CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

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DIFFUSION OF IONS IN A PLASMA WITH APPLICATIONS TO THE IONOSPHERE

Corso tenuto a Varenna dal 19 al 27 settembre 1966

DIFFUSION OF IONS IN A PLASMA WITH APPLICATIONS TO THE IONOSPHERE

by

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I. Derivation of the diffusion equations in plasmas

1. The term 'plasma' was first used by Langmuir for the state of a gas which is fully ionised (for example, the high solar atmosphere) or only partially ionised, (for example, the ionosphere). Our main interest in this course will be the diffusion of ions in such a plasma, arising from non-uniformity of composition, of pressure gradients or electric fields.

We begin by considering the simple case of a fully ionised gas and for simplicity restrict ourselves to the case when only one type of ion and electrons are present.

2. The velocity distribution function

We make the familiar assumption of molecular chaos, in which it is supposed that particles having velocity resolutes lying in a certain range are, at any instant, distributed at random. It is therefore most convenient to use six dimensional space in which the coordinates are the resolutes of the position vector $\underline{r}$ and velocity $\underline{v}$. The state of the plasma can then be specified by the distribution functions $f_\alpha(t, \underline{r}, \underline{v})$, where t is the time, that characterise each particle component α, for example, the ions or the electrons The quantity

$$f_\alpha(t, \underline{r}, \underline{v})\ d\underline{r}\ d\underline{v} \tag{1}$$

then represents the number of particles in the six dimensional volume element $d\underline{r}\ d\underline{v}$. In the simplest case, the plasma consists of single ions (α = i) and electrons (α = e) . In more complicated cases, the plasma may consist of several ion species in addition to neutral particles (α = n) such as atoms, molecules, exited atoms, etc. The total number of particles of constituent α in the element $d\underline{r}$ is obtained by integrating (1) throughout the velocity space. This number is, by hypothesis, $n_\alpha \, d\underline{r}$ and thus $n_\alpha = \int f_\alpha(t, \underline{r}, \underline{v}_\alpha) \, d\underline{v}_\alpha$ (2) . The behaviour of the ionised gas is described by a system of equations (Boltzmann equations) which can be derived as follows. Suppose that each particle of mass m_α is acted on by force $m_\alpha \underline{F}_\alpha$, then in a time dt in which the particles of constituent α suffer no collisions, the same particles that occupy the volume of phase space $d\underline{r} \, d\underline{v}_\alpha$ at time t would occupy the volume of phase $(\underline{r} + \underline{v}_\alpha dt)(\underline{v} + F_\alpha dt)$ at time t + dt . The number in this set is

$$f_\alpha(t + dt, \ r + \underline{v}_\alpha dt, \ v + \underline{F}_\alpha dt)$$

and the difference

$$f_\alpha(t + dt, \underline{r} + \underline{v}_\alpha dt, \underline{v}_\alpha + \underline{F}_\alpha dt) - f_\alpha(t, \underline{r}, \underline{v}_\alpha) \, d\underline{r} \, d\underline{v}_\alpha$$

therefore represent the difference in the gain of particles by collisions to this final set and the loss of the particle to the original set in time dt. This must be proportional to $d\underline{r} \, d\underline{v}_\alpha dt$; and we denote it by $C_\alpha d\underline{r} \, d\underline{v}_\alpha dt$. Taking the limit as $dt \rightarrow 0$, we arrive at Boltzmann's equation for f_α , viz

$$\frac{f_\alpha}{t} + (\underline{v}_\alpha . \nabla) f_\alpha + (\underline{F}_\alpha . \nabla_{\underline{v}_\alpha}) f_\alpha = C_\alpha \tag{3}$$

where $\nabla_{\underline{v}_\alpha}$ stands for the gradient operator $\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, \frac{\partial}{\partial w_\alpha}$ in velocity space.

3. Charge neutrality and the Debye distance

In general a plasma will rapidly attain a state of electrical neutrality; this is because the potential energy of the particle resulting from any space charge would otherwise greatly exceed its thermal energy. Small departures from strict neutrality will occur over small distances whose order of magnitude can be obtained as follows. The electrostatic potential V satisfies Poisson's equation.

$$\nabla^2 V = -4\pi(Zn_i - n_e)e \tag{4}$$

Here Ze is the charge on an ion and $-e$ that of the electrons. In thermodynamic equilibrium , the number densities of the ions and elections respectively are given by

$$n_i = n_i^{(o)} \exp(-ZeV/kT_i), \quad n_e = n_e^{(o)} \exp(eV/kT_e), \tag{5}$$

where k is the Boltzmann constant, T_i, T_e are the ion and electron temperatures and $n_i^{(o)}$ and $n_e^{(o)}$ are the values of n_i and n_e for strict neutrality so that $n_e^{(o)} = Zn_i^{(o)}$. In general, departures from neutrality are small so that we may expand the exponential to the first power of the arguments only. We have approximately

$$Zn_i - n_e \simeq Zn_i^{(o)}\left(1 - \frac{ZeV}{uT_i}\right) - n_e^{(o)}\left(1 + \frac{eV}{uT_e}\right)$$

and hence

$$\nabla^2 V = \frac{V}{D^2} \quad , \tag{6}$$

where

$$D = \left\{\frac{kT_eT_i}{4\pi Ze^2(n_i^{(o)}T_i + n_e^{(o)}Te)}\right\}^{1/2} \tag{7}$$

The quantity D has the dimensions of a length and is called the Debye distance. The solution of (6) for spherical symmetry is

$$V = \frac{e_\alpha}{r} \exp\left(-\frac{r}{D}\right) \tag{8}$$

where e_α is the charge on the particle. For small distances r from the origin ($r << D$), (8) reduces to the pure Coulomb potential of the charged particle. For large distances($r >> D$), $V \rightarrow 0$ exponentially. Thus in a neutral plasma in thermodynamical equilibrium the Coulomb field of the individual charge is cut off (shielded) at a distance of order D. Hence, we may assume that the particles do not interact in collisions for which the impact parameter is greater than D. The Debye shielding is not established instantaneously; oscillations of the space charge will have a frequency $\omega_o = \left(4\pi n_e^2/m_e\right)^{\frac{1}{2}}$(since the displacemente of the electrons (or ions) bodily by a distance x gives rise to an electric field of intensity $4\pi n_e ex$ lending to restore neutrality). Thus the time required to establish shielding is of the order

$$\tau \sim \frac{1}{\omega_o}$$

4. Diffusion of test particles in a plasma

A particular particle, which we call 'test particle', in a plasma will suffer collisions with the other particles in the plasma, which we call 'field particles'. Electrostatic forces between the particles have a greater range than the forces between neutral molecules in an ordinary gas. Consequently, the cumulative effect of distant encounters will be far more important than the effect of close collisions, which change completely the particle velocities. We shall therefore suppose that the deflections which the test particles undergo are mostly small. The motion of the test particle is most conveniently descibed in the velocity space, i.e., a space in which the velocity vector $\underline{v}$ is taken as the position vector and the apex of this vector is called the velocity point of the particle. Referred to Cartesian coordinates the coordinates of these points will be denoted by v_x, v_y, v_z .

As the test particle changes its position in ordinary space, its position in velocity space changes either continuously or discontinuously due to encounter with fixed particles. In general the displacement is complicated.(Fig. 1)

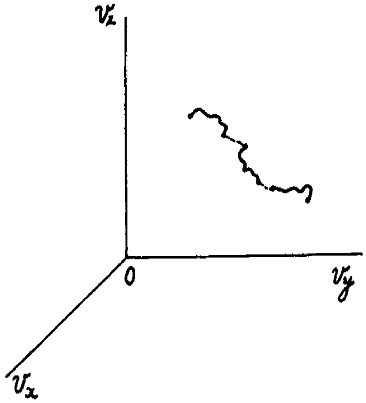

FIG. 1.

It is clearly impossible, and indeed futile, to trace the motion of a single particle and we are forced to consider a statistical description of the motion. In this, instead of a single particle, we consider an assembly containing a large number of test particles which have the same velocity $\underline{v}_o$ initially.

Suppose these are concentrated around the point $\underline{v}_o$ in the velocity space . At subsequent times the cloud will spread, changing both its size and shape, as a result of successive encounters.

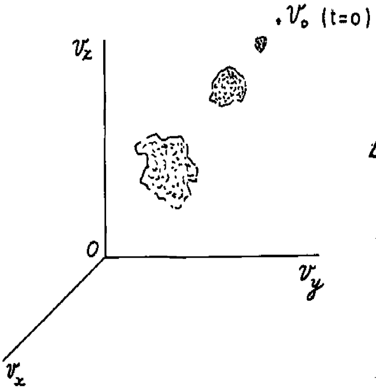

FIG. 2.

We now require to find quantities which will adequately describe the process. One such quantity is the change in velocity $\Delta \underline{v}$ of a test particle produced by the encounters. Suppose that $\underline{v}_o$ is parallel to the z-axis and consider the resolutes $\Delta v_x, \Delta v_y, \Delta v_z$ of $\Delta \underline{v}$. Suppose that $(\Delta v_x)_i$ is the change in Δv_x produced by the ith encounter. Then after N encounters,

$$\Delta v_x = \sum_{i=1}^{N} (\Delta v_x)_i$$

We assume that all the encounters are random, but as we have already seen, we cannot predict the change Δv_x for a single test particles. However, we can define an average value of Δv_x, say $\overline{\Delta v_x}$ for the large assembly of particles under consideration. If the distribution of velocities is isotropic, then $\overline{\Delta v_x} \equiv 0$, by symmetry, and likewise $\overline{\Delta v_y} \equiv 0$. But $\overline{\Delta v_z}$ need not vanish since the assembly (or cloud) has an initial velocity in the z-direction. However the mean square of Δv_x^2 will not vanish. This mean value will contain terms of the form $\overline{(\Delta v_x)_i^2}$ and $\overline{(\Delta v_x)_i(\Delta v_x)_j}$: If the collisions are small we may expect that successive collisions will produce, on the average, the same average change as the first collisions. Thus the N terms $\overline{(\Delta v_x)_i^2}$ are all equal. But the mixed products $\overline{(\Delta v_x)_i(\Delta v_x)_j}$ will vanish when averaged over all particles considered since successive collisions are uncorrelated. Hence

(9) $$\overline{\Delta v_x^{\,2}} = N\overline{(\Delta v_x)_i^2}$$

The dispersion of the points in Fig. 2 will therefore increase like $\sqrt{N}$, but not, in general, equally in all directions. But the centre of gravity may be displaced by an amount proportional to N. (Fig. 2)

The dispersion of the points in the velocity space produced by collisions of the test particles with the field particles is analogous to the diffusion of particles in an ordinary gas. To measure the rate of diffusion in the v_x direction, we consider the average value of (9) per unit time. The resultant value of $\overline{\Delta v_x^{\,2}}$, measuring the increase of velocity of dispersion of a group of particles per second, will be denoted by $\langle \Delta v_x^2 \rangle$ and called a 'diffusion coefficient', a term due to Spitzer . If the velocity distribution of the field particles is isotropic, the diffusion coefficients $\langle \Delta v_x \rangle$ and $\langle \Delta v_x \Delta v_y \rangle$ vanish identically,

The encounters between test and field particles which we are considering are assumed to be binary encounters only .(*)
Let $\underline{v}$ be the velocity of a field particle relative to a test particle. Then there will be only three independent diffusion coefficients, namely, $\langle \Delta v_\parallel \rangle$, $\langle \Delta v_\parallel^2 \rangle$ and $\langle \Delta v_\perp^2 \rangle$, where $v_\parallel$ and $v_\perp$ are measured respectively parallel and perpendicular to $\underline{v}$. Their values will depend on the velocity distribution function of the field particles.

(*) The justification for this will be given in Section 7.

5. Binary encounter of two charged particles

(Hyperbolic orbit)

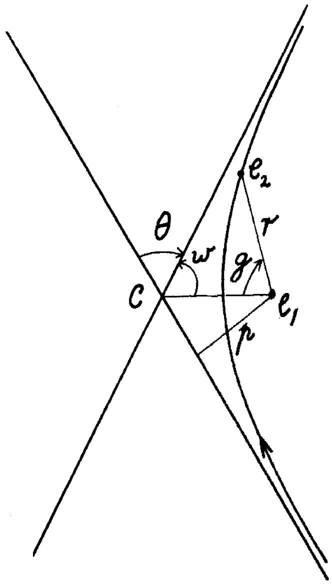

FIG. 3.

Consider the motion of charge e_2 relative to charge e_1 ; let $\underline{r}_1$ and $\underline{r}_2$ be the position vectors of e_1 and e_2 relative to a Newtonian origin. Then the equation of motion of the charges are respectively

$$m_1\ddot{\underline{r}}_1 = + \frac{e_1 e_2 \underline{r}}{r^3}, \quad m_2\ddot{\underline{r}}_2 = - \frac{e_1 e_2 \underline{r}}{r^3}$$

where $\underline{r} = \underline{r}_2 - \underline{r}_1$ and m_1 and m_2 are the masses of the charges. Hence

$$\ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = -e_1 e_2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \frac{\underline{r}}{r^3}$$

that is, the relative motion is the same as that of a particle under a central force at A varying inversely as the square of the distance whose strength is $\frac{e_1 e_2}{m_{12}}$, where

$m_{12} = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. (Fig. 3).

Let v_∞ be the relative velocity of the charges at infinity and p the impact parameter. The energy integral is, with the usual notation,

$$v^2 = \frac{e_1 e_2}{m_{12}} \left(\frac{2}{r} + \frac{1}{a}\right)$$

whence

(10) $$v_\infty^2 = \frac{e_1 e_2}{m_{12} a}$$

The polar equation of the orbit is

(11) $$r = \frac{\ell}{1 + e\cos\varphi}$$

where ℓ is the semi-latus rectum and e the eccentricity. As $r \to \infty$, $\varphi \to \pi - w$ so that (11) gives

$$\cos w = \frac{1}{e}$$

Also $AC = ae$; hence

$$\sin w = \frac{p}{ae}$$

Thus $1 = \sin^2 w + \cos^2 w = \frac{1}{e^2} + \frac{p^2}{a^2 e^2}$ or $e^2 = 1 + \frac{p^2}{a^2}$ giving

$$\cos w = \frac{1}{\sqrt{1 + \frac{p^2}{a^2}}}, \quad \sin w = \frac{p}{a\sqrt{1 + \frac{p^2}{a^2}}}, \quad \tan w = \frac{p}{a}$$

or using (1)

(12) $$\tan w = \frac{p v_\infty^2 m_{12}}{e_1 e_2}$$

6. Calculation of diffusion coefficients

Consider the scattering of test particles (α) by a flux of field particles (β) . The spatial density of the latter is

$$dn_\beta = f_\beta(\underline{v}')\, d\underline{v}'$$

where $\underline{v}'$ is the velocity of the particles and f_β the distribution function of the field particles. Consider the collision of a test particle α with a field particle β of this flux. Then the velocity $\underline{v}_\alpha$ of the test particle is related to the velocity $\underline{v}_g$ of the centre of mass of the two particles and their relative velocity $\underline{u}$ by

$$\underline{v}_\alpha = \underline{v}_g + \frac{m_\beta}{m_\alpha + m_\beta}\,\underline{u}\ ;$$

hence, since $\underline{v}_g$ is unaltered by the encounter,

$$\Delta\underline{v}_\alpha = \frac{m_{\alpha\beta}}{m_\alpha}\Delta\underline{u} \tag{13}$$

where $m_{\alpha\beta}$ is the reduced mass of m_α and m_β, and $\Delta\underline{u}$ the change in $\underline{u}$ produced by the encounter.

Also, in taking the average of the change of velocity over the test particles in the assembly, the summation reduces to a summation over all particles of the flux incident on a <u>fixed</u> scattering centre. The number of particles moving through an area $dA = p\,dp\,d\varphi$ of a plane π perpendicular to $\underline{u}$ in unit time is

$$dn_\beta \left|\underline{u}\right| dA = f_\beta(\underline{v}')\,d\underline{v}'\,u\,dA \tag{14}$$

Multiply this by the components of the vector $\Delta\underline{v}_\alpha$ given by (13) and integrate over all the plane π and then over the velocities of the field particles we find

$$<\Delta v_k> = \int f_\beta(\underline{v}')\,w_k\,d\underline{v}' \quad (k = x, y, z) \tag{15}$$

where

$$w_k = \frac{m_{\alpha\beta}}{m_\alpha}\int_{\text{plane}} \Delta u_k\ u\,dA\ ,$$

$$<\Delta v_k\,\Delta v_\ell> = \int f_\beta(\underline{v}')\,w_{k\ell}\,d\underline{v}'\ , \quad (k, l = x, y, z) \tag{16}$$

where

$$w_{k\ell} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \Delta u_k\,\Delta u_\ell\ u\,dA$$

(15) and (16) are the diffusion coefficients. It will be convenient to compute these integrals relative to a coordinate system in which the z-axis

is along $\underline{u}$.

For referring to Fig. a, and Fig. 4b .

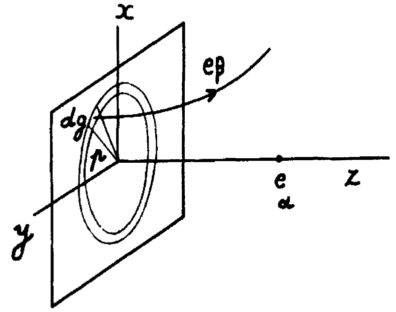

FIG. 4 a

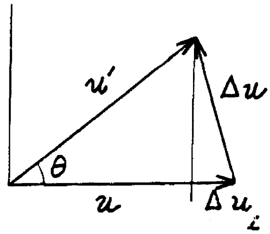

FIG. 4b

we have

$$\Delta u_x = u \sin\theta \cos\varphi$$
$$\Delta u_y = u \sin\theta \sin\varphi$$
$$\Delta u_z = -u(1 - \cos\theta)$$

Also from (12) and the fact that $\theta = \pi - 2w$ we have

(17) $$\tan\frac{\theta}{2} = \frac{p_\perp}{p}$$

where
$$p_\perp = \frac{e_\alpha e_\beta}{m_{\alpha\beta} u^2}$$

Then $$\Delta u_x = 2u \frac{p\, p_\perp}{p^2 + p_\perp^2} \cos\varphi \;, \quad \Delta u_y = 2\, u \frac{p\, p_\perp}{p^2 + p_\perp^2} \sin\varphi$$
$$\Delta u_z = -2u \frac{p_\perp^2}{p^2 + p_\perp^2}$$

Integration with respect to p and φ over the plane gives

$$w_x = 0 = w_y$$

whilst

$$w_z = \frac{m_{\alpha\beta}}{m_\alpha} \int_{\text{plane}} \Delta u_z \, u \, d\, A .$$

If the limits of integration for p are 0 and ∞, the integral diverges; however, we have already seen that the Coulomb field of individual charges is cut off at distance of order D, the Debye distance. Hence we can take D as the upper limit for p in the integral. We then find

$$\mathcal{W}_z = - \frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \int_0^{\mathcal{D}} \frac{p\,dp}{p^2+p_\perp^2}$$

(18)

$$= -\lambda \frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2$$

where

(19)
$$\lambda = \int_0^{\mathcal{D}} \frac{p\,dp}{p^2+p_\perp^2} = \log_e \frac{(D^2+p_\perp^2)^{\frac{1}{2}}}{p_\perp}$$

It has been tacitly assumed that $D \gg p_\perp$; to illustrate that this is likely to be the case in general, let us take $T_i = T_e =$ $= 1$ ke V (10^{7o} K) , $n_i = n_e = 10^5$ cm^{-3}, and $Z = 1$, then

$$D = \left(\frac{kT}{8\pi n e^2}\right)^{\frac{1}{2}} \sim \frac{1}{2} \times 10^{-3} \text{ cm}$$

and

$$p_\perp = \frac{e^2}{3kT} \sim \frac{1}{2} \times 10^{-10} \text{ cm}$$

so that $D/p_\perp \sim 10^7$. Hence in (19) we may neglect $p_\perp$ compared with D in the numerator and write

(19a)
$$\lambda \sim \log_e \frac{D}{p_\perp} = \log_e \left\{ \frac{3}{2e^3}\left(\frac{k^3T^3}{2\pi n}\right)^{\frac{1}{2}} \right\}$$

Likewise

$$w_{xx} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int (\Delta u_x)^2\, u\, dA = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \left(2u \frac{p\, p_\perp}{p^2+p_\perp^2} \cos\varphi\right)^2 updpd\varphi$$

$$= \frac{1}{4\pi u}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \int_0^D \frac{p^3\, dp}{(p^2+p_\perp^2)^2}$$

$$\simeq (\lambda - \tfrac{1}{2})\, \frac{1}{4\pi u}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \quad \text{neglecting terms of order} \left(\frac{p_\perp}{D}\right)^2$$

Again $w_{yy} = w_{xx}$ and $w_{\alpha\beta} = 0$, $\alpha \neq \beta$: Finally

$$w_{zz} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int (\Delta u_z)^2\, u\, dA = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \left(-2u\frac{p_\perp^2}{p^2+p_\perp^2}\right)^2 updp\, d\varphi$$

and the integration with respect to p can, in fact, be carried out from 0 to ∞ since the integral is finite ;
we find

$$w_{zz} = 4\pi\left(\frac{e_\alpha e_\beta}{m_\alpha u}\right)^2$$

Since this is λ times smaller than w_{xx} or w_{yy} it may be set to zero . We can now express w_k and $w_{k\ell}$ as a vector or tensor respectively. In fact,

$$w_k = -\lambda \frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \frac{u_k}{u} \tag{20}$$

$$w_{k\ell} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} - \begin{pmatrix} 0, & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}$$

$$= A\left(\delta_{k\ell} - \frac{u_k u_\ell}{u^2}\right) \tag{21}$$

where

(22) $$A = (\lambda - \frac{1}{2}) \frac{1}{4\pi u} \left(\frac{4\pi e_\alpha e_\beta}{m_\alpha} \right)^2$$

Finally

(23) $$< \Delta v_k > = -(1 + \frac{m_\alpha}{m_\beta}) Q_{\alpha\beta} \int \frac{u_k}{u^3} f_\beta(\underline{v}') \, d\underline{v}'$$

(24) $$< \Delta v_k \Delta v_\ell > = \frac{Q_{\alpha\beta}}{4\pi} \int \left(\frac{\delta_{k\ell}}{u} - \frac{u_k u_\ell}{u^3} \right) f_\beta(\underline{v}') \, d\underline{v}'$$

where $\underline{u} = \underline{v} - \underline{v}'$

(25) $$Q_{\alpha\beta} = \lambda \left(\frac{4\pi e_\alpha e_\beta}{m_\alpha} \right)^2 .$$

It can be shown that the third and higher diffusion coefficients $< \Delta v_k \Delta v_\ell \Delta v_m .. >$ are smaller than the first two diffusion coeffients by a factor of λ. This means that the motion of Coulomb particles can be visualized as a diffusion in velocity space. The approximation in which only the first two diffusion coefficients are considered is called the Fokker- Planck approximation.

7. Justification of the assumption of binary encounters in the theory.

The assumption is certainly justified for short-range force. If the interaction range d(effective diameter of the molecule) is much smaller thatn the mean distance between the particles, $n^{-1/3}$, where n is the density of the gas, the sphere of action, of volume $\sim d^3$, will contain only a small number of particles N_d, that is

$$N_d = nd^3 \ll 1.$$

Under these conditions the probability of multiple collisions, involving three or more particles simultaneously, is very small. A description in terms of binary collisions is adequate.

Coulomb forces acting between particles of a plasma are not short-range forces. The potential energy between two such charges e_1 and e_2 is

$$\frac{e_1 e_2}{r} \exp\left(-\frac{r}{D}\right) \tag{26}$$

where r is the distance apart of the charges. Thus the interaction between them extends at least as far as the Debye distance D, and for conditions in which we are interested $D \gg n^{-1/3}$ and the sphere of action contains many particles, i.e.,

$$N_D = nD^3 \gg 1 \tag{27}$$

In this case a given particle will interact simultaneously with many particles and the results derived earlier on the basis of binary collisions is suspect. A rigorous analysis shows that the formulae derived yield logarithmic accuracy. However, a non-rigorous, but plausible discussion can be given along the following lines.

Let us consider a test particle moving through the plasma and suppose that it is so massive that its velocity can be treated as constant. Draw a cylinder of radius p with the trajectory as axis (Fig. 5)

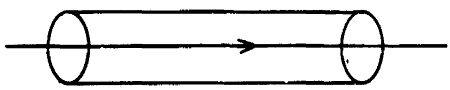

FIG. 5.

Collisions of the test particles with field particles for which $p >> n^{-1/3}$ will be many-body collisons. Those characterized by immact parameters $p << n^{-1/3}$ are binary collisions . We shall show that the method used to treat binary collisions need not be restricted to collisions with impact parameters $p << n^{-1/3}$, but can be extended to parameters $>> n^{-1/3}$.

Now, when $r << D$, the potential energy between the charges is simply $e_1 e_2/r$ so that the presence of other particles has no effect on the interaction between two particles separated by a distance smaller than D. Thus results derived on the hypothesis of binary collisions apply for all impact parameters smaller than the Debye radius, i.e., $p << D$. Because $D >> n^{-1/3}$, in the present case the collisions can be regarded as binary interactions even when $p >> n^{-1/3}$ as long as $p << D$. Accordingly, even if $p \sim D$, the difference between the exact interaction formulae which takes account of other particles, and a pure Coulomb interaction, is small (by a factor of order 1) . Thus, cutting off the Coulomb interaction for the impact parameter $p = D$ provides an approximate method of taking into account the effect of multiple collisions for which $p >> n^{-1/3}$.

8. Diffusion in velocity space.

From a microscopic point of view , the change of spatial coordinates of a particle during a collision can always be neglected . Hence, as far as the spatial part of the phase space is concerned, the motion of a particle corresponds to a continuous point to point variation.

On the other hand, collisions have a marked effect on the continuity of motion in the velocity space. The velocity can be changed abruptly by a single near collision, essential in a vanishingly small time interval. Hence, a particular velocity point $\underline{v}$ in a cloud of particles in velocity space can be 'annihilated' by a collision and 'recreated' at some remote point without passing through intermediate points in the velocity space. Thus in general the effect of collisions cannot be expressed in the kinetic theory by introducing a term describing the divergence of flux in velocity space. But this will certainly only be the case for near collisions in which the velocity of the particle is changed abruptly. In the case of coulomb forces, the change in velocity, characterised by the quantities $\langle \Delta v_k \rangle$ and $\langle \Delta v_k \Delta v_\ell \rangle$ is due to the

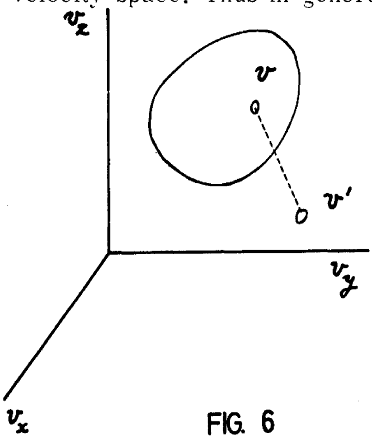

FIG. 6

effect of remote interactions and the changes in velocity are small. For example, if $\lambda = 15$, then the relative change in particle velocity

$$\frac{|\Delta \underline{v}|}{v} = \frac{p_\perp}{p} = e^{-\lambda} \frac{D}{p} \sim 10^{-6} \left(\frac{D}{p}\right),$$

and so very small.

If these interactions are referred in velocity space, the whole process may be regarded as a form of diffusion. The motion can be regarded as nearly continuous.

9. Calculation of the diffusion coefficient for a Maxwellian distribution of velocities.

The expressions (23) and (24) may be expressed more conveniently by introducing the super-potentials. In fact, since

$$u = \sqrt{(\underline{v}_k - \underline{v}'_k)(\underline{v}_k - \underline{v}'_k)}$$

we have

$$\frac{\partial}{\partial v_k}\frac{1}{u} = -\frac{u_k}{u^3}$$

$$\frac{\partial^2}{\partial v_k \partial v_\ell} u = \frac{\partial}{\partial v_k}\frac{1}{2}\frac{2}{u}(v_\ell - v'_\ell) = \frac{\partial}{\partial v_k}\frac{v_\ell - v'_\ell}{u} = \frac{\partial}{\partial v_k}\frac{u_\ell}{u}$$

$$= -\frac{u_k u_\ell}{u^3} + \delta_{k\ell}\frac{1}{u}$$

Hence (23) and (24) can be written

(28) $$\langle \Delta v_k \rangle = -\left(1 + \frac{m_\alpha}{m_\beta}\right) Q_{\alpha\beta} \frac{\partial \varphi_\beta}{\partial v_k}$$

(29) $$\langle \Delta v_k \Delta v_\ell \rangle = -2 Q_{\alpha\beta} \frac{\partial^2 \psi_\beta}{\partial v_k \partial v_\ell}$$

where

(30) $$\varphi_\beta = -\frac{1}{4\pi}\int \frac{f_\beta(\underline{v}')\, d\underline{v}'}{|\underline{v} - \underline{v}'|}$$

(31) $$\psi_\beta = -\frac{1}{8\pi}\int |\underline{v} - v'| f_\beta(\underline{v}')\, d\underline{v}'$$

which have been termed 'super-potentials' by Rosenbluth et al. For a Maxwellian distribution function

$$f(\underline{v}') = n\left(\frac{m}{2\pi n T}\right)^{3/2} e^{-\frac{m v'^2}{2kT}}$$

Chandrasekhar found that

(32) $$\langle \Delta v_{\parallel} \rangle = -\frac{1}{2} n_\beta Q_{\alpha\beta}\left(1 + \frac{m_\alpha}{m_\beta}\right) G\left(\frac{m_\beta v}{2kT_\beta}\right)$$

$$\langle \Delta v_{\perp}^2 \rangle = \langle \Delta v_{xx}^2 \rangle + \langle \Delta v_{yy}^2 \rangle$$

(33) $$= \frac{1}{2v} n_\beta \, Q_{\alpha\beta} \left\{ \Phi\left(\frac{m_\beta v}{2kT_\beta}\right) - G\left(\frac{m_\beta v}{2kT_\beta}\right) \right.$$

(34) where $$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \, d\xi$$

is the usual error function and

(35) $$G(x) = \frac{\Phi(x) - x\Phi'(x)}{2x^2}$$

Values of G and $\Phi - G$ are given by Spitzer and others.

10. Relaxion times. (Collision interval)

The term "relaxation time" is used to denote the time in which collisions will alter the original velocity distribution; or again , the time that the ions and electrons in a gas will attain, through collisions, a Maxwellian distribution.

Various relaxation times can be defined ; the time between collisions (collision interval or the reciprocal of the collision frequency) may be defined as the time in which small deflections will deflect test particles through 90° . More precisely, if τ_D is the 'deflection time', we have

(36) $$\langle \Delta v_{\perp}^2 \rangle \tau_D = v^2$$

Substituting from (33) we find

(37) $$\tau_D = \frac{2 v^3}{n_\beta Q_{\alpha\beta} (\Phi_\beta - G_\beta)}$$

An energy exchange time τ_E can likewise be defined by the relation

(38) $$<\Delta E^2> \tau_E = E^2 \,;$$

the change of energy

(39) $$\Delta E = \frac{1}{2} m(2v\Delta v_{11} + \Delta v^2_{11} + \Delta v_\perp^2)$$

If only dominant terms are required

$$<\Delta E^2> = m^2 v^2 <\Delta v^2_{11}>$$

and (38) gives

(40) $$\tau_E = \frac{v^3}{4 n_\beta Q_{\alpha\beta} G_\beta}$$

An important special case is that of a group of ions, or a group of electrons, interacting amongst themselves. If we consider such a group whose velocity has the root mean square value for the group, then $\left(\frac{mv^2}{2nT}\right)^{\frac{1}{2}} = 1.225.$

In this case we find that $\tau_D / \tau_E = 1{,}14$ so that $\tau_D \sim \tau_E$ and is a measure of both the time required to reduce substantially any anisotropy in the velocity distribution function and the time for the kinetic energies to approach a Maxwellian distribution. We shall call this particular value of τ_D the 'self-collision interval' for a group of particles and will be denoted by τ_c From (37) we have

(41) $$\tau_c = \frac{m^{\frac{1}{2}} (3\, k\, T)^{3/2}}{5.71\, \pi n e^4 Z^4 \log_e \lambda}$$

where T is in degrees K, m is the mass of a typical particle of the group. It may be written Am_H where m_H is the mass of a proton . For electrons, $A = \frac{1}{1825}$ so that the self-collision time for electrons is $\frac{1}{43}$ that for protons, provided the ions and electrons have the same temperatures.

We consider next the approach to equilibrium of a two component plasma; to fix our ideas we consider the case when the constituents are ions and electrons. There are three stages involved in the process. First, collisions between ions and electrons lead to an isotropic velocity distribution of electrons, and the same time collisions between electrons themselves establishes a Maxwellian distribution. Secondly, collisions between the ions themselves establishes an isotropic velocity distribution amongst the ions. Thirdly, the ions and electrons which have already attained Maxwellian distribution, but possibly at different temperatures T_i and T_e, will be brought to the same temperature by collisions between the ions and electrons.

To consider the last process we require the equation of energy ($\mathcal{E}_\alpha = \frac{1}{2} m_\alpha v^2$)

$$\frac{d\mathcal{E}_\alpha}{dt} = \frac{1}{2} m_\alpha \frac{d}{dt} \overline{v_i v_i} = m_\alpha \left(\frac{1}{2} \langle \Delta v_i \Delta v_i \rangle + v_i \langle \Delta v_i \rangle\right) \tag{42}$$

using

$$\overline{\Delta v_i \Delta v_i} = \overline{v_i v_i} - \bar{v}_i \bar{v}_i \tag{43}$$

Using (28) and (29) we find

$$\frac{d\mathcal{E}_\alpha}{dt} = - m_\alpha Q_{\alpha\beta} \left[\varphi_\beta + \left(1 + \frac{m_\alpha}{m_\beta}\right) \underline{v} \cdot \nabla \varphi_\beta \right] \tag{44}$$

since

$$\langle \Delta v_i \Delta v_i \rangle = - 2Q_{\alpha\beta} \nabla^2 \psi_\beta = - 2Q_{\alpha\beta} \varphi_\beta \tag{45}$$

Since the distribution of velocities are Maxwellian, this may be rewritten

(46) $$\frac{d\mathcal{E}_\alpha}{dt} = - \frac{2\mathcal{E}_\alpha}{\tau_{\alpha\beta}(\mathcal{E}_\alpha)}\left\{ \frac{m_\alpha}{m_\beta}\mu(x_\beta) - \mu'(x_\beta)\right\}$$

where $x_\beta = \frac{1}{2}\ \frac{m_\beta v^2}{kT_\beta} = \frac{m_\beta}{m_\alpha}\ \frac{\mathcal{E}_\alpha}{kT_\beta}$, and $\mu(x) = \Phi(x) - x\,\Phi'(x)$

Also

(47) $$\tau_{\alpha\beta}(\mathcal{E}_\alpha) = \frac{4\pi v^3}{n_\beta Q_{\alpha\beta}} = \frac{(\frac{1}{2} m_\alpha)^{\frac{1}{2}}\ \mathcal{E}_\alpha^{3/2}}{\pi e_\alpha^2\, e_\beta^2\, n_\beta \log\lambda}$$

where $\mathcal{E}_\alpha = k(T_\alpha + \frac{m_\alpha}{m_\beta} T_\beta)$. After some algebra, (46) can be reduced to

(48) $$\frac{dT_\alpha}{dt} = \frac{T_\beta - T_\alpha}{\tau^*_{\alpha\beta}}$$

where

(49) $$\tau^*_{\alpha\beta} = \frac{3\, m_\alpha\, m_\beta\, k^{3/2}}{8(2\pi)^{\frac{1}{2}}\, n_\beta e_\alpha^2\ e_\beta^2 \log\lambda}\left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta}\right)^{3/2}$$

It is easily verified that

$$\tau^*_{ee} : \tau^*_{ii} : \tau^*_{ei} : \tau^*_{ie} = 1 : \sqrt{\frac{M}{m}} : \frac{M}{m} : \frac{M}{m}$$

where $T_\alpha \sim T_\beta$ and where M is the mass of the ion and m the electronic mass.

Equation (48) was first given by Spitzer ; it shows that if the mean square relative velocity, which is $\propto \left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta}\right)$,does not change appreciably, , $\tau^*_{\alpha\beta}$ is nearly constant and departure from equipartitions decrease exponentially.

11. Relaxation towards the steady state

The solution of Boltzmann's equation for non-uniform gases is found by successive approximation. We write

$$f = f_o(1 + \mathcal{E}),$$

where f_o is the Maxwellian distribution function and $\mathcal{E}$ is small compared with unity . We have seen that in a plasma of two constituents each constituent will approach its Maxwellian distribution in a time equal to the relaxation time $\tau^*_{\alpha\alpha}$ and the two constituents will attain equal temperatures in a relaxation time $\tau^*_{\alpha\beta}$. As a first approximation , therefore, we can take the collision term C_α to be of the form

$$-\frac{f - f_o}{\tau^*} \tag{50}$$

so that if f is the distribution function at time $t = 0$ and f_o the Maxwellian distribution function, then departure from a Maxwellian state $f - f_o \rightarrow 0$ with time as $e^{-t/\tau}$.

12. Equations of continuity and motion for a fully ionized gas

We consider the plasma to be a mixture of positive ions (i) and electrons (e) and denote their number densities by n_i and n_e , and their velocities by $\underline{v}_i$ and $\underline{v}_e$ respectively. Then

$$n_i = \int f_i \, d\underline{v}_i \, , \qquad n_e = \int f_e \, d\underline{v}_e \tag{51}$$

where f_i and f_e denote the velocity distribution functions for the ions and electrons respectively and $d\underline{v}_i$ and $d\underline{v}_e$ denote an element of volume in the velocity space for ions and electrons, respectively. Denoting their masses by m_i and m_e and the densities of the ion and electron gas by ρ_i and ρ_e respectively, we have

$$\rho_i = n_i m_i \, , \qquad \rho_e = n_e m_e \tag{52}$$

Denote by $\bar{\underline{v}}_i$ and $\bar{\underline{v}}_e$ the mean velocities of the ion and electron gas in a volume element of the plasma, then

(53) $$n_i \bar{\underline{v}}_i = \int \underline{v}_i f_i \, d\underline{v}_i, \qquad n_e \bar{\underline{v}}_e = \int \underline{v}_e f_e \, d\underline{v}_e$$

It is convenient to introduce the total number density n_o and total mass density ρ_o, defined as $n_o = n_i + n_e$

(54) $$\rho_o = \rho_i + \rho_e$$

and the mean velocity $\underline{v}_o$ of the plasma element defined by

(55) $$\rho_o \underline{v}_o = \rho_i \bar{\underline{v}}_i + \rho_e \bar{\underline{v}}_e$$

Let $\underline{V}_i$ and $\underline{V}_e$ be the peculiar or thermal velocities of the ions and electrons, respectively, defined by

(56) $$\underline{V}_i = \underline{v}_i - \underline{v}_o, \qquad \underline{V}_e = \underline{v}_e - \underline{v}_o$$

Then it follows from (55) that

(57) $$\rho_i \bar{\underline{V}}_i + \rho_e \bar{\underline{V}}_e = 0$$

The partial pressure for the ion and electron gases, and total pressures defined in a frame of reference moving with the mean velocity $\underline{v}_o$ are respectively given by

(58) $$p_i = \rho_i \overline{\underline{V}_i \underline{V}_i}, \qquad p_e = \rho_e \overline{\underline{V}_e V_e}, \qquad p_o = p_i + p_e$$

The hydrostatic partial pressures for ions and electrons are defined by

(59) $$p_i = \frac{1}{3} \rho_i \overline{V_i^2}, \qquad p_e = \frac{1}{3} \rho_e \overline{V_e^2}$$

and the corresponding mean kinetic temperature by

(60) $$p_i = k n_i T_i, \qquad p_e = k n_e T_e$$

Boltzmann's equation for the two distribution functions f_i and f_e are

(61) $$\frac{\partial f_\alpha}{\partial t} + (\underline{v}_\alpha . \nabla) f_\alpha + (\underline{F}_\alpha . \nabla_{\underline{v}_\alpha}) f_\alpha = C_\alpha, \alpha = i, e$$

where $m_i \underline{F}_i$ and $m_e \underline{F}_e$ are the forces acting on an ion and electron respectively. If these are produced by an electric field $\underline{E}$ and magnetic field $\underline{B}$, then

(62) $$\underline{F}_i = \frac{e_i}{m_i} (\underline{E} + \underline{v}_i \times \underline{B}), \quad \underline{F}_e = \frac{e_e}{m_e} (\underline{E} + \underline{v}_e \times \underline{B})$$

where e_i and e_e are the charges carried by an ion and electron respectively.

We next form the moment equations; if $\varphi_\alpha(\underline{v}_\alpha)$ be any function of molecular properties for the constituent α of the plasma, then by multiplying equation (61) by φ_α, integrating partially and remembering that

(63) $$n_\alpha \bar{\varphi}_\alpha = \int \varphi_\alpha f_\alpha d\underline{v}_\alpha$$

we find

(64) $$\frac{\partial (n_\alpha \bar{\varphi}_\alpha)}{\partial t} + \nabla \cdot (n_\alpha \overline{\varphi_\alpha \underline{v}_\alpha}) - n_\alpha \underline{F}_\alpha . \overline{\nabla_{v_\alpha} \varphi_\alpha} = \int \varphi_\alpha C_\alpha d\underline{v}_\alpha$$

The right-hand side represents the change of the mean value of φ_α due to collisions. This vanishes if $\varphi_\alpha = 1$ and (64) gives

(65) $$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \underline{v}_o) + \nabla \cdot (n_\alpha \underline{\bar{V}}_\alpha) = \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \underline{\bar{v}}_\alpha) = 0$$

which is the equation of continuity for the component α. Multiplying the equations of continuity for the ions and electrons (65) by m_i and

m_e respectively and adding we have the equation of continuity for the plasma as whole,

$$\frac{\partial \rho_o}{\partial t} + \nabla \cdot (\rho_o \underline{v}_o) = 0 \tag{66}$$

If we set $\varphi_\alpha = m_\alpha \underline{v}_\alpha$, then , after some simplification and using (65), one obtains

$$\rho_\alpha \frac{d\underline{v}_o}{dt} + (\nabla \cdot p_\alpha - \rho_\alpha \underline{F}_\alpha) + \frac{d(\rho_\alpha \overline{\underline{V}}_\alpha)}{dt} + \rho_\alpha (\overline{\underline{V}}_\alpha \cdot \nabla)\underline{v}_o$$
$$+ \rho_\alpha \overline{V}_\alpha \nabla \cdot \underline{v}_o = \int m_\alpha \underline{v}_\alpha C_\alpha \, d\underline{v}_\alpha \tag{67}$$

Adding the equations for the ions and electrons and noting that the total momentum of the ions and electrons in the element is unaltered by collisions, we get

$$\rho_o \frac{d\underline{v}_o}{dt} = - \nabla \cdot p_o + \rho_i \underline{F}_i + \rho_e \underline{F}_e \tag{68}$$

which is the equation of mass motion. Equation (67) refers to an element of either constituent following the mass-motion of the plasma. An equation can also be obtained referred to the local mean velocity of the constituent, $\overline{\underline{v}}_\alpha$. Denoting by d_α / dt the time derivative in this case, so that

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + \overline{\underline{v}}_\alpha \cdot \nabla \tag{69}$$

we find after some rearrangement of terms that

$$\rho_\alpha \frac{d_\alpha \overline{\underline{v}}_\alpha}{dt} + \nabla \cdot (p_\alpha - \rho_\alpha \overline{\underline{V}}_\alpha \overline{\underline{V}}_\alpha) - \rho_\alpha \underline{F}_\alpha = \int m_\alpha \underline{v}_\alpha C_\alpha \, d\underline{v}_\alpha \tag{70}$$

where

$$P_\alpha = p_\alpha - \rho_\alpha \overline{\underline{V}}_\alpha \overline{\underline{V}}_\alpha = \rho_\alpha \overline{\underline{V}_\alpha \underline{V}_\alpha} - \rho_\alpha \overline{\underline{V}}_\alpha \overline{\underline{V}}_\alpha \tag{71}$$

is the relative pressure tensor. It is easily shown that this is equal to

(72) $$\mathrm{P}_{\alpha} = \rho_{\alpha}\,\overline{(\underline{v}_{\alpha} - \overline{\underline{v}}_{\alpha})\,(\underline{v}_{\alpha} - \overline{\underline{v}}_{\alpha})} = \rho_{\alpha}\,\overline{\underline{u}_{\alpha}\,\underline{u}_{\alpha}}$$

when $\underline{u}_{\alpha} = \underline{v}_{\alpha} - \overline{\underline{v}}_{\alpha}$ is the velocity of a particle relative to the mean velocity of the element. Thus (70) can now be written

(73) $$\rho_{\alpha}\frac{d_{\alpha}\overline{\underline{v}}_{\alpha}}{dt} + \nabla\cdot\mathrm{P}_{\alpha} - \rho_{\alpha}\underline{F}_{\alpha} = \int m_{\alpha}\,\underline{v}_{\alpha}C_{\alpha}\,d\underline{v}_{\alpha}$$

which is the equation of motion of the constituent α referred to the mean velocity $\overline{\underline{v}}_{\alpha}$ of this constituent.

13. Approximate calculation of the collision term

Since particles of one constituent can collide with each other and with particles from another constituent, the collision term in Boltzmann's equation (3) may be written

(74) $$C_{\alpha} = \sum_{\beta} C_{\alpha\beta}(f_{\alpha}, f_{\beta})$$

where $C_{\alpha\beta}$ gives the change per unit time in the distribution function for particles of the constituent α due to collisions with particles of contituent β, $C_{\alpha\beta}$ depend on the respective distribution functions f_{α}, f_{β}. Certain properties of the collision terms are immediately obvious and do not depend on the explicit form of the $C_{\alpha\beta}$. Thus

(75) $$\int C_{\alpha\beta}\,d\underline{v}_{\alpha} = 0$$
$$\int m_{\alpha}\,v_{\alpha}\,C_{\alpha\alpha}\,d\underline{v}_{\alpha} = 0$$

neglecting processes which may convert particles of one constituent into that of another, e.g., ionization, dissociation , etc.

We have alredady noted in section 11 that as a rough approximation we may write

$$C_\alpha = \frac{f_\alpha^{(o)} - f_\alpha}{\tau} \tag{76}$$

where $f_\alpha^{(o)}$ is the Maxwellian distribution. On account of the second relation in (75), we may take τ to be $\tau_{\alpha\beta}^*$ and that, if departures from equilibrium are small, we may treat $\tau_{\alpha\beta}^*$ as constant. We can now evaluate approximately the collision term in (73). Write

$$\underline{v}_\alpha = \underline{v}_o + \underline{V}_\alpha , \tag{77}$$

where $\underline{v}_o$ is the mean mass velocity of the two constituent plasma. Then

$$\int m_\alpha \underline{v}_\alpha C_\alpha d\underline{v}_\alpha = \int m_\alpha \underline{v}_o C_\alpha d\underline{v}_\alpha + \int m_\alpha \underline{V}_\alpha C_\alpha d\underline{v}_\alpha \tag{78}$$

Since $\underline{v}_o$ is a constant in the first integral, this vanishes by virtue of the first equation in (75). Hence (78) reduces to

$$\int m_\alpha \underline{V}_\alpha C_\alpha d\underline{v}_\alpha . \tag{79}$$

Substituting (76) in (79) and noting that $d\underline{v}_\alpha = d\underline{V}_\alpha$, this reduces to

$$\int m_\alpha \underline{V}_\alpha \frac{f_\alpha^{(o)}}{\tau} d\underline{V}_\alpha - \int m_\alpha \underline{V}_\alpha \frac{f_\alpha}{\tau} d\underline{V}_\alpha . \tag{80}$$

But $\int m_\alpha \underline{v}_\alpha f_\alpha^{(o)} d\underline{V}_\alpha$ vanishes identically; thus (80) reduces to

$$-n_\alpha m_\alpha \underline{\bar{V}}_\alpha / \tau \tag{81}$$

Here $\tau \sim \tau_{\alpha\beta}^*$ denotes effectively the electron-ion scattering time and we may interpret this result as follows. The electrons lose their ordered velocity with respect to the ions in a time of the order τ and hence lose momentum $m_\alpha \underline{V}_\alpha$ per particle α which is communicated to the particle β . This implies that the particles are subjected to a frictional force $n_\alpha m_\alpha \bar{\underline{V}}_\alpha / \tau$. This is equal and opposite to the force exerted on the particle β . In fact, since $\tau^*_{\alpha\beta} = \tau^*_{\beta\alpha}$ we have, adding to (81) the corresponding equation for the particle β ,

$$n_\alpha m_\alpha \bar{\underline{V}}_\alpha + n_\beta m_\beta \bar{\underline{V}}_\beta = 0$$

Using this relation, (81) may be expressed in terms of the mean relative velocity , namely,

$$- \frac{\rho_\alpha \rho_\beta}{\rho_0} (\bar{\underline{v}}_\alpha - \bar{\underline{v}}_\beta) / \tau \tag{82}$$

since $\bar{\underline{v}}_\alpha - \bar{\underline{v}}_\beta = \bar{\underline{V}}_\alpha - \bar{\underline{V}}_\beta$. Hence equation (73) can finally be written

$$\rho_\alpha \frac{d_\alpha \bar{\underline{v}}_\alpha}{dt} + \nabla . P_\alpha - \rho_\alpha \underline{F}_\alpha = - \frac{\rho_\alpha \rho_\beta}{\rho_0 \tau} (\bar{\underline{v}}_\alpha - \bar{\underline{v}}_\beta) \tag{83}$$

14. Rate of diffusion of the two constituents

Dividing this equation by ρ_0 and substracting from it the corresponding equation for the β-constituent we obtain an expression for the differential or diffusion velocity

$$\bar{\underline{v}}_\alpha - \bar{\underline{v}}_\beta = -\tau \left\{ \frac{d_\alpha \bar{\underline{v}}_\alpha}{dt} - \frac{d_\beta \bar{\underline{v}}_\beta}{dt} + \frac{1}{\rho} \nabla . P_\alpha - \frac{1}{\rho_\beta} \nabla . P_\beta - (\underline{F}_\alpha - \underline{F}_\beta) \right\} . \tag{84}$$

It is convenient at this stage to introduce the coefficient of diffusion of the other two constituents, namely,

$$D_{\alpha\beta} = kT \left(\frac{m_0}{m_\alpha \; m_\beta} \right) \tau \tag{85}$$

where $m_0 = m_\alpha + m_\beta$. We find, after some algebra, that (84) can be written

$$\bar{\underline{v}}_{\alpha} - \bar{\underline{v}}_{\beta} = D_{\alpha\beta}\left\{\frac{m_{\alpha}\, m_{\beta}}{k\,T\,m_{o}}\,(\underline{f}_{\alpha} - \underline{f}_{\beta}) - \frac{m_{\alpha} - m_{\beta}}{m_{o}}\nabla \log P_{o} + \frac{n_{o}}{n_{\alpha}\, n_{\beta}}\,\nabla\cdot\,\frac{n_{\alpha}}{n_{o}} - \frac{m_{\alpha}\, m_{\beta}}{k\,T\,m_{o}}\,(\underline{F}_{\alpha} - \underline{F}_{\beta})\right. \tag{86}$$

where $P_{o} = P_{\alpha} + P_{\beta}$ is the total pressure, and we have written $\underline{f}_{\alpha}$ for $\dfrac{d_{\alpha}\underline{v}_{\alpha}}{dt}$, etc. The four terms inside the bracket (86) correspond to components of the relative velocity of diffusion due respectively to (1) the relative acceleration, (2) the pressure gradient, (3) a concentration gradient, and (4) external forces. Note that gravitational forces do not contribute to the velocity of diffusion. These component velocities of diffusion tend to have the following effects: (1) and (4) have indeed the same effect and tend to separate the constituents in the direction of the relative acceleration or forces. (2) tends to make the composition uniform and (3) tends to increase the proportion of the heavier constitution in the regions of higher pressure.

15. Three-constituent plasma. (Partially ionized gas) .

We shall consider a partially ionized gas consisting of electrons one kind of ions and one kind of neutral particles. The velocity of each constituent will be denoted by $\underline{v}_{e}$, $\underline{v}_{i}$, $\underline{v}_{n}$ respectively. Because of their much smaller mass, the momentum of the electrons may be neglected in defining the mean mass velocity $\underline{v}_{o}$, which is thus approximately

$$\underline{v}_{o} = \frac{1}{\rho_{o}}\,(n_{i} m_{i} \underline{v}_{i} + n_{n} m_{n} \underline{v}_{n}) \,, \tag{87}$$

where $\rho_{o} = n_{i} m_{i} + n_{n} m_{n}$.

However, as in the case of a two-constituent plasma, it is more convenient to derive the moment equation of the Boltzmann equation of each constituent relative to axes moving with the mean velocity $\bar{\underline{v}}$ of that component. The equation of continuity (65) will hold as before, and the equation of momentum will likewise be the same as before, except for the collision term C_α.
It is clear that the collision of particles of each constituent with those of the other two constituents will yield a collision term of the form (82); however, we can no longer drop the suffixes so that denoting by ρ_e, ρ_i, ρ_n the mass derivatives of the electrons, ions, and neutrals, we have

(88) $$\int m_e \underline{v}_e C_e \, d\underline{v}_e = -\frac{\rho_e \rho_i}{\rho_i + \rho_e} (\bar{\underline{v}}_e - \bar{\underline{v}}_i)/\tau_{ei} - \frac{\rho_e \rho_n}{\rho_e + \rho_n} (\bar{\underline{v}}_e - \bar{\underline{v}}_n)/\tau_{en}$$

(89) $$\int m_i \underline{v}_i C_i \, d\underline{v}_i = -\frac{\rho_i \rho_e}{\rho_i + \rho_e} (\bar{\underline{v}}_i - \bar{\underline{v}}_e)/\tau_{ie} - \frac{\rho_i \rho_n}{\rho_i + \rho_n} (\bar{\underline{v}}_i - \bar{\underline{v}}_n)/\tau_{in}$$

(90) $$\int m_n \underline{v}_n C_n \, d\underline{v}_n = -\frac{\rho_n \rho_e}{\rho_n + \rho_e} (\bar{\underline{v}}_n - \bar{\underline{v}}_e)/\tau_{ne} - \frac{\rho_n \rho_i}{\rho_n + \rho_i} (\bar{\underline{v}}_n - \bar{\underline{v}}_i)/\tau_{ni}$$

The τ' s are called the 'collision intervals' by analogy with what has been said previously. We have $\tau_{ei} = \tau_{ie}$, $\tau_{in} = \tau_{ni}$, $\tau_{en} = \tau_{ne}$, so that there are effectively only three 'collision intervals' ; writing

(91) $$\theta_{\alpha\beta} = \frac{\rho_\alpha \rho_\beta}{\rho_\alpha + \rho_\beta} \frac{1}{\tau_{\alpha\beta}} \quad (\alpha, \beta = e, i, n)$$

the equation of motions for the ions, electrons and neutral particles are respectively

(92) $$\rho_i \frac{d_i \bar{\underline{v}}_i}{dt} = -\nabla \cdot P_i + Z n_i e (\underline{E} + \bar{\underline{v}}_i \times \underline{B}) + \rho_i \underline{F}_i - \theta_{ie} (\bar{\underline{v}}_i - \bar{\underline{v}}_e) - \theta_{in} (\bar{\underline{v}}_i - \bar{\underline{v}}_n)$$

$$(93)\quad \rho_e \frac{d\bar{v}_e}{dt} = -\nabla\cdot P_e - n_e e(\underline{E} + \underline{v}_e \times \underline{B}) + \rho_e \underline{F}_e - \theta_{ie}(\bar{\underline{v}}_e - \bar{\underline{v}}_i) - \theta_{en}(\bar{\underline{v}}_e - \bar{\underline{v}}_n)$$

$$(94)\quad \rho_n \frac{d\bar{v}_n}{dt} = -\nabla\cdot P_n + \rho_n \underline{F}_n - \theta_{in}(\bar{\underline{v}}_n - \bar{\underline{v}}_i) - \theta_{en}(\bar{\underline{v}}_n - \bar{\underline{v}}_e)$$

where n_i, n_e, n_n are the number densities of the ions, electrons and neutrals, P_n is the relative partial pressure of the neutral gas, and Ze, $-e$ are the charges on a positive ion and electron respectively.

16. Diffusive equilibrium in a fully ionized plasma in a magnetic field

The equations of motions for the ions (assumed to be singly ionized for simplicity) and electrons may be written respectively (dropping the bar over the velocities),

$$(95)\quad \begin{aligned} &\frac{d\underline{v}_i}{dt} + \frac{1}{\rho_i}\nabla\cdot P_i - \underline{F}_i - \frac{e}{m_i}\underline{E} - \underline{v}_i \times \underline{\omega}_i = -\frac{\rho_e}{\rho_o \tau}(\underline{v}_i - \underline{v}_e) \\ &\frac{d\underline{v}_e}{dt} + \frac{1}{\rho_e}\nabla\cdot P_e - \underline{F}_e + \frac{e}{m_e}\underline{E} + \underline{v}_e \times \underline{\omega}_e = -\frac{\rho_i}{\rho_o \tau}(\underline{v}_e - \underline{v}_i) \end{aligned}$$

where $\underline{\omega}_i = e\underline{B}/m_i e$ and $\underline{\omega}_e = e\underline{B}/m_e c$ are the cyclotron frequencies for the ions and electrons. Now a plasma is electrically neutral to a high degree of approximation so that $n_i/n_e \simeq 1$. Also $m_e/m_i \ll 1$ so that $\rho_e/\rho_o \simeq m_e/m_i$ and $\rho_i/\rho_o \simeq 1$. With these approximations the equations may be rewritten

$$(96)\quad \frac{d\underline{v}_i}{dt} + \frac{m_e}{m_i \tau}\underline{v}_i - \frac{m_e}{m_i \tau}\underline{v}_e - \underline{v}_i \times \omega_i \underline{u} = \frac{e}{m_i}(\underline{E} - \frac{1}{n_i e}\nabla\cdot P_i) + \underline{F}_i \equiv \underline{G}_i$$

$$(97)\quad \frac{d\underline{v}_e}{dt} + \frac{1}{\tau}\underline{v}_e - \frac{\underline{v}_i}{\tau} + \underline{v}_e \times \omega_e \underline{u} = -\frac{e}{m_e}(\underline{E} + \frac{1}{n_e e}\nabla\cdot P_e) + \underline{F}_e \equiv G_e$$

where $\underline{u}$ is a unit vector along $\underline{B}$, $\omega_i = |\underline{\omega}_i|$, $\omega_e = |\underline{\omega}_e|$, and $\underline{F}_i$ and $\underline{F}_e$ are extraneous forces other that the electric and magnetic forces or pressure gradients.

An inspection of these equations shows that there exists a transient part of the solution for $\underline{v}_i$ and $\underline{v}_e$ which decays exponentially with time approximately as $e^{-t/\tau}$, i.e., in a time of the order of the relaxation time. The plasma therefore attains what is termed a state of <u>diffusive equilibrium</u> in which the acceleration terms in (96) and (97) can be neglected, yielding the approximate diffusive equations

(98) $$\underline{v}_i - \underline{v}_e - \underline{v}_i \times \Omega\underline{u} = \underline{G}_i \frac{m_i}{m_e} \tau$$

(99) $$\underline{v}_e - \underline{v}_i + \underline{v}_e \times \Omega\underline{u} = \underline{G}_e \tau$$

where $\Omega = \omega_e \tau$. Solving these vector equations we obtain

(100) $$\underline{v}_i = \lambda_i \underline{u} + \frac{1}{\Omega^2}\left[\left(\frac{m_i}{m_e} \underline{G}_i + \underline{G}_e\right)\tau + \Omega\tau \frac{m_i}{m_e} \underline{G}_i \times \underline{u}\right]$$

(101) $$\underline{v}_e = \lambda_e \underline{u} + \frac{1}{\Omega^2}\left[\left(\frac{m_i}{m_e} \underline{G}_i + \underline{G}_e\right)\tau + \Omega\tau \underline{G}_e \times \underline{u}\right]$$

where λ_i and λ_e are arbitrary parameters. However, multiplying (98) and (99) scalarly by $\underline{n}$ and adding the resulting equations have

(102) $$\left(\frac{m_i}{m} \underline{G}_i + \underline{G}_e\right) \cdot \underline{u} = 0$$

Using the approximation $n_i/n_e \simeq 1$ this reduces to

(103) $$(\nabla \cdot P_o + \rho_i \underline{F}_i + \rho_e \underline{F}_e) \cdot \underline{u} = 0$$

Thus , along a magnetic line of force, the pressure gradient in that direction balances the total external force in the same direction. In the case of the protonosphere this implies that, along a line of force, the pressure decreases exponentially with height, the temperature being approximately constant in this region. The effect of the magnetic field depends roughly on the magnitude of Ω, that is, the product $\omega_e \tau$, the ratio of the cyclotron frequency to the collision frequency. If the electrons are able to spiral many times between collisions, then $\Omega \gg 1$ and we see from (100) and (101) that, whilst motion of the ions and electrons along the lines of force is unimpeded, the component at right angles is of order Ω^{-1} and thus becomes vanishingly small as $\Omega \to \infty$; we note also that the drift of the ions at right angles to the magnetic field due to the pressure gradient is greater than the corresponding drift for the electrons.

These results, of course, are to be expected from general consideration of the motion of charged particles in a magnetic field.

V. C. A. Ferraro

II. Application to the Ionosphere

17. The atmosphere

The scientific study of the upper atmosphere is nowadays called aeronomy - a term due to Chapman. The discussion of the atmosphere is based largely on chemical composition and temperature and the various layers into which the atmosphere can so be divided are referred to as 'spheres' . The upper boundaries are referred to as '-pause'. Thus the troposphere denotes the layer extending from ground level upwards in which the temperature decreases with height. Above this layer is the stratosphere in which , for many kilometres the temperature remains constant. The tropopause is the upper level of the troposhere and separates it from the stratosphere. The height of the the tropopause varies with latitude, being about 11 km in mid-latitudes. Above the stratosphere the temperature increases and the region of higher temperature is the mesosphere; the temperature then decreases again and reaches its lowest value (about 180° K) at a height of 80-85 km. The temperature rapidly rises above this level (mesopause) to about 300 km where it attains a temperature of over 1000° K. This region is called the thermosphere. Above this level the atmosphere is maintained in isothermal equilibrium. This is the exosphere and here collisions between the molecules are so rare that they move in free orbits under gravity. The various regions are illustrated in figure 7 .

18. The ionosphere, heliosphere and protonosphere

Sunlight of wave lengths less than about 2900 A (1 A = 10^{-8} cm) is capable of ionizing oxygen and nitrogen. This radiation cannot be observed at ground level owing to strong absorption by atmospheric ozone. Its emission from the sun has been detected by satellites and other space vehicles and, prior to this, its presence was indicated by the existence of several ionized layers in the atmosphere.

Radio methods of observations of the ionosphere have revealed the existence of two main layers, the E and F layers, the former at about 120 km height and the other at 180-300 km. The F layer is thicker than the E layer and separates into two parts during daytime, the lower called F_1 and the upper F_2. At night they partly merge and become indistinguishable. Below the E-layer there is another at about 70 km called the D-layer. The thickness of the F-layer diminishes rather slowly with height and has no well defined upper boundary. It merges into the <u>heliosphere</u> where neutral and ionized helium are present to a height of about 1000 km and above this we have many ionized hydrogen or protons. This highest part of the ionosphere is called the <u>protonosphere</u> but its limits are difficult to define. It seems likely that this region is in diffusive equilibrium. The various layers are illustrated in fig. 8.

19. Processes in the ionosphere

We shall here deal only with the large-scale structure of the ionospheric layers, and particularly our attention will be directed towards the effect of diffusion on the distribution of ionization of the F_2 region.

The ionization in the ionosphere is due essentially to the production of ion-electron pairs by the absorption of solar U.V. and X-ray radiation - at least in middle and low latitudes. At higher latitudes, ionization can also be produced by collisions between high-energy charged particles, precipitated in the atmosphere, with neutral atmospheric molecules

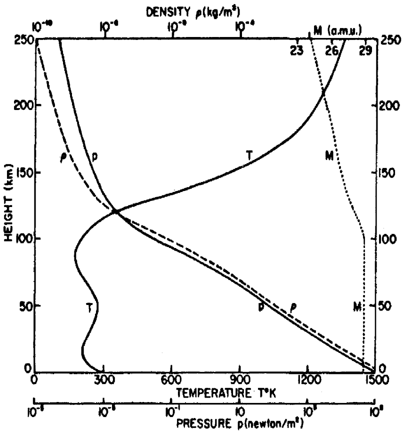

FIG. 7. U.S. STANDARD ATMOSPHERE (Government Printing Office, Washington, D.C., 1962). Vertical distribution of pressure p, density ρ, temperature T and mean molecular mass M to 250 km. The composition is assumed constant up to 100 km.

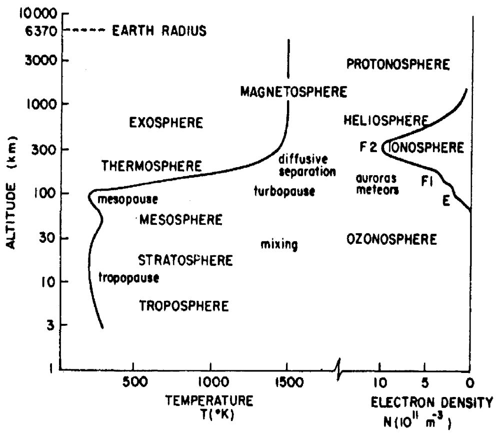

FIG. 8. REGIONS OF THE ATMOSPHERE, SHOWING CONVENTIONAL NAMES DESCRIPTIVE OF LEVELS, PHYSICAL REGIMES, AND CHARACTERISTIC CONSTITUENTS. The temperature profile is taken from the U.S. Standard Atmosphere and the electron density profile represents average daytime conditions for middle latitudes, high solar activity.

where

(2.3.61) $$J_n^*(r) = \int_{-1}^{1} E(r,\tau)\,(1-\tau^2)^n\, d\tau.$$

Bergman considers a partial differential equation of the form:

(2.3.62) $$\Delta_3 \psi + F(y, z)\,\psi = 0 .$$

We introduce the variables : $X = x$, $Z = (z+iy)/2$, $Z^* = -(z-iy)/2$, and express the function $F(y, z)$ appearing in Eq. (2.3.62) as a function of Z and Z^* ; we also use the symbol F for this new function. The equation (2.3.62) then assumes the form :

(2.3.63) $$\psi_{XX} - \psi_{ZZ^*} + F(Z, Z^*)\,\psi = 0$$

We proceed to obtain particular solutions of (2.3.63) which are polynomials in X, as follows. Let $\tilde{\gamma}(Z, Z^*)$ be any solution of the equation:

(2.3.64) $$-\tilde{\gamma}_{ZZ^*} + F\,\tilde{\gamma} = 0$$

and let the polynomials $P^{(N,k,\alpha)}$ be defined as follows:

$$P^{(N,k,k-2\nu)} \equiv \binom{N}{k-\nu}\cdot\binom{k-\nu}{\nu}\; Z^{N-k+\nu} Z^{*\nu} ,$$

(2.3.65) $N=0, 1, 2, \cdots$; $k=0, 1, 2, \cdots, 2N$; $\nu=k, k-2, \cdots, k-2\left[\frac{k}{2}\right]$.

Let the functions $\pi^{(N,k,\alpha)}(Z, Z^*)$ satisfy the equations :

(2.3.66) $$-N\,\tilde{\gamma}_{Z^*} P^{(N-1,k,k)} - \pi^{(N,k,k)}_{ZZ^*} + F\,\pi^{(N,k,k)} = 0$$

$$- N\tilde{\gamma}_{Z^*} P^{(N-1,k,\nu)} - N\tilde{\gamma}_{Z}\, P^{(N-1,k-2,\nu)} +$$

(2.3.67) $$+ (\nu+2)(\nu+1)\,\pi^{(N,k,\nu+2)} - \pi^{(N,k,\nu)}_{ZZ^*} + F\,\pi^{(N,k,\nu)} = 0, \quad \nu<k.$$

Negative ions may be formed in the lower ionosphere by attachment of elec trons.

The important losses of ionization arise from atomic ion-electron (radiative) recombination, molecular ion and electron (dissociative) recombination and , in the lower ionosphere, by the attachment of an electron to a neutral molecule.

Ionization may also be affected by transport processes; the ions and electrons (plasma) in the ionosphere may be thought as a minor constituent of the atmosphere. It is acted on by gravity and by pressure gradients in the plasma. Unlike the neutral constituent, the ions and elec trons are acted on by electric and magnetic forces. As we have seen in section 14, the plasma tends to diffuse through the neutral air if the forces acting on it are not in equilibrium. Ions and electrons diffuse together, since any tendency to separate the positive and negative charges w: give rise to a large electric field opposing this separation. This is called 'ambipolar' or 'plasma' diffusion and proceeds rapidly in the F region but not in the lower ionosphere. In this region the plasma tends to be set in motion by movements of the neutral air, which may be due to large scale wind-system or to temperature.

The various processes outlined above which modify the ionization in the layers of the ionosphere must balance and this balance can be expressed as an equation of continuity . If the transport processes (wind and diffusion) result in a net drift velocity $\underline{v}$, and we denote the electron density by n , the rate of production and loss of ionization by q and L respectively, the equation of continuity is

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\underline{v}) = q - L \ . \qquad (104)$$

In the absence of production and loss of ionization (104) reduces to (65) already found in of section 12 .

Various wavelengths in the radiation from the sun are responsible for the production of ionization. It would be outside the scope of these lectures to go into more than a few details. A major part of the E-region ionization arises from the wavelength band 911-1027 A which ionizes 0_2 to 0_2^{+}. In the F-region the wavebands 170-796 Å and 796 Å to 911 Å are mainly responsible for the ionization, in this case the ions formed being 0^+ and N_2^+ .

Amongst the loss processes we may note the following :

(a) Ion-ion recombination (coefficient α_i)

$$X^+ + Y^- \rightarrow X + Y .$$

(b) Electron-ion recombination (coefficient α_e)

(i) Three-body : $X^+ + e + M \rightarrow X + M$

Here M denotes a neutral particle which exchanges energy and momentum but does not take part in the chemical reaction.

(ii) Radiative : $X^+ + e \rightarrow X^* \rightarrow X + h\nu$.

(iii) Dissociative : $XY^+ + e \rightarrow X^* + Y^*$

Here X^* denotes an atom left in the excited state.

Process (i) can occur in the lower D region but is rare at greater heights.

Process (ii) is likely to be the fastest loss process only in the uppermost levels of the F regions. Elsewhere in the E and F regions the dissociative recombination process are important.

(c) Ion-atom interchange

$$A^{+} + XY \longrightarrow XY^{+} + A \quad .$$

Ion-atom interchange (c) followed by dissociative recombination b (iii) is the principal loss process in the E and F regions. There is still considerable controversy as to precisely which reactions are important.

The rates for processes b (iii) and (c) are given , in terms of the reaction constants K_b and K_c , by the expressions

$$\text{(105)} \qquad \frac{dn(e)}{dt} = -K_b n(XY^{+})\, n\,(e)$$

$$\text{(106)} \qquad \frac{dn(A^{+})}{dt} = -K_c n(A^{+}) n(XY) \quad .$$

If we suppose that the atmosphere is electrically neutral,

$$\text{(107)} \qquad n(A^{+}) + n(XY^{+}) = n(e)$$

and we suppose further that the ionization is in equilibrium . Then if the electrons and positive ions are produced by incident radiation at the rate q per unit volume

$$\text{(108)} \qquad q = K_b n(XY^{+}) n(e) = K_c n(A^{+}) n(XY) \quad .$$

Eliminating $n(A^{+})$ and $n(XY^{+})$ by using (107) we find

$$\text{(109)} \qquad q = \frac{K_b K_c n(XY) n^{2}(e)}{K_c n(XY) + K_b n(e)} \quad .$$

If $K_e n(XY) >> K_b n(e)$, this reduces to

(110) $$q = K_b n^2(e)$$

which corresponds to a quadratic law of recombination αn^2, the coefficient of recombination α being equal to K_b. This law holds very nearly in the E region where $\alpha \simeq 10^{-8}$ cm^3 sec^{-1}.

If $K_c n(XY) \ll K_b n(e)$, then (109) reduces to

(111) $$q = K_c n(XY) n(e)$$

which corresponds to an 'attachment' law of the form $\beta\, n(e)$ with an 'attachment' coefficient

(112) $$\beta = K_c n(XY)$$

If, as is usually the case, n(XY) decreases upwards, so will β

20. Chapman's Theory

We consider the simple case of ionization by absorption of monochromatic radiation in an atmosphere of uniform composition and temperature. Such an atmosphere will be distributed exponentially; in fact if h denotes the height, n the number density, g the acceleration of gravity, m the mean molecular mass of the gas, the statical equation is

(113) $$\frac{dp}{dh} = -nmg \quad .$$

(114) Also $$p = knT$$

where k is Boltzmann's constant (1.38×10^{-16} cgs) and T is the temperature, so that eliminating p between these equations we find

(115)
$$\frac{d\log n}{dh} = -\frac{mg}{kT} = -\frac{1}{H}$$

where H is a quantity having the dimensions of a length, called the <u>scale height</u>. Integration of (115) now gives

(116)
$$n = n_0 e^{-h/H}$$

that is, an exponential distribution of density. Let I denote the intensity of the radiation at the height h and I_∞ the intensity of the incident solar radiation. Let χ denote the zenith distance of the sun at the height h at any time, then the decrease in I by absorption over the path of length $ds = (\sec\chi)dh$ between the levels $h + dh$ and h is given by

(117)
$$dI = -I\sigma n\,(\sec\chi)dh\,,$$

where σ is the absorption cross-section of the molecules. Using (116) we have

$$\frac{dI}{I} = -(\sigma n_0 \sec\chi)\,e^{-h/H}\,dh$$

which can be integrated to give

(118)
$$\log_e\left(\frac{I}{I_\infty}\right) = -(\sigma n_0 H\sec\chi)e^{-h/H}\quad dh$$

since $I \to I_\infty$ as $h \to \infty$. Hence

(119)
$$I = I_\infty \exp\left(-\sigma n_0\, He^{-h/H}\sec\chi\right)\,.$$

The absorption of radiant energy per unit volume of the atmosphere is $dI/ds = (dI/dh)\cos\chi$ and if β ions are produced by the absorption of unit quantity of energy, the rate of production of ions per unit volume is

$$q(h) = \beta I_\infty n_o \sigma \exp\left(-h/H - n_o \sigma H e^{-h/H} \sec\chi\right) . \tag{120}$$

The total number of ions produced in a vertical column of air of unit area of cross-section by the complete absorption of the incident radiation is clearly $\beta I_\infty \cos\chi$.

The rate of ion-production q has a maximum q_m at a height h_m where

$$e^{h_m/H} = n_o \sigma H \sec\chi \tag{121}$$

giving

$$q_m = (\beta I_\infty \cos\chi)/He . \tag{122}$$

Denote the values of h_m and q_m for the overhead sun ($\chi = 0$) by h_o and q_o ; then

$$e^{h_o/H} = n_o \sigma H, \qquad I_o = \beta I_\infty / eH \tag{123}$$

whence

$$h_m = h_o + H \log_e \sec\chi \tag{124}$$

$$q_m = q_o \cos\chi . \tag{125}$$

In terms of q_o and h_o, we may now write (120) as

$$q(h) = q_o \exp\left(1 - \frac{h-h_o}{H} - e^{-\frac{h-h_o}{H}} \sec\chi\right) \tag{126}$$

or measuring heights in terms of H as unit from the level h_o, writing

$$z = \frac{h - h_o}{H} \tag{127}$$

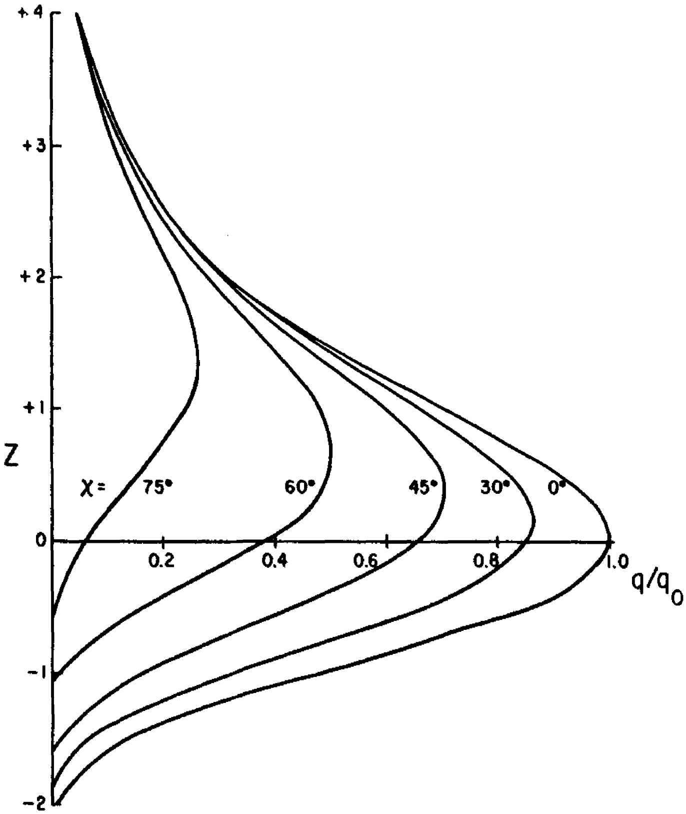

FIG. 9. NORMALIZED CHAPMAN PRODUCTION FUNCTION $q(z,\chi)/q_0 = \exp\left(1 - z - e^{-z} \sec \chi\right)$. Values at several reduced heights are shown as a function of zenith angle. The broken line is the envelope, $q_m/q_o = \cos \chi$.

Once we know the relation between the density ρ and velocity q, we can integrate Eqs. (2.6.17) and (2.6.18) to yield the sonic line in the physical (x, y)-plane.

For stream functions ψ defined by corresponding $\chi_1(\theta)$, $\chi_2(\theta)$ given in (2.6.16), the sonic line in the physical (x, y)-plane is given in terms of the parameter θ by :

(2.6.19) $$x = (1/2)\,(6/5)^3\left[a-(6b/5)\right]\cos^2\theta\,,$$

(2.6.20) $$y = (6/5)^3\left\{(1/2)\left[a+(6b/5)\right](\theta-\pi/2) + (1/4)\left[a-(6b/5)\right]\sin 2\theta\right\}\,,$$

provided we take the origin in the physical plane as the image of the point $q=(5/6)^{1/2}$, $\theta=\pi/2$ on the sonic line in the hodograph plane. The value of q is calculated by $M^2 = q^2/\left[1-(\gamma-1)\,q^2/2\right]$ with $\gamma = 1.4$.

Example (I) : sonic line in physical plane is circular.

If we consider the stream function ψ defined by *(2.6.16) with* a, b related according to :

(2.6.21) $$a + (6b/5) = 0\,,$$

then (2.6.19) and (2.6.20) yield for parametric equations of the sonic line in the physical plane :

(2.6.22) $$x = a(6/5)^3\cos^2\theta\,, \qquad y = (1/2)a(6/5)\sin 2\theta\,.$$

Elimination of θ gives :

(2.6.23) $$x^2 + y^2 - a(6/5)^3\,x = 0\,,$$

(called the reduced height), we have

$$q(h) = q_o \exp\,(1 - z - e^{-z} \sec \chi\,) \,. \tag{128}$$

This is called the Chapman-function (Chapman, 1931). It has the interesting property that as χ varies, its shape is unchanged, its peak is shifted to the level $z_m = \log_e(\sec\chi)$ and its amplitute is scaled by the factor $\cos\chi$. This can be seen by writing the above equation in the form

$$q = (q_o \cos\chi)\,\exp\left[\,1 - (z-z_m) - e^{z_m - z}\right]. \tag{129}$$

The ratio q/q_o is shown in Fig. 9.

In the actual ionosphere the production formula is considerably more complicated, partly because there are different atmospheric gases, differently distributed, and the ionizing radiation is not monochromatic but consists of a range of wave lengths and σ depends on the wavelength.

The above theory neglects the curvature of the earth; Chapman has considered the modifications introduced by taking this into account. This correction is only important near sunrise and sunset.

The theory can also be extended to deal with gases which are not at the same temperature at all heights. If the temperature is proportional to the height, so that

$$H = K_o + \gamma\, h \tag{130}$$

then it can be shown that (125) takes the modified form

$$q_m = q_o\,(\cos\chi\,)^{1+\gamma} \tag{131}$$

21. Plasma diffusion

The F2 peak of ionization is observed at about 300 Km. No mechanism seemed capable of causing a peak of production at such a height and Bradbury (1938) suggested that the production peak occurred at a lower height (the F1 region, in fact) and he attributed the upward increase in electron density to a rapid upward decrease of a linear loss coefficient. The hypothesis is unsatisfactory, for even if β varies as

$$\exp\left[-(h-h_o)/H_\beta\right]$$

where H_β is the scale height , then the electron density well above the production peak is approximately $q/\beta \propto \exp\left[+(h-h_o)(\frac{1}{H_\beta} - \frac{1}{H_a})\right]$, where H_a is the scale height of the ionizable gas.
Since $H_a < H_\beta$, then q/β increases indefinitely upwards, and we must find some other explanation for the peak in the electron density . We require some transport process to limit the value of the electron density at great heights. One such process is plasma , or ambipolar diffusion Attention to the probable importance of diffusion in the ionosphere was first directed by Hulburt in 1928. I considered the problem in greater detail in 1945 and showed that diffusion was unimportant in the E and F_1 region of the ionosphere but that it might become important in the F_2 region and above it. Mariani (1956) drew similar conclusions but Yonezawa first discussed in detail the problem of the formation of the F_2 region and showed that diffusion could provide an explanation .

In deriving the equation of diffusion for the ionization, we shall neglect, for the present, the earth's magnetic field, so that the only forces acting are gravity $\underline{g}$, the electric field $\underline{E}$, and the frictional forces due to collisions. The full equations for a three constituent plasma, (92-94), , have been derived in section 15. However, in the absence of any external electric fields, the slightest separation of the ions and electrons will give rise to large electrostatic fields opposing any further separation so that we many set $n_i = n_e$ and $\underline{v}_i = \underline{v}_e$ in these equations. Also, if we assume that the neutral air is at rest, $\underline{v}_n = 0$. Furthermore, the relaxation times in the F2 region are small , so that the steady state is quickly attained. That is, we may neglect the acceleration of the ions and electrons in (92-94) . Writing $\nu_{in} = 1/\tau_{in}$, etc. , for the collision frequencies, these become

$$(132) \qquad -\nabla P_i + ne\,\underline{E} + nm_i\underline{g} - nm_i \nu_{in} \underline{v}_D = 0$$

$$(133) \qquad -\nabla P_e - ne\underline{E} + nm_e\underline{g} - nm_e \nu_{en} \underline{v}_D = 0$$

$$(134) \qquad -\nabla P_n + n_n m_n \underline{g} + nm_i \nu_{in} \underline{v}_0 + nme \nu_{en} \underline{v}_D = 0$$

where we have written $\underline{v}_D$ for the common velocity of diffusion , of the the ions and electrons, and n for their number density.

Further simplifications can be made by noting that $m_e \ll m_i$, $m_i \nu_{in} \gg m_e \nu_{en}$ (i.e. collision with the neutral particles are important for ions but not for electrons). Also, if T_i and T_e are the temperature of the ionic and electronic constituent, we have

$$(135) \qquad P_i = kn_i T_i , \qquad P_e = kn_e T_e .$$

Using these equations, on adding (132) and (133) and solving for the velocity of diffusion of the ion-electron component we find, if h denotes the height at any level,

(136) $$-v_D = \frac{1}{m_i \nu_{in}} \left\{ \frac{1}{n} \frac{\partial}{\partial n} \left[nk(T_i + T_e) + m_i g \right] \right\}$$

where v_D is measured positively upwards. In the F region, the ion, electron and neutral air temperatures may all be different. It seems likely that $T_i = T$, the temperature of the neutral air.

Then, if we introduce the neutral air scale height $H = kT/m_n g$, and write $\mu = m_i/2m_n$, $\tau = T_e/T_i = T_e/T$, and introduce the coefficient of diffusion of ions through the neutral gas

(137) $$D_{in} = kT/m_i \nu_{in}$$

equation (136) becomes

(138) $$-v_D = D_{in}(1+\tau) \left[\frac{1}{n} \frac{\partial n}{\partial h} + \frac{1}{T} \frac{\partial T}{\partial h} + \frac{2\mu}{(1+\tau) H} + \frac{\partial \tau / \partial h}{1+\tau} \right.$$.

If $T_i = T_e = T$ at all heights, $\tau = 1$, and

(139) $$-v_D = D\left(\frac{1}{n} \frac{\partial n}{\partial h} + \frac{1}{T} \frac{\partial T}{\partial h} + \frac{\mu}{H}\right)$$

where we have written $D = 2D_{in}$.

The contribution of diffusion to the continuity equation (104) is to add the term $\partial (nv_D)/\partial h$ to the left hand side of equation. In general

(140) $$\frac{\partial (nv_D)}{\partial h} = D \mathscr{D} n$$

where $\mathscr{D}$ is a differential operator of the second order.

The value of D is of the form b/n_n, where the factor b depends on the temperature. It is now believed that in the F2 layer the neutral gas is mainly atomic oxygen and the ions mostly O^+. The value of b is affected by charge exchange and its value not exactly known. The value derived from kinetic theory, taking account of the effect of electrostatic induction by the charged ions on the neutral ions, gives n D of order 10^{19}. But this may well be too large by a factor of 10 because of charge exchange. Also $D \propto 1/n \propto e^{(h-h_0)/H}$; if we assume that the temperature of the gas is uniform, then

$$(141) \qquad -v_D = D\left(\frac{1}{n}\frac{\partial n}{\partial h} + \frac{1}{2H}\right),$$

and the diffusion term in the continuity equation on the right hand side takes the form

$$(142) \qquad D\mathcal{D}\, n = D\left(\frac{\partial^2 n}{\partial h^2} + \frac{3}{2H}\frac{\partial n}{\partial h} + \frac{n}{2H^2}\right),$$

(Ferraro 1945). The first term in this equation is characteristic of diffusion formulae. The second and third arise from the effect of gravity and the height dependence of D.

22. The equation of diffusion for the F2 region

We shall ignore for the present the geomagnetic field; we shall also assume that we are considering a locality on the equator so that $\chi = \varphi$, the time in radians measured from sunrise. It is related to the actual time by the equation

$$(143) \qquad t = k\varphi,$$

where $k = 1.37 \times 10^4 \sec^{-1}$. The rate of production of electrons q will be taken to be represented by the Champan function

$$(144) \qquad q = q_0 \exp\left[1 - \frac{h-h_0}{H} - (\operatorname{cosec}\varphi)\, e^{-\frac{h-h_0}{H}}\right] \quad 0 \leq \varphi \leq \pi,$$

$= 0 \qquad\qquad \pi \leq \varphi \leq 2\pi$

where h_o refers to the level of maximum rate of production of ions q_o. Electrons are assumed lost in the F2 region according to an attachment-type law (119) with a coefficient which decreases at a rate proportional to the density of the neutral particles. Thus

(145) $$L = Kn$$

where

$$K = \beta_o \exp\left(-\frac{h-h_o}{H}\right)$$

Then the equation of diffusion to be solved is

(146) $$\frac{\partial n}{\partial t} = q - \beta_o n \exp\left(-\frac{h-h_o}{H}\right) + D\left(\frac{\partial^2 n}{\partial h^2} + \frac{3}{2H}\frac{\partial n}{\partial h} + \frac{n}{2H^2}\right)$$

(147) where $D = b/n_n$ and $n_n = N_o \exp\left(-\frac{h-h_o}{H}\right)$

Since $D = (b/n_n)$ increases exponentially upwards, whereas both q and L decrease exponentially with height, it follows that at some level the diffusion term will be the dominant term in equations (146) which then reduces to

(148) $$\frac{\partial^2 n}{\partial h^2} + \frac{3}{2H}\frac{\partial n}{\partial h} + \frac{n}{2H^2} = 0 \, .$$

Solving this equation we find

(149) $$n = A_1 e^{-\frac{h}{2H}} + A_2 e^{-\frac{h}{H}}$$

and from equation (141) it follows that the first term corresponds to diffusive equilibrium, with $v_D = 0$, in which the ionization has a scale height twice that of the neutral gas. For the second term $v_D \neq 0$, and is positive if $A_2 > 0$. This represents a boundary condition of a <u>finite</u> flux of ionization at $h \approx \infty$, which is upwards if $A_2 > 0$. If ionization is gained or lost at the top of the ionosphere, then a term of this type must be included in the solution.

23. <u>Solution of the diffusion equation</u>

Writing

$$x = \frac{h - h_o}{H} \tag{150}$$

and expressing t in terms of φ by (143), equation (146) can be written in the non-dimensional form

$$\frac{\partial n}{\partial \varphi} = kq - \beta e^{-z} n + \frac{e^{z}}{\gamma}\left(\frac{\partial^2 n}{\partial z^2} + \frac{3}{2}\frac{\partial n}{\partial z} + \frac{n}{2}\right) \tag{151}$$

where

$$\beta = k\beta_o \qquad \text{and} \qquad \gamma = \frac{N_o H^2}{Kb} \tag{152}$$

are non-dimensional parameters. The solution of (151) thus depends only on these two parameters and the boundary conditions. One of these, as we have already seen above, depends on the value of the flux of ionization at $z = +\infty$. The other requires that $n \to 0$ as $z \to -\infty$.

Assuming that q is given by the Chapman function (144) , we require a solution of (151) which is periodic in φ, with period 2π. The method of solution of this, and related, equations has been given by Gliddon (1959) and consists in determining a suitable Green's function for this equation. The analysis is involved and reference must be made to the original papers . Typical solutions indicating the variation of the ionization at various heights from the level of maximum ion production are illustrated in Figs. 10 a and 10 b . Fig. 11 illustrates the variation of the level of maximum electron density for two distinct cases. From such solutions, the following general daytime characteristics of the behaviour of the ionization may be deduced.

(i) The F2 maximum electron density occurs at a level where diffusion and loss are comparable , i.e., where $K_m \simeq D_m/H^2$ and the subscript m refers to the maximum.

(ii) At the maximum and below it, the electron density is approximately given by $n \simeq q/K$, that is , balance between production and loss, as in the absence of diffusion.

(iii) Well above the maximum, diffusion becomes important and n varies as $e^{-(1/2)z}$ as already noted in section 22.

(iv) The level of maximum ionization falls rapidly at sunrise because of the rapid production of ionization in the lower F regions. It reaches a minimum height of one or one and a half scale heights above the level h_o of maximum ion-production which remains at about the same level until late afternoon. Therefore, the level rises again steadily to a height of about three scale heights above h_o after sunset.(See Fig. 11)

The night-time decay of the ionization has been studied by Martyn (1956) , Duncan (1956) and Dungey (1956) . This is also affected by vertical electromagnetic drifts .

24. Effect of a magnetic field on diffusion in the ionosphere

We now consider the effect of a magnetic field on diffusion of ions in the ionosphere. We are here dealing with a thernary mixture of neutral molecules, ions and electrons, of which the ionized particles form a minor

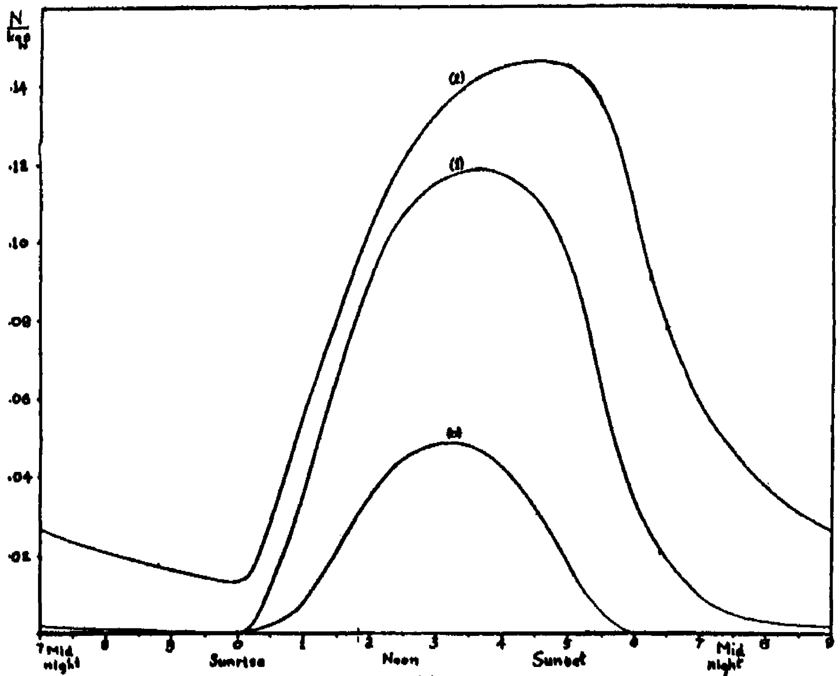

Fig. 10 a

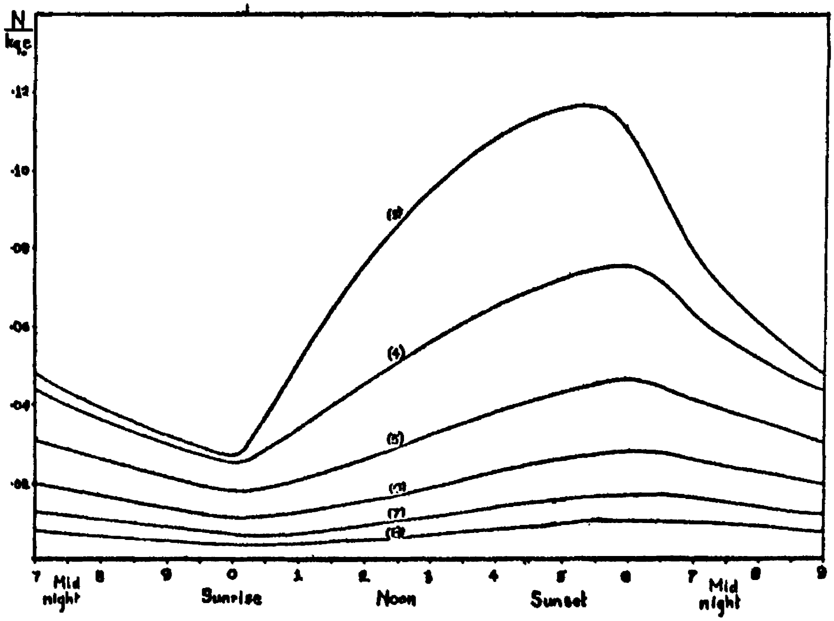

Fig.10 b Diurnal variation of electron density at intervals of one scale height. Case II ($\beta = 10$, $\gamma = 20$)

below (2.7.22) :

(2.7.24) $\quad f_n(r, t) = P_n^{(\nu-1/2,\mu-1/2)}(t)\,(kr)^{-\mu-\nu} J_{\mu+\nu+2n}(kr), n = 0, 1, \cdots,$

where $t = \cos 2\theta$, $r^2 = x^2 + y^2$ and $P_n^{(\alpha,\beta)}$ stands for the Jacobi Polynomials (see [16]) :

$$P_n^{(\nu-1/2,\mu-1/2)}(t)\,(kr)^{-\mu-\nu} J_{\mu+\nu+2n}(kr) \equiv f_n(r,t) =$$

$$= \Gamma(\nu+n+1/2)\left[\Gamma(\nu)\Gamma(1/2)\Gamma(n+1)\right]^{-1}\int_0^{\pi}\left\{(k\sigma)^{-\mu-\nu} J_{\mu+\nu+2n}(k\sigma)\right\} \cdot$$

$$\cdot\left\{(\sigma/x)^{\mu}\,\Phi_2(\mu, 1-\mu, \nu; \xi^1 - \eta^1)(\sin\Phi)^{2\nu-1}\, d\Phi\right\},$$

$$\sigma = x + iy\cos\Phi, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \xi^1 = -y^2\sin^2\Phi/(4x\sigma),$$

$$\eta^1 = -k^2y^2\sin^2\Phi/4, \quad k, \mu, \nu, > 0, \eta = 0, 1, 2, \cdots.$$

An arbitrary solution of the class S may be represented in a series form [39] :

$$w(r, \theta) = (kr)^{-\mu-\nu}\sum_{n=0}^{\infty} a_{2n}\; n!\left[\Gamma(n+\nu+1/2)\right]^{-1}.$$

(2.7.25) $$\cdot P_n^{(\nu-1/2,\mu-1/2)}(\cos 2\theta)\, J_{\mu+\nu+2n}(kr),$$

and an even analytic function regular about the origin may be expressed as

(2.7.26) $$f(\sigma) = \sigma^{-\mu-\nu}\sum_0^{\infty} a_{2n}\, J_{\mu+\nu+2n}(\sigma).$$

Hence for r sufficiently small it follows that the class of analytic functions (2.7.26) is mapped onto the class of solutions (2.7.25) by an operator of the form :

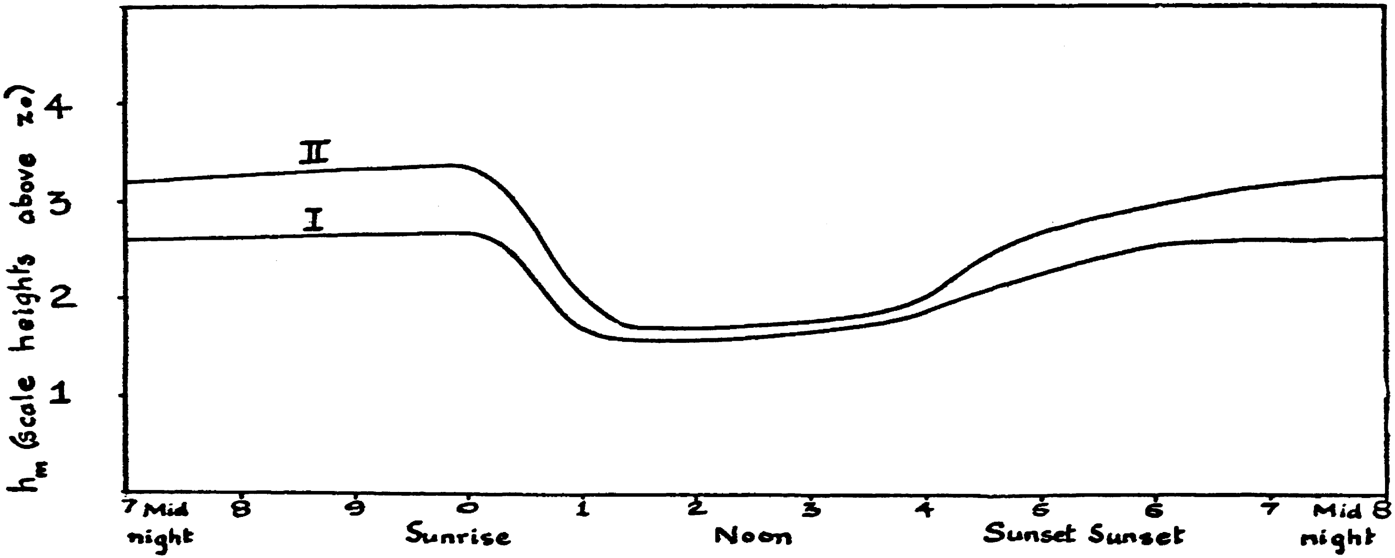

Fig. 11 . Comparison of height of maximum electron density for two distinct cases : case I for , case II for.

$$\int_{-1}^{+1}(1-\xi)^{\nu-1/2}(1+\xi)^{\mu+1/2}P_n^{(\nu-1/2,\mu-1/2)}(\xi)\,P_m^{(\nu-1/2,\mu-1/2)}(\xi)\,d\xi$$

$$=\delta_{nm}\;2^{\mu+\nu}\Gamma(n+\nu+1/2)\Gamma(n+\mu+1/2)\,\cdot$$

$$\cdot\left[(2n+\mu+\nu)\Gamma(n+1)\Gamma(n+\nu+\mu)\right]^{-1}. \tag{2.7.30}$$

Thus, if we define :

$$K_n(\sigma,\ r,\ \xi)\quad 2^{-(\mu+\nu)}b_n(kr/k\sigma)^{\mu+\nu}J_{\mu+\nu+2n}(k\sigma)\,(J_{\mu+\nu+2n}(kr))^{-1}.$$

$$\cdot\,P_n^{(\nu-1/2,\mu-1/2)}(\xi)\,(1-\xi)^{\nu-1/2}(1+\xi)^{\mu-1/2}, \tag{2.7.31}$$

where

$$b_n=(2n+\mu+\nu)\,\Gamma(n+\mu+\nu)\;\;(\Gamma(n+\mu+1/2))^{-1},$$

we have

$$a_{2n}(k\sigma)^{-(\mu+\nu)}J_{\mu+\nu+2n}(k\sigma)\;=\int_{1}^{+1}K_n\,(\sigma,\ r,\ \xi)\,.$$

$$\cdot\,u\,(\,r((\,1+\xi\,)/2)^{1/2},\ r\,(\,(1-\xi\,)/2)^{1/2})\,d\xi\,. \tag{2.7.32}$$

Hence

$$f(k\sigma)=\int_{-1}^{+1}K\,(\sigma,\ r,\ \xi\,)\,u\left[\,r((1+\xi)/2)^{1/2},\ \ r((1-\xi)/2)^{1/2}\ \ d\xi\,, \right. \tag{2.7.33}$$

where

$$K(\sigma,r,\ \xi\,)=2^{-(\mu+\nu)}(r/\sigma\,)^{\mu+\nu}(1-\xi\,)^{\nu-1/2}\,(1+\xi\,)^{\mu-1/2}.$$

$$\cdot\;\sum_{0}^{\infty}(2n+\mu+\nu\,)\Gamma\,(n+\nu+\mu)\;\;(\Gamma(n+\mu+\nu))^{-1}$$

$$J_{\mu+\nu+2n}\;(k\sigma)\left[\,J_{\mu+\nu+2n}(kr)\right]^{-1}\,P_n^{(\nu-1/2,\mu-1/2)}(\,\xi\,)\,.$$

To verify that these formal calculations are justified and that K constituent. The relevant equations of the problem have been given in section 15, namely equations (92-94). However, for much the same reason as stated in section 21, we may neglect the acceleration of the ions and electrons, and assume that the neutral air is at rest. It should be stated, however, that Yonezawa (1958) and Dougherty (1960) have suggested the possibility that the neutral atoms and molecules are accelerated by the flow of the ions through the neutral air, in the comparatively short time of 20 minutes. Nevertheless we shall restrict ourselves to the case when $v_n \simeq 0$; Again quite apart from the fact that the collision frequencies for encounter between ions and electrons exceed those for collisions of the ions or electrons with the neutral atoms, the differential velocity $\underline{v}_i - \underline{v}_e$ between the ions and electrons must remain small otherwise large electric fields would develop because of the consequent large separation of charges of opposite sign. Likewise, we must have, very nearly, overall charge neutrality, so that $n_i \simeq n_e$. Assuming also that the ion and electron temperatures are equal , equations (92) and (93) then reduce to

$$(153) \qquad -\nabla P_i + ne(\underline{E} + \underline{v}_i \times \underline{B}/c) + \rho_i \underline{g} - \rho_i \nu_{in} \underline{v}_i = 0$$

$$(154) \qquad -\nabla P_e - ne(\underline{E} + \underline{v}_e \times \underline{B}/c) + \rho_e \underline{g} - \rho_e \nu_{en} \underline{v}_e = 0$$

where n is the number density of the ions or electrons, $\underline{B}$ the magnetic field intensity, ν_{in} and ν_{en} the collision frequencies for ion-neutral atom and electron-neutral atom encounters. Again, in the F2 region, $m_i \nu_{in} \gg m_e \nu_{en}$ so that collision with neutral particles are important for the ions but not for electrons. In fact, we may neglect this term and also the weight of the electrons in (146) because of the small mass of the electrons. This now becomes approximately,

$$(155) \qquad -\nabla P_e - ne(\underline{E} + \underline{v}_e \times \underline{B}/c) = 0 .$$

Also $P_e = knT(\simeq P_i)$ and if, as is legitimate, T is assumed constant, this equation can be written

$$(156) \qquad kT\nabla \log n + e(\underline{E} + \underline{v}_e \times \underline{B}/c) = 0 \quad .$$

Hence ,

$$(157) \qquad \text{curl } \underline{E} + \text{curl } (\underline{v}_e \times \underline{B}/c) = 0$$

or, using Maxwell's equation, curl $E = -\dfrac{1}{c}\dfrac{\partial \underline{B}}{\partial t}$, this becomes

$$(158) \qquad \frac{\partial \underline{B}}{\partial t} = \text{curl } (\underline{v}_e \times \underline{B})$$

showing that the magnetic field lines are frozen in the electron gas. Since the magnetic pressure of the geomagnetic field greatly exceeds the electron gas pressure in the F2 region, the electrons can only move freely along the magnetic lines of force, but not at right angles to them. Thus $\underline{v}_e$ is parallel to $\underline{B}$ and (148) now gives the electric field as

$$(159) \qquad \underline{E} = -\frac{kT}{e}\nabla \log n$$

or

$$(160) \qquad n \propto e^{-eV/kT}$$

where V is the electrostatic potential. This implies that the electron density attains its thermodynamic equilibrium a each instant . Substituting (159) in (153) now gives an equation to determine the velocity of diffusion of the ions, namely,

$$(161) \qquad -2kT\nabla \log n + e\underline{v}_i \times \underline{B}/c + m_i\underline{g} - m_i \nu_{in} \underline{v}_i = 0.$$

The solution of this vector equation is

(162) $$\underline{v}_i(1 + \Omega^2) = \underline{C} + (\underline{\Omega} \cdot \underline{C})\underline{\Omega} + \underline{C} \times \underline{\Omega}$$

where

(163) $$\underline{\Omega} = \frac{e\underline{B}}{m_i c \nu_{in}}$$

and

$$\underline{C} = \frac{1}{m_i \nu_{in}}(-2kT\nabla \log n + m_i\underline{g}) = D(\nabla \log n + \frac{1}{2H}\underline{k}) ,$$

$\underline{k}$ being a unit vector along the downward vertical, and D the coefficient of diffusion defined earlier, In general , $\nabla \log n$ will also be a vector directed along the vertical so that $\underline{C}$ is vertical.

Equation (162) shows that the velocity of diffusion of the ions has three components, one along $\underline{C}$ and the other two along and perpendicular to the magnetic field. Again, in the F2-region , $|\Omega|$ is large, being of the order of 200 at a level of 300 km. Hence, except over the magnetic equator where $\underline{C}$ and $\underline{\Omega}$ are in general perpendicular, the largest component of the diffusion velocity is along the magnetic lines of force and the smallest, that along $\underline{C}$. Hence, very nearly, we may write

(164) $$\underline{v}_i = (\underline{b} . \underline{C})\underline{b} + \frac{1}{\Omega}\underline{C} \times \underline{b}$$

where $\underline{b}$ is a unit vector along $\underline{B}$ (or $\underline{\Omega}$) . This result is easily interpreted; a large value of Ω implies that the ions can spiral around a line of force many times between collisions and hence its velocity is along the line of force except when changed abruptly by a collision. The net results of such collisions is to give rise to the 'Hall' component of the velocity of diffusion given by the second term in (164) .

The equation of diffusion can now be obtained by using the equation of continuity for the ions

(165) $$\frac{\partial n}{\partial t} = q - L - \operatorname{div}\,(n\,\underline{v}_i)$$

and substituting for $\underline{v}_i$ the expression given in (164).

For a plane stratified atmosphere, I showed in 1945 that the diffusion term in (165) can be expressed in the approximate form $D\mathcal{D}n$, where

(166) $$\mathcal{D}n = D(\sin^2 I)\left(\frac{\partial^2 n}{\partial h^2} + \frac{3}{2H}\frac{\partial n}{\partial h} + \frac{n}{2H^2}\right) ,$$

I is the inclination of the lines of force above the horizontal (magnetic dip angle) and h the vertical height; (168) is the same as equation (142) except for the extra factor $\sin^2 I$. This approach is incorrect if the neutral air is in motion, as may well be the case. Near the magnetic equator the 'correction factor' $\sin^2 I$ fails to give the correct result, and a more careful derivation of the diffusion operator D is required. This was derived by Kendall (1962) and by Lyon (1963). Assuming that the geomagnetic field is that of a centred dipole, Kendall found that one form of the operator is given by

$$\mathcal{D}'n = \frac{\sin^2 I}{2H^2}n + \frac{3\cot\theta}{aH(1+3\cos^2\theta)}\frac{\partial n}{\partial\theta} + \frac{1}{a^2\sin^2\theta(1+3\cos^2\theta)}\frac{\partial^2 n}{\partial\theta^2}$$

$$+ \left(\frac{15\cos^4\theta + 10\cos^2\theta - 1}{\sin^2\theta(1+3\cos^2\theta)^2}\right) n ,$$

where θ is the magnetic colatitude and a the distance of the highest point from the centre of the earth.

The geomagnetic field in effect reduces the coefficient of diffusion from D to $D\sin^2 I$ and this diminishes the velocity of diffusion in the ration $1 : \sin^2 I$. The correction is small, amounting to $\frac{1}{2}$ for $\theta = 45^o$; but near the equator, where I is small , vertical diffusion is clearly negligible.

One final remark needs to be made; although we have been able to determine the velocity of the ions, that of the electrons cannot be determined. However, as we have already mentioned, the velocities of the ions and electrons at any one point cannot differ greatly, and their components along a magnetic line of force must be very nearly equal.

25. Diffusive equilibrium in the upper ionosphere

The equations of diffusion of ions and electrons , (132) and (133), are also useful in discussing the equilibrium of charged particles in the topside F region. At these heights photochemical processes are negligible and the collision frequencies ν_{in}, ν_{en} so small that they can also be neglected. Assuming also that the ions are stratified horizontally, that they are singly ionized and all at the same temperature T_i, the equatior of equilibrium for the jth species of ions and electrons are respectively

$$\frac{d(n_j kT_i)}{dh} = -n_j m_j g + n_j eE \qquad (j = 1, 2, \ldots) \tag{167}$$

$$\frac{d(n_e kT_e)}{dh} = -n_e m_e g - n_e eE \tag{168}$$

where E is the elctric field and T the electron temperature. Since there must be very nearly overall charge neutrality we must have further

(169) $$\sum_j n_j = n_e$$

so that adding the equations (167) and (168) we can eliminate the electric field . Defining the mean ionic mass m_+ by the equation

(170) $$\sum_j n_j m_j = m_+ n_e$$

and neglecting the electronic mass, we find after elimination of E that

(171) $$-\frac{d \log n_e}{dh} = \frac{m_+ g}{kT_2(1+\tau)}$$

where $\tau = T_e / T_i$. Substituting this equation in (168) determines E which when inserted in (167) gives the equation for the distribution of ions as

(172) $$-\frac{d \log n_j}{dh} = (m_j - \frac{m_+ \tau}{1+\tau}) \frac{g}{kT_i} \quad .$$

If there is only one species of ion present, and $T_e = T_i$ (so that $\tau = 1$) , the mean ionic mass is equal to m_i so that the effective scale height is twice that of the neutral atomic mass, as before. However, a light atom for which $m_j < \frac{\tau}{1+\tau} m_+$ actually has a negative scale height, so that n_j increases upwards , as first pointed out by Mange (1957) . The solution of the equation (171) and (172) has been effected by Hanson (1962) taking account of the variation of g with height. A typical equilibrium distribution computed for a mixture of 0^+ , H_e^+ , H^+ ions is shown in Fig. 12 . The reduced scale height z refers to the level z = 0 at which the ionic concentrations of 0^+ and H_e^+ are equal.

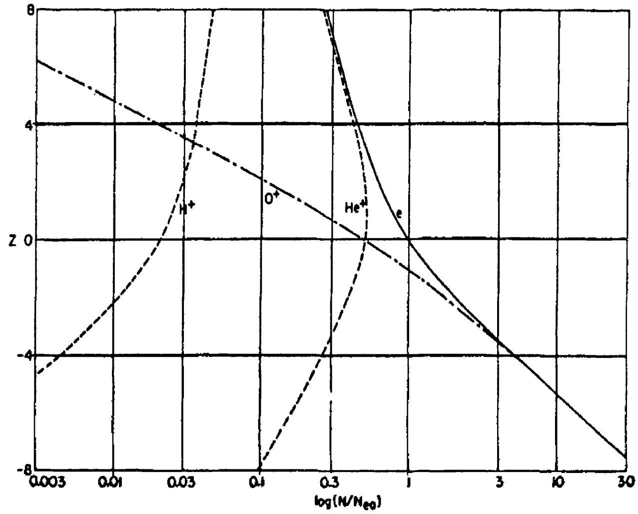

FIG. 12 IDEALIZED DISTRIBUTIONS OF ELECTRONS (e) AND OF O^+, He^+ and H^+ IONS, COMPUTED BY SOLVING EQS. (III-59), (III-60). Electron and ion concentrations are given in terms of the electron density N_{ea} at the level $z = 0$, at which height the ionic composition is taken to be $[O^+] = [He^+] = 49\%$, $[H^+] = 2\%$. The level at which the He^+ and H^+ concentrations are equal is near $z = 17$. The unit of reduced height z is the scale height of neutral atomic oxygen.

Fig. 12

$$f_1 = a^m \cdot f_2 = a^n ; \quad g = x^p ; \quad n + mp = 0 ; \quad p = - n m^{-1} ;$$

$$f_1 = \exp (ma) ; \quad f_2 = \exp(n a) ; \quad g = x^p ; \quad n+mp=0 ; \quad p = -n m^{-1} ;$$

(3.4.4) $f_1 = \exp (i ma) ; f_2 = \exp (i na) ; g=x^p ; n+mp = 0 ; \quad p = -n m^{-1} ;$

where in the last proposition only the real parts Re $\{ f_i \}$ should be taken into account. One may construct easily more complicated functions f_i ($i = 1, 2$) and g :

(3.4.5) $\bar{x} = ma + x ; \bar{y} = y \exp (na) ; \bar{y} = y \exp (p\bar{x}) = y \exp (px)$,

which results in $p = - n/m$, or :

(3.4.6) $\bar{x} = ma + x ; \bar{y} = y \exp (a \exp n) ; \bar{y} = y \exp (p\bar{x}) = y \exp (px)$,

which results in $p = - m^{-1} \exp n$. The absolute invariants of the group are :

(3.4.7) $\eta = y x^p ; \qquad \eta = y \exp (px)$.

In a similar way we deal with the dependent variables; thus , as an example one may choose:

(3.4.8) $\bar{y}_\delta \equiv \bar{k}_\delta = f_3(a) y \equiv f_3 k_\delta$.

For illustrative purposes assume a function $g(x)$ equal to x^r, say . Then it should be :

(3.4.9) $k_\delta x^r = \bar{k}_\delta \bar{x}_\delta^r \qquad f_3 f_1^r = 1$, etc

The procedure is identical with the one, explained above. The absolute invariants of the group (3.4.8) are :

(3.4.10) $g_\delta = k_\delta x^r$.

Hence by virtue of equation :

V. C. A. Ferraro

ATMOSPHERIC DYNAMO

26. Introduction

Balfour Stewart in 1882 first put forward the hypothesis that the daily variations of the earth's magnetic field could be ascribed to electric currents induced in conducting regions of the upper atmosphere by its motion across the geomagnetic field. The motion of the atmosphere was attributed to tidal forces due to the sun and moon. The theory, generally known as the dynamo theory, was developed mathematically by Schuster in 1908 and later by Chapman. However, their theory encountered certain difficulties connected with the reduction of the electrical conductivity, by the geomagnetic field, the required value appearing to the too large to be reconciled with theoretical values of the estimates of the tidal motion. The difficulty was eventually resolved by Martyn and Hirono, almost simultaneously, who showed that because of the horizontal and vertical variations of the inducting polarization electric charges are set up which tend to restore the full electrical conductivity. We shall begin by calculating the electrical conductivity in a highly ionized gas .

27. The electrical conductivities

These were derived in formal manner by Baker and Martyn (1952-3) and reviewed by Chapman (1956) ; we shall denote, as before, all quantities referring to ions, electrons and neutral particles by suffixes i, e, n. Also the region of the ionosphere in which the dynamo currents flow are now known to lie at a height of about 110 km, that is, in the E-region . In this region, the number of ions and electrons is about 10^5 per cc. Thus, in the equations (92)-(94) we have $\rho_n \gg \rho_i \gg \rho_e$ so that

$\theta_{ie} \simeq \rho_e \nu_{ie}$, $\theta_{ie} \simeq \rho_i \nu_{in}$, $\theta_{en} \simeq \rho_e \nu_{en}$ and these equations become approximately

$$\rho_i \frac{d\underline{v}_i}{dt} = -\nabla p_i + n_i e(\underline{E} + \underline{v}_i \times \underline{B}/c) + \rho_i \underline{F} - \rho_e \nu_{ie}(\underline{v}_i - \underline{v}_e) - \rho_i \nu_{in}(\underline{v}_i - \underline{v}_n) \tag{173}$$

$$\rho_e \frac{d\underline{v}_e}{dt} = -\nabla p_e - n_e e(\underline{E} + \underline{v}_e \times \underline{B}/c) + \rho_e \underline{F} - \rho_e \nu_{ie}(\underline{v}_e - \underline{v}_i) - \rho_e \nu_{en}(\underline{v}_e - \underline{v}_n) \tag{174}$$

$$\rho_n \frac{d\underline{v}_n}{dt} = -\nabla p_n + \rho_n \underline{F}_n - \rho_i \nu_{in}(\underline{v}_n - \underline{v}_i) - \rho_e \nu_{en}(\underline{v}_n - \underline{v}_e) \, . \tag{175}$$

In the dynamo region it is legitimate to neglect the acceleration of the particles; it is also convenient to introduce the mean mass velocity $\underline{v}_o$ and the electric current density $\underline{j}$ as new variables, where

$$\rho_o \underline{v}_o = \rho_i \underline{v}_i + \rho_e \underline{v}_e + \rho_n \underline{v}_n,$$

$$\rho_o = \rho_i + \rho_e + \rho_n \tag{176}$$

$$\underline{j} = \frac{e}{c}(n_i \underline{v}_i - n_e \underline{v}_e) \, ,$$

where ρ_o is the total mass density. The variables are thus $\underline{v}_o$, $\underline{v}_n$ and $\underline{j}$; assuming that $n_i \simeq n_e = n$ and neglecting the ratio m_e/m_i, we find approximately

$$\underline{v}_i = \underline{v}_o + \frac{n_n}{n}(\underline{v}_o - \underline{v}_n) + \frac{m_e}{m_i}\frac{c}{ne}\underline{j}$$

$$\underline{v}_e = \underline{v}_o + \frac{n_n}{n}(\underline{v}_o - \underline{v}_n) - \frac{c}{ne}\underline{j} \, . \tag{177}$$

From (173) (175) we then find the approximate equation of mass equilibrium

$$(178) \qquad 0 = -\nabla p + \underline{j} \times \underline{B} + \rho_i \underline{F} + m_e(\nu_{en} - \nu_{in})\frac{c}{e}\underline{j} - (\nu_{in} + \nu_{en}\frac{me}{mi})\rho_o(\underline{v}_o - \underline{v}_n)$$

whilst it can be shown that the approximate equation for the electric current density $\underline{j}$ is

$$(179) \qquad 0 = -ne\underline{E}_o + \underline{j} \times \underline{B} + \frac{mec}{e}(\nu_{ie} + \nu_{en})\underline{j} - \frac{e}{c}\left(\frac{-\nabla p + \underline{j} \times \underline{B}}{m_i \nu_{in}}\right) \times \underline{B}$$

where we have taken $m_n \simeq m_i$ and $m_i \nu_{in} \gg m_e \nu_{en}$. Here

$$(180) \qquad \underline{E}_o = \underline{E}' + \underline{E}''$$

where $\underline{E}' = \underline{E} + \underline{v}_o \times \underline{B}/c$ is the electric field following the mean motion and $E'' = (\nabla p_e)/n_e$, the 'equivalent electric field produced by the electron pressure gradient. Equation (179) can be solved for $\underline{j}$ as a linear vector function of $\underline{E}_o$ and ∇p. Writing

$$(181) \qquad \underline{\omega}_e = \frac{e\underline{B}}{mc}, \qquad \underline{\omega}_i = \frac{e\underline{B}}{m_e c}, \qquad \underline{B} = B\,\underline{b}$$

so that $\underline{\omega}_i$ and $\underline{\omega}_e$ are the ion and electron cyclotron frequencies an $\underline{b}$ is a unit vector along $\underline{B}$, and defining the conductivity

$$(182) \qquad \sigma = \frac{ne^2}{m_e \nu_e}, \qquad \nu_e = \nu_{ie} + \nu_{en}$$

we find

$$(A^2+D^2)\underline{j} = A\sigma\underline{E}_o + \sigma(AC+D^2)(\underline{E}_o.\underline{b})\underline{b} + \sigma D E_o \times \underline{b} -$$

$$(183) \qquad AC\frac{\nabla p}{B} \times \underline{b} + CD\left(\frac{\nabla p}{B} \times \underline{b}\right) \times \underline{b}$$

where

(184) $$A = 1 + C, \qquad C = \frac{\omega_e \omega_i}{\nu_e \nu_{in}}, \qquad D = \frac{\omega_e}{\nu_e}.$$

In the case when $\nabla p = 0$, (173) can be simplified. In fact, if we write

(185) $$\underline{E}_o = \underline{E}_{11} + \underline{E}_{\perp}$$

where E_{11} and $E_{\perp}$ are the components of the total electric field $\underline{E}_o$ parallel and perpendicular to $\underline{B}$, we find

(186) $$\underline{E}_{11} = (\underline{E}_o . \underline{b})\underline{b}, \qquad \underline{E} = \underline{b} \times (\underline{E}_o \times \underline{b})$$

so that (183) can now be written

(187) $$\underline{j} = \sigma_0 (\underline{E}_o . \underline{b})\underline{b} + \sigma_1 \underline{b} \times (\underline{E}_o \times \underline{b}) + \sigma_2 \underline{b} \times \underline{E}_o .$$

Here σ_o, σ_1, σ_2 are respectively the direct transverse and Hall conductivities and it is easily verified that

(188) $$\sigma_o = \sigma$$

(189) $$\sigma_1 = \frac{A}{A^2 + D^2} \sigma$$

(190) $$\sigma_2 = \frac{D}{A^2 + D^2} \sigma$$

where σ is given by (182). These formulae agree with those found by Chapman by a somewhat different approach. If $B = 0$, then $C = D = 0$ and $\sigma_1 = \sigma_o$ and $\sigma_2 = 0$ as should be the case. Equation (187) can be then written.

(191) $$\underline{j} = \sigma_o \underline{E}_o$$

the usual form of Ohm's Law. If $B \neq 0$, then (187) shows that the electrical conductivity is anisotropic, and if we take cartesian axes $0(xyz)$ with $0x$ along $\underline{b} \times \underline{E}_0$, $0y$ along $\underline{E}$ and $0z$ along $\underline{E}_{11}$ we may wirte (187) in the form

$$\underline{j} = \sigma \cdot \underline{E}_0 \tag{192}$$

where σ is the second order tensor

$$\begin{bmatrix} \sigma_1 & -\sigma_2 & 0 \\ \sigma_2 & \sigma_1 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \tag{193}$$

(the right hand side of (192) denoting the contracted product of σ and $\underline{E}_0$) . Formulae (189), (190) show that the transverse and Hall conductivities are reduced respectively in the ratio

$$\frac{A}{A^2 + D^2}, \qquad \frac{D}{A^2 + D^2} \tag{194}$$

If the magnetic field is large, A and D will be large , the reduction of the electrical conductivities σ_1 and σ_2 will also be large, and the electric currents will flow mainly along the lines of force.

Let us suppose that an electric field is set up which prevents any further flow of the Hall current. Such a field must be in the direction of $\underline{b} \times \underline{E}_0$ so that the total electric field is now $\underline{E}_0 + \lambda \underline{b} \times \underline{E}_0$, where λ is a constant. Substituting in (187) we find that the Hall current (parallel to $\underline{b} \times \underline{E}_o$) vanishes provided $\lambda = \sigma_2 / \sigma_1$, and that

$$\underline{j} = \sigma_o (\underline{E}_o . \underline{b})\underline{b} + \sigma_3 \underline{b} \times (\underline{E}_o \times \underline{b}) \tag{195}$$

where

$$\sigma_3 = \sigma_1 + \frac{\sigma_2^2}{\sigma_1} \tag{196}$$

is called the Cowling conductivity. In a fully ionized gas, $\sigma_3 = \sigma_o$, that is, the conductivity is the same as in the absence of a magnetic field. In a partially ionized gas; however, although σ_3 exceeds both σ_1 and σ_2, it is in general smaller than σ_o.

28. Numerical Illustrations

A review of the electrical conductivities in the ionosphere has been given by Chapman . Values were given for a hot (1500 °K) and a cool (850 °K) ionosphere and in the table below the values of ν_{ie}, ν_{in}, ν_{em} are taken from a corresponding table in Chapman's paper. In the table below we also give numerical values of the conductivities for the higher temperature. The conducting σ_o increases upwards over the range considered (100- 300 km) and the conductivities $\sigma_1, \sigma_2, \sigma_3$ each have one peak in this range.

29. Effective conductivities in the ionosphere

Baker and Martyn have shown that because of the limited vertical extent of the conducting layer of the E-region , the flow of current is nearly horizontal. In fact, if $\underline{j}$ contains a vertical compnnent, charges will accummulate on the boundaries of the layer because this current cannot flow in the region of low conductivity. These charges are called polarisation

Height in Km

	90	100	125	150	175	200	250	300
ν_{ie}	1.43×10^{2}	9.35×10^{2}	7.63×10^{2}	6.66×10^{2}	4.84×10^{2}	3.74×10^{2}	3.08×10^{2}	2.68×10^{2}
ν_{en}	6.83×10^{5}	1.61×10^{5}	1.83×10^{4}	2.81×10^{3}	8.80×10^{2}	3.57×10^{2}	4.50×10	3.74×10
ν_{in}	4.12×10^{4}	8.59×10^{3}	6.16×10^{2}	1.12×10^{2}	3.20×10	1.21×10	2.90	1.08
w_e/ν_e	1.26×10	5.32×10	5.92×10^{2}	2.48×10^{3}	6.33×10^{3}	1.18×10^{4}	2.14×10^{4}	2.82×10^{4}
w_i/ν_i	3.92×10^{-3}	1.88×10^{-2}	.282	1.67	6.10	1.69×10	7.69×10	2.29×10^{2}
σ_o	4.12×10^{-15}	1.74×10^{-13}	2.91×10^{-12}	1.62×10^{-11}	4.14×10^{-11}	7.72×10^{-11}	1.76×10^{-10}	2.79×10^{-10}
σ_1	2.70×10^{-17}	1.23×10^{-16}	1.29×10^{-15}	2.88×10^{-15}	1.04×10^{-15}	3.87×10^{-16}	1.07×10^{-16}	4.31×10^{-17}
σ_2	3.25×10^{-16}	3.27×10^{-15}	4.54×10^{-15}	1.72×10^{-15}	1.71×10^{-16}	2.29×10^{-17}	1.38×10^{-18}	1.87×10^{-19}
σ_3	3.93×10^{-15}	8.70×10^{-14}	1.73×10^{-14}	3.91×10^{-15}	1.07×10^{-15}	3.88×10^{-16}	1.07×10^{-16}	4.31×10^{-17}

Table I

Collision frequencies, electric conductivities, and the ratio of cyclotron to collision frequencies for a model ionosphere at a temperature of 1480 ^{o}K .

charges. Because the flow is horizontal we can replace the 3×3 tensor σ by a 2×2 tensor σ^* representing the layer conductivity whose components depend on the magnetic dip angle I. It will be convenient to use coordinate x, y for the magnetic southward and eastward directions. Then we can write

$$\sigma^* = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{bmatrix} \tag{197}$$

where

$$\sigma_{xx} = \frac{\sigma_0 \sigma_1}{\sigma_0 \sin^2 I + \sigma_1 \cos^2 I} \simeq \frac{\sigma_1}{\sin^2 I}$$

$$\sigma_{xy} = \frac{\sigma_0 \sigma_2 \sin I}{\sigma_0 \sin^2 I + \sigma_1 \cos^2 I} \simeq \frac{\sigma_2}{\sin I} \tag{198}$$

$$\sigma_{yy} = \frac{\sigma_2^2 \cos^2 I}{\sigma_0 \sin^2 I + \sigma_1 \cos^2 I} + \sigma_1 \simeq \sigma_1$$

The approximations given on the right arise generally since $\sigma_0 \gg \sigma_1$ or σ_2, but they are not valid near the magnetic equator where I=0. Here we have

$$\sigma_{xx} = \sigma_0, \quad \sigma_{xy} = 0, \quad \sigma_{yy} = \sigma_1 + \frac{\sigma_2^2}{\sigma_1} = \sigma_3 \,. \tag{199}$$

Two consequences follow immediately from (199) ; firstly , the high conductivity σ_0 along the magnetic lines of force ensures that these are very nearly electric equipotentials . Secondly, the east-west conductivity σ_{yy} at the equator is very large, being the Cowling conductivity σ_3

which is comparable with, though smaller than, σ_0. This highly conducting strip along the magnetic equator carries a large current, known as the equatorial electrojet and is confined to a belt a few degrees in widths about the magnetic equator, where $\sigma_0 \sin^2 I \ll \sigma_1 \cos^2 I$. Outside these belts the electric current falls rapidly. The simple model of the dynamo region is therefore a relatively horizontally stratified layer. The magnetic variations observed at the ground which are produced by these currents are best calculated by considering the layer as a current-sheet, with integrated conductivities,

$$\Sigma_1 = \int \sigma_1 \, dh, \qquad \Sigma_2 = \int \sigma_2 \, dh \quad ,$$

where the integrals are taken over the thickness of the horizontal layer. If Σ^* denotes the tensor $\int \sigma^* dh$, we can summarise the electrical equations as

$$\underline{J} = \Sigma^* \cdot \underline{E}_t \, , \qquad \underline{J} = \int \underline{j} \, dh \tag{200}$$

where

$$\underline{E}_t = \underline{w} \times \underline{B} - \nabla \varphi \, , \tag{201}$$

$\underline{w}$ being the velocity of the neutral air and φ the electrostatic potential. The first term in (201) represents the induced field, whilst the electrostatic field $-\nabla \varphi$ forces the electric currents to flow horizontally.

cannot be zero . If in the last equation of (3.6.2) , $f^4 = 0$, then $\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial u} = 0$ in which case $\left|\frac{\partial(\bar{x}, \bar{y}, \bar{u})}{\partial(x, y, u)}\right| = 0$, contrary to the original assumption. Therefore it is necessary that $f^4 \neq 0$.

Similarly, the subgroup :

$$(3.6.10) \qquad S_{A_1} : \bar{x} = f^1(x, y ; a) , \qquad \bar{y} = f^2(x, y ; a) ,$$

must have an inverse and therefore neither the Jacobian determinant associated with S_{A_1} :

$$(3.6.11) \qquad \left|\frac{\partial(\bar{x}, \bar{y})}{\partial(x, y)}\right| = \begin{vmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} \end{vmatrix} ,$$

nor the Jacobian determinant :

$$(3.6.12) \qquad \left|\frac{\partial(x, y)}{\partial(\bar{x}, \bar{y})}\right| = \begin{vmatrix} \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial \bar{y}} \\ \frac{\partial y}{\partial \bar{x}} & \frac{\partial y}{\partial \bar{y}} \end{vmatrix} ,$$

associated with the inverse transformation may be equal to zero . From (3.6.12) , it follows that not both $\frac{\partial x}{\partial \bar{y}}$ and $\frac{\partial y}{\partial \bar{y}}$ are equal to zero.. If $\frac{\partial u}{\partial x} \neq 0$, then from (3.6.8) :

$$(3.6.13) \qquad \frac{\partial x}{\partial \bar{y}} = 0 ,$$

and therefore

$$(3.6.14) \qquad \frac{\partial y}{\partial \bar{y}} \neq 0 .$$

Eqs. (3.6.6) , (3.6.7) , (3.6.8) and the above remarks imply that :

16. D.F. Martyn , Processes controlling ionization distribution in the F2 reginn, Aust J.Phys. , 9 , 161 -165 , (1956) .

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18. T. Yonezawa, A ŋew theory of the formation of the F2 layer, J. Radio Res. Lab. , 3, 1 - 16 , (1956).

19. T. Yonezawa, On the influence of electron-ion diffusion exerted upon the formation of the F2 layer, J. Radio Res. Lab., 5 , 165 - 187 (1958).

and

$$\text{(vi)} \qquad \rho C_v \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) +$$

$$+ p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) -$$

$$- 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] - \mu \left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \right.$$

$$+ \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 +$$

$$\left. + \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 \right] = 0 \quad \text{(energy)}, \tag{3.6.21}$$

where μ , R , C_v and k are constants

The above system of six partial differential equations in six unknown functions of x, y, z and t will be referred to as the system " A " . In order to find the conditions on a group A_1 such that " A " is conformally invariant under A_1 , it is necessary to find the system corresponding to " A " and A_1. A_1 is a transformation group defined as :

$$A_1 : \bar{x} = f^1(x, y, z; a) , \quad \bar{y} = f^2(x, y, z; a), \quad \bar{z} = f^3(x, y, z; a) ,$$
$$\bar{u} = f^4(u; a) \equiv f^5(u; a)\, u + f^6(a) .$$

Since no new techniques are involved in finding the system in question, this problem will be left to the reader. The present paper will confine its attention to one such group under which " A " is conformally invariant. It is the following group :

$$P^1 : \bar{x} = x + \gamma_1 a , \quad \bar{y} = y - \gamma_2 a , \quad z = \bar{z} + \gamma_3 a ,$$

$$\bar{t} = t - \gamma_4 a , \quad \bar{u} = u , \quad \bar{v} = v, \; \bar{w} = w ,$$