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Interval and Fuzzy Functional and their Variation

The writer put forward the concept of an interval and a fuzzy (value) functional variation on base of the classic function and functional variation in 1991 [Cao91a]. In 1992, he extended the research of convex function and convex functional into the consideration of the interval and the fuzzy environment. Later he processed the research for a conditional extremum variation problem in interval and fuzzy-valued functional [Cao01e] and the functional variation with fuzzy functions [Cao99a].

In this chapter, interval and fuzzy-valued functional variation are discussed as follows.

Section 1, Interval functional and its variation

Section 2, Fuzzy-valued functional and its variation

Section 3, Convex interval and fuzzy function and functional

Section 4, Convex fuzzy-valued function and functional

Section 5, Variation of interval and fuzzy-valued functional condition extremum

Section 6, Variation of condition extremum on functional with fuzzy functions

10.1 Interval Functional and Its Variation

In chapter 1, we can find the definition of interval numbers, its operation and interval functional. In this section, we discuss some properties of interval functional and its variation, introduce extremely valued condition for interval functional.

Definition 10.1.1. If each function $\bar{y}(x)$ is a certain interval function $\bar{y}(x) = [y^-(x), y^+(x)]$ and $\bar{\Pi}$ has some interval value corresponding to it, the interval variable is called a functional dependent on function $\bar{y}(x)$, written as $\bar{\Pi} = \Pi[y^-(x), y^+(x)]$.

Definition 10.1.2. Suppose that the interval functional $\Pi(\bar{y}(x))$ is defined on interval $[a, b]$, and if point $x \in [a, b]$, $\delta y^- = y^-(x) - y_1^-(x)$, and $\delta y^+ =$

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$y^+(x) - y_1^+(x)$ exist, where $y_1^-(x)$, $y_1^+(x)$ and $y^-(x)$, $y^+(x)$ are ordinary functions which belong to the domain of a functional, the functional $\Pi[\bar{y}(x)]$ is called the interval-model-variable variationableness, and $[\min(\delta y^-, \delta y^+), \max(\delta y^-, \delta y^+)]$ is called variation at x for $\bar{y}$, i.e.,

$$\delta\bar{y} \triangleq [\min(\delta y^-, \delta y^+), \max(\delta y^-, \delta y^+)],$$

at $\delta y^-(x_0) \leqslant \delta y^+(x_0)$ (or $\delta y^-(x_0) \geqslant \delta y^+(x_0)$), $\bar{y}$ is the same order (or antitone) variationableness at x_0. For $\forall x \in [a,b]$, $\delta\bar{y} = [\delta y^-, \delta y^+]$ (or $\delta\bar{y} = [\delta y^+, \delta y^-]$) is called the same order (or antitone) variation on $[a,b]$.

Definition 10.1.3. If for $\forall\varepsilon > 0, \exists\delta > 0$, and when $d_H(\bar{y}^{(k)}(x), \bar{y}_0^{(k)}(x)) \subset (-\delta, \delta)$, we have $d_H(\Pi\bar{y}(x), \Pi\bar{y}_0(x)) \subset (-\varepsilon, \varepsilon)$, calling the functional $\Pi\bar{y}(x)$ a k-th approaching continual interval functional on $\bar{y}_0(x)$, where, d_H denotes a Hausdorff metric.

Definition 10.1.4. For $\forall x \in [a,b]$,

$$y_1^-(x) \leqslant y_1^+(x), y^-(x) \leqslant y^+(x), y^{-'}(x) \leqslant y^{+'}(x), \cdots, y^{-(n)}(x) \leqslant y^{+(n)}(x)$$

if and only if functional $\Pi(y^-(x))$ and $\Pi(y^+(x))$ approach to continue by k-th on $y^-(x) = y_1^-(x)$ and $y^+(x) = y_1^+(x)$ [Ail52], respectively, $\Pi\bar{y}(x)$ is called an k-th approaching continual interval functional on $\bar{y} = \bar{y}_1$.

Definition 10.1.5. For the interval functional $\Pi(\bar{y}(x))$, $\frac{\partial}{\partial\vartheta}\Pi(\bar{y} + \vartheta\delta\bar{y})|_{\vartheta=0}$ is called the 1st variation for the interval functional and, by $\delta\bar{\Pi}$, we deduce

$$\begin{aligned}\delta\bar{\Pi} \triangleq [\min\{&\frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}, \frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}\}, \\ \max\{&\frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}, \frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}\}].\end{aligned} \tag{10.1.1}$$

At $\frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0} \leqslant \frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}$, the functional $\bar{\Pi}$ is the same order variationableness, and

$$\delta\bar{\Pi} = [\frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}, \frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}]$$

represents the same order variation of functional.

At $\frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0} \leqslant \frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}$, the functional $\bar{\Pi}$ is an antitone variationableness, and

$$\delta\bar{\Pi}' = [\frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}, \frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}]$$

represents the antitone variation of functional.

The next step is only to consider the same order variation since the antitone variation can be done in the same way. Therefore, (10.1.1) can be simplified into

$$\delta\bar{\Pi} \triangleq [\frac{\partial}{\partial\vartheta}\Pi(y^- + \vartheta\delta y^-)|_{\vartheta=0}, \frac{\partial}{\partial\vartheta}\Pi(y^+ + \vartheta\delta y^+)|_{\vartheta=0}].$$

Definition 10.1.6. For the interval functional $\bar{\Pi} = \Pi(\bar{y}(x))$, $\frac{\partial^n}{\partial\vartheta^n}\Pi(\bar{y} + \vartheta\delta\bar{y})|_{\vartheta=0}$ is called n-th variation for the interval functional and, by sign $\delta^n\bar{\Pi}$, writing $\delta^n\bar{\Pi} = \frac{\partial^n}{\partial\vartheta^n}\Pi(\bar{y} + \vartheta\delta\bar{y})|_{\vartheta=0}$. We call $\delta^n\bar{\Pi}$ $(n \geqslant 2)$ a higher variation.

Definition 10.1.7. We call $F(x, \bar{y}(x), \bar{y}'(x))$ an interval-compound function, writing

$$F(x, \bar{y}(x), \bar{y}'(x)) \triangleq [F^-(x, y^-(x), y^{-'}(x)), F^+(x, y^+(x), y^{+'}(x))].$$

Definition 10.1.8. For the interval-compound function $F(x, \bar{y}(x), \bar{y}'(x))$, if fixing variable x, we define

$$\delta\bar{F} = \frac{\partial F}{\partial\vartheta}(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta(\delta\bar{y})')|_{\vartheta=0}$$

as the 1st-interval variation of the interval-compound function $\bar{F}$,

$$\delta^n\bar{F} = \frac{\partial F^n}{\partial\vartheta^n}(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta(\delta\bar{y})')|_{\vartheta=0}$$

is called its n-th interval variation.

Similarly, we can define the variation of

$$F[x, \bar{y}(x), \bar{y}'(x), \cdots, \bar{y}^{(n)}(x)],\ \phi[x, \bar{y}(x), \bar{z}(x)].$$

Theorem 10.1.1. *For interval model variation, we have*

$$(1)\ \ (\delta\bar{y})' = \delta\bar{y}', (\delta\bar{y})^{(n)} = \delta\bar{y}^{(n)}; \qquad (2)\ \ \delta(\delta\bar{y}) = 0.$$

Proof: (1) Let the interval-compound function be

$$\bar{F} = F[x, \bar{y}(x), \bar{y}'(x)] = \bar{y}';\ F[x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta(\delta\bar{y})'] = \bar{y}' + \vartheta(\delta\bar{y})'.$$

Under the meaning of the same order variationableness, we have

$$\frac{\partial F}{\partial\vartheta}(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta(\delta\bar{y})') = (\delta\bar{y})'.$$

Therefore, we obtain

$$\delta\bar{F} = \delta\bar{y}' = \frac{\partial F}{\partial\vartheta}(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta(\delta\bar{y})')|_{\vartheta=0} = (\delta\bar{y})'.$$

Similarly, we can prove that the formula holds under the meaning of an antitone variationableness, hence

$$(\delta\bar{y})' = \delta\bar{y}'.$$

In the proper order, we can prove

$$(\delta\bar{y})^{(n)} = \delta\bar{y}^{(n)}.$$

(2) Let the interval-compound function be

$$F(x, \bar{y}(x)) = \delta\bar{y},\ F(x, \bar{y} + \vartheta\delta\bar{y}) = \delta\bar{y}.$$

Then

$$\frac{\partial F}{\partial\vartheta}(x, \bar{y} + \vartheta\delta\bar{y}) = 0 \Rightarrow \delta\bar{F} = \frac{\partial F}{\partial\vartheta}(x, \bar{y} + \vartheta\delta\bar{y})|_{\vartheta=0} = 0.$$

So, the theorem holds.

Theorem 10.1.2. *Let $\bar{F}, \bar{F}_1, \bar{F}_2$ be interval-compound functions with the variationable same order. Then*

(1) $\delta(\bar{F}_1 \pm \bar{F}_2) = \delta\bar{F}_1 \pm \delta\bar{F}_2,$
(2) $\delta(\bar{F}_1 \cdot \bar{F}_2) = \bar{F}_1\delta\bar{F}_2 + \bar{F}_2\delta\bar{F}_1,$
(3) $\delta(k \cdot \bar{F}) = k\delta\bar{F},$
(4) $\delta(\frac{\bar{F}_1}{\bar{F}_2}) = \frac{\bar{F}_2\delta\bar{F}_1 - \bar{F}_1\delta\bar{F}_2}{\bar{F}_2^2}(\bar{F}_2 \neq 0),$
(5) $\delta\bar{F}^n = n\bar{F}^{n-1}\delta\bar{F},$
(6) $\delta\int_a^b \bar{F}dx = \int_a^b \delta\bar{F}dx.$

Proof: We only prove (2) and (6):

(2) Let $F(x, \bar{y}(x), \bar{y}'(x)) = F_1(x, \bar{y}(x), \bar{y}'(x)) \cdot F_2(x, \bar{y}(x), \bar{y}'(x))$. Then

$$\begin{aligned}\frac{\partial F}{\partial\vartheta}&(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')\\ &= \frac{\partial}{\partial\vartheta}\{F_1(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')\}F_2(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')\\ &+ F_1(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')\frac{\partial}{\partial\vartheta}F_2(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}').\end{aligned} \tag{10.1.2}$$

From this, we obtain

$$\delta\bar{F} = \bar{F}_1\delta\bar{F}_2 + \bar{F}_2\delta\bar{F}_1.$$

(6) Let $\bar{F} = F(x, \bar{y}(x), \bar{y}'(x))$. Then

$$\begin{aligned}\delta\int_a^b F(x, \bar{y}(x), \bar{y}'(x))dx &= \frac{\partial}{\partial\vartheta}\int_a^b F(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')|_{\vartheta=0}dx\\ &= \int_a^b \frac{\partial}{\partial\vartheta}F(x, \bar{y} + \vartheta\delta\bar{y}, \bar{y}' + \vartheta\delta\bar{y}')|_{\vartheta=0}dx\\ &= \int_a^b \delta F(x, \bar{y}(x), \bar{y}'(x))dx.\end{aligned}$$

Lemma I. (Basic interval variation lemma) *The interval function* $\bar{\phi}(x)$ *remains continuous on* (a, b) *and as for arbitrary function* $\eta(x)$*, it satisfies with the following.*

(1) $\eta(x)$ *has* k*th continuous derivative on (a, b);*
(2) $\eta(a) = \eta(b)$*;*
(3) $|\eta(x)| < \epsilon, |\eta^{(1)}(x)| < \epsilon, \cdots, |\eta^{(k)}(x)| < \epsilon$*, where* ϵ *is a small arbitrary positive number,*

we have $\int_a^b \bar{\phi}(x)\eta(x)dx = 0$*, then* $\bar{\phi}(x) \equiv 0$ *on* $[a, b]$.

Proof: Because

$$\int_a^b \bar{\phi}(x)\eta(x)dx = [\int_a^b \phi^-(x)\eta(x)dx, \int_a^b \phi^+(x)\eta(x)dx] = 0,$$

by condition, we notice $\bar{\phi}(x) = [\phi^-(x), \phi^+(x)]$.

If by applying the variation lemma [Ail52] into $\phi^-(x)$, $\int_a^b \phi^-(x)\eta(x)dx$ and $\phi^+(x)$, $\int_a^b \phi^+(x)\eta(x)dx$, we have $\phi^-(x) = 0$ and $\phi^+(x) = 0$, respectively. Hence, the lemma holds.

The extreme value of the interval functional.

Definition 10.1.9. If the value of interval functional $\Pi(\bar{y}(x))$ in any curve approaching to $\bar{y} = \bar{y}_0(x)$ is smaller than $\Pi(\bar{y}_0(x))$, i.e., if $\Delta\bar{\Pi} = \Pi(\bar{y}(x)) - \Pi(\bar{y}_0(x)) \subset 0(\text{or} = 0)$, functional $\Pi(\bar{y}_0(x))$ reaches the maximum (or a strict one) on $\bar{y} = \bar{y}_0$.

The minimum (or a strict one) of $\Pi(\bar{y}(x))$ can be defined by imitation, and maximum (or minimum) of $\Pi(\bar{y}(x))$ is called an extreme value.

Theorem 10.1.3. *If the interval functional* $\Pi(\bar{y}(x))$ *with variation reaches maximum (or minimum) on* $\bar{y} = \bar{y}_0(x)$*, then, on* $\bar{y} = \bar{y}_0(x)$ *there exists* $\delta\bar{\Pi} = 0$.

Proof: Consider

$$\begin{aligned} &\Pi(\bar{y}_0(x) + \vartheta\delta\bar{y}) = \bar{\phi}(\vartheta) \\ \Longleftrightarrow\ &[\Pi(y_0^-(x) + \vartheta\delta y^-), \Pi(y_0^+(x) + \vartheta\delta y^+)] = [\phi^-(\vartheta), \phi^+(\vartheta)], \end{aligned}$$

when $y_0^-(x)$ and δy^-, $y_0^+(x)$ and δy^+ are fixed, respectively, $\bar{\phi}(\vartheta)$ is an interval function of ϑ. By assumption, $\bar{\phi}(0)$ is taken for extreme value $\Longleftrightarrow$ $\phi^-(0), \phi^+(0)$ is.

Therefore, $\quad \phi^{-'}(0) = 0, \phi^{+'}(0) = 0,$
i.e.,

$$\bar{\phi}'(0) = 0 \Longrightarrow \delta\Pi(\bar{y}_0(x)) = 0.$$

It is not difficult to extend the results above into the interval functional dependent upon multi-model-variable $\Pi(\bar{y}_1(x), \bar{y}_2(x), \cdots, \bar{y}_n(x))$ and upon a model interval functional of multi-variable or upon its variety of model interval functionals

$$\Pi(\bar{y}(x_1, x_2, \cdots, x_n));$$

or

$$\varPi(\bar{z}_1(x_1, x_2, \cdots, x_n), \bar{z}_2(x_1, x_2, \cdots, x_n), \cdots, \bar{z}_n(x_1, x_2, \cdots, x_n)).$$

Theorem 10.1.4. *If the interval functional $\varPi(\bar{y}(x))$ with 1st and 2nd interval variation $\delta\bar{\varPi}$, $\delta^2\bar{\varPi}$, and when $\bar{y} = \bar{y}_0(x)$, $\delta\varPi(\bar{y}_0(x)) = 0$, and $\delta^2\varPi(\bar{y}_0(x)) \neq 0$ hold, then an extreme value is taken for functional $\varPi(\bar{y}(x))$ on $\bar{y} = \bar{y}_0(x)$. When $\delta^2\varPi(\bar{y}_0(x)) \subset 0$, maximum exists and when $\delta^2\varPi(\bar{y}_0(x)) \supset 0$, minimum exists.*

Proof: Let the interval functional be $\bar{\phi}(\vartheta) = \varPi(\bar{y}_0(x)+\vartheta\delta\bar{y})$, if $\delta\varPi(\bar{y}_0(x)) = 0$, $\delta^2\varPi(\bar{y}_0(x)) \subset 0$, then

$$\begin{aligned}
\bar{\phi}'(0) = 0, \bar{\phi}''(0) \subset 0 &\Longrightarrow \begin{cases} {\phi^{-}}'(0) = 0, {\phi^{-}}''(0) < 0 \\ {\phi^{+}}'(0) = 0, {\phi^{+}}''(0) < 0 \end{cases} \\
&\Longrightarrow \begin{cases} \phi^{-}(0), & \text{maximal value is taken} \\ \phi^{+}(0), & \text{maximal value is taken} \end{cases} \\
&\Longleftrightarrow \bar{\phi}(0), \quad \text{maximal value is taken,}
\end{aligned}$$

i.e.,

$$\varPi(\bar{y}_0(x) + \vartheta\delta\bar{y}) \leqslant \varPi(\bar{y}_0(x)).$$

Therefore, maximal value is taken for $\varPi(\bar{y}_0(x))$.

Similarly, we can prove the states of the minimum.

10.2 Fuzzy-Valued Functional and Its Variation

10.2.1 Introduction

We aim at extending conception of functional variation under interval meaning into fuzzy state, having put forward the conception of fuzzy variation. In this section, we discuss some properties of fuzzy-valued functional and its variation, educe extremely valued condition for interval functional.

10.2.2 Variation of Fuzzy-Value Functional at Ordinary Point

In [DPr78] and [Luo84a,b], we can find some definitions of fuzzy number, its operation, and those of the fuzzy-value function.

Definition 10.2.1. Let

(1) $\tilde{y} : [a, b] \to \mathscr{R}, x \mapsto \tilde{y}(x), \tilde{y}(x)$ is a fuzzy-value function defined on $[a, b]$;

(2) $\bar{y}_\alpha : [a, b] \to E_{\mathscr{R}} = \{[e, f] | e \leqslant f; e, f \in \mathscr{R}\}$,

$$x \mapsto \bar{y}_\alpha(x) \triangleq (\tilde{y}(x))_\alpha = [y_\alpha^-(x), y_\alpha^+(x)].$$

Then $\bar{y}_\alpha$ is called an α-cuts function for $\tilde{y}$, which is an interval function defined on [a,b].

Definition 10.2.2. If for a kind of fuzzy-value function $\tilde{y}(x)$, each function $\tilde{y}(x)$ has some fuzzy numbers $\Pi(\tilde{y}(x))$ corresponding to it, then $\Pi(\tilde{y}(x))$ is called a fuzzy-valued functional of such function $\tilde{y}(x)$, and we write it down as $\tilde{\Pi} = \Pi(\tilde{y}(x))$.

Definition 10.2.3. Let the fuzzy-valued functional be defined on $[a,b]$. If for $\forall\alpha \in (0,1]$, there exists $\delta\bar{y}_\alpha = \bar{y}_\alpha(x) - \bar{y}_{1\alpha}(x)$, such that

$$\bigcup_{\alpha\in(0,1]} \alpha\delta\bar{y}_\alpha = \bigcup_{\alpha\in(0,1]} \alpha(\bar{y}_\alpha(x) - \bar{y}_{1\alpha}(x)),$$

then it is called a fuzzy-model-variable variation in functional $\Pi(\tilde{y}(x))$, written as

$$\delta\tilde{y} = \tilde{y}(x) - \tilde{y}_1(x).$$

Definition 10.2.4. Let $\tilde{y}(x)$ be defined on $[a,b]$, for $\forall\alpha \in (0,1]$, $\bar{y}_\alpha(x)$ is the same order (or antitone) variationableness, $\delta\tilde{y}(x) = \bigcup\limits_{\alpha\in(0,1]} \alpha\delta\bar{y}_\alpha$ is called the same order (or antitone) variation for $\tilde{y}(x)$.

Definition 10.2.5. Let

$$\bar{y}_\alpha : [a,b] \to E_{\mathscr{R}}, \quad x \mapsto \bar{y}_\alpha(x),$$
$$\bar{\Pi}_\alpha : E_{\mathscr{R}} \to [g,h], \quad \bar{y}_\alpha \mapsto \Pi(\bar{y}_\alpha(x)).$$

Then $\Pi(\bar{y}_\alpha)$ is called an α-cuts functional of $\Pi\tilde{y}$, if only if for $\forall\alpha \in (0,1]$, when $\Pi(\bar{y}_\alpha)$ approaches to continue by kth on $\bar{y}_\alpha = \bar{y}_{0\alpha}(x)$, a fuzzy-valued functional $\Pi\tilde{y}$ can be called a kth approaching continuation on $\tilde{y} = \tilde{y}_0(x)$.

Definition 10.2.6. For the fuzzy-valued functional $\Pi(\tilde{y}(x))$, we call $\dfrac{\partial}{\partial\vartheta}\Pi(\tilde{y} + \vartheta\delta\tilde{y})|_{\vartheta=0}$ the 1st variation of a fuzzy-valued functional, by sign $\delta\tilde{\Pi}$, then

$$\delta\tilde{\Pi} \triangleq \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial}{\partial\vartheta}\Pi(\bar{y}_\alpha + \vartheta\delta\bar{y}_\alpha)|_{\vartheta=0},$$

call $\dfrac{\partial^2}{\partial\vartheta^2}\Pi(\tilde{y} + \vartheta\delta\tilde{y})|_{\vartheta=0}$ the 2nd variation of a fuzzy-valued functional, and by sign $\delta^2\tilde{\Pi}$, then

$$\delta^2\tilde{\Pi} \triangleq \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial^2}{\partial\vartheta^2}\Pi(\bar{y}_\alpha + \vartheta\delta\bar{y}_\alpha)|_{\vartheta=0}.$$

Definition 10.2.7. For a fuzzy-valued functional of type $\tilde{\Pi} = \Pi(\tilde{y}(x), \tilde{z}(x))$, $\tilde{\Pi} = \Pi(\tilde{u}(x,y))$, whose 1st variation is

$$\delta\tilde{\Pi} = \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial}{\partial\vartheta}\Pi(\bar{y}_\alpha + \vartheta\delta\bar{y}_\alpha, \bar{z}_\alpha + \vartheta\delta\bar{z}_\alpha)|_{\vartheta=0},$$

$$\delta\tilde{\Pi} = \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial}{\partial\vartheta}\Pi(\bar{u}_\alpha + \vartheta\bar{u}_\alpha)|_{\vartheta=0}.$$

Similarly, we can define the 2nd variation.

Definition 10.2.8. We call function like $F(x, \tilde{y}(x), \tilde{y}'(x))$ a fuzzy-valued compound function with

$$F(x, \tilde{y}(x), \tilde{y}'(x)) \triangleq \bigcup_{\alpha\in(0,1]} \alpha F(x, \bar{y}_\alpha(x), \bar{y'}_\alpha(x)).$$

Definition 10.2.9. If fixing variable x for a fuzzy-valued compound function $F(x, \tilde{y}(x)), \tilde{y}'(x))$, we define

$$\delta\tilde{F} = \frac{\partial}{\partial\vartheta}F(x, \tilde{y} + \vartheta\delta\tilde{y}, \tilde{y}' + \vartheta(\delta\tilde{y})')|_{\vartheta=0}$$
$$\triangleq \bigcup_{\alpha\in(0,1]} \alpha\delta\bar{F}_\alpha = \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial}{\partial\vartheta}F(x, \bar{y}_\alpha + \vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha + \vartheta\delta\bar{y}'_\alpha)|_{\vartheta=0},$$

which is called the 1st fuzzy-valued variation.

And then

$$\delta^n\tilde{F} = \frac{\partial^n}{\partial\vartheta^n}F(x, \tilde{y} + \vartheta\delta\tilde{y}, \tilde{y}' + \vartheta(\delta\tilde{y})')|_{\vartheta=0}$$
$$\triangleq \bigcup_{\alpha\in(0,1]} \alpha\delta^n\bar{F}_\alpha = \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial^n}{\partial\vartheta^n}F(x, \bar{y}_\alpha + \vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha + \vartheta\delta\bar{y}'_\alpha)|_{\vartheta=0},$$

which is called an nth fuzzy-value variation.

In the same way, we can define the fuzzy-valued variation of

$$F(x, \tilde{y}(x), \tilde{y}'(x), \cdots, \tilde{y}^{(n)}(x)); \quad \phi(x, \tilde{y}(x), \tilde{z}(x)).$$

Theorem 10.2.1. *For fuzzy valued-model-variable variation, we have*

$$(1)\ \ (\delta\tilde{y})' = \delta\tilde{y}',\ \ (\delta\tilde{y})^{(n)} = \delta\tilde{y}^{(n)}; \tag{10.2.1}$$

$$(2)\ \ \delta(\delta\tilde{y}) = 0. \tag{10.2.2}$$

Proof:

a) Let the fuzzy-valued compound function be

$$\tilde{F} = F(x, \tilde{y}(x), \tilde{y}'(x)) = \tilde{y}'; \quad F(x, \tilde{y} + \vartheta\delta\tilde{y}, \tilde{y}' + \vartheta(\delta\tilde{y})') = \tilde{y}' + \vartheta(\delta\tilde{y})'.$$

Under the same order variationable meaning, we have

$$\frac{\partial}{\partial\vartheta}F(x,\tilde{y}+\vartheta\delta\tilde{y},\tilde{y}'+\vartheta(\delta\tilde{y})')$$
$$\triangleq \bigcup_{\alpha\in(0,1]}\alpha\frac{\partial}{\partial\vartheta}F(x,\bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha,\bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha)=\bigcup_{\alpha\in(0,1]}\alpha(\delta\bar{y}_\alpha)'\triangleq(\delta\tilde{y})'.$$

Therefore, we obtain

$$\delta\tilde{F}=\delta\tilde{y}'=\frac{\partial}{\partial\vartheta}F(x,\tilde{y}+\vartheta\delta\tilde{y},\tilde{y}'+\vartheta(\delta\tilde{y})')|_{\vartheta=0}=(\delta\tilde{y})'.$$

We can prove similarly that the formula holds under the variationable meaning in antitone, so $(\delta\tilde{y})'=\delta\tilde{y}'$.

By mathematical induction, we can prove $(\delta\tilde{y})^{(n)}=\delta\tilde{y}^{(n)}$ in proper order.

So, (10.2.1) holds.

b) Let the fuzzy-valued compound function be

$$F(x,\tilde{y}(x))=\delta\tilde{y},\quad F(x,\tilde{y}+\vartheta\delta\tilde{y})=\delta\tilde{y}.$$

Then

$$\frac{\partial}{\partial\vartheta}F(x,\tilde{y}+\vartheta\delta\tilde{y})\triangleq\bigcup_{\alpha\in(0,1]}\alpha\frac{\partial}{\partial\vartheta}F(x,\bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha)=0.$$

Therefore

$$\delta\tilde{F}=\frac{\partial}{\partial\vartheta}F(x,\tilde{y}+\vartheta\delta\tilde{y})|_{\vartheta=0}=0,$$

i.e., (10.2.2) holds.

Theorem 10.2.2. Let $\tilde{F},\tilde{F}_1,\tilde{F}_2$ be fuzzy-valued compound functions with the same order variationable. Then

(1) $\delta(\tilde{F}_1\pm\tilde{F}_2)=\delta\tilde{F}_1\pm\delta\tilde{F}_2$,
(2) $\delta(\tilde{F}_1\cdot\tilde{F}_2)=\tilde{F}_1\delta\tilde{F}_2+\tilde{F}_2\delta\tilde{F}_1$,
(3) $\delta(k\cdot\tilde{F})=k\delta\tilde{F}$,
(4) $\delta(\frac{\tilde{F}_1}{\tilde{F}_2})=\frac{\tilde{F}_2\delta\tilde{F}_1-\tilde{F}_1\delta\tilde{F}_2}{\tilde{F}_2^2},(\tilde{F}_2\neq 0)$,
(5) $\delta\tilde{F}^n=n\tilde{F}^{n-1}\delta\tilde{F}$,
(6) $\delta\int_a^b\tilde{F}dx=\int_a^b\delta\tilde{F}dx$.

Proof: Only (2) and (6) are proved, and the others can be proved similarly.

(2) Let $F(x,\tilde{y}(x),\tilde{y}'(x))=F_1(x,\tilde{y}(x),\tilde{y}'(x))\cdot F_2(x,\tilde{y}(x),\tilde{y}'(x))$. Then

$$\begin{aligned}
&\frac{\partial F}{\partial \vartheta}(x, \tilde{y}+\vartheta\delta\tilde{y}, \tilde{y}'+\vartheta\delta\tilde{y}') \\
&= \frac{\partial}{\partial \vartheta}\{F_1(x, \tilde{y}+\vartheta\delta\tilde{y}, \tilde{y}'+\vartheta\delta\tilde{y}')F_2(x, \tilde{y}+\vartheta\delta\tilde{y}, \tilde{y}'+\vartheta\delta\tilde{y}')\} \\
&\Longleftrightarrow \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial F}{\partial \vartheta}(x, \bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha) \\
&= \bigcup_{\alpha\in(0,1]} \alpha\frac{\partial F_1}{\partial \vartheta}\{(x, \bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha)\}F_2(x, \bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha) \\
&\quad + F_1(x, \bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha)\frac{\partial F_2}{\partial \vartheta}(x, \bar{y}_\alpha+\vartheta\delta\bar{y}_\alpha, \bar{y}'_\alpha+\vartheta\delta\bar{y}'_\alpha).
\end{aligned}$$

(6) The conclusion holds from proof of (6) in Theorem 10.1.2 and from expressive theorem of fuzzy numbers.

Lemma II. (Basic fuzzy variation lemma) *The fuzzy-valued function $\tilde{\phi}(x)$ continues on (a,b), and an arbitrary function $\eta(x)$ satisfies with a classical variation lemma condition [Ail52], i.e.,*

1^0 *$\eta(x)$ with a kth continuous derivative on (a, b),*

2^0 *$\eta(a) = \eta(b)$,*

3^0 *$|\eta(x)| < \epsilon$, $|\eta^{(1)}(x)| < \epsilon, \cdots, |\eta^{(k)}(x)| < \epsilon$, where ϵ is a small arbitrary positive number, we have $\int_a^b \tilde{\phi}(x)\eta(x)dx = 0$, then $\tilde{\phi}(x) = 0$ on [a,b].*

Proof: Since $\int_a^b \tilde{\phi}(x)\eta(x)dx \triangleq \bigcup\limits_{\alpha\in(0,1]} \alpha\int_a^b \bar{\phi}(x)\eta(x)dx$, for an arbitrary $\alpha \in (0,1]$, we have $\tilde{\phi}(x) = 0, x \in [a,b]$ from variation Lemma I and expressive theorem of fuzzy numbers.

The extreme value of the fuzzy-valued functional is below.

Definition 10.2.10. If the fuzzy-valued functional $\Pi(\tilde{y}(x))$ is smaller than $\Pi(\tilde{y}_0(x))$ on an arbitrary curve near to $\tilde{y} = \tilde{y}_0$, i.e., if $\Delta\tilde{\Pi} = \Pi(\tilde{y}(x)) - \Pi(\tilde{y}_0(x)) \subset 0(\text{or} = 0)$, functional $\Pi(\tilde{y}(x))$ is known to reach the maximum (or a strict maximum) on curve $\tilde{y} = \tilde{y}_0(x)$.

The minimum valued curve can be defined similarly as above.

Theorem 10.2.3. *If the fuzzy-valued functional $\Pi(\tilde{y}(x))$ with variation reaches maximum (or minimum) on $\tilde{y} = \tilde{y}_0(x)$, then, there is $\delta\tilde{\Pi} = 0$ on $\tilde{y} = \tilde{y}_0(x)$.*

Proof: As

$$\begin{aligned}
&\Pi(\tilde{y}_0(x)+\vartheta\delta\tilde{y}) = \tilde{\phi}(\vartheta) \\
\Leftrightarrow &\bigcup_{\alpha\in(0,1]} \alpha\Pi(\bar{y}_{0\alpha}(x)+\vartheta\delta\bar{y}_\alpha) = \bigcup_{\alpha\in(0,1]} \alpha\bar{\phi}_\alpha(\vartheta),
\end{aligned}$$

for an arbitrary $\alpha \in (0,1]$ holds, then the conclusion holds from Theorem 10.1.3.

It is not difficult to extend the results above into the fuzzy-valued functional of other types.

Theorem 10.2.4. *If the fuzzy-valued functional $\Pi(\tilde{y}(x))$ has the 1st and 2nd fuzzy-valued variation $\delta\tilde{\Pi}$ and $\delta^2\tilde{\Pi}$, and, at $\tilde{y} = \tilde{y}_0(x)$,*

$$\delta\Pi(\tilde{y}_0(x)) = 0, \quad \delta^2\Pi(\tilde{y}_0(x)) \neq 0,$$

holds, then the extreme value is taken for the fuzzy-valued functional $\Pi(\tilde{y}(x))$ on $\tilde{y} = \tilde{y}_0(x)$. At $\delta^2\Pi(\tilde{y}_0(x)) \subset 0$, maximum exists, and at $\delta^2\Pi(\tilde{y}_0(x)) \supset 0$, minimum exists.

Proof: Let the fuzzy-valued functional be $\tilde{\phi}(\vartheta) = \Pi(\tilde{y}_0(x) + \vartheta\delta\tilde{y})$. Whlie

$$\begin{aligned}\tilde{\phi}(\vartheta) &= \Pi(\tilde{y}_0(x) + \vartheta\delta\tilde{y})\\ &\triangleq \bigcup_{\alpha\in(0,1]} \alpha\bar{\phi}_\alpha(\vartheta) = \bigcup_{\alpha\in(0,1]} \alpha\Pi(\bar{y}_{0\alpha}(x) + \vartheta\delta\bar{y}_\alpha),\end{aligned}$$

for an arbitrary $\alpha \in (0,1]$, the conclusion holds from Theorem 10.1.4.

With L-R fuzzy functional variation discussed, we can obtain the same conclusion corresponding to the results above.

10.2.3 Variation of Ordinary or Fuzzy-Valued Functional at Fuzzy Points

Let Πy be a variationable ordinary functional on [a,b] and $\delta\Pi y$ be variation of Πy. Suppose that $\tilde{X}$ is a fuzzy point, i.e., it is a convex fuzzy set on $\mathscr{R}$, and a support of $\tilde{X}$ is

$$s(\tilde{X}) = \{x \in \mathscr{R} | \mu_{\tilde{X}}(x) > 0\} \subset [a,b].$$

Since $\delta\Pi y$ is also a function on [a,b], and by using one-place extension principle, we have the following.

Definition 10.2.11. Suppose that $\delta\Pi y(\tilde{X}) = \bigcup\limits_{\alpha\in(0,1]} \alpha\delta\Pi y(X_\alpha)$ is the 1st variation of ordinary functional at fuzzy point $\tilde{X}$, where $\delta\Pi y(X_\alpha) = \{z | \exists x \in X_\alpha; \delta\Pi y(x) = z\}$, and its membership function is

$$\mu_{\delta\Pi y(\tilde{X})}(z) = \bigvee_{\delta\Pi y(x)=z} \mu_{\tilde{X}}(x),$$

we call $\dfrac{\partial^2}{\partial\vartheta^2}\Pi(y(\tilde{X}) + \vartheta\delta y(\tilde{X}))|_{\vartheta=0}$ the 2nd variation of ordinary functional at fuzzy points, writing $\delta^2\Pi y(\tilde{X})$, i.e.,

$$\delta^2(\Pi y(\tilde{X})) \triangleq \bigcup_{\alpha\in(0,1]} \alpha \frac{\partial^2}{\partial\vartheta^2}\Pi(y(X_\alpha)+\vartheta\delta y(X_\alpha))|_{\vartheta=0}.$$

The variation property of an ordinary functional at ordinary points can be extended into the state of an ordinary functional variation at fuzzy points by Definition 10.2.11.

Definition 10.2.12. Let $\Pi\tilde{y}$ be one-place fuzzy-valued functional, which can be variationable on [a, b], where variation $\delta\Pi\tilde{y}$ is mapping from [a, b] to $\mathscr{F}(\mathscr{R})$. By the extended principle, let $\tilde{X}$ be a fuzzy point and $S(\tilde{X}) \subset [a,b]$ be a support. Then the variation of $\Pi\tilde{y}$ at fuzzy point $\tilde{X}$ can be defined by

$$\delta\Pi\tilde{y}(\tilde{X}) = \bigcup_{\alpha\in(0,1]} \alpha\delta\Pi\tilde{y}(X_\alpha) \in \mathscr{F}(\mathscr{F}(\mathscr{R})),$$

where $\delta\Pi(\tilde{y}(X_\alpha)) = \{\tilde{\gamma} \in \mathscr{F}(\mathscr{R}) | \exists x \in X_\alpha; \delta\Pi\tilde{y}(x) = \tilde{\gamma}\}$, its membership function represents

$$\mu_{\delta\Pi\tilde{y}(\tilde{X})}(\tilde{\gamma}) = \bigvee_{\delta\Pi\tilde{y}(x)=\tilde{\gamma}} \mu_{\tilde{X}}(x),$$

we call $\frac{\partial^2}{\partial\vartheta^2}\Pi(\tilde{y}(\tilde{X})+\vartheta\delta\tilde{y}(\tilde{X}))|_{\vartheta=0}$ the 2nd variation of the fuzzy-valued functional at fuzzy points, writing $\delta^2\tilde{\Pi}\tilde{y}(\tilde{X})$ as

$$\delta^2\tilde{\Pi}\tilde{y}(\tilde{X}) \triangleq \bigcup_{\alpha\in(0,1]} \alpha \frac{\partial^2}{\partial\vartheta^2}\Pi(\tilde{y}(X_\alpha)+\vartheta\delta\tilde{y}(X_\alpha))|_{\vartheta=0}.$$

The corresponding results of Section 10.1 and Section 10.2.2 can be used into the state of ordinary or fuzzy-valued functional on fuzzy-pointed variation, which is omitted here.

10.2.4 Conclusion

The author has put forward the basic conception and properties of variation for interval and fuzzy functional in this section, and discussed its further results which will be widely used in fuzzy physics, engineering theory and approximate calculation. The variational calculus on border and direct algorithms in variational problems under fuzzy environment will be discussed next.

10.3 Convex Interval and Fuzzy Function and Functional

10.3.1 Introduction

On the foundation of interval and fuzzy function, we introduce a concept on convex interval and convex fuzzy function with functional, give the definition of a convex function and convex functional about an interval and a ordinary function at fuzzy points, and judge their convexity condition.

10.3.2 Convex Interval Function with Functional

1. *Convex interval function*

See Ref.[Cen87] about the definition of interval function.

Definition 10.3.1. Let $\bar{J}(y) = [J^-(y), J^+(y)](J^-(y) \leqslant J^+(y))$ be an interval function defined on $[a, b] \subset D \subset \mathscr{R}$ (D is a convex region and $\mathscr{R}$ a real field). If $\forall \lambda \in [0, 1]$ and $y, z \in D$, there always exist

$$J^-(\lambda y + (1-\lambda)z) \leqslant \lambda J^-(y) + (1-\lambda)J^-(z)$$

and

$$J^+(\lambda y + (1-\lambda)z) \leqslant \lambda J^+(y) + (1-\lambda)J^+(z),$$

i.e.,

$$\bar{J}(\lambda y + (1-\lambda)z) \subseteq \lambda \bar{J}(y) + (1-\lambda)\bar{J}(z), \tag{10.3.1}$$

we call $\bar{J}(y)$ a convex interval function.

For interval function $\bar{J}(y)$, if $\bar{J}$ is convex, then $-\bar{J}(y) \triangleq [-J^+(y), -J^-(y)]$ is a concave function.

Definition 10.3.2. Suppose $\bar{J}(y)$ to be an interval function and if at $y_0 \in [a, b]$, there exists common nth derivatives $J^{-(n)}(y_0)$ and $J^{+(n)}(y_0)(n = 1, 2)$, meaning that $\bar{J}(y)$ has nth derivable at y_0, and

$$[\min\{J^{-(n)}(y_0), J^{+(n)}(y_0)\}, \max\{J^{-(n)}(y_0), J^{+(n)}(y_0)\}]$$

is nth interval derivative in $\bar{J}(y)$ at y_0.

When $J^{-(n)}(y_0) \leqslant J^{+(n)}(y_0)$, $[J^{-(n)}(y_0), J^{+(n)}(y_0)]$ is nth interval same order derivative in $\bar{J}(y)$ at y_0. Otherwise, $[J^{+(n)}(y_0), J^{-(n)}(y_0)]$ is nth interval antitone derivative in $\bar{J}(y)$ at y_0.

We assume the function to be all the same order derivable in the book.

In the binary situation ($n(\geqslant 3)$-variate circumstance is discussed similarly), we call

$$\frac{\partial^2 \bar{J}(y_i, y_k)}{\partial y_i \partial y_k} = \{\frac{\partial^2 J^-(y_i, y_k)}{\partial y_i \partial y_k}, \frac{\partial^2 J^+(y_i, y_k)}{\partial y_i \partial y_k}\}$$

the 2nd partial derivative in binary interval function $\bar{J}$.

It is not difficulty to get the definition of interval matrix and interval Taylor theorem [JM61] by using the definition of interval function.

Theorem 10.3.1. *If $\bar{J}(y)$ is the 2nd differentiable interval function, with an interval matrix being $(\frac{\partial^2 \bar{J}}{\partial y_i \partial y_k}) \supseteq 0$, then $\bar{J}$ is a convex interval function.*

Proof: According to the proof in Ref. [JM61], we suppose

$$\bar{f}(t) = \bar{J}(ty + (1-t)z),$$

since

$$\bar{f}^{''}(t) = \sum_{i,k}(y_i - z_i)(y_k - z_k)(\frac{\partial^2 \bar{J}}{\partial y_i \partial y_k})|_{ty+(1-t)z},$$

the right is non-negative, such that $\bar{f}^{''}(t) \geqslant 0$. As for functions $\bar{f}^{-''}(t)$ and $\bar{f}^{+''}(t)$, we apply Taylor theorem [JM61], respectively and get

$$\bar{f}(1) - \bar{f}(\lambda) = (1-\lambda)\bar{f}'(\lambda) + \frac{1}{2}(1-\lambda)^2 \bar{f}''(\lambda') \supseteq (1-\lambda)\bar{f}'(\lambda), \quad (10.3.2)$$

where λ' is a number between 1 and λ.

Similarly,

$$\bar{f}(0) - \bar{f}'(\lambda) \supseteq -\lambda \bar{f}'(\lambda). \quad (10.3.3)$$

$\lambda \times (10.3.2) + (1-\lambda) \times (10.3.3)$, then

$$\lambda \bar{f}(1) + (1-\lambda)\bar{f}(0) - \bar{f}'(\lambda) \supseteq 0,$$

this is (10.3.1), $\bar{J}$ being a convex function by Definition 10.3.1. The theorem is certificated.

Note 10.3.1. The interval function derivative is no more an interval number [WL85].

2. Convex interval functional

Definition 10.3.3. Let

$$\begin{aligned} &\bar{\Pi}(y, y') = \int_{\lambda_0}^{\lambda_1} \bar{F}(x, y, y')dx \triangleq \\ &[\Pi^-(y, y'), \Pi^+(y, y')] = [\int_{\lambda_0}^{\lambda_1} F^-(x, y, y')dx, \int_{\lambda_0}^{\lambda_1} F^+(x, y, y')dx]. \end{aligned} \quad (10.3.4)$$

Then we call (10.3.4) an interval functional, where $\bar{F}$ is an interval function.

Definition 10.3.4. Let $\bar{\Pi}$ be an interval functional defined in convex region D. If for $0 \leqslant \lambda \leqslant 1; y, y'; z, z' \in D$, we always have

$$\bar{\Pi}[\lambda y + (1-\lambda)z, \lambda y' + (1-\lambda)z'] \subseteq \lambda \bar{\Pi}(y, y') + (1-\lambda)\bar{\Pi}(z, z'), \quad (10.3.5)$$

calling the interval functional $\bar{\Pi}$ a convex in D.

If $\bar{\Pi}(y, y')$ is a convex interval functional, then $-\bar{\Pi}(y, y') \triangleq [-\Pi^+(y, y'), -\Pi^-(y, y')]$ is a concave one.

Theorem 10.3.2. *Let $\bar{F}_{y'y'} \supseteq 0$ and $\bar{F}_{yy}\bar{F}_{y'y'} - (\bar{F}_{yy'})^2 \supseteq 0$. Then $\bar{F}(x, y, y')$ is a convex interval function concerning two variable numbers $y(x), y'(x)$. If*

$y(x), y'(x)$ are regarded as two independent functions, then $\bar{\Pi}(y, y')$ is called a convex interval functional in Definition 10.3.3.

Proof: It is similar with Formal (10.3.1), for $0 \leqslant \lambda \leqslant 1; y, y'; z, z' \in D$, (10.3.5) always holds. Similarly to the proof in Theorem 10.3.1, we only prove

$$\frac{\partial^2}{\partial t^2}\bar{\Pi}(ty + (1-t)z, ty' + (1-t)z') \supseteq 0. \tag{10.3.6}$$

But, from Formal (10.3.4) in Definition 10.3.3, we can see that the left of Formal (10.3.6) is

$$\int [(\bar{F}_{yy})(y-z)^2 + 2(\bar{F}_{yy'})(y-z)(y'-z') + (\bar{F}_{y'y'})(y'-z')^2]dx, \tag{10.3.7}$$

where $(\bar{F}_{yy})$, etc., represents $\bar{F}_{yy}(x, ty + (1-t)z, ty' + (1-t)z')$ etc., and by an assumption, we know

$$(F^-_{yy})(y-z)^2 + 2(F^-_{yy'})(y-z)(y'-z') + (F^-_{y'y'})(y'-z')^2 \geqslant 0,$$
$$(F^+_{yy})(y-z)^2 + 2(F^+_{yy'})(y-z)(y'-z') + (F^+_{y'y'})(y'-z')^2 \geqslant 0.$$

Therefore

$$(\bar{F}_{yy})(y-z)^2 + 2(\bar{F}_{yy'})(y-z)(y'-z') + (\bar{F}_{y'y'})(y'-z')^2 \supseteq 0,$$

i.e., (10.3.7) $\supseteq 0$, such that (10.3.6) holds.

10.3.3 Convex Function with Functional at Fuzzy Points

1. *Convex function at fuzzy points*

Suppose J to be an ordinary differentiable function defined on $[a, b]$, and $\tilde{x}$ to be a fuzzy point (i.e., a convex fuzzy set on $\mathscr{R}$), and its support is

$$S(\tilde{x}) = \{x \in \mathscr{R} | \mu_{\tilde{x}}(x) > 0\} \subseteq [a, b].$$

Suppose again $y(\tilde{x})$ means also a fuzzy point, and its support is

$$S(y(\tilde{x})) = \{y(x) \in \mathscr{R} | \mu_{y(\tilde{x})}(y(x)) > 0\} \subseteq [c, d],$$

then we have the following by an extension principle.

Suppose J to be a one-place function defined on $[a, b]$, if $S(y(\tilde{x})) \subset [c, d]$, then we define

$$J(y(\tilde{x})) \triangleq \bigcup_{\alpha \in (0,1]} \alpha J(y(\bar{x}_\alpha)).$$

Definition 10.3.5. Let $J(y(\tilde{x}))$ be an ordinary function defined on $[a, b]$. Then we call $J(y(\tilde{x}))$ a convex function at fuzzy point $\tilde{x}$ if for $\forall \lambda, \alpha \in [0, 1]$ and $y(\tilde{x}), z(\tilde{x}) \in \mathscr{R}$, we have

$$\begin{aligned} &J(\lambda y(\tilde{x})+(1-\lambda)z(\tilde{x}))\subseteq \lambda J(y(\tilde{x}))+(1-\lambda)J(z(\tilde{x}))\\ &\triangleq \bigcup_{\alpha\in(0,1]}\alpha\{J(\lambda y(\bar{x}_\alpha)+(1-\lambda)z(\bar{x}_\alpha))\}\\ &\subseteq \bigcup_{\alpha\in(0,1]}\alpha\{\lambda J(y(\bar{x}_\alpha)+(1-\lambda)J(z(\bar{x}_\alpha))\}. \end{aligned} \tag{10.3.8}$$

Definition 10.3.6. Let $J(y(\tilde{x}))$ be an ordinary function defined on $[a,b]$. If the derivative $J^{(n)}(y(\bar{x}_{0\alpha}))(n=1,2)$ there exists $\forall\alpha\in(0,1]$ at point $y(\bar{x}_{0\alpha})\in\mathscr{R}$, then we call n-th derivative of $J(y(\tilde{x}))$ existence at fuzzy point $y(\tilde{x}_0)$, written as

$$J^{(n)}(y(\tilde{x}_0))=\bigcup_{\alpha\in(0,1]}\alpha J^{(n)}(y(\bar{x}_{0\alpha})),$$

where

$$J^{(n)}(y(\bar{x}_{0\alpha}))=\{\gamma|\exists y(x_0)\in y(\bar{x}_{0\alpha}),J^{(n)}(y(x_0))=\gamma\},$$

its membership function is

$$\mu_{J^{(n)}(y(\tilde{x}_0))}(\gamma)=\bigvee_{J^{(n)}(y(x_0))=\gamma}\mu_{y(\tilde{x}_0)}(y(x_0)).$$

In the binary situation ($n(\geqslant 3)$-variate circumstance is discussed similarly), we call

$$\frac{\partial^2 J(y_i(\tilde{x}),y_k(\tilde{x}))}{\partial y_i\partial y_k}=\bigcup_{\alpha\in(0,1]}\alpha\frac{\partial^2 J(y_i(\bar{x}_\alpha),y_i(\bar{x}_\alpha))}{\partial y_i\partial y_k}$$

the 2nd partial derivative in a binary ordinary function at fuzzy points, and its membership function is

$$\mu_{\frac{\partial^2 J(y_i(\tilde{x}),y_k(\tilde{x}))}{\partial y_i\partial y_k}}(\gamma)=\bigvee_{\frac{\partial^2 J(y_i(x),y_k(x))}{\partial y_i\partial y_k}=\gamma}\{\mu_{y_i(\tilde{x})}(y_i)\bigwedge\mu_{y_k(\tilde{x})}(y_k)\}.$$

Theorem 10.3.3. *Let $y(\tilde{x})$ be a fuzzy point. If J is the 2-nd differentiable ordinary function, with a matrix being $(\frac{\partial^2 J}{\partial y_i\partial y_k})\supseteq 0$, then $J(y(\tilde{x}))$ is a convex function at fuzzy points.*

Proof: According to the assumption and definition of fuzzy numbers, let

$$f(t)=J(ty(\tilde{x})+(1-t)z(\tilde{x}))$$

be only the function concerning t. Then

$$f''(t)=\sum_{i,k}(y_i(\tilde{x})-z_i(\tilde{x}))(y_k(\tilde{x})-z_k(\tilde{x}))(\frac{\partial^2 J}{\partial y_i\partial y_k})|_{ty(\tilde{x})+(1-t)z(\tilde{x})},$$

and the right end is not negative because the right end of

$$\sum_{i,k}(y_i(\tilde{x})-z_i(\tilde{x}))(y_k(\tilde{x})-z_k(\tilde{x}))(\frac{\partial^2 J}{\partial y_i \partial y_k})|_{ty(\tilde{x})+(1-t)z(\tilde{x})} =$$

$$\bigcup_{\alpha\in(0,1]} \alpha\{\sum_{i,k}(y_i(\bar{x}_\alpha)-z_i(\bar{x}_\alpha))(y_k(\bar{x}_\alpha)-z_k(\bar{x}_\alpha))(\frac{\partial^2 J}{\partial y_i \partial y_k})|_{ty(\tilde{x})+(1-t)z(\tilde{x})}\},$$

obviously it is not negative, hence $f^{''}(t) \geqslant 0$. From the extension principle and by applying Taylor theorem, we get

$$f(1)-f(\lambda) = (1-\lambda)f'(\lambda) + \frac{1}{2}(1-\lambda)^2 f''(\lambda') \supseteq (1-\lambda)f'(\lambda), \qquad (10.3.9)$$

where λ' is a number between 1 and λ.

Similarly,

$$f(0)-f'(\lambda) \supseteq -\lambda f'(\lambda). \qquad (10.3.10)$$

$\lambda \times (10.3.9) + (1-\lambda) \times (10.3.10)$, then

$$\lambda f(1) + (1-\lambda)f(0) - f'(\lambda) \supseteq 0,$$

i.e., (10.3.8). Hence $J(y(\tilde{x}))$ is a convex function at fuzzy points in Definition 10.3.5 and the theorem holds.

2. *Convex functional at fuzzy points*

Definition 10.3.7. Suppose Π to be an ordinary functional and $\tilde{x}$ to be a fuzzy point at $\mathscr{R}$, then we call

$$\begin{aligned} &\Pi(y(\tilde{x}), y'(\tilde{x})) = \int_{\lambda_0}^{\lambda_1} F(\tilde{x}, y(\tilde{x}), y'(\tilde{x}))dx \triangleq \\ &\bigcup_{\alpha\in(0,1]} \alpha\Pi(y(\bar{x}_\alpha), y'(\bar{x}_\alpha)) = \bigcup_{\alpha\in(0,1]} \alpha \int_{\lambda_0}^{\lambda_1} F(\bar{x}_\alpha, y(\bar{x}_\alpha), y'(\bar{x}_\alpha))dx \end{aligned} \qquad (10.3.11)$$

a functional at fuzzy points, where F is an ordinary function.

Definition 10.3.8. Let Π be an ordinary functional defined in convex region D. If in fuzzy points $y(\tilde{x}), z(\tilde{x}) \in \mathscr{R}$ for arbitrarily $\lambda \in [0,1]$, there is

$$\begin{aligned} &\Pi(\lambda y(\tilde{x}) + (1-\lambda)z(\tilde{x}), \lambda y'(\tilde{x}) + (1-\lambda)z'(\tilde{x})) \\ &\quad \subseteq \lambda\Pi(y(\tilde{x}), y'(\tilde{x})) + (1-\lambda)\Pi(z(\tilde{x}), z'(\tilde{x})) \triangleq \\ &\bigcup_{\alpha\in(0,1]} \alpha\{\Pi(\lambda y(\bar{x}_\alpha) + (1-\lambda)z(\bar{x}_\alpha), \lambda y'(\bar{x}_\alpha) + (1-\lambda)z'(\bar{x}_\alpha))\} \\ &\quad \subseteq \bigcup_{\alpha\in(0,1]} \alpha\{\lambda\Pi(y(\bar{x}_\alpha), y'(\bar{x}_\alpha)) + (1-\lambda)\Pi(z(\bar{x}_\alpha), z'(\bar{x}_\alpha))\}, \end{aligned} \qquad (10.3.12)$$

then Π is called a convex functional at fuzzy points in D.

Theorem 10.3.4. *Let $F_{y'y'} \supset 0$ and $F_{yy}F_{y'y'} - (F_{yy'})^2 \supseteq 0$. Then $F(\tilde{x}, y(\tilde{x}), y'(\tilde{x}))$ is a convex function concerning two fuzzy variable numbers $y(\tilde{x})$ and $y'(\tilde{x})$. If $y(\tilde{x})$ and $y'(\tilde{x})$ are regarded as two independent fuzzy functions, then we call $\Pi(y(\tilde{x}), y'(\tilde{x}))$ a convex functional at fuzzy points defined by (10.3.11).*

Proof: It is similar with Formal (10.3.8), for $0 \leqslant \lambda \leqslant 1$, we always have (10.3.12) hold. Similarly to the proof in Theorem 10.3.3, we only prove

$$\frac{\partial^2}{\partial t^2}\Pi(ty(\tilde{x}) + (1-t)z(\tilde{x}), ty'(\tilde{x}) + (1-t)z'(\tilde{x})) \supseteq 0. \tag{10.3.13}$$

But, from Formal (10.3.11) in Definition 10.3.7, we can see that the left end of Formal (10.3.13) is

$$\begin{aligned}\int [(F_{yy})(y(\tilde{x}) - z(\tilde{x}))^2 &+ 2(F_{yy'})(y(\tilde{x}) - z(\tilde{x}))(y'(\tilde{x}) - z'(\tilde{x})) \\ &+ (F_{y'y'})(y'(\tilde{x}) - z'(\tilde{x}))^2]dx,\end{aligned} \tag{10.3.14}$$

where (F_{yy}), etc., represents $F_{yy}(\tilde{x}, ty(\tilde{x}) + (1-t)z(\tilde{x}), ty'(\tilde{x}) + (1-t)z'(\tilde{x}))$, etc. And by an assumption, we know

$$\begin{aligned}&(F_{yy})(y(x^-) - z(x^-))^2 + 2(F_{yy'})(y(x^-) - z(x^-))(y'(x^-) - z'(x^-)) \\ &\qquad + (F_{y'y'})(y'(x^-) - z'(x^-))^2 \geqslant 0, \\ &(F_{yy})(y(x^+) - z(x^+))^2 + 2(F_{yy'})(y(x^+) - z(x^+))(y'(x^+) - z'(x^+)) \\ &\qquad + (F_{y'y'})(y'(x^+) - z'(x^+))^2 \geqslant 0.\end{aligned}$$

Therefore, there is

$$\begin{aligned}&\bigcup_{\alpha\in(0,1]} \alpha(F_{yy})(y(\bar{x}_\alpha) - z(\bar{x}_\alpha))^2 + 2(F_{yy'})(y(\bar{x}_\alpha) - z(\bar{x}_\alpha))(y'(\bar{x}_\alpha) - z'(\bar{x}_\alpha)) \\ &\qquad + (F_{y'y'})(y'(\bar{x}_\alpha) - z'(\bar{x}_\alpha))^2 \supseteq 0, \\ \Rightarrow\ &(F_{yy})(y(\tilde{x}) - z(\tilde{x}))^2 + 2(F_{yy'})(y(\tilde{x}) - z(\tilde{x}))(y'(\tilde{x}) - z'(\tilde{x})) \\ &\qquad + (F_{y'y'})(y'(\tilde{x}) - z'(\tilde{x}))^2 \supseteq 0,\end{aligned}$$

i.e., (10.3.14) $\supseteq 0$, such that (10.3.13) holds.

10.3.4 Conclusion

In this section, we expand the concept of a classic convex, establish the theory frame of the convex interval and fuzzy functions with convex functionals. In the next section, we will advance cove fuzzy-value function and functional. Under this frame, we can develop a lot of researches to optimizing problems concerning static, more static and dynamic cases under interval and fuzzy environment. The work concerning this aspect will be researched continuously.

10.4 Convex Fuzzy-Valued Function and Functional

In this section, on the foundation of fuzzy-valued function and functional variation, we put forward the next [Cao09].

(1) Developing a concept on convex fuzzy-valued function with functional.

(2) Discussing the convexity in fuzzy-valued function and functional at ordinary and fuzzy points, respectively.

10.4.1 Convex Fuzzy-Valued Function and Functional at Ordinary Points

1. *Convex fuzzy-Valued Function at Ordinary Points*

Fuzzy-valued function with functional can be defined similarly as the above section.

Definition 10.4.1. Suppose $\tilde{J}(y)$ to be a fuzzy-valued function defined at $[a,b]$, and

$$\tilde{J}(y) \triangleq \bigcup_{\alpha\in(0,1]} \alpha \bar{J}_\alpha(y) = \bigcup_{\alpha\in(0,1]} \alpha[J_\alpha^-(y), J_\alpha^+(y)],$$

if for $\forall\lambda \in [0,1]$ and $y, z \in \mathscr{R}$, we have

$$\tilde{J}(\lambda y + (1-\lambda)z) \subseteq \lambda\tilde{J}(y) + (1-\lambda)\tilde{J}(z), \tag{10.4.1}$$

then we call $\tilde{J}(y)$ the convex fuzzy-valued function.

Here

$$(10.4.1) \triangleq \bigcup_{\alpha\in(0,1]} \alpha\{\bar{J}_\alpha(\lambda y + (1-\lambda)z)\} \leqslant \bigcup_{\alpha\in(0,1]} \alpha\{\lambda\bar{J}_\alpha(y) + (1-\lambda)\bar{J}_\alpha(z)\}$$

$$\Longleftrightarrow \bigcup_{\alpha\in(0,1]} \alpha\{J_\alpha^-(\lambda y + (1-\lambda)z)\} \leqslant \bigcup_{\alpha\in(0,1]} \alpha\{\lambda J_\alpha^-(y) + (1-\lambda)J_\alpha^-(z)\}$$
$$\bigcup_{\alpha\in(0,1]} \alpha\{J_\alpha^+(\lambda y + (1-\lambda)z)\} \leqslant \bigcup_{\alpha\in(0,1]} \alpha\{\lambda J_\alpha^+(y) + (1-\lambda)J_\alpha^+(z)\}.$$

If $\tilde{J}(y)$ is a convex fuzzy-valued function, then $-\tilde{J}(y) = \bigcup_{\alpha\in(0,1]} \alpha[-J_\alpha^+(y),$ $-J_\alpha^-(y)]$ is a concave one.

Definition 10.4.2. Let $\tilde{J}(y)$ be a fuzzy-valued function defined at interval $[a,b]$. If at some point $y_0 \in (a,b]$, there exists nth interval derivative $\bar{J}_\alpha^{(n)}(y_0)(n=1,2)$ for $\forall\alpha \in (0,1]$, then we call that nth fuzzy-valued derivative exists in $\tilde{J}(y)$ at y_0, written down as

$$\tilde{J}^{(n)}(y_0) = \bigcup_{\alpha\in(0,1]} \alpha \bar{J}_\alpha^{(n)}(y_0) = \bigcup_{\alpha\in(0,1]} \alpha[J_\alpha^{-(n)}(y_0), J_\alpha^{+(n)}(y_0)],$$

its membership function being

$$\mu_{\tilde{J}^{(n)}(y_0)}(\gamma) = \bigvee\{\alpha | J_\alpha^{-(n)}(y_0) = \gamma, \text{ or } J_\alpha^{+(n)}(y_0) = \gamma\}.$$

As for the binary situation ($n(\geqslant 3)$-variate circumstance is discussed similarly), we call

$$\frac{\partial^2 \tilde{J}(y_i, y_k)}{\partial y_i \partial y_k} = \bigcup_{\alpha\in(0,1]} \alpha \frac{\partial}{\partial y_k}(\frac{\partial \bar{J}_\alpha(y_i, y_k)}{\partial y_i})$$
$$= [\bigcup_{\alpha\in(0,1]} \alpha\{\frac{\partial^2 J_\alpha^-(y_i, y_k)}{\partial y_i \partial y_k}\}, \bigcup_{\alpha\in(0,1]} \alpha\{\frac{\partial^2 J_\alpha^+(y_i, y_k)}{\partial y_i \partial y_k}\}]$$

2nd partial derivative in binary fuzzy-valued function $\tilde{J}$, its membership function being

$$\mu_{\frac{\partial^2 \tilde{J}(y_i, y_k)}{\partial y_i \partial y_k}}(\gamma) = \bigvee\{\alpha | \frac{\partial^2 J_\alpha^-(y_i, y_k)}{\partial y_i \partial y_k} = \gamma, \text{ or } \frac{\partial^2 J_\alpha^+(y_i, y_k)}{\partial y_i \partial y_k} = \gamma\}.$$

Theorem 10.4.1. *If $\tilde{J}(y)$ is the 2nd differentiable fuzzy-valued function, with a fuzzy-valued matrix being $(\frac{\partial^2 \tilde{J}}{\partial y_i \partial y_k}) \supseteq 0$, then $\tilde{J}$ is a convex fuzzy-valued function.*

Proof: According to the assumption and definition of a fuzzy-valued function, let

$$\tilde{f}(t) = \tilde{J}(ty + (1-t)z).$$

Because the right of $\tilde{f}''(t) = \sum\limits_{i,k}(y_i - z_i)(y_k - z_k)(\frac{\partial^2 \tilde{J}}{\partial y_i \partial y_k})|_{ty+(1-t)z}$ is not negative, such that $\tilde{f}''(t) \geqslant 0$, from an extension principle and by applying Taylor theorem, we get

$$\tilde{f}(1) - \tilde{f}(\lambda) = (1-\lambda)\bar{f}'(\lambda) + \frac{1}{2}(1-\lambda)^2 \tilde{f}''(\lambda') \supseteq (1-\lambda)\tilde{f}'(\lambda), \quad (10.4.2)$$

where λ' is a number between 1 and λ. Similarly,

$$\tilde{f}(0) - \tilde{f}'(\lambda) \supseteq -\lambda \bar{f}'(\lambda). \quad (10.4.3)$$

$\lambda \times (10.4.2) + (1-\lambda) \times (10.4.3)$, then

$$\lambda \tilde{f}(1) + (1-\lambda)\tilde{f}(0) - \tilde{f}'(\lambda) \supseteq 0,$$

which is (10.4.1). Hence $\tilde{J}$ is a convex fuzzy-valued function by Definition 10.4.1 and the theorem is certificated.

Note 10.1. The derivative of fuzzy-valued function is not necessarily a fuzzy number [WL85].

2. *Convex fuzzy-valued functional at ordinary points*

Definition 10.4.3. We call the formal

$$\begin{aligned}\tilde{\Pi}(y,y') &= \int_{\lambda_0}^{\lambda_1} \tilde{F}(x,y,y')dx \\ &\triangleq \bigcup_{\alpha\in(0,1]} \alpha\bar{\Pi}_\alpha(y,y') = \bigcup_{\alpha\in(0,1]} \alpha[\Pi_\alpha^-(y,y'),\Pi_\alpha^+(y,y')] \\ &= \bigcup_{\alpha\in(0,1]} \alpha\int_{\lambda_0}^{\lambda_1} \bar{F}_\alpha(x,y,y')dx\end{aligned} \tag{10.4.4}$$

a fuzzy-valued functional, where $\tilde{F}$ is a fuzzy-valued function.

Definition 10.4.4. Let $\tilde{\Pi}(y,y')$ be a fuzzy-valued functional defined in convex region D. If $\forall\lambda\in[0,1]; y,y';z,z'\in D$, we always have

$$\begin{aligned}&\tilde{\Pi}(\lambda y+(1-\lambda)z,\lambda y'+(1-\lambda)z') \subseteq \lambda\tilde{\Pi}(y,y')+(1-\lambda)\tilde{\Pi}(z,z') \\ &\triangleq \bigcup_{\alpha\in(0,1]} \alpha\bar{\Pi}_\alpha(\lambda y+(1-\lambda)z,\lambda y'+(1-\lambda)z') \\ &\subseteq \bigcup_{\alpha\in(0,1]} \alpha\{\lambda\bar{\Pi}_\alpha(y,y')+(1-\lambda)\bar{\Pi}_\alpha(z,z')\},\end{aligned} \tag{10.4.5}$$

calling the fuzzy-valued functional $\tilde{\Pi}(y,y')$ convex in D.

If $\tilde{\Pi}(y,y')$ is a convex fuzzy-valued functional, then $-\tilde{\Pi}(y,y') = \bigcup_{\alpha\in(0,1]} \alpha[-\Pi_\alpha^+(y,y'),-\Pi_\alpha^-(y,y')]$ is a concave one.

Theorem 10.4.2. *Let $\tilde{F}_{y'y'}\supset 0$ and $\tilde{F}_{yy}\tilde{F}_{y'y'}-(\tilde{F}_{yy'})^2\supseteq 0$. Then $\tilde{F}(x,y,y')$ is a convex fuzzy-valued function concerning two variable numbers $y(x)$ and $y'(x)$. If $y(x)$ and $y'(x)$ are regarded as two independent functions, then we call $\tilde{\Pi}(y,y')$ a convex fuzzy-valued functional by Definition 10.4.3.*

Proof: It is similar with Formal (10.4.1), for $0\leqslant\lambda\leqslant 1; y,y';z,z'\in D$, we always have (10.4.5) hold. Similarly to the proof in Theorem 10.4.1, we only prove

$$\frac{\partial^2}{\partial t^2}\tilde{\Pi}(ty+(1-t)z,ty'+(1-t)z')\supseteq 0. \tag{10.4.6}$$

But, from Formal (10.4.4) in Definition 10.4.3, we can see that the left of the Formal (10.4.6) is

$$\int[(\tilde{F}_{yy})(y-z)^2+2(\tilde{F}_{yy'})(y-z)(y'-z')+(\tilde{F}_{y'y'})(y'-z')^2]dx, \tag{10.4.7}$$

where $(\tilde{F}_{yy})$, etc., represents $\tilde{F}_{yy}(x, ty+(1-t)z, ty'+(1-t)z')$, etc., and by an assumption, we know

$$(\bar{F}_\alpha)_{yy}(y-z)^2 + 2(\bar{F}_\alpha)_{yy'}(y-z)(y'-z') + (\bar{F}_\alpha)_{y'y'}(y'-z')^2 \supseteq 0,$$

therefore,

$$\bigcup_{\alpha\in(0,1]} \alpha\{(\bar{F}_\alpha)_{yy}(y-z)^2 + 2(\bar{F}_\alpha)_{yy'}(y-z)(y'-z') + (\bar{F}_\alpha)_{y'y'}(y'-z')^2\} \supseteq 0,$$

$$\Rightarrow \ (\tilde{F}_{yy})(y-z)^2 + 2(\tilde{F}_{yy'})(y-z)(y'-z') + (\tilde{F}_{y'y'})(y'-z')^2 \supseteq 0,$$

i.e., (10.4.7) $\supseteq 0$, such that (10.4.6) holds.

10.4.2 Convex Fuzzy-Valued Function and Functional at Fuzzy Points

1. *Convex fuzzy-valued function at fuzzy points*

Suppose that $\tilde{J}$ is a one-place fuzzy-valued function defined at $[a, b]$. By extension principle, if $y(\tilde{x})$ is a fuzzy point and its support is $S(y(\tilde{x})) \subset [c, d]$, then

$$\tilde{J}(y(\tilde{x})) \triangleq \bigcup_{\alpha\in(0,1]} \alpha\tilde{J}(y(\bar{x}_\alpha)) \in \mathscr{F}(\mathscr{F}(\mathscr{R}))$$

is a fuzzy-valued function defined at the fuzzy points, where $\tilde{J}(y(\tilde{x})) = \{\tilde{\gamma} \in \mathscr{F}(\mathscr{R}) | \exists y(x) \in y(\bar{x}_\alpha), \tilde{J}(y(x)) = \tilde{\gamma}\}$, its membership function being

$$\mu_{\tilde{J}(y(\tilde{x}))}(\tilde{\gamma}) = \bigvee_{\tilde{J}(y(x))=\tilde{\gamma}} \mu_{y(\tilde{x})}(y(x)).$$

Definition 10.4.5. If $\forall\lambda \in [0, 1]$ and fuzzy points $y(\tilde{x}), z(\tilde{x}) \in \mathscr{R}$, there is

$$\begin{aligned}
&\tilde{J}[\lambda y(\tilde{x}) + (1-\lambda)\tilde{J}(z(\tilde{x}))] \subseteq \lambda\tilde{J}(y(\tilde{x})) + (1-\lambda)\tilde{J}(z(\tilde{x}))\\
&\triangleq \bigcup_{\alpha\in(0,1]} \alpha\{\tilde{J}[\lambda y(\bar{x}_\alpha) + (1-\lambda)z(\bar{x}_\alpha)]\}\\
&\quad \subseteq \bigcup_{\alpha\in(0,1]} \alpha\{\lambda\tilde{J}(y(\bar{x}_\alpha)) + (1-\lambda)\tilde{J}(z(\bar{x}_\alpha))\},
\end{aligned}$$

then we call $\tilde{J}$ a convex fuzzy-valued function at fuzzy points.

Definition 10.4.6. Suppose $\tilde{J}(y(x))$ to be the fuzzy-valued function defined at interval $[a, b]$, if $\forall\alpha \in (0, 1], \tilde{J}^{(n)}(y(\bar{x}_{0\alpha}))(n = 1, 2)$ exist at certain point $y(\bar{x}_{0\alpha}) \in \mathscr{R}$, then we call that nth derivative of $\tilde{J}(y(x))$ exists at fuzzy point $y(\tilde{x}_0)$, written down as

$$\tilde{J}^{(n)}(y(\tilde{x}_0)) = \bigcup_{\alpha\in(0,1]} \alpha\tilde{J}^{(n)}(y(\bar{x}_{0\alpha})) \in \mathscr{F}(\mathscr{F}(\mathscr{R})),$$

where $\tilde{J}^{(n)}(y(\bar{x}_{0\alpha})) = \{\tilde{\gamma} \in \mathscr{F}(\mathscr{R}) | \exists y(x_0) \in y(\bar{x}_{0\alpha}), \tilde{J}^{(n)}(y(x_0)) = \tilde{\gamma}\}$, its membership function being

$$\mu_{\tilde{J}^{(n)}(y(\tilde{x}_0))}(\tilde{\gamma}) = \bigvee_{\tilde{J}^{(n)}(y(x_0))=\tilde{\gamma}} \mu_{y(\tilde{x}_0)}(y(x_0)).$$

In the binary situation ($n(\geqslant 3)$-variate circumstance are discussed similarly), we call

$$\frac{\partial^2 \tilde{J}(y_i(\tilde{x}), y_k(\tilde{x}))}{\partial y_i \partial y_k} = \bigcup_{\alpha \in (0,1]} \alpha \frac{\partial^2 \tilde{J}(y_i(\bar{x}_\alpha), y_k(\bar{x}_\alpha))}{\partial y_i \partial y_k} \in \mathscr{F}(\mathscr{F}(\mathscr{R}))$$

the 2nd partial derivative in binary fuzzy-valued function at fuzzy points, where

$$\frac{\partial^2 \tilde{J}(y_i(\bar{x}_\alpha), y_k(\bar{x}_\alpha))}{\partial y_i \partial y_k} = \{\tilde{\gamma} | \exists (y_i(x), y_k(x)) \in y_i(\bar{x}_\alpha) \times y_k(\bar{x}_\alpha), \frac{\partial^2 \tilde{J}(y_i(x), y_k(x))}{\partial y_i \partial y_k} = \tilde{\gamma}\},$$

its membership function being

$$\mu_{\frac{\partial^2 \tilde{J}(y_i(\tilde{x}), y_k(\tilde{x}))}{\partial y_i \partial y_k}}(\tilde{\gamma}) = \bigvee_{\frac{\partial^2 \tilde{J}(y_i(x), y_k(x))}{\partial y_i \partial y_k} = \tilde{\gamma}} \{\mu_{y_i(\tilde{x})}(y_i(x)) \bigwedge \mu_{y_k(\tilde{x})}(y_k(x))\}.$$

Theorem 10.4.3. *Let $y(\tilde{x})$ be a fuzzy point. If $\tilde{J}$ is a 2nd differentiable fuzzy-valued function, with a fuzzy-valued matrix being $(\frac{\partial^2 \tilde{J}}{\partial y_i \partial y_k}) \supseteq 0$, then $\tilde{J}$ is a convex fuzzy-valued function at fuzzy points.*

Combine Theorem 10.3.1 with Theorem 10.3.3 and we can get a proof immediately in this theorem.

2. *Convex fuzzy-valued functional at fuzzy points*

Definition 10.4.7. Suppose $\tilde{\Pi}$ to be a fuzzy-valued functional and $\tilde{x}$ to be a fuzzy point in $\mathscr{R}$, then we call

$$\tilde{\Pi}(y(\tilde{x}), y'(\tilde{x})) = \int_{\lambda_0}^{\lambda_1} \tilde{F}(\tilde{x}, y(\tilde{x}), y'(\tilde{x}))dx \triangleq$$

$$\bigcup_{\alpha \in (0,1]} \alpha \tilde{\Pi}(y(\bar{x}_\alpha), y'(\bar{x}_\alpha)) = \bigcup_{\alpha \in (0,1]} \alpha \int_{\lambda_0}^{\lambda_1} \tilde{F}(\bar{x}_\alpha, y(\bar{x}_\alpha), y'(\bar{x}_\alpha))dx$$

a fuzzy-valued functional at fuzzy points.

Definition 10.4.8. Let $\tilde{\Pi}$ be a fuzzy-valued functional defined in convex region D. If in fuzzy point $\tilde{x} \in \mathscr{R}$ for arbitrarily $\lambda \in [0,1]$, there is

$$\begin{aligned}
&\tilde{\Pi}(\lambda y(\tilde{x})+(1-\lambda)z(\tilde{x}),\lambda y'(\tilde{x})+(1-\lambda)z'(\tilde{x}))\\
&\quad\subseteq \lambda\tilde{\Pi}(y(\tilde{x}),y'(\tilde{x}))+(1-\lambda)\tilde{\Pi}(z(\tilde{x}),z'(\tilde{x}))\\
&\quad\triangleq \bigcup_{\alpha\in(0,1]}\alpha\{\tilde{\Pi}(\lambda y(\bar{x}_\alpha)+(1-\lambda)z(\bar{x}_\alpha),\lambda y'(\bar{x}_\alpha)+(1-\lambda)z'(\bar{x}_\alpha))\}\\
&\quad\subseteq \bigcup_{\alpha\in(0,1]}\alpha\{\lambda\tilde{\Pi}(y(\bar{x}_\alpha),y'(\bar{x}_\alpha))+(1-\lambda)\tilde{\Pi}(z(\bar{x}_\alpha),z'(\bar{x}_\alpha))\},
\end{aligned}$$

then we call $\tilde{\Pi}$ a convex fuzzy-valued functional at fuzzy points in D.

Theorem 10.4.4. *Let $\tilde{F}_{y'y'} \supset 0$ and $\tilde{F}_{yy}\tilde{F}_{y'y'} - (\tilde{F}_{yy'})^2 \supseteq 0$. Then $\tilde{F}(\tilde{x}, y(\tilde{x}), y'(\tilde{x}))$ is a convex fuzzy-valued function concerning two fuzzy variable numbers $y(\tilde{x})$ and $y'(\tilde{x})$. If $y(\tilde{x})$ and $y'(\tilde{x})$ are regarded as two independent fuzzy functions, then we call $\tilde{\Pi}(y(\tilde{x}), y'(\tilde{x}))$ in Definition 10.4.7 a convex fuzzy-valued functional at fuzzy points.*

Combine Theorem 10.3.2 with Theorem 10.3.4 and we can get a proof in this theorem immediately.

10.5 Variation of Condition Extremum on Interval and Fuzzy-Valued Functional

10.5.1 Introduction

In this section the interval and fuzzy valued variation is going to be extended into a functional condition extremum, developing that of an interval and fuzzy-valued functional and verifying an effectiveness of the extension with a numerical example.

10.5.2 Variation of Condition Extremum in Interval Functional

Definition 10.5.1. We call

$$\begin{aligned}
\bar{\Pi} &= \int_{x_0}^{x_1} \bar{F}(x, y; y')dx\\
&= [\int_{x_0}^{x_1} F^-(x, y; y')dx, \int_{x_0}^{x_1} F^+(x, y; y')dx]
\end{aligned} \tag{10.5.1}$$

an interval functional dependent on n unknown functions, where $y = y_1, y_2, \cdots, y_n; y' = y'_1, y'_2 \cdots y'_n$.

In [Cao91a] [Cao01e] and [Luo84a,b] you can find the definition on interval value and its functional variation.

Theorem 10.5.1. *Suppose that functions $y_1, y_2, \cdots, y_n$ enable extremum to exist in the interval functional (10.5.1) under the condition*

$$\bar{\varphi}_i(x,y)=0 \qquad (i=1,2,\cdots,m;m<n) \tag{10.5.2}$$

with (10.5.2) independent, i.e., in m-order interval function determinants, only one determinant is not zero, i.e.,

$$\frac{D(\bar{\varphi}_1,\bar{\varphi}_2,\cdots,\bar{\varphi}_m)}{D(y_1,y_2,\cdots,y_m)}\neq 0, \tag{10.5.3}$$

then proper chosen factors $\bar{\lambda}_i(x)$ and $y_j(i=1,\cdots,m;j=1,2,\cdots,n)$ tally with Euler equation determined by interval functional

$$\bar{\Pi}^*=\int_{x_0}^{x_1}(\bar{F}+\sum_{i=1}^{m}\bar{\lambda}_i(x)\bar{\varphi}_i)dx=\int_{x_0}^{x_1}\bar{F}^*dx, \tag{10.5.4}$$

while functions $\bar{\lambda}_i(x)$ and $y_j(x)(i=1,2,\cdots,m;j=1,2,\cdots,n)$ are determined by the interval Euler equations and interval ones

$$\bar{F}^*_{y_j}-\frac{d}{dx}\bar{F}^*_{y_j'}=0\ (j=1,2,\cdots,n), \tag{10.5.5}$$

$$\bar{\varphi}_i=0\ (i=1,2,\cdots,m) \tag{10.5.6}$$

respectively. If $y_1,y_2,\cdots,y_n$ and $\bar{\lambda}_1(x),\bar{\lambda}_2(x),\cdots,\bar{\lambda}_m(x)$ are all regarded as model-variable of interval functional $\bar{\Pi}^*$, then (10.5.6) can be considered as Euler equations of internal functional $\bar{\Pi}^*$, where

$$(10.5.3)\triangleq\frac{D(\varphi_1^-,\cdots,\varphi_m^-)}{D(y_1,\cdots,y_m)}\neq 0,\frac{D(\varphi_1^+,\cdots,\varphi_m^+)}{D(y_1,\cdots,y_m)}\neq 0.$$

Proof: According to interval definition [Cao01e] and basic condition of extremum (Ref.[Cao91a]. Theorem 1.1), we have

$$\begin{aligned}\delta\bar{\Pi}^*=0\Leftrightarrow&\int_{x_0}^{x_1}\sum_{j=1}^{n}[\frac{\partial\bar{F}}{\partial y_j}+\sum_{i=1}^{m}\bar{\lambda}_i(x)\frac{\partial\bar{\varphi}_i}{\partial y_j}-\frac{d}{dx}\frac{\partial\bar{F}}{\partial y_j'}]\delta y_j dx=0\\ \Rightarrow&\int_{x_0}^{x_1}\sum_{j=1}^{m}(\bar{F}^*_{y_j}-\frac{d}{dx}\bar{F}^*_{y_j'})\delta y_j dx=0\\ \Rightarrow&\bar{F}^*_{y_j}-\frac{d}{dx}\bar{F}^*_{y_j'}=0\quad(j=1,2,\cdots,m),\end{aligned} \tag{10.5.7}$$

where $\bar{F}^*=\bar{F}+\sum\limits_{i=1}^{m}\bar{\lambda}_i(x)\bar{\varphi}_i$. Besides, since (10.5.7) represents an interval linear group with respect to $\bar{\lambda}_i$ and when (10.5.3) holds, we have the solution $\bar{\lambda}_i(x)=[\lambda_i^-(x),\lambda_i^+(x)]\ (i=1,2,\cdots,m)$. As for such $\bar{\lambda}_i(x)$, the necessary condition of the extremum in $\int_{x_0}^{x_1}\sum\limits_{j=1}^{n}(\bar{F}^*_{y_j}-\frac{d}{dx}\bar{F}^{*'}_{y_j})\delta y_j dx=0$ can be changed into $\int_{x_0}^{x_1} \sum_{j=m+1}^{n} (\bar{F}^*_{y_j} - \frac{d}{dx}\bar{F}^*_{y'_j})\delta y_j dx = 0$. Because of the arbitrary of $\delta y_j (j = m+1, \cdots, n)$, all of items are made to be zero, except one of them, by turns, and by applying basic variation Lemma I in Section 10.1, we have

$$\bar{F}^*_{y_j} - \frac{d}{dx}\bar{F}^*_{y'_j} = 0 \ (j = m+1, \cdots, n). \tag{10.5.8}$$

By combination of (10.5.7) and (10.5.8), we enable a condition extremum function of functional required by Π_i and factor $\bar{\lambda}_i(x)$ all to tally with (10.5.5) and (10.5.6).

Theorem 10.5.2. *Suppose that functions $y_1, y_2, \cdots, y_n$ enable extremum to exist in the interval functional (10.5.1) under the condition*

$$\bar{\psi}_i(x, y; y') = 0 \ (i = 1, 2, \cdots, m; m < n) \tag{10.5.9}$$

with (10.5.9) independent, i.e., there exists an m-order interval function determinant

$$\frac{D(\bar{\psi}_1, \bar{\psi}_2, \cdots, \bar{\psi}_m)}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0, \tag{10.5.10}$$

then proper chosen factors $\bar{\lambda}_i(x)$ and $y_j (i = 1, \cdots, m; j = 1, 2, \cdots, n)$ enable the interval functional in (10.5.1) to reach the condition extremum curve, i.e., its extremum curve

$$\bar{\Pi}^* = \int_{x_0}^{x_1} (\bar{F} + \sum_{i=1}^{m} \bar{\lambda}_i(x)\bar{\psi}_i)dx = \int_{x_0}^{x_1} \bar{F}^*_1 dx,$$

where $\bar{F}^*_1 = \bar{F} + \sum_{i=1}^{m} \bar{\lambda}_i(x)\bar{\psi}_i$, and

$$(10.5.10) \triangleq \frac{D(\psi_1^-, \psi_2^-, \cdots, \psi_m^-)}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0, \frac{D(\psi_1^+, \psi_2^+, \cdots, \psi_m^+)}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0.$$

Proof: The theorem can be proved as Theorem 10.5.1.

10.5.3 Variation on Fuzzy-Valued Functional Condition Extremum at Ordinary Points

Definition 10.5.2. Call

$$\tilde{\Pi} = \int_{x_0}^{x_1} \tilde{F}(x, y; y')dx = \bigcup_{a \in (0.1]} \alpha \int_{x_0}^{x_1} \bar{F}_\alpha(x, y; y')dx \tag{10.5.11}$$

a fuzzy-valued functional depending upon n unknown functions, here $\bar{F}_\alpha(x, y; y') = [F_\alpha^-(x, y; y'), F_\alpha^+(x, y; y')]$.

The definition can be found in Ref.[Cao91a] [Cao01e] and [Luo84a,b] with respect to fuzzy value and its functional variation.

Theorem 10.5.3. *Suppose that functions $y_j (j = 1, 2, \cdots, n)$ make an extremum exist in (10.5.11) under the condition*

$$\bar{\varphi}_i(x, y_1, \cdots, y_n) = 0 \ (i = 1, 2, \cdots, m; m < n) \tag{10.5.12}$$

with (10.5.12) independent, i.e, there exists an m-order fuzzy-valued function determinant

$$\frac{D(\tilde{\varphi}_1, \tilde{\varphi}_2, \cdots, \tilde{\varphi}_m)}{D(y_1, y_2, \cdots, y_m)} \neq 0, \tag{10.5.13}$$

then, the proper chosen factors $\bar{\lambda}_i(x)$ and $y_j (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ satisfy Euler equation determined by fuzzy-valued functional

$$\tilde{\Pi}^* = \int_{x_0}^{x_1} (\tilde{F} + \sum_{i=1}^{m} \tilde{\lambda}_i(x)\tilde{\varphi}_i)dx = \int_{x_0}^{x_1} \tilde{F}^* dx, \tag{10.5.14}$$

while functions $\tilde{\lambda}_i(x)$ and $y_j(x)$ are determined by fuzzy-valued Euler equations and a fuzzy-valued one

$$\tilde{F}^*_{y_j} - \frac{d}{dx}\tilde{F}^*_{y'_j} = 0 \ (j = 1, 2, \cdots, n), \tag{10.5.15}$$

$$\tilde{\varphi}_i = 0 \ (i = 1, 2, \cdots, m), \tag{10.5.16}$$

respectively. If we regard $\tilde{\lambda}_i(x)$ and $y_j (i = 1, \cdots, m; j = 1, \cdots, n)$ as the variables of fuzzy-valued functional $\tilde{\Pi}^*$, we can regard (10.5.16) as Euler equations of fuzzy functional $\tilde{\Pi}^*$, where

$$(10.5.13) \triangleq \bigcup_{a \in (0,1]} \alpha \frac{D(\bar{\varphi}_{1\alpha}, \bar{\varphi}_{2\alpha}, \cdots, \bar{\varphi}_{m\alpha}}{D(y_1, y_2, \cdots, y_m)} \neq 0.$$

Proof: According to fuzzy-valued (or fuzzy-valued functional) definition [Cao01e] and its basic condition of extremum ([Cao91a], Theorem 2.1), we have

$$\begin{aligned} &\delta\tilde{\Pi}^* = 0 \\ &\Leftrightarrow \bigcup_{\alpha \in (0,1]} \alpha \int_{x_0}^{x_1} \sum_{j=1}^{n} (\bar{F}^*_{y_j\alpha} - \frac{d}{dx}\bar{F}^*_{y'_j\alpha})\delta y_j dx = 0 \\ &\Rightarrow \bigcup_{\alpha \in (0,1]} \alpha(\bar{F}^*_{y_j\alpha} - \frac{d}{dx}\bar{F}^*_{y'_j\alpha}) = 0 \ (j = 1, 2, \cdots, n), \end{aligned} \tag{10.5.17}$$

where $\bar{F}^*_\alpha = \bar{F}_\alpha + \sum_{i=1}^{m} \bar{\lambda}_i(x)\bar{\varphi}_{i\alpha}$. Besides, (10.5.17) is a fuzzy valued linear group with respect to $\bar{\lambda}_{i\alpha}$. When (10.5.13) holds, as for a certain α, we can get the solution $\bar{\lambda}_{i\alpha}(x) = [\lambda^-_{i\alpha}(x), \lambda^+_{i\alpha}(x)](i = 1, 2, \cdots, m)$ by the proof in Theorem 10.5.1. As for such $\bar{\lambda}_{i\alpha}(x)$, the necessary condition of fuzzy extremum

$$\bigcup_{\alpha\in(0,1]} \alpha \int_{x_0}^{x_1} \sum_{j=1}^{n} (\bar{F}^*_{y_j\alpha} - \frac{d}{dx}\bar{F}^*_{y'_j\alpha})\delta y_j d\alpha = 0$$

is turned into

$$\bigcup_{\alpha\in(0,1]} \alpha \int_{x_0}^{x_1} \sum_{j=m+1}^{n} (\bar{F}^*_{y_j\alpha} - \frac{d}{dx}\bar{F}^*_{y'_j\alpha})\delta y_j d\alpha = 0.$$

Because δy_j is arbitrary, all of items are made to be zero, except one of them, in turns and, by the application of basic variation Lemma II in Section 10.2, we have

$$\bigcup_{\alpha\in(0,1]} \alpha(\bar{F}^*_{y_j\alpha} - \frac{d}{dx}\bar{F}^*_{y'_j\alpha}) = 0 \ (j = m+1, \cdots, n). \tag{10.5.18}$$

By combining (10.5.17) and (10.5.18), as for arbitrary $\alpha \in (0, 1]$, the condition extremum obtained by fuzzy-valued functional $\tilde{\Pi}$ and factor $\bar{\lambda}_i(x)$ should all meet with (10.5.15) and (10.5.16) from Theorem 10.5.1. Now the theorem holds.

Theorem 10.5.4. *Suppose that functions $y_j(j = 1, 2, \cdots n)$ enable extremum to exist in (10.5.11) under the condition*

$$\tilde{\psi}_i(x, y; y') = 0 \ (i = 1, 2, \cdots, m; m < n) \tag{10.5.19}$$

with (10.5.19) independent, i.e., there exists an m-order fuzzy-valued function determinant

$$\frac{D(\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_m)}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0, \tag{10.5.20}$$

then proper chosen factors $\tilde{\lambda}_i(x)$ and $y_j(i = 1, \cdots, m; j = 1, 2, \cdots, n)$, enable the fuzzy-valued functional in (10.5.11) to reach the condition extremum curve, i.e., its extremum curve

$$\tilde{\Pi}^* = \int_{x_0}^{x_1} (\tilde{F} + \sum_{i=1}^{m} \tilde{\lambda}_i(x)\tilde{\psi}_i)dx = \int_{x_0}^{x_1} \tilde{F}^*_1 dx,$$

where $\tilde{F}^*_1 = \tilde{F} + \sum_{i=1}^{m} \tilde{\lambda}_i(x)\tilde{\psi}_i$, and

$$(10.5.20) \triangleq \bigcup_{\alpha\in(0,1]} \alpha \frac{D(\psi^-_{1\alpha}, \psi^-_{2\alpha}, \cdots, \psi^-_{m\alpha})}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0,$$

$$\bigcup_{\alpha\in(0,1]} \alpha \frac{D(\psi^+_{1\alpha}, \psi^+_{2\alpha}, \cdots, \psi^+_{m\alpha})}{D(y'_1, y'_2, \cdots, y'_m)} \neq 0.$$

Proof: The theorem can be proved like Theorem 10.5.3.

10.5.4 Numerical Example

Example 10.5.1: Find fuzzy functional extremum $\tilde{S} = \int_{x_0}^{x_1} \tilde{y}dx$ under the equal circumference

$$\int_{x_0}^{x_1} \widetilde{\sqrt{1+y'^2}}dx = \tilde{l}.$$

Make a supplementary functional $\tilde{\Pi}^* = \int_{x_0}^{x_1}(\tilde{y} + \tilde{\lambda}\widetilde{\sqrt{1+y'^2}})dx$, and fuzzy Euler equation is $\tilde{F} - y'\tilde{F}_{y'} = \tilde{C}_1$, i.e.,

$$\tilde{y} + \tilde{\lambda}\widetilde{\sqrt{1+y'^2}} - \frac{\tilde{\lambda}y'^2}{\widetilde{\sqrt{1+y'^2}}} = \tilde{C}_1. \tag{10.5.21}$$

For a certain determined α, (10.5.21) is

$$\bigcup_{\alpha\in(0,1]} \alpha\{\bar{y}_\alpha + \bar{\lambda}_\alpha(\overline{\sqrt{1+y'^2}})_\alpha - \frac{\bar{\lambda}_\alpha\bar{y}'^2_\alpha}{(\overline{\sqrt{1+y'^2}})_\alpha}\} = \bigcup_{\alpha\in(0,1]} \alpha\{\bar{C}_{1\alpha}\}.$$

We first find

$$\bar{y}_\alpha - \bar{C}_{1\alpha} = -\frac{\bar{\lambda}_\alpha}{\sqrt{1+\bar{y}'^2_\alpha}}$$

by introducing parameter t, such that $\bar{y}' = \overline{\text{tg}}\ t$, then

$$\bar{y}_\alpha - \bar{C}_{1\alpha} = -\bar{\lambda}_\alpha\overline{\cos}\ t;$$

$\dfrac{d\bar{y}_\alpha}{x} = \overline{\text{tg}}\ t$, therefore, $d\bar{x}_\alpha = \dfrac{d\bar{y}_\alpha}{\overline{\text{tg}}\ t} = \dfrac{\bar{\lambda}_\alpha\overline{\sin}\ tdt}{\overline{\text{tg}}\ t} = \bar{\lambda}_\alpha\overline{\cos}\ tdt;$

$$\bar{x}_\alpha = \bar{C}_{2\alpha} + \bar{\lambda}_\alpha\overline{\sin}\ t.$$

Then, when extremal equation is represented by parameter form, we have

$$\begin{cases} \bar{x}_\alpha - \bar{C}_{2\alpha} = \bar{\lambda}_\alpha\overline{\sin}\ t \\ \bar{y}_\alpha - \bar{C}_{1\alpha} = -\bar{\lambda}_\alpha\overline{\cos}\ t \end{cases}$$

and by canceling t, then

$$(\bar{x}_\alpha - \bar{C}_{2\alpha})^2 + (\bar{y}_\alpha - \bar{C}_{2\alpha})^2 = \bar{\lambda}^2_\alpha,$$

such that

$$(\tilde{x} - \tilde{C})^2 + (\tilde{y} - \tilde{C})^2 = \tilde{\lambda}^2.$$

It is curve variety of functional extremum we find, where $C_{i\alpha}^-, C_{i\alpha}^+, \lambda_\alpha^-, \lambda_\alpha^+ (i = 1, 2)$ are constants and parameters; $\bar{C}_{i\alpha}, \bar{\lambda}_\alpha (i = 1, 2)$ are interval constants and parameters; $\tilde{C}, \tilde{\lambda}$ are fuzzy constant and parameter.

10.5.5 Conclusion

The functional condition extremum problem mentioned in this section contains more information than a classical one. We notice that for $\alpha \in (0, 1]$, it is difficult for us to find all the curves. But, in practical application, we find a solution to some α (or finite α) according to the requirement. It is worth mentioning that we can obtain the more satisfactory result in 0.618 searching way.

The result discussed here can be easily extended into a condition extremum variation of ordinary or fuzzy-valued functional with fuzzy function $\tilde{y}_j (j = 1, \cdots, n)$.

10.6 Variation of Condition Extremum on Functional with Fuzzy Function

10.6.1. Introduction

By Definition "nest of set", a condition extremum variation problem of an ordinary and fuzzy functionals on ordinary function are extended into function being a fuzzy state. In this section, we discuss the first condition extremum variation of functional with function, and extend it to variation of fuzzy-valued functional condition extremum with fuzzy function.

10.6.2 Condition Extremum Variation of Functional with Fuzzy Function

Let F be an ordinary differentiable functional defined on $[x_0, x_1] \subseteq \mathscr{R}$, $\tilde{y}_j (j = 1, 2, \cdots, n)$ be fuzzy functions (i.e., a convex fuzzy set on $\mathscr{R}$) and the support of $\tilde{y}_j$ be

$$s(\tilde{y}_j) = \{x \in \mathscr{R} | \mu_{\tilde{y}_j}(x) > 0\} \subseteq [a_j, b_j].$$

By the extension principle, we have the following.

Definition 10.6.1. Let's call

$$\tilde{\Pi} = \int_{x_0}^{x_1} F(x, \tilde{y}; \tilde{y}') dx = \int_{x_0}^{x_1} \bigcup_{\alpha \in [0,1]} \alpha F(x, \bar{y}_\alpha; \bar{y}'_\alpha) dx \qquad (10.6.1)$$

an ordinary functional depending on n fuzzy functions, where $\bar{y}_{j\alpha} = [y_{j\alpha}^-, y_{j\alpha}^+]$, $\tilde{y}'_{j\alpha} = [\min(y'^-_{j\alpha}, y'^+_{j\alpha}),\ \max(y'^-_{j\alpha}, y'^+_{j\alpha})]$, and $\tilde{y}'_{j\alpha}$, $\bar{y}'_{j\alpha}$, $y'^-_{j\alpha}$, $y'^+_{j\alpha}$ denote a fuzzy derivative, an interval value one and interval value left and right ones of $\tilde{y}_j$, respectively, and

$$F(x,\tilde{y},\tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha F(x,\bar{y}_\alpha,\bar{y}'_\alpha),$$

its membership function is

$$\mu_{F(x,\tilde{y},\tilde{y}')}(\gamma) = \bigvee_{F(x,y,y')=\gamma} \{\mu_{\tilde{y}}(y)\wedge\mu_{\tilde{y}'}(y)\},$$

where

$$F(x,\bar{y}_\alpha,\bar{y}'_\alpha) = \{\gamma|\exists(x,y,y')\in X\times\bar{Y}_\alpha\times\bar{Y}'_\alpha, F(x,y,y')=\gamma\},$$
$$\tilde{y}=(\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_n), \tilde{y}'=(\tilde{y}'_1,\tilde{y}'_2,\cdots,\tilde{y}'_n);$$
$$\bar{y}_\alpha=(\bar{y}_{1\alpha},\bar{y}_{2\alpha},\cdots,\bar{y}_{n\alpha}), \bar{y}'_\alpha=(\bar{y}'_{1\alpha},\bar{y}'_{2\alpha},\cdots,\bar{y}'_{n\alpha}).$$

From Definition 10.6.1, we know that the ordinary functional dependent on n fuzzy functions can be changed into an interval one for a certain determined α value. Therefore, it is easy to find definitions of the functional variation according to Ref.[Cao91a] [Ail52] and [Cao91b].

Definition 10.6.2. Let us call

$$F_{y_j}(x,\tilde{y},\tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha F_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha),$$
$$F_{y'_j}(x,\tilde{y},\tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha F_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)\ (j=1,2,\cdots,n)$$

a partial derivatives of ordinary functional F on fuzzy points $(x,\tilde{y};\tilde{y}')$ with respect to $\tilde{y}_j$ and $\tilde{y}'_j$ $(j=1,2,\cdots,n)$, respectively, whose membership functions are respectively

$$\mu_{F_{y_j}(x,\tilde{y},\tilde{y}')}(\gamma) = \bigvee_{F_{y_j}(x,y,y')=\gamma} [\mu_{\tilde{y}}(y)\wedge\mu_{\tilde{y}'}(y')] =$$
$$\bigvee_{F_{y_j}(x,y_1,\cdots,y_n;y'_1,\cdots,y'_n)=\gamma} \left[(\mu_{\tilde{y}_1}(y_1)\wedge\cdots\wedge\mu_{\tilde{y}_n}(y_n))\wedge(\mu_{\tilde{y}'_1}(y'_1)\wedge\cdots\wedge\mu_{\tilde{y}'_n}(y'_n))\right],$$
$$\mu_{F_{y'_j}(x,\tilde{y},\tilde{y}')}(\gamma) = \bigvee_{F_{y'_j}(x,y,y')=\gamma} [\mu_{\tilde{y}}(y)\wedge\mu_{\tilde{y}'}(y')] =$$
$$\bigvee_{F_{y'_j}(x,y_1,\cdots,y_n;y'_1,\cdots,y'_n)=\gamma} \left[\mu_{\tilde{y}_1}(y_1)\wedge\cdots\wedge\mu_{\tilde{y}_n}(y_n)\wedge(\mu_{\tilde{y}'_1}(y'_1)\wedge\cdots\wedge\mu_{\tilde{y}'_n}(y'_n))\right],$$

where $x,\gamma\in\mathscr{R}$, $y=(y_1,y_2,\cdots,y_n)$ and $y'=(y'_1,y'_2,\cdots,y'_n)$ are real function vectors on real region $\mathscr{R}$.

Theorem 10.6.1. *Suppose that fuzzy functions $\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_n$ enable ordinary functional (10.6.1) under the conditions*

$$\phi_i(x,\tilde{y})=0\ (i=1,2,\cdots,m;m<n)) \tag{10.6.2}$$

to realize extremum and (10.6.2) are independent, i.e., there is an m-order fuzzy function determinant with fuzzy function

$$\frac{D(\phi_1(x,\tilde{y}),\phi_2(x,\tilde{y}),\cdots,\phi_m(x,\tilde{y}))}{D(\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_m)}\neq 0, \tag{10.6.3}$$

then the proper chosen factors $k_i(x)$ and $\tilde{y}_j(i=1,2,\cdots,m;j=1,2,\cdots,n)$ satisfy Euler equation given by an ordinary functional with fuzzy function

$$\begin{aligned}\Pi^*&=\int_{x_0}^{x_1}[F(x,\tilde{y},\tilde{y}')+\sum_{i=1}^{m}k_i(x)\phi_i(x,\tilde{y})]dx\\&=\int_{x_0}^{x_1}F^*(x,\tilde{y},\tilde{y}')dx,\end{aligned} \tag{10.6.4}$$

while functions $k_i(x)$ and $\tilde{y}_j(x)$ are determined by Euler equations with fuzzy function

$$F^*_{y_j}(x,\tilde{y},\tilde{y}')-\frac{d}{dx}F^*_{y'_j}(x,\tilde{y},\tilde{y}')=0\ (j=1,2,\cdots,n) \tag{10.6.5}$$

and by Equations with fuzzy function

$$\phi_i(x,\tilde{y})=0\ (i=1,2,\cdots,m). \tag{10.6.6}$$

If $\tilde{y}_j$ and $k_i(x)(j=1,2,\cdots,n;i=1,2,\cdots,m)$ are regarded as fuzzy-model-variable of functional Π^*, (10.6.6) is regarded as Euler equation of Π^* with fuzzy function.

Proof: If for an arbitrary $\alpha\in[0,1]$, the basic condition of extremum is

$$\begin{aligned}&\delta\Pi^*=0\Longleftrightarrow\bigcup_{\alpha\in[0,1]}\alpha\delta\bar{\Pi}^*_\alpha=0\\&\Longleftrightarrow\int_{x_0}^{x_1}\bigcup_{\alpha\in[0,1]}\alpha\sum_{j=1}^{n}[F_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)\delta y_j-\frac{d}{dx}F_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)\delta y'_j]dx=0,\end{aligned}$$

or we intergrade by part the second item in each middle bracket. And by using the definition of the interval value (or function) in Ref. [Cao93c], and the basic condition of extremum in Theorem 1.1 in Ref. [Cao91a], we have

$$\int_{x_0}^{x_1}\bigcup_{\alpha\in[0,1]}\alpha\sum_{j=1}^{n}[F_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)-\frac{d}{dx}F_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)]\delta y_j dx=0, \tag{10.6.7}$$

$\bar{y}_\alpha=(\bar{y}_{1\alpha},\bar{y}_{2\alpha},\cdots,\bar{y}_{n\alpha})$, obeys the independent constraints of m

$$\bigcup_{\alpha\in[0,1]}\alpha\phi_i(x,\bar{y}_\alpha)=0\ (i=1,2,\cdots,m).$$

As for this formula, we find variation with

$$\bigcup_{\alpha\in[0,1]} \alpha\sum_{j=1}^{n}\frac{\partial\phi_i(x,\bar{y}_\alpha)}{\partial y_j}\delta y_j = 0\ (i=1,2,\cdots,m),$$

where there are $n-m$ variation arbitrations in δy_j, for example, say $\delta y_{m+1},\cdots,$ δy_n arbitration, then

$$\int_{x_0}^{x_1} k_i(x)[\bigcup_{\alpha\in[0,1]} \alpha\sum_{j=1}^{n}\frac{\partial\phi_i(x,\bar{y}_\alpha)}{\partial y_j}\delta y_j]dx = 0\ (i=1,2,\cdots,m).$$

Add them with satisfied equation (10.6.7) from admitted variation δy_j, respectively, then

$$\int_{x_0}^{x_1}\bigcup_{\alpha\in[0,1]} \alpha\sum_{j=1}^{n}[\frac{\partial F(x,\bar{y}_\alpha,\bar{y}'_\alpha)}{\partial y_j}+\sum_{i=1}^{m}k_i(x)\frac{\partial\phi_i(x,\bar{y}_\alpha)}{\partial y_j}-$$

$$\frac{d}{dx}\frac{\partial F(x,\bar{y}_\alpha,\bar{y}'_\alpha)}{\partial y'_j}]\delta y_j dx = 0,$$

$$\xrightarrow{F^*(x,\bar{y}_\alpha,\bar{y}'_\alpha)=F(x,\bar{y}_\alpha,\bar{y}'_\alpha)+\sum_{i=1}^{m}k_{i\alpha}(x)\phi_i(x,\bar{y}_\alpha)}$$

$$\int_{x_0}^{x_1}\bigcup_{\alpha\in[0,1]} \alpha\sum_{j=1}^{n}[F^*_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)-\frac{d}{dx}F^*_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)]\delta y_j dx = 0,$$

and then we change it into

$$\int_{x_0}^{x_1}\bigcup_{\alpha\in[0,1]} \alpha\sum_{j=m+1}^{n}[F^*_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)-\frac{d}{dx}F^*_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)]\delta y_j dx = 0. \quad (10.6.8)$$

Again because $\delta y_j(j=m+1,m+2,\cdots,n)$ is arbitrary, we make all of above function equations be zero except one of them by turns. For $\forall\alpha\in[0,1]$, by applying basic Lemma I in variation of Ref.[Cao91a], we have

$$\bigcup_{\alpha\in[0,1]}\alpha[F^*_{y_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)-\frac{d}{dx}F^*_{y'_j}(x,\bar{y}_\alpha,\bar{y}'_\alpha)]=0 \quad (10.6.9)$$
$$(j=m+1,m+2,\cdots,n).$$

Combine (10.6.8) and (10.6.9), we enable the function of conditional extremum realized by functional Π^* and factor $k_i(x)$ to all satisfy equations (10.6.5) and (10.6.6), so, Theorem 10.6.1 holds.

Theorem 10.6.2. *If we change (10.6.2) in Theorem 10.6.1 into differential equation with fuzzy function*

$$\phi_1(x,\tilde{y},\tilde{y}')=0\ (i=1,\cdots,m;m<n),$$

while the other conditions are unchanged, the conclusion is still true.

10.6.3 Variation of Fuzzy-Valued Functional Condition Extremum with Fuzzy Function

Definition 10.6.3. We call

$$\begin{aligned}\tilde{\Pi}^* &= \int_{x_0}^{x_1} \tilde{F}(x, \tilde{y}; \tilde{y}')dx \in \mathscr{F}(\mathscr{F}(\mathscr{R})) \\ &= \bigcup_{\alpha\in[0,1]} \alpha \int_{x_0}^{x_1} \tilde{F}(x, \bar{y}_\alpha, \bar{y}'_\alpha)dx\end{aligned} \tag{10.6.10}$$

a fuzzy-valued functional dependendent on n-fuzzy functions, where $\bar{y}_\alpha$ and $\bar{y}'_\alpha$ are defined by Definition 10.6.1, with

$$\tilde{F}(x, \tilde{y}, \tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha\tilde{F}(x, \bar{y}_\alpha, \bar{y}'_\alpha),$$
$$\tilde{F}(x, \bar{y}_\alpha, \bar{y}'_\alpha) = \{\tilde{\gamma} \in \mathscr{F}(\mathscr{R}) | \exists (x, y, y') \in X \times \bar{Y}_\alpha \times \bar{Y}'_\alpha, \tilde{F}^*(x, y, y') = \tilde{\gamma}\},$$

and its membership function is

$$\mu_{\tilde{F}(x,\tilde{y},\tilde{y}')}(\tilde{\gamma}) = \bigvee_{\tilde{F}(x,y,y')=\tilde{\gamma}} \{\mu_{\tilde{y}}(y) \wedge \mu_{\tilde{y}'}(y')\}.$$

Definition 10.6.4. Let's call

$$\tilde{F}_{y_j}(x, \tilde{y}; \tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha\tilde{F}^*_{y_j}(x, \bar{y}_\alpha; \bar{y}'_\alpha) \in \mathscr{F}(\mathscr{F}(\mathscr{R})),$$
$$\tilde{F}_{y'_j}(x, \tilde{y}; \tilde{y}') = \bigcup_{\alpha\in[0,1]} \alpha\tilde{F}^*_{y'_j}(x, \bar{y}_\alpha; \bar{y}'_\alpha) \in \mathscr{F}(\mathscr{F}(\mathscr{R}))(j = 1, 2, \cdots, n)$$

a partial derivation of fuzzy-valued functional $\tilde{F}$ on fuzzy point $(x, \tilde{y}; \tilde{y}')$ with respect to y and y', respectively, where

$$\tilde{F}^*_{y_j}(x, \bar{y}_\alpha; \bar{y}'_\alpha) = \{\tilde{\gamma} | \exists (x, y; y') \in X \times \bar{Y}_\alpha \times \bar{Y}'_\alpha, \tilde{F}^*_{y_j}(x, y; y') = \tilde{\gamma}\},$$
$$\tilde{F}^*_{y'_j}(x, \bar{y}_\alpha; \bar{y}'_\alpha) = \{\tilde{\gamma} | \exists (x, y; y') \in X \times \bar{Y}_\alpha \times \bar{Y}'_\alpha, \tilde{F}^*_{y'_j}(x, y; y') = \tilde{\gamma}\},$$

their membership functions are

$$\mu_{\tilde{F}_{y_j}(x,\tilde{y};\tilde{y}')}(\tilde{\gamma}) = \bigvee_{\tilde{F}_{y_j}(x,y;y')=\tilde{\gamma}} (\mu_{\tilde{y}}(y) \bigwedge \mu_{\tilde{y}'}(y')),$$
$$\mu_{\tilde{F}_{y'_j}(x,\tilde{y};\tilde{y}')}(\tilde{\gamma}) = \bigvee_{\tilde{F}_{y'_j}(x,y;y')=\tilde{\gamma}} (\mu_{\tilde{y}}(y) \bigwedge \mu_{\tilde{y}'}(y')).$$

Theorem 10.6.3. *Suppose that fuzzy functions $\tilde{y}_j (j = 1, 2, \cdots, n)$ make when fuzzy-valued functional (10.6.10) with fuzzy function under the condition*

$$\phi_i(x, \tilde{y}) = 0 \ (i = 1, 2, \cdots, m; m < n) \tag{10.6.11}$$

reached extremum, and (10.6.11) is independent, i.e., there is an m-order fuzzy-valued function determinant with fuzzy function

$$\frac{D(\phi_1(x, \tilde{y}), \phi_2(x, \tilde{y}), \cdots \phi_m(x, \tilde{y}))}{D(\tilde{y})} \neq 0. \tag{10.6.12}$$

Then, the proper chosen factors $k_i(x)(i = 1, 2, \cdots, m)$ *and* $\tilde{y}_j(j = 1, 2, \cdots, n)$ *satisfy Euler equation obtained by fuzzy functional with fuzzy function*

$$\tilde{\Pi}^* = \int_{x_0}^{x_1} [(\tilde{F}(x, \tilde{y}, \tilde{y}') + \sum_{i=1}^{m} k_i(x)\phi_i(x, \tilde{y}))]dx = \int_{x_0}^{x_1} \tilde{F}^*(x, \tilde{y}, \tilde{y}')dx,$$

while functions $k_i(x)$ *and* $y_j(x)$ *are determined by fuzzy-valued Euler equations*

$$\tilde{F}^*_{y_j}(x, \tilde{y}, \tilde{y}') - \frac{d}{dx}\tilde{F}^*_{y'_j}(x, \tilde{y}, \tilde{y}') = 0 \ (j = 1, 2, \cdots, n)$$

and by fuzzy-valued equations:

$$\phi_j(x, \tilde{y}) = 0 \ (i = 1, 2, \cdots, m). \tag{10.6.13}$$

If we regard $k_i(x)$ *and* $\tilde{y}_j(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ *as the model variable of fuzzy-valued functional* $\tilde{\Pi}^*$*, we can write (10.6.13) as Euler equations of fuzzy-valued functional* $\tilde{\Pi}^*$*, where*

$$(10.6.12) \triangleq \bigcup_{\alpha \in [0,1]} \alpha \frac{D(\bar{\phi}_{1\alpha}(x, \bar{y}_\alpha), \bar{\phi}_{2\alpha}(x, \bar{y}_\alpha), \cdots, \bar{\phi}_{m\alpha}(x, \bar{y}_\alpha))}{D(\bar{y}_\alpha)} \neq 0.$$

Theorem 10.6.4. *If we change (10.6.11) in Theorem 10.6.3 into fuzzy differential equations with fuzzy functions*

$$\phi_i(x, \tilde{y}; \tilde{y}') = 0 \ (i = 1, 2, \cdots, m; m < n)$$

with the other conditions unchanged, the conclusion holds.

10.6.4 Conclusion

The condition extremum problem, functional, ordinary or fuzzy-valued, with fuzzy function is advanced, and a method to it is obtained for them containing fuzzy functions by the aid of the variational methods. This model will contain more information than a classical one and will be of extensive use in engineering act fields, the application examples remains to be completed by readers.