3

Regression and Self-regression Models with Fuzzy Variables

In 1989, based on the theory of Zadeh fuzzy sets [Zad65a], self-regression forecast model with T-fuzzy variables was advanced [Cao89b],[Cao89c],[Cao 90a], and, again in 1992, a linearizable non-linear regression model with T-fuzzy variables [Cao95c] was developed. The application appears vastly extensive because of much wider information in models.

1) Make use of a fuzzy distance, follow the classic regression analytical method with a beeline(or curve) imitation.

2) Ascertain the regression model with fuzzy variables under a cone and platform index. Because fuzzy regression analysis is an interval estimation, a kind of analytical methods become much useful.

This chapter introduces T-fuzzy variables, $(\cdot, c)$ fuzzy variables and flat (or trapezoidal) fuzzy variables into regression models, and builds more practical kinds of way to the model determination. Meanwhile, their application is discussed.

3.1 Regression Model with T- Fuzzy Variables

3.1.1 Basic Property

As for definition and property of T-fuzzy number, we can read Ref. [TUA82]. It is easy to prove that this kind of fuzzy numbers are regular and convex fuzzy subsets.

Definition 3.1.1. Let $\tilde{x} = (m(x), c_1), \tilde{y} = (m(y), c_2)$. Then the distance on $\mathcal{T}(\mathscr{R})$, T-fuzzy number set ($\mathscr{R}$ is a real number set), is defined as

$$d(\tilde{x}, \tilde{y})^2 = D_2(\text{Supp}(\tilde{x}), \text{Supp}(\tilde{y}))^2 + (m(\tilde{x}) - m(\tilde{y}))^2,$$

where Supp$(\cdot)$ denotes the support interval of $(\cdot)$ and $m(\cdot)$ denotes its modal value.

B.-Y. Cao: Optimal Models & Meth. with Fuzzy Quantities, STUDFUZZ 248, pp. 63–94.
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Lemma 3.1.1.

$$d(\tilde{y}_i,\tilde{y}_j)^2 = 2d(\tilde{y}_i,\tilde{x})^2 + 2d(\tilde{x},\tilde{y}_j)^2 - 4d(\tilde{x},\frac{\tilde{y}_i+\tilde{y}_j}{2})^2.$$

Proof: From parallelogram rule, we can get:

$$\begin{aligned}2(y_i-x)^2+2(x-y_j)^2 &= [(y_i-x)-(x-y_j)]^2+[(y_i-x)+(x-y_j)]^2\\ &= (y_i-y_j)^2+[2x-(y_i+y_j)]^2.\end{aligned}$$

In addition, if we establish $\tilde{y}_i=(y_i,\underline{\eta}_i,\overline{\eta}_i)_T, \tilde{y}_j=(y_j,\underline{\eta}_j,\overline{\eta}_j)_T, \tilde{x}=(x,\underline{\xi},\overline{\xi})_T$, and let

$$\begin{aligned}\underline{F}&=y_i-\underline{\eta}_i-(x-\underline{\xi}), & \underline{G}&=x-\underline{\xi}-(y_j-\underline{\eta_j}),\\ \overline{F}&=y_i+\overline{\eta}_i-(x+\overline{\xi}), & \overline{G}&=x+\overline{\xi}-(y_j+\overline{\eta_j}).\end{aligned}$$

Because

$$\begin{aligned}&2(\underline{F}^2+\underline{G}^2)+2(\overline{F}^2+\overline{G}^2)=(\underline{F}-\underline{G})^2+(\overline{F}-\overline{G})^2+(\underline{F}+\underline{G})^2+(\overline{F}+\overline{G})^2\\ &\quad=2(\underline{F}^2+\overline{F}^2)+2(\underline{G^2}+\overline{G^2})=2[y_i-x-(\underline{\eta_i}-\underline{\xi})]^2+2[y_i-x+(\overline{\eta}_i-\overline{\xi})]^2\\ &\qquad+2[x-y_j-(\underline{\xi}-\underline{\eta_j})]^2+2[x-y_j+(\overline{\xi}-\overline{\eta}_j)]^2\\ &\quad=2[(y_i-\underline{\eta_i})-(x-\underline{\xi})]^2+2[(x-\underline{\xi})-(y_j-\underline{\eta_j})]^2\\ &\qquad+2[(y_i+\overline{\eta}_i)-(x+\overline{\xi})]^2+2[(x+\overline{\xi})-(y_j+\overline{\eta}_j)]^2\\ &\quad=[(y_i-\underline{\eta_i})-(y_j-\underline{\eta_j})]^2+[2(x-\underline{\xi})-(y_i-\underline{\eta_i}+y_j-\underline{\eta_j})]^2\\ &\qquad+[(y_i+\overline{\eta}_i)-(y_j-\overline{\eta}_j)]^2+[2(x+\overline{\xi})-(y_i-\overline{\eta}_i+y_j-\overline{\eta}_j)]^2\\ &\Rightarrow\quad [(y_i-y_j)-(\underline{\eta_i}-\underline{\eta_j})]^2+[(y_i-y_j)+(\overline{\eta}_i+\overline{\eta}_j)]^2\\ &\quad=2[(y_i-x)-(\underline{\eta_i}-\underline{\xi})]^2+2[(x-y_j)-(\underline{\xi}-\underline{\eta_j})]^2\\ &\qquad+2[(y_i-x)+(\overline{\eta}_i-\overline{\xi})]^2+2[(x-y_j)+(\overline{\xi}-\overline{\eta}_j)]^2\\ &\qquad-4[(x-\underline{\xi})-\frac{y_i+y_j-\underline{\eta_i}-\underline{\eta_j}}{2}]^2-4[(x+\overline{\xi})-\frac{y_i+y_j-\overline{\eta}_i-\overline{\eta}_j}{2}]^2,\end{aligned}$$

i.e.,

$$\begin{aligned}&D_2(\text{Supp}\widetilde{y_i},\text{Supp}\widetilde{y_j})^2=2D_2(\text{Supp}\widetilde{y_i},\text{Supp}\widetilde{x})^2\\ &\quad+2D_2(\text{Supp}\widetilde{x},\text{Supp}\tilde{y}_j)^2-4D_2\{\text{Supp}\widetilde{x},\text{Supp}\frac{\widetilde{y_i}+\widetilde{y_j}}{2}\}^2.\end{aligned}$$

Theorem 3.1.1. *Let V be a closed cone in $P(\mathscr{R})$ (subspace of $\mathcal{T}(\mathscr{R})$). Then for any $\tilde{x}$ in $P(\mathscr{R})$, there exists the unique T-fuzzy number $\tilde{y}_0$ in V, such that for all of $\tilde{y}$ in V, we have $d(\tilde{x},\tilde{y}_0)\leqslant d(\tilde{x},\tilde{y})$, and a necessary and sufficient condition where $\widetilde{y_0}$ being unique minimizing fuzzy number in V is that $\widetilde{x}$ is $\tilde{y}_0-$orthogonality to V.*

Proof: *Sufficiency.* Because of

$$\begin{aligned}
d(\widetilde{x},\widetilde{y})^2 &= [x-y-(\underline{\xi}-\underline{\eta})]^2+[x-y+(\overline{\xi}-\overline{\eta})]^2+(x-y)^2\\
&= [x-y_0-(\underline{\xi}-\underline{\eta_0})]^2+[x-y_0+(\overline{\xi}-\overline{\eta})]^2+(x-y_0)^2\\
&\quad+[y_0-y-(\underline{\eta_0}-\underline{\eta})]^2+[y_0-y+(\overline{\eta}_0-\overline{\eta})]^2\\
&\quad+(y_0-y)^2+2[y_0-y-(\underline{\eta_0}-\underline{\eta})][x-y_0-(\underline{\xi}-\underline{\eta})]\\
&\quad+2[y_0-y+(\overline{\eta}_0-\overline{\eta})][x-y_0+(\overline{\xi}-\overline{\eta}_0)]+2(y_0-y)(x-y_0)\\
&\geqslant D(\widetilde{x},\widetilde{y_0})^2+D(\widetilde{y_0},\widetilde{y})^2+(x-y_0)^2+(y_0-y)^2\\
&= d(\widetilde{x},\widetilde{y_0})^2+d(\widetilde{y_0},\widetilde{y})^2,
\end{aligned}$$

again because of $d(\widetilde{y_0},\widetilde{y})^2>0$, hence $d(\widetilde{x},\widetilde{y})^2>d(\widetilde{x},\widetilde{y_0})^2$ for $\widetilde{y}\neq\widetilde{y_0}$ holds true.

Necessity. If for some $\widetilde{y}$ in V, such that for $\lambda\in(0,1)$, we have

$$\begin{aligned}
&[y_0-y-(\underline{\eta_0}-\underline{\eta})][x-y_0-(\underline{\xi}-\underline{\eta_0})]\\
&+[y_0-y+(\overline{\eta}_0-\overline{\eta})][x-y_0+(\overline{\xi}-\overline{\eta}_0)]+(y_0-y)(x-y_0)=-\lambda.
\end{aligned}$$

Suppose $d(\widetilde{y},\widetilde{y_0})=1$ and, in order not to lose generality, we consider $\widetilde{y_1}=(1-\lambda)\widetilde{y_0}+\lambda\widetilde{y}$; it is known in V from the convex. Then

$$\begin{aligned}
d(\widetilde{x},\widetilde{y})^2 &= d(\widetilde{x},\widetilde{y_0})^2+\lambda^2 d(\widetilde{y},\widetilde{y_0})^2+\lambda[2(y_0-y-(\underline{\eta_0}-\underline{\eta}))(x-y_0-(\underline{\xi}-\underline{\eta_0}))\\
&\quad+2(y_0-y+(\overline{\eta}_0-\overline{\eta}))(x-y_0+(\overline{\xi}-\overline{\eta}_0))+2(y_0-y)(x-y_0)]\\
&= d(\widetilde{x},\widetilde{y_0})^2-\lambda^2,
\end{aligned}$$

hence $\tilde{y}_0$ is not a minimum element in V. Contradiction.

Therefore, unique, sufficient and necessary condition of $\widetilde{y_0}$-orthogonality can get certificated.

The following proof exists in $\tilde{y}_0$ again.

If $\widetilde{x}\in V$, then existence is proved. If $\widetilde{x}\notin V$, then define $\delta=\inf\{d(\widetilde{x},\widetilde{y})|\widetilde{y}\in V\}$. Suppose $\tilde{y}_i$ to be fuzzy sequence in V, such that $d(\widetilde{x},\widetilde{y_i})\to\delta$. From equality

$$d(\widetilde{y_i},\widetilde{y_j})^2=2d(\widetilde{y_i},\widetilde{x})^2+2d(\widetilde{x},\widetilde{y_j})^2-4d(\widetilde{x},\frac{\widetilde{y_i}+\widetilde{y_j}}{2})^2,$$

and because $\forall i,j$, $\frac{\widetilde{y_i}+\widetilde{y_j}}{2}$ in V, but cone V is convex, hence, $d(\widetilde{x},\frac{\widetilde{y_i}+\widetilde{y_j}}{2})\geqslant\delta$, then

$$d(\widetilde{y_i},\widetilde{y_j})^2\leqslant 2d(\widetilde{y_i},\widetilde{x})^2+2d(\widetilde{x},\widetilde{y_j})^2-4\delta^2,$$

and when $i,j\to\infty$, $d(\widetilde{y_i},\widetilde{y_j})\to 0$, $\{\tilde{y}_i\}$ is a Cauchy sequence. Again, because $(\mathcal{T}(\mathscr{R}),d)$ is complete, and V is closure, $\widetilde{y_0}=\lim\widetilde{y_i}$ in V.

Corollary 3.1.1. *Let N be a positive integer. If V is a close cone in $P(\mathscr{R})^N$, the measurement in $P(\mathscr{R})^N$ is represented by d_N, which is defined* *as* $d_N(\tilde{x},\tilde{y})^2 = \sum_{i=1}^{N} d(\tilde{x}_i,\tilde{y}_i)^2$*, where* $\tilde{x}_i,\tilde{y}_i \in P(\mathscr{R})(i = 1,2,\cdots,N)$ *are the component in* $N-$*dimensional fuzzy vector* $\tilde{x},\tilde{y} \in P(\mathscr{R})^N$*, then for arbitrary* $\tilde{x}$ *in* $P(\mathscr{R})^N$*, the unique vector* $\tilde{y}_0$ *exists in* V*, such that*

$$d_N(\tilde{x},\tilde{y}_0)^2 \leqslant d_N(\tilde{x},\tilde{y})^2$$

is established for all $\tilde{y}$ *in* V*.*

3.1.2 Regression Model with T-Fuzzy Variables

Consider

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n + \varepsilon,$$

we call it a regression model, where $E, \beta_p (p = 0,1,\cdots,n)$ are ordinary real number, $x_p (p = 1,\cdots,n)$ and y are ordinary real variables.

Definition 3.1.2. If

$$\tilde{y} = \beta_0 \mathbb{e} + \beta_1\tilde{x}_1 + \cdots + \beta_n\tilde{x}_n + \varepsilon, \tag{3.1.1}$$

$\tilde{x}_p(p = 1,2,\cdots,n)$ is the T-fuzzy variable, $\tilde{y}$ is T-fuzzy function variable, $\mathbb{e}$ is $n-$vector represented by all $e = (1,0,0)$, $\beta_0,\beta_1,\cdots,\beta_n \in \mathscr{R}$ and ε is an error. We call (3.1.1) a regression model with T-fuzzy variables.

The concept about T-fuzzy variable is shown as Section 1.7 in Chapter 1.

Definition 3.1.3. Assume that $P(\mathscr{R})$ is a subspace consisting of the support $\mathcal{T}(\mathscr{R})$ of all non-negative elements. For each $(x,\underline{\xi},\overline{\xi}) \in P(\mathscr{R})$ and $x - \underline{\xi} \geqslant 0, P(\mathscr{R})$ is a cone in $\mathcal{T}(\mathscr{R})$ and also a closed convex subset of $\mathcal{T}(\mathscr{R})$ with respect to the topology induced by d. Here

$$d(\widetilde{x},\widetilde{y})^2 = \frac{[x-y-(\underline{\xi}-\underline{\eta})]^2 + [x-y+(\overline{\xi}-\overline{\eta})]^2 + (x-y)^2}{3},$$
$$\widetilde{x},\widetilde{y} \in P(\mathscr{R})^N, \widetilde{x}_i,\widetilde{y}_i \in P(\mathscr{R}).$$

Assume that the test of data sets $\tilde{x}_{1i},\tilde{x}_{2i},\cdots,\tilde{x}_{ni}$ and $\widetilde{y}_i$ are given by a linear regression equation

$$\widetilde{y}_i = \beta_0 + \beta_1\tilde{x}_{1i} + \cdots + \beta_n\tilde{x}_{ni}, \tag{3.1.2}$$

$\tilde{x}_{pi} = (x_{pi},\underline{\xi}_{pi},\overline{\xi}_{pi})(p = 0,1,\cdots,n; i = 1,\cdots,N)$ a fuzzy independent variable, and $\tilde{y}_i = (y_i,\underline{\eta}_i,\overline{\eta}_i)$ an affine function from $P(\mathscr{R})^N$ to $\mathcal{T}(\mathscr{R})$. If again

$$(M) \qquad r(\beta_0,\beta) = \sum_{i=1}^{N} d(\beta_0 + \beta_1\widetilde{x}_{1i} + \cdots + \beta_n\widetilde{x}_{ni};\widetilde{y}_i)^2,$$

then $\beta_p(p = 0,1,\cdots,n)$ is determined by applying the least square method, it is a pity that income β_p are all T- fuzzy numbers rather than real numbers, so that the classical least square method can not be directly applied, therefore conversion should be made. For this reason, we induce definitions and properties first as follows.

Different expressions arise for $r(\beta_0, \beta)$ according as some of the β_p are positive and negative because $\beta_p x_p = (\beta_p x_p, \beta_p \underline{\xi}_p, \beta_p \overline{\xi}_p)$ if $\beta_p \geqslant 0$ and $\beta_p x_p = (\beta_p x_p, \beta_p \overline{\xi}_p, \beta_p \underline{\xi}_p)$ and when $\beta_p < 0$. So if negative β appears in (M), "mixed" upper and lower spreads occur in each summand as can easily be seen from above form. Consequently, in order to derive analogues of the normal equations it is necessary to specify certain cones in which to seek minimizing solution to (M). Then we define the following.

Definition 3.1.4. Assume that $\tilde{x}_i = (\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni})$ $(i = 1, 2, \cdots, N)$. If partition the set of nature numbers $\{1, 2, \cdots, n\}$ into two exhaustive, mutually exclusive subsets $J(-), J(+)$, one of which may be empty. To each of such partition associate a binary multi-index $\mathcal{J} = (J_1, J_2, \cdots, J_n)$ defined by $j_p = \begin{cases} 0, & \text{if } \ p \in J(+), \\ 1, & \text{if } \ p \in J(-). \end{cases}$ Denoted by the cone $C(\mathcal{J})$ in $\mathcal{T}(\mathscr{R})^n$

$$C(\mathcal{J}) = \{\beta_0 \mathbb{e} + \beta_1 x_1 + \cdots + \beta_n x_n | \beta_p \geqslant 0, \text{if } j_p = 0; \beta_p < 0, \text{if } j_p = 1\},$$

we call $\mathcal{J}$ a cone index, and $C(\mathcal{J})$ is a cone determined by it.

Proposition 3.1.1. *For a given cone index $\mathcal{J}$, then the problem of minimizing in cone $(M(\mathcal{J}))$*

$$(M(\mathcal{J})) \qquad r(\beta_0(\mathcal{J}), \beta(\mathcal{J})) = \sum_{i=1}^{N} d(\beta_0 + \beta_1 \widetilde{x}_{1i} + \cdots + \beta_n \widetilde{x}_{ni}, \widetilde{y}_i)^2 \quad (3.1.3)$$

has a unique parameter solution $\beta_0(\mathcal{J}), \beta_1(\mathcal{J}), \cdots, \beta_n(\mathcal{J})$.

Definition 3.1.5. Assume fuzzy data to be $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}; \tilde{y}_i$, and we call the system $S(\mathcal{J})$ consisting of $n+1$ equations

$$\frac{\partial r(\beta_0(\mathcal{J}), \beta(\mathcal{J}))}{\partial \beta_p} = 0 \ (p = 0, 1, \cdots, n), \tag{3.1.4}$$

and write it as

$$S(\mathcal{J}) \qquad \begin{pmatrix} N & \sum_{i=1}^{N} x_{1i}(\mathcal{J}) & \cdots & \sum_{i=1}^{N} x_{ni}(\mathcal{J}) \\ \sum_{i=1}^{N} x_{1i}(\mathcal{J}) & \sum_{i=1}^{N} x_{1i}^2(\mathcal{J}) & \cdots & \sum_{i=1}^{N} x_{1i}(\mathcal{J}) x_{ni}(\mathcal{J}) \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{N} x_{ni}(\mathcal{J}) & \sum_{i=1}^{N} x_{ni}(\mathcal{J}) x_{1i}(\mathcal{J}) & \cdots & \sum_{i=1}^{N} x_{ni}^2(\mathcal{J}) \end{pmatrix} \cdot \begin{pmatrix} \beta_0(\mathcal{J}) \\ \beta_1(\mathcal{J}) \\ \vdots \\ \beta_n(\mathcal{J}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{N} y_i(\mathscr{J}) \\ \sum_{i=1}^{N} x_{1i}(\mathscr{J})y_i(\mathscr{J}) \\ \vdots \\ \sum_{i=1}^{N} x_{ni}(\mathscr{J})y_i(\mathscr{J}) \end{pmatrix}.$$

If $S(\mathscr{J})$ has a solution $\beta_0(\mathscr{J}), \beta_1(\mathscr{J}), \cdots \beta_n(\mathscr{J})$, such that $\beta_p > 0$ at $j_p = 0; \beta_p < 0$ at $j_p = 1$, then we call (3.1.3) a $\mathscr{J}-$compatible with the data.

If the unconstraint minimization of $S(\mathscr{J})$ is compatible with $\tilde{Y}_i = \beta_0 \mathscr{E} + \beta_1 \tilde{x}_{1i} + \cdots + \beta_n \tilde{x}_{ni}$ in $C(\mathscr{J})$, then the model is called compatibleness.

Theorem 3.1.2. *Let the data $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}; \tilde{y}_i, (i = 1, 2, \cdots, N)$ satisfy both Equation (3.1.2), for all of cone index $\mathscr{J}$, there exists a unique solution $\beta_0(\mathscr{J}), \beta_1(\mathscr{J}), \cdots, \beta_n(\mathscr{J})$ in system (3.1.4).*

Proof: Catalogue $\{\tilde{x}_{pi}\}$ by subscription.

For $i = 1, 2, \cdots, N, w_i = y_i$ and for each $p, z_{pi} = x_{pi}$, for $i = N + 1, \cdots, 2N, w_i = y_i - \underline{\eta}_i$, and for each p, $z_{pi} = \begin{cases} x_{pi} - \underline{\xi}_{pi}, \text{ if } j_p = 0, \\ x_{pi} + \overline{\xi}_{pi}, \text{ if } j_p = 1; \end{cases}$ for $i = 2N + 1, \cdots, 3N, w_i = y_i + \overline{\eta}_i$ and for each p, $z_{pi} = \begin{cases} x_{pi} + \overline{\xi}_{pi}, \text{ if } j_p = 0, \\ x_{pi} - \underline{\xi}_{pi}, \text{ if } j_p = 1. \end{cases}$

Then it is not difficult to see that $S(\mathscr{J})$ is the same system as the crisp normal equations for the least squares fitting model

$$w = \beta_0 + \beta_1 z_1 + \cdots + \beta_n z_n \tag{3.1.5}$$

to the data $w_i, z_{1i}, z_{2i}, \cdots, z_{ni}$.

By using the classical least square method, it is easier for us to find a unique optimal solution $\beta_p (p = 0, 1, \cdots, n)$ in (3.1.5) concerning to a cone index $\mathscr{J}$.

3.1.3 Regression Model with T-Fuzzy Data

We call (3.1.1) regression model with T-fuzzy parameters.

According to the theory above, the modeling steps in Model (3.1.1) can be concluded as follows:

1^0 Work out a sequence table by observation data and classify the data by Definition 3.1.4.

2^0 Change the observation date $\tilde{x}_{pi}$ and the dependent variable $\tilde{y}_i$ into nonfuzziness.

Then fuzzy data are changed into ordinary data before (3.1.1) is changed into a classical linear regression model (3.1.5).

3^0 From the knowledge of Theorem 3.1.1, the model has a unique solution $\widehat{\beta}_0, \widehat{\beta}_1, \cdots, \widehat{\beta}_n$ in it, replaced in (3.1.5), it can be testified by a classical determination method.

Calculate $r_p(p = 1, \cdots, n)$ and s:

$$r_p = \frac{N\sum_{i=1}^{N} z_{pi}w_i - \sum_{i=1}^{N} z_{pi}\sum_{i=1}^{N} w_i}{\sqrt{[N\sum_{i=1}^{N} z_{pi}^2 - (\sum_{i=1}^{N} z_{pi})^2][N\sum_{i=1}^{N} w_i^2 - (\sum_{i=1}^{N} w_i)^2]}}, \quad (3.1.6)$$

$$s = \sqrt{\frac{\sum_{i=1}^{N} z_{pi}^2(\sum_{i=1}^{N} z_{pi})^2/N - (\sum_{i=1}^{N} z_{pi}w_{pi} - (\sum_{i=1}^{N} z_{pi})(\sum_{i=1}^{N} w_{pi})/N)}{(N-2)}}$$

$(p = 0, 1, \ldots, n)$.

4^0 Decision.

If $|r_p| > r_{0.05}$, then a test goes through.

5^0 A forecast model is obtained as follows,

$$w' = \hat{\beta}_0^1 - 2S + \hat{\beta}_1 z, \;\; w'' = \hat{\beta}_0^2 + 2S + \hat{\beta}_1 z.$$

Example 3.1.1: The needed petroleum is arranged for a western developed country during 1965 and 1981 as follows.

Table 3.1.1. Needed Arrangement of Petroleum in Developed Country

Years	1965	1967	1969
Demand(Ktoe)	(8.05, 0.020.03)	(8.28, 0.02, 0.02)	(8.5, 0.02, 0.01)
	1971	1973	1975
	(8.7, 0.01, 0.03)	(8.94, 0.05, 0.03)	(9, 0, 0.01)
	1977	1979	1981
	(9.04, 0.01, 0.02)	(9.18, 0.02, 0.03)	(9.28, 0.03, 0.04)

Try to forecast the country's petroleum demanded in 1998.

From the data in Table 3.1.1, we know that each datum represents a cone and its figure constructed by its top and a linear distribution, therefore, applying the method above to it, we obtain the following:

1^0 Divide the T-fuzzy data annually into two, one is $\{65, 69, 73, 75, 81\}$, denoting $J(-)$, and the other is $\{67, 71, 77, 79\}$, denoting $J(+)$.

2^0 Nonfuzzify it. Classify the data into three parts:

One is

$$(8.5, 0.02, 0.01), (9, 0, 0.01), (9.28, 0.03, 0.04),$$

to which the year corresponding is $\{69, 75, 81\}$.

Another is

$$(8.28, 0.01, 0.02), (8.94, 0.05, 0.03), (9.18, 0.02, 0.03),$$

to which the year corresponding is $\{67, 73, 79\}$;
and the other is

$$(8.05, 0.02, 0.03), (8.7, 0.01, 0.03), (9.04, 0.01, 0.02),$$

to which the year corresponding is $\{65, 71, 77\}$.

By the expression of z_{pi}, the needed petroleum in Table 3.1.1 can be turned into

Table 3.1.2. Needed Petroleum Crisp Value

Years	1965	1967	1969	1971	1973	1975	1977	1979	1981
Demand(ktoe)	8.03	8.27	8.5	8.73	8.97	9	9.06	9.16	9.28

3^0 List Table

Table 3.1.3. Unary Regression Simplify Table

t	0	1	2	3	4	5	6	7	8	$\sum = 36$
w	8.03	8.27	8.5	8.73	8.97	9	9.06	9.16	9.28	$\sum = 79$
t^2	0	1	4	9	16	25	36	49	64	$\sum = 204$
w^2	64.48	68.39	72.25	76.21	80.46	81	82.08	83.9	86.1	$\sum = 694.87$
tw	0	8.27	17	26.19	35.88	45	54.36	63.42	74.24	$\sum = 324.36$
$\bar{t}=\sum t/9=4$, $\bar{t}^2=16$, $\bar{w}=\sum w/9 \approx 8.778$										

and estimate parameter $\hat{\beta}_0$, $\hat{\beta}_1$:

$$\hat{\beta}_1 = \frac{\sum t_i w_i - n\bar{t}\bar{w}}{\sum t_i^2 - n\bar{t}^2} \approx 0.1392, \hat{\beta}_0 = \bar{w} - \hat{\beta}_1 \bar{t} \approx 8.2212.$$

Substitute them for (3.1.5), then

$$\hat{w} = 8.2212 + 0.1392t.$$

4^0 Test

From (3.1.6), we calculate $r = 1.652$, at $r_{0.05} = 0.666$, we have $r > r_{0.05}$. Then, a test goes through.

Again $s = \sqrt{\dfrac{1.426 - 0.1392 \times 8.352}{7}} \approx 0.194$, then

$$w' = \hat{\beta}_0^1 + 0.1392t = 7.8332 + 0.1392t,$$
$$w'' = \hat{\beta}_0^2 + 0.1392t = 8.6092 + 0.1392t.$$

5^0 Forecast

$$w''_{1998} = 7.8332 + 0.1392 \times 16.5 = 10.13,$$
$$w''_{1998} = 8.6092 + 0.1392 \times 16.5 = 10.906.$$

Such that

$$y = (\frac{w''_{1998} + w''_{1998}}{2}, 0.382 \times \frac{w''_{1998} - w''_{1998}}{2}, 0.618 \times \frac{w''_{1998} - w''_{1998}}{2}) = (10.518, 0.1482, 0.2398),$$

i.e., the petroleum needed for the country in 1998 is a bit more than 10.518($ktoe$), which tallies with practice.

3.2 Self-regression Model with T-Fuzzy Variables

If we modify (3.1.1) for

$$\widetilde{Y}_t = \beta_0 + \beta_1 \widetilde{Y}_{t-1} + \cdots + \beta_n \widetilde{Y}_{t-n} + \varepsilon_t, \tag{3.2.1}$$

then call (3.2.1) an n-order self-regression model with T-fuzzy variables, where $\beta_0, \beta_1, \cdots, \beta_n$ are awaiting-evaluation parameters, $\widetilde{Y}_t$ is fuzzy correlated variable, $\widetilde{Y}_{t-p} = (Y_{t-p}, \underline{\eta}_{t-p}, \overline{\eta}_{t-p})(p = 1, \cdots, n)$ is an independent variable in p period changed backward, with ε being an error.

Theorem 3.2.1. *Assume that the data set is $\widetilde{Y}_{(t-1)_i}, \cdots, \widetilde{Y}_{(t-n)_i}$ and $\widetilde{Y}_{t_i}$ is given by model*

$$Y_{t_i} = \widehat{\beta}_0 + \widehat{\beta}_1 \widetilde{Y}_{(t-1)_i} + \cdots + \widehat{\beta}_n \widetilde{Y}_{(t-n)_i} (i = 1, \cdots, 3N),$$

the system

$$\frac{\partial \overline{r}[\widehat{\beta}_0(\mathcal{J}), \widehat{\beta}(\mathcal{J})]}{\partial \widehat{\beta}_p} = 0 \ (p = 0, \cdots, n)$$

has a unique solution $\widehat{\beta}_0(\mathcal{J}), \widehat{\beta}_1(\mathcal{J}), \cdots, \widehat{\beta}_n(\mathcal{J})$ for all cone indices.

Proof: Similar to the proof of Theorem 3.1.2, the formula corresponding to (3.1.5) is

$$\widehat{Z}_t = \widehat{\beta}_0 + \widehat{\beta}_1 Z_{t-1} + \widehat{\beta}_2 Z_{t-2} + \cdots + \widehat{\beta}_n Z_{t-n}, \tag{3.2.2}$$

then

$$\overline{r}(\widehat{\beta}_0, \widehat{\beta}) = \sum_{i=1}^{3N} d[\widehat{\beta}_0 + \widehat{\beta}_1 Z_{(t-1)_i} + \cdots + \widehat{\beta}_p Z_{(t-n)_i}; Z_t]^2.$$

The normal equation $S(\mathcal{J})$ is simplified into a classical form

$$\begin{cases} \sum\limits_{i=1}^{N} Z_{t_i} = n\widehat{\beta}_0(\mathcal{J}) + \sum\limits_{i=1}^{N} Z_{(t-1)_i}\widehat{\beta}_1(\mathcal{J}) + \cdots + \sum\limits_{i=1}^{N} Z_{(t-n)_i}\widehat{\beta}_n(\mathcal{J}), \\ \sum\limits_{i=1}^{N} Z_{(t-1)_i}\widehat{\beta}_0(\mathcal{J}) + \sum\limits_{i=1}^{N} Z^2_{(t-1)_i}\widehat{\beta}_1(\mathcal{J}) + \cdots + \sum\limits_{i=1}^{N} Z_{(t-1)_i}Z_{(t-n)_i}\widehat{\beta}_n(\mathcal{J}) \\ \qquad = \sum\limits_{i=1}^{N} Z_t Z_{(t-1)_i}, \\ \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\ \sum\limits_{i=1}^{N} Z_{(t-n)_i}\widehat{\beta}_0(\mathcal{J}) + \sum\limits_{i=1}^{N} Z_{(t-1)_i}Z_{(t-n)_i}\widehat{\beta}_1(\mathcal{J}) + \cdots + \sum\limits_{i=1}^{N} Z^2_{(t-n)_i}\widehat{\beta}_n(\mathcal{J}) \\ \qquad = \sum\limits_{i=1}^{N} Z_t Z_{(t-n)_i}. \end{cases}$$

Equations contain a unique solution $\widehat{\beta}_p(p = 0, 1, \cdots, n)$, and this theorem is certified.

Hereby, the modeling steps in Model (3.2.1) can be concluded as follows.

1^0 Design a self-dependent sequence table by tested data $\widetilde{Y}_{(t-p)_i} = (Y_{(t-p)_i}, \underline{\eta}_{(t-p)_i}, \overline{\eta}_{(t-p)_i})$ and classify the data in the table by means of Definition 3.1.4.

Table 3.2.1. Self-related Sequence Table

Q	1983 sale	Sequence	Y_t	move	backward
	Y_t	Y_{t-1}	Y_{t-2}	$\cdots$	Y_{t-p}
I					$(y_{(t-p)_1}, \underline{\eta}_{(t-p)_1}, \overline{\eta}_{(t-p)_1})$ $y'_{t-p,1}$
II				$\cdots$	$(y_{(t-p)_2}, \underline{\eta}_{(t-p)_2}, \overline{\eta}_{(t-p)_2})$ $y'_{t-p,2}$
III			$(y_{(t-2)_1}, \underline{\eta}_{(t-2)_1}, \overline{\eta}_{(t-2)_1})$ $y'_{(t-2)_1}$	$\cdots$	$(y_{(t-p)_3}, \underline{\eta}_{(t-p)_3}, \overline{\eta}_{(t-p)_3})$ $y'_{(t-p)_3}$
IV		$(y_{(t-1)_1}, \underline{\eta}_{(t-1)_1}, \overline{\eta}_{(t-1)_1})$ $y'_{(t-1)_1}$	$(y_{(t-2)_2}, \underline{\eta}_{(t-2)_2}, \overline{\eta}_{(t-2)_2})$ $y'_{(t-2)_2}$	$\cdots$	$(y_{(t-p)_4}, \underline{\eta}_{(t-p)_4}, \overline{\eta}_{(t-p)_4})$ $y'_{(t-p)_4}$
I	$(y_{t_1}, \underline{\eta}_{t_1}, \overline{\eta}_{t_1})$ y'_{t_1}	$(y_{(t-1)_2}, \underline{\eta}_{(t-1)_2}, \overline{\eta}_{(t-1)_2})$ $y'_{(t-1)_2}$	$(y_{(t-2)_3}, \underline{\eta}_{(t-2)_3}, \overline{\eta}_{(t-2)_3})$ $y'_{(t-2)_3}$	$\cdots$	
II	$(y_{t_2}, \underline{\eta}_{t_2}, \overline{\eta}_{t_2})$ y'_{t_2}	$(y_{(t-1)_3}, \underline{\eta}_{(t-1)_3}, \overline{\eta}_{(t-1)_3})$ $y'_{(t-1)_3}$	$(y_{(t-2)_4}, \underline{\eta}_{(t-2)_4}, \overline{\eta}_{(t-2)_4})$ $y'_{(t-2)_4}$		
III	$(y_{t_3}, \underline{\eta}_{t_3}, \overline{\eta}_{t_3})$ y'_{t_3}	$(y_{(t-1)_4}, \underline{\eta}_{(t-1)_4}, \overline{\eta}_{(t-1)_4})$ $y'_{(t-1)_4}$			
IV	$(y_{t_4}, \underline{\eta}_{t_4}, \overline{\eta}_{t_4})$ y'_{t_4}				

Q—Quarter.

2^0 Change fuzzy $\widetilde{Y}_{(t-p)_i}$ and the dependent variable $\widetilde{Y}_{t_i}$ into nonfuzziness by the proof of Theorem 3.1.2.

3^0 Calculate self-dependent coefficients, and let

$$\gamma_p = \frac{N\sum\limits_{i=1}^{N} Z_{(t-p)_i} Z_{t_i} - \sum\limits_{i=1}^{N} Z_{(t-p)_i} \sum\limits_{i=1}^{N} Z_{t_i}}{\sqrt{[N\sum\limits_{i=1}^{N} Z^2_{(t-p)_i} - (\sum\limits_{i=1}^{N} Z_{(t-p)_i})^2][N\sum\limits_{i=1}^{N} Z^2_{t_i} - (\sum\limits_{i=1}^{N} Z_{t_i})^2]}}. \tag{3.2.3}$$

Calculate quarterly self-related coefficients by moving backwards $i(i=1,\cdots,N)$, and, by taking $\gamma_K = \max\{\gamma_p | p = 1,\cdots,n\}$, it is proper to determine the model set up on benchmark time series Z_t by moving backwards n quarters.

4^0 $\widehat{\beta}_p(\mathcal{J})(p = 0,1,\cdots,n)$ is determined by $S(\mathcal{J})$, planted into (3.2.2). Let

$$IC = \frac{\sqrt{\frac{1}{K}\sum\limits_{i=1}^{N}(\widehat{Z}_{t_i} - Z_{t_i})^2}}{\sqrt{\frac{1}{K}\sum\limits_{i=1}^{N}\widehat{Z}^2_{t_i}} + \sqrt{\frac{1}{K}\sum\limits_{i=1}^{N} Z^2_{t_i}}} \quad (\widehat{Z}^2_{t_i} + Z^2_{t_i} = 1). \tag{3.2.4}$$

If $0 \leqslant IC \leqslant 1$, then it is an effective forecast, and $IC \to 0, \widehat{Z}_{t_i} \to Z_{t_i}$ is a perfect case, while $IC = 1$, the forecast is most uncorrect. Therefore, when IC is a smaller positive number, the fuzzy self-regression forecast model determined by $\beta_p(\mathcal{J})(p = 0,1,\cdots,n)$ can be used in an actual forecast.

Example 3.2.1: If the candy sale quantity of 1980-1983 in certain place shows below

Table 3.2.2. Candy Sale of 1980-1983 in Certain Place (10000/unit)

Quarters	1980	1981	1982	1983
I	$(23, 0.1, 0)$	$(25, 0.1, 0.1)$	$(26, 0.1, 0.2)$	$(27, 0.2, 0.1)$
II	$(11, 0.6, 0.8)$	$(11, 0.8, 0.5)$	$(12, 0.3, 1)$	$(13, 1.2, 0.3)$
III	$(11, 0.9, 0.4)$	$(12, 1, 0.5)$	$(14, 0.7, 0.9)$	$(15, 1.1, 0.8)$
IV	$(15, 0.8, 1)$	$(16, 0.6, 0.3)$	$(18, 0.1, 0.4)$	$(20, 0.4, 1)$

Try to forecast the candy sale quantity of 1984 Quarter I,II.

1. Choose a 1-order fuzzy self-regression model, and we list table according to the data in Table 3.2.2.

Note. The ordinary real data under blanks are obtained by taking a main value of fuzzy numbers in first quarter; at odd numbers, $Z_{(t-p)_i} = Y_{(t-p)_i} + \overline{\eta}_{(t-p)_i}$ is taken; at even numbers, $Z_{(t-p)_i} = Y_{(t-p)_i} - \underline{\eta}_{(t-p)_i}$ is taken, on two diagonals of the former. Let $Z_{(t-p)_i} = Y_{(t-p)_i} - \underline{\eta}_{(t-p)_i}$ at odd numbers while $Z_{(t-p)_i} = Y_{(t-p)_i} + \overline{\eta}_{(t-p)_i}$ at even numbers on the two diagonals of the latter.

Table 3.2.3. Self-related Sequence Table

Quarters	1983 sale	The	sale	sequence	move	backward
	Y_t	Y_{t-1}	Y_{t-2}	Y_{t-3}	Y_{t-4}	Y_{t-5}
IV						(16,0.6,0.3) 15.4
I					(26,0.1,0.2) 26	(26,0.1,0.2) 26
II				(12,0.3,1) 11.7	(12,0.3,1) 11.7	(12,0.3,1) 13
III			(14,0.7,0.9) 14.9	(14,0.7,0.9) 14.9	(14,0.7,0.9) 13.3	(14,0.7,0.9) 13.3
IV		(18,0.1,0.4) 17.9	(18,0.1,0.4) 17.9	(18,0.1,0.4) 18.4	(18,0.1,0.4) 18.4	
I	(27,0.2,0.1) 27	(27,0.2,0.1) 27	(27,0.2,0.1) 27	(27,0.2,0.1) 27		
II	(13,1.2,0.3) 11.8	(13,1.2,0.3) 13.3	(13,1.2,0.3) 13.3			
III	(15,1.1,0.8) 13.9	(15,1.1,0.8) 13.9				
IV	(20,0.4,1) 21					

Q		The	sale	sequence	move	backward	
	Y_{t-6}	Y_{t-7}	Y_{t-8}	Y_{t-9}	Y_{t-10}	Y_{t-11}	Y_{t-12}
II							(23,0.1,0) 23
III						(11,0.6,0.8) 10.4	(11,0.6,0.8) 10.4
IV					(11,0.9,0.4) 11.4	(11,0.9,0.4) 11.4	(11,0.9,0.4) 10.1
I				(15,0.8,1) 14.2	(15,0.8,1) 14.2	(15,0.8,1) 16	(15,0.8,1) 16
II			(25,0.8,1) 25	(25,0.8,1) 25	(25,0.8,1) 25	(25,0.8,1) 25	
III		(11,0.8,0.5) 10.2	(11,0.8,0.5) 10.2	(11,0.8,0.5) 11.5	(11,0.8,0.5) 11.5		
IV	(12,1,0.5) 12.5	(12,1,0.5) 12.5	(12,1,0.5) 11	(12,1,0.5) 11			
I	(16,0.6,0.3) 15.4	(16,0.6,0.3) 16.3	(16,0.6,0.3) 16.3				
II	(26,0.1,0.2) 26	(26,0.1,0.2) 26					
III	(12,0.3,1) 13						

2. By means of (3.2.3):

$$\gamma_p = \frac{4\sum_{i=1}^{4} Z_{(t-p)_i} Z_{t_i} - \sum_{i=1}^{4} Z_{(t-p)_i} Z_{t_i}}{\sqrt{[4\sum_{i=1}^{4} Z^2_{(t-p)_i} - (\sum_{i=1}^{4} Z_{(t-p)_i})^2][4\sum_{i=1}^{4} Z^2_{t_i} - (\sum_{i=1}^{4} Z_{t_i})^2]}} (p = 1, \cdots, 12),$$

the self-related coefficients calculated are:

$$\begin{aligned} R =&\{\gamma_1, \gamma_2, \cdots, \gamma_{12}\} = \{-0.378, -0.618, -0.088, -0.990, -0.506, \\ &-0.601, -0.015, -0.980, -0.496, -0.595, -0.274, -0.802\}, \end{aligned}$$

then $\gamma_4 = \max|R| = 0.990$. Therefore, the sequence by moving 4 quarters backwards as follows

$$\widehat{Z}_t = \widehat{\beta}_0 + \widehat{\beta}_1 Z_{t-4}.$$

Through the normal equations $S(\mathcal{J})$, we can get

$$\widehat{\beta}_0(\mathcal{J}) = \frac{\sum_{i=1}^{4} Z_{t_i} \sum_{i=1}^{4} Z^2_{(t-4)_i} - \sum_{i=1}^{4} Z_{(t-4)_i} \sum_{i=1}^{4} Z_{t_i} Z_{(t-4)_i}}{4\sum_{i=1}^{4} Z^2_{(t-4)_i} - (\sum_{i=1}^{4} Z_{(t-4)_i})^2} \approx -0.059,$$

$$\widehat{\beta}_1(\mathcal{J}) = \frac{4\sum_{i=1}^{4} Z_{t_i} Z_{(t-4)_i} - \sum_{i=1}^{4} Z_{(t-4)_i} \sum_{i=1}^{4} Z_{t_i}}{4\sum_{i=1}^{4} Z^2_{(t-4)_i} - (\sum_{i=1}^{4} Z_{(t-4)_i})^2} \approx -1.0675.$$

Therefore

$$\hat{Z}_{t_i} = -0.059 + 1.0675 Z_{(t-4)_i}. \tag{3.2.5}$$

3. Testification

Nonfuzzification sale data are $Z_{t_i} = \{27, 11.8, 13.9, 21\}$ in 1983 checked in Table 3.2.2, replaced into (3.2.5), we obtain $\hat{Z}_{t_i} = \{27.6591, 12.3939, 14.1019, 19.5461\}$, from (3.2.4), then

$$IC = \frac{\sqrt{\frac{1}{4}\sum_{i=1}^{4}(\widehat{Z}_{t_i} - Z_{t_i})^2}}{\sqrt{\frac{1}{4}\sum_{i=1}^{4}\widehat{Z}^2_{t_i}} + \sqrt{\frac{1}{4}\sum_{i=1}^{4} Z^2_{t_i}}} \approx 0.022,$$

so the forecast is very accurate. Therefore $\widehat{Y}_t = \widehat{\beta}_0(\mathcal{J}) + \widehat{\beta}_1(\mathcal{J}) Y_{t-4} = -0.059E + 1.0675 Y_{t-4}$ can be used to forecast the sale in Quarter I, II in 1984, that is

$$Y_{t-1} = (28.7266, 0.2135, 0.10675),$$

$$Y_{t-2} = (13.7816, 1.281, 0.03203).$$

3.3 Regression Model with $(\cdot, c)$ Fuzzy Variables

3.3.1 Determination of the Modal with $(\cdot, c)$ Fuzzy Variables

Consider

$$\tilde{y} = \beta_0\mathscr{E} + \beta_1\tilde{x}_1 + \cdots + \beta_n\tilde{x}_n + \varepsilon, \tag{3.3.1}$$

where $\tilde{x}_p(p = 1, 2, \cdots, n)$ is the $(\cdot, c)$ fuzzy variable, $\tilde{y}$ is $(\cdot, c)$ fuzzy function variable; $\mathscr{E}$ is $n-$vector represented by all $E = (1, 0)$, $\beta_0, \beta_1, \cdots, \beta_n \in \mathscr{R}$ and ε is an error.

Definition 3.3.1. We call (3.3.1) a regression model with $(\cdot, c)$ fuzzy variables.

If $\tilde{x} = (x, c_1)$ and $\tilde{y} = (y, c_2)$, then their metric d on $\mathcal{T}(\mathscr{R})$ is defined by

$$d(\tilde{x}, \tilde{y})^2 = \frac{(x - y - (c_1 - c_2))^2 + (x - y + (c_1 - c_2))^2 + (x - y)^2}{3}.$$

Certainty path is researched for Model (3.3.1) as follows.

Definition 3.3.2. Suppose that $\tilde{x} = (x, c) \in P(\mathscr{R})$ for each $\tilde{x}, x \geqslant c$, then $P(\mathscr{R})$ is one of the cone $\mathcal{T}(\mathscr{R})$, and is a close convex subset of $\mathcal{T}(\mathscr{R})$ relevant with topology induced by distance d.

Suppose test data to be $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}; \tilde{y}_i$, for the Model (3.3.1), $\beta_p(p = 1, 2, \cdots, n)$ to be an ordinary real number, $\tilde{x}_{pi}$ a $(\cdot, c)$ fuzzy variable, and $\tilde{y}_i$ a $(\cdot, c)$ affine function from $P(\mathscr{R})^N$ to $\mathcal{T}(\mathscr{R})$, where $\tilde{x}_{pi} = (x_{pi}, c_{pi}), \tilde{y}_i = (y_i, c_i')$ $(i = 1, 2, \cdots, N; p = 1, 2, \cdots, n)$.

Let

$$(M) \qquad r(\beta_0, \beta) = \sum_{i=1}^{N} d(\beta_0 + \beta_1\tilde{x}_{1i} + \cdots + \beta_n\tilde{x}_{ni}, \tilde{y}_i)^2.$$

Then β_i (where $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$) determined by applying the least square method is a $(\cdot, c)$ fuzzy number rather than a real number. Similarly to method of Section 3.1, we induce definitions and properties first as follows.

Definition 3.3.3. Suppose $\tilde{x}_i = (\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni})$ $(i = 1, 2, \cdots, N)$. If partition the set of nature numbers $\{1, 2, \cdots, n\}$ into two exhaustive, mutually exclusive subsets $J(-), J(+)$, one of which may be empty set, and then contacts a binary multi-index $J = (J_1, J_2, \cdots, J_n)$ defined by $\mathcal{T}_i = \{0, \text{ if } i \in J(+);\ 1, \text{ if } i \in J(-)\}$ for this division.

Especially, we write

$$J_0 = (0, 0, \cdots, 0), \quad J_1 = (1, 1, \cdots, 1).$$

Definition 3.3.4. Use

$$C(J) = \{\beta_0\mathscr{E} + \beta_1\tilde{x}_{i1} + \cdots + \beta_n\tilde{x}_{in} | \beta_p \geqslant 0, \text{if } j_p = 0;\ \beta_p < 0, \text{if } j_p = 1\}$$

to represent a cone in $\mathcal{T}(\mathscr{R})^N$ and we call it a determined cone from the cone index J.

Proposition 3.3.1. *For a given cone index J, the minimization model*

$$r(\beta_0(J), \beta(J)) = \sum_{i=1}^{N} d(\beta_0 E + \beta_1 \tilde{x}_{1i} + \cdots + \beta_n \tilde{x}_{ni}, \tilde{y}_i)^2 \tag{3.3.2}$$

has a unique parameter solution $\beta_0(J), \beta(J)$ in cone $C(J)$, where $\beta(J) = (\beta_1(J), \beta_2(J), \cdots, \beta_n(J))$.

Definition 3.3.5. Suppose fuzzy data to be $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}; \tilde{y}_i$, and we call the system $S(J)$ consisting of $n+1$ equations

$$\frac{\partial r(\beta_0(J), \beta(J))}{\partial \beta_p} = 0 (p = 0, 1, \cdots, n).$$

If $S(J)$ has a solution $\beta_0(J), \beta_1(J), \cdots, \beta_n(J)$, such that $\beta_p > 0$, when $j_p = 0$; $\beta_p < 0$ when $j_p = 1$, then we call (3.3.2) $J-$compatible with the data.

If the unconstraint minimization of $S(J)$ is $J-$compatible with data. Then a model is $J-$compatible if the formal equations $S(J)$ for unconstrained minimization are compatible with $\beta_0 \mathscr{E} + \beta_1 \tilde{x}_1 + \cdots + \beta_n \tilde{x}_n$ lying in $C(J)$.

Theorem 3.3.1. *Let the data set $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni};\ \tilde{y}_i, (i = 1, 2, \cdots, N)$ satisfy Equation (3.3.2), for all of cone indices J, there exists a unique solution $\beta_0(J), \beta_1(J), \cdots, \beta_n(J)$ in system $S(J)$.*

Proof: Catalogue $\{\tilde{x}_{pi}\}$ by subscription: $i = 1, 2, \cdots, N$ is one type; $i = N+1, \cdots, 3N$ the other type. Hence, when $i = 1, 2, \cdots, N$, $w_i = y_i$ to each $p, z_{pi} = x_{pi}$; when $i = N+1, \cdots, 2N$, we have $w_i = y_i - c'_i$. But when $i = 2N+1, \cdots, 3N$, we have $w_i = y_i + c'_i$ to each p,

$$z_{pi} = \begin{cases} x_{pi} - c_{pi}, \text{ if } j_p = 0, \\ x_{pi} + c_{pi}, \text{ if } j_p = 1. \end{cases}$$

From here we can get a classical regression model with cone index J corresponding to (3.3.1), suitable to data $w_i, z_{pi} (i = 1, 2, \cdots, 3N)$. Now, mark it

$$w = \beta_0 + \beta_1 z_1 + \cdots + \beta_n z_n. \tag{3.3.3}$$

By using the classical least square method, it is easier for us to find out a unique optimal solution $\beta_p (p = 0, 1, \cdots, n)$ in (3.3.3) concerning to a cone index J.

Accordingly, it is of practical value for us to approach Model (3.3.1) by using a crisp model in (3.3.3).

3.3.3 Obtaining $(\cdot, c)$ Fuzzy Data

Data in actuality are most random and fuzzy. The so-called "precision" data are almost approximation of a true value. By using fuzzy data, we can obviously get more information in objects. Therefore, it is most important for us to obtain fuzzy data by usual methods as follows.

A. Direct obtainment.

Record experiments or the measurement data as fuzzy numbers according to its character.

B. Fitting.

Fit the collected fuzzy data into a distributing function with the known fuzzy numbers; the closed one is what we long for.

C. The assignment of information.

D. Structure method and etc.

Below is only the structure of the $(\cdot, c)$ fuzzy number to be introduced. But historical data are fuzzy. Because of variety of reasons, suppose what we record is a group of real numbers $x_1, x_2, \cdots, x_n$ and $(\cdot, c)$ fuzzy number can be constructed by the group of "accurate" number, then take fuzzy time series analysis. Its steps as follows.

1^0 Let

$$M_t = \max\{x_{t-1}, x_t, x_{t+1}\}, m_t = \min\{x_{t-1} x_t, x_{t+1}\}.$$

Suppose that the data are influenced by the front and back data each (or two each) at t period, then

$$M_t \geqslant m_t, \quad \text{at } t = 2, 3, \cdots, N-1$$

and

$$M_t = \max\{x_1, x_2\}, m_t = \min\{x_1, x_2\}, \text{ at } t = 1;$$
$$M_t = \max\{x_{N-1}, x_N\}, m_t = \min\{x_{N-1}, x_N\}, \text{ at } t = N.$$

2^0 Let

$$\mu_{\tilde{y}_t}(x) = \begin{cases} 1 - \dfrac{1}{c_t}|x - \alpha_t|, & x \in [m_t, M_t], \\ 0, & x \notin [m_t, M_t]. \end{cases}$$

Here

$$\alpha_t = \frac{1}{2}(M_t + m_t), c_t = \frac{1}{2}(M_t - m_t), t = 1, 2, \cdots, N.$$

3^0 $\tilde{y}_t = (\alpha_t, c_t)$ is composition by α_t with c_t, which is a $(\cdot, c)$ fuzzy number.

In the steps above, c_t may be a fixed positive number. In application, as for free-fixed t value within interval $[1, N]$ according to practical situation, what we seek after is to choose c_t value corresponding to t.

According to the method discussed in this section, we can design a series of systems such as breakdown diagnosis in computer, future forecasting and recent identification with $(\cdot, c)$ fuzzy variables.

3.4 Self-regression with $(\cdot, c)$ Fuzzy Variables

Consider

$$\widetilde{Y}_t = \widetilde{A}_0 \mathscr{E} + \widetilde{A}_1 \widetilde{Y}_{t-1} + \cdots + \widetilde{A}_n \widetilde{Y}_{t-n} + \varepsilon_t \tag{3.4.1}$$

and

$$\widetilde{y}_t = \widetilde{f}_t(\tilde{y}_{t-1}, \tilde{y}_{t-2}, \cdots, \tilde{y}_{t-n}) + \varepsilon_t, \tag{3.4.2}$$

where data $\widetilde{Y}_{t-p}, \tilde{y}_{t-p} (p = 1, \cdots, n)$ and dependent sequence $\widetilde{Y}_t, \tilde{y}_t$ are all $(\cdot, c)$ fuzzy data, respectively. $\widetilde{Y}_t, \widetilde{f}_t$ is a fuzzy linear function and a fuzzy nonlinear function to be linearized, respectively, and ε_t is error.

We call (3.4.1) a linear self-regression model with $(\cdot, c)$ fuzzy variables and (3.4.2) a nonlinear self-regression model with $(\cdot, c)$ fuzzy variables.

3.4.1 Linear Model

For a linear self-regression model with $(\cdot, c)$ fuzzy variables (3.4.1), we discuss it determinedly.

Definition 3.4.1 Let $P(\mathscr{R})$ be a subspace consisting of the support $\mathcal{T}(\mathscr{R})$ of all non-negative elements. For each $(Y_{t-p}, \eta_{t-p}) \in P(\mathscr{R}), Y_{t-p} - \eta_{t-p} \geqslant 0, P(\mathscr{R})$ is a cone of $\mathcal{T}(\mathscr{R})$, which is a closed convex subset corresponding to topology induced by d. When $\widetilde{Y}_{t-p} = (Y_{t-p}, \eta_{t-p}), \widetilde{Y}_t = (Y_t, \eta_t)$,

$$d(\widetilde{Y}_{t-p}, \widetilde{Y}_t)^2 = \frac{[Y_{t-p} - Y_t - (\eta_{t-p} - \eta_t)]^2 + [Y_{t-p} - Y_t + (\eta_{t-p} - \eta_t)]^2 + (Y_{t-p} - Y_t^2)}{3},$$
$$\widetilde{Y}_t, \widetilde{Y}_{t-p} \in P(\mathscr{R})^N, \widetilde{Y}_{t_i}, \widetilde{Y}_{t-p_i} \in P(\mathscr{R}), (p = 1, 2, \cdots, n; i = 1, 2, \cdots, N).$$

Definition 3.4.2. Let $\widetilde{Y}_{t-p} = (\widetilde{Y}_{t-p,1}, \widetilde{Y}_{t-p,2}, \cdots, \widetilde{Y}_{t-p,N})$. Then we partition the set of natural numbers $\{1,2,\cdots,n\}$ into two exhaustive, mutually exclusive subsets $J(-)$ and $J(+)$, one of which may be empty. Each partition associates a binary multi-index $J = (J_1, J_2, \cdots, J_n)$ defined by $j_p = \begin{cases} 0, & \text{if } p \in J(+), \\ 1, & \text{if } p \in J(-). \end{cases}$

Especially, $J_0 = (0, 0, \cdots, 0), J_1 = (1, 1, \cdots, 1)$.

Denote by $C(J)$ the cone in $\mathcal{T}(\mathscr{R})$,

$$C(J) : \{A_0\mathscr{E} + A_1\widetilde{Y}_{t-1} + \cdots + A_n\widetilde{Y}_{t-n} | A_p > 0, \text{if } j_p = 0; A_p < 0, \text{if } j_p = 1\}$$
$$(p = 1, 2, \cdots, n)$$

is the cone of $\mathcal{T}(\mathscr{R})^n$ and it is determined by cone index J.

Proposition 3.4.1. *For a given cone index J, the model of minimizing in the cone $C(J)$:*

$$r(A_0(J), A(J)) = \sum_{i=1}^{N} d(A_0 + A_1\widetilde{Y}_{(t-1)_i} + \cdots + A_n\widetilde{Y}_{(t-n)_i}; \widetilde{Y}_{t_i})^2 \tag{3.4.3}$$

has a unique parameter set $A_0(J), A_1(J), \cdots, A_n(J)$.

Definition 3.4.3. Let the system $\dfrac{\partial r(A_0(J), A((J))}{\partial A_p} = 0 (p = 0, 1, \cdots, n)$ written $S(J)$. If $S(J)$ has a solution A_p, such that $A_p(J) > 0$ at $j_p = 0$; and $A_p(J) < 0$ at $j_p = 1$, then Model (3.4.3) is $J-$compatible with the data. If the minimization of the unconstrained normal equations $S(J)$ is compatible with $A_0\mathscr{E} + A_1\widetilde{Y}_{t-1} + \cdots + A_n\widetilde{Y}_{t-n}$ lying in $C(J)$, we call the model $J-$compatible.

Theorem 3.4.1. *Suppose that the data set* $\widetilde{Y}_{(t-1)_i}, \cdots, \widetilde{Y}_{(t-n)_i}$ *and* $\widetilde{Y}_{t_i}$ *is given by model* $\widetilde{Y}_{t_i} = A_0 + \sum\limits_{p=1}^{n} A_p\widetilde{Y}_{(t-p)_i} (i = 1, 2, \cdots, N)$, *then* $S(J)$ *has unique solution* $A_p(J)(p = 0, 1, \cdots, n)$ *for all cone indexes.*

Proof: Classify the observation data by subscripts and we might as well let $i = 1, 2, \cdots, N$ corresponding to the small fluctuating data, and the other data corresponding to $i = N + 1, \cdots, 3N$. Then $W_{t_i} = Y_{t_i}$ to each p, $Z_{(t-p)_i} = Y_{(t-p)_i}$ at $i = 1, 2, \cdots, N; W_{t_i} = Y_{t_i} - \eta_{t_i}$ to each p, at $i = N + 1, \cdots, 2N$. But at $i = 2N + 1, \cdots, 3N$, we have $W_{t_i} = Y_{t_i} + \eta_{t_i}$ to each p,

$$Z_{(t-p)_i} = \begin{cases} Y_{(t-p)_i} - \underline{\xi}_{(t-p)_i}, & \text{if } j_p = 0, \\ Y_{(t-p)_i} + \overline{\xi}_{(t-p)_i}, & \text{if } j_p = 1. \end{cases}$$

Hence determining self-regression model is turned into determining one $W_{t_i} = \widehat{A}_0 + \widehat{A}_1 Z_{(t-1)_i} + \cdots + \widehat{A}_n Z_{(t-n)_i}$.

Let

$$r(\widehat{A}_0, \widehat{A}) = \sum_{i=1}^{3N} d(\widehat{A}_0 + \sum_{p=1}^{n} \widehat{A}_p Z_{(t-p)_i}; W_{t_i})^2,$$

and $\dfrac{\partial r(A_0(J), A((J))}{\partial A_p} = 0$. Then we obtain the formal equations and the unique solution to $\widehat{A}_p(J)(p = 0, 1, 2, \cdots, n)$ after solving the equations.

So the modeling steps can be concluded as follows.

Step 1. Work out a self-dependent sequence table by observation data and classify the data by Definition 3.4.2.

Step 2. Change the observation value $\widetilde{Y}_{(t-p)_i}$ and the dependent variable $\widetilde{Y}_{t_i}$ into nonfuzziness by the proof of Theorem 3.4.1.

Step 3. Calculate

$$r_p = \frac{N\sum_{i=1}^{N} Z_{(t-p)_i} W_{t_i} - \sum_{i=1}^{N} Z_{(t-p)_i} \sum_{i=1}^{N} W_{t_i}}{\sqrt{[N\sum_{i=1}^{N} Z_{(t-p)_i}^2 - (\sum_{i=1}^{N} Z_{(t-p)_i})^2][N\sum_{i=1}^{N} W_{t_i}^2 - (\sum_{i=1}^{N} W_{t_i})^2]}}$$

$$(p = 1, 2, \cdots, n)$$

and take $|r_K| = \max\{r_p | p = 1, 2, \cdots, n\}$, so the best model is determined as

$$\widehat{W_t} = \widehat{A}_0 + \sum_p \widehat{A}_p Z_{t-p}.$$

Step 4. Decision

Let

$$IC = \frac{\sqrt{\frac{1}{K}\sum_{i=1}^{N}(\widehat{W}_{t_i} - W_{t_i})^2}}{\sqrt{\frac{1}{K}\sum_{i=1}^{N}\widehat{W}_{t_i}} + \sqrt{\frac{1}{K}\sum_{i=1}^{N} W_{t_i}^2}} \quad (\widehat{W}_{t_i}^2 + W_{t_i}^2 = 1).$$

Then the forecast is an efficient one at $IC \in (0, 1)$, a perfect one at $IC = 0$ and an inefficient one at $IC = 1$.

So $\widehat{\widehat{Y}}_{t+q} = \widehat{A}_0 + \sum_p \widehat{A}_p Y_{t-(p+q)}$ is determined, and the state at q moment can be estimated as

$$Y_{t+q}^* \in [Y_{t+q} - 0.382\underline{\eta}_{t+q}, Y_{t+q} + 0.618\overline{\eta}_{t+q}].$$

We are satisfied with the result after forecasting the sale of candies with Model (3.4.1) in some places in the first half year in 1984. The methods mentioned above can be developed into complicated ones by computers.

3.4.2 Non-linear Model

In this section, non-fuzzifying problem of (3.4.2) is resolved under cone index J, making it linearized with transformation.

Proposition 3.4.2. *Suppose the model like (3.4.2), for a fix cone index J, the minimized model in cone $C(J)$*

$$r(\widetilde{\beta}_0(J), \widetilde{\beta}(J)) = \sum_{i=1}^{N} d(\widetilde{f}_t(\widetilde{y}_{(t-1)_i}, \cdots, \widetilde{y}_{(t-p)_i}), \widetilde{f}_t(\widetilde{y}_{t_i}))^2$$

has a unique parameter solution $\widetilde{\beta}_0(J), \widetilde{\beta}_1(J), \cdots, \widetilde{\beta}_n(J)$.

Definition 3.4.4. Like $\dfrac{\partial r(\widetilde{\beta}_0(J),\widetilde{\beta}(J))}{\partial \widetilde{\beta}_p}=0$ and $\dfrac{\partial r_1(\widetilde{\beta}'_0(J),\widetilde{\beta}'(J))}{\partial \widetilde{\beta}'_p}=0$, the systems are written as $S(J)$ and $S_1(J)$.

Theorem 3.4.2. *Suppose data sets* $\widetilde{y}_{(t-1)_i},\cdots,\widetilde{y}_{(t-n)_i};\widetilde{y}_{t_i}$ *are all given by model* $\widetilde{y}_t=\widetilde{f}_t(y_{(t-1)_i},\cdots,\widetilde{y}_{(t-p)_i})(i=1,2,\cdots,N)$, *for all cone index* $J, S(J)$ *has a unique solution* $\widetilde{\beta}_0(J),\widetilde{\beta}_1(J),\cdots,\widetilde{\beta}_n(J)$.

Proof: Prove the following like Definition 3.1 in [Cao89b]. When data fluctuate little, we take $i=1,2,\cdots,N$, at this time, $w_{t_i}=y_{t_i}$, to each $p, z_{(t-p)_i}=y_{(t-p)_i}$; at $i=N+1,\cdots,2N$, $w_{t_i}=y_{t_i}-\eta_{t_i}$. But at $i=2N+1,\cdots,3N$, we have $w_{t_i}=y_{t_i}+\eta_{t_i}$, to each p,

$$z_{(t-p)_i}=\begin{cases} y_{(t-p)_i}-\eta_{(t-p)_i}, & \text{if } j_p=0,\\ y_{(t-p)_i}+\eta_{(t-p)_i}, & \text{if } j_p=1.\end{cases}$$

Therefore, a deterministic self-regression model is gained as follows:

$$w_t=f_t(z_{t-1},z_{t-2},\cdots,z_{t-n}).$$

By variable replacement, it is linearized, then

$$L(w_t)=L[f_t(z_{t-1},z_{t-2},\cdots,z_{t-n})],$$

i.e.,

$$U_t=\widetilde{\beta}_0(J)+\sum_{p=1}^{n}\widetilde{\beta}_j(J)z_{t-p}.$$

It is not difficult to get a conclusion by using least square principle for U_t.

Proposition 3.4.3. *As for fix cone index* J, *the minimum model in cone* $C(J)$

$$r_1(\widetilde{\beta}'_0(J),\widetilde{\beta}'(J))=\sum_{i=1}^{N}\widetilde{D}_i^2 d[\widetilde{f}_t(\widetilde{y}_{(t-1)_i},\cdots,\widetilde{y}_{(t-p)_i}),\widetilde{f}_t(\widetilde{y}_{t_i})]^2$$

has a unique parameter solution $\widetilde{\beta}'_0(J),\widetilde{\beta}'_1(J),\cdots,\widetilde{\beta}'_n(J)$.

Theorem 3.4.3. *Suppose a datum set* $\widetilde{y}_{(t-1)_i},\cdots,\widetilde{y}_{(t-n)_i};\widetilde{y}_{t_i}$ *is all given by Model (3.4.2), then to all cone indexes* $J,S_1(J)$ *has a unique solution* $\widetilde{\beta}'_0(J),\widetilde{\beta}'_1(J),\cdots,\widetilde{\beta}'_n(J)$.

Proof: Similarly to the proof of Theorem 3.4.1, we only notice $S_1(J)$, i.e.,

$$\begin{pmatrix} N\sum\limits_{i=1}^{N} D_i & \sum\limits_{i=1}^{N} D_i z_{(t-1)_i} & \cdots & \sum\limits_{i=1}^{N} D_i z_{(t-p)_i} \\ \sum\limits_{i=1}^{N} D_i z_{(t-1)_i} & \sum\limits_{i=1}^{N} D_i z_{(t-1)_i}^2 & \cdots & \sum\limits_{i=1}^{N} D_i z_{(t-1)_i} z_{(t-p)_i} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{i=1}^{N} D_i z_{(t-p)_i} & \sum\limits_{i=1}^{N} D_i z_{(t-1)_i} z_{(t-p)_i} & \cdots & \sum\limits_{i=1}^{N} D_i z_{(t-p)_i}^2 \end{pmatrix} \begin{pmatrix} \widetilde{\beta}_0'(J) \\ \widetilde{\beta}_1'(J) \\ \vdots \\ \widetilde{\beta}_n'(J) \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{i=1}^{N} D_i U_{t_i} \\ \sum\limits_{i=1}^{N} D_i z_{(t-1)_i} U_{t_i} \\ \vdots \\ \sum\limits_{i=1}^{N} D_i z_{(t-p)_i} U_{t_i} \end{pmatrix}. \tag{3.4.4}$$

Obviously, (3.4.4) has a unique solution $\widetilde{\beta}_0'(J), \widetilde{\beta}_1'(J), \cdots, \widetilde{\beta}_n'(J)$.

It is also verified that, by adopting weight least square method to determine fuzzy nonlinear self-regression model, the forecasting is more accurate.

The construction of models is induced as follows.

1. By observation data $(y_{(t-p)_i}, \eta_{(t-p)_i})$, we authorize a table in fuzzy self-related sequence table like Table 3.2.1.

2. Nonfuzzify (3.4.2) (or by variable replacement), and change it into deterministic nonlinear model (or linear fuzzy model).

3. By variable replacement (or fuzzfication), and change the corresponding model into a classical self-regression model

$$U_t = \widetilde{\beta}_0'(J) + \sum_{p=1}^{n} \widetilde{\beta}_p'(J) z_{t-p}.$$

4. Determine r_α by checking the table in critical value of related coefficient.

Suppose

$$r_p = \frac{N\sum\limits_{i=1}^{N} z_{(t-p)_i} U_{t_i} - \sum\limits_{i=1}^{N} z_{(t-p)_i} \sum\limits_{i=1}^{N} U_{t_i}}{\sqrt{[N\sum\limits_{i=1}^{N} z_{(t-p)_i}^2 - (\sum\limits_{i=1}^{N} z_{(t-p)_i})^2][N\sum\limits_{i=1}^{N} U_{t_i}^2 - (\sum\limits_{i=1}^{N} U_{t_i})^2]}}$$

and calculate the self-related coefficient in quarter by moving backwards $p (p = 1, 2, \cdots, n)$.

If $|r_p| > r_\alpha$, the linear relation is marked between p period backwards and norm time sequence in building a self-regression model.

Again take $|r_K| = \max\{r_p|p = 1, 2, \cdots, n\}$, and the model is best, which is built on the norm time series U_t backwards to K quarter.

The model is

$$\widehat{U}_t(\widetilde{\beta}'_0, \widetilde{\beta}') = \beta'_0 + \sum_{p=1}^{n} \widetilde{\beta}'_p z_{t-p}. \tag{3.4.5}$$

5. The parameter in (3.4.5) $\widetilde{\beta}'_0(J), \widetilde{\beta}'_1(J), \cdots, \widetilde{\beta}'_n(J)$ is determined by using the classical least squares method, placed into (3.4.5). This is what we find.

6. Testify.

Let

$$IC = \frac{\sqrt{\frac{\sum_{i=1}^{N}(\widehat{U}_{t_i} - U_{t_i})}{K}}}{\sqrt{\frac{\sum_{i=1}^{N}\widehat{U}^2_{t_i}}{K}} + \sqrt{\frac{\sum_{i=1}^{N}U^2_{t_i}}{K}}} \ (\widehat{U}^2_{t_i} + U^2_{t_i} = 1).$$

It is an effective forecast at $IC \in [0, 1)$, a prefect forecast at $IC = 0$ and an ineffective forecast at $IC = 1$.

7. Regeneration.

According to $\widetilde{\beta}'_p(p = 1, 2, \cdots, n)$ determine coefficient in (3.4.2) before supposing the best model solved in nonlinear infinite regression problem to be

$$\widehat{U'}_t(\widetilde{\beta}'_0, \widetilde{\beta}') = \widetilde{f}_t(\widetilde{y}_{t-1}, \cdots, \widetilde{y}_{t-n}).$$

Given

$$\widehat{U'}_{t+q}(\widetilde{\beta}'_0, \widetilde{\beta}') = \widetilde{f}_{t+q}(\widetilde{y}_{t-(1+q)}, \cdots, \widetilde{y}_{t-(n+q)}),$$

we can forecast the constant statement at q.

Example 3.4.1: Suppose $U_t = \beta'_0(J) + \sum\limits_{p=1}^{n} \widetilde{\beta}'_p(J) z_{t-p}$. Let $U_t = \ln U'_t$. Then

$$U'_t = e^{\widetilde{\beta}'_0(J) + \sum\limits_{p=1}^{n} \widetilde{\beta}'_p(J) z_{t-p}}$$

$$\Rightarrow U'_{t-(n+q)} = e^{\widetilde{\beta}'_0(J) + \sum\limits_{p=1}^{n} \widetilde{\beta}'_p(J) z_{t-p}}.$$

Therefore

$$U'_{t-(n+q)} = e^{\widetilde{\beta}'_0 + \sum\limits_{p=1}^{n} \widetilde{\beta}'_p \widetilde{y}_{t-(n+q)}}$$

is what we find.

If there exist parameters in the formula, the parameters need determining by an optimization method.

8. Determine the region of forecasting evaluation [Cao89c].

Since $U'_{t+q} = (U'_{t+q}, \underline{\theta}_{t+q}, \overline{\theta}_{t+q})_T$, the forecasting region is

$$U^*_{t+q} \in [U'_{t+q} - 0.382\underline{\theta}_{t+q}, U'_{t+q} + 0.618\overline{\theta}_{t+q}].$$

3.5 Nonlinear Regression with T-fuzzy Data to be Linearized

3.5.1 Introduction

Consider a nonlinear model as follows:

$$\widetilde{y} = \widetilde{f}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) + \varepsilon, \tag{3.5.1}$$

where $\widetilde{f}$ is a fuzzy nonlinear function to be linearized, ε is an error, $\tilde{y} = (y, \underline{\eta}, \overline{\eta})$ and $\tilde{x}_p = (x_p, \underline{\xi}_p, \overline{\xi}_p)(p = 1, 2, \cdots, n)$ denote T-fuzzy correlated variables and independent variables, respectively. We call (3.5.1) a nonlinear regression model with T-fuzzy variables. A classical model is regarded as its especial example.

In this section, non-T-fuzzified problem of (3.5.1) is resolved under cone index J, making it linearized with transformation. Meanwhile, the theories of this model are demonstrated in confirmation and linearized, and a method is advanced to this problem.

3.5.2 Prepare Theorem and Property

Seen in Section 1.7 is a fuzzy number of definition and property relevant to T-fuzzy number. It is easy to certificate that this T-fuzzy datum $\tilde{x} = (x, \underline{\xi}, \overline{\xi})$ is regular and convex fuzzy subset.

Definition 3.5.1. If $\tilde{x} = (m(x), L_1, R_1), \tilde{y} = (m(y), L_2, R_2)$, then the distance definition on the T-fuzzy number set $\mathcal{T}(\mathscr{R})$ is

$$d(\widetilde{x}, \widetilde{y})^2 = D_2(\text{Supp}(\tilde{x}), \text{Supp}(\tilde{y}))^2 + (m(\tilde{x}) - m(\tilde{y}))^2,$$

where Supp$(\cdot)$ denotes a support of $(\cdot)$, $m(\cdot)$ denotes a main value of $(\cdot)$.

Especially, when $\widetilde{x} = (x, \underline{\xi}, \overline{\xi}), \tilde{y} = (y, \underline{\eta}, \overline{\eta})$, then

$$d(\widetilde{x}, \widetilde{y})^2 = \frac{(x - y - (\underline{\xi} - \underline{\eta}))^2 + (x - y + (\overline{\xi} - \overline{\eta}))^2 + (x - y)^2}{3}.$$

Lemma 3.5.1. $d(\tilde{y}_i, \widetilde{y}_j)^2 = 2d(\tilde{y}_i, \tilde{x})^2 + 2d(\tilde{x}, \tilde{y}_j)^2 - 4d(\tilde{x}, \dfrac{(\tilde{y}_i + \tilde{y}_j)}{2})^2.$

Proof: Similar to Lemma 3.1.1, this lemma can be proved.

Theorem 3.5.1. *Let V be a closed cone in $P(\mathscr{R})$. Then for any $\widetilde{x}$ in $P(\mathscr{R})$, a unique T-fuzzy number $\widetilde{y}_0$ exists in V, such that $d(\widetilde{x}, \widetilde{y}_0) \leqslant d(\widetilde{x}, \widetilde{y})$ for all $\widetilde{y}$ in V. A necessary and sufficient condition, where $\widetilde{y}_0$ is the unique minimizing fuzzy number in V, is that $\widetilde{x}$ is $\tilde{y}_0-$orthogonality to V.*

Proof: Similar to Theorem 3.1.1, this theorem is not difficult to prove.

3.5.3 Two Kinds of Non-T-Fuzzily Approach and Its Equivalence

Based on the above-mentioned theories, by taking Model (3.5.1) for example, we inquire into a method to the conversion of a non-fuzzy linear model.
I. T-fuzzifying before making variable replacement linearized

Definition 3.5.2. For the given cone index $\mathcal{J}$, the measurement is defined between the fuzzy data and regression curve as

$$Q(\widetilde{r}_0(\mathcal{J}), \widetilde{r}(\mathcal{J})) = \sum_{i=1}^{N} d(\widetilde{f}(\widetilde{x}_{1i}, \widetilde{x}_{2i}, \cdots, \widetilde{x}_{ni}); \widetilde{y}_i)^2. \tag{3.5.2}$$

Theorem 3.5.2. *Suppose that T-fuzzy data $\widetilde{x}_{1i}, \widetilde{x}_{2i}, \cdots, \widetilde{x}_{ni}, \widetilde{y}_i$ are all given from model $\widetilde{y}_i = \widetilde{f}(\widetilde{x}_{1i}, \widetilde{x}_{2i}, \cdots, \widetilde{x}_{ni})(i = 1, 2, \cdots, N)$ for all of cone index $\mathcal{J}$, there exists a unique solution $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$ in a normal equations* $\dfrac{\partial Q(\widetilde{r}_0(\mathcal{J}), \widetilde{r}(\mathcal{J}))}{\partial \widetilde{r}_p} = 0(p = 0, 1, \cdots, n).$

Proposition 3.5.1. *For a given cone index $\mathcal{J}$, the minimization Model* (3.5.2) *has a unique parameter solution set $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$ in cone $C(\mathcal{J})$.*

In fact,we can take a list of T-fuzzy samples

$$((x_1, \underline{\xi}_1, \overline{\xi}_1), (y_1, \underline{\eta}_1, \overline{\eta}_1)), \cdots, ((x_n, \underline{\xi}_n, \overline{\xi}_n), (y_n, \underline{\eta}_n, \overline{\eta}_n)),$$

for the smaller sample in fluctuation, let $w_i = y_i$, and to each $p, z_{pi} = x_{pi}$, at $i = 1, 2, \cdots, N$; to the rest sample, it can be handled as follows.

On the one hand, let $w_i = y_i - \underline{\eta}_i$. To each p,

$$z_{pi} = \begin{cases} x_{pi} - \underline{\xi}_{pi}, & if \quad j_p = 0, \\ x_{pi} + \overline{\xi}_{pi}, & if \quad j_p = 1, \end{cases}$$

at $i = N + 1, N + 2, \cdots, 2N$. On the other hand, let $w_i = y_i + \overline{\eta}_i$. To each p,

$$z_{pi} = \begin{cases} x_{pi} + \overline{\xi}_{pi}, & if \quad j_p = 0, \\ x_{pi} - \underline{\xi}_{pi}, & if \quad j_p = 1, \end{cases}$$

at $i = 2N + 1, 2N + 2, \cdots, 3N$.

Therefore (3.5.1) can be changed into a classically expressive type as follows:

$$w_i = f(z_{1i}, z_{2i}, \cdots, z_{ni})(i = 1, 2, \cdots, N).$$

Through an appropriate linear transformation L, the linear regression model is then acquired below:

$$U = \widetilde{r}_0(\mathcal{J}) + \sum_{p=1}^{n} \widetilde{r}_p(\mathcal{J}) z_p. \tag{3.5.3}$$

Thereout, it is easy to obtain a result in Proposition 3.5.1 (or in Theorem 3.5.1)

Corollary 3.5.1. *Under the condition of Theorem 3.5.2, for a given cone index $\mathcal{J}$, (3.5.3) there exists a group of unique coefficients $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$.*

II. Variables replacement before non-T-fuzzified

Suppose $\widetilde{y}$ like (3.5.1), it can be changed into a linear function with T-fuzzy variables through an appropriate variable replacement:

$$\widetilde{s} = \widetilde{r}'_0 + \sum_{p=1}^{n} \widetilde{r}'_p \widetilde{u}_{p_i}. \tag{3.5.4}$$

Theorem 3.5.3. *Under the condition of Theorem 3.5.2, for a given cone index $\mathcal{J}$, (3.5.4) has a unique coefficient $\widetilde{r}'_p(\mathcal{J})(p = 0, 1, \cdots, n)$.*

Proof: Because the coefficients in (3.5.4) $\widetilde{r}'_p(p = 0, 1, \cdots, n)$ are all confirmed by T-fuzzy data $\widetilde{u}_{p_i}$, according to Theorem 3.5.2 and proof in Proposition 3.5.1, the theorem also gets true.

Thereout it is known that, (3.5.4) can be changed into a deterministic linear model

$$V = \widetilde{r}'_0(\mathcal{J}) + \sum_{p=1}^{n} \widetilde{r}'_p(\mathcal{J}) z'_p. \tag{3.5.5}$$

Theorem 3.5.4. *Under the condition of Theorem 3.5.2, in the same fix cone index $\mathcal{J}$, the determined T-fuzzy data regression Equation (3.5.3) is equivalent to (3.5.5).*

Proof: Because under the same fix cone index $\mathcal{J}$, original T-fuzzy datum $\widetilde{x}_p$ is determined in cone $C(\mathcal{J})$, hence first to (3.5.1), we implement non-T-fuzzification: $N(\widetilde{y})$; carry out again the linearized: $L(W)$ (N, L mean the implement of non-T-fuzzication and linearized, respectively) before getting z_p. Or towards (3.5.1) we carry out the linearized first, then non-T-fuzzification and get z'_p. Acquisition of the independent variable sequence should be equal accordingly, i.e., $z_p = z'_p$. Again because, in the above cone $C(\mathcal{J})$, normal equations corresponding to (3.5.3) and (3.5.5)

$$\frac{\partial Q(\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J}))}{\partial \widetilde{r}_p} = 0$$

and

$$\frac{\partial Q(\widetilde{r}'_0(\mathcal{J}), \widetilde{r}'_1(\mathcal{J}), \cdots, \widetilde{r}'_n(\mathcal{J}))}{\partial \widetilde{r}'_p} = 0 (p = 0, 1, \cdots, n)$$

contain a unique parameter solution, respectively, $\widetilde{r}_p(\mathcal{J})$ and $\widetilde{r}'_p(\mathcal{J})$, and again according to $z_p = z'_p$, we have $\widetilde{r}_p(\mathcal{J}) = \widetilde{r}'_p(\mathcal{J})(p = 0, 1, \cdots, n)$. Hence, (3.5.3) $\Longleftrightarrow$ (3.5.5).

3.5.4 Weight of Linearized Nonlinear Regression with T-Fuzzy Variables

T-fuzzy data reflect more objectively observation ones in different positions in the whole test. In a convex cone, the center value is regarded as main value, with the value distributed at both sides of left and right. Consider the influence degree of a data pair $\widetilde{y};\widetilde{x}_1,\widetilde{x}_2,\cdots,\widetilde{x}_n$, it is effective for us to handle a linear regression problem with T-fuzzy variables by adopting non-T-fuzzifying [Cao93e]. But it is not necessarily the best to handle a non-linear regression with T-fuzzy variables by adopting the above two replacements before determining regression coefficient with a least squares principle. Therefore, we need fuzzy weight processing for the error item $\widetilde{y}_i-\widetilde{\widehat{y}}_i$. Because, at different points $\widetilde{y}_i(i=1,\cdots,N)$, when the similar deviation is transformed to the original T-Fuzzy variables, the transform makes the direct proportion between partial difference rate and fuzzy difference $(\frac{\triangle\widetilde{y}}{\triangle\widetilde{s}})_i$ or fuzzy derivative $(\frac{d\widetilde{y}}{d\widetilde{s}})_i$. It is known that the model handled by weight is more accurate than the non-weighted one handled in a practical operation.

Assume that we write fuzzy difference or fuzzy derivative as $\widetilde{D}_i=(\frac{\triangle\widetilde{y}}{\triangle\widetilde{s}})_i$ or $\widetilde{D}_i=(\frac{d\widetilde{y}}{d\widetilde{s}})_i$. Let

$$\begin{aligned}Q_1(\widetilde{r}_0,\widetilde{r})&=\sum_{i=1}^{N}[(\frac{d\widetilde{y}}{d\widetilde{s}})_i(\widetilde{s}_i-\widetilde{\widehat{s}}_i)]^2(\sum_{i=1}^{N}[(\frac{\triangle\widetilde{y}}{\triangle\widetilde{s}})_i(\widetilde{s}_i-\widetilde{\widehat{s}}_i)]^2)\\&=\sum_{i=1}^{N}[\widetilde{D}_i(\widetilde{s}_i-\widetilde{\widehat{s}}_i)]^2=\sum_{i=1}^{N}\widetilde{D}_i^2(\widetilde{s}_i-(r_0+\sum_{i=1}^{N}r_p\widetilde{u}_{p_i}))^2.\end{aligned}\tag{3.5.6}$$

Then we discuss the following by using Method II in 3.5.3 (If based on Method I, we can get the similar result).

Proposition 3.5.2. *For the given cone index $\mathcal{J}$, in cone $C(\mathcal{J})$, then*

$$(\frac{\triangle\widetilde{y}}{\triangle\widetilde{s}})_i\Rightarrow\ (\frac{\triangle y(z(\mathcal{J}))}{\triangle s(z(\mathcal{J}))})_i,\quad (\frac{d\widetilde{y}}{d\widetilde{s}})_i\Rightarrow\ (\frac{dy(z(\mathcal{J}))}{ds(z(\mathcal{J}))})_i$$

and (3.5.6) can be changed into

$$Q_1(\widetilde{r}_0,\widetilde{r})=\sum_{i=1}^{N}\widetilde{D}_i^2(V_i-(\widetilde{r}_0+\sum_{i=1}^{N}\widetilde{r}_pz_{p_i}))^2,\tag{3.5.7}$$

where, D_i denotes $(\frac{\triangle y(z(\mathcal{J}))}{\triangle s(z(\mathcal{J}))})_i$ or $(\frac{dy(z(\mathcal{J}))}{ds(z(\mathcal{J}))})_i$.

Proposition 3.5.3. *For the given cone index $\mathcal{J}$, the minimized model in $C(\mathcal{J})$ (3.5.6) has a unique parameter solution $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$.*

Theorem 3.5.5. *Let T-fuzzy data $\widetilde{x}_{1i}, \widetilde{x}_{2i}, \cdots, \widetilde{x}_{ni}; \widetilde{y}_i$ be all given by model $\widetilde{y}_i = f_i(\widetilde{x})(i = 1, 2, \cdots, N), \widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_n)$. Then to the given cone index $\mathcal{J}$, a normal equations $\dfrac{\partial Q_1(\widetilde{r}_0, \widetilde{r})}{\partial \widetilde{r}_p} = 0$ contains a unique solution $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$.*

Proof: By following the proof of Proposition 3.5.1 and Theorem 3.5.4, (3.5.6) can be changed into (3.5.7) and the normal equations with respect to (3.5.7) into

$$\begin{pmatrix} N\sum\limits_{i=1}^{N} D_i & \sum\limits_{i=1}^{N} D_i Z_{1i} & \cdots & \sum\limits_{i=1}^{N} D_i z_{ni} \\ \sum\limits_{i=1}^{N} D_i z_{1i} & \sum\limits_{i=1}^{N} D_i z_{1i}^2 & \cdots & \sum\limits_{i=1}^{N} D_i z_{1i} z_{ni} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{i=1}^{N} D_i z_{ni} & \sum\limits_{i=1}^{N} D_i z_{ni} z_{1i} & \cdots & \sum\limits_{i=1}^{N} D_i z_{ni}^2 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{r}_0(\mathcal{J}) \\ \widetilde{r}_1(\mathcal{J}) \\ \vdots \\ \widetilde{r}_n(\mathcal{J}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{i=1}^{N} D_i V_i \\ \sum\limits_{i=1}^{N} D_i z_{1i} V_i \\ \vdots \\ \sum\limits_{i=1}^{N} D_i z_{ni} V_i \end{pmatrix}, \tag{3.5.8}$$

i.e.,

$$(Dz^T z)\tilde{r}(\mathcal{J}) = Dz^T V.$$

Therefore a unique solution $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$ exists in (3.5.8).

Correspondingly, we can get a testifying formula related to the regression equation [Guj86]

$$\sum_{i=1}^{N} [D_i(V_i - \widehat{V}_i)]^2 = \sum_{i=1}^{N} D_i^2 (V_i - \overline{V}_i) \Big[1 - \frac{\sum\limits_{i=1}^{N} D_i^2 (\widehat{V}_i - \overline{V})^2}{\sum\limits_{i=1}^{N} D_i^2 (V_i - \overline{V})^2}\Big].$$

Obviously,

$$\widehat{R}^2 = \frac{\sum\limits_{i=1}^{N} D_i^2 (\widehat{V}_i - \overline{V})^2}{\sum\limits_{i=1}^{N} D_i^2 (V_i - \overline{V})^2} \leqslant 1,$$

i.e.,

$$|\widehat{R}| = \sqrt{\frac{\sum\limits_{i=1}^{N} D_i^2(\widehat{V}_i - \overline{V})^2}{\sum\limits_{i=1}^{N} D_i^2(V_i - \overline{V})^2}} \leqslant 1, \tag{3.5.9}$$

calling $\widehat{R}$ a weighted related-coefficient. At $|\widehat{R}| \to 1$, it represents more linear related between V and z. If $\widehat{R} > R_\alpha$ (determined by checking related coefficient table), then the linear relation of regression equation $V = \widetilde{r}_0 + \sum\limits_{p=1}^{n} \widetilde{r}_p \widetilde{z}_p$ is significant.

The test of significance in regression coefficients is shown as follows.

Let

$$\begin{aligned} G &= \frac{\dfrac{\sum\limits_{i=1}^{N} D_i^2(\widehat{V}_i - V)^2}{K}}{\dfrac{\sum\limits_{i=1}^{N} D_i^2(V_i - \widehat{V}_i)^2}{N-K-1}}, \\ F &= \frac{\dfrac{\widehat{R}_p^2}{c_{pp}}}{\dfrac{\sum\limits_{i=1}^{N} D_i^2(V_i - \widehat{V}_i)^2}{N-K-1}} \quad (p = 1, 2, \cdots, n). \end{aligned} \tag{3.5.10}$$

Then negate $H_0 : \widetilde{r}_j = 0$ at $F^{(j)} > F_\alpha(1, N-K-1)$, where c_{pp} is p-element on the main diagonal of matrix $(Dz^T z)^{-1}$. If some p exists, such that $F^{(p)} < F_\alpha(1, N-K-1)$, then it shows that z_p influences V little, omitted here.

3.5.5 Numeric Example

Example 3.5.1: Suppose that a non-linear fuzzy regression model as follows:

$$\widetilde{y} = A_0 + be^{-\frac{c}{\widetilde{x}}},$$

where A_0, b, c are all constants, and then, by its non-T-fuzzification, we have

$$W = A_0 + be^{-\frac{c}{z}}.$$

Besides, z is a geometrical sequence, and suppose $\Delta = \dfrac{z_{k+1}}{z_k}$, then

$$W_k = A_0 + be^{-\frac{c}{z_k}}, W_{k+1} = A_0 + be^{-\frac{c}{z_{k+1}}},$$

hence

$$W_{k+1} = A_0 + be^{-\frac{c}{z_k}\Delta} = A_0 + \Big(\frac{W_k - A_0}{b}\Big)^{-\frac{1}{\Delta}},$$

which can be turned into

$$v = r_0 + r_1 u,$$

where $v = \ln(W_{k+1} - A_0), u = \ln(W_k - A_0), r_0 = \frac{1}{\Delta}\ln|b|$ and $r_1 = -\frac{1}{\Delta}$ contain parameters, which should be evaluated by an optimum seeking method.

Therefore, The modeling steps of (3.5.1) should be concluded as follows:

1) Replacement. (3.5.1) is replaced variably (or dealing by non-T-fuzzification), and it is linearized (or changed into deterministic non-linearity type).

2) Change. Non-T-fuzzify (or variable replacement), and the problem is changed into a linear deterministic model:

$$V = \widetilde{r}_0(\mathcal{J}) + \widetilde{r}_1(\mathcal{J})z_1 + \cdots + \widetilde{r}_n(\mathcal{J})z_n. \tag{3.5.11}$$

3) Determination. Determine $\widetilde{r}_0(\mathcal{J}), \widetilde{r}_1(\mathcal{J}), \cdots, \widetilde{r}_n(\mathcal{J})$ by solving (3.5.8), i.e., it is a regression coefficient of (3.5.11).

4) Calculation. Calculate (3.5.9) and (3.5.10), and testify (3.5.11) by an ordinary method.

5) Forecast. Coefficient in (3.5.1) is determined by its solution $\widetilde{r}_p(\mathcal{J})(p = 0, 1 \cdot \cdot, n)$, and then (3.5.1) can be used to forecast, the choice of q moment in forecasting region is similar to Ref.[Cao89c]. If $\widetilde{y}_q = (y_q, \underline{\eta}_q, \overline{\eta}_q)$, then $y_q^* \in [y_q - 0.328\underline{\eta}_q, y_q + 0.618\overline{\eta}_q]$.

3.5.6 Conclusion

The method can be programmed for operation on computers, thus the model mentioned here is more accurate, more effective and better practical than the models which clear and non-weight nonlinear.

3.6 Regression and Self-regression Models with Flat Fuzzy Variables

3.6.1 Introduction

Because $(\cdot, c)$ fuzzy data contain L-R fuzzy variables, T-fuzzy variables and the flat fuzzy variables (or trapezoid fuzzy variables), we can further more apply the flat fuzzy variables $\tilde{x}_{*i} = (x_{*i}^-, x_{*i}^+, \underline{\xi}_{*i}, \overline{\xi}_{*i}), \tilde{y}_* = (y_*^-, y_*^+, \underline{\eta}_*, \overline{\eta}_*)$ to the regression and self-regression models in this section.

3.6.2 Determination of the Model with Flat Fuzzy Variables

Definition 3.6.1. Suppose that the models are

$$\tilde{y} = \beta_0\mathbb{E} + \beta_1\tilde{x}_1 + \cdots + \beta_n\tilde{x}_n + \varepsilon \tag{3.6.1}$$

and

$$\tilde{y}_t = \beta_0\mathbb{E} + \beta_1\tilde{x}_{t-1} + \cdots + \beta_n\tilde{x}_{t-n} + \varepsilon_t, \tag{3.6.2}$$

where $\tilde{x}_p, \tilde{x}_{t-p}(p=1,2,\ldots,n)$ are flat fuzzy variables, and $\tilde{y}, \tilde{y}_t$ are flat fuzzy function variables. We call (3.6.1) and (3.6.2) a regression model and a self-regression one with flat fuzzy variables, respectively, $\mathbb{E}$ is an $n-$vector represented by all $\mathcal{E}=(1,1,0,0)$, and $\varepsilon, \varepsilon_t$ are errors, and t is time.

Because the variables in (3.6.1) and (3.6.2) are fuzzy, it is impossible to obtain a meaningful result by a classical least square method. Therefore, determination path is researched to model (3.6.1)(3.6.2) as follows.

Definition 3.6.2. Let $\tilde{x}=(x^-, x^+, \underline{\xi}, \overline{\xi}) \in P(\mathscr{R})$ for each $\tilde{x}, x^- \geqslant \underline{\xi}, x^+ \geqslant \overline{\xi}$. Then $P(\mathscr{R})$ is one of the platform $\mathcal{T}(\mathscr{R})$, and is a convex close subset of $\mathcal{T}(\mathscr{R})$ relevant with topology induced by distance d.

Suppose test data to be $\tilde{x}_{*1}, \tilde{x}_{*2}, \ldots, \tilde{x}_{*N}; \tilde{y}_*$, where $\tilde{x}_{*i}=(x_{*i}^-, x_{*i}^+, \underline{\xi}_{*i}, \overline{\xi}_{*i})$ $(i=1,2,\ldots,N), \tilde{y}_*=(y_*^-, y_*^+, \underline{\eta}_*, \overline{\eta}_*)$, and when the model is a regression model with flat fuzzy variables, "$*$" is taken to p; when the model is a self-regression model with flat fuzzy variables, "$*$" taken to t-p. Hence for the model (3.6.1) and (3.6.2), $\beta_i (i=1,2,\ldots,N)$ is a ordinary real number, $\tilde{x}_{*i}$ is a flat fuzzy variable, $\tilde{y}_*$ is a flat affine function from $P(\mathscr{R})^N$ to $\mathcal{T}(\mathscr{R})$.

Let

$$r(\beta_0,\beta)=\sum_{i=1}^{N} d_i(\tilde{x}_{*i},\tilde{y}_*)^2=\sum_{i=1}^{N}[\tilde{y}_*-(\beta_0+\beta_1\tilde{x}_{*1}+\ldots+\beta_n\tilde{x}_{*N})]^2.$$

Then β_p determined by applying the least square method is a flat fuzzy number rather than a real number, where $\beta=(\beta_1,\beta_2,\ldots,\beta_n)$, so that a classical least square method can't be directly applied, and a conversion should be made. Similarly to method of Section 3.1, we induce definitions and properties below.

Definition 3.6.3. Assume $\tilde{x}_{*i}=(\tilde{x}_{*i},\tilde{x}_{*i},\ldots,\tilde{x}_{*i})$ $(i=1,2,\ldots,N)$. If partition the set of nature numbers $\{1,2,\cdots,n\}$ into two exhaustive, mutually exclusive subsets $T(-), T(+)$, one of which may be empty set ϕ. Then to each such partition associate a binary multi-index $T=(\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_n)$ defined by $\mathcal{T}_i=\{0, \text{if } i\in T(+); 1, \text{if } i \in T(-)\}$. Especially, we write $\mathcal{T}_0=(0,0,\cdots,0), \mathcal{T}_1=(1,1,\cdots,1)$. Use

$$C(T)=\{\beta_0\mathbb{E}+\beta_1\tilde{x}_1+\ldots+\beta_n\tilde{x}_n|\beta_p\geqslant 0, \text{if } j_p=0; \beta_p\leqslant 0, \text{if } j_p=1\}$$

to represent a platform in $\mathcal{T}(\mathscr{R})^N$, we call it a determined platform from the platform index T.

Proposition 3.6.1. *For a given platform index T, there exists a unique parameter solution $\beta_0(T),\beta_1(T),\ldots,\beta_n(T)$ of minimum model*

$$r(\beta_0(T),\beta(T))=\sum_{i=1}^{n} d(\beta_0+\beta_1x_{1i}+\ldots+\beta_nx_{ni},y_i)]^2 \tag{3.6.3}$$

in platform $C(T)$, where $\beta(T)=(\beta_1(T),\beta_2(T),\ldots,\beta_n(T))$.

Definition 3.6.4. Suppose data to be $\tilde{x}_{*1}, \tilde{x}_{*2}, \ldots, \tilde{x}_{*n}; \tilde{y}_*$, and we call the system $S(T)$ consisting of $n+1$ equation

$$\frac{\partial r(\beta_0(T), \beta(T))}{\partial \beta_p} = 0(p = 0, 1, \ldots, n).$$

If $S(T)$ has a solution $\beta_0(T), \beta_1(T), \ldots, \beta_n(T)$, such that $\beta_p > 0$ at $j_p = 0; \beta_p < 0$ at $j_p = 1$, then we call (3.6.3) T-compatible with the data.

If un-constraints least value of $S(T)$ is compatible with $\beta_0 \mathbb{E} + \sum\limits_{p=1}^{n} \beta_p \tilde{x}_p$ in $C(T)$, then this model is called compatibleness.

Theorem 3.6.1. *Suppose that flat fuzzy data $\tilde{x}_{1i}, \tilde{x}_{2i}, \ldots, \tilde{x}_{ni}, \tilde{y}_i$ satisfy (3.6.1) and (3.6.2), respectively, then, for all of the platform index T, there exists a unique solution $\beta_0(T), \beta_1(T), \ldots, \beta_n(T)$ in system*

$$\frac{\partial r(\beta_0(T), \beta(T))}{\partial \beta_p} = 0 \ (p = 0, 1, \ldots, n).$$

Proof: Suppose that flat fuzzy data are $\tilde{x}_{*i} = (x_{*i}^-, x_{*i}^+, \underline{\xi}_{*i}, \overline{\xi}_{*i}), \tilde{y}_* = (y_*^-, y_*^+, \underline{\eta}_*, \overline{\eta}_*)$, and "$*$" taken to p, or "$*$" taken to t-p. Catalogue $\{\tilde{x}_{*i}\}$ by subscription.

For $i = 1, 2, \ldots, N$, take

$$w_* = \frac{\overline{\eta}_* y_*^- + \underline{\eta}_* y_*^+ + \underline{\eta}_* \overline{\eta}_*}{\underline{\eta}_* + \overline{\eta}_*} + \frac{\underline{\eta}_* + \overline{\eta}_*}{2},$$

to each $*$,

$$z_{*i} = \frac{\overline{\xi}_{*i} x_{*i}^- + \underline{\xi}_{*i} x_{*i}^+}{\underline{\xi}_{*i} + \overline{\xi}_{*i}} + \frac{\underline{\xi}_{*i} + \overline{\xi}_{*i}}{2}$$

for $i = N+1, \ldots, 2N$, let $w_* = y_*^- - \underline{\eta}_*$. To each $*$,

$$z_{*i} = \begin{cases} \dfrac{\overline{\xi}_{*i} x_{*i}^- + \underline{\xi}_{*i} x_{*i}^+}{\underline{\xi}_{*i} + \overline{\xi}_{*i}} - \underline{\xi}_{*i}, & j_* = 0, \\ \dfrac{\overline{\xi}_{*i} x_{*i}^- + \underline{\xi}_{*i} x_{*i}^+}{\underline{\xi}_{*i} + \overline{\xi}_{*i}} + \overline{\xi}_{*i}, & j_* = 1, \end{cases}$$

and for $i = 2N+1, \ldots, 3N$, let $w_* = y_*^+ - \overline{\eta}_*$. To each $*$,

$$z_{*i} = \begin{cases} \dfrac{\overline{\xi}_{*i} x_{*i}^- + \underline{\xi}_{*i} x_{*i}^+}{\underline{\xi}_{*i} + \overline{\xi}_{*i}} - \overline{\xi}_{*i}, & j_* = 0, \\ \dfrac{\overline{\xi}_{*i} x_{*i}^- + \underline{\xi}_{*i} x_{*i}^+}{\underline{\xi}_{*i} + \overline{\xi}_{*i}} + \underline{\xi}_{*i}, & j_* = 1. \end{cases}$$

Under the given platform index T, let $z_{*i} = \sum_{i=1}^{3N} \frac{z_{*i}}{3N}$ and we can change the regression or self-regression model with flat fuzzy variables into a determined one with platform index T:

$$w = \beta_0 + \beta_1 z_1 + \ldots + \beta_n z_n, \tag{3.6.4}$$

$$w_t = \beta_0 + \beta_1 z_{t-1} + \ldots + \beta_n z_{t-n}. \tag{3.6.5}$$

From here we can get a classical regression and self-regression model with platform index T corresponding to (3.6.1) and (3.6.2). By using the classical least square method, it is easier for us to find out an optimal solution to the unique $\beta_p (p = 0, 1, \ldots, n)$ in (3.6.4) or (3.6.5).

Accordingly, it is of value for us to approach Model (3.6.1) and (3.6.2) by using crisp models.

3.6.3 Conclusion

According to paper [Cao93e], the model in this section can be generalized into a model of nonlinear regression and time series. If we integrate the model and method here with Data Mining, we can search for an easier acquisition of fuzzy data in those characteristic problems, which are difficult to be described by numerical value. At the same time, we can design a series of systems such as fault diagnosis in computer, future forecasting, resent identification with $(\cdot, c)$ fuzzy data [YL99] and as well.