Degrees of Truth, Ill-Known Sets and Contradiction

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Abstract. In many works dealing with knowledge representation, there is a temptation to extend the truth-set underlying a given logic with values expressing ignorance and contradiction. This is the case with partial logic and Belnap bilattice logic with respect to classical logic. This is also true in three-valued logics of rough sets. It is found again in interval-valued, and type two extensions of fuzzy sets. This paper shows that ignorance and contradiction cannot be viewed as additional truth-values nor processed in a truth-functional manner, and that doing it leads to weak or debatable uncertainty handling approaches.

1 Introduction

From the inception of many-valued logics, there have been attempts to attach an epistemic flavor to truth degrees. Intermediary truth-values between true and false were often interpreted as expressing a form of ignorance or partial belief (less often, the idea of contradiction). However, multiple-valued logics are generally truth-functional. The trouble here is that, when trying to capture the status of any unknown proposition by a truth-value, the very assumption of truth-functionality (building truth-tables for all connectives) is debatable. Combining two propositions whose truth-value is unknown sometimes results in tautological or contradictory statements, whose truth-value can be asserted from the start, even without any prior knowledge. As long as p can only be either true or false, even if this truth-value cannot be computed or prescribed as of to-day, the proposition $p \wedge \neg p$ can be unmistakably at any time predicted as being false and $p \vee \neg p$ as being true while $p \wedge p$ and $p \vee p$ remain contingent. So there is no way of defining a sensible truth-table that accounts for the idea of *possible* : belief is never truth-functional [21]. Mixing up truth and belief has led to a very confusing situation in traditional many-valued logics, and has probably hampered the development of applications of these logics. The

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epistemic understanding of truth-functional many-valued logics has been criticized by some scholars quite early, for instance by Urquhart [41]. Fuzzy logic is likewise often attacked because it is truth-functional. A well-known example is by Elkan [25] criticising the usual fuzzy connectives $\max, \min, 1-$, as leading to an inconsistent approach. Looking at these critiques more closely, it can be seen that the root of the controversy also lies in a confusion between degrees of truth and degrees of belief. Fuzzy logic is not specifically concerned with belief representation, only with gradual (not black or white) concepts [33]. However this misunderstanding seems to come a long way. For instance, a truth-value strictly between true and false was named "possible" [38], a word which refers to uncertainty modelling and modalities. We claimed in [18] that we cannot consistently reason under incomplete or conflicting information about propositions by augmenting the set of Boolean truth-values *true* and *false* with epistemic notions like "unknown" or "contradictory", modeling them as additional genuine truth-values of their own, as done in partial logic and Belnap's allegedly useful four-valued logic.

After reminding how uncertainty due to incompleteness is handled within propositional logic, the paper summarizes the critical discussion on partial logic previously proposed in [18], showing the corresponding extension of sets to ill-known sets, whose connectives are closely related to Kleene 3-valued logic. Two examples of ill-known sets are exhibited, especially rough sets. The debatable assumption behind some three-valued logics of rough sets is laid bare. Next, a critical discussion on Belnap logic is given, borrowing from [18]. Finally, we consider the case of truth-functional extensions of fuzzy set algebras, such as interval-valued fuzzy sets and membership/nonmembership pairs of Atanassov, as well as type two fuzzy sets where truth-functionality is also taken for granted, and that suffer from the same kind of limitations.

2 Truth vs. Belief in Classical Logic

In the following, 1 stands for *true* and 0 stands for *false*. In a previous paper [22] we pointed out that while classical (propositional) logic is always presented as the logic of the true and the false, this description neglects the epistemic aspects of this logic. Namely, if a set $\mathscr{B}$ of well-formed Boolean formulae is understood as a set of propositions believed by an intelligent agent (a *belief base*) then the underlying uncertainty theory is ternary and not binary. The three situations are:

1. p is believed (or known), which is the case if $\mathscr{B}$ implies p;
2. its negation is believed (or known), which is the case if $\mathscr{B}$ implies $\neg p$;
3. neither p nor $\neg p$ is believed, which is the case if $\mathscr{B}$ implies neither $\neg p$ nor p.

In this setting belief is Boolean, in the sense that a proposition is believed or not. We can define a belief assessment procedure to propositions, by means of a certainty function N assigning value 1 to p whenever $\mathscr{B}$ implies p ($N(p) = 1$) and 0 otherwise. The third situation above indicates a proposition that is neither believed nor is disbelieved by a particular agent. N is not a truth-assignment: one may have $N(p) = N(\neg p) = 0$, when p is unknown. The N function encodes a necessity-like modality. Indeed it is not fully compositional; while $N(p \wedge q) = \min(N(p), N(q))$, $N(p \vee q) \neq \max(N(p), N(q))$, generally, and $N(p)$ is not $1 - N(\neg p)$. The latter is the possibility function, in agreement with the duality between possibility and necessity in modal logic. So even Boolean belief is not compositional.

It is clear that belief refers to the notion of validity of p in the face of $\mathscr{B}$ and is a matter of consequencehood, not truth-values. The property $N(p \wedge q) = \min(N(p), N(q))$ just expresses that the intersection of deductively closed knowledge bases is closed, while $N(p \vee q) \neq \max(N(p), N(q))$ reminds us that the union of deductively closed propositional bases is not closed.

Through inference, we can check what are the possible truth-values left for propositions when constraints expressed in the belief base are taken into account. In fact, belief is represented by means of *subsets of possible truth-values* enabled for p when taking propositions in $\mathscr{B}$ for granted. Full belief in p corresponds to the singleton $\{1\}$ (only the truth-value "true" is possible); full disbelief in p corresponds to the singleton $\{0\}$; the situation of total uncertainty relative to p for the agent corresponds to the set $\{0,1\}$. This set is to be understood disjunctively (both truth-values for p remain possible due to incompleteness, but only one is correct). Under such conventions, the characteristic function of $\{0,1\}$ is viewed as a possibility distribution π (Zadeh [48]). Namely, $\pi(0) = \pi(1) = 1$ means that both 0 and 1 are possible. It contrasts with other uses of subsets of truth-values, interpreted conjunctively, whereby $\{0,1\}$ is understood as the *simultaneous* attachment of "true" and "false" to p (expressing a contradiction, see Dunn [24]). This convention is based on necessity degrees $N(0) = 1 - \pi(1); N(1) = 1 - \pi(0)$. Then clearly, $N(0) = 1 = N(1)$ indicates a strong contradiction. But this convention cannot be easily extended beyond two-valued truth sets, so we shall not use it.

It must be emphasized that $\{0\}, \{1\}$, and $\{0,1\}$ are not truth-values of propositions in $\mathscr{B}$. They express what can be called *epistemic valuations* whereby the agent believes p, believes $\neg p$, or is ignorant about p respectively. It makes it clear at the mathematical level that confusing truth-values and epistemic valuations comes down to confusing elements of a set and singletons contained it it, let alone subsets.

Clearly, the negation of the statement *p is believed* (inferred from $\mathscr{B}$) is not the statement *¬p is believed* , it is *p is not believed*. However, the statement *p is not believed* cannot be written in propositional logic because its syntax does not allow for expressing ignorance in the object language. The latter requires a modal logic, since in classical logic, if one interprets $p \in \mathscr{B}$ as a belief, $\neg p \in \mathscr{B}$ means that $\neg p$, is believed, not that p is just not believed. Likewise, $p \vee q \in \mathscr{B}$ is believed does not mean that either p is believed or q is believed. Assigning epistemic valuation $\{1\}$ to $p \vee q$ is actually weaker than assigning $\{1\}$ to one of p or q. In the case of ignorance about p, $\{0,1\}$ should be assigned to p and to $\neg p$. However only $\{1\}$ can be attached to their disjunction (since it is a tautology). This fact only reminds us that the union of deductively closed belief sets need not be deductively closed.

In order to capture the lack of belief or ignorance at the object level, formulas of propositional logic can be embedded within a modal-like system (Dubois, Hájek, and Prade [17], Banerjee and Dubois [6]). This embedding of classical logic into a modal logic is not the usual one: usually, propositional logic is a fragment (without modalities) of a modal logic. In the system MEL developed in [6], all wffs are made of classical propositions p prefixed by $\Box$, or their combination by means of classical connectives (formulas α of the form $\Box p$ are in MEL, and so are $\neg\alpha$, $\alpha \wedge \beta$). Boxed formulas $\Box p$ are new atoms of a (higher order) propositional logic MEL satisfies modal axoms K and D, but does not allow for nested modalities. The boxed fragment of MEL is isomorphic to propositional calculus. Any modal logic where the K axiom $\Box(p \to q) \to (\Box p \to \Box q)$ holds verifies this embedding property. So MEL is not at all a standard modal logic, in the sense that it encapsulates propositional calculus but it does not extend it. Philosophically, MEL modalities are understood *de dicto*, and not *de re*, contrary to the tradition of XXth century logic. Namely, $\Box p$ concerns the certainty of being able to assert p, not the certainty that an event referred to as p has occurred, is really true: MEL forbids direct access to the "real world", and this is consistent with the fact that propositional formulas like p (stating that p is true) cannot be expressed in MEL.

In fact, the truth-value of $\Box p$ tells whether p is believed or not: $\Box p$ *is true* precisely means that the agent's beliefs enforce $\{1\}$ as the subset of truth-values left to p, i.e. *it is true that p is believed (to be true)*. So, what belief internally means may be captured by a kind of external truth-set, say $\{\mathbf{0}, \mathbf{1}\}$. Mind that the value $\mathbf{1}$ in $t(\Box p) = \mathbf{1}$ and the value 1 in $t(p) = 1$ refer to different truth-sets (and different propositions). This trick can be used for probability theory and other non-compositional uncertainty theories (see Godo, Hájek et al. [30, 29]) and leads to a better way of legitimating the use of many-valued logics for uncertainty management: the lack of compositionality of belief is captured in the object language. For instance, the degree of probability $Prob(p)$ can be modeled as the truth-value of the proposition "Probable(p)" (which expresses the statement that p is probably true), where *Probable* is a many-valued predicate, but $Prob(p)$ is not the (allegedly) multivalued truth-value of the (Boolean) proposition p.

3 From Partial Logic to Ill-Known Sets

Partial logic starts from the claim that not all propositional variables need to be assigned a truth-value, thus defining partial interpretations and that such undefinedness may stem from a lack of information. This program is clearly in the scope of theories of uncertainty and partial belief, introduced to cope with limited knowledge. Other interpretations of partiality exist, that are not considered here. From a historical perspective, the formalism of partial logic is not so old, but has its root in Kleene [35]'s three-valued logic, where the third truth-value expresses the impossibility to decide if a proposition is true or false. The reader is referred to the dissertation of Thijsse [40] and a survey paper by Blamey [11].

3.1 Connectives of Partial Logic

At the semantic level, the main idea of partial logic is to change interpretations $s \in S$ into partial interpretations, also called *coherent situations* (or *situations,* for short) obtained by assigning a Boolean truth-value to some (but not all) of the propositional variables forming a set $Prop = \{a, b, c, \ldots\}$. A coherent situation can be represented as any conjunction of literals pertaining to distinct propositional variables. Denote by σ a situation, $\mathscr{S}$ the set of such situations, and $V(a, \sigma)$ the partial function from $Prop \times \mathscr{S}$ to $\{0, 1\}$ such that $V(a, \sigma) = 1$ if a is true in σ, 0 if a is false in σ, and is undefined otherwise. Then, two relations are defined for the semantics of connectives, namely *satisfies* ($\models_T$) and *falsifies* ($\models_F$):

- $\sigma \models_T a$ if and only if $V(\sigma, a) = 1$; $\sigma \models_F a$ if and only if $V(\sigma, a) = 0$;
- $\sigma \models_T \neg p$ if and only if $\sigma \models_F p$; $\sigma \models_F \neg p$ if and only if $\sigma \models_T p$;
- $\sigma \models_T p \wedge q$ if and only if $\sigma \models_T p$ and $\sigma \models_T q$;
- $\sigma \models_F p \wedge q$ if and only if $\sigma \models_F p$ or $\sigma \models_F q$
- $\sigma \models_T p \vee q$ if and only if $\sigma \models_T p$ or $\sigma \models_T q$;
- $\sigma \models_F p \vee q$ if and only if s $\models_F p$ and $\sigma \models_F q$.

In partial logic a coherent situation can be encoded as a truth-assignment t_σ mapping each propositional variable to the set $\{0, \frac{1}{2}, 1\}$, understood as a partial Boolean truth-assignment in $\{0, 1\}$. Let $t_\sigma(a) = 1$ if atom a appears in σ, 0 if $\neg a$ appears in σ, and $t_\sigma(a) = \frac{1}{2}$ if a is absent from σ. The basic partial logic can thus be described by means of a three-valued logic, where $\frac{1}{2}$ (again) means *unknown*. The connectives can be expressed as follows: $1 - x$ for the negation, max for disjunction, min for the conjunction, and $\max(1 - x, y)$ for the implication. Note that if $t_\sigma(p) = t_\sigma(q) = \frac{1}{2}$, then also $t_\sigma(p \vee q) = t_\sigma(p \wedge q) = t_\sigma(p \to q) = \frac{1}{2}$ in this approach.

3.2 Supervaluations

Since these definitions express truth-functionality in a three-valued logic, this logic fails to satisfy all classical tautologies. But this anomaly stems from the same difficulty again, that is, no three-element set can be endowed with Boolean algebra structure! (nor is the set **3** of non-empty intervals on $\{0, 1\}$). A coherent situation σ can be interpreted as a special set $A(\sigma) \subseteq S$ of standard Boolean interpretations, and can be viewed as a disjunction thereof. A coherent situation can be encoded as a formula whose set of models $A(\sigma)$ can be built just completing σ by all possible assignments of 0 or 1 to variables not assigned yet. It represents an epistemic state reflecting a lack of information. If this view is correct, the equivalence $\sigma \models_T p \vee q$ if and only if $\sigma \models_T p$ or $\sigma \models_T q$ cannot hold under classical model semantics. Indeed $\sigma \models_T p$ supposedly means $A(\sigma) \subseteq [p]$ and $\sigma \models_F p$ supposedly means $A(\sigma) \subseteq [\neg p]$, where $[p]$ is the set of interpretations where p is true. But while $A(\sigma) \subseteq [p \vee q]$ holds whenever $A(\sigma) \subseteq [p]$ or $A(\sigma) \subseteq [q]$ holds, the converse is invalid!

This is the point made by Van Fraassen [42] who first introduced the notion of supervaluation to account for this situation. A supervaluation SV over a coherent situation σ is (in our terminology) a function that assigns, to each proposition in the language and each coherent situation σ, the *super-truth-value* $SV(p, \sigma) = 1$ (0) to propositions that are true (false) for all Boolean completions of σ. It is clear that p is "super-true" ($SV(p, \sigma) = 1$) if and only if $A(\sigma) \subseteq [p]$, so that supervaluation theory recovers missing classical tautologies by again giving up truth-functionality: $p \vee \neg p$ is always super-true, but $SV(p \vee q, \sigma)$ cannot be computed from $SV(p, \sigma)$ and $SV(q, \sigma)$. The term "super-true" in the sense of Van Fraassen stands for "certainly true" in the terminology of possibilistic belief management in classical logic. The belief calculus at work in propositional logic covers the semantics of partial logic as a special case. It exactly coincides with the semantics of the supervaluation approach. Assuming compositionality of epistemic annotations by means of Kleene three-valued logic provides only an imprecise approximation of the actual Boolean truth-values of complex formulas [14].

3.3 Ill-Known Sets

Besides, the algebra underlying this (Kleene-like) three-valued logic is isomorphic to the set **3** of non-empty intervals on $\{0,1\}$, equipped with the interval extension of classical connectives. Consider $\frac{1}{2}$ as the set $\{0,1\}$ (understood as an interval such that $0 < 1$), the other intervals being the singletons $\{0\}$ and $\{1\}$. Indeed this comes down to computing the following cases:

- For conjunction : $\{0\} \wedge \{0,1\} = \{0 \wedge 0, 0 \wedge 1\} = \{0\}$;
  $\{1\} \wedge \{0,1\} = \{1 \wedge 0, 1 \wedge 1\} = \{0,1\}$, etc.
- For disjunction : $\{0\} \vee \{0,1\} = \{0 \vee 0, 0 \vee 1\} = \{0,1\}$;
  $\{1\} \vee \{0,1\} = \{1 \vee 0, 1 \vee 1\} = \{1\}$, etc.
- For negation: $\neg\{0,1\} = \{\neg 0, \neg 1\} = \{0,1\}$.

It yields the following tables for connectives $\vee$ and $\wedge$:

Table 1 Kleene disjunction for interval-valued sets

$\vee$	$\{0\}$	$\{0,1\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{0,1\}$	$\{1\}$
$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{1\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Table 2 Kleene conjunction for interval-valued sets

$\wedge$	$\{0\}$	$\{0,1\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{0,1\}$	$\{0\}$	$\{0,1\}$	$\{0,1\}$
$\{1\}$	$\{0\}$	$\{0,1\}$	$\{1\}$

This remark suggests that sets could be extended to *ill-known subsets* of a set S, assigning to elements $s \in S$ one of the three non-empty subsets of $\{0,1\}$. It is tempting to model them by three-valued sets denoted $\hat{A}$ whose characteristic function ranges on $\mathbf{3} = 2^{\{0,1\}} - \emptyset$ with the following conventions

$$\mu_{\hat{A}}(s) = \begin{cases} \{1\} & \text{if } s \text{ belongs for sure to the set } A \\ \{0\} & \text{if } s \text{ for sure does not belongs to the set } A \\ \{0,1\} & \text{if it is unknown whether } s \text{ belongs or not to the set } A \end{cases}$$

It encodes a pair of nested sets (A_*, A^*), A_* containing the sure elements, $A^* \setminus A_*$ being the elements with unknown membership. This is called an interval-set by Yao [45]. It is possible to extend the standard set theoretic operations to such three-valued sets using Kleene three-valued logic, equivalent to the interval operations to connectives defined in Tables 1 and 2 ($\hat{A}$ looks like a kind of fuzzy set). Equivalently, one may, as done by Yao [45], consider the interval extension of Boolean connectives to interval sets $A_* \subset A \subset A^*$. Note that while the subsets of $\{0,1\}$ form a Boolean algebra (under set inclusion), the set of "intervals" $\mathbf{3} = \{\{0\},\{1\},\{0,1\}\}$ of $\{0,1\}$ form a 3-element chain, a different structure, hence the loss of tautologies, if it is used as a new truth set, so that ill-known sets have properties different from sets. However it should be clear that this algebraic structure does not address (but in a very approximate way) the issue of reasoning about the ill-known set A. For instance, the complement of $\hat{A}$ is $\hat{A}^c$ obtained by switching $\{0\}$ and $\{1\}$ in the above definition, i.e. yields the pair $((A^*)^c, (A_*)^c)$. Hence $\hat{A} \cap \hat{A}^c$ corresponds to the pair $(A_* \cap (A^*)^c, A^* \cap (A_*)^c)$, where $A_* \cap (A^*)^c = \emptyset$ while $A^* \cup (A_*)^c \neq \emptyset$, generally [45]. However, the fuzzy set $\hat{A}$ is not an object in itself, it is a representation of the incomplete knowledge of an agent about a set A, of which all that is known is that $A_* \subset A \subset A^*$. But despite the fact that A is ill-known, $A \cap A^c = \emptyset$ regardless of what is known or not, and this information is lost by the Kleene setting, considering subsets of truth-values as truth-values and acting compositionally. Kleene's three valued logic is more naturally truth-functional when viewed as a simplified variant of *fuzzy logic*, where the third truth-value means half-true. The loss of classical tautologies then looks more acceptable.

For instance, let a one-to-many mapping $\Phi : S \to 2^V$ represent an imprecise observation of some attribute $f : S \to V$. Namely, for each object $s \in S$, all that is known about the attribute value $f(s)$ is that it belongs to the set $\Phi(s) \subseteq V$. Suppose we want to describe the set $f^{-1}(T)$ of objects that satisfy a property T, namely $\{s \in S : f(s) \in S \subset V\}$. Because of the incompleteness of the information, the subset $f^{-1}(T) \subseteq S$ is an "ill-known set" [20]. In other words, $f^{-1}(T)$ can be approximated from above and from below, respectively by upper and lower inverses of A via Φ:

- $\Phi^*(T) = \{s \in S \text{ s.t. } \Phi(s) \cap A \neq \emptyset\}$ is the set of objects that possibly belong to $f^{-1}(T)$.
- $\Phi_*(T) = \{s \in S \text{ s.t. } \Phi(s) \subseteq A\}$ is the set of objects that surely belong to $f^{-1}(T)$.

The pair $(\Phi_*(T), \Phi^*(T))$ is such that $\Phi_*(T) \subseteq f^{-1}(T) \subseteq \Phi^*(T)$ and defines an ill-known set. The multi-valued mappings Φ^* and Φ_* are respectively upper and lower inverses of Φ. Clearly, connectives will not be not truth-functional, since in general, inclusions $\Phi^*(T \cap U) \subset \Phi^*(T) \cap \Phi^*(U)$ and $\Phi_*(T) \cup \Phi_*(T) \subset \Phi_*(T) \cup \Phi_*(U)$ will be strict.

4 Rough Sets and 3-Valued Logic

Another typical example of ill-known set is a rough set. Here, uncertainty takes the form of a partition of the universe S of objects, say $S_1, \ldots S_k$. For instance objects are described by an insufficient number of attributes so that some objects have the same description. All that is known about any object in S is which subset of the partition it belongs to. So each subset A of S is only known in terms of its upper and lower approximations, a pair (A_*, A^*) such that

$$A^* = \cup\{S_i, S_i \cap A \neq \emptyset\}$$

and

$$A_* = \cup\{S_i, S_i \subseteq A\}.$$

It is clear that truth-functionality fails again as $(A \cap B)^* \subset A^* \cap B^*$ and $A_* \cup B_* \subset (A \cup B)^*$, in general (e.g. Yao [46]).

4.1 Three-Valued Settings for Rough Sets

However, various authors have tried to capture the essential features of rough sets by means of a three-valued compositional calculus (for instance Banerjee [7, 5], Itturioz [34], etc.). This is due to the existence of several points of view on rough sets, some of which are compatible with a less stringent interpretation. The most standard view is to call *rough relatively to an equivalence relation* R a subset A of S such that $A_* \neq A^*$; on the contrary, a set A such that $A_* = A^*$ is said to be *exact*. The next definition considers rough sets as equivalence classes *of subsets* of S that have the same upper and lower approximations. In this view, two sets A and B such that $A_* = B_*$ and $A^* = B^*$ are considered indistinguishable, and one is led to study nested pairs of exact sets (E, F) with $E \subseteq S$ as primitive objects representing equivalence classes of indistinguishable sets. Note that (E, F) is indeed a pair of upper and lower approximation only if $F \setminus E$ does not contain any singleton of S (since such a singleton can never overlap a subset of S without being included in it). So defining a rough set as any nested pair of exact sets (E, F) is not really faithful to the basic framework.

It is nevertheless tempting to see approximation pairs of subsets as naturally 3-valued entities. The basic justification for this move is the existence of some underlying sets C, D such that (Bonikowski [12]):

$$C^* = A^* \cap B^* \text{ and } D_* = A_* \cup B_* \tag{1}$$

that depend on the original sets A, B. Moreover $C_* = A_* \cap B_*$ and $D^* = A^* \cup B^*$ as well. Banerjee and Chakraborty [7] use

$$C = A \sqcap B = (A \cap B) \cup (A \cap B^* \cap (A \cap B)^{*c})$$

and

$$D = A \sqcup B = (A \cup B) \cap (A \cup B_* \cup (A \cup B)^c_*).$$

Likewise, noticing that $(A^c)_* = (A_*)^c$, an implication $\Rightarrow$ can be defined such that $(E,F) \Rightarrow (E',F')$ holds if and only if $E \subseteq E'$ and $F \subseteq F'$, namely, if $E = A_*$, $F = A^*$, $E' = B_*$, $F' = B^*$, the pair of nested exact sets $(E,F) \Rightarrow (E',F')$ is made of the upper and lower approximations of $((A_*)^c \cup B_*) \cap ((A^*)^c \cup B^*)$. This framework for rough equivalence classes is the one of what Banerjee and Chakraborty [7] call prerough algebra. It is shown to be equivalent to a 3-valued Łukasiewicz algebra by Banerjee [5].

4.2 On the Language-Dependent Definition of Sets

However, it must be noticed that the sets C and D as defined by Banerjee and Chakraborty [7] (also Iturrioz [34]) so as to ensure the validity of equation (1) do not depend on operands A and B only: since C is defined using an upper approximation and D involves a lower approximation, C and D depend on the partition used to define exact sets. In fact Bonikowski [12] shows that the set C is always of the form $(A_* \cap B_*) \cup Y$ where Y is obtained as follows: Let the exact set $(A^* \cap B^*) \setminus (A_* \cap B_*)$ be made of union $S_1 \cup \cdots \cup S_k$ of equivalence classes of objects in S, and consider proper non-empty subsets T_i of $S_i, i = 1 \ldots k$. Then take $Y = T_1 \cup \cdots \cup T_k$. Note that by construction $Y_* = \emptyset$, while $Y^* = S_1 \cup \cdots \cup S_k$. These properties ensure that $C^* = A^* \cap B^*$ while $C_* = A_* \cap B_*$. Besides, this construct makes it clear that no such equivalence class S_i should be a singleton of the original set S (otherwise S_i has no proper non-empty subset, and $Y_* = \emptyset$ may be impossible.)

Rough sets are induced by the existence of several objects that cannot be told apart because of having the same description in a certain language used by an observer. However, subsets of S defined in extension exist independently of whether they can be described exactly or not in this language. The set C, laid bare above, whose upper and lower approximations are $A^* \cap B^*$ and $A_* \cap B_*$ depends on the number of attributes used to describe objects. Moreover, this set is not even uniquely defined. Here lies the questionable assumption: sets A and B are intrinsically independent of the higher level language: they are given subsets of objects that can be defined in extension (perhaps using a lower level more precise language). On the contrary, their upper and lower approximations depend on the higher level language used to describe these sets: the more attributes the finer the descriptions. In other words, pairs of exact sets (A_*, A^*) and (B_*, B^*) are not existing entities, they are mental constructs representing A and B using attributes. They are observer-dependent, while A and B can be viewed as actual subsets. On the contrary the above discussion shows that C and D are not actual entities, as these subsets are observer-dependent as well, and can be chosen arbitrarily to some extent. Adding one attribute will not affect A nor B but it will change the equivalence relation, hence the partition, hence C and D as well. So the algebraic construct leading to a three-valued logic does away with the idea that approximation pairs stem from well-defined intrinsic subsets of the original space, and that logical combinations of such approximation pairs should reflect the corresponding combination of lower level ("objective") entities, that should not be affected by the discrimination power of the observer or the higher level language used to describe the objects : to be objective entities, C and D should be well-defined and depend only on A and B in S.

But then as recalled earlier, truth-functionality is lost, i.e. we cannot exactly represent the combination of subsets of S by the combination of their approximations.

The pre-rough algebras and the corresponding 3-valued logic studied by Banerjee and Chakraborty [7] are tailored for manipulating equivalence classes of subsets of S, all consisting of *all* sets having the same upper and lower approximations, without singling out any of them as being the "real" one in each such equivalence class. More recently, Avron and Konikowska [3] have tried to suggest a more relaxed three-valued setting for rough sets using non-deterministic truth-tables, that accommodate the inclusions $(A \cap B)^* \subset A^* \cap B^*$ and $A_* \cup B_* \subset (A \cup B)^*$, admitting the idea that if an element belongs to both boundaries of two upper approximations, it may or not belong to the boundary of the upper approximation of their intersection.

5 Belnap Four-Valued Logic

Two seminal papers of Belnap [9, 10] propose an approach to reasoning both with incomplete and with inconsistent information. It relies on a set of truth-values forming a bilattice, further studied by scholars like Ginsberg [28] and Fitting [26] (see Konieczny et al. [36] for a recent survey). Belnap logic, considered as a system for reasoning under imperfect information, suffers from the same difficulties as partial logic, and for the same reason. Indeed one may consider this logic as using the three epistemic valuations already considered in the previous sections (*certainly true, certainly false and unknown*), along with an additional one that accounts for epistemic conflicts.

5.1 The Contradiction-Tolerant Setting

Belnap considers an artificial information processor, fed from a variety of sources, and capable of anwering queries on propositions of interest. In this context, inconsistency threatens, all the more so as the information processor is supposed never to subtract information. The basic assumption is that the computer receives information about atomic propositions in a cumulative way from outside sources, each asserting for each atomic proposition whether it is true, false, or being silent about it. The notion of *epistemic set-up* is defined as an assignment, of one of four values denoted $\mathbf{T}, \mathbf{F}, \mathbf{BOTH}, \mathbf{NONE}$, to each atomic proposition $a, b, \ldots$:

1. Assigning **T** to a means the computer has only been told that a is true.
2. Assigning **F** to a means the computer has only been told that a is false.
3. Assigning **BOTH** to a means the computer has been told at least that a is true by one source and false by another.
4. Assigning **NONE** to a means the computer has been told nothing about a.

In view of the previous discussion, the set $\mathbf{4} = \{\mathbf{T}, \mathbf{F}, \mathbf{BOTH}, \mathbf{NONE}\}$ coincides with the power set of $\{0, 1\}$, namely $\mathbf{T} = \{1\}$, $\mathbf{F} = \{0\}$, the encoding of the other values depending on the adopted convention: under Dunn Convention, $\mathbf{NONE} = \emptyset$;

BOTH $= \{0,1\}$. It expresses accumulation of information by sources. This convention uses Boolean necessity degrees, i.e. **BOTH** means $N(0) = N(1) = 1$, **NONE** means $N(0) = N(1) = 0$. According to the terminology of possibility theory, **NONE** $= \{0,1\}$; **BOTH** $= \emptyset$. These subsets represent constraints, i.e., mutually exclusive truth-values, one of which is the right one. **NONE** means $\pi(0) = \pi(1) = 1$, **BOTH** means $\pi(0) = \pi(1) = 0$. Then $\emptyset$ corresponds to no solution.

The approach relies on two orderings in **4**:

- *The information ordering*, $\sqsubset$, such that **NONE** $\sqsubset$ **T** $\sqsubset$ **BOTH**; **NONE** $\sqsubset$ **F** $\sqsubset$ **BOTH**. This ordering reflects the inclusion relation of the sets $\emptyset$, $\{0\}, \{1\}$, and $\{0,1\}$, using Dunn convention. It intends to reflect the amount of (possibly conflicting) data provided by the sources. **NONE** is at the bottom because (to quote) "it gives no information at all". **BOTH** is at the top because (following Belnap) it gives too much information.
- *The logical ordering*, $\prec$, according to which **F** $\prec$ **BOTH** $\prec$ **T** and **F** $\prec$ **NONE** $\prec$ **T** each reflecting the truth-set of Kleene's logic. It corresponds to the idea of "less true than", even if this may sound misleadingly suggesting a confusion with the idea of graded truth. In fact **F** $\prec$ **BOTH** $\prec$ **T** canonically extends the ordering $0 < 1$ to the set **3** of non-empty intervals on $\{0,1\}$, under Dunn convention and **F** $\prec$ **NONE** $\prec$ **T** does the same under possibility degree convention.

Then, connectives of negation, conjunction and disjunction are defined truth-functionally on the bilattice. The set **4** is isomorphic to $2^{\{0,1\}}$ equipped with two lattice structures:

- *the information lattice*, a Scott approximation lattice based on union and intersection of sets of truth-values using Dunn convention. For instance, in this lattice the maximum of **T** and **F** is **BOTH**;
- *the logical lattice*, based on the interval extension of min, max and $1-$ from $\{0,1\}$ to $2^{\{0,1\}} \setminus \{\emptyset\}$ respectively under Dunn Convention (for **BOTH**) and possibility degree convention (for **NONE**).

These logical connectives respect the following constraints:

1. They reduce to classical negation, conjunction and disjunction on $\{\mathbf{T},\mathbf{F}\}$;
2. They are monotonic w.r.t. the information ordering $\sqsubset$;
3. $p \wedge q = q$ if and only if $p \vee q = p$;
4. They satisfy commutativity, associativity of $\vee, \wedge$, De Morgan laws.

For instance, the first property enforces $\neg$**T** $=$ **F** and $\neg$**F** $=$ **T** and then, the monotonicity requirement forces the negation $\neg$ to be such that $\neg$**BOTH** $=$ **BOTH** and $\neg$**NONE** $=$ **NONE**. It can be shown that the restrictions of all connectives to the subsets $\{\mathbf{T},\mathbf{F},\mathbf{NONE}\}$ and $\{\mathbf{T},\mathbf{F},\mathbf{BOTH}\}$ coincide with Kleene's three-valued truth-tables, encoding **BOTH** and **NONE** as $\frac{1}{2}$. The conjunction and disjunction operations $\vee$ and $\wedge$ exactly correspond to the lattice meet and joint for the logical lattice ordering. In fact, **BOTH** and **NONE** cannot be distinguished by the logical ordering $\prec$ and play symmetric roles in the truth-tables. The major new point is the result of combining conjunctively and disjunctively **BOTH** and **NONE**.

The only possibility left for such combinations is that **BOTH** $\wedge$ **NONE** $=$ **F** and **BOTH** $\vee$ **NONE** $=$ **T**. This looks intuitively surprising but there is no other choice and this is in agreement with the information lattice.

5.2 Is It How a Computer Should Think ?

Belnap's calculus is an extension of partial logic to the truth-functional handling of inconsistency. In his paper, Belnap does warn the reader on the fact that the four values are not ontological truth-values but epistemic ones. They are qualifications referring to the state of knowledge of the agent (here the computer). The set-representation of Belnap truth-values after Dunn [24] rather comforts the idea that these are not truth-values. Again, $\{1\}$ is a subset of $\{0,1\}$ while 1 is an element thereof.

Belnaps explicitly claims that the systematic use of the truth-tables of **4** "tells us how the computer should answer questions about complex formulas, based on a set-up representing its 'epistemic state"([9], p. 41). However, since the truth-tables of conjunction and disjunction extend the ones of partial logic so as to include the value **BOTH**, Belnap's logic inherits all difficulties of partial logic regarding the truth-value **NONE**. Moreover, equalities **BOTH** $\wedge$ **BOTH** $=$ **BOTH**, **BOTH** $\vee$ **BOTH** $=$ **BOTH** are hardly acceptable when applied to propositions of the form p and $\neg p$, if it is agreed that these are classical propositional formulas.

Another issue is how to interpret the results **BOTH** $\wedge$ **NONE** $=$ **F** and **BOTH** $\vee$ **NONE** $=$ **T**. One may rely on bipolar reasoning and argumentation to defend that when p is **BOTH** and q is **NONE**, $p \wedge q$ should be **BOTH** $\wedge$ **NONE** $=$ **F**. Suppose there are two sources providing information, say S_1 and S_2. Assume S_1 says p is true and S_2 says it is false. This is why p is **BOTH**. Both sources say nothing about q, so q is **NONE**. So one may consider that S_1 would have nothing to say about $p \wedge q$, but one may legitimately assert that S_2 would say $p \wedge q$ is false. In other words, $p \wedge q$ is **F**: one may say that there is one reason to have $p \wedge q$ false, and no reason to have it true. However, suppose two atomic propositions a and b with $E(a) =$ **BOTH** and $E(b) =$ **NONE**. Then $E(a \wedge b) =$ **F**. But since Belnap negation is such that $E(\neg a) =$ **BOTH** and $E(\neg b) =$ **NONE**, we also get $E(\neg a \wedge b) = E(a \wedge \neg b) = E(\neg a \wedge \neg b) =$ **F**. Hence $E((a \wedge b) \vee (\neg a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge \neg b)) =$ **F** that is, $E(\top) =$ **F** which is hardly acceptable again. See Fox [27] for a related critique.

More recently Avron et al. [2] have reconsidered the problem of a computer collecting and combining information from various sources in a wider framework, where sources may provide information about complex formulas too. The combination of epistemic valuations attached to atoms or formulas is dictated by rules that govern the properties of connectives and their interaction with valuation assignments in a more transparent way than Belnap truth-tables. Various assumptions on the combination strategy and the nature of propositions to be inferred (the possibly true ones or the certainly true ones) lead to recover various more or less strong logics, including Belnap formalism. The proposed setting thus avoids making the confusion between truth-values (that can be Boolean or not in the proposed approach, according to the properties chosen) and epistemic valuations.

6 Interval-Valued Fuzzy Sets

IVFs were introduced by Zadeh [50], along with some other scholars, in the seventies (see [23] for a bibliography), as a natural truth-functional extension of fuzzy sets. Variants of these mathematical objects exist, under various names (vague sets [13] for instance). The IVF calculus has become popular in the fuzzy engineering community of the USA because of many recent publications by Jerry Mendel and his colleagues [39]. This section points out the fact that if intervals of membership grades are interpreted as partial ignorance about precise degrees, the calculus of IVFs suffers from the same flaw as partial logic, and the truth-functional calculus of ill-known sets, of which it is a many-valued extension.

6.1 Definitions

An interval-valued fuzzy set is defined by an interval-valued membership function. Independently, Atanassov [1] introduced the idea of defining a fuzzy set by ascribing a membership function and a non-membership function separately, in such a way that an element cannot have degrees of membership and non-membership that sum up to more than 1. Such a pair was given the misleading name of "Intuitionistic Fuzzy Sets" as it seems to be foreign to intuitionism [23]. It also corresponds to an intuition that differs from the one behind IVFs, although both turned out to be mathematically equivalent notions (e.g. G. Deschrijver, E. Kerre [15]).

An IVF is defined by a mapping F from the universe S to the set of closed intervals in $[0,1]$. Let $F(s) = [F_*(s), F^*(s)]$. The union, intersection and complementation of IVF's are obtained by canonically extending fuzzy set-theoretic operations to interval-valued operands in the sense of interval arithmetic. As such operations are monotonic, this step is mathematically obvious. For instance, the most elementary fuzzy set operations are extended as follows, for conjunction $F \cap G$, disjunction $F \cup G$ and negation F^c, respectively:

$$[F \cap G](s) = [\min(F_*(s), G_*(s)), \min(F^*(s), G^*(s))];$$

$$[F \cup G](s) = [\max(F_*(s), G_*(s)), \max(F^*(s), G^*(s))];$$

$$F^c(s) = [1 - F^*(s), 1 - F_*(s)].$$

Considering IVFs as a calculus of intervals on $[0,1]$ equipped with such operations, they are a special case of L-fuzzy sets in the sense of Goguen [31], so as mathematical objects, they are not of special interest. An IVF is also a special case of type two fuzzy set (also introduced by Zadeh [49]). Of course all connectives of fuzzy set theory were extended to interval-valued fuzzy sets and their clones. IFVs are being studied as specific abstract algebraic structures [16], and a multiple-valued logic was recently proposed for them, called the triangle logic [43].

6.2 The Paradox of Truth-Functional Interval-Valued Connectives

Paradoxes of IVFs are less blatant than those of Kleene and Łukasiewicz three-valued logics (when the third truth-value refers to ideas of incomplete knowledge) because in the latter case, the lack of excluded-middle law on Boolean propositions is a striking anomalous feature. In the case of fuzzy logic, some laws of classical logic are violated anyway. However, the fact that interval-valued fuzzy sets have a weaker structure than the fuzzy set algebra they extend should act as a warning. Indeed, since fuzzy sets equipped with fixed connectives have a given well-defined structure, this structure should be valid whether the membership grades are known or not.

For instance, the fact that $\min(F(s), F^c(s)) \leq 0.5$ should hold whether $F(s)$ is known or not. This is a weak form of the contradiction law. However, applying the truth-tables of interval-valued fuzzy sets to the case when $F(s) = [0,1]$ (total ignorance) leads to $\min(F(s), 1 - F(s)) = [0,1]$, which means a considerable loss of information. The same feature appears with the weak excluded middle law, where again $\max(F(s), F^c(s)) = [0,1]$ is found, while $\max(F(s), F^c(s)) \geq 0.5$ should hold in any case. More generally, if the truth-value $t(p) = F(s)$ is only known to belong to some subinterval $[a,b]$ of the unit interval, the truth-functional calculus yields $t(p \wedge \neg p) = \min(F(s), 1 - F(s)) \in [\min(a, 1-b), \min(b, 1-a)]$, sometimes not included in $[0, \frac{1}{2}]$.

In fact, treating fuzzy sets with ill-known membership functions as a truth-functional calculus of IFVs is similar to the paradoxical calculus of ill-known sets based on Kleene's three-valued logics, where the third truth value is interpreted as total ignorance. Indeed, as shown above, operations on ill-known sets as well as partial logic are debatably construed as an interval-valued truth-functional extension of Boolean logic that is isomorphic to Kleene logic. Ill-known sets are to classical sets what IVFs are to fuzzy sets.

The basic point is that IVFs lead to a multiple-valued logic where the truth set [0, 1] is turned into the set of intervals on [0, 1], i.e. *intervals are seen as genuine truth-values*. This approach does not address the issue of ill-known membership grades, where the latter are nevertheless supposed to be precise, even if out of reach. Choosing intervals for truth-values is a matter of adopting a new convention for truth, while reasoning about ill-known membership grades does not require a change of the truth set. When reasoning about ill-known membership grades, the truth set remains [0, 1] and truth-values obey the laws of some multiple-valued calculus, while intervals model epistemic states about truth-values, just like elements in Belnap **4**. A logic that reasons about ill-known membership grades cannot be truth-functional. It should handle weighted formulas where the weight is an interval representing our knowledge about the truth-value of the formula, similar to Pavelka's logic [33], Lehmke's weighted fuzzy logic [37]. Then, the algebraic properties of the underlying logic should be exploited as constraints. Interval-weighted formulas are also signed formulas in many-valued logic. Reasoning about ill-known membership grades is then a matter of constraint propagation, especially interval analysis, and not only simple interval arithmetics on connectives. Automated reasoning methods based on signed formulae in multiple-valued logics follow this line and turn inference into optimization problems [32].

6.3 Reasoning about Ill-Known Truth-Values

The generic reasoning problem in interval-valued fuzzy logic is of the following form: Given a set of weighted many-valued propositional formulas $\{p_i, [a_i, b_i]), i = 1, \ldots, n\}$, the problem of inferring another proposition p comes down to finding the most narrow interval $[a, b]$ such that $(p, [a, b])$ can be deduced from $\{p_i, [a_i, b_i]), i = 1, \ldots, n\}$. It corresponds to the following optimization problem:

maximize (resp. minimize) $t(p)$ under the constraints $t(p_i) \in [a_i, b_i], i = 1, \ldots, n$.

This problem cannot be solved by a truth-functional interval-valued fuzzy logic. A simpler instance of this problem is the one of finding the membership function of a complex combination of IVFs. It comes down to finding the interval containing the truth-value of a many-valued formula, given intervals containing the truth-values of its atoms. For instance, using the most basic connectives, finding the membership function of $F \cap F^c$ when F is an IVF comes down to solving for each element of the universe of discourse the following problem:

maximize (resp. minimize) $f(x) = \min(x, 1 - x)$ under the constraint $x \in [a, b]$.

Since the function f is not monotonic, the solution is obviously not (always) the interval $[\min(a, 1 - b), \min(b, 1 - a)]$ suggested by IVF connectives, it is as follows:

$$f(x) \in [a, b] \text{ if } b \leq 0.5;$$
$$f(x) \in [\min(a, 1 - b), 0.5] \text{ if } a \leq 0.5 \leq b;$$
$$f(x) \in [1 - a, 1 - b] \text{ if } a \geq 0.5.$$

Only the first and the third case match the IVF connectives solution.

In Łukasiewicz logic, using the bounded sum and linear product connectives, inferring in the interval-valued setting comes down to solving linear programming problems [32]. Especially the condition $F \cap F^c = \emptyset$ is always trivially valid using linear product, even if F is an IFV, since $\max(0, x + (1 - x) - 1) = 0$.

6.4 Type 2 Fuzzy Sets vs. Fuzzy Truth-Values

The next step beyond interval-valued fuzzy sets is the case of type two fuzzy sets. It is then assumed that the truth value $F(s)$ of element $s \in S$ is changed into a fuzzy set of the unit interval. Generally, it is supposed to be a fuzzy interval on the unit interval, that for clarity we can denote by $\tilde{F}(s)$, with membership function $\mu_{\tilde{F}(s)}$ for each $s \in S$. The rationale for such a notion is again the idea that membership grades to linguistic concepts are generally ill-known, or that several different persons will provide different membership grades. On such a basis connectives for fuzzy sets are extended to type two fuzzy sets using the extension principle [50, 19], for instance using extended versions of min, max and $1 - \cdot$:

$$\mu_{\tilde{F}(s)\cap\tilde{G}(s)}(t) = \sup_{t=\min(t',t")} \min(\mu_{\tilde{F}(s)}(t'), \mu_{\tilde{G}(s)}(t''))$$

$$\mu_{\tilde{F}(s)\cup\tilde{G}(s)}(t) = \sup_{t=\max(t',t")} \min(\mu_{\tilde{F}(s)}(t'), \mu_{\tilde{G}(s)}(t''))$$

$$\mu_{\tilde{F}^c(s)}(t) = \mu_{\tilde{F}(s)}(1-t)$$

See [44] for a careful study of connectives for type two fuzzy sets; their results apply as well to the special case of IVFs. An operational setting where this truth-functional calculus makes sense is yet to come.

In fact, this calculus is partially at odds with the most usual interpretation of type two membership grades, namely fuzzy truth-values proposed by Zadeh [49]. It corresponds to a fuzzification of the ill-known attribute situation of section 3.3. Bellman and Zadeh [8] defined the *fuzzy truth-value* of a fuzzy statement "x is F" given that another one, "x is B", is taken for granted. When $B = \{s_0\}$, i.e. "$x = s_0$", the degree of truth of "x is F" is simply $F(s_0)$, the degree of membership of s_0 to the fuzzy set F. More generally, the information on the degree of truth of "x is F" given "x is B" will be described by a fuzzy set $\tau(F;B)$ of the unit interval with membership function:

$$\mu_{\tau(F;B)}(t) = \begin{cases} \sup\{B(s) \mid F(s) = t\}, & \text{if } F^{-1}(t) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \qquad (2)$$

for all $t \in [0,1]$. As can be checked, $\tau(S;D)$ is a fuzzy subset of truth-values and $\mu_{\tau(F;B)}(t)$ is the degree of possibility, according to the available information B, that there exists an interpretation that makes "x is F" true at degree t.

We can apply this approach to interpret type two fuzzy sets as stemming from an ill-known attribute f described by means of a *fuzzy* mapping $\Phi : S \to V$ such that $\Phi(s)$ is a fuzzy subset of possible values of the actual attribute $f(s)$. The degree $F(s)$ to which an element $s \in S$ satisfies a prescribed fuzzy property P_F defined on V is ill-known and can be represented by a fuzzy membership grade $\tilde{F}(s) = \tau(P_F, \Phi(s))$. Again, it will not be possible to apply the truth-functional calculus of type two fuzzy sets to this case where membership grades are ill-known. Generally, considering the case of two fuzzy properties P_F, P_G on V, the fuzzy truth-value $\tau(P_F \cap P_G, \Phi(s))$ is not a function of $\tau(P_F, \Phi(s))$ and $\tau(P_G, \Phi(s))$; $\tau(P_F \cup P_G, \Phi(s))$ is not a function of $\tau(P_F, \Phi(s))$ and $\tau(P_G, \Phi(s))$. This lack of compositionality is one more proof that fuzzy truth-values are not full-fledged truth-values in the sense of a compositional many-valued logic.

7 Conclusion

In conclusion, there is a pervasive confusion between truth-values and the epistemic valuations an agent may use to describe a state of knowledge: the former are compositional by assumption, the latter cannot be consistently so. This paper suggests that such difficulties appear in partial logic, three-valued logics of rough sets, Belnap logic, interval-valued and type two fuzzy logic. In logical approaches to incompleteness and contradiction, the goal of preserving tautologies of the underlying logic (classical or multivalued) should supersede the goal of maintaining a truth-functional setting. Considering subsets or fuzzy subsets of a truth-set as genuine truth-values leads to new many-valued logics that do not address the issue of uncertain reasoning on the underlying original logic. Such "powerset logics" are special cases of lattice-valued logic that need another motivation than reasoning under uncertainty. Our critique encompasses the truth-functional calculus of type two fuzzy sets [39] as well, since it again considers fuzzy sets of truth-values as truth-values. In that respect, the meaning of "fuzzy truth-values" proposed in [49] is sometimes misunderstood, as they cannot be at the same time genuine truth-values and ill-known ones.

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