$

$$\left.+\frac{W'h\,r^2\sin\phi}{2xa^2\Lambda_2\Delta^2\sin^4\theta}\left\{(x^2-1)\ \Delta^2 e^{-ha/H} + \sin^4\theta\ e^{-hr/H}\right\}\right]\frac{\partial V}{\partial x}$$

$$- W'\ \sin\phi\left[\frac{6rH(1+\cos^2\theta)}{a^2(1+3\cos^2\theta)^2} - \frac{r^2}{2a^2(1+3\cos^2\theta)}\right]V$$

$$- \beta\ e^{-\lambda z/H}\ V$$

$$+ \Upsilon\ q_o\ e\ \exp\left(-\frac{z}{2H} - \frac{e^{-z/H}}{\cos\chi}\right) \tag{44}$$

We see from lecture (3) that this can be now integrate numerically.

P. C. Kendall

Lecture 5

Equilibrium solutions

Diffusive equilibrium (and comparison of theoretical and satellite results)

When the velocity of the ions is zero, i.e.

(1) $$\underline{V}_i = 0$$

we have a situation known as diffusive equilibrium.
You may be surprised that such a simple situation could occur at all; however all this means is that the ion gas is in hydrostatic equilibrium. Production Q and loss L of electrons could only be occurring if Q = L , but this is unlikely to occur simultaneously with (1) . The ion-electron gas has half the mean molecular weight of the ion gas and so if the temperature is uniform

(2) $$N \propto \exp(-z/2H)$$

where z is the height above some fixed reference level and H is the scale height of the ions. It follows that (2) should satisfy equation (44) of the previous lecture. The student should verify this as an exercise. Note that this is also a useful check of complicated diffusion equations. Thus , what we are in fact obtaining when we use (1) is a solution of equation (44) of the previous lecture, for the case of no production (night-time) and no loss (β =0) . It follows that in investigating solutions of th full diffusion equation we first study diffusive equilibrium. There is another reason for doing this. We can include the effects of a height varying temperature field quite easily. Consider a constituent characterised by ionic mass m_i , partial pressures p_i, p_e for ions and electrons, and electric field $\underline{E}$. For equilibrium of the ions along a field line where I is th dip angle

$$\frac{dp_i}{ds} = - N m_i \; g \; \sin I \; + Ne \; E_s \; ,$$

where s denotes arc length and E_s is the component of $\underline{E}$ along a field line. For equilibrium of electrons, neglecting the electron mass,

$$\frac{dp_e}{ds} = - Ne \; E_s \; .$$

Thus, as $p_i = NkT = p_e$, where $T = T(z)$ is the temperature, z being the height, we have <u>along a field line</u>

$$\frac{dp_i}{ds} = - \frac{p_i \sin I}{2H(z)}$$

where

$$H(z) = kT(z)/m_i g$$

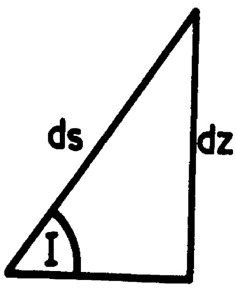


Fig. 15

But along a field line (Fig. 15)

$$dz = ds \; \sin \; I$$

Thus

$$\frac{dp_i}{dz} = - \frac{p_i}{2H(z)}$$

giving

$$p_i \propto \exp - \int \frac{dz}{2H} .$$

Thus <u>along a field line</u>

(3) $$N \propto \frac{1}{T} \exp - \int \frac{dz}{2H} .$$

If α is the latitude, the equation of a field line in spherical polar coordinates, r, θ, ϕ (θ = colatitude, ϕ = longtitude) is

(4) $$r = a \sin^2\theta = a \cos^2\alpha$$

we see that

(5) $$N = f(a) \quad g(z)$$

where f is some function of a only and

(6) $$g(z) = \frac{1}{T} \exp - \int \frac{dz}{2H} .$$

On these assumptions, (i.e. ignoring variations of H with respect to latitude) we can compare theory and experiment.

Satellite Aloutte I results* appear in the form of Fig. 16 , which shows electron density versus latitude, each curve being for a fixed height above the ground (shown in Km) .

* Taken from a private communication from Dr J.W. King

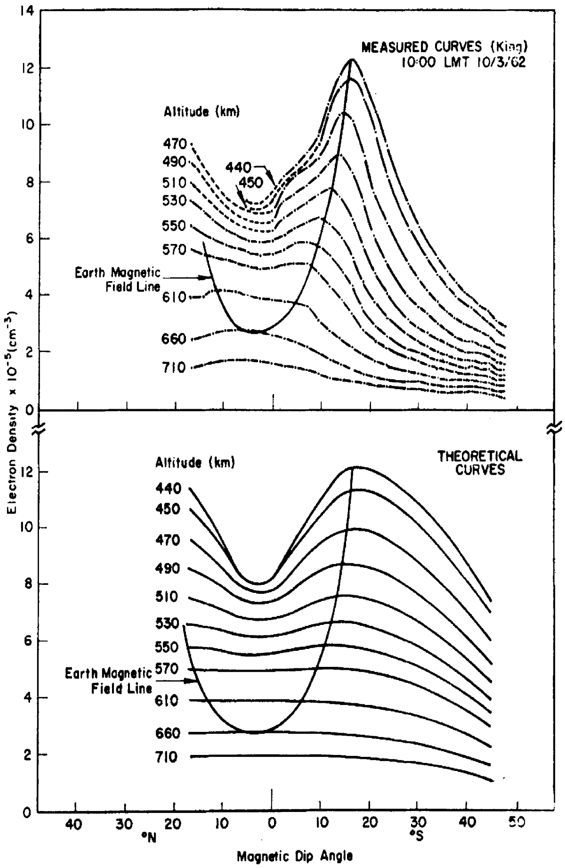


Figure 16

Figure 17

A clue as to the procedure is seen when we take K = constant and f(a) = any function with a single maximum. Then we obtain Fig. 17 (see Goldberg, Kendall and Schmerling, 1964, Journal of Geophysical Research 69 , 417-427) .

This is so similar to the previous diagram that further investigation is worthwhile. Baxter and Kendall, 1965 , Journal of Atmospheric and

Terrestrial Physics 27, 129-132 have made quantitative comparisons between theory and experiment. Thus, if the lowest of the constant height experimental curves is assumed to be given together with the equatorial profile, we can deduce all the other curves using this theory. As the experimental curves are also available this is an interesting comparison between theory and experiment. Thus, at fixed height $z = z_1$ (say) we have

$$N = N(z_1, \alpha) = \text{given function},$$

where α = latitude. Thus, putting $r_1 = r_e + z_1$, where r_e is the earth's radius and using (4), (5) gives

$$f(r_1 \sec^2 \alpha) = N[z_1, \alpha] / g(z_1)$$

giving

$$f(a) = N[z_1, \cos^{-1} \sqrt{(r_1/a)}] / g(z_1) .$$

We also know that at the equator

$$N = N[z, 0] = \text{given function}.$$

Thus from (5)

$$g(z) = N[z, 0] / f(r)$$

$$= N[z, 0] / f(r_e + z)$$

If follows that at a general point

$$N = f(a) \quad g(z)$$

$$= \frac{N[z_1, \cos^{-1} \sqrt{(r_1 \cos^2 \alpha / r)}] N[z, 0]}{N[z_1, \cos^{-1} \sqrt{(r_1/r)}]}$$

The results of the investigation are shown in Fig. 18 . The solid lines are observations, the broken lines are theory.

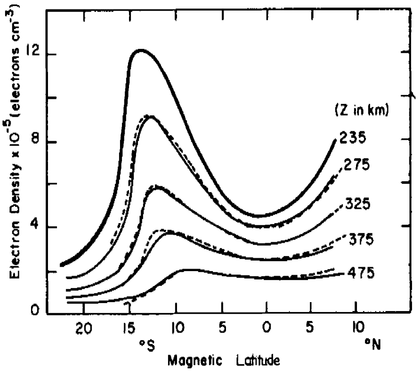


Figure 18

All we can conclude without further information on the temperature and composition is that mathematically the distribution N is almost separable in the variables z and a.

Effects of electrodynamic drift on the topside ionosphere

By expanding in a power series Baxter, Kendall and Windle, Journal of Atmospheric and Terrestrial Physics 1965 27 , 1263-1273 have studied the disturbance of this type of diffusive equilibrium by an upwards and outwards Hall drift. They find that the ionization adjusts itself so as to compensate for the Hall motion at right angles to the field lines. The diagrams will not be reproduced in these notes. The *ad hoc* methods they used to solve the diffusion equation show that there is room for mathematical work in this field.

Equilibrium solutions and the effects of electrodynamic drift

Bramley and Peart (1966) and Hanson and Moffett (1966) have both integrated the equilibrium equations with realistic production and loss terms. The word "equilibrium" is taken to mean $\partial/\partial t \equiv 0$. Thus we take

ϕ = constant and put $\partial/\partial\phi = 0$ in equation (44). Then, using obvious abbreviations

$$\frac{\partial V}{\partial a_o} = f_1(x, a_o) + f_2(x, a_o)V + f_3(x, a_o)\frac{\partial V}{\partial x} + f_3(x, a_o)\frac{\partial^2 V}{\partial x^2} ,$$

where

$$a_o = a/H$$

It should be noted that these authors did not use the mathematical transformations of Lecture 4. The problem is clearly readily tractable numerically. The calculation is started from the field line $a_o = (6550 \text{ km})/H$ and continued as far as may be required. The boundaries of the system are shown in Fig. 19

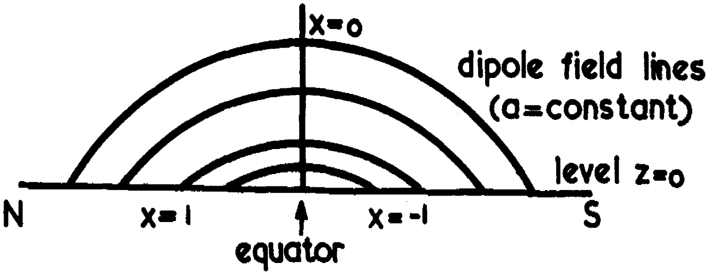


Fig. 19

The problem in the a_o,x plane is straightforward (Fig. 20)

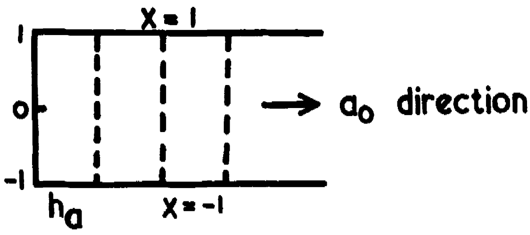


Figure 20

h_a = steplength of numerical process in a_o direction

On the boundaries (which happen also to be $z = 0$, we put $V = - f_1(x, a_o) / f_2(x, a_o)$. The results obtained are sket ched in Fig. 21.

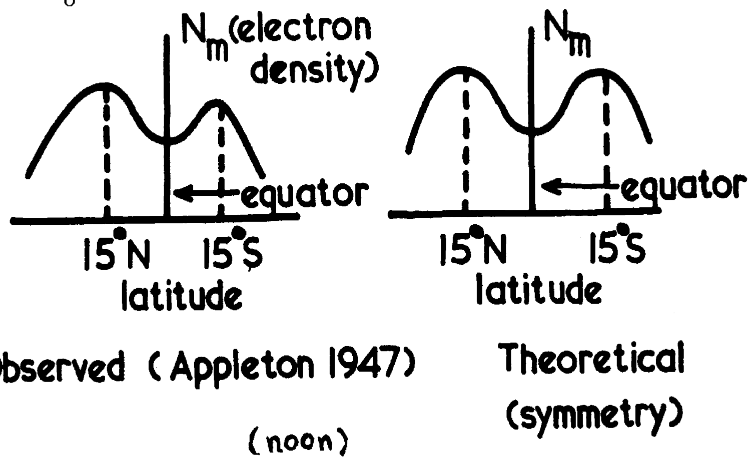


Figure 21

A drift upwards of about 10m/s is needed, with typical F2 layer parameters, to produce a suitable "Appleton anomoly" in the case of symmetry. Hanson and Moffett find that a SN wind of 60 m/s could cause the observed asymmetry, but do not claim this as explanation.

P. C. Kendall

Lecture 6

Time dependent solutions of the F2 region diffusion equation (with Dr. R.G. Baxter) Using obvious abbreviations, the full diffusion equation of Lecture 4 (equ.44) becomes approximately, if $W'H/a \ll 1$,

$$\frac{\partial V}{\partial \phi} + f(\phi)\frac{\partial V}{\partial a}$$

(1)
$$= f_1(x, a, \phi) + f_2(x, a, \phi)V + f_3(x, a, \phi)\frac{\partial V}{\partial x} + f_4(x, a, \phi)\frac{\partial^2 V}{\partial x^2}$$

The operator on the right hand side may be reduced by integrating the equation

$$\frac{da}{d\phi} = f(\phi)$$

Thus , as the integral is

$$a = a_o + \int_0^{\phi} f \, d\phi ,$$

where a_o is a constant of integration, the required transformation is

(2)
$$\phi' = \phi$$
$$a' = a - \int_0^{\phi} f \, d\phi$$

That this algorithm works can be seen at once, for

$$\frac{\partial}{\partial \phi} = \frac{\partial \phi'}{\partial \phi}\frac{\partial}{\partial \phi'} + \frac{\partial a'}{\partial \phi}\frac{\partial}{\partial a'}$$
$$= \frac{\partial}{\partial \phi'} - f\frac{\partial}{\partial a'}$$

Also

$$\frac{\partial}{\partial a} = \frac{\partial}{\partial a'}$$

Thus

$$\frac{\partial}{\partial \phi'} = \frac{\partial}{\partial \phi} + f \frac{\partial}{\partial a}$$

Using variables

$$x,\ a',\ \phi\ ,$$

so that

$$f_i(x, a, \phi) = f_i(x,\ a' + \int_0^{\phi'} f d\phi'\ ,\ \phi') \quad i = 1, 2, \dots 4,$$

equation (1) becomes

$$\frac{\partial V}{\partial \phi'} = f_1 + f_2 V + f_3 \frac{\partial V}{\partial x} + f_4 \frac{\partial^2 V}{\partial x^2} \tag{3}$$

This is the form of equation dealt with successfully in lecture 3 by numerical methods. Mathematically we simply choose a value of a', then integrate in ϕ' as usual, using the Crank-Nicolson method to advance

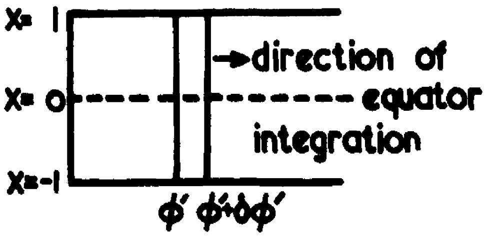


Fig. 22

the integration from ϕ' to $\phi' + \delta\phi'$ (Fig. 22) . On the boundaries we put

$$V = - f_1 / f_2 \quad \text{(boundary condition)}$$

The integration is started at $\phi' = h_{\phi'}$, where $h_{\phi'}$, is the step length in the ϕ' direction. This avoids difficulty with the singularity at $\phi' = 0$.

We can thus develop the values

$$V = V(x, a', \phi')$$

Hence

$$V = V(x, a, \phi)$$

Then the electron density is

$$N = e^{-z/2H} V,$$

where z denotes the height , and H is the constant scale height.

Physical explanation

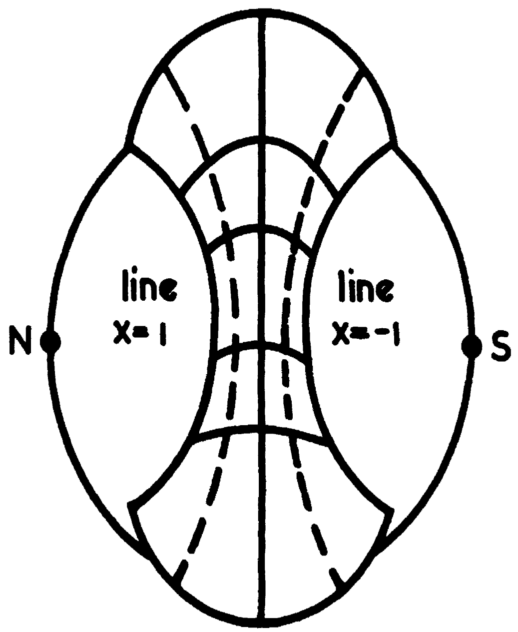


Figure 23

Fig. 23 shows how the Hall drift $\underline{E} \times \underline{B}/B^2$ combines with the earth's rotation so that a magnetic field line moves up and down and also rotates about the axis of symmetry. We have simply chosen coordinates moving with the field lines. The diagram shows the surface described by a moving field line as it passes round the earth.

Processing of results

The results appear as tables of N, the electron density, against x for a given ϕ' and a'. This immediately gives us

$$a = a' + \int_0^{\phi} f \, d\phi,$$

so we then know which field line

$$r = a \sin^2 \theta$$

we are on. Knowing a, to each value of x equation (33) of lecture 4 gives one value of r. Thus we can make the computer print out

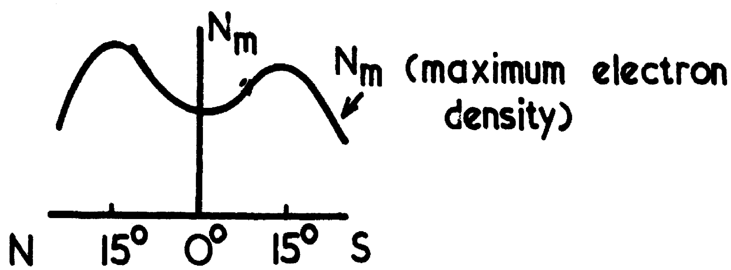
	x	N	r	θ
	0	-	-	-
	0.1	-	-	-
(say)	0.2	-	-	-
	...			
	1.0	-	-	-

The problem can thus be readily converted back into the original spherical polar coordinates r, θ, ϕ. (of course ϕ measures the time, because $t = \tau\phi$). As a final grace we can make the computer interpolate to $\theta = 90^{\circ}, 89^{\circ}, 88^{\circ} \ldots$ and also $\frac{z}{H} = 0, 0.5, 1.0 \ldots$ This enables us to draw curves of N at fixed latitude or fixed height.

Physical data

The measurements of electron density in the ionosphere are made by ionosonde from either the ground, thus dealing with the 'bottomside' of the ionosphere, or from a satellite, thus dealing with the 'topside' ionosphere. A radio wave travelling into the ionosphere is reflected at a point where the electron density has a given value which depends on the wave frequency. Thus, in a layer with a single maximum of electron density there is a critical frequency f_oF_2 (in the ordinary mode of propagation) beyond which radio waves pass through without reflection. There is another mode of wave propagation, the extraordinary mode, with the critical frequency f_EF_2 . [See Ratcliffe's book]

Thus an analysis using the critical frequency f_oF2, from many ground based stations gives results such as those of Appleton (1947) (Fig. 24) and Martyn (1959) (Fig. 25)

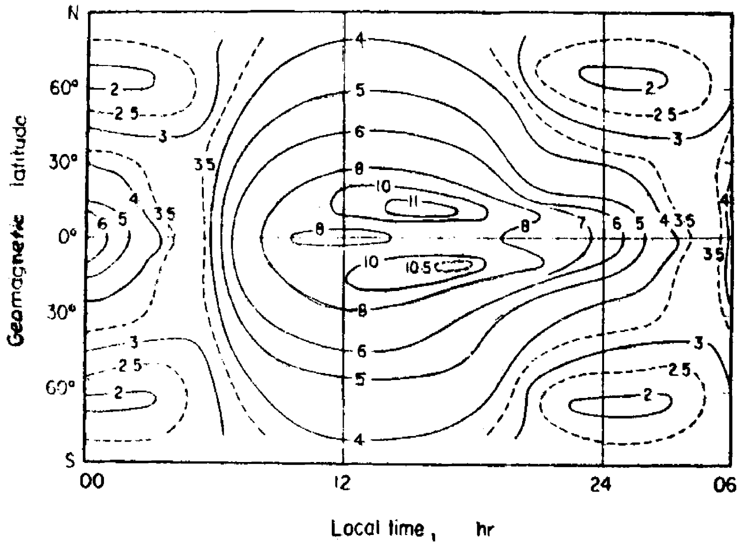


Appleton anomaly

(noon)

Figure 24

Figure 25



World curves of f_0F2 for the equinox, sunspot minimum year 1943–1944 (after MARTYN).

In lecture 5 we saw some curves (resulting from a satellite) of N at fixed height as a function of latitude. These may be found in King et al (1964). There are now many of these results available as printed tables e.g. from the Canadian Defence Telecommunications Laboratory, Ottawa. Ground based results have been given by Croom et al for N at fixed height.

Results. Figures 26 and 27 show results for typical F2 region parameters and drift amplitudes of 7.3 m/s and 73 m/s respectively. Note the development of the Appleton anomaly. These results are only preliminary.

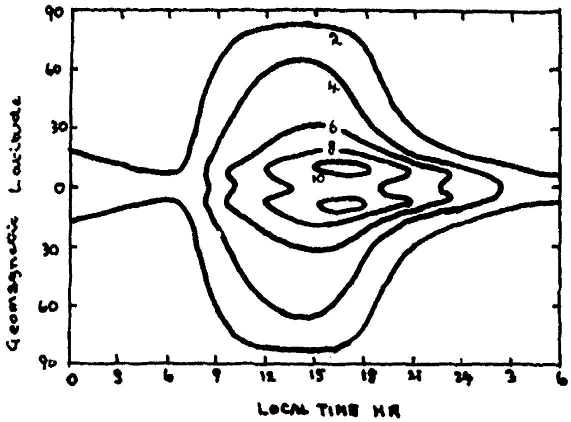


FIG 26 Nmax world curves - Isolines of Nmax $(100/\tau q_0 e)$
drift amplitude = 7.3 m/s

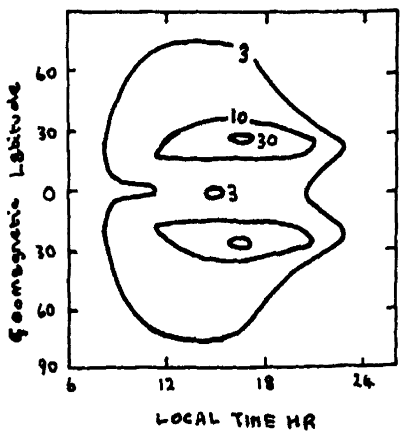


FIG 27 Nmax world curves
- Isolines of Nmax $(100/\tau q_0 e)$
drift amplitude = 73 m/s

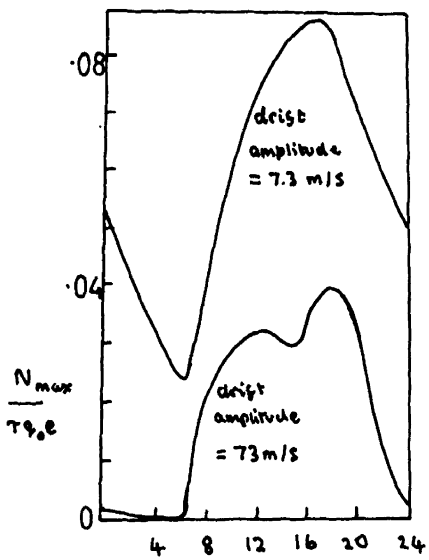


FIG 29
Diurnal variation of Nmax $/\tau q_0 e$
at equater

Clearly 73 m/s is too large, as the trough is too deep. However, the results do show that Martyn (1947) had a good idea as to the real cause of the Appleton anomaly. He proposed that Hall drift would transport ionization away from the equator, thus producing the trough.

Near the dip equator, at stations such as Huancayo, the diurnal variation of N_m has a double maximum, as sketched in Fig. 28

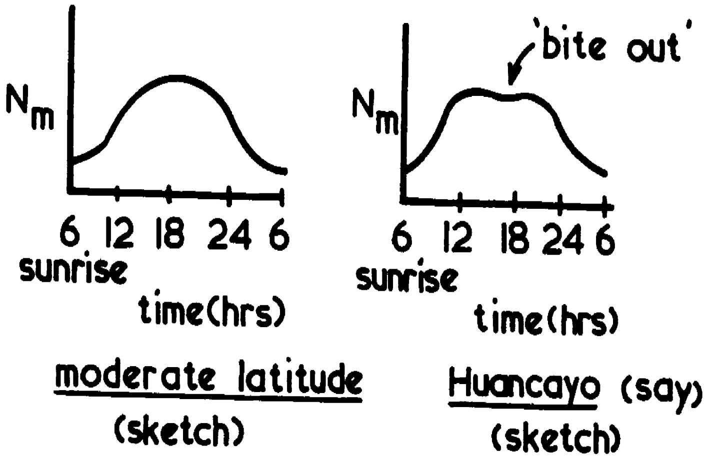


Fig. 28

This is reproduced by our theoretical results for a drift amplitude of 73 m/s as shown in Figure 29. It should be stressed that 73 m/s is too large and that these results are only preliminary.

P. C. Kendall

Lecture 7

Motions of the neutral air induced by ion-drag (Nith Mr. W. M. Pickering) The ion-drag problem Cowling (1945), Hirono (1953), Baker and Martyn (1953), Hirono and Kitamura (1956) and Dougherty (1961) have suggested that the motion of the plasma (in the F2 region) could itself set the neutral air in motion. Consider the case of moderate latitude, with a horizontally stratified atmosphere, as in lecture 2 (see Fig. 30)

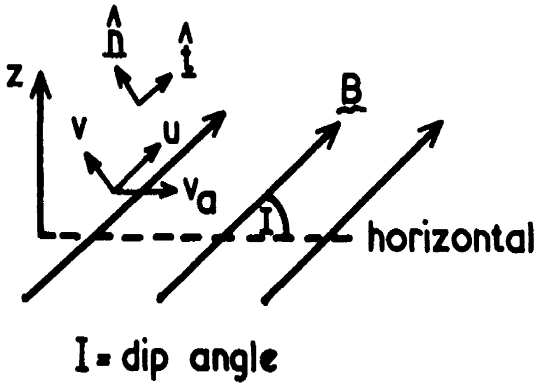


Fig. 30

Let u be the diffusion velocity (along a field line) and v be the specified Hall drift. Then if the neutral air has achieved the same horizontal velocity as the ions,

$$v_a = u \cos I - v \sin I$$

But along a field line the plasma equation of motion is

$$\frac{2NkT}{D}(u - v_a \cos I) = - \sin I \frac{\partial(p_i + p_e)}{\partial z} - Nm_i g \sin I$$

But $p_i = p_e = NkT$ (we assume) and $H = kT/m_i g$. Thus

$$u = -v\cot I - \frac{D}{N\sin I}\left(\frac{\partial N}{\partial z} + \frac{N}{2H}\right)$$

and if $\hat{\underline{t}}$ is the unit tangent along $\underline{B}$ and $\hat{\underline{n}}$ is the unit normal shown, the continuity equation becomes

$$\frac{\partial N}{\partial t} = Q - L - \underline{\nabla}.N(u\hat{\underline{t}} + v\hat{\underline{n}})$$

$$= Q - L - \hat{\underline{t}}.\underline{\nabla}(Nu) - v\,\hat{\underline{n}}.\underline{\nabla}\,N$$

But $\hat{\underline{t}} = (\cos\ I,\ \sin I)$ and $\hat{\underline{n}} = (-\sin I, \cos I)$. Thus

$$\frac{\partial N}{\partial t} = Q - L - \sin I\,\frac{\partial}{\partial z}(Nu) - v\cos I\,\frac{\partial N}{\partial z}$$

$$= Q - L - \frac{\partial}{\partial z}\,D\left(\frac{\partial N}{\partial z} + \frac{N}{2H}\right)$$

Thus as

$$D \propto \exp(z/H)$$

we have

$$\frac{\partial N}{\partial t} = Q - L + D\mathfrak{D}^{*}N,$$

where

$$\mathfrak{D}^{*} \equiv \left(\frac{\partial^2}{\partial z^2} + \frac{3}{2H}\frac{\partial}{\partial z} + \frac{1}{2H^2}\right)$$

Looking back to lecture 2 we there obtained

$$\mathfrak{D} \equiv \sin^2 I\ \ \mathfrak{D}^{*}$$

We also see that the electrodynamic (Hall) drift term v has been completely cancelled . The consequences of including neutral air coupling may thus be serious.

P. C. Kendall

The temperature gradient problem

Apart from the problem of ion-drag there is another problem. King and Kohl (1965) have suggested that as there is a temperature gradient in the upper atmosphere (see their references) it might drive a neutral wind through the pressure gradient. They showed that in the F-region the major force on the neutrals, apart from this pressure gradient, is due to their collisions with ions. It is clearly of interest to investigate the coupling between the plasma and neutral air motions. A given temperature gradient would certainly set the neutral air in motion; however, the velocity would be coupled with the ion velocity; giving rise to a coupled system of equations. If the ions were at rest, the problem of working out the neutral air velocity $\underline{v}_a$ would be relatively simple. The ions are, however, free to move along the field lines. In the F1 layer, for example, diffusion is regarded as unimportant and Geisler (1966) has shown how the problem of neutral air motion can be treated. We here consider the F2 layer where slipping between the moving plasma and the neutral atmosphere is believed to occur. The configuration in Fig. 3 will be used. The magnetic field $\underline{B}$ is taken to be horizontal. The x-axis is also horizontal. The y-axis is vertical. All variables are functions of x and y only, being, in particular, independent of time. The plane x = 0 is taken to be the equatorial plane, and there is supposed to be symmetry about this plane.

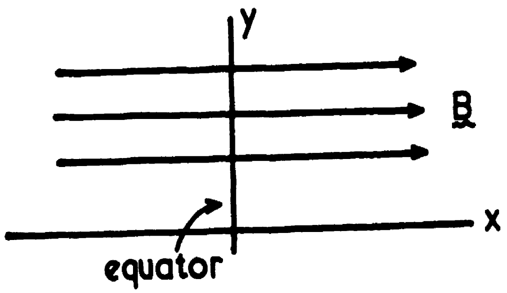


Fig. 31

Notation

n_a = equilibrium neutral air density

$= n_o \exp(-y/H)$

T = equilibrium temperature , assumed uniform

H = scale height $= kT/m_i g$

m_i = mass of ion (= mass of neutral atom)

k = Boltzmann constant

g = acceleration due to gravity

$\underline{v}_a$ = velocity of neutral air

v_{a1} = horizontal velocity of neutral air

v_{a2} = vertical velocity of neutral air

$\underline{v}_i$ = ion velocity

u = ion velocity along a field line

v = Hall drift $\perp^r$ to a field line = constant

T' = artificially maintained small temperature perturbation

D = twice coefficient of diffusion ions through neutrals

n'_a = perturbation in neutral air density

Q = rate of production of electrons

L = rate of loss of electrons

In the model we assume that all temperature gradients, velocities and their effects are small.

We also ignore the self viscosity of the neutral air.

Equations of the problem

The horizontal and vertical linearized equations of motion for the neutral air are

(1) $$\frac{2NkT}{D}(v_{a1} - u) = -\frac{\partial}{\partial x}(n'_a kT) - \frac{\partial}{\partial x}(n_a kT')$$

and

(2) $$\frac{2NkT}{D}(v_{a2} - v) = -\frac{\partial}{\partial y}(n_a' kT) - \frac{\partial}{\partial y}(n_a kT') - m_i n_a' g$$

The linearized horizontal equation of motion of the plasma is

(3) $$\frac{2NkT}{D}(u - v_{a1}) = -\frac{\partial}{\partial x}(2NkT)$$

We note that there is no vertical equation of motion of the plasma, as it moves vertically with the given Hall drift v, assumed constant. Adding (1) and (3) gives

$$\frac{\partial}{\partial x}(n'_a kT + 2NkT + n_a kT') = 0$$

Thus

$$n'_a + 2N + n_a T'/T = F(y) \quad ,$$

where F is an arbitrary function of y. We shall assume that

$$n'_a \to 0 \quad N \to 0 \quad \text{and} \quad T' \to 0 \quad \text{as} \quad x \to \infty \ ,$$

giving

(4) $$n'_a + 2N + n_a T'/T = 0$$

Substituting back we obtain

(5) $$v_{a1} = u + \frac{D}{N}\frac{\partial N}{\partial x}$$

(6) $$v_{a2} = v + \frac{D}{N}\frac{\partial N}{\partial y} + \frac{D}{H} + \frac{D n_a T'}{2HNT}$$

Assuming that $N \ll n_a$, so that production and loss of neutral air molecules are negligible, the linearized continuity equation for the neutral air is

$$\operatorname{div} n_a \underline{v}_a = 0$$

Thus, substituting for $\underline{v}_a$ from (5) and (6),

$$\frac{\partial}{\partial x}(n_a u) + \frac{\partial}{\partial x}\left(\frac{n_a D}{N}\frac{\partial N}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{n_a D}{N}\frac{\partial N}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{n_a D}{H}\right)$$
$$+ \frac{\partial}{\partial y}\left(\frac{D n_a^{*} T'}{2HNT}\right) + \frac{\partial}{\partial y}(n_a v) = 0$$

Whence, under conditions of symmetry ($u = 0$ at $x = 0$), and assuming that

$$D \propto n_a^{-1}$$

We obtain an expression for the horizontal velocity of the ions, namely,

$$u = -D\int_0^x \nabla^2 \log N \, dx + \frac{vx}{H} - D\int_0^x \frac{\partial}{\partial y}\left(\frac{n_a T'}{2HNT}\right) dx \qquad (7)$$

This enables us to form the diffusion equation from the continuity equation for electrons, $Q - L = \operatorname{div}(N\underline{v}_i)$. Thus

$$0 = Q - L + \frac{\partial}{\partial x} ND \int_0^x \nabla^2 \log N \, dx - v\left(\frac{\partial N}{\partial y} + \frac{1}{H}\frac{\partial}{\partial x}(xN)\right)$$
$$+ \frac{\partial}{\partial x} ND \int_0^x \frac{\partial}{\partial y}\left(\frac{n_a T'}{2HNT}\right) dx \qquad (8)$$

Thus, even from a very simple model we have produced a non-linear diffusion equation.

3. Conclusion

It is of interest to compare equation (8) with the the simpler diffusion equation which might be obtained by making earlier approximations. For example, if we had argued that for the purpose of calculating $\underline{v}_a$ the ions might as well be assumed stationary, and the pressure gradient might as well be assumed to be $(\partial/\partial x)(n_a kT')$, we would have obtained

$$(9) \qquad v_{\alpha 1} = -\frac{D}{2N}\frac{\partial}{\partial x}\left(\frac{n_a T'}{T}\right) .$$

Then the diffusion equation would have become

$$(10) \qquad 0 = Q - L + D\frac{\partial^2 N}{\partial x^2} - v\frac{\partial N}{\partial y} + \frac{\partial^2}{\partial x^2}\left(\frac{Dn_a T'}{2T}\right)$$

This equation is linear, and has terms corresponding in turn to each term in (8). It is, nevertheless, incorrect. We note, however, that a temperature which is higher at the equator than alsewhere would give rise to terms of the same sign in (7) and (9), corresponding to transport of plasma away from the equator.

We conclude that although current F2 layer theory looks promising there are still questions to be answered concerning the coupling between the plasma and neutral air motions.

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