6

Fuzzy Linear Programming

In this chapter, based on the general fuzzy linear programming, we first aim at discussing how to solve an optimal judge problem of Zimmermann arithmetic; then we put forward "the more-for-less paradox" of fuzzy linear programming, inquiry into the one with various fuzzy coefficients, and study a new linear programming model with T- fuzzy variables. Finally we make some extension to fuzzy line programming.

6.1 Fuzzy Linear Programming and Its Algorithm

Suppose that $x = (x_1, x_2, \cdots, x_n)^T$ is an n-dimensional decision vector, $c = (c_1, c_2, \cdots, c_n)$ is an n-dimensional objective coefficient vector, $A = (a_{ij})(1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n)$ is an $m \times n$-dimensional constraint coefficient matrix, $b = (b_1, b_2, \cdots, b_m)^T$ is an m-dimensional constant vector, and fuzzify objective and constraint function in the ordinary linear programming, then

$$\begin{aligned} &\widetilde{\max}\ (\text{or } \widetilde{\min})\ z = cx \\ &\text{s.t. } Ax \lesssim b, \\ &\quad\ x \geqslant 0, \end{aligned} \tag{6.1.1}$$

we call it a fuzzy linear programming. Let the rank(A)=m. "$\lesssim$" denotes the fuzzy version of "$\leqslant$" and has the linguistic interaction "essentially smaller than or equal to" [Zim76][LL01]. $\widetilde{\max}$ represents fuzzy maximizing, written as $cx = \sum\limits_{j=1}^{n} c_j x_j, Ax = (\sum\limits_{j=1}^{n} a_{ij} x_j)_{m\times n} (1 \leqslant i \leqslant m)$.

B.-Y. Cao: Optimal Models & Meth. with Fuzzy Quantities, STUDFUZZ 248, pp. 139–191.
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The membership function of fuzzy objective $\tilde{g}(x)$ is

$$\mu_{\tilde{G}}(x) = \tilde{g}(\sum_{j=1}^{n} c_j x_j) = \begin{cases} 0, & \text{when } \sum\limits_{j=1}^{n} c_j x_j \leqslant z_0, \\ \dfrac{1}{d_0}(\sum\limits_{j=1}^{n} c_j x_j - z_0), & \text{when } z_0 < \sum\limits_{j=1}^{n} c_j x_j \leqslant z_0 + d_0, \\ 1, & \text{when } \sum\limits_{j=1}^{n} c_j x_j > z_0 + d_0, \end{cases} \tag{6.1.2}$$

written as $t_0 = \sum\limits_{j=1}^{n} c_j x_j$, the image of $\tilde{g}(t_0)$ is shown as Figure 6.1.1.

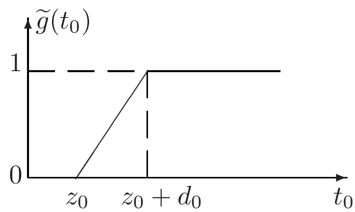

Fig. 6.1.1. Image of $\tilde{g}(t_0)$

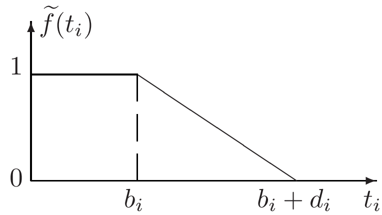

Fig. 6.1.2. Image of $\tilde{f}(t_i)$

The membership functions of fuzzy constraints $f(x)$ are:

$$\mu_{\tilde{S}_i}(x) = \tilde{f}(\sum_{j=1}^{n} a_{ij} x_j) = \begin{cases} 1, & \text{when } \sum\limits_{j=1}^{n} a_{ij} x_j \leqslant b_i, \\ 1 - \dfrac{1}{d_i}(\sum\limits_{j=1}^{n} a_{ij} x_j - b_i), & \text{when } b_i < \sum\limits_{j=1}^{n} a_{ij} x_j \leqslant b_i + d_i, \\ 0, & \text{when } \sum\limits_{j=1}^{n} c_j x_j > b_i + d_i, \end{cases} \tag{6.1.3}$$

written as $t_i = \sum\limits_{j=1}^{n} a_{ij} x_j$, the image of $\tilde{f}(t_i)$ is shown as Figure 6.1.2, where $d_i \geqslant 0 (0 \leqslant i \leqslant m)$ is a flexible index by an appropriate choice.

Consider a symmetric form fuzzy linear programming (6.1.1), written as $\mu_{\widetilde{S}} = \widetilde{S}_f$ and $\mu_{\widetilde{G}} = \widetilde{M}_f$, and we call it condition and unconditional fuzzy superiority set of f concerning constraint $\widetilde{S}$, respectively.

6.1.1 Replacement Solution Method in Fuzzy Linear Programming

Theorem 6.1.1. *For a symmetric type programming, we have*

$$\max_{x\in X} \mu_{\tilde{D}}(x) = \max_{\alpha\in[0,1]} (\alpha \wedge \max_{x\in S_\alpha} \mu_{\tilde{G}}(x)). \tag{6.1.4}$$

Proof: From Decomposition Theorem, we denote fuzzy constraint $\tilde{S}$ for $\mu_{\tilde{S}}(x) = \bigvee\limits_{\alpha\in[0,1]} \alpha \bigwedge S_\alpha(x)$, then

$$\begin{aligned}\mu_{\tilde{D}(x)} &= \mu_{\tilde{G}(x)} \bigwedge \mu_{\tilde{S}}(x) = \mu_{\tilde{G}}(x) \bigwedge [\bigvee_{\alpha\in[0,1]} (\alpha \bigwedge S_\alpha(x))] \\ &= \bigvee_{\alpha\in[0,1]} [\mu_{\tilde{G}}(x) \bigwedge (\alpha \bigwedge S_\alpha(x))],\end{aligned}$$

where $S_\alpha(x) = \begin{cases} 1, & x \in S_\alpha, \\ 0, & x \notin S_\alpha. \end{cases}$ Hence

$$\begin{aligned}\max_{x\in X} \mu_{\tilde{D}}(x) &= \bigvee_{x\in X} \bigvee_{\alpha\in[0,1]} [\mu_{\tilde{G}}(x) \bigwedge (\alpha \bigwedge S_\alpha(x))] \\ &= \bigvee_{\alpha\in[0,1]} \{\alpha \bigwedge [\bigvee_{x\in X} (\mu_{\tilde{G}}(x) \bigwedge S_\alpha(x))]\},\end{aligned}$$

while

$$\begin{aligned}\bigvee_{x\in X} [\mu_{\tilde{G}}(x) \bigwedge S_\alpha(x)] &= \{\bigvee_{x\in S_\alpha} [\mu_{\tilde{G}}(x) \bigwedge S_\alpha(x)]\} \bigvee \{\bigvee_{x\notin S_\alpha} [\mu_{\tilde{G}}(x) \bigwedge S_\alpha(x)]\} \\ &= \bigvee_{x\in S_\alpha} \mu_{\tilde{G}}(x).\end{aligned}$$

Therefore, (6.1.4) is certificated.

For the sake of the convenience, let

(1) $\varphi:\ [0,1] \to [0,1],\ \varphi(\alpha) = \max\limits_{x\in S_\alpha} \mu_{\tilde{G}}(x)$;
(2) $\psi:\ [0,1] \to [0,1],\ \psi(\alpha) = \alpha \bigwedge \varphi(\alpha)$.

Obviously, φ has the properties:

1^0 $\varphi(0) = \max\limits_{x\in X} \mu_{\tilde{G}}(x)$;
2^0 φ is a gradually decreasing function.

Asai, Tanaka et al have given φ a sufficiency condition of continuity [TOA73]:
If fuzzy constraint $\tilde{S}$ is a strict convex fuzzy set, then function φ is a continuous function in [0,1].

Theorem 6.1.2. *If φ continues in [0,1], then φ has a unique fixed point.*

Proof: If $f(\alpha) = \alpha - \varphi(\alpha)$, we know $\varphi(\alpha)$ is a continuous function in [0,1], $f(\alpha)$ also is a continuous function in [0,1].

Because $f(1) = 1 - \varphi(1) \geqslant 0$, which comes from value region of $\varphi(\alpha)$ decision at [0,1], similarly, we have $f(0) = 0 - \varphi(0) < 0$.

Therefore, point α^* at least exists in the continuous function $\varphi(\alpha)$ in [0,1], such that $f(\alpha^*) = 0$, i.e., $\alpha^* = \varphi(\alpha^*)$.

Now prove uniqueness. In reverse suppose of α_1^*, α_2^*, all satisfy $\varphi(\alpha_1^*) = \alpha_1^*, \varphi(\alpha_2^*) = \alpha_2^*$, when $\alpha_1^* \leqslant \alpha_2^*$, then $\varphi(\alpha_1^*) \leqslant \varphi(\alpha_2^*)$. This is impossible, and because of $\varphi(\alpha)$ definition, we have $\alpha_1^* \leqslant \alpha_2^* \Longleftrightarrow \varphi(\alpha_1^*) \geqslant \varphi(\alpha_2^*)$; hence $\alpha_1^* = \alpha_2^*$.

Theorem 6.1.3. *The fixed point α^* of the continuous function $\varphi(\alpha)$ is all the fixed point of the function $\psi(\alpha)$, i.e., $\psi(\alpha) = \alpha^*$.*

Proof: $\psi(\alpha^*) = \alpha^* \bigwedge \varphi(\alpha^*) = \alpha^* \bigwedge \alpha^* = \alpha^*$.

Theorem 6.1.4. *If φ is continuous, then*

$$\max_{x \in X} \mu_{\tilde{D}}(x) = \psi(\alpha^*) = \alpha^*$$

to fuzzy adjudge $\mu_{\tilde{D}}(x)$, where α^* is the fixed point in φ.

Proof: Because $\max\limits_{x \in X} \mu_{\tilde{D}}(x) = \max\limits_{\alpha \in [0,1]} \psi(\alpha), \psi(\alpha^*) = \alpha^* \bigwedge \varphi(\alpha^*) = \alpha^*$, it only proves $\max\limits_{\alpha \in [0,1]} \psi(\alpha) = \psi(\alpha^*)$.

(1) When $\alpha \leqslant \alpha^*, \varphi(\alpha) \geqslant \varphi(\alpha^*) = \alpha^* \geqslant \alpha$, then

$$\psi(\alpha) = \alpha \bigwedge \varphi(\alpha) = \alpha \leqslant \alpha^* = \psi(\alpha^*).$$

(2) When $\alpha \geqslant \alpha^*, \varphi(\alpha) \leqslant \varphi(\alpha^*) = \alpha^* \leqslant \alpha$, then

$$\psi(\alpha) = \alpha \bigwedge \varphi(\alpha) = \varphi(\alpha) \leqslant \alpha^* \bigwedge \varphi(\alpha^*) = \psi(\alpha^*).$$

Therefore,

$$\forall \alpha \in [0, 1], \psi(\alpha) \leqslant \psi(\alpha^*),$$

i.e.,

$$\psi(\alpha^*) = \max_{\alpha \in [0,1]} \psi(\alpha).$$

Theorem 6.1.5. *If α^* is a fixed point of the continuous function $\psi(\alpha)$, then $\alpha^* = \max\limits_{x \in X} \mu_{\tilde{D}}(x)$, that is, the fixed point α^* of $\psi(\alpha)$ is a determination optimal judgment value x^*.*

From Theorem 6.1.4, it easily gets $\alpha^* = \max\limits_{x \in S_\alpha} \mu_{\tilde{A}_0}(x) = \max\limits_{x \in X} \mu_{\tilde{D}}(x)$.

Thus, we converse fuzzy linear programming into a process to solve an ordinary linear programming.

In (6.1.1), we only discuss finding maximum problem in objective function $f(x)$ (to find a fuzzy minimum problem, we can convert it into finding a fuzzy maximum of $-f(x)$).

Concrete steps of solution to (6.1.1) shown follows.

1^0 Solve two linear programmings

$$(I) \quad \begin{aligned} &\min \ cx \\ &\text{s.t.} \ Ax \leqslant b, \\ &\quad\ \ x \geqslant 0, \end{aligned}$$

$$(II) \quad \begin{aligned} &\max \ cx \\ &\text{s.t.} \ Ax \leqslant b, \\ &\quad\ \ x \geqslant 0. \end{aligned}$$

Find the minimum $m = \min cx$ and maximum $M = \max cx$ are obtained, respectively. If zero stays in the feasible region of Problem (I), and coefficient c is all not negative, then $m = 0$ can be got directly.

2^0 Determine replacement accuracy $\varepsilon > 0$.

According to Theorem 6.1.1, we take $\alpha_1 \in (0, 1)$, suppose $k = 1$, and change the problem into finding a linear programming

$$\begin{aligned} &\max \mu_{\tilde{A}_0}(x) \\ &\text{s.t.} \ Ax \leqslant b_{\alpha_k}, \\ &\quad\ \ x \geqslant 0, \end{aligned}$$

where $\mu_{\tilde{A}_0}(x) = \dfrac{cx - m}{M - m}, b_{\alpha_k} = \{(1 - \alpha_k)p_1 + b_1, (1 - \alpha_k)p_2 + b_2, \cdots, (1 - \alpha_k)p_m + b_m\}$.

We can get a maximum $g_k = \max\limits_{x \in S_{\alpha_k}} \mu_{\tilde{A}_0}(x)$.

3^0 A calculation error: $\varepsilon_k = g_k - \alpha_k$.

If $|\varepsilon_k| < \varepsilon$, then to Step 4^0. Otherwise, suppose $\alpha_{k+1} = \alpha_k + \gamma_k \varepsilon_k$, where γ_k is a replacement modifying coefficient, it needs appropriately selecting, such that $0 \leqslant \alpha_{k+1} \leqslant 1$. Then, we change k into $k + 1$, and turn to Step 2^0.

4^0 Let $\alpha^* = \alpha_k$. Then solve the linear programming

$$\begin{aligned} &\max \mu_{\tilde{A}_0}(x) \\ &\text{s.t.} \ Ax \leqslant b_{\alpha}, \\ &\quad\ \ x \geqslant 0. \end{aligned}$$

From the knowledge of Theorem 6.1.5, the obtained optimal solution sets is an optimal solution to (6.1.1)(determination judgement).

Theoretically, there exists uncountably infinite α in Step 3^0 at [0,1]. In fact, it can't be compared with one by one calculation. In order to solve the problem, we shall apply the concept and theory of a fixed point.

6.1.2 Zimmermann Algorithm to Fuzzy Linear Programming

Reconsider problem in (6.1.1). In order to find an optimal solution to a fuzzy objective function under the fuzzy constraint, we can convert a fuzzy objective function into a fuzzy constraint condition $cx \gtrsim z_0$, correspondingly, it has a fuzzy set $\widetilde{G} \in \mathscr{F}(x)$ (the fuzzy objective set) in X, its membership function is (6.1.2), and for every constraint condition $\sum\limits_{j=1}^{n} a_j x_j \lesssim b_j$, a fuzzy set $\widetilde{S}_i$ in X corresponds to it and its membership function is (6.1.3).

Let $\tilde{S} = \tilde{S}_1 \bigcap \tilde{S}_2 \bigcap \cdots \bigcap \tilde{S}_m \in \mathscr{F}(X)$. Then we call it fuzzy constraint set corresponding to constraint condition $Ax \lesssim b, x \geqslant 0$, when $d_i = 0 (1 \leqslant i \leqslant m)$, $\tilde{S}$ is changed into an ordinarily constraint set S, and at this time, "$\lesssim$" is changed into "$\leqslant$" in constraint equations.

Definition 6.1.1. Suppose $\mu_{\tilde{G}}(x), \mu_{\tilde{S}_i}(x)$ is in turns the membership function of fuzzy objective and i-th fuzzy constraint, then we call fuzzy set $\tilde{D}$ satisfying $\mu_{\tilde{D}}(x) = \mu_{\tilde{G}}(x) \bigwedge (\bigwedge\limits_{i=1}^{n} \mu_{\tilde{S}_i}(x)), x \geqslant 0$ is fuzzy decision in (6.1.1), but point x^* satisfying $\mu_{\tilde{D}}(x^*) = \bigvee\limits_{x \in X} \mu_{\tilde{D}}(x)$ is an optimal solution in (6.1.1).

Fuzzy programming (6.1.1) can be written as

$$\begin{cases} -cx \lesssim -z_0, \\ Ax \lesssim b, \\ x \geqslant 0, \end{cases} \tag{6.1.5}$$

where z_0 is an expecting value for objective and it is a constant. We can see it easily at $\mu_{\tilde{S}}(x) = 1, \mu_{\tilde{G}}(x) = 0$, and hope to make the objective value bigger than z_0, but must be lower than $\mu_{\tilde{S}}(x)$, caring for fuzzy constraint set $\tilde{S}$ with a fuzzy objective set $\tilde{G}$ at both sides, according to the definition we can use fuzzy judgement $\tilde{D} = \tilde{G} \bigcap \tilde{S}$, i.e.,

$$\begin{aligned} \mu_{\tilde{D}}(x) &= \mu_{\tilde{G}}(x) \bigwedge \mu_{\tilde{S}}(x) = \mu_{\tilde{G}}(x) \bigwedge [\bigwedge_{i=1}^{m} \mu_{\tilde{S}_i}(x)] \\ &= \bigwedge_{i=0}^{m} [\mu_{\tilde{S}_i}(Bx)] = \min_{0 \leqslant i \leqslant m} (\frac{b'_i - (Bx)_i}{d_i}), \end{aligned} \tag{6.1.6}$$

where $(Bx)_i$ denotes an element of matrix (Bx) in i-th row. $B = (-c,\ A)^T$, $b' = (-z_0,\ b)^T$.

Let $\alpha = \min\limits_{0 \leqslant i \leqslant m} (\frac{b'_i - (Bx)_i}{d_i})$, then $\mu_{\tilde{D}}(x) = \alpha$, hence we can get the following.

Theorem 6.1.6. *Maximization $\mu_{\tilde{D}}(x)$ is equivalent to linear programming*

$$\begin{aligned} \max \quad & G = \alpha \\ \text{s.t.} \quad & 1 - \frac{1}{d_i} (\sum_{j=1}^{n} a_{ij} x_j - b_j) \geqslant \alpha \quad (1 \leqslant i \leqslant m), \\ & \frac{1}{d_0} (\sum_{j=1}^{n} c_j x_j - z_0) \geqslant \alpha, \\ & 0 \leqslant \alpha \leqslant 1, x_1, \cdots, x_n \geqslant 0. \end{aligned} \tag{6.1.7}$$

Again according to Definition 6.1.1 and Theorem 6.1.6, and obviously.

Theorem 6.1.7. *Suppose $\bar{x}^* = (x_1^*, x_2^*, \cdots, x_n^*; \alpha^*)^T$ is an optimal solution in (6.1.7), then $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ is an optimal solution in (6.1.1), and they have constraint and optimization level of α.*

Zimmermann initiated an arithmetic to Problem (6.1.1)[Zim78]. Here we introduce its solution as follows:

1^0 First find an ordinary linear programming

$$\begin{aligned} &\max z = cx \\ &\text{s.t. } Ax \leqslant b \\ &\quad x \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} &\max z = cx \\ &\text{s.t. } Ax \leqslant b + d, \\ &\quad x \geqslant 0, \end{aligned}$$

we obtain a maximum value z_0 and $z_0 + d_0$, where $b + d = (b_1 + d_1, \cdots, b_m + d_m)^T$. Here, z_0 is an object function maximum under the constraint condition $Ax \leqslant b$ obeyed strictly (the membership degree is $\mu_{\tilde{S}}(x) = 1$ at this time). $z_0 + d_0$ is an object function maximum when the constraint condition to be relaxed as $Ax \leqslant b + d$ (the membership degree $\mu_{\tilde{S}}(x) = 0$ at this time). z_0 and $z_0 + d_0$ corresponds to two extreme cases $\mu_{\tilde{S}}(x) = 1$ and $\mu_{\tilde{S}}(x) = 0$, which can adequate lowers membership degree $\mu_{\tilde{S}}(x)$, such that the optimal value is improved, lying between z_0 and $z_0 + d_0$.

2^0 Construct a fuzzy object set $\tilde{G} \in \mathscr{F}(x)$, its membership function is like (6.1.2), hence, fuzzy judgement in (6.1.5) is that in (6.1.6). Then finding the optimal point x^*, such that

$$\mu_{\tilde{D}}(x^*) = \mu_{\tilde{G}}(x^*) \bigwedge \mu_{\tilde{S}}(x^*) = \bigvee_{x \in X} \mu_{\tilde{G}}(x) \bigwedge \mu_{\tilde{S}}(x)).$$

3^0 Let

$$\begin{aligned} \tilde{G} \circ \tilde{S} &= \bigvee_{x \in X} (\mu_{\tilde{G}}(x) \bigwedge \mu_{\tilde{S}}(x)) \\ &= \bigvee \{\alpha | \mu_{\tilde{G}}(x) \geqslant \alpha, \mu_{\tilde{S}}(x) \geqslant \alpha, (0 \leqslant \alpha \leqslant 1)\} \\ &= \bigvee_{x \in X} \{\alpha | \mu_{\tilde{G}}(x) \geqslant \alpha; \mu_{\tilde{S}_1}(x) \geqslant \alpha, \cdots, \mu_{\tilde{S}_m}(x) \geqslant \alpha, (0 \leqslant \alpha \leqslant 1)\}. \end{aligned}$$

According to Theorem 6.1.6, this is the ordinarily linear programming with the parameter

$$
\begin{aligned}
&\max\ G=\alpha\\
&\text{s.t.}\ \sum_{j=1}^{n} a_{ij}x_j + d_i\alpha \leqslant b_i + d_i, (1 \leqslant i \leqslant m),\\
&\qquad \sum_{j=1}^{n} c_j x_j - d_0\alpha \geqslant z_0,\\
&\qquad 0 \leqslant \alpha \leqslant 1, x_1, \cdots, x_n \geqslant 0.
\end{aligned}
$$

We find its optimal solution $x^* = (x_1^*, x_2^*, \cdots, x_n^*; \alpha^*)^T$ by use of a simplex method, thus optimal point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ in (6.1.1) is obtained by Theorem 6.1.7; corresponding, the objective function value is $z^* = cx^*$, the optimal level is $\mu_{\tilde{D}}(x^*) = \alpha^*$.

6.2 Expansion on Optimal Solution of Fuzzy Linear Programming

6.2.1 Introduction

We consider a form of linear programming with fuzzy constraint in (6.1.1) being

$$
(\widetilde{LP}) \qquad \begin{aligned}
&\max\ z = cx\\
&\text{s.t.}\ Ax \lesssim b,\\
&\qquad x \geqslant 0,
\end{aligned}
$$

its corresponding parameter linear programming presents as follows:

$$
(LP_\alpha) \qquad \begin{aligned}
&\max\ \ c^T x\\
&\text{s.t.}\ \ Ax \leqslant b + (1-\alpha)d,\\
&\qquad x \geqslant 0,
\end{aligned}
$$

where $\alpha \in [0, 1]$. Let $x^{(\alpha)}, \alpha \in [0, 1]$ denotes an optimal solution to linear programming (LP_α), and B_α denotes an optimal basis, z_α denotes an optimal value.

After Zimmermann's algorithm [Zim78] has been given, people always simplify it and obtain the optimal value at $\alpha^* = 0.5$ [Fu90], [Pan87], [LL97]. However, the above results hold in the case that the optimal basis of (LP_0) is identical to that of (LP_1). If the optimal basis of (LP_0) is not identical to that of (LP_1), what is the value of α^* when its optimal solution is obtained?

6.2.2 Relevant Theorems of Parameter Linear Programming (LP_α)

Lemma 6.2.1. *Assume* $x^{(1)} = (x_1^{(1)}, \cdots, x_m^{(1)}, 0, \cdots, 0)^T$*, its corresponding optimal basis is* B_1*, consisting of the first* m *columns of* A*. If* $B_1^{-1}(b+d) \geqslant 0$ *does not hold, there will be* $\begin{pmatrix} B_1 & N \\ 0 & I \end{pmatrix} x^{(0)} \neq \begin{pmatrix} b+d \\ 0 \end{pmatrix}$.

Corollary 6.2.1. *Suppose* $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$, *without loss of generality, let* $x^{(\alpha_2)} = (x_1^{(\alpha_2)}, \cdots, x_m^{(\alpha_2)}, 0, \cdots, 0)^T$. *Its corresponding optimal basis is* B_{α_2}, *which consists of the first* m *columns of* A. *If* $B_{\alpha_2}^{-1}(b+(1-\alpha_2)d) \geqslant 0$ *does not hold, there will be* $\begin{pmatrix} B_{\alpha_1} & N \\ 0 & I \end{pmatrix} x^{(\alpha_1)} \neq \begin{pmatrix} b+(1-\alpha_1)d \\ 0 \end{pmatrix}$.

Theorem 6.2.1. *Let* $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$. *Suppose that the optimal solution to linear programming* (L_{α_2}) *is* $x^{(\alpha_2)} = (x_1^{(\alpha_2)}, \cdots, x_m^{(\alpha_2)}, 0, \cdots, 0)^T$, *and the corresponding optimal basis is* B_{α_2}, *there exits*

(1) If $B_{\alpha_2}^{-1}(b + (1-\alpha_2)d) \geqslant 0$, *then* $\overline{x} = \begin{pmatrix} B_{\alpha_2}^{-1}(b+(1-\alpha_1)d) \\ 0 \end{pmatrix}$ *is the optimal solution to* (L_{α_1}).

(2) If $B_{\alpha_2}^{-1}(b + (1-\alpha_2)d) \geqslant 0$ *does not hold, there is* $c^T\overline{x} > z_{\alpha_1}$.

Proof:

(1) It can be immediately proved by a simplex method of the linear programming.

(2) Without loss of generality, we only consider $\alpha_1 = 0, \alpha_2 = 1$, that is, we only prove that if $B_1^{-1}(b+d) \geqslant 0$ does not hold, there exists

$$c^T \begin{pmatrix} B_1^{-1}(b+d) \\ 0 \end{pmatrix} > z_0 = c^T x^{(0)}.$$

Transforming x by

$$\xi = \begin{pmatrix} B_1 & N \\ 0 & I \end{pmatrix} x = \begin{cases} \sum\limits_{j=1}^{n} a_{ij}x_j, & (1 \leqslant i \leqslant m), \\ 0, & (m+1 \leqslant i \leqslant n). \end{cases}$$

Since B_1 is a feasible basis of (LP_1), this transformation is full rank. Therefore, objective function can be transformed into function with respect to ξ as follows:

$$f(x) = c^T x = c^T \begin{pmatrix} B_1^{-1} & -B_1^{-1}N \\ 0 & I \end{pmatrix} \xi = c_1\xi_1 + \cdots + c_n\xi_n.$$

According to the Theorem 2 of [Fu90], we have

$$c_i \geqslant 0, (1 \leqslant i \leqslant m), c_i \leqslant 0, (m+1 \leqslant i \leqslant n).$$

Now we consider linear programming (LP_0).

By Lemma 6.2.1, we obtain

$$\begin{pmatrix} B_1 & N \\ 0 & I \end{pmatrix} x^{(0)} \neq \begin{pmatrix} b+d \\ 0 \end{pmatrix}.$$

Since $x^{(0)}$ is the optimal solution to (LP_0), there is $Ax^{(0)} \leqslant b+d$.

We obtain it because, at least, one of the following inequality holds,

$$\xi_i^{(0)} = \sum_{j=1}^{n} a_{ij} x_j^{(0)} < b_i + d_i \ (1 \leqslant i \leqslant m);$$
$$\xi_i^{(0)} = x_i^{(0)} > 0 \ (m+1 \leqslant i \leqslant n).$$

Such that we have

$$f(x^{(0)}) = \sum_{i=1}^{n} c_i \xi_i^{(0)} < \sum_{i=1}^{m} c_i (b_i + d_i) + \sum_{i=m+1}^{n} c_i \ 0$$
$$= c^T \begin{pmatrix} B_1^{-1} & -B_1^{-1} N \\ 0 & I \end{pmatrix} \begin{pmatrix} b+d \\ 0 \end{pmatrix} = c^T \begin{pmatrix} B_1^{-1}(b+d) \\ 0 \end{pmatrix}.$$

Therefore,

$$z_0 = f(x^{(0)}) = c^T x^{(0)} < c^T \begin{pmatrix} B_1^{-1}(b+d) \\ 0 \end{pmatrix},$$

the proof is completed.

Theorem 6.2.2. *The optimal value z_α of the linear programming (LP_α) has a linear relation with α and the slope increases with the decrease of α.*

Proof: It follows from Theorem 6.2.1 that

$$c_{B_1}^T B_1^{-1} b \leqslant c_{B_\alpha}^T B_\alpha^{-1} b$$

and

$$c_{B_\alpha}^T B_\alpha^{-1} (b + (1-\alpha) d) \leqslant c_{B_1}^T B_1^{-1} (b + (1-\alpha) d),$$

so that

$$-c_{B_1}^T B_1^{-1} d \leqslant -c_{B_\alpha}^T B_\alpha^{-1} d.$$

Similarly, if $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$, then

$$-c_{B_{\alpha_2}}^T B_{\alpha_2}^{-1} d \leqslant -c_{B_{\alpha_1}}^T B_{\alpha_1}^{-1} d.$$

Furthermore, the optimal value Z_α of (LP_α) with respect to parameter, α is

$$z_\alpha = c_{B_\alpha}^T B_\alpha^{-1} (b + (1-\alpha) d) = c_{B_\alpha}^T B_\alpha^{-1} (b+d) - \alpha c_{B_\alpha}^T B_\alpha^{-1} d.$$

Therefore, the results hold, which completes the proof.

From the theorems we can get the z_α diagram as Figure 6.2.1 and Figure 6.2.2.

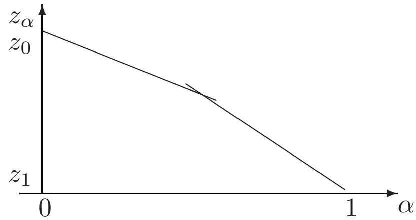

Fig. 6.2.1. $B_1^{-1}(b+d) \ngeqslant 0$ and $B_0^{-1}b \ngeqslant 0$

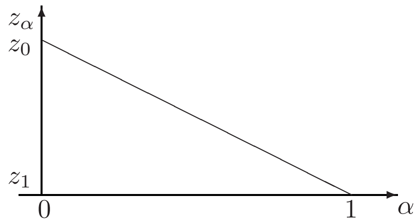

Fig. 6.2.2. $B_1^{-1}(b+d) \geqslant 0$ or $B_0^{-1}b \geqslant 0$

6.2.3 Optimal Solution and Algorithm for Fuzzy Linear Programming

From the methods of Zimmermann, its representative method is to transform fuzzy linear programming $(\widetilde{LP})$ into general linear programming as follows:

$$(LP) \qquad \begin{aligned} &\max \quad \alpha \\ &\text{s.t.} \;\; Ax + d\alpha \leqslant b + d \\ &\qquad c^T x - d_0\alpha \geqslant z_1, \\ &\qquad x \geqslant 0, 0 \leqslant \alpha \leqslant 1, \end{aligned}$$

where $d_0 = z_0 - z_1$.

Suppose (α^*, x^*) denotes an optimal solution to (LP), we discuss the relation between (α^*, x^*) and (LP_{α^*}), then the results are obtained as follows.

Theorem 6.2.3. *If (α^*, x^*) is the optimal solution to (LP), then x^* is the optimal solution to (LP_{α^*}).*

Proof: Disproof. Suppose x^* is not an optimal solution to (LP_{α^*}). Since it is a feasible solution, we have $c^T x^* < z_\alpha$. By Theorem 6.2.1 and Theorem 6.2.2, there exists $\alpha_1 \in [\alpha^*, 1]$ such that

$$c^T x^* < z_{\alpha_1} = c^T x^{(\alpha_1)} < z_{\alpha^*}$$

and

$$Ax^{(\alpha_1)} = b + (1 - \alpha_1)d.$$

Let $d_0 = z_0 - z_1$. Since z_0, z_1 is the optimal solution to (LP_0) and (LP_1), respectively, we have $d_0 > 0$, then

$$\frac{c^T x^{(\alpha_1)} - z_1}{d_0} > \frac{c^T x^* - z_1}{d_0} \geqslant \alpha^*,$$

$$\frac{z_{\alpha^*} - z_1}{d_0} > \frac{c^T x^* - z_1}{d_0} \geqslant \alpha^*.$$

Let $\alpha_2 = \dfrac{z_{\alpha^*} - z_1}{d_0}$, and take $\bar{\alpha} = \min(\alpha_1, \alpha_2)$. Then

$$Ax^{(\alpha_1)} = b + (1 - \alpha_1)d \leqslant b + (1 - \bar{\alpha})d$$

and

$$\frac{c^T x^{(\alpha_1)} - z_1}{d_0} \geqslant \bar{\alpha} > \alpha^*,$$

i.e., $\qquad Ax^{(\alpha_1)} + \bar{\alpha}d \leqslant b + d,\ c^T x^{(\alpha_1)} - d_0\bar{\alpha} \geqslant z_1.$

So $(\bar{\alpha}, x^{(\alpha_1)})$ is a feasible solution, but $\bar{\alpha} > \alpha^*$, which contradicts with the conditions in this theorem, which completes the proof.

Therefore, we only consider the optimal solution α and optimal value z_α to the linear programming (LP_α) for the fuzzy linear programming. That is, we only consider the function z_α.

Moreover, a membership function of fuzzy objective sets is defined as C_α : $z_\alpha = z_1 + d_0\alpha$, where $d_0 = z_0 - z_1$, which is a simple fuzzy number, and its image is a straight line. Therefore, for the fuzzy linear programming, when the 'intersection' operations denote the fuzzy decision, their optimal solution equals the intersection point of Figure 6.2.1 (or Figure 6.2.2) and the straight line. It is the intersection point of object function $S_\alpha : z_\alpha = z_1 + d_0\alpha$ and the constraint function

$$z_\alpha = c_{B_\alpha}^T B_\alpha^{-1}(b + (1 - \alpha)d).$$

From above results, we have the following conclusions.

Theorem 6.2.4. *Suppose that the B_0 and B_1 are optimal basis of (LP_0) and (LP_1), respectively.*

1) If $B_0^{-1}b \geqslant 0$, or $B_1^{-1}(b+d) \geqslant 0$, then fuzzy decision of $(\widetilde{LP})$ is $\alpha = 0.5$.
2) If $B_0^{-1}b \ngeqslant 0$ and $B_1^{-1}(b+d) \ngeqslant 0$, then fuzzy decision of $(\widetilde{LP})$ is $\alpha > 0.5$.

Proof: 1) When $B_0^{-1}b \geqslant 0$, from the Theorem 6.2.2, the relation between z_α and α in linear programming (LP_α) presents as follows:

$$z_\alpha = c_{B_\alpha}^T B_\alpha^{-1}(b + d) - \alpha c_{B_\alpha}^T B_\alpha^{-1} d = c_{B_1}^T B_1^{-1}(b + d) - \alpha c_{B_1}^T B_1^{-1} d.$$

Its intersect with the object set

$$S_\alpha : z_\alpha = z_1 + d_0\alpha \Longrightarrow z_\alpha = c_{B_1}^T B_1^{-1} b + \alpha c_{B_1}^T B_1^{-1} d$$

is $\alpha = 0.5, z_\alpha = c_{B_1}^T B_1^{-1}(b + 0.5d)$.

As a similar argument,we can prove that the optimal solution is $\alpha = 0.5$ when $B_1^{-1}(b+d) \geqslant 0$.

2) When the two conditions in 1) are dissatisfied, from the Theorem 6.2.2, the function

$$z_\alpha = c_{B_\alpha}^T B_\alpha^{-1}(b+d) - \alpha c_{B_\alpha}^T B_\alpha^{-1} d$$

is a fold line whose slope increases with α decreases.

Moreover, it is a cave function and the membership function of fuzzy objective sets is a monotonously increased line segment whose slope is d_0, i.e., the line segment $\overline{AB}$ in the Figure 6.2.3. Therefore, z_α intersection with membership function of objective set is shown in Figure 6.2.3

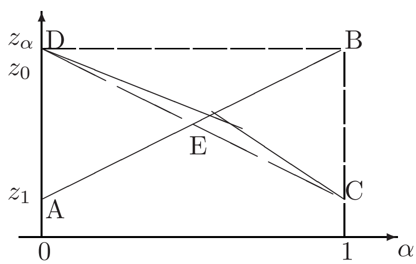

Fig. 6.2.3. $B_1^{-1}(b+d) \ngeqslant 0$ and $B_0^{-1}b \ngeqslant 0$

Link $(0, z_0)$ and $(1, z_1)$, its line segment $\overline{CD}$ stands under the fold line $\widehat{CD}$. Obviously, $\overline{CD}$ intersect $\overline{AB}$ in the point $E(0.5, \dfrac{z_0+z_1}{2})$. Therefore, when $\alpha \leqslant 0.5$, we have that $\overline{AB}$ and fold line $\widehat{CD}$ have no intersection joint. Otherwise, contradict with the concave property of z_α. It follows that their intersection point satisfies $\alpha > 0.5$, i.e., fuzzy decision $\alpha > 0.5$. So the proof is complete.

It is easy to know from Theorem 6.2.1 and Theorem 6.2.2 that the function

$$z_\alpha = c_{B_\alpha}^T B_\alpha^{-1}(b+d) - \alpha c_{B_\alpha}^T B_\alpha^{-1} d$$

is a sectional function. Then the function can be expressed as

$$z_\alpha = c_{B_1}^T B_1^{-1}(b+d) - \alpha c_{B_1}^T B_1^{-1} d \tag{6.2.1}$$

and

$$z_\alpha = c_{B_0}^T B_0^{-1}(b+d) - \alpha c_{B_0}^T B_0^{-1} d, \tag{6.2.2}$$

when the function z_α cross through $(0, z_0)$ and $(1, z_1)$, respectively. The intersection point of the above straight line is

$$\alpha' = \frac{c_{B_1}^T B_1^{-1}(b+d) - c_{B_0}^T B_0^{-1}(b+d)}{c_{B_0}^T B_0^{-1} d - c_{B_1}^T B_1^{-1} d}.$$

Suppose that $B_0^{-1}(b + (1-\alpha')d) \geqslant 0$, then we have $B_1^{-1}(b + (1-\alpha')d) \geqslant 0$.

In fact, if $B_1^{-1}(b + (1-\alpha')d) \ngeqslant 0$, and since $B_0^{-1}(b + (1-\alpha')d) \geqslant 0, z_{\alpha'}$ is an optimal solution to $(LP_{\alpha'})$.

By Theorem 6.2.1 and Theorem 6.2.2, we obtain $c^T\bar{x} > z_{\alpha'}$, i.e.,

$$z_{\alpha'} = c_{B_1^T}(B_1^{-1}(b + (1-\alpha')d) > z_{\alpha'},$$

which self-contradicts, so

$$B_1^{-1}(b + (1-\alpha')d) \geqslant 0.$$

Therefore, the function z_α is subsection function below:

$$z_\alpha = \begin{cases} c_{B_0}^T B_0^{-1}(b+d) - \alpha c_{B_0}^T B_0^{-1} d, & 0 \leqslant \alpha \leqslant \alpha', \\ c_{B_1}^T B_1^{-1}(b+d) - \alpha c_{B_1}^T B_1^{-1} d, & \alpha' \leqslant \alpha \leqslant 1. \end{cases}$$

It follows from the above theorems that the optimal solution is the intersection point of S_α: $z_\alpha = z_1 + d_0\alpha$ and z_α.

However, when $B_0^{-1}(b+(1-\alpha')d) \ngeqslant 0$, the method to $(\widetilde{LP})$ represents very complicated. We can obtain the optimal solution to fuzzy linear programming by solving the corresponding linear programming (LP).

We suggest the algorithm to $(\widetilde{LP})$ below:

1^0 Obtain the optimal solution $x^{(0)}, x^{(1)}$ to linear programming (LP_0), (LP_1). We denote their corresponding optimal basis as B_0, B_1, its corresponding objective function coefficient as c_{B_0}, c_{B_1} and its optimal value as z_0, z_1, respectively.

2^0 Compute the intersection point of two straight lines z_α crossing through $(0, z_0)$ and $(1, z_1)$, respectively, denoted by α'.

3^0 Determination. If $B_0^{-1}(b+(1-\alpha')d) \geqslant 0$, go to 4^0; otherwise, go to 7^0.

4^0 Compute the intersection point of the function z_α and S_α, we obtain (α_1, z_{α_1}). If $\alpha' \geqslant \alpha_1$, go to 5^0; if $\alpha' < \alpha_1$, then go to 6^0.

5^0 Write $\alpha = \dfrac{z_0 - z_1}{z_0 - z_1 + c_{B_0}^T B_0^{-1} d}$, the optimal solution to programming $(\widetilde{LP})$ is $x = \begin{pmatrix} B_0^{-1}(b+(1-\alpha)d) \\ 0 \end{pmatrix}$, the optimal value equals to $z_{\alpha_1} = c_{B_0}^T B_0^{-1}(b + (1-\alpha)d)$. It ends.

6^0 Write $\alpha = \dfrac{c_{B_1}^T B_1^{-1} d}{z_0 - z_1 + c_{B_1}^T B_1^{-1} d}$, the optimal solution to programming $(\widetilde{LP})$ is $x = \begin{pmatrix} B_1^{-1}(b+(1-\alpha)d) \\ 0 \end{pmatrix}$, the optimal value is $z_{\alpha_1} = c_{B_1}^T B_1^{-1}(b + (1-\alpha)d)$. It ends.

7^0 Solve linear programming $(\widetilde{LP})$ and we obtain optimal solution x of the $(\widetilde{LP})$ and the optimal value z. It ends.

If the intersection point α' satisfies the condition $B_0^{-1}(b+(1-\alpha')d) \geqslant 0$, it is easy to get a conclusion as follows.

Theorem 6.2.5. *Assumption condition of 2) holds in Theorem 6.2.4, α' is an intersection point of (6.2.1) and (6.2.2). If $B_0^{-1}(b+(1-\alpha')d) \geqslant 0$, then the linear programming $(LP_{\alpha'})$ is degenerative [Cao91c], and that is the basic variable $B_0^{-1}(b+(1-\alpha')d)$ in optimal basic solution with a zero value.*

6.2.4 Example

Example 6.2.1: Find

$$\begin{aligned} \max\ \ & x_1 + x_2 \\ \text{s.t.}\ \ & x_1 + 2x_2 \lesssim 100, \\ & x_1 \lesssim 50, \\ & x_2 \lesssim 20, \\ & x_1 \geqslant 0, x_2 \geqslant 0. \end{aligned} \tag{6.2.3}$$

Step 1. Let $d = (0,5,5)^T$. Then optimal basic of corresponding (LP_1) is B_1, and $B_1^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c_{B_1^{-1}} = (0,1,1)$.

Solve linear programming

$$\begin{aligned} \max\ \ & x_1 + x_2 \\ \text{s.t.}\ \ & x_1 + 2x_2 \leqslant 100 \\ & x_1 \leqslant 55 \\ & x_2 \leqslant 25 \\ & x_1 \geqslant 0, x_2 \geqslant 0 \end{aligned}$$

and we get an optimal solution $x^{(1)} = (x_1^{(1)}, x_2^{(2)}, y_3) = (55, 22.5, 2.5)$ as well as an optimal value $z_1 = 77.5$.

Let $d = (0,2,2)^T$. Then optimal basic of corresponding (LP_0) is B_0, and

$$B_0^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{pmatrix}, c_{B_0^{-1}} = (1,1,0),$$

and optimal value is $z_0 = 70$.

Step 2. Solve

$$\alpha_1 = \frac{c_{B_1}^T B_1^{-1}(b+d) - c_{B_0}^T B_0^{-1}(b+d)}{c_{B_0}^T B_0^{-1} d - c_{B_1}^T B_1^{-1} d},$$

we get $\alpha_1 = \dfrac{1}{3}$.

Step 3. Since

$$B^{-1}(b+(1-\alpha_1)d)=\begin{pmatrix}1&-1&-2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}100\\50+\frac{5}{3}\\20+\frac{5}{3}\end{pmatrix}=\begin{pmatrix}0\\50+\frac{5}{3}\\20+\frac{5}{3}\end{pmatrix},$$

we turn to Step 4.

Step 4. Given the optimal value z_α in parametric linear programming (LP_α) with respect to the function of parameter α as follows:

$$z_\alpha=\begin{cases}80-10\alpha, & \frac{1}{3}\leqslant\alpha\leqslant 1\\ 77.5-2.5\alpha, & 0\leqslant\alpha\leqslant\frac{1}{3}\end{cases}$$

and the objective function is $S_\alpha=70+7.5\alpha$. Solve the intersection of the two functions, we obtain the interaction point $(\frac{4}{7},\frac{520}{7})$. Since $\frac{4}{7}\geq\frac{1}{3}$, we have

$$x_{B_1}=B_1^{-1}(b+(1-\alpha)d)=(\frac{365}{7},\frac{155}{7},\frac{25}{7}).$$

Therefore, the optimal solution in fuzzy linear programming (6.2.3) is obtained as $\alpha=\dfrac{4}{7}, x_1=\dfrac{365}{7}, x_2=\dfrac{155}{7}$, and optimal value denotes $z=\dfrac{520}{7}$.

6.2.5 Conclusion

From the relation between linear programming $(\widetilde{LP})$ and parameter α, we know that optimal problem $(\widetilde{LP})$ can be transformed into solving the intersection of two linear functions. It is a fuzzy optimal solution obtained directly from the optimal solutions $x^{(1)}, x^{(0)}$ of programmings (LP_1) and (LP_0) and optimal basis B_1 and B_0, so that it is unnecessary to calculate the more complex linear programming than (LP_1) and (LP_0).

6.3 Discussion of Optimal Solution to Fuzzy Constraints Linear Programming

6.3.1 Introduction

In this section, we focus on the fuzzy constraint linear programming. First we discuss the properties of an optimal solution vector and of an optimal value in the corresponding parametric programming, and propose a method to the critical values. Then we present a new algorithm to the fuzzy constraint linear programming by associating an object function with an optimal value of parametric programming.

The normal form of a linear programming with fuzzy constraint is $(\widetilde{LP})$ as Section 6.2.1

$$(\widetilde{LP}) \qquad \begin{aligned} &\max\ z = cx \\ &\text{s.t. } Ax \lesssim b, \\ &\quad x \geqslant 0, \end{aligned}$$

the representative method to $(\widetilde{LP})$ is to turn it into a classical linear programming [Cao02a]. We will try to explain the number that its fuzzy decision usually is 0.5 found by Researchers [Cao02a][Fu90][LC02][Pan87]. Here we shall propose another algorithm to $(\widetilde{LP})$.

6.3.2 Analysis of Fuzzy Linear Programming

Suppose x_α denotes an optimal solution to (LP_α), B_α and z_α an optimal basis matrix and an optimal value of (LP_α), respectively, and then we consider

$$(LP_\alpha) \qquad \begin{aligned} &\max\ z = c^T x \\ &\text{s.t. } Ax \leqslant b + (1-\alpha)d, \\ &\quad x \geqslant 0, \end{aligned}$$

where α is a parameter on the interval [0,1], $d \geqslant 0$, $b + (1-\alpha)d$ will vary with parameter α. Its optimal solution is $B_\alpha^{-1}(b+(1-\alpha)d)$. If we solve (LP_α) by using a simplex method, there is no relationship between discriminate number $\sigma = c_N - c_B B^{-1} N$ and parameter α, so the variation of an optimal basis matrix is decided only by x_α.

6.3.2.1 Properties of the Parametric Linear Programming

Definition 6.3.1. Let B be one of the optimal basic matrix of (LP_α). If an interval $[\alpha_1, \alpha_2]$ exists, satisfying that B is an optimal basic matrix of $(LP_\alpha)(\forall \alpha \in [\alpha_1, \alpha_2])$ while B is not an optimal matrix for each $\alpha \overline{\in} [\alpha_1, \alpha_2]$, we call that α_1 and α_2 critical values of (LP_α) and $[\alpha_1, \alpha_2]$ a characteristic interval.

Theorem 6.3.1. *(LP_α) has a finite characteristic interval on the interval [0,1].*

Proof: Let us assume B is an optimal basis matrix of (LP_α), and there are two characteristic intervals $[\alpha_{i-1}, \alpha_i]$ and $[\alpha_{i+1}, \alpha_{i+2}], (\alpha_i < \alpha_{i+1})$ corresponding to B. The optimal solution to (LP_α) is $(x_B, x_N)^T$, where

$$\begin{aligned} &x_B = B^{-1}(b + (1-\alpha)d) \geqslant 0, \alpha \in [\alpha_{i-1}, \alpha_i] \cup [\alpha_{i+1}, \alpha_{i+2}], \\ &x_N = 0, \alpha_i < \alpha_{i+1}. \end{aligned}$$

So $x_B = B^{-1}(b + (1-\alpha)d) \geqslant 0$ when $\alpha \in [\alpha_i, \alpha_{i+1}]$, this means an optimal matrix of (LP_α) is also B on the interval $[\alpha_i, \alpha_{i+1}]$. Therefore the characteristic interval where the optimal matrix keeps invariant is $[\alpha_{i-1}, \alpha_{i+2}]$. So the optimal matrix has only one corresponding characteristic interval. Because the coefficient matrix of (LP_α) keeps invariant on the interval [0,1], and an optimal matrix is finite, the number of characteristic intervals is finite. This means (LP_α) has finite characteristic interval on the interval [0,1].

Theorem 6.3.2. *Let B be an optimal basis matrix of (LP_α) on a characteristic interval $[\alpha_1, \alpha_2]$. If $(B^{-1}b)_i \neq 0 (1 \leqslant i \leqslant m)$, then*

$$\alpha_1 = \max\Big[\frac{[B^{-1}(b+d)]_i}{(B^{-1}d)_i}, 0 \mid (B^{-1}d)_i < 0 (1 \leqslant i \leqslant m)\Big], \tag{6.3.1}$$

$$\alpha_2 = \min\Big[\frac{[B^{-1}(b+d)]_i}{(B^{-1}d)_i}, 0 \mid (B^{-1}d)_i > 0 (1 \leqslant i \leqslant m)\Big] \tag{6.3.2}$$

is derived, where $(B^{-1}(b+d))_i$ and $(B^{-1}d)_i$ are the i-th components of $B^{-1}(b+d)$ and $B^{-1}d$, respectively.

Proof: We can use partitioned matrices to represent the simplex method to a linear programming [LL97]. Since

$$\begin{pmatrix} B & N & b+(1-\alpha)d \\ c_B & c_N & z \end{pmatrix} \Longrightarrow \begin{pmatrix} I & B^{-1}N & B^{-1}(b+(1-\alpha)d) \\ c_B & c_N & z \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} I & B^{-1}N & B^{-1}(b+(1-\alpha)d) \\ 0 & c_N - c_B B^{-1}N & z - c_B B^{-1}(b+(1-\alpha)d) \end{pmatrix},$$

where N is a non-basis matrix corresponding to B, there is no relationship between variable α and the discriminate number. $B^{-1}(b+(1-\alpha)d) \geqslant 0$ is only required in order to make the optimal matrix of (LP_α) invariant. This means

$$\forall\ i, [B^{-1}(b+(1-\alpha)d)]_i \geqslant 0,$$

i.e.,

$$\forall\ i, [B^{-1}(b+d)]_i - \alpha(B^{-1}d)_i \geqslant 0.$$

By solving this inequality, we can obtain $\alpha \in [\alpha_1, \alpha_2]$, where α_1 and α_2 are represented with (6.3.1) and (6.3.2). It is obvious that the optimal matrix of (LP_α) will change at $\alpha > \alpha_2$ or $\alpha < \alpha_1$. Therefore the characteristic interval, corresponding to the optimal basis matrix B, is $[\alpha_1, \alpha_2]$.

Based on the above conclusion, we can easily get the properties of optimal value function Z_α as follows.

Property 6.3.1. *Let B be an optimal matrix of (LP_α) on the characteristic interval $[\alpha_i, \alpha_j]$. Then $x_\alpha = B^{-1}(b+(1-\alpha)d)(\alpha_i \leqslant \alpha \leqslant \alpha_j)$ is a linear vector function about variable α. The optimal value function $z_\alpha = c_B B^{-1}(b+(1-\alpha)d)$ is a linear function about variable α and decreases with the increase of variable α.*

Property 6.3.2. *The optimal value function z_α of (LP_α) continues on the interval [0,1].*

6.3.2.2 Optimal Solution to Fuzzy Linear Programming

Theorem 6.3.3. *Let $\tilde{S}$ be the fuzzy constraint, $\tilde{G}$ the fuzzy objective function on domain X, then the optimal solution x^* to the fuzzy optimal set $\tilde{D} = \tilde{G} \wedge \tilde{S}$ satisfies*

$$\begin{aligned}\mu_{\tilde{D}}(x^*) &= \max_{x \in X} \mu_{\tilde{D}}(x) \\ &= \max_{0 \leqslant \alpha \leqslant 1} \{\alpha \wedge \max_{x \in S_\alpha} \mu_{\tilde{S}}(x)\},\end{aligned}$$

where $S_\alpha = \{x | x \in X, \mu_{\tilde{G}}(x) \geqslant \alpha\}$ [Cao02a].

The fuzzy objective function can be defined as $G_\alpha : z_\alpha = z_1 + d_0\alpha$, we can use the intersection of the fuzzy objective function $G_\alpha : z_\alpha = z_1 + d_0\alpha$ and fuzzy constraints $S_\alpha : z_\alpha = c_{B\alpha}B_\alpha^{-1}(b + (1-\alpha)d)$ to find an optimal decision of $(\widetilde{LP})$, shown as in Figure 6.3.1.

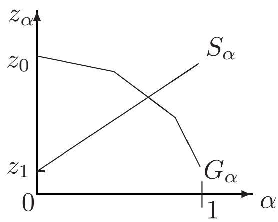

Fig. 6.3.1. The Intersection of G_α and S_α

6.3.3 Algorithm to Fuzzy Linear Programming

Let z_1 be an optimal value of (LP_1), and z_0 be an optimal value of (LP_0), $d_0 = z_0 - z_1 > 0$. Based on the above conclusions, we give a new algorithm to fuzzy linear programming as follows.

Step 1. Solve linear programmings (LP_0) and (LP_1).

Let the optimal solutions be x_0, x_1, the optimal values be z_0, z_1, and the optimal matrix of (LP_0) be B_0.

Step 2. Solve

$$[{B_0}^{-1}(b + (1-\alpha)d)]_i = 0.$$

Assume the solutions as

$$\alpha_1, \cdots, \alpha_{n-1}, (0 < \alpha_1 < \cdots < \alpha_{n-1} < 1).$$

Let $\alpha_0 = 0, \alpha_n = 1, \alpha = \alpha_1, k = 1$.

Step 3. Solve (LP_α).

Let the optimal value be z_α. If $z_\alpha \leqslant z_1 + d_0\alpha$, turn to Step 4, otherwise let $k = k+1, \alpha = \alpha_k$, turn to Step 3.

Step 4. Solve the optimal decision

$$\alpha^* = \frac{z_{1\alpha_k} - z_{1\alpha_{k-1}} - z_{\alpha_{k-1}\alpha_k} + z_{\alpha_k\alpha_{k-1}}}{z_{\alpha_k} - z_{\alpha_{k-1}} - \alpha_k d_0 + \alpha_{k-1} d_0}.$$

Step 5. Solve linear programming (LP_{α^*}), and we can obtain an optimal solution x_{α^*} and an optimal value z_{α^*}.

Example 6.3.1: Calculate

$$\begin{aligned} \max\ & 3x_1 + 5x_2 \\ \text{s.t.}\ & 7x_1 + 2x_2 \lesssim 66, \\ & 5x_1 + 3x_2 \lesssim 61, \\ & x_1 + x_2 \lesssim 16, \\ & x_1 \lesssim 8, \\ & x_2 \lesssim 5, \\ & x_i \geqslant 0 (i = 1, 2), \end{aligned} \tag{6.3.3}$$

where $d_1 = d_2 = d_3 = d_4 = 0, d_5 = 7$ is a flexible value of a object and constraint function, respectively.

We obtain $z_0 = 72, z_1 = 49, d_0 = 23$ by calculating (LP_0) and (LP_1) corresponding to (6.3.3), respectively. The inverse matrix of the optimal matrix in (LP_0) is $B_0^{-1} = (b_1, \cdots, b_5)$, where

$$\begin{aligned} & b_1 = (1, 0, 0, 0, 0)^T, b_2 = (0, 1, 0, 0, 0)^T, b_3 = (-7, -5, 1, -1, 0)^T, \\ & b_4 = (0, 0, 0, 1, 0)^T, b_5 = (5, 2, -1, 1, 1)^T. \end{aligned}$$

By calculating the equations

$$[B_0^{-1}(66, 61, 16, 8, 12 - 7\alpha)^T]_i = 0, (i = 1, \cdots, 5),$$

respectively, we obtain $\alpha_1 = \dfrac{5}{14}, \alpha_2 = \dfrac{2}{5}, \alpha_3 = \dfrac{4}{7}$.

Assume $\alpha_0 = 0, \alpha_4 = 1$, and we use *Lindo* software to solve the linear programming (LP_{α_1}) before we obtain an optimal value $z_{\alpha_1} = z_{\frac{5}{14}}$=67.

Because $Z_{\frac{5}{14}} > Z_1 + \frac{5}{14} d_0$, we must continue to solve the linear programming (LP_{α_2}).

By calculating linear programming (LP_{α_2}), we obtain an optimal value $z_{\alpha_2} = z_{\frac{2}{5}}$=66.04. Because $z_{\frac{2}{5}} > z_1 + \frac{2}{5} d_0$, we must continue to solve the linear programming (LP_{α_3}).

By calculating linear programming (LP_{α_3}), we gain an access to an optimal value $z_{\alpha_3} = z_{\frac{4}{7}}$=61.429. Because $z_{\frac{4}{7}} < z_1 + \frac{4}{7} d_0$, the optimal decision is $\alpha^* = 0.557$.

By calculating $(LP_{0.557})$, we obtain

$$x_{0.557} = (6.8606, 8.8990)^T$$

and

$$Z_{0.557} = 65.0768.$$

So the optimal solution to the example is $x^* = (6.8606, 8.8990)^T$ and the optimal value is $z^* = 65.078$.

6.3.4 Conclusion

We know that optimal decision of the fuzzy constraint linear programming does not necessarily equal 0.5, and the optimal value function figure of $(\widetilde{LP})$ is not necessarily a segment. Based on the properties of the optimal value function, we have proposed a new algorithm to fuzzy constraint linear programming.

6.4 Relation between Fuzzy Linear Programming and Its Dual One

6.4.1 Introduction

Let a linear programming primal problem like

$$\begin{aligned} \min z &= cx \\ \text{s.t. } Ax &= b, \\ x &\geqslant 0, \end{aligned} \tag{6.4.1}$$

while

$$\begin{aligned} \max\ & yb \\ \text{s.t. } & yA \leqslant c \\ & y \geqslant 0 \end{aligned} \tag{6.4.2}$$

is a dual linear programming in (6.4.1)[Dan63], where $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, $y = (y_1, y_2, \cdots, y_m), c = (c_1, c_2, \cdots, c_n), b = (b_1, b_2, \cdots, b_m)^{\mathrm{T}}$ is a variable and constant vector, respectively, $A = (a_{ij})_{m\times n}$ is an $m \times n$ matrix.

We discuss relation between them as follow.

6.4.2 Case with Fuzzy Coefficients

Consider a linear programming with fuzzy coefficient to be

$$\begin{aligned} \min \tilde{z} &= \tilde{c}x \\ \text{s.t. } Ax &= b, \\ x &\geqslant 0, \end{aligned} \tag{6.4.3}$$

where $\tilde{c}$ is a fuzzy coefficient; its dual form is

$$\begin{aligned} \max\ & w = \tilde{y}b \\ \text{s.t. } & \tilde{y}A \leqslant \tilde{c}, \\ & \tilde{y} \geqslant 0, \end{aligned} \tag{6.4.4}$$

where $\tilde{y}$ denotes a fuzzy variable vector.

Lemma 6.4.1. *The dual form of (6.4.3) is (6.4.4). If there exists an optimum solution in one, then there exists an optimum solution in the other, with there existing the same fuzzy optimum value in (6.4.3) and (6.4.4) for a continuous and strictly monotone function* $\tilde{\phi}$.

Proof: According to formula (1.5.3) in Section 1.5, (6.4.1) is turned into the following problem for solution:

$$\begin{aligned} \min\ & cx, \\ \text{s.t.}\ & \mu_{\tilde{\phi}}(c) \geqslant 1-\alpha, \alpha \in [0,1], \\ & Ax = b, c \in \mathscr{R}^n, \\ & x \geqslant 0. \end{aligned}$$

If we define [Ver84]: $\forall c \in \mathscr{R}^n, \mu_{\tilde{\phi}}(c) = \inf\limits_j \mu_{\tilde{\phi}_j}(c_j)(1 \leqslant i \leqslant l, l \leqslant n), c = (c_1, c_2, \cdots, c_n)$. But if $\mu_{\tilde{\phi}}(c) \geqslant 1-\alpha$, then

$$\begin{aligned} \inf_j \mu_{\tilde{\phi}_j}(c_j) \geqslant 1-\alpha \Longleftrightarrow & \mu_{\tilde{\phi}_j}(c_j) \geqslant 1-\alpha (1 \leqslant j \leqslant l) \\ \Longleftrightarrow & c_j \geqslant \mu_{\tilde{\phi}_j^{-1}}(1-\alpha). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \min\ & \sum_{j=1}^n c_j x_j \\ \text{s.t.}\ & c_j \geqslant \mu_{\tilde{\phi}_j^{-1}}(1-\alpha)\ (1 \leqslant j \leqslant n), \\ & Ax = b, \alpha \in [0,1], \\ & x \geqslant 0. \end{aligned}$$

This problem is equivalent to

$$\begin{aligned} \min\ & \sum_{j=1}^n c_j x_j \\ \text{s.t.}\ & c_j = \mu_{\tilde{\phi}_j^{-1}}(1-\alpha) \\ & Ax = b, \alpha \in [0,1] \\ & x \geqslant 0 \end{aligned}$$

$$\begin{aligned} \Longleftrightarrow \min\ & \mu_{\tilde{\phi}^{-1}}(\beta)x \\ \text{s.t.}\ & Ax = b, \beta \in [0,1], \\ & x \geqslant 0, \end{aligned} \tag{6.4.5}$$

where $\beta = 1-\alpha$; the dual form of (6.4.5) is

$$\begin{aligned} \max\ & yb \\ \text{s.t.}\ & yA = \mu_{\tilde{\phi}^{-1}}(\beta), \beta \in [0,1] \\ & y \geqslant 0 \end{aligned} \tag{6.4.6}$$

$$
\begin{aligned}
\Longleftrightarrow\ & \max yb \\
& \text{s.t. } \mu_{\tilde{\phi}}(c) \geqslant \beta \\
& \qquad yA = c \\
& \qquad y \geqslant 0, \beta \in [0,1] \\
\Longleftrightarrow\ & (6.4.4).
\end{aligned}
$$

We can see that the lemma holds because of the same parameter solutions to (6.4.4) as well as to (6.4.6), by the equivalence of (6.4.5) with (6.4.3), and (6.4.6) with (6.4.4), and by the mutual dual problems of (6.4.3) and (6.4.4).

6.4.3 Case with Fuzzy Variables

Consider fuzzification of linear programming

$$
\begin{aligned}
\min\ & \tilde{z} = c\tilde{x} \\
\text{s.t. } & A\tilde{x} \geqslant \tilde{b}, \\
& \tilde{x} \geqslant 0,
\end{aligned}
\tag{6.4.7}
$$

called a linear programming with fuzzy variable [AMA93], where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^{\mathrm{T}}$ an $n-$dimensional fuzzy variable vector, $0 \leqslant c \in \mathscr{R}^n, \tilde{b} \in (\mathscr{F}(\mathscr{R}))^m$ a fuzzy vector, respectively, and $A \in \mathscr{R}^{m\times n}$ represents an $m \times n$ matrix.

The dual problem of (6.4.7) is denoted by

$$
\begin{aligned}
\max\ & \tilde{w} = y\tilde{b} \\
\text{s.t. } & yA \leqslant c, \\
& y \geqslant 0,
\end{aligned}
\tag{6.4.8}
$$

where $c \in \mathscr{R}^n$, $A \in \mathscr{R}^{m\times n}$, $y \in \mathscr{R}^m$, $\tilde{b} \in (\mathscr{F}(\mathscr{R}))^m$.

$\tilde{x}$ is said to be a fuzzy feasible solution to (6.4.7) if and only if $\tilde{x}$ satisfies the constraints of the problem. By an optimal fuzzy solution to (6.4.7) we denote a fuzzy feasible solution, say $\tilde{x}^0$, such that $c\tilde{x}^0 \leqslant c\tilde{x}$ for all $\tilde{x}$ belong to the set of all fuzzy feasible solutions to (6.4.7).

The relation between fuzzy linear programming (6.4.7) and its dual programming (6.4.8) is as follow.

In order to solve programming (6.4.7), we shall find an optimal solution to problem (6.4.8). However (6.4.8) is, in fact, a linear programming with fuzzy coefficient, and we already know how to solve this. It follows that we shall discuss the relationships between the primary and dual programmings.

Lemma 6.4.2. *If $\tilde{x}$ is any fuzzy feasible solution to (6.4.7) and y is any feasible one to (6.4.8), then $y\tilde{b} \leqslant c\tilde{x}$.*

Proof: Straightforward.

Lemma 6.4.3. *If $\tilde{x}^0$ is a fuzzy feasible solution to (6.4.7) and y^0 is a feasible one to (6.4.8), such that $y^0\tilde{b} = c\tilde{x}^0$, then y^0 is an optimal solution to (6.4.8) and $\tilde{x}^0$ is a fuzzy optimal one to (6.4.7).*

Proof: Straightforward.

Theorem 6.4.1. *If the dual problem (6.4.8) has an optimal solution, then problem (6.4.7) has a fuzzy optimal solution.*

Proof: We first transform (6.4.7) into the form

$$\begin{aligned} &\max\ \tilde{w} = y\tilde{b} \\ &\text{s.t. } yA + y_s I = c, \\ &\quad\ y, y_s \geqslant 0, \end{aligned} \tag{6.4.9}$$

where $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n)$, y_s represents a slack variable and I is a unit matrix.

Let $A' = (A, I)^{\mathrm{T}}, y' = (y, y_s), c' = (c, 0)^{\mathrm{T}}$. Formula (6.4.9) is simplified as follows:

$$\begin{aligned} &\min\ \tilde{w}' = y'\tilde{b} \\ &\text{s.t. } y'A' = c', \\ &\quad\ y' \geqslant 0. \end{aligned} \tag{6.4.10}$$

Let y'_B be an optimal basic solution to (6.4.10). Such that $\tilde{w}_j - \tilde{b}_j \geqslant 0$ for all j; thus, $\tilde{b}_B B^{-1} A \geqslant \tilde{b}$, where B is a basic matrix corresponding to A.

If we write $\tilde{x} = \tilde{b}_B B^{-1}$, we can see that $\tilde{x}$ is a fuzzy feasible solution to (6.4.7). On the other hand, we have

$$\tilde{z} = c\tilde{x} = \tilde{b}_B B^{-1} c = y_B \tilde{b}_B = \tilde{w}.$$

Hence, $\tilde{x}$ is an optimal solution to (6.4.7).

Lemma 6.4.4. *If problem (6.4.8) has an unbounded solution, then problem (6.4.7) has no fuzzy feasible solution.*

Proof: Straightforward.

We conclude that, in order to solve a linear programming with fuzzy variables, it is sufficient to solve its dual problem. We can then obtain the fuzzy optimal solution to our problem by using the theorem and lemmas of this section, and vice versa.

Let $\mu_{\tilde{\phi}}$ be (1.5.3) in Section 1.5. If a fuzzified form of (6.4.1) is (6.4.3), its primal programming with parameter is

$$\begin{aligned} &\min\ (m + \beta d n^{-1})x \\ &\text{s.t. } Ax = b, \beta \in [0, 1], \\ &\quad\ x \geqslant 0, \end{aligned} \tag{6.4.11}$$

where m, n are real numbers, with $c \geqslant m + \beta d n^{-1} \Longleftrightarrow c = m + \beta d n^{-1}$, d denoting a flexible index, $\beta = 1 - \alpha$, c is freely fixed in the value interval $[m, n]$, while the dual problem in (6.4.11) is

$$\begin{aligned} &\max yb \\ &\text{s.t. } yA = m + \beta d n^{-1}, \beta \in [0, 1], \\ &\quad\ y \geqslant 0. \end{aligned} \tag{6.4.12}$$

Theorem 6.4.2. *Let $\mu_{\tilde{\phi}}: \mathscr{R} \to [0,1]$ be a continuous and strictly monotone membership function. x_0 is a unique solution to (6.4.1) if and only if x_0 remains a parameter solution to (6.4.5) $\forall \beta \in [0,1] (\beta = 1-\alpha)$.*

Proof: Similar to the proof of Ref.[Man79], then
x_0 is a unique solution to (6.4.1)

$$\begin{aligned}
\Longleftrightarrow\ & \forall d/n \in \mathscr{R}^n, \exists x, y, \alpha \in \mathscr{R}^{n+m+1} \\
& Ax - b\alpha \geqslant 0, -yA + c\alpha = 0, y \geqslant 0 \\
& yb - cx \geqslant 0, -dn^{-1}x + dn^{-1}x_0\alpha > 0, \alpha > 0 \\
\Longleftrightarrow\ & \forall dn^{-1} \in \mathscr{R}^n, \exists x, u, y, \kappa, r \in \mathscr{R}^{n+m+k+2} \\
& -Ax + b\kappa + u = 0 \\
& yA - (\kappa c + \beta dn^{-1}) = 0, -yb + cx + dn^{-1}x_0\beta + r = 0 \\
& u, \kappa, r \geqslant 0, \beta \in [0,1], \beta + r > 0 \\
\Longleftrightarrow\ & \forall dn^{-1} \in \mathscr{R}^n, \exists x, y, \kappa \in \mathscr{R}^{n+m+1} \\
& Ax \geqslant b\kappa, cx = \kappa c x_0, \kappa \geqslant 0 \\
& yA = \kappa c + \beta dn^{-1}, yb \geqslant (\kappa c + \beta dn^{-1})x_0, \beta \in [0,1] \\
\Longleftrightarrow\ & \forall dn^{-1} \in \mathscr{R}^n, \exists y, \kappa \in \mathscr{R}^{m+1} \\
& yA = \kappa c + \beta dn^{-1}, yb = (\kappa c + \beta dn^{-1})x_0 \\
& \kappa \geqslant 0, \beta \in [0,1] \\
& (cx + \beta dn^{-1}x_0 \leqslant yb \leqslant yAx_0 = (\kappa c + \beta dn^{-1})x_0) \\
\Longleftrightarrow\ & \forall dn^{-1} \in \mathscr{R}^n, \exists y \in \mathscr{R}^m \\
& yA = c + \beta dn^{-1}, yb = (c + \beta dn^{-1})x_0 \\
& \beta \in [0,1], \text{ and let } \kappa = 1 \\
\Longleftrightarrow\ & \forall dn^{-1} \in \mathscr{R}^n, \exists \beta \in [0,1],
\end{aligned}$$

so a solution to (6.4.5) is found to be x_0.

Because x_0 denotes a feasible solution to (6.4.5), $\bar{y}$ is a feasible one to dual problem (6.4.6) coming from (6.4.5), with $\bar{y}b = (m + \beta dn^{-1})x_0$, where $\bar{y} = y(\beta)$.

But, the fuzzy solution to (6.4.7) is given by an optimal solution to the parametric linear problem [Ver84], therefore, the theorem holds.

Similar to a corollary in Ref.[Man79], we can confirm the following.

Corollary 6.4.1. *The dual optimal solution y is unique to (6.4.2) associated with a primal optimal solution x_0 to (6.4.1), if and only if, for a continuous and strictly monotone membership function $\mu_{\tilde{\phi}}: \mathscr{R} \to [0,1]$, such that $\forall \beta \in [0,1]$, y remains a dual optimal parameter solution to the perturbed linear programming (6.4.12).*

Theorem 6.4.3. *Let $\mu_{\tilde{\phi}}: \mathscr{R} \to [0,1]$ be a continuous and strictly monotone membership function. A solution x_0 is unique to linear programming (6.4.1) if only if x_0 is still a fuzzy optimal solution to fuzzy linear programming (6.4.7).*

Proof: From Lemma 6.4.1, we know (6.4.7) $\Longleftrightarrow$ (6.4.5), so, from the result where Theorem 6.4.2 is applied to (6.4.5), x_0 is a unique solution to (6.4.1) if and only if x_0 remains a parameter optimal solution to (6.4.5). But the minimization in (6.4.7) is equivalent to that in (6.4.5), and x_0 is a parameter optimum solution to (6.4.5) if and only if x_0 is a fuzzy optimal solution to (6.4.4).

Corollary 6.4.2. *The dual optimal solution y to (6.4.2) corresponding to the primal optimum solution x_0 to (6.4.1) is unique, if and only if, for a continuous and strictly monotone membership function $\mu_{\tilde{\phi}} : \mathscr{R} \to [0,1], \tilde{y}$ is still a dual optimum solution to the programming (6.4.3).*

Proof: Let $\mu_{\tilde{\phi}}$ be Formula (1.5.3) in Section 1.5. Then we have

$$\begin{aligned}(6.4.3) \Longleftrightarrow \min \quad & z = cx \\ \text{s.t.} \quad & Ax = b, \mu_{\tilde{\phi}}(c) \geqslant \beta, \beta \in [0,1] \\ & x \geqslant 0 \\ \Longleftrightarrow (6.4.11) &\end{aligned}$$

by Ref. [Ver84]. Apply Corollary 6.4.1 to (6.4.11) and the conclusion holds.

Definition 6.4.1. Let $\mu_{\tilde{A}_0}(x), \mu_{\tilde{F}}(x)$ be membership functions of fuzzy objection and fuzzy constraint. Then we call a fuzzy set $\tilde{D}$ satisfying $\mu_{\tilde{D}}(x) = \mu_{\tilde{A}_0}(x) \wedge \mu_{\tilde{F}}(x), x \geqslant 0$ a fuzzy decision for the programming

$$\begin{aligned} & \tilde{c}x \gtrsim b_0 \\ \text{s.t.} \ & Ax \lesssim b, \\ & x \geqslant 0, \end{aligned} \tag{6.4.13}$$

while we call a point x satisfying $\mu_{\tilde{D}}(x^*) = \max\limits_{x \geqslant 0}\{(1 - \mu_{\tilde{A}_0}(x)) \wedge \mu_{\tilde{F}}(x)\}$ an optimal solution to (6.4.13).

Theorem 6.4.4. *The maximization of $\mu_{\tilde{D}}(x)$ is equivalent to linear programming*

$$\begin{aligned} \min \quad & (m + \beta M_0 n^{-1})x \\ \text{s.t.} \quad & Ax \leqslant b_1 + B\beta b_2^{-1} + d\alpha, \alpha, \beta \in [0,1], \\ & x \geqslant 0, \end{aligned} \tag{6.4.14}$$

d denoting a flexible index; M_0 and B representing the length in intervals $[m, n]$ and $[b_1, b_2]$, respectively.

Proof: From Formulas (1.5.3) (1.5.4) and (1.5.5) in Section 1.5, we have

$$\begin{aligned} \max \mu_{\tilde{D}}(x) \Longleftrightarrow \max \quad & (-\tilde{c})x \\ \text{s.t.} \quad & Ax \lesssim \tilde{b} \\ & x \geqslant 0 \\ \Longleftrightarrow \min \quad & \tilde{c}x \\ \text{s.t.} \quad & \mu_{\tilde{\phi}}(c) \geqslant \beta \\ & Ax \leqslant b + d\alpha, \mu_{\tilde{\phi}}(b) \geqslant \beta \\ & \beta \in [0,1] \\ & x \geqslant 0 \\ \Longleftrightarrow (6.4.14), & \end{aligned}$$

where $\tilde{c}, \tilde{b}$ can be freely fixed in the close value interval $[m, n]$ and $[b_1, b_2]$, its degree of accomplishment is determined by Formula (1.5.3).

6.4.4 Conclusion

The method in this chapter indicates that we can change a linear programming with fuzzy variables into a dual programming with fuzzy coefficients for solution, such that the problem is solved easily.

6.5 Antinomy in Fuzzy Linear Programming

6.5.1 Introduction

In 1971, Charnes and Klingman initiated the more-for-less paradox or the less-for-more paradox of the allotment model [Ck71]. If a constant b in (6.4.1) increases by $d(>0)$, then an objective value z decreases instead. If constant c in (6.4.2) decreases by $d(>0)$, then an objective value yb increases instead. Such a strange phenomenon is called "antinomy" in mathematics. Again in 1986, Lin discussed antinomy of the general linear programming [Lin86] by taking its expansion. In 1987, Charnes, Duffuaa and Ryan also discussed "more-for-less paradox" of the general linear programming [CDR87]. In 1991, Yang and Jing again put forward another sufficiency and necessary condition in antinomy of the general linear programming and condition of the non-linear programming, where antinomy appeared [YJ91]. In 1991, author initially used the method of fuzzy sets to study antinomy of the linear programming [Cao91c].

We introduce antinomy problem in fuzzy linear programming, and present a fuzzy set method for its investigation.

6.5.2 Reason for Antinomy Emergence

Definition 6.5.1. Suppose x^0 is a basic feasible solution to (6.4.1). If its basic variable value are all positive, then we call x^0 a nondegeneration basic feasible solution; if there are some basic variable value equalling to zero, then we call x^0 a degeneration basic feasible solution.

If all basic feasible solutions in the linear programming are nondegeneration, we call them nondegeneration.

Example 6.5.1: Consider finding

$$\begin{aligned} \min \quad & z = 2x_1 + 3x_2 + x_3 + 2x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = b'_1, \\ & 4x_2 + 2x_3 + 6x_4 = b'_2, \\ & 5x_1 + 6x_2 + 5x_3 + 4x_4 = b'_3, \\ & x_1, x_2, x_3, x_4 \geqslant 0, \end{aligned}$$

where $b'_i = b_i + d_i (i = 1, 2, 3)$.

If we suppose x_1, x_2, x_3 to be basic variables, accordingly, a basis matrix B as well as an inverse matrix B^{-1} is denoted respectively by

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 5 & 6 & 5 \end{pmatrix}$$

and

$$B^{-1} = \begin{pmatrix} -4 & \frac{-1}{2} & 1 \\ -5 & 0 & 1 \\ 10 & \frac{1}{2} & -2 \end{pmatrix}.$$

When assignment volume of three products $b = (9, 12, 46)^{\mathrm{T}}$ is increases to $b' = (10, 18, 50)^{\mathrm{T}}$, the minimum cost $z = c_B x_B$ decreases from $z = 15$ to $z = 11$. Why? If the problem nondegenerates and when a negative component exists in $y = c_B B^{-1}$ or in a certain evaluation coefficient $z_s < 0$, then the objective function is

$$\begin{aligned} c_B B^{-1} b' = yb' &= yb + yd \\ &< yb = cx^* \end{aligned}$$

or

$$\begin{aligned} c_B x_B = c_B B^{-1} b' &= c_B B^{-1} b + c_B B^{-1} P_s \\ &= cx^* + \delta z_s < cx^*, \end{aligned}$$

so that antinomy appears.

Therefore, we have a discussion as follows [CDR87][Lin86]:

Corollary 6.5.1. *Let a basic solution $x^* = (x_B, x_N)$ in (6.4.1) be a nondegeneration optimum solution. If $\exists j_0 : z_{j_0} < 0$, then antinomy takes shape in (6.4.1).*

Proposition 6.5.1. *Let a basic solution $x^* = (x_B, x_N)$ in (6.4.1) be a nondegeneration optimum solution. Antinomy arises if and only if a negative component exists in $y = c_B B^{-1}$.*

Does the conclusion above hold if programming (6.4.1) degenerates?

Proposition 6.5.2. *If definition (6.4.1) denotes a degeneration linear programming, then*

$$\min\{cx | Ax = b + \sum_{j=1}^{n} \varepsilon^j P_j = b(\varepsilon), x \geqslant 0, \varepsilon > 0 \text{ sufficiently small}\} \qquad (6.5.1)$$

is a linear programming of nondegeneration.

Theorem 6.5.1. *If any basic feasible solution is $\varepsilon = 0$ in (6.5.1) when ε is sufficiently small, a basic feasible solution can be obtained to a degenerated linear programming (6.4.1).*

Proposition 6.5.3. *If a basic solution $x^*(0) = (x_B(0), x_N)$ denotes a degeneration optimum solution to linear programming (6.4.1), then there exists antinomy if and only if a negative component exists in $y = c_B B^{-1}$.*

Proof: Take a basic solution $x^*(\varepsilon) = (x_B(\varepsilon), x_N)$ in (6.5.1) into consideration, where

$$x_B(\varepsilon) = B^{-1}b_0 + \varepsilon_B + B^{-1}N\varepsilon_N - B^{-1}Nx_N = B^{-1}b(\varepsilon) - B^{-1}Nx_N$$

is nondegeneration. According to Proposition 6.5.1, if $x_B(\varepsilon)$ is a nondegeneration optimum solution, then there appears antinomy if and only if a negative component exists in $y = c_B B^{-1}$. When ε is sufficiently small and if we suppose $\varepsilon = 0$ in any basic feasible solution to (6.5.1), we can obtain a basic feasible solution to (6.4.1). If (6.5.1) is solved when ε is sufficiently small, we can get a list of basic feasible solutions $x(\varepsilon) = \{x^0(\varepsilon), x^1(\varepsilon), \cdots\}$ until we have an optimal solution $x^*(\varepsilon)$. If $\varepsilon = 0$, we also have a list of basic feasible solutions $x(0) = \{x^0(0), x^1(0), \cdots\}$ to (6.4.1). Since the coefficient matrices and the objective functions are all equal in (6.5.1) and (6.4.1), accordingly, the test numbers are identical in basic feasible solutions $x^i(\varepsilon)$ and $x^i(0)$. Therefore, $x^*(0)$ is also an optimal solution to (6.4.1). This demonstrates that $x^*(\varepsilon)$ serving as a nondegeneration optimum solution to (6.5.1) is equivalent to $x^*(0)$ serving as a degeneration optimum one to (6.4.1). At this time, the objective function denoted by

$$\begin{aligned} c_B B^{-1}b'(\varepsilon) &= yb'(\varepsilon) - yNx_N \\ &= yb_0(\varepsilon) + \delta yT - yNx_N \\ &< yb_0(\varepsilon) = cx^*(\varepsilon) \end{aligned}$$

holds when a negative component exists in y and there exists $c_B B^{-1}b'(0) < cx^*(0)$ for $\varepsilon = 0$. Therefore the proposition holds.

Corollary 6.5.2. *Antinomy arises in (6.4.1) under the condition of Proposition 6.5.3 and in the event of $\exists j_0 : z_{j_0} < 0$.*

Proof: Because they have identical coefficient matrices and objective functions, and (6.4.1) and (6.5.1) have the same test numbers in their basic feasible solutions $x^i(\varepsilon)$ as well as $x^i(0)$, we know a negative component must exist in y, in the event of $z_{j_0} = c_B B^{-1}P_j = yP_j < 0$, with

$$\begin{aligned} c_B x_B(\varepsilon) &= c_B B^{-1}b_0(\varepsilon) + \delta c_B B^{-1}P_s - c_B B^{-1}Nx_N \\ &= cx^*(\varepsilon) + \delta z_s < cx^*(\varepsilon) \end{aligned}$$

($\delta > 0$ sufficiently small), so, $c_B x_B(0) < cx^*(0)$ for $\varepsilon = 0$. Therefore, the corollary holds.

In conclusion, whether a classical linear programming degenerates or not results in the fact that antinomy comes into being. If we try to keep antinomy from being contrary, we only change the equal-sign into an inequality sign in constraint condition.

Proposition 6.5.4. *Let $\mu_{\tilde{\phi}}$ be a continuous and strictly monotone function. If a basic solution $x^* = (x_B, x_N)^T$ nondegenerates in fuzzy linear programming*

$$\begin{aligned} &\min \tilde{z} = \tilde{c}x \\ &s.t.\ Ax = b, \\ &\quad\ \ x \geqslant 0, \end{aligned} \tag{6.5.2}$$

then antinomy arises if and only if a negative component exists in a fuzzy shadow price $\tilde{y} = \tilde{c}_B B^{-1}$.

Proof: *Necessity.* Since (6.5.2)$\Longleftrightarrow$(6.4.5) and (6.4.4) $\Longleftrightarrow$(6.4.6), if $\tilde{y} = \tilde{c}_B B^{-1} \geqslant \tilde{0} \Longleftrightarrow y(\beta) = (\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1} \geqslant 0$, and then to $\forall T \geqslant 0$, when $b \to b + T$, for any feasible solution in this problem with a soft constraint, we know

$$\begin{aligned} \mu_{\tilde{D}}(x) = \mu_{\tilde{\phi}^{-1}}(\beta)x &\geqslant y(b+T) \\ &\geqslant yb = \mu_{\tilde{\phi}^{-1}}(\beta)x^* = \mu_{\tilde{D}}(x^*) \end{aligned}$$

from a dual theorem of ordinary parameter linear programming, such that $\tilde{c}x \geqslant \tilde{y}b = \tilde{c}x^*$.

Sufficiency. If there exists a negative component in $\tilde{y}$, then $\exists T \geqslant 0, \sum\limits_{i=1}^{m} t_i > 0$, such that

$$\tilde{y}T<0 \Longleftrightarrow \mu_{\tilde{\phi}^{-1}}(\beta)B^{-1}T \leqslant 0.$$

Let $b' = b + \delta T (\delta > 0)$. Then the problem with a soft constraint concerning a basic solution in basis B is

$$\begin{aligned} &x_B = B^{-1}b' = B^{-1}b + \delta B^{-1}T, \\ &x_N = 0. \end{aligned}$$

x^* nondegenerates on the proposition assumption, $x_B \geqslant 0$ means a basic feasible solution having test numbers unchangeable when $\delta > 0$ is sufficiently small. Therefore it also belongs to an optimal solution with soft constraints and $\tilde{y} = \tilde{c}_B B^{-1}$ is still a fuzzy optimal solution to the dual problem. But the objective value $\forall \beta \in [0,1]$ is denoted by formula below, i.e.,

$$\begin{aligned} (\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1}b' = y(\beta)b' &= y(\beta)b + \delta y(\beta)T \\ &< y(\beta)b = cx^* \\ \Longleftrightarrow \tilde{c}_B B^{-1}b' = \tilde{y}b' &= \tilde{y}b + \delta\tilde{y}T \\ &< \tilde{y}b = \tilde{c}x^*, \end{aligned}$$

such that antinomy arises in (6.5.2).

Corollary 6.5.3. *Let $\mu_{\tilde{\phi}}$ be a continuous and strictly monotone function. If a basic solution $x^* = (x_B, x_N)$ to (6.5.2) denotes a nondegeneration optimum solution, then the condition where antinomy arises is $\exists j_0$, such that $\tilde{z}_{j_0} < \tilde{0}$.*

Proof: From the proof of Proposition 6.5.4, we know

$$\begin{aligned}\sup_{Ax=b} \mu_{\tilde{\phi}_0}(x) &= \sup \mu_{\tilde{\phi}_0}(x_B)\\ &= \sup_{\beta\in[0,1]} (\mu_{\tilde{\phi}^{-1}}(\beta))_B x_B\\ &= \sup_{\beta\in[0,1]} \{(\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1}b + P_j\delta(\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1}P_jT\}\\ &< \sup_{\beta\in[0,1]} (\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1}bP_j.\end{aligned}$$

(Because $z_j = (\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1}P_j < 0$, where $x_B = B^{-1}b + \delta B^{-1}T, x_N = 0$, we know there must exist a negative component in $y(\beta) = (\mu_{\tilde{\phi}^{-1}}(\beta))_B B^{-1})$.

It is equivalent that there must be a fuzzy negative component in $\tilde{y}$ from the knowledge of $\tilde{z}_j = \tilde{c}_B B^{-1}P_j = \tilde{y}P_j < \tilde{0}$, such that we have

$$\tilde{c}_B x_B = \tilde{c}x^* + \delta\tilde{z}_j < \tilde{c}x^*.$$

Proposition 6.5.5. *If definition (6.5.2) serves as a degeneration fuzzy linear programming, then*

$$\begin{aligned}&\min \tilde{z} = \tilde{c}x\\ &s.t.\ \ Ax = b + \sum_{j=1}^{n} \varepsilon^j P_j = b(\varepsilon)\\ &\qquad x \geqslant 0\end{aligned} \tag{6.5.3}$$

serves as a nondegeneration fuzzy linear programming, where ε^j is a sufficiently small positive number.

Proposition 6.5.6. *Let $\mu_{\tilde{\phi}}$ be a continuous and strictly monotone function. If a basic solution $x^*(0) = (x_B(0), x_N)$ to (6.5.2) denotes a degeneration optimum solution, then antinomy appears if and only if a fuzzy negative component exists in $\tilde{y} = \tilde{c}_B B^{-1}$.*

Corollary 6.5.4. *Let $\mu_{\tilde{\phi}}$ be a continuous and strictly monotone function. Suppose that a basic solution $x^*(0) = (x_B(0), x_N)$ to (6.5.2) is a degeneration optimum solution, then the condition where antinomy arises is $\exists j_0$, such that $\tilde{z}_{j_0} < \tilde{0}$.*

In fact, because

$$\begin{aligned}&\min\ \ \tilde{c}x\\ &\text{s.t.}\ \ Ax = b(\varepsilon)\\ &\qquad x \geqslant 0\end{aligned} \tag{6.5.4}$$

is equivalent to

$$\begin{aligned}&\min\ \ \mu_{\tilde{\phi}^{-1}}(\beta)x\\ &\quad\text{s.t.}\ \ Ax = b(\varepsilon), \beta \in [0,1],\\ &\qquad\ x \geqslant 0,\end{aligned} \tag{6.5.5}$$

the dual form of (6.5.5) is

$$\begin{aligned}
\max\ \ & yb(\varepsilon) \\
\text{s.t.}\ \ & yA = \mu_{\tilde{\phi}^{-1}}(\beta), \beta \in [0,1], \\
& y \geqslant 0.
\end{aligned}$$

Therefore, (6.5.4) is equivalent to

$$\begin{aligned}
\max\ \ & \tilde{y}b(\varepsilon) \\
\text{s.t.}\ \ & \tilde{y}A = \tilde{c}, \\
& \tilde{y} \geqslant \tilde{0}.
\end{aligned}$$

From Proposition 6.5.3 and Corollary 6.5.1, we know properties as follows:

a. If there exists a degeneration optimum basic solution $x^*(0,\beta)$ in the classical linear programming (6.5.5) with parameter variable β, then the antinomy appears if and only if a negative component exists in $\tilde{y} = \tilde{c}_B B^{-1}$.

b. Under the condition of a, if $\exists j_0 < 0$, then antinomy appears in (6.5.5).

6.5.3 Example

Example 6.5.2: The fuzzy linear programming corresponding to Example 6.5.1 in this section is

$$\begin{aligned}
\min\ \ & z = 2x_1 + 3x_2 + x_3 + 2x_4 \\
\text{s.t.}\ \ & x_1 + x_2 + x_3 + x_4 \lesssim 9, \\
& 4x_2 + 2x_3 + 6x_4 \lesssim 12, \\
& 5x_1 + 6x_2 + 5x_3 + 4x_4 \lesssim 46, \\
& x_i \geqslant 0 (i = 1, \cdots, 4).
\end{aligned} \tag{6.5.6}$$

Assume $d_1 = 1, d_2 = 6, d_3 = 4$ and we make a parameter programming, then (6.5.6) is turned into

$$\begin{aligned}
\min\ \ & z = 2x_1 + 3x_2 + x_3 + 2x_4 \\
\text{s.t.}\ \ & x_1 + x_2 + x_3 + x_4 + x_5 = 10, \\
& 4x_2 + 2x_3 + 6x_4 + x_6 = 18, \\
& 5x_1 + 6x_2 + 5x_3 + 4x_4 + x_7 = 50, \\
& x_i \geqslant 0 (i = 1, \cdots, 7).
\end{aligned}$$

Under the unchangeable condition of basis matrix B, an optimal parameter solution denotes

$$x = B^{-1}b(\alpha) = (4 - 3\frac{1}{2}\alpha, 1 - 4\alpha, 4 + 8\frac{1}{2}\alpha)^{\mathrm{T}},$$

and optimal value is

$$z = 15 - 10\frac{1}{2}\alpha.$$

When b is added from $b = (9, 12, 46)^{\mathrm{T}}$ to $b' = (10, 18, 50)^{\mathrm{T}}$, z decreases from $z = 15$ to 11. Therefore, antinomy comes into being because negative components exist in a solution vector for $\alpha > \dfrac{1}{4}$.

6.5.4 Conclusion

On the whole, no matter whether (6.5.2) is a generation or nongeneration fuzzy linear programming, antinomy appears in both of them. If we prevent antinomy in fuzzy linear programming from being contrary, the constraint equal-sign can be turned into a soft constraint.

Overall, if antinomy is changed into formula (6.4.12) for solution, the antinomy of a fuzzy linear programming is not contrary. If the optimal solution in a primal linear programming is unique, then antinomy does not exist, which can be concluded as solutions to fuzzy linear programming (6.5.2). In the light of Theorem 6.4.4, an ordinary linear programming is only a particular example of fuzzy linear programming (6.4.12) for $d_i = 0$. Therefore, it can be changed into finding solutions by a fuzzy set method no matter whether it is antinomy of linear programming or fuzzy linear programming.

6.6 Fuzzy Linear Programming Based on Fuzzy Numbers Distance

6.6.1 Introduction

In the section, we discuss the constraint conditions with fuzzy coefficients, whose standard form is:

$$\begin{aligned} &\max\ z = cx \\ &\text{s.t.}\quad \widetilde{A}x \leqslant \widetilde{b}, \\ &\qquad\ \ x \geqslant 0, \end{aligned}$$

where $c = (c_1, c_2, \cdots, c_n)$ is an n-dimensional clear row vector, $\widetilde{A} = (\widetilde{a}_{ij})$ is an $m \times n$ fuzzy number matrix, $\widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_m)^T$ is an m-dimensional fuzzy line vector and $x = (x_1, x_2, \cdots, x_n)^T$ is a decisive vector. Solving this kind of fuzzy linear programming is based on an order relation between fuzzy numbers, by which we can transform fuzzy linear programming into clear linear programming.

6.6.2 Distance

A. Distance between Interval Numbers

Assume $a = [a_1, a_2]$ and $b = [b_1, b_2]$ to be two interval numbers, $a = b \Longleftrightarrow a_1 = b_1$ and $a_2 = b_2$.

Similarly to Ref. [LiuH04], we also consider the different value between corresponding point and point in the intervals, giving a new definition on the distance between interval numbers.

Definition 6.6.1. Let $a = [a_1, a_2]$ and $b = [b_1, b_2]$ be two interval numbers. Then define

$$d(a,b) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |[\ \frac{a_1+a_2}{2} + x(a_2-a_1)] - [\ \frac{b_1+b_2}{2} + x(b_2-b_1)]|dx \quad (6.6.1)$$

as the distance between a and b.

Regarding the distance $d(a,b)$ between interval numbers as a proposition, we can verify satisfaction of the three conditions in distance.

In fact, let

$$f(x) = |[\ \frac{a_1+a_2}{2} + x(a_2-a_1)] - [\ \frac{b_1+b_2}{2} + x(b_2-b_1)]|.$$

Since $f(x)$ is a simple function about x, it concludes that $f(x)$ is continuous, so $d(a,b)$ is integrable.

(1) Since $f(x) \geqslant 0$, also by the continuity and integrable character of $f(x)$, we have $d(a,b) \geqslant 0$. If $d(a,b) = 0$, then $f(x) = 0$.

When $f(x) = 0$, we have $[\frac{a_1+a_2}{2} + x(a_2-a_1)] - [\ \frac{b_1+b_2}{2} + x(b_2-b_1)] = 0$, i.e., $(\frac{a_1+a_2}{2} - \frac{b_1+b_2}{2}) + x[(a_2-a_1)-(b_2-b_1)] = 0$ $(\forall x \in [-\frac{1}{2}, \frac{1}{2}])$, which satisfies $a_1 = b_1, a_2 = b_2$, hence $a = b$.

On the contrary, when $a = b$, i.e., $a_1 = b_1, a_2 = b_2$, we have $f(x) = 0$. Thus

$$d(a,b) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = 0.$$

(2) $d(a,b) = d(b,a)$ holds obviously.

(3) For any interval number c, where $c = [c_1, c_2]$, denote $a_x = \frac{a_1+a_2}{2} + x(a_2-a_1), b_x = \frac{b_1+b_2}{2} + x(b_2-b_1), c_x = \frac{c_1+c_2}{2} + x(c_2-c_1)$. Then $0 \leqslant |a_x - b_x| \leqslant |a_x - c_x| + |c_x - b_x|$ satisfies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |a_x - b_x|dx \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} |a_x - c_x|dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} |c_x - b_x|dx.$$

It follows that $d(a,b) \leqslant d(a,c) + d(c,b)$ holds.

In the distance formula, the integralled function $f(x) = |[\ \frac{a_1+a_2}{2} + x(a_2 - a_1)] - [\ \frac{b_1+b_2}{2} + x(b_2-b_1)]|$ is the distance function between corresponding point and point in two intervals. At $x = -\frac{1}{2}$, $f(-\frac{1}{2})$ is the distance between left endpoints of the two interval numbers; at $x = \frac{1}{2}$, $f(\frac{1}{2})$ is the distance between right endpoints of the two interval numbers.

B. Distance between Fuzzy Numbers

Definition 6.6.2 [TD02]. The fuzzy set $\widetilde{A}$ in the real number set is called an L-R fuzzy number. If its membership function is:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\dfrac{a_2 - x}{a_2 - a_1}), & a_1 \leqslant x \leqslant a_2, \\ 1, & a_2 \leqslant x \leqslant a_3, \\ R(\dfrac{x - a_3}{a_4 - a_3}), & a_3 \leqslant x \leqslant a_4, \\ 0, & x < a_1, x > a_4, \end{cases} \tag{6.6.2}$$

where L, R are strictly decreasing functions in $[0, 1]$, and satisfy

$$L(x) = R(x) = 1 (x \leqslant 0); \quad L(x) = R(x) = 0 (x \geqslant 1),$$

the fuzzy number is denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Especially, when $L(x) = R(x) = 1 - x$, fuzzy number defined in (6.6.2) is a trapeziform fuzzy number, denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4)$; when $L(x) = R(x) = 1 - x$ and $a_2 = a_3$, fuzzy number defined in (6.6.2) is a triangular fuzzy number, denoted by $\widetilde{A} = (a_1, a_2, a_3)$.

$\forall \alpha \in [0, 1]$, α-level curve of fuzzy number $\widetilde{A}$ denotes an interval number:

$$\widetilde{A}_\alpha = [A_L(\alpha), A_R(\alpha)],$$

where $A_L(\alpha) = a_2 - (a_2 - a_1) L_A^{-1}(\alpha); A_R(\alpha) = a_3 + (a_4 - a_3) R_A^{-1}(\alpha)$.

By the distance between interval numbers, we define the distance between fuzzy numbers as follows.

Definition 6.6.3. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers, and

$$\widetilde{A}_\alpha = [A_L(\alpha), A_R(\alpha)] = [a_2 - (a_2 - a_1) L_A^{-1}(\alpha),\ a_3 + (a_4 - a_3) R_A^{-1}(\alpha)],$$
$$\widetilde{B}_\alpha = [B_L(\alpha), B_R(\alpha)] = [b_2 - (b_2 - b_1) L_B^{-1}(\alpha),\ b_3 + (b_4 - b_3) R_B^{-1}(\alpha)].$$

Then we define the distance between $\widetilde{A}$ and $\widetilde{B}$ by

$$D(\widetilde{A}, \widetilde{B}) = \int_0^1 d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) d\alpha, \tag{6.6.3}$$

where

$$d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |[\frac{a_L(\alpha) + a_R(\alpha)}{2} + x(a_R(\alpha) - a_L(\alpha))] \\ -[\frac{b_L(\alpha) + b_R(\alpha)}{2} + x(b_R(\alpha) - b_L(\alpha)]| dx.$$

In fact, let

$$f(x,\alpha) = |[\frac{a_L(\alpha)+a_R(\alpha)}{2} + x(a_R(\alpha)-a_L(\alpha))] \\ -[\frac{b_L(\alpha)+b_R(\alpha)}{2} + x(b_R(\alpha)-b_L(\alpha)]|.$$

Since $f(x,\alpha)$ is a simple function about $x, f(x,\alpha)$ is continues; it concludes that $d(\widetilde{A}_\alpha, \widetilde{B}_\alpha)$ is also continuous, so $D(\widetilde{A}, \widetilde{B})$ is integrable.

(1) Since $d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) \geqslant 0$, also by the continuity and integrable character of $d(\widetilde{A}_\alpha, \widetilde{B}_\alpha)$, we have $D(\widetilde{A}, \widetilde{B}) \geqslant 0$. If $D(\widetilde{A}, \widetilde{B}) = 0$, it satisfies $d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) = 0$.

When $d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) = 0$, by the distance definition between interval numbers, we know $a_L(\alpha) = b_L(\alpha),\ \ a_R(\alpha) = b_R(\alpha)$, then $\widetilde{A} = \widetilde{B}$.

On the contrary, when $\widetilde{A} = \widetilde{B}$, i.e., $a_L(\alpha) = b_L(\alpha)$, $a_R(\alpha) = b_R(\alpha)$. $d(\widetilde{A}_\alpha\ \widetilde{B}_\alpha) = 0$, the result holds clearly true. So $D(\widetilde{A}, \widetilde{B}) = \int_0^1 d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) d\alpha = 0$.

(2) $D(\widetilde{A}, \widetilde{B}) = D(\widetilde{B}, \widetilde{A})$ holds obviously.

(3) For any fuzzy number $\tilde{C}$, by the distance definition between interval numbers:

$$0 \leqslant d(\widetilde{A}_\alpha, \widetilde{B}_\alpha) \leqslant d(\widetilde{A}_\alpha, \widetilde{C}_\alpha) + d(\widetilde{B}_\alpha, \widetilde{C}_\alpha),$$

where $\widetilde{C}_\alpha = [C_L(\alpha), C_R(\alpha)] = [C_2-(C_2-C_1)L_C^{-1}(\alpha), C_3+(C_4-C_3)R_C^{-1}(\alpha)]$, so

$$\int_0^1 d(\widetilde{A}_\alpha, \widetilde{B}_\alpha)d\alpha \leqslant \int_0^1 d(\widetilde{A}_\alpha, \widetilde{C}_\alpha)d\alpha + \int_0^1 d(\widetilde{C}_\alpha, \widetilde{B}_\alpha)d\alpha$$

holds.

6.6.3 Ranking Fuzzy Numbers

Here, we present a ranking idea about fuzzy numbers: before ranking fuzzy numbers, we fix a real number M as refereing object (M is supremum about support set of $\widetilde{A}$ and support set of $\widetilde{B}$). The nearer a fuzzy number to M, the larger it is; that is, the smaller the distance to M, the larger a fuzzy number is.

Definition 6.6.4. If $M = \sup(s(A) \cup s(B))$, we call M the supremum of $\widetilde{A}$ and $\widetilde{B}$, where $s(A)$ and $s(B)$ are the support sets of $\widetilde{A}$ and $\widetilde{B}$, respectively.

By Definition 6.6.3, we can obtain the distance from fuzzy number $\widetilde{A}$ to M:

$$D(\widetilde{A}, M) = \int_0^1 \{\int_{-\frac{1}{2}}^{\frac{1}{2}} \{M - [\frac{a_L(\alpha)+a_R(\alpha)}{2} + x(a_R(\alpha)-a_L(\alpha))]\}dx\}d\alpha,$$

$\forall \alpha \in [0,1]$. Coordinate:

$$D(\widetilde{A}, M) = M - \frac{a_2+a_3}{2} + \frac{a_2-a_1}{2}\int_0^1 L_A^{-1}(\alpha) - \frac{a_4-a_3}{2}\int_0^1 R_A^{-1}(\alpha). \quad (6.6.4)$$

Similarly

$$D(\widetilde{B}, M) = M - \frac{b_2 + b_3}{2} + \frac{b_2 - b_1}{2} \int_0^1 L_B^{-1}(\alpha) - \frac{b_4 - b_3}{2} \int_0^1 R_B^{-1}(\alpha). \quad (6.6.5)$$

Thus, we can obtain the definition of ranking fuzzy numbers as follows in light of this.

Definition 6.6.5. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers, and M be the supremum of $\widetilde{A}$ and $\widetilde{B}$. Then

(1) When $D(\widetilde{A}, M) < D(\widetilde{B}, M)$, we call $\widetilde{A} > \widetilde{B}$;
(2) When $D(\widetilde{A}, M) = D(\widetilde{B}, M)$, we call $\widetilde{A} = \widetilde{B}$;
(3) When $D(\widetilde{A}, M) > D(\widetilde{B}, M)$, we call $\widetilde{A} < \widetilde{B}$.

Especially, when $\widetilde{A}$ and $\widetilde{B}$ are trapeziform fuzzy numbers or triangular numbers, respectively, we can get concrete expressions:

1^0 When $\widetilde{A}$ and $\widetilde{B}$ are trapeziform fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4), \widetilde{B} = (b_1, b_2, b_3, b_4)$, then

$$D(\widetilde{A}, M) = M - \frac{a_1 + a_2 + a_3 + a_4}{4}; \quad D(\widetilde{B}, M) = M - \frac{b_1 + b_2 + b_3 + b_4}{4}.$$

By Definition 6.6.5:

(1) $a_1 + a_2 + a_3 + a_4 > b_1 + b_2 + b_3 + b_4 \Leftrightarrow \widetilde{A} > \widetilde{B}$;
(2) $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4 \Leftrightarrow \widetilde{A} = \widetilde{B}$;
(3) $a_1 + a_2 + a_3 + a_4 < b_1 + b_2 + b_3 + b_4 \Leftrightarrow \widetilde{A} < \widetilde{B}$.

2^0 When $\widetilde{A}$ and $\widetilde{B}$ are triangular numbers $\widetilde{A} = (a_1, a_2, a_3), \widetilde{B} = (b_1, b_2, b_3)$, then

$$D(\widetilde{A}, M) = M - \frac{a_1 + 2a_2 + a_3}{4}; \quad D(\widetilde{B}, M) = M - \frac{b_1 + 2b_2 + b_3}{4}.$$

By Definition 6.6.5:

(1) $a_1 + 2a_2 + a_3 > b_1 + 2b_2 + b_3 \Leftrightarrow \widetilde{A} > \widetilde{B}$;
(2) $a_1 + 2a_2 + a_3 = b_1 + 2b_2 + b_3 \Leftrightarrow \widetilde{A} = \widetilde{B}$;
(3) $a_1 + 2a_2 + a_3 < b_1 + 2b_2 + b_3 \Leftrightarrow \widetilde{A} < \widetilde{B}$.

6.6.4 Linear Programming in Constraint with Fuzzy Coefficients

Assume that the linear programming in constraint with fuzzy coefficient is defined as follows:

$$\begin{aligned} \max\ & z = cx \\ \text{s.t.}\ & \widetilde{A}x \leqslant \widetilde{b}, \\ & x \geqslant 0, \end{aligned} \quad (6.6.6)$$

denoted as:

$$
\begin{aligned}
& \max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
& \text{s.t.} \quad \widetilde{a}_{i1}x_1 + \widetilde{a}_{i2}x_2 + \cdots + \widetilde{a}_{in}x_n \leqslant \widetilde{b}_i, \\
& \qquad\; x_1,x_2,\cdots,x_n \geqslant 0 \quad i = 1, 2, \cdots, m,
\end{aligned} \tag{6.6.7}
$$

where fuzzy numbers are triangular fuzzy ones, i.e.,

$$
\begin{aligned}
& \widetilde{a}_{i1} = (a_{i11}, a_{i12}, a_{i13}), \widetilde{a}_{i2} = (a_{i21}, a_{i22}, a_{i23}), \cdots, \\
& \widetilde{a}_{in} = (a_{in1}, a_{in2}, a_{in3}); \widetilde{b}_i = (b_{i1}, b_{i2}, b_{i3}).
\end{aligned}
$$

By Zadeh's extension principle, the sum of any triangular fuzzy numbers is still a triangular one. Formula (6.6.7) is equivalent to the format as follows:

$$
\begin{aligned}
& \max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
& \text{s.t.} \ (a_{i11}x_1 + a_{i21}x_2 + \cdots + a_{in1}x_n, a_{i12}x_1 + a_{i22}x_2 + \cdots + a_{in2}x_n, \\
& \qquad a_{i13}x_1 + a_{i23}x_2 + \cdots + a_{in3}x_n) \leqslant (b_{i1}, b_{i2}, b_{i3}), \\
& \qquad x_1,x_2,\cdots,x_n \geqslant 0 \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{6.6.8}
$$

By Definition 6.6.5 and the method to ranking triangular fuzzy numbers, we transform Formula (6.6.8) into a linear programming as follows:

$$
\begin{aligned}
& \max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
& \text{s.t.} \ (a_{i11}x_1 + a_{i21}x_2 + \cdots + a_{in1}x_n) \\
& \qquad +2(a_{i12}x_1 + a_{i22}x_2 + \cdots + a_{in2}x_n) \\
& \qquad +(a_{i13}x_1 + a_{i23}x_2 + \cdots + a_{in3}x_n) \leqslant b_{i1} + 2b_{i2} + b_{i3}, \\
& \qquad x_1,x_2,\cdots,x_n \geqslant 0 \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{6.6.9}
$$

6.6.5 Numerical Example

Example 6.6.1: Find solution to the linear programming in constraint with fuzzy coefficients:

$$
\begin{aligned}
& \max z = 3x_1 + 4x_2 \\
& \text{s.t.} \ \widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 \leqslant \widetilde{b}_1, \\
& \qquad \widetilde{a}_{21}x_1 + \widetilde{a}_{22}x_2 \leqslant \widetilde{b}_2, \\
& \qquad x_1, x_2 \geqslant 0,
\end{aligned}
$$

where $\widetilde{a}_{11} = (3, 4, 4), \widetilde{a}_{12} = (20, 20, 21), \widetilde{a}_{21} = (11, 12, 13), \widetilde{a}_{22} = (5.4, 6.4, 7.4)$, $\widetilde{b}_1 = (4500, 4600, 4800), \widetilde{b}_2 = (4600, 4800, 5250)$.

Solution: By Formula (6.6.9), transform the fuzzy linear programming into a clear linear programming

$$
\begin{aligned}
& \max z = 3x_1 + 4x_2 \\
& \text{s.t.} \ (3x_1 + 20x_2) + 2(4x_1 + 20x_2) + (4x_1 + 21x_2) \\
& \qquad\qquad \leqslant 4500 + 2 \times 4600 + 4800, \\
& \qquad (11x_1 + 5.4x_2) + 2(12x_1 + 6.4x_2) + (13x_1 + 7.4x_2) \\
& \qquad\qquad \leqslant 4600 + 2 \times 4800 + 5250, \\
& \qquad x_1, x_2 \geqslant 0,
\end{aligned} \tag{6.6.10}
$$

we obtain $x_1^* = 315 \quad x_2^* = 170 \quad z_1^* = 1625$.

By the ranking idea from Ref. [LiR02], we transform the fuzzy linear programming into a clear linear programming as follow:

$$\begin{aligned} \max\ & z = 3x_1 + 4x_2 \\ \text{s.t.}\ \ & 3x_1 + 20x_2 \leqslant 4500, \\ & 4x_1 + 20x_2 \leqslant 4600, \\ & 4x_1 + 21x_2 \leqslant 4800, \\ & 11x_1 + 5.4x_2 \leqslant 4600, \\ & 12x_1 + 6.4x_2 \leqslant 4800, \\ & 13x_1 + 7.4x_2 \leqslant 5250, \\ & x_1, x_2 \geqslant 0, \end{aligned} \tag{6.6.11}$$

and we obtain $x_1^* = 308 \ \ x_2^* = 168 \ \ z_2^* = 1598$.

Obviously $z_1^* > z_2^*$, at the same time, the number of constraint conditions in (6.6.10) reduces by four times compared to the numbers in (6.6.11), which indicates the ranking rule in the paper is superior to the ranking rule in Ref. [LiR02], consequently we gain a better optimal value of the linear programming in constraint with fuzzy coefficients.

6.6.6 Conclusion

We propose a new distance between fuzzy numbers based on the distance between interval numbers. In sighting the ranking idea, we get a ranking rule about fuzzy numbers. On the basis of the ranking rule, we gain a new approach to linear programming in triangular fuzzy numbers with coefficients. At the same time, we use the simplicity of triangular fuzzy numbers in solving the problem. But it remains to research the linear programming in general fuzzy coefficients.

6.7 Linear Programming with L-R Coefficients

6.7.1 Introduction

Consider linear programming

$$\begin{aligned} \widetilde{\max}\ & \tilde{z} = \tilde{c}x \\ \text{s.t.}\ \ & \tilde{A}x \leqslant \tilde{b}, \\ & x \geqslant 0, \end{aligned} \tag{6.7.1}$$

where $\tilde{c} = (\tilde{c}_1, \cdots, \tilde{c}_n), \tilde{b} = (\tilde{b}_1, \cdots, \tilde{b}_m)^T$ are L-R vectors, $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ an L-R matrix, $\tilde{c}_j = (c_j, \underline{c_j}, \ \overline{c_j})_{LR}, \ \tilde{b}_i = (b_i, \underline{b_i}, \ \overline{b_i})_{LR}$ and $\tilde{a}_{ij} = (a_{ij}, \ \underline{a_{ij}}, \ \overline{a_{ij}})_{LR}$ L-R numbers, and $x = (x_1, x_2, \cdots, x_n)^T$ an ordinarily variable vector.

Now two kinds of situations are discussed as follows respectively.

6.7.2 Linear Programming in Constraints with L-R Coefficients

Consider

$$\begin{aligned} \max\ & z = cx \\ \text{s.t.}\ & \tilde{A}x \leqslant \tilde{b}, \\ & x \geqslant 0. \end{aligned} \tag{6.7.2}$$

Because $\tilde{a}_{ij}, \tilde{b}_i (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ are all L-R numbers, and $x_j \geqslant 0$, then

$$\sum_{j=1}^{n} \widetilde{a}_{ij} x_j = (\sum_{j=1}^{n} a_{ij} x_j,\ \sum_{j=1}^{n} \underline{a}_{ij} x_j,\ \sum_{j=1}^{n} \overline{a}_{ij} x_j)_{LR}$$

is still an L-R number, hence

$$\begin{aligned} & \sum_{j=1}^{n} \widetilde{a}_{ij} x_j \leqslant \widetilde{b_i} \\ \Longleftrightarrow & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \\ & \sum_{j=1}^{n} \underline{a}_{ij} x_j \geqslant \underline{b}_i, \\ & \sum_{j=1}^{n} \overline{a}_{ij} x_j \leqslant \overline{b}_i, \end{aligned} \tag{6.7.3}$$

written down as $A = (a_{ij}), \underline{A} = (\underline{a}_{ij}), \overline{A} = (\overline{a}_{ij}), b = (b_1, b_2, \cdots, b_m)^T, \underline{b} = (\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_m)^T$, $\overline{b} = (\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_m)^T$. Therefore (6.7.2) can be rewritten as the following ordinary linear programming with $3m$ linear inequality constraints, i.e.,

$$\begin{aligned} \max\ & z = cx \\ \text{s.t.}\ & Ax \leqslant b, \\ & \underline{A}x \geqslant \underline{b}, \\ & \overline{A}x \leqslant \overline{b}, \\ & x \geqslant 0. \end{aligned} \tag{6.7.4}$$

It is worthwhile to point out that turning (6.7.2) into (6.7.4) is irrelevant with choice of concrete appearance in reference function L and R in L-R number.

We only consider two variable linear programming, and illustrate a method to it (may also use a simplex method to it).

Example 6.7.1: A person on a business trip, needs to take two kinds of goods, each wrapped heavy by "6 kg possibility more" of Goods A (denoted as $\tilde{6} = (6, 0, 1)_{LR}$), worth 20 dollars. Goods B wrapped heavy by "2 kg or so" (denoted as $\tilde{2} = (2, 1, 1)_{LR}$), worth 10 dollars. This person wishes to take "about 21 kg" at most once (it can be denoted as $\tilde{21} = (21, 1, 5)_{LR}$), hoping the total value of goods he takes is the greatest.

Solution: Suppose that Goods A he takes is package x_1, and B is package x_2, then the problem involves finding a solution to linear programming in constraints with fuzzy coefficients as follows:

$$\begin{array}{ll} \max & z = 20x_1 + 10x_2 \\ \text{s.t.} & \tilde{6}x_1 + \tilde{2}x_2 \leqslant \tilde{21}, \\ & x_1 \geqslant 0, \\ & x_2 \geqslant 0. \end{array} \tag{6.7.5}$$

It is equivalent to a solution to an ordinary linear programming

$$\begin{array}{ll} \max & z = 20x_1 + 10x_2 \\ \text{s.t.} & 6x_1 + 2x_2 \leqslant 21, \\ & \qquad\quad x_2 \geqslant 1, \\ & x_1 + x_2 \leqslant 5 \\ & x_1 \geqslant 0, \\ & x_2 \geqslant 0. \end{array}$$

Use an illustrating method to the problem (see Figure 6.7.1), the optimal solution is to get $x_1^* = \dfrac{11}{4}, x_2^* = \dfrac{9}{4}$, the optimal value is $z^* = \dfrac{310}{4} = 77\dfrac{1}{2}$.

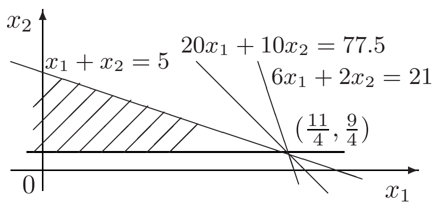

Fig. 6.7.1. Illustrating Method to (6.7.5)

If the goods allow to be torn open, then Goods A he takes can be $2\dfrac{3}{4}$ packages, B $2\dfrac{1}{4}$ packages, total worth 77.5 dollars. If the goods must be taken by whole packages, it needs taking an integral for the restrict x_1, x_2, and that is, to solve it with an integral programming method. The result is that Goods A he would take is 2 packages, B is 3 packages (or A is 3, B is 1), the total value amounting to 70 dollars.

6.7.3 Linear Programming in Object with L-R Coefficient

Consider the problem as follows:

$$\begin{aligned} \widetilde{\max}\ \tilde{z} &= \tilde{c}x \\ \text{s.t.}\ \ Ax &\leqslant b, \\ x &\geqslant 0. \end{aligned} \tag{6.7.6}$$

Because

$$\tilde{c} = (\tilde{c}_1, \cdots, \tilde{c}_n), \tilde{c}_j = (c_j, \underline{c_j}, \overline{c_j})_{LR},$$

$$\tilde{z} = (z, \underline{z},\ \overline{z})_{LR} = (\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} \underline{c}_j x_j, \sum_{j=1}^{n} \overline{c}_j x_j)_{LR}$$

are all L-R numbers, and according to approximately formula of $\widetilde{\max}$, (6.7.6) is approximately equivalent to a linear programming with 3 objectives

$$\begin{aligned} &\max(z = \sum_{j=1}^{n} c_j x_j = cx) \\ &\min(\underline{z} = \sum_{j=1}^{n} \underline{c}_j x_j = \underline{c}x) \\ &\max(\overline{z} = \sum_{j=1}^{n} \overline{c}_j x_j = \overline{c}x) \\ &\text{s.t. } Ax \leqslant b, \\ &\qquad x \geqslant 0. \end{aligned}$$

Example 6.7.2: Find fuzzy linear programming as follow:

$$\begin{aligned} \widetilde{\max}\ \tilde{z} &= \tilde{20}x_1 + \tilde{10}x_2 \\ \text{s.t.}\quad 6x_1 + 2x_2 &\leqslant 21, \\ x_1 &\geqslant 0, \\ x_2 &\geqslant 0, \end{aligned}$$

where $\tilde{20} = (20, 3, 4)_{LR}, \tilde{10} = (10, 2, 1)_{LR}$.

This problem is approximately equivalent to

$$\begin{aligned} &\max z = 20x_1 + 10x_2 \\ &\min \underline{Z} = 3x_1 + 2x_2 \\ &\max \overline{Z} = 4x_1 + x_2 \\ &\text{s.t. } 6x_1 + 2x_2 \leqslant 21, \\ &\qquad x_1 \geqslant 0, \\ &\qquad x_2 \geqslant 0. \end{aligned}$$

Find an optimal solution to each objective respectively

$$x_1^{(1)}=0,\ x_2^{(1)}=10.5,\ Z^{(1)}=105, \text{ when } \underline{Z}^{(1)}=21, \overline{Z}^{(1)}=10.5.$$
$$x_1^{(2)}=0,\ x_2^{(2)}=0,\ Z^{(2)}=0, \text{ when } \underline{Z}^{(2)}=\overline{Z}^{(2)}=0.$$
$$x_1^{(3)}=3.5,\ x_2^{(3)}=0,\ Z^{(3)}=14, \text{ when } \underline{Z}^{(3)}=70, \overline{Z}^{(3)}=10.5.$$

Subjectively give a flex index for $d_1=5, d_2=20, d_3=4$, and we construct three fuzzy objective sets $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$: $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$:

$$\begin{aligned}
\mu_{\tilde{M}_1}(x) &= f_1(20x_1+10x_2)\\
&= \begin{cases} 0, & 20x_1+10x_2<100,\\ 1-\dfrac{1}{5}(105-20x_1-10x_2), & 100\leqslant 20x_1+10x_2<105,\\ 1, & 20x_1+10x_2\geqslant 105;\end{cases}\\
\mu_{\tilde{M}_2}(x) &= f_2(3x_1+2x_2)\\
&= \begin{cases} 0, & 3x_1+2x_2\geqslant 20,\\ 1-\dfrac{1}{20}(3x_1+2x_2), & 0<3x_1+2x_2<20;\end{cases}\\
\mu_{\tilde{M}_3}(x) &= f_3(4x_1+x_2)\\
&= \begin{cases} 0, & 4x_1+x_2<10,\\ 1-\dfrac{1}{4}(14-4x_1-x_2), & 10\leqslant 4x_1+x_2<14,\\ 1, & 4x_1+x_2\geqslant 14.\end{cases}
\end{aligned}$$

Let $\tilde{M} = \tilde{M}_1 \bigcap \tilde{M}_2 \bigcap \tilde{M}_3$. Then the problem is changed into an ordinarily linear programming

$$\begin{aligned}
&\max\ \alpha\\
&\text{s.t.}\ \ 1-\frac{1}{5}(105-20x_1-10x_2)\geqslant \alpha,\\
&\qquad 1-\frac{1}{20}(3x_1+2x_2)\geqslant \alpha,\\
&\qquad 1-\frac{1}{4}(14-4x_1-x_2)\geqslant \alpha,\\
&\qquad 6x_1+2x_2\leqslant 21,\\
&\qquad 0\leqslant \alpha\leqslant 1,\\
&\qquad x_1\geqslant 0,\\
&\qquad x_2\geqslant 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\max\ \alpha\\
&\text{s.t.}\ \ 20x_1 + 10x_2 - 5\alpha \geqslant 100,\\
&\qquad 3x_1 + 2x_2 + 20\alpha \leqslant 20,\\
&\qquad 4x_1 + x_2 - 4\alpha \geqslant 10,\\
&\qquad 6x_1 + 2x_2 \leqslant 21,\\
&\qquad 0 \leqslant \alpha \leqslant 1,\\
&\qquad x_1, x_2 \geqslant 0.
\end{aligned}$$

The optimal solution

$$x_1^* = 0.488, x_2^* = 9.035, \alpha^* = 0.022$$

is obtained, correspondingly,

$$z^* = 100.11, \underline{z}^* = 19.534, \overline{z}^* = 10.987.$$

And then the approximately fuzzy optimal value is

$$\tilde{z}^* = (100.11, 19.534, 10.987)_{LR}.$$

6.7.4 Conclusion

As for the object and constraint with L-R coefficient in the linear programming, we integrate the method above, which can also be changed into to a fuzzy optimal solution to multi-object linear programming.

Meanwhile, determination of this model may cause the constraint field of linear programming to be empty sets after subjectively the flexible indexes d_1, d_2, d_3 are given. At this time, the problem has no optimal solution in them, so this needs to be adjusted appropriately to a flexible index, in order to guarantee the existence of an optimal solution.

6.8 Linear Programming Model with *T*-Fuzzy Variables

6.8.1 Introduction

Theoretically, we build a new linear programming model on the basis of T-fuzzy numbers, study its dual form, nonfuzzify it under a cone index $\mathcal{J}$, and turn a linear programming with T-fuzzy variables into a linear programming depending on a cone index $\mathcal{J}$. In such a theoretical framework, we can transplant many results of the linear programming into a linear programming with T-fuzzy variables [Cao96a].

6.8.2 Linear Programming with T-Fuzzy Variables

Definition 6.8.1. Let fuzzy linear programming be

$$(\widetilde{LP}) \qquad \begin{aligned} &\widetilde{\min}\ c\tilde{x} \\ &\text{s.t.}\ \ A\tilde{x} \leqslant \tilde{b}, \\ &\qquad \tilde{x} \geqslant 0, \end{aligned} \tag{6.8.1}$$

where c is a real $1\times n$ matrix, A a real $m\times n$ matrix, $\tilde{x}$ a real $n-$dimensional T-fuzzy variable vector, and $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m)^{\mathrm{T}}$ a real $m-$dimensional T-fuzzy vector.

If $\tilde{x}$ and $\tilde{b}$ are T-fuzzy data defined as in Ref. [Cao89b,c],[Dia87] and [DPr80], i.e., $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^{\mathrm{T}}$; here $\tilde{x}_l = (x_l, \underline{\xi}_l, \overline{\xi}_l)(1 \leqslant l \leqslant n), \tilde{1} = (1, \underline{1}, \overline{1})$ and (6.8.1) is called a linear programming with T-fuzzy variables.

But we call

$$(LP(\mathcal{J})) \qquad \begin{aligned} &\min\ \sum_{l=1}^{n} c_l U_l \\ &\text{s.t.}\ \sum_{l=1}^{n} a_{il} U_l \leqslant b_i(\mathcal{J})(1 \leqslant i \leqslant m), \\ &\qquad U \geqslant 0 \end{aligned}$$

a linear programming depending on a cone index $\mathcal{J}$, where $U = (U_1, U_2, \cdots, U_n)^{\mathrm{T}}$ is an $n-$dimensional vector, $U_l = \sum\limits_{i=1}^{3M} \dfrac{U_{il}}{3M}$ and $b_i(\mathcal{J})$ is a number depending on a cone index $\mathcal{J}$.

Theorem 6.8.1. *Let the linear programming be given from T-fuzzy variables as $(\widetilde{LP})$. Then $(\widetilde{LP})$ is equivalent to $(LP(\mathcal{J}))$ for a given cone index $\mathcal{J}$, and $(LP(\mathcal{J}))$ has an optimal solution depending on a cone index $\mathcal{J}$, equivalent to $(\widetilde{LP})$ with a T-fuzzy optimal one.*

Proof: Let $\{\tilde{x}_{il}\}$ be a column T-fuzzy variable satisfying $(\widetilde{LP})$, where $\tilde{x}_{il} = (x_l, \underline{\xi}_{il}, \overline{\xi}_{il})_T (1 \leqslant i \leqslant m; 1 \leqslant l \leqslant n)$. We classify vectors of the column by subscripts, and might as well let $l = 1, \cdots, N$ correspond to a smaller fluctuating variable, and the other variables correspond to $l = N+1, \cdots, 3N$. Then for $i = 1, \cdots, M$ and each l, $U_{il} = x_i + \dfrac{\underline{\xi}_{il} + \overline{\xi}_{il}}{2}$; for $i = M+1, \cdots, 2M$ and each l, $U_{il} = \begin{cases} x_l - \underline{\xi}_{il}, & j_l = 0, \\ x_l + \overline{\xi}_{il}, & j_l = 1; \end{cases}$ for $i = 2M+1, \cdots, 3M$ and each l, $U_{il} = \begin{cases} x_l + \overline{\xi}_{il}, & j_l = 0, \\ x_l - \underline{\xi}_{il}, & j_l = 1. \end{cases}$ So, under a given cone index $\mathcal{J}$, $(\widetilde{LP})$ is changed into $LP(\mathcal{J}))$.

From the equivalence of $(\widetilde{LP})$ and $(LP(\mathcal{J}))$, we know that $(LP(\mathcal{J}))$ has an optimal solution depending on a cone index $\mathcal{J}$, which is equivalent to $(\widetilde{LP})$ with an optimal T-fuzzy solution. Therefore, the theorem holds.

Theorem 6.8.1 shows us that $(\widetilde{LP})$ can be turned into an ordinary parametric linear programming $(LP(\mathcal{J}))$ depending on a cone index $\mathcal{J}$, where $(LP(\mathcal{J}))$ has many methods and an optimal one to it can be found in any literature on linear programming.

6.8.3 Dual Problem

For the linear programming with T-fuzzy variables, there always exits a dual linear programming with T-fuzzy parameter corresponding to it.

Let $U_l = x_l + \sum\limits_{i=1}^{3M} \dfrac{\xi'_{il}}{3M}$. Then

$$\begin{aligned}
&(LP(\mathcal{J})) \;\; \Leftrightarrow \\
&\min \;\; \sum_{l=1}^{n} c_l\Big(x_l + \sum_{i=1}^{3M} \frac{\xi'_{il}}{3M}\Big) \\
&\text{s.t.} \;\; \sum_{l=1}^{n} a_{il}\Big(x_l + \sum_{i=1}^{3M} \frac{\xi'_{il}}{3M}\Big) \leqslant b_i(\mathcal{J}), \\
&\qquad x_l \geqslant 0 \; (1 \leqslant i \leqslant m; 1 \leqslant l \leqslant n),
\end{aligned} \tag{6.8.2}$$

where ξ'_{il} is $\underline{\xi}'_{il}$ (resp. $-\underline{\xi}'_{il}$) or $\overline{\xi}'_{il}$ (resp. $-\overline{\xi}'_{il}$).

Substitute $x'_l = x_l + \sum\limits_{i=1}^{3M} \dfrac{\xi'_{il}}{3M}$, and then we might as well let $x_l \geqslant \sum\limits_{i=1}^{3M} \dfrac{\xi'_{il}}{3M}$, and turn (6.8.2) into

$$\begin{aligned}
&\min \;\; \sum_{l=1}^{n} c_l x'_l \\
&\text{s.t.} \;\; \sum_{l=1}^{n} a_{il} x'_l \leqslant b_i(\mathcal{J}), \\
&\qquad x'_l \geqslant 0 \; (1 \leqslant i \leqslant m; 1 \leqslant l \leqslant n),
\end{aligned} \tag{6.8.3}$$

i.e.,

$$\begin{aligned}
&\min \;\; cx' \\
&\text{s.t.} \;\; Ax' \leqslant b(\mathcal{J}), \\
&\qquad x' \geqslant 0,
\end{aligned}$$

while the dual form of (6.8.3) is

$$\begin{aligned}
&\max \;\; yb(\mathcal{J}) \\
&\text{s.t.} \;\; A^{\mathrm{T}} y \geqslant c, \\
&\qquad y \geqslant 0.
\end{aligned} \tag{6.8.4}$$

Theorem 6.8.2. *Suppose linear programming $(\widetilde{LP})$ is deduced from T-fuzzy variables. Its dual form is*

$$\begin{aligned}&\widetilde{\max}\ y\tilde{b}\\ &\text{s.t.}\ \ A^{\mathrm{T}}y \geqslant c\\ &\qquad y \geqslant 0\end{aligned} \tag{6.8.5}$$

and $(\widetilde{LP})$ has an optimal T-fuzzy solution equivalent to (6.8.5) having an optimal solution, and $(\widetilde{LP})$ has the same optimal T-fuzzy values as (6.8.5).

Proof: As $(\widetilde{LP})$ can be changed into $(LP(\mathcal{J}))$ under above cone index $\mathcal{J}$, and the dual form of $(LP(\mathcal{J}))$ is equivalent to (6.8.4), then (6.8.5) can be changed into (6.8.4) under the cone index $\mathcal{J}$ above. Again, $(\widetilde{LP})$ is known to be mutually dual with (6.8.5) due to the equivalence of $(\widetilde{LP})$ with $(LP(\mathcal{J}))$, and (6.8.5) with (6.8.4), and the mutual duality of $(LP(\mathcal{J}))$ and (6.8.4).

Again, $(LP(\mathcal{J}))$ and (6.8.4) are, respectively, an ordinary primal linear programming and a dual linear programming depending on the same cone index $\mathcal{J}$. As for $(LP(\mathcal{J}))$ and (6.8.4), applying Theorem 2 in Section 4.2 in Ref. [GZ83], we know that if one of them has an optimal solution, so has the other. They contain the same optimal values, therefore the theorem holds from the arbitrariness of the cone index $\mathcal{J}$.

Theorem 6.8.3. *Suppose that $(\widetilde{LP})$ is deduced from T-fuzzy variables, then dual programming $(\widetilde{LP})$ and (6.8.5) have optimal T-fuzzy solutions and optimal solutions, respectively, if and only if they have T-fuzzy feasible ones and feasible ones, respectively, at the same time.*

Proof: Necessity is apparent and sufficiency is proved as follows.

$(\widetilde{LP})$ can be changed into $(LP(\mathcal{J}))$ and (6.8.5) into (6.8.4) under the given cone index $\mathcal{J}$. Meanwhile $(LP(\mathcal{J}))$ with (6.8.4) is mutually dual under the same cone index $\mathcal{J}$. In a similar way to the proof of Theorem 1 in Section 4.2 in Ref. [GZ83], we can prove that $(LP(\mathcal{J}))$ and (6.8.4) have feasible solutions depending on a cone index $\mathcal{J}$ if and only if they contain optimal solutions depending on a cone index $\mathcal{J}$. Again, we know the theorem holds because of the equivalence of $(\widetilde{LP})$ and $(LP(\mathcal{J}))$, and (6.8.5) and (6.8.4), and the duality of $(\widetilde{LP})$ and (6.8.5).

Corollary 6.8.1. *If $\tilde{x}^0$ is a feasible T-fuzzy solution to $(\widetilde{LP})$ and y^0 is a feasible solution to (6.8.5), with $c\tilde{x}^0 = y^0\tilde{b}$, then $\tilde{x}^0$ is an optimal T-fuzzy solution to $(\tilde{LP})$ and y^0 is an optimal solution to (6.8.5).*

Proof: Straightforward.

6.8.4 Numerical Example

Example 6.8.1: Find

$$\begin{aligned}&\widetilde{\max}\ (3\tilde{x}_1 - \tilde{x}_2)\\ &\text{s.t.}\ \ 2\tilde{x}_1 - \tilde{x}_2 \leqslant \tilde{2}, \quad \text{where } \tilde{2} = (2,0,0),\\ &\qquad \tilde{x}_1 \leqslant \tilde{4}, \qquad\qquad \text{where } \tilde{4} = (4,0,0),\\ &\qquad \tilde{x}_1, \tilde{x}_2 \geqslant 0, \qquad\ \text{where } 0 = (0,0,0),\end{aligned}$$

and give a column of T-fuzzy data:

$$\begin{aligned}
&\tilde{x}_1: \ 1.\ (x_1, 0.5, 1.2), \quad && 2.\ (x_1, 0.8, 1), \quad && 3.\ (x_1, 1, 1.4);\\
&\tilde{x}_2: \ 4.\ (x_2, 0, 0.4), \quad && 5.\ (x_2, 0.6, 1), \quad && 6.\ (x_2, 1.5, 0.9).
\end{aligned}$$

Solution:

(i) Number the data by means of 1–6

Group the data into three parts from Definition 3.1.4: I, No. 1,4; II, No. 2,5; $j_2 = 0, j_5 = 1$; and III. No. 3,6; $j_3 = 1, j_6 = 0$, here $j_l = 1$ for odd numbers and $j_l = 0$ for even numbers.

(ii) Nonfuzzification

Let x_1, x_2 be

$$\frac{(x_1 + 0.85) + (x_1 - 0.8) + (x_1 + 1.4)}{3} = x_1 + 0.483,$$
$$\frac{(x_2 + 0.2) + (x_2 + 1) + (x_2 - 1.5)}{3} = x_2 - 0.1.$$

(iii) Obtain a linear programming corresponding to (6.8.2) as follows:

$$\begin{aligned}
&\max\ (3x_1 - x_2 + 1.55)\\
&\text{s.t. } 2x_1 - x_2 + 1.07 \leqslant 2\\
&\qquad x_1 + 0.483 \leqslant 4\\
&\qquad x_1, x_2 \geqslant 0\\
\Rightarrow &\max\ (3x_1 - x_2 + 1.55)\\
&\text{s.t. } 2x_1 - x_2 \leqslant 0.93,\\
&\qquad x_1 \leqslant 3.52,\\
&\qquad x_1, x_2 \geqslant 0.
\end{aligned}$$

The optimal solution depending on a cone index $\mathcal{J}$ is $x_1 = 3.52, x_2 = 6.11$, and the optimal value is 6.00. If x_1 stands for an expensive resource, then x_2 stands for a cheap resource. Decrease x_1 and increase x_2 properly and we obtain the same optimal value as in the non-crisp case. Obviously it decreases its cost.

6.8.5 Conclusion

The linear programming with T-fuzzy variables can always be turned into a parameter programming for solution, which is called a prime problem for fuzzy linear programming.

Since a close connection exists between the prime problem and the dual one, we can find an answer to the latter more easily than the former.

6.9 Multi-Objective Linear Programming with T-Fuzzy Variables

6.9.1 Introduction

There are a lot of fuzzy and undetermined phenomena in the realistic world. If we describe such phenomena with T-fuzzy numbers [Cao90][Dia87], we can get more information. Here we will extend the model in [Cao96a] into a multi-objective linear programming with T-fuzzy variables, and discuss its algorithm, which tests the effectiveness of the model and method by a numerical example.

6.9.2 Building of Model

Consider an ordinary multi-objective linear programming:

$$\begin{aligned} V-\max\ & c^{(j)}x \quad (1 \leqslant j \leqslant r) \\ \text{s.t.}\ & Ax \leqslant b, \\ & x > 0, \end{aligned} \tag{6.9.1}$$

where, $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ is an n dimension vector, $b = (b_1, b_2, \cdots, b_m)^{\mathrm{T}}$ an m dimension constant vector, $c^{(j)}$ and A denote $r \times n$ and $m \times n$ matrix, respectively.

Because of practical problems, we extend (6.9.1) into the problem of a linear programming with T-fuzzy variables. Introducing T-fuzzy data into (6.9.1), then

$$\begin{aligned} V-\max\ & c^{(j)}\tilde{x} \quad (1 \leqslant j \leqslant r) \\ \text{s.t.}\ & A\tilde{x} \lesssim \tilde{b}, \\ & \tilde{x} \geqslant 0, \end{aligned} \tag{6.9.2}$$

we call (6.9.2) a multi-objective linear programming model with T-fuzzy variables, where, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^{\mathrm{T}}$ is an n dimension T-fuzzy vector, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m)^{\mathrm{T}}$ an m dimension T-fuzzy constant vector, $\tilde{x}_l = (x_l, \underline{\xi}_l, \overline{\xi}_l)$ a T-fuzzy variable, and $\tilde{b}_i = (b, \underline{b}_i, \overline{b}_i)$ a T-fuzzy number.

6.9.3 Non Fuzzification of Model

Theorem 6.9.1. *If (6.9.2) is given by T-fuzzy variables, then, to the given cone index $\mathcal{J}$, (6.9.2) can be turned into*

$$\begin{aligned} V-\max\ & c^{(j)}U(\mathcal{J})(1 \leqslant j \leqslant r) \\ \text{s.t.}\ & AU(\mathcal{J}) \leqslant b(\mathcal{J}), \\ & U(\mathcal{J}) > 0, \end{aligned} \tag{6.9.3}$$

where

$$c^{(j)}U(\mathcal{J})=\sum_{l=1}^{n}c_l^{(j)}U_l(1\leqslant j\leqslant r);$$
$$AU(\mathcal{J})=\sum_{l=1}^{n}a_{il}U_l(1\leqslant i\leqslant m);$$
$$U(\mathcal{J})=(U_1(\mathcal{J}),U_2(\mathcal{J}),\cdots,U_n(\mathcal{J}))^{\mathrm{T}}$$

and $U_{il}(\mathcal{J})=\dfrac{\sum_{i=1}^{3M}U_{il}(\mathcal{J})}{3M}$ *are a vector and a variable with cone index* $\mathcal{J}$*, respectively.* $b(\mathcal{J})=(b_1(\mathcal{J}),b_2(\mathcal{J}),\cdots,b_m(\mathcal{J}))^{\mathrm{T}}$ *and* $b_i(\mathcal{J})$ *are a constant vector and a constant with cone index* $\mathcal{J}$*. And (6.9.3) has a satisfactory solution depending on cone index* $\mathcal{J}$*, which is equivalent that (6.9.2) has a fuzzy satisfactory one.*

Proof: Let $\{\tilde{x}_{il}\}$ be a column T-fuzzy variables tallying with (6.9.2), where $\tilde{x}_{il}=(x_{il},\underline{\xi}_{il},\overline{\xi}_{il})(1\leqslant i\leqslant m;1\leqslant l\leqslant n)$. We classify vectors of the column by subscripts, and might as well let $l=1,2,\cdots,N$ correspond to smaller fluctuating variables, while the other variables correspond to $l=N+1,\cdots,3N$, then to $i=1,2,\cdots,M$ and each l,

$$U_{il}=x_{il}+\frac{\underline{\xi}_{il}+\overline{\xi}_{il}}{2};$$

to $i=M+1,\cdots,2M$ and each l,

$$U_{il}=\begin{cases}x_{il}+\underline{\xi}_{il}, & if\quad j_l=0,\\ x_{il}-\overline{\xi}_{il}, & if\quad j_l=1,\end{cases}$$

to $i=2M+1,\cdots,3M$, and each l,

$$U_{il}=\begin{cases}x_{il}-\underline{\xi}_{il}, & if\quad j_l=0,\\ x_{il}+\overline{\xi}_{il}, & if\quad j_l=1.\end{cases}$$

Then, under the given cone index $\mathcal{J}$, (6.9.2) is turned into (6.9.3), such that (6.9.3) can be found out.

Since (6.9.2) is equivalent to (6.9.3), a parameter optimal solution in (6.9.3) depending on cone index $\mathcal{J}$ is equivalent to an optimal T- fuzzy one in (6.9.2).

We conclude the solutions to Model (6.9.2) as follows.

1^0 To the given T-fuzzy variables $\tilde{x}_l$, we partition natural number set $\{1,2,\cdots,n\}$ into three parts by subscription.

I: $U_{il}=x_{il}+\dfrac{\underline{\xi}_{il}+\overline{\xi}_{il}}{2},i=1,2,\cdots,M$ and each l,

II:

$$U_{il}=\begin{cases}x_{il}-\underline{\xi}_{il}, & if\quad j_l=0,\\ x_{il}+\overline{\xi}_{il}, & if\quad j_l=1,\end{cases}$$

$i=M+1,\cdots,2M$ and each l.

III:

$$U_{il} = \begin{cases} x_{il} + \overline{\xi}_{il}, & if \quad j_l = 0, \\ x_{il} - \underline{\xi}_{il}, & if \quad j_l = 1, \end{cases}$$

$i = 2M+1, \cdots, 3M$, and each l.

2^0 Nonfuzzified $\tilde{x}_l$.

We take

$$U_{il} = x_{il} + \sum_{l=1}^{3N} \frac{\xi_{il}^*}{3N},$$

where ξ_{il}^* be $(\underline{\xi}_{il} + \frac{\overline{\xi}_{il}}{2}), or \pm \underline{\xi}_{il}$, or $\pm \overline{\xi}_{il}$.

3^0 Substitute U_{il} for $\tilde{x}_l$ in (6.9.2) and we can get (6.9.3).

4^0 Determine a satisfactory (or effective) solution to problem (6.9.3) by the aid of solution to an ordinary multi-objective linear programming and we can get a fuzzy satisfactory solution to (6.9.2).

There are a lot of methods to finding satisfactory (effective) solutions to programming (6.9.3). Here, we advance two ways to nonfuzzification (6.9.2):

1) Nonfuzzification before a weighted method

Turn (6.9.2) into a linear programming (6.9.3) with cone index $\mathcal{J}$.

Give weight to r objective functions

$$f_j(U) = \sum_{l=1}^{n} c_l^{(j)} (\sum_{i=1}^{3M} \frac{U_{il}(\mathcal{J})}{3M}),$$

we have

$$f(U) = \gamma_1 f_1(U) + \gamma_2 f_2(U) + \cdots + \gamma_r f_r(U),$$

where $\gamma_j (j = 1, \cdots, r)$ is a factor of weight, satisfying with $0 \leqslant \gamma_j \leqslant 1$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_r = 1$.

Turn (6.9.3) into a single objective linear programming

$$\begin{aligned} \max \quad & f(U(\mathcal{J})) \\ \text{s.t.} \quad & AU(\mathcal{J}) \leqslant b(J), \\ & U(\mathcal{J}) \geqslant 0. \end{aligned} \tag{6.9.4}$$

2) By a weighted method before nonfuzzification

Consider (6.9.2), and r fuzzy objective functions to (6.9.2) are weighted:

$$f(\tilde{x}) = \gamma_1 f_1(\tilde{x}) + \gamma_2 f_2(\tilde{x}) + \cdots + \gamma_r f_r(\tilde{x}),$$

Programming (6.9.2) is changed into

$$\begin{aligned} \max \quad & f(\tilde{x}) \\ \text{s.t.} \quad & A\tilde{x} \lesssim \tilde{b}, \\ & \tilde{x} \geqslant 0. \end{aligned} \tag{6.9.5}$$

Now nonfuzzify (6.9.5) by the method mentioned, and we can obtain (6.9.4).

6.9.4 Finding Solution

We have many algorithms, such as genetic and simulated annealing algorithm (algorithm process is omitted) by which, to single objective linear programmings (6.9.4) and (6.9.5), we can finally get a satisfactory solution with practical value. Now we are searching for a better algorithm since the overall optimum constraint by a single algorithm code fails to show the result and to ensure its convergence for the optimum solution.

Assume that there are computer programmings for solution in (6.9.2) or (6.9.3) in order to discuss theoretically the searching (omitted here), we consider the following example.

Example 6.9.1: Find

$$\begin{aligned} \max\ & (\tilde{z}_1, \tilde{z}_2) \\ & \tilde{z}_1 = 5\tilde{x}_1 + \tilde{x}_2 \\ & \tilde{z}_2 = \tilde{x}_1 + \tilde{x}_2 \\ \text{s.t.}\ & \tilde{x}_1 + \tilde{x}_2 \lesssim \tilde{6}, \\ & \tilde{0} \leqslant \tilde{x}_1 \lesssim \tilde{5}, \tilde{x}_2 \geqslant 0, \end{aligned}$$

where $\tilde{5} = (5, 0, 0), \tilde{6} = (6, 00)$.

We take T-fuzzy variables as follows:

$$\begin{array}{llll} \tilde{x}_1: & 1.\ (x_1, 1, 0), & 2.\ (x_1, 0, 1), & 3.\ (x_1, 2, 1). \\ \tilde{x}_2: & 4.\ (x_2, 0, 1); & 5.\ (x_2, 1, 0); & 6.\ (x_2, 2, 2). \end{array}$$

Now, divide the data into there groups including No.1,4; No.2,5 and No.3,6. As for data No.1,4, we get a value by Formula I. For the rest, we use the formulas corresponding to $j_p = 1$ and $j_p = 0$ in Formula II and III, when odd numbers and even numbers appear, respectively.

So, we can nonfuzzified $\tilde{x}_1, \tilde{x}_2$

$$\begin{array}{l} \tilde{x}_1: \ [(x_1 + 0.5) + (x_1 - 0) + (x_1 + 1)]/3 = x_1 + 0.5, \\ \tilde{x}_2: \ [(x_2 + 0.5) + (x_2 - 0) + (x_3 - 2)]/3 = x_2 - 0.5, \\ f(\tilde{x}) = \gamma_1 \tilde{z}_1 + \gamma_2 \tilde{z}_2 : f(U(J)) = 6x_1 + 5x_2 \text{ when } \gamma_1 = \gamma_2 = 1. \end{array}$$

Such that, a linear programming corresponding to (6.9.5) appears as follows:

$$\begin{aligned} \max\ & z = 6x_1 + 5x_2 \\ \text{s.t.}\ & x_1 + x_2 \leqslant 6, \\ & 0 \leqslant x_1 \leqslant 5, x_2 \geqslant 0. \end{aligned}$$

Its corresponding superior solution is $x_1 = 5, x_2 = 1, z_1 = 26, z_2 = 9$.

6.9.5 Conclusion

Therefore, we know that we can turn (6.9.2) into an ordinary multi-objective parameter linear programming (6.9.3) depending on cone index $\mathcal{J}$. And as to (6.9.3), we adopt the methods to multi-objective programming, such as methods by which we change a multi-objective majorized problem into a single one or series of single ones.