Algorithms for Trapezoidal Approximations of Fuzzy Numbers Preserving the Expected Interval

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Abstract. Fuzzy number approximation by trapezoidal fuzzy numbers which preserves the expected interval is discussed. New algorithms for calculating the proper approximations are proposed. It is shown that the adequate approximation operator is chosen with respect both to the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number.

Keywords: fuzzy numbers, approximation of fuzzy numbers, expected interval, expected value, width, ambiguity, value of a fuzzy number.

1 Introduction

Trapezoidal approximation of fuzzy numbers was considered by many authors (see, e.g. [1]– [4], [7], [14]– [18], [22]– [25]). In [16] a list of criteria which trapezoidal approximation operators should possess was formulated and a new approach to trapezoidal approximation that lead to, so-called, the nearest trapezoidal approximation operator preserving the expected interval was suggested. Then in [17] a corrected version of that operator was given but the ultimate shape of that approximation operator was presented in [4] and [23]. It appears that the form of the nearest trapezoidal approximation operator preserving the expected interval depends on the particular shape of a fuzzy number to be approximated. Actually, a given fuzzy number might be approximated by one of the four admissible approximation operators. Which one

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should be used depends on parameters that characterize the location and spread of a fuzzy number, i.e. on its value, weighted expected value, ambiguity and width, respectively. These conditions together with natural algorithms for computing the nearest trapezoidal approximation preserving the expected interval were given in [14].

One may ask why the preservation of the expected interval is so exposed. There are many reasons to do so starting from the important properties of the expected interval itself (see, e.g. [10,20]). But there are also some other interesting properties of fuzzy numbers which remain invariant under approximation provided the expected interval remains unchanged (see [16]).

In the present paper we suggest modified conditions for choosing the proper approximation operator. They are both simpler than discussed previously and they have more natural interpretation. One of the suggested algorithms indicates that the adequate approximation operator is chosen with respect both to the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number. Moreover, another mathematical formulae for the operators under discussion are proposed.

2 Concepts and Notations

Let A denote a *fuzzy number*, i.e. such fuzzy subset A of the real line $\mathbb{R}$ with membership function $\mu_A : \mathbb{R} \to [0,1]$ which is (see [9]):

- normal (i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$),
- fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$, $\forall x_1, x_2 \in \mathbb{R}$, $\forall \lambda \in [0,1]$),
- μ_A is upper semicontinuous,
- supp$A$ is bounded, where supp$A = cl(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and cl is the closure operator.

A space of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Moreover, let $A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\}$, $\alpha \in (0,1]$, denote an α-cut of a fuzzy number A. As it is known, every α-cut of a fuzzy number is a closed interval, i.e. $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\} \tag{1}$$

$$A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}. \tag{2}$$

The *expected interval* $EI(A)$ of a fuzzy number A is given by (see [10], [20])

$$EI(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \tag{3}$$

The middle point of the expected interval given by

$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right) \tag{4}$$

is called the *expected value* of a fuzzy number and it represents the typical value of the fuzzy number A (see [10], [20]). Sometimes its generalization, called *weighted expected value*, might be interesting. It is defined by

$$EV_q(A) = (1-q)\int_0^1 A_L(\alpha)d\alpha + q\int_0^1 A_U(\alpha)d\alpha, \tag{5}$$

where $q \in [0,1]$ (see [11]).

Another useful parameter characterizing the nonspecifity of a fuzzy number is called the *width* of a fuzzy number (see [6]) and is defined by

$$\begin{aligned} w(A) &= \int_{-\infty}^{\infty} \mu_A(x)dx \\ &= \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha. \end{aligned} \tag{6}$$

To simplify the representation of fuzzy numbers Delgado et al. [7] suggested two parameters – value and ambiguity – which represent some basic features of fuzzy numbers and hence they were called a canonical representation of fuzzy numbers. The first notion

$$Val(A) = \int_0^1 \alpha(A_L(\alpha) + A_U(\alpha))d\alpha \tag{7}$$

is called the *value* of fuzzy number A and might be seen as a point that corresponds to the typical value of the magnitude that the fuzzy number A represents. The next index, called the *ambiguity* is given by

$$Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha, \tag{8}$$

and it characterizes the global spread of the membership function and hence is a measure of vagueness of fuzzy number A.

For two arbitrary fuzzy numbers A and B with α-cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$d(A,B) = \sqrt{\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha} \tag{9}$$

is the distance between A and B (for more details we refer the reader to [11]). There are, of course, some other measures of the distance between fuzzy numbers (e.g. [5]), however (9) is not only very popular but it seems to be especially useful in relation with the expected interval (see [12]).

3 Trapezoidal Approximation

Suppose we want to substitute a fuzzy number A by a "suitable" trapezoidal fuzzy number $T(A)$, i.e. by a fuzzy number with linear sides and the membership function having the following form

$$\mu_{T(A)}(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \le x < t_2, \\ 1 & \text{if} \quad t_2 \le x \le t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 < x \le t_4, \\ 0 & \text{if} \quad t_4 < x. \end{cases} \tag{10}$$

A family of all trapezoidal fuzzy number will be denoted by $\mathbb{F}^T(\mathbb{R})$.

Here one may ask a natural question: *Why are we interested in trapezoidal approximation of fuzzy numbers?*

Besides the immediate answer that it is a quite interesting mathematical problem there are, of course, many practical reasons for such approximations. The most important motivation is to simplify the representation of fuzzy numbers which

- makes calculations easier
- simplifies computer applications
- gives more intuitive and more natural interpretation
- enables the first step of the defuzzification process, i.e.

$$A \in \mathbb{F}(\mathbb{R}) \Rightarrow T(A) \in \mathbb{F}^T(\mathbb{R}) \Rightarrow C(T(A)) \in \mathbb{P}(\mathbb{R}) \Rightarrow W(C(T(A))) \in \mathbb{R} \tag{11}$$

  where $\mathbb{P}(\mathbb{R})$ is a family of all intervals, while C and W denote an operator that produces an interval approximation and defuzzification operator, respectively.

If we agree that such trapezoidal approximation is worth of trouble, then the following important question arises: *How to construct an optimal trapezoidal approximation of a fuzzy number?* And here one can suggest many possible solutions. However, as it was motivated in [16], a suitable operator should possess some desired properties and should fulfill some necessary and minimal requirement. For the broad list of such postulated characteristics we refer the reader to [16]. Moreover, in this paper and in some further ones (see, e.g., [17, 4, 23]) is was shown that the approximation operator which guarantees many desired properties can be obtained as the operator T which produces a trapezoidal fuzzy number $T(A)$ that is the closest with respect to distance (9) to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one. More precisely, we get the following problem.

Problem:

Find such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$, which minimizes

$$d(A, T(A)) = \sqrt{\int_0^1 (A_\alpha^L - T_\alpha^L(A))^2 d\alpha + \int_0^1 (A_\alpha^U - T_\alpha^U(A))^2 d\alpha} \tag{12}$$

and preserves the expected interval of a fuzzy number, i.e. fulfills the following condition:

$$EI(T(A)) = EI(A). \tag{13}$$

However, since a trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core, our goal reduces to finding such real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$. Such operator is called *the nearest trapezoidal approximation operator preserving the expected interval* (actually, T is d-nearest trapezoidal approximation operator, where D is given by (9), however further on we call it, in brief, just the nearest one).

The solution of the above mentioned problem was suggested firstly in [16] and later it was improved in [17]. Although operators given in these papers generally produce proper approximations, one can construct such fuzzy number that they do not work correctly and the output is not a trapezoidal fuzzy number. Therefore, Ban [4] and Yeh [23] removed this gap and proposed a final solution containing four possible operators $T_i(A) = T_i(t_1, t_2, t_3, t_4)$, $i = 1, \ldots, 4$:

(a) If

$$-\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \leq 0 \tag{14}$$

then the solution $T_1(A) = T_1(t_1, t_2, t_3, t_4)$ is given by

$$t_1 = 4\int_0^1 A_L(\alpha)\,d\alpha - 6\int_0^1 \alpha A_L(\alpha)\,d\alpha \tag{15}$$

$$t_2 = -2\int_0^1 A_L(\alpha)\,d\alpha + 6\int_0^1 \alpha A_L(\alpha)\,d\alpha \tag{16}$$

$$t_3 = -2\int_0^1 A_U(\alpha)\,d\alpha + 6\int_0^1 \alpha A_U(\alpha)\,d\alpha \tag{17}$$

$$t_4 = 4\int_0^1 A_U(\alpha)\,d\alpha - 6\int_0^1 \alpha A_U(\alpha)\,d\alpha \tag{18}$$

(b) If

$$-\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha > 0 \tag{19}$$

and

$$2\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \leq 0 \tag{20}$$

$$-\int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 \alpha A_L(\alpha)\,d\alpha + 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \leq 0 \tag{21}$$

then we get $T_2(A) = T_2(t_1, t_2, t_3, t_4)$, where

$$t_1 = 3\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \tag{22}$$

$$t_2 = -\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 \alpha A_L(\alpha)\,d\alpha + 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \tag{23}$$

$$t_3 = t_2 \tag{24}$$

$$t_4 = \int_0^1 A_L(\alpha)d\alpha + 3\int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha \tag{25}$$

(c) If

$$2\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\,d\alpha - 3\int_0^1 \alpha A_U(\alpha)\,d\alpha > 0 \tag{26}$$

then we get $T_3(A) = T_3(t_1, t_2, t_3, t_4)$ given by

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha)d\alpha \tag{27}$$

$$t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha \tag{28}$$

(d) If

$$-\int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 \alpha A_L(\alpha)\,d\alpha + 3\int_0^1 \alpha A_U(\alpha)\,d\alpha > 0 \tag{29}$$

then we obtain $T_4(A) = T_4(t_1, t_2, t_3, t_4)$ such that

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha \tag{30}$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha. \tag{31}$$

Therefore, we have received four different operators providing the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number, where T_1 leads to trapezoidal (but not triangular) fuzzy number, T_2 stands for the operator that leads to triangular fuzzy number with two sides, while T_3 and T_4 produce triangular fuzzy numbers with the right side only or with the left side only, respectively (note, that in [17] operators T_1 and T_2 were given only).

Which operator should be used in a particular situation depends on a given fuzzy number, i.e. it depends on conditions (14), (19)–(21), (26) or (29) that seem to be very artificial and technical. Hence there was a great need for some further considerations to make these conditions more clear and to find their better interpretation. It was done by Grzegorzewski [14] who simplified the requirements for choosing the proper approximation operators. According to [14] we get the following algorithm for computing the nearest trapezoidal approximation preserving the expected interval.

Algorithm 1

Step 1. If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (15)-(18), else
Step 2. if $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ then apply operator T_2 given by (22)-(25), else
Step 3. if $Val(A) < EV_{\frac{1}{3}}(A)$ then apply operator T_3 given by (27)-(28), else
Step 4. apply operator T_4 given by (30)-(31).

As it is seen we approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided ambiguity of this fuzzy number is greater than one third of its width. Otherwise, we approximate A by a triangular number. It means that for less vague fuzzy numbers the solution is always a triangular fuzzy number.

Thus, to sum up, the distinction between possible solutions - either trapezoidal $T_1(A)$ or triangular $T_2(A)$ - depends on the relationship between two parameters of the original fuzzy number that describe its dispersion. In other words, to approximate a fuzzy number A we apply operator T_2 provided A has only slight ambiguity and its typical value is located neither close to the left nor to the right border of its support. However, a fuzzy number with its value *Val* located close to the left border of its support would be approximated by a triangular fuzzy number with the right side only, produced by operator T_3, while a fuzzy number with its value *Val* located close to the right border of its support would be approximated by a triangular fuzzy number with the left side only, produced by operator T_4.

We can also obtain an equivalent algorithm for choosing a proper approximation operator using parameter $\overline{y}(A)$ called the y-coordinate of the centroid point of a fuzzy number A. In [21] the authors showed that

$$\overline{y}(A) = \frac{\int_0^1 \alpha \left(A_U(\alpha) - A_L(\alpha)\right) d\alpha}{\int_0^1 \left(A_U(\alpha) - A_L(\alpha)\right) d\alpha}. \tag{32}$$

It is easily seen that

$$\overline{y}(A) = \frac{Amb(A)}{w(A)}. \tag{33}$$

Therefore, we get immediately that our condition (14) is equivalent to the following one

$$\overline{y}(A) \geq \frac{1}{3}. \tag{34}$$

It means that we approximate a fuzzy number A by the trapezoidal approximation operator T_1 if the y-coordinate of the centroid point of A is not smaller than one third. Otherwise, we apply operator T_2 or T_3 or T_4. The consecutive steps for choosing a suitable operator remains as before. Thus we get another algorithms (which are in fact a conjunction of our Algorithm 1 and 2 and the algorithm given in [23]).

Algorithm 2

Step 1. If $\overline{y}(A) \geq \frac{1}{3}$ then apply operator T_1 given by (15)-(18), else
Step 2. if $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ then apply operator T_2 given by (22)-(25), else
Step 3. if $Val(A) < EV_{\frac{1}{3}}(A)$ then apply operator T_3 given by (27)-(28), else
Step 4. apply operator T_4 given by (30)-(31).

4 Discussion and New Algorithms

Although the given above explanation of the conditions that delimits situations corresponding to different approximation operators is correct, yet it sounds slightly insufficient. Especially conditions related to the location parameters do not have clear interpretation. However, it appears that we can propose equivalent conditions which seem to be more satisfactory and more natural.

Since by (5)

$$EV_{\frac{2}{3}}(A) = \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha + \frac{2}{3}\int_0^1 A_U(\alpha)d\alpha$$
$$EV_{\frac{1}{3}}(A) = \frac{2}{3}\int_0^1 A_L(\alpha)d\alpha + \frac{1}{3}\int_0^1 A_U(\alpha)d\alpha$$

then according to (4) and (6) we get

$$EV_{\frac{2}{3}}(A) + EV_{\frac{1}{3}}(A) = \int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha = 2EV(A)$$

and

$$EV_{\frac{2}{3}}(A) - EV_{\frac{1}{3}}(A) = \frac{1}{3}\int_0^1 A_U(\alpha)d\alpha - \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha = \frac{1}{3}w(A).$$

Hence

$$EV_{\frac{2}{3}}(A) = EV(A) + \frac{1}{6}w(A)$$

and

$$EV_{\frac{1}{3}}(A) = EV(A) - \frac{1}{6}w(A).$$

Thus conditions (20)-(21) is equivalent to

$$|EV(A) - Val(A)| \leq \frac{1}{6} w(A). \tag{35}$$

Similarly, requirement (26) might be replaced by

$$Val(A) < EV(A) - \frac{1}{6} w(A), \tag{36}$$

while (29) is equivalent to

$$Val(A) > EV(A) + \frac{1}{6} w(A). \tag{37}$$

Thus we get another algorithm.

Algorithm 3

Step 1. If $Amb(A) \geq \frac{1}{3} w(A)$ then apply operator T_1 given by (15)-(18), else
Step 2. if $|EV(A) - Val(A)| \leq \frac{1}{6} w(A)$ then apply operator T_2 given by (22)-(25), else
Step 3. if $Val(A) > EV(A) + \frac{1}{6} w(A)$ then apply operator T_4 given by (30)-(31), else
Step 4. apply operator T_3 given by (27)-(28).

As it is seen now to find a proper approximation using Algorithm 2 we have to calculate at most 4 parameters, while in previous Algorithm 1 we had 5 parameters.

To emphasize much more that the distinction among operators T_2, T_3 and T_4 is based on the asymmetry of spread of the membership function let us introduce the following notions.

Definition 1. *The left-hand ambiguity of a fuzzy number A with α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ is defined by*

$$Amb_L(A) = \int_0^1 \alpha [EV(A) - A_L(\alpha)] d\alpha, \tag{38}$$

while the right-hand ambiguity of a fuzzy number A is given by

$$Amb_U(A) = \int_0^1 \alpha [A_U(\alpha) - EV(A)] d\alpha. \tag{39}$$

One may notice that our definition of the left-hand and right-hand ambiguity differs from the definitions proposed in [8] where the center point of the core of a fuzzy number is placed instead of $EV(A)$. Assuming that the expected value of a fuzzy number $EV(A)$ characterizes its typical value the left-hand and right-hand ambiguity describe the spread of the the left-hand and right-hand part of a fuzzy number, respectively. Moreover, as it is easily seen, both characteristics give the total spread of a fuzzy number, i.e.

$$Amb_L(A) + Amb_U(A) = Amb(A). \tag{40}$$

For our further considerations the following notion would be useful.

Definition 2. *The difference between the left-hand and right-hand ambiguity of a fuzzy number A is defined by*

$$\Delta Amb(A) = Amb_U(A) - Amb_L(A) \tag{41}$$

By (7) and (4) we get immediately that

$$\Delta Amb(A) = Val(A) - EV(A). \tag{42}$$

Therefore we get another condition equivalent both to (20)-(21) and (35), i.e.

$$|\Delta Amb(A)| \leq \frac{1}{6}w(A). \tag{43}$$

Similarly, condition

$$\Delta Amb(A) > \frac{1}{6}w(A) \tag{44}$$

is equivalent to (29) and (37), while condition

$$\Delta Amb(A) < -\frac{1}{6}w(A), \tag{45}$$

i.e.

$$Amb_L(A) - Amb_U(A) > \frac{1}{6}w(A) \tag{46}$$

is equivalent to (26) and (36).

Using these requirements we obtain another algorithm for computing the nearest trapezoidal approximation preserving the expected interval.

Algorithm 4

Step 1. If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (15)-(18), else
Step 2. if $|\Delta Amb(A)| \leq \frac{1}{6}w(A)$ then apply operator T_2 given by (22)-(25), else
Step 3. if $\Delta Amb(A) > \frac{1}{6}w(A)$ then apply operator T_4 given by (30)-(31), else
Step 4. apply operator T_3 given by (27)-(28).

It is worth noticing that Algorithm 4 utilizes 3 parameters only and a final decision for the proper choice of the approximation operator depends both on the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number.

Last of all let us notice that we may express formulae describing operators $T_1 - T_4$ using suitable parameters describing fuzzy numbers instead of relevant integrals given in Section 3. In particular we obtain a very natural formulae especially for operators T_3 and T_4, namely:

- $T_1(A) = T_1(t_1, t_2, t_3, t_4)$, where

$$t_1 = EV(A) - 2w(A) + 6Amb_L(A) \tag{47}$$
$$t_2 = EV(A) + w(A) - 6Amb_L(A) \tag{48}$$
$$t_3 = EV(A) - w(A) + 6Amb_U(A) \tag{49}$$
$$t_4 = EV(A) + 2w(A) - 6Amb_U(A). \tag{50}$$

- $T_2(A) = T_2(t_1, t_2, t_3, t_4)$, where

$$t_1 = EV(A) - w(A) - 3\Delta Amb(A) \tag{51}$$
$$t_2 = t_3 = EV(A) - \Delta Amb(A) \tag{52}$$
$$t_4 = EV(A) + w(A) - 3\Delta Amb(A) \tag{53}$$

- $T_3(A) = T_3(t_1, t_2, t_3, t_4)$ is given by

$$t_1 = t_2 = t_3 = EV(A) - \frac{1}{2}w(A) \tag{54}$$
$$t_4 = EV(A) + \frac{3}{2}w(A) \tag{55}$$

- $T_4(A) = T_4(t_1, t_2, t_3, t_4)$ is given by

$$t_1 = EV(A) - \frac{3}{2}w(A) \tag{56}$$
$$t_2 = t_3 = t_4 = EV(A) + \frac{1}{2}w(A). \tag{57}$$

As it is seen, in all cases the crucial point of the trapezoidal fuzzy number obtained as the approximation is the expected value the original fuzzy number which is invariant under approximation (see [14]). Then all points $t_1 - t_4$ that describe the trapezoidal fuzzy number are obtained by adding or subtracting some multiplicities of different measures of spread of the original fuzzy number.

5 Properties

Before we discuss the properties of our trapezoidal approximation operators let us notice that we can consider the family $\mathbb{F}(\mathbb{R})$ of all fuzzy numbers as a union of four subfamilies $\mathbb{F}_i(\mathbb{R})$ corresponding to different approximation operators to be used. Namely, we may say that a fuzzy number A belongs to subfamily $\mathbb{F}_i(\mathbb{R})$ if and only if T_i $(i = 1, \ldots, 4)$ is an appropriate operator that should be used for getting a proper trapezoidal approximation. Thus, according to the previous sections we can prove the following lemmas (which are extended versions of those given in [14]).

Lemma 1. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_1(\mathbb{R})$,
(b) *condition (14) holds,*
(c) $Amb(A) \geq \frac{1}{3}w(A)$,
(d) $\overline{y}(A) \geq \frac{1}{3}$.

Lemma 2. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_2(\mathbb{R})$,
(b) *conditions (19), (20) and (21) hold,*
(c) $Amb(A) < \frac{1}{3}w(A)$ *and* $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$,
(d) $\overline{y}(A) < \frac{1}{3}$ *and* $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$,
(e) $Amb(A) < \frac{1}{3}w(A)$ *and* $|EV(A) - Val(A)| \leq \frac{1}{6}w(A)$,
(f) $Amb(A) < \frac{1}{3}w(A)$ *and* $|\Delta Amb(A)| \leq \frac{1}{6}w(A)$.

Lemma 3. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_3(\mathbb{R})$,
(b) *condition (26) holds,*
(c) $Val(A) < EV_{\frac{1}{3}}(A)$,
(d) $Val(A) < EV(A) + \frac{1}{6}w(A)$,
(e) $\Delta Amb(A) < \frac{1}{6}w(A)$.

Lemma 4. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_4(\mathbb{R})$,
(b) *condition (29) holds,*
(c) $Val(A) > EV_{\frac{2}{3}}(A)$,
(d) $Val(A) > EV(A) + \frac{1}{6}w(A)$,
(e) $\Delta Amb(A) > \frac{1}{6}w(A)$.

One may notice that subfamilies $\mathbb{F}_1(\mathbb{R}), \ldots, \mathbb{F}_4(\mathbb{R})$ form a partition of a family of all fuzzy numbers $\mathbb{F}(\mathbb{R})$. Actually, by lemmas given above, we may conclude immediately that

$$\mathbb{F}_1(\mathbb{R}) \cup \ldots \cup \mathbb{F}_4(\mathbb{R}) = \mathbb{F}(\mathbb{R}) \tag{58}$$

and

$$\mathbb{F}_i(\mathbb{R}) \cap \mathbb{F}_j(\mathbb{R}) = \emptyset \quad \text{for} \quad i \neq j. \tag{59}$$

Introducing this useful notation we can now turn back to properties of the trapezoidal approximation operators. It can be shown that for $A \in \mathbb{F}_i(\mathbb{R})$ the nearest trapezoidal approximation operator T_i, $i = 1, \ldots, 4$, preserving expected interval is invariant to translations and scale invariant, is monotonic and fulfills identity criterion, preserves the expected value and the weighted expected value and fulfills the nearness criterion with respect to metric (9) in the subfamily of all trapezoidal fuzzy numbers with fixed expected interval. Moreover, it is continuous and compatible with the extension principle, is order invariant with respect to some preference fuzzy relations, is correlation invariant and it preserves the width. For more details we refer the reader to [4], [16] and [17]. It has been also shown ([14]) that T_1 and T_2 preserve the value of a fuzzy number, while $Val(T_3(A)) < Val(A)$ and $Val(T_4(A)) > Val(A)$. However the ambiguity is preserved only by T_1 and $Amb(T_i(A)) > Amb(A)$ for $i = 2,3,4$.

6 Conclusions

In the present contribution we have continued the discussion on the problem of trapezoidal approximation of fuzzy numbers showing another algorithms for computing the proper nearest trapezoidal approximation preserving the expected interval. It seems that these new algorithms are simpler and have more clear interpretation than the algorithms proposed before. Especially Algorithm 4 is the most concise and shows that the choice of the adequate approximation operator depends both on the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of the original fuzzy number.

Trapezoidal approximation, of course, is not the only possible way for simplifying the shape of the membership function of the fuzzy numbers under study. Even greater simplification can be obtained through the interval approximation. The readers interested in this approach are referred to [6, 12, 13]. On the other hand one may need a nonlinear approximation. Such attempt was proposed in [19].

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