1

Prepare Knowledge

This chapter represents definitions on fuzzy sets and its operation properties, $\alpha-$cut sets and convex fuzzy sets. Besides, based on the elaborative fuzzy relation, it introduces a fuzzy operator related to this chapter, and it exhibits fuzzy function. At the same time, it describes fuzzy mathematics of three mainstream theorems—expansion principle, decomposition theorem and representation theorem. Finally, it inquires into five-type fuzzy numbers and its operations.

1.1 Fuzzy Sets

In order to give fuzzy sets concept, the chapter first describes a foundation concept on a fuzzy sets theory—universe. The so-called universe means all of the object involved is a set commonly, usually by writing English alphabets X,Y,Z and etc. The fuzzy sets differ from classic ones with a strict mathematics definition. We give its mathematics description as follows.

Definition 1.1.1. A so-called fuzzy subset $\tilde{A}$ in set X is a set

$$\tilde{A} = \{(\mu_{\tilde{A}}(x), x)|x \in X\},$$

where $\mu_{\tilde{A}}(x)$ is a real number in interval $[0,1]$, called a membership degree from point x to $\tilde{A}$. This function is defined in the interval [0, 1]

$$\mu_{\tilde{A}} : X \longrightarrow [0,1],$$
$$x \mapsto \mu_{\tilde{A}}(x)$$

called a membership function in fuzzy set $\tilde{A}$.

At the same time, fuzzy subsets are also often called fuzzy sets.

From Definition 1.1.1 of the fuzzy sets, there exist few next conclusions, obviously:

B.-Y. Cao: Optimal Models & Meth. with Fuzzy Quantities, STUDFUZZ 248, pp. 1–32.
springerlink.com

(1) The concept of fuzzy sets is an expansion concept of classical sets. If $\mathscr{F}(X)$ means all fuzzy sets on X, i.e.,

$$\mathscr{F}(X) = \{\tilde{A} | \tilde{A} \text{ is a fuzzy set on } X\},$$

then $P(X) \subset \mathscr{F}(X)$, where $P(X)$ is the power sets on X, i.e.,

$$P(X) = \{A | A \text{ is a classic set on } X\},$$

that is, if the membership function of fuzzy set $\tilde{A}$ takes only 0 and 1, two values, then $\tilde{A}$ is exuviated into the classic sets of X.

(2) The concept of the membership function is the expansion of the characteristic function concept.

When $A \in P(X)$ is an ordinary subset in X, the characteristic function of A is

$$\chi_A = \begin{cases} 1, x \in A & (\text{membership degree of } x \text{ for } A \text{ is } 1), \\ 0, x \notin A & (\text{membership degree of } x \text{ for } A \text{ is } 0). \end{cases}$$

This means in fuzzy sets, the nearer the membership degree $\mu_{\tilde{A}}(x)$ in fuzzy set $\tilde{A}$ is to 1, the bigger x belonging to $\tilde{A}$ degree is; whereas, the nearer $\mu_{\tilde{A}}(x)$ is to 0, the smaller x belonging to $\tilde{A}$ degree is. If the value region of $\mu_{\tilde{A}}(x)$ is $\{0, 1\}$, then fuzzy set $\tilde{A}$ is an ordinary set A, but membership function $\mu_{\tilde{A}}(x)$ is a characteristic function $\chi_A(x)$.

(3) We call fuzzy sets in $\mathscr{F}(X) \diagdown P(X)$ true fuzzy sets.

Several representation methods to fuzzy sets are shown as follows.

1^0 A representation method to fuzzy set by Zadeh

If set X is a finite set, let universe $X = \{x_1, x_2, \cdots, x_n\}$. The fuzzy set is

$$\tilde{A} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \cdots + \frac{\mu_{\tilde{A}}(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i},$$

here symbol "Σ" is no longer a numerical sum, $\frac{\mu_{\tilde{A}}(x_i)}{x_i}$ is not a fraction; it only has the sign meaning, that is, only membership degree of the point x_i with respect to fuzzy set $\tilde{A}$ is $\mu_{\tilde{A}}(x_i)$.

If X is an infinite set, a fuzzy set on X is

$$\tilde{A} = \int_{x \in X} \frac{\mu_{\tilde{A}}(x)}{x}.$$

Similarly, the sign "$\int$" is not an integral any more, only means an infinite logic sum, but the meaning of $\frac{\mu_{\tilde{A}}(x)}{x}$ is in accordance with the finite case.

2^0 When the universe X is a finite set, the fuzzy set represented in Definition 1.1.1 is

$$\tilde{A} = \{(\mu_{\tilde{A}}(x_1), x_1), (\mu_{\tilde{A}}(x_2), x_2), \cdots, (\mu_{\tilde{A}}(x_n), x_n)\}.$$

3^0 When the universe X is a finite set, the fuzzy set represented according to a vector form is

$$\tilde{A} = (\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2), \cdots, \mu_{\tilde{A}}(x_n)).$$

Remarkably, X and ϕ also can be seen as fuzzy set in X, if membership functions $\mu_{\tilde{A}}(x) \equiv 1$ and $\mu_{\tilde{A}}(x) \equiv 0$, then $\tilde{A}$ is a complete set X and an empty set ϕ, respectively.

An element that the membership degree is 1 definitely belongs to this fuzzy set; an element that the membership degree is 0 does not belong to this fuzzy set definitely. But the membership function value in (0, 1) forms a distinct boundary, also calling distinct subsets of fuzzy sets.

When a fuzzy object is described by using the fuzzy set, choice of its membership function is a key. Now we give three membership functions basically:

1. Partial minitype (abstains up, Figure 1.1.1)

$$\mu_{\tilde{A}}(x) = \begin{cases} [1 + (a(x-c))^b]^{-1}, & \text{when } x \geqslant c, \\ 1, & \text{when } x < c, \end{cases}$$

where $c \in X$ is an arbitrary point, $a > 0, b > 0$ are two parameters.

2. Partial large-scale (abstains down, Figure 1.1.2)

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{when } x \leqslant c, \\ [1 + (a(x-c))^{-b}]^{-1}, & \text{when } x > c, \end{cases}$$

where $x \in X$ is an arbitrary point, $a > 0, b > 0$ are two parameters.

3. Normal type (middle type, Figure 1.1.3)

$$\mu_{\tilde{A}}(x) = e^{-\left(\frac{x-a}{b}\right)^2},$$

where $a \in X$ is an arbitrary value, $b > 0$ is a parameter.

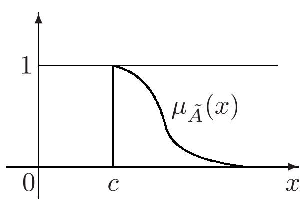

Figure 1.1.1

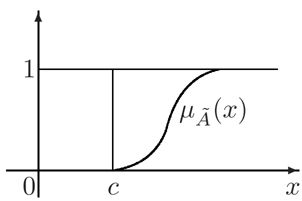

Figure 1.1.2

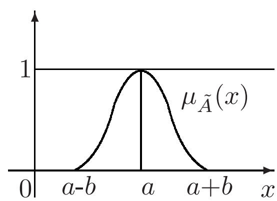

Figure 1.1.3

Obviously, Type 1 and 2 is dual, and its meaning shows clear at a glance. Type 3 is a fuzzy set $\tilde{A}$, which is " sufficiently near to number set of a", then this membership function in $\tilde{A}$ is defined on a center type according to the definition.

Example 1.1.1: Let $X \subseteq \mathscr{R}^+$($\mathscr{R}^+$ is a non-negative real number set). Regard age as universe and take X=[0,100]. Zadeh gave "oldness" $\tilde{O}$ and " youth" $\tilde{Y}$, these two membership functions respectively are

$$\mu_{\tilde{O}}(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 50, \\ \left[1 + \left(\dfrac{x-50}{5}\right)^{-2}\right]^{-1}, & 50 < x \leqslant 100, \\ 1, & x > 100, \end{cases}$$

and

$$\mu_{\tilde{Y}}(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 25, \\ \left[1 + \left(\dfrac{x-25}{5}\right)^{2}\right]^{-1}, & 25 < x \leqslant 100, \\ 0, & x > 100. \end{cases}$$

If some person's age is 28, then his membership degree belongings to "youth" or "oldness" respectively is

$$\left[1 + \left(\frac{28-25}{5}\right)^{2}\right]^{-1} = 0.735 \text{ and } 0.$$

If some person's age is 55, then his membership degree belongings to "youth" or "oldness" respectively is

$$\left[1 + \left(\frac{55-25}{5}\right)^{2}\right]^{-1} = 0.027$$

and

$$\left[1 + \left(\frac{55-50}{5}\right)^{-2}\right]^{-1} = 0.5.$$

According to the three-type membership functions mentioned above, we can, to a certain, calculate its membership degree by concrete object x. When its accuracy is not required high, for simple account, we can determine the membership degree by adopting evaluation.

Example 1.1.2: Suppose X={1,2,3,4}, these four elements constitute a small number set. Obviously, element 1 is standardly a small number, it should belong to this set, and its membership degree is 1; element 4 is not a small number, and it should not belong to this set, its membership degree being 0. Element 2 "also returns small" or make "eighty percent small", its membership degree being 0.8; element 3 probably is "force small", or makes "two percent small"; its membership degree being 0.2. The fuzzy sets written in small numbers as $\tilde{A}$, its elements still are 1,2,3,4, at the same time, and a membership degree of element in $\tilde{A}$ is given, denoted by

Zadeh's representation is $\tilde{A} = \frac{1}{1} + \frac{0.8}{2} + \frac{0.2}{3} + \frac{0}{4}$.
An order dual representation is $\tilde{A}$={(1,1),(0.8,2),(0.2,3),(0,4)}.
A vector method simply shows as $\tilde{A}$=(1,0.8,0.2,0).

1.2 Operations in Fuzzy Sets

Because the value region in membership function of fuzzy sets corresponding to clear-subset characteristic function is extended from $\{0,1\}$ to $[0,1]$. Similar to the characteristic function to demonstrate the relation between a distinctive subset, we have the following.

Definition 1.2.1. Let $\tilde{A}, \tilde{B} \in \mathscr{F}(X)$. If to arbitrary $x \in X$, we have
Inclusion: $\tilde{A} \subseteq \tilde{B} \Longleftrightarrow \mu_{\tilde{A}}(x) \leqslant \mu_{\tilde{B}}(x)$.
Equality: $\tilde{A} = \tilde{B} \Longleftrightarrow \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$.

From Definition 1.2.1, $\tilde{A} = \tilde{B} \Longleftrightarrow \tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$. That is to say, the inclusion relation is a binary relation on fuzzy power set $\mathscr{F}(X)$ with following properties, i.e.,

(1) $\tilde{A} \subseteq \tilde{A}$ (reflection).
(2) $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A} \Longrightarrow \tilde{A} = \tilde{B}$ (symmetry).
(3) $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{C} \Longrightarrow \tilde{A} \subseteq \tilde{C}$ (transitivity).

Since relation " $\subseteq$ " constitutes an order relation on $\mathscr{F}(X), (\mathscr{F}(X), \subseteq)$ stands for a partially ordered set. Again as $\phi,\ X \in \mathscr{F}(X)$, hence $\mathscr{F}(X)$ contains maximum element X and minimum element ϕ.

Definition 1.2.2. Let $\tilde{A}, \tilde{B} \in \mathscr{F}(x)$. Then we define
Union: $\tilde{A} \bigcup \tilde{B}$, whose membership function is

$$\mu_{(\tilde{A} \bigcup \tilde{B})}(x) = \mu_{\tilde{A}}(x) \bigvee \mu_{\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

Intersection: $\tilde{A} \bigcap \tilde{B}$, whose membership function is

$$(\mu_{\tilde{A} \bigcap \tilde{B})}(x) = \mu_{\tilde{A}}(x) \bigwedge \mu_{\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$

Complement: $\tilde{A}^c$, whose membership function is

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x).$$

Their images show like Figure 1.2.1—Figure 1.2.3:

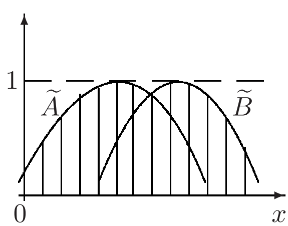

Figure 1.2.1 $\widetilde{A} \bigcup \widetilde{B}$

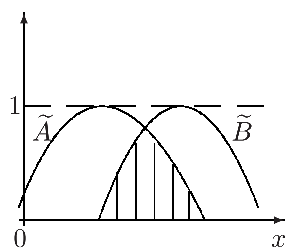

Figure 1.2.2 $\widetilde{A} \bigcap \widetilde{B}$

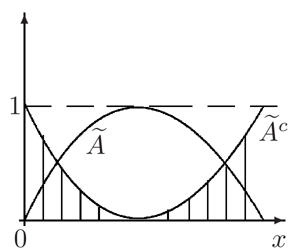

Figure 1.2.3 $\widetilde{A}^c$

Comparing operation of union, intersection and complement in distinctive set, we discover immediately that the fuzzy sets operation is exactly a parallel definition of the distinct set operation, $\tilde{A} \bigcup \tilde{B}$ is a minimum fuzzy set embodying $\tilde{A}$ and embodied again in $\tilde{B}$. $\tilde{A}\bigcap\tilde{B}$ is a maximum fuzzy set embodying $\tilde{A}$ and embodied again in $\tilde{B}$.

According to the two kinds of cases, where the universe X is finite or infinite, the calculation formula of union, intersection and complement in fuzzy sets $\tilde{A}$ and $\tilde{B}$ can be represented, respectively, like the following:

(1) The universe is $X = \{x_1, x_2, \cdots, x_n\}$, and $\tilde{A} = \sum\limits_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i}, \tilde{B} = \sum\limits_{i=1}^{n} \frac{\mu_{\tilde{B}}(x_i)}{x_i}$, then

$$\tilde{A}\bigcup\tilde{B} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i) \vee \mu_{\tilde{B}}(x_i)}{x_i},$$

$$\tilde{A}\bigcap\tilde{B} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i) \wedge \mu_{\tilde{B}}(x_i)}{x_i},$$

$$\tilde{A}^c = \sum_{i=1}^{n} \frac{1 - \mu_{\tilde{A}}(x_i)}{x_i}.$$

(2) X is an infinite set, and $\tilde{A} = \int_{x\in X} \frac{\mu_{\tilde{A}}(x)}{x}, \quad \tilde{B} = \int_{x\in X} \frac{\mu_{\tilde{B}}(x)}{x}$, then

$$\tilde{A}\bigcup\tilde{B} = \int_{x\in X} \frac{\mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x)}{x},$$

$$\tilde{A}\bigcap\tilde{B} = \int_{x\in X} \frac{\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x)}{x},$$

$$\tilde{A}^c = \int_{x\in X} \Big(1 - \frac{\mu_{\tilde{A}}(x)}{x}\Big).$$

Example 1.2.1: Suppose $X = \{x_1, x_2, x_3, x_4\}$; $\tilde{A} = \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$; $\tilde{B} = \frac{0}{x_1} + \frac{0.2}{x_2} + \frac{0.8}{x_3} + \frac{0}{x_4}$, then

$$\begin{aligned}
\tilde{A}\bigcup\tilde{B} &= \frac{1 \vee 0}{x_1} + \frac{0.8 \vee 0.2}{x_2} + \frac{0.2 \vee 0.8}{x_3} + \frac{0 \vee 0}{x_4} \\
&= \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{0.8}{x_3} + \frac{0}{x_4}. \\
\tilde{A}\bigcap\tilde{B} &= \frac{1 \wedge 0}{x_1} + \frac{0.8 \wedge 0.2}{x_2} + \frac{0.2 \wedge 0.8}{x_3} + \frac{0 \wedge 0}{x_4} \\
&= \frac{0}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}. \\
\tilde{A}^c &= \frac{1 - 1}{x_1} + \frac{1 - 0.8}{x_2} + \frac{1 - 0.2}{x_3} + \frac{1 - 0}{x_4} \\
&= \frac{0}{x_1} + \frac{0.2}{x_2} + \frac{0.8}{x_3} + \frac{1}{x_4}.
\end{aligned}$$

Example 1.2.2: Compute union, intersection and complement of the fuzzy sets $\tilde{Y}$ and $\tilde{O}$ in Example 1.1.1 in Section 1.1.

From the definition, we have

$$\tilde{Y}\bigcup\tilde{O}=\int_{x\in X}\frac{\mu_{\tilde{Y}}(x)\bigvee\mu_{\tilde{O}}(x)}{x}$$

$$=\int_{0\leqslant x\leqslant 25}\frac{\frac{1}{x}+\int_{25<x\leqslant x^*}\left[1+\left(\frac{x-25}{5}\right)^2\right]^{-1}}{x}$$

$$+\int_{x^*<x\leqslant 100}\frac{\left[1+\left(\frac{x-50}{5}\right)^{-2}\right]^{-1}}{x}+\int_{x>100}\frac{1}{x},$$

where $x^*\approx 51$;

$$\tilde{Y}\bigcap\tilde{O}=\int_{50<x\leqslant x^*}\frac{\left[1+\left(\frac{x-50}{5}\right)^{-2}\right]^{-1}}{x}$$

$$+\int_{x^*<x\leqslant 100}\frac{\left[1+\left(\frac{x-25}{5}\right)^{2}\right]^{-1}}{x};$$

$$\tilde{O}^c=\int_{0\leqslant x\leqslant 50}\frac{1}{x}+\int_{50<x\leqslant 100}\frac{1-\left[1+\left(\frac{x-50}{5}\right)^{-2}\right]^{-1}}{x};$$

$$\tilde{Y}^c=\int_{25\leqslant x\leqslant 100}\frac{1-\left[1+\left(\frac{x-25}{5}\right)^{2}\right]^{-1}}{x}+\int_{x>100}\frac{1}{x}.$$

The union, intersection and complement operation in fuzzy set can be extended to several fuzzy sets.

Definition 1.2.3. Suppose T to be an index set, $\tilde{A}_t\in\mathscr{F}(\mathrm{X})\ \ (t\in T)$, then

$$\mu_{\bigcup\limits_{t\in T}\tilde{A}_t}(x)=\bigvee_{t\in T}\mu_{\tilde{A}_t}(x)=\sup_{t\in T}\mu_{\tilde{A}_t}(x),\ x\in X,$$

$$\mu_{\bigcap\limits_{t\in T}\tilde{A}_t}(x)=\bigwedge_{t\in T}\mu_{\tilde{A}_t}(x)=\inf_{t\in T}\mu_{\tilde{A}_t}(x),\ x\in X.$$

Obviously,

$$\bigcup_{t\in T}\tilde{A}_t,\bigcap_{t\in T}\tilde{A}_t\in\mathscr{F}(X).$$

In particular, when T is a finite set,

$$\mu_{\bigcup\limits_{t\in T}\tilde{A}_t}(x)=\max_{t\in T}\mu_{\tilde{A}_t}(x),\ x\in X,$$

$$\mu_{\bigcap\limits_{t\in T}\tilde{A}_t}(x)=\min_{t\in T}\mu_{\tilde{A}_t}(x),\ x\in X.$$

Theorem 1.2.1. *($\mathscr{F}(X),\bigcup,\bigcap,c$) satisfies the following properties:*

(1) Idempotent law $\tilde{A}\bigcup\tilde{A}=\tilde{A},\quad \tilde{A}\bigcap\tilde{A}=\tilde{A}.$
(2) Commutative law $\tilde{A}\bigcup\tilde{B}=\tilde{B}\bigcup\tilde{A},\quad \tilde{A}\bigcap\tilde{B}=\tilde{B}\bigcap\tilde{A}.$
(3) Associative law

$$(\tilde{A}\bigcup\tilde{B})\bigcup\tilde{C}=\tilde{A}\bigcup(\tilde{B}\bigcup\tilde{C}),$$
$$(\tilde{A}\bigcap\tilde{B})\bigcap\tilde{C}=\tilde{A}\bigcap(\tilde{B}\bigcap\tilde{C}).$$

(4) Absorptive law $(\tilde{A}\bigcup\tilde{B})\bigcap\tilde{A}=\tilde{A},\quad (\tilde{A}\bigcap\tilde{B})\bigcup\tilde{A}=\tilde{A}.$
(5) Distributive law

$$(\tilde{A}\bigcup\tilde{B})\bigcap\tilde{C}=(\tilde{A}\bigcap\tilde{C})\bigcup(\tilde{B}\bigcap\tilde{C}),$$
$$(\tilde{A}\bigcap\tilde{B})\bigcup\tilde{C}=(\tilde{A}\bigcup\tilde{C})\bigcap(\tilde{B}\bigcup\tilde{C}).$$

(6) 0-1 law

$$\tilde{A}\bigcap X=\tilde{A},\quad \tilde{A}\bigcap\phi=\phi,$$
$$\tilde{A}\bigcup X=X,\quad \tilde{A}\bigcup\phi=\tilde{A}.$$

(7) Restore original law $(\tilde{A}^c)^c=\tilde{A}.$
(8) Dual law $(\tilde{A}\bigcup\tilde{B})^c=\tilde{A}^c\bigcap\tilde{B}^c,\quad (\tilde{A}\bigcap\tilde{B})^c=\tilde{A}^c\bigcup\tilde{B}^c.$

Proof: Proved by taking Property (8) for example, the rest can be verified directly.

From $\forall x\in X$, we have

$$\begin{aligned}\mu_{(\tilde{A}\bigcup\tilde{B})^c}(x)&=1-\mu_{\tilde{A}\bigcup\tilde{B}}(x)\\&=1-\max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}=\min\{1-\mu_{\tilde{A}}(x),1-\mu_{\tilde{B}}(x)\}\\&=\min\{\mu_{\tilde{A}^c}(x),\mu_{\tilde{B}^c}(x)\}=\mu_{\tilde{A}^c\bigcap\tilde{B}^c}(x).\end{aligned}$$

Hence

$$(\tilde{A}\bigcup\tilde{B})^c=\tilde{A}^c\bigcap\tilde{B}^c.$$

Similarly, we can prove

$$(\tilde{A}\bigcap\tilde{B})^c=\tilde{A}^c\bigcup\tilde{B}^c.$$

It is pointed out that the operation in a fuzzy set no longer satisfies the excluded-middle law. Namely, under circumstance generally, we have

$$\tilde{A}\bigcup\tilde{A}^c\neq X,\ \tilde{A}\bigcap\tilde{A}^c\neq\phi.$$

But we have

$$\tilde{A}\bigcup \tilde{A}^c \geqslant \frac{1}{2},\ \tilde{A}\bigcap \tilde{A}^c \leqslant \frac{1}{2}.$$

Example 1.2.3: If $\mu_{\tilde{A}}(x) \equiv 0.5,\ \mu_{\tilde{A}^c}(x) \equiv 0.5$, then

$$\mu_{\tilde{A}\bigcup \tilde{A}^c}(x) = \max\{0.5,\ 0.5\} = 0.5 \neq 1,$$
$$\mu_{\tilde{A}\bigcap \tilde{A}^c}(x) = \min\{0.5,\ 0.5\} = 0.5 \neq 0.$$

1.3 $\alpha-$Cut and Convex Fuzzy Sets

1.3.1 α–Cut Set

Definition 1.3.1. Suppose $\tilde{A} \in \mathscr{F}(X), \forall \alpha \in [0,1]$, we write

$$(\tilde{A})_\alpha = A_\alpha = \{x|\mu_{\tilde{A}}(x) \geqslant \alpha\},$$

then A_α is said to be an α–cut set of fuzzy set $\tilde{A}$. Again, we write

$$(\tilde{A})_{\underset{\cdot}{\alpha}} = A_{\underset{\cdot}{\alpha}} = \{x|\mu_{\tilde{A}}(x) > \alpha\},$$

$A_{\underset{\cdot}{\alpha}}$ is called a strong α–cut set of fuzzy set $\tilde{A}$, α a confidence level, and

$$(\tilde{A})_{\underset{\cdot}{0}} = A_{\underset{\cdot}{0}} = \{x|\mu_{\tilde{A}}(x) > 0\} = \text{supp}\tilde{A},$$

$A_{\underset{\cdot}{0}}$ is called a support of fuzzy set $\tilde{A}$.

If this support supp $\tilde{A} = \{x\}$ is a single point set, then $\tilde{A}$ is called a fuzzy point on X.

Audio-visually, the meaning in A_α is that if x to the membership degree of $\tilde{A}$ attains or exceeds the level α, at last it has the qualified member; since all of these qualified members constitute A_α, it is a classical subset in X.

Example 1.3.1: Suppose $\tilde{A} = \dfrac{0.5}{x_1} + \dfrac{0.7}{x_2} + \dfrac{0}{x_3} + \dfrac{0.9}{x_4} + \dfrac{1}{x_5}$, then

at $\alpha = 1$, $A_1 = \{x_5\}$, $A_{\underset{\cdot}{1}} = \phi$,

at $\alpha = 0.9$, $A_{0.9} = \{x_4,\ x_5\}$, $A_{\underset{\cdot}{0.9}} = \{x_5\}$,

at $\alpha = 0.7$, $A_{0.7} = \{x_2,\ x_4,\ x_5\}$, $A_{\underset{\cdot}{0.7}} = \{x_4,\ x_5\}$,

at $\alpha = 0.5$, $A_{0.5} = \{x_1,\ x_2,\ x_4,\ x_5\}$, $A_{\underset{\cdot}{0.5}} = \{x_2,\ x_4,\ x_5\}$,

at $\alpha = 0$, $A_0 = X$, $A_{\underset{\cdot}{0}} = \{x_1,\ x_2,\ x_4,\ x_5\}$.

α-cut set has the following properties.

Property 1.3.1

$$(1)\quad (\tilde{A}\bigcup \tilde{B})_\alpha = A_\alpha \bigcup B_\alpha,\quad (\tilde{A}\bigcap \tilde{B})_\alpha = A_\alpha \bigcap B_\alpha.$$

$$(2)\quad (\tilde{A}\bigcup \tilde{B})_{\underset{\cdot}{\alpha}} = A_{\underset{\cdot}{\alpha}} \bigcup B_{\underset{\cdot}{\alpha}},\quad (\tilde{A}\bigcap \tilde{B})_{\underset{\cdot}{\alpha}} = A_{\underset{\cdot}{\alpha}} \bigcap B_{\underset{\cdot}{\alpha}}.$$

Proof: We prove only the first formula in (1).

$$\begin{aligned}(\tilde{A}\bigcup\tilde{B})_\alpha &= \{x|\mu_{\tilde{A}\bigcup\tilde{B}}(x)\geqslant\alpha\} = \{x|\mu_{\tilde{A}}(x)\vee\mu_{\tilde{B}}(x)\geqslant\alpha\}\\ &= \{x|\mu_{\tilde{A}}(x)\geqslant\alpha\}\bigcup\{x|\mu_{\tilde{B}}(x)\geqslant\alpha\} = A_\alpha\bigcup B_\alpha.\end{aligned}$$

Proof of the other formulas is the same.

Property 1.3.2

$$(1)\ \ (\bigcup_{t\in T}\tilde{A}_t)_\alpha \supseteq \bigcup_{t\in T}(\tilde{A}_t)_\alpha,\ (\bigcap_{t\in T}\tilde{A}_t)_\alpha = \bigcap_{t\in T}(\tilde{A}_t)_\alpha,\ (\tilde{A}^c)_\alpha = (A_{\underset{\cdot}{1-\alpha}})^c.$$

$$(2)\ \ (\bigcup_{t\in T}\tilde{A}_t)_{\underset{\cdot}{\alpha}} = \bigcup_{t\in T}(\tilde{A}_t)_{\underset{\cdot}{\alpha}},\ (\bigcap_{t\in T}\tilde{A}_t)_{\underset{\cdot}{\alpha}} \subseteq \bigcap_{t\in T}(\tilde{A}_t)_{\underset{\cdot}{\alpha}},\ (\tilde{A}^c)_{\underset{\cdot}{\alpha}} = (A_{1-\alpha})^c.$$

Proof in Property 1.3.2 is easy, readers themselves can prove it.

It must be pointed out that the first formula in (1) and the second formula in (2) can't be changed for the equation.

Example 1.3.2: Let $\mu_{\tilde{A}_n}(x)\equiv\frac{1}{2}(1-\frac{1}{n}),\ n=1,\ 2,\cdots$. Then $\mu_{\bigcup\limits_{n=1}^{\infty}\tilde{A}_n}(x)\equiv\frac{1}{2}$, so that $(\bigcup\limits_{n=1}^{\infty}\tilde{A}_n)_{0.5}=X$. But

$$(\tilde{A}_n)_{0.5}=\phi\ \ (n\geqslant 1),$$

so that

$$\bigcup_{n=1}^{\infty}(\tilde{A}_n)_{0.5}=\phi.$$

Therefore $$(\bigcup_{n=1}^{\infty}\tilde{A}_n)_{0.5}\neq\bigcup_{n=1}^{\infty}(\tilde{A}_n)_{0.5}.$$

Similarly, let $\mu_{\tilde{B}_n}(x)\equiv\frac{1}{2}(1+\frac{1}{n}),\ n=1,\ 2,\cdots$. We can prove

$$(\bigcap_{n=1}^{\infty}\tilde{B}_n)_{\underset{\cdot}{0.5}}\neq\bigcap_{n=1}^{\infty}(\tilde{B}_n)_{\underset{\cdot}{0.5}}.$$

Definition 1.3.2. Suppose $\tilde{A}\in\mathscr{F}(\mathrm{X})$, set

$$\mathrm{Ker}\ \tilde{A}=\{x|\mu_{\tilde{A}}(x)=1\}$$

is called a kernel of fuzzy set $\tilde{A}$ and $\tilde{A}$ is a normal fuzzy set if Ker $\tilde{A}\neq\phi$.

1.3.2 Convex Fuzzy Sets

Recall the first concept of ordinary convex sets. Suppose $X=\mathscr{R}^n$ to be n-dimensional Euclidean space, A is an ordinary subset in X. If $\forall x_1, x_2\in A$, and $\forall\lambda\in[0,1]$, we have

$$\lambda x_1 + (1-\lambda)x_2 \in A,$$

and then call A convex sets.

Before introduction of the convex fuzzy set concepts, we prove first result below.

Theorem 1.3.1. *Suppose $\tilde{A}$ to be a fuzzy set in X, if $\alpha \in [0,1], A_\alpha = \{x|\mu_{\tilde{A}}(x) \geqslant \alpha\}$ are all convex sets if and only if $\forall x_1, x_2 \in X, \lambda \in [0,1]$, there is*

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2). \quad (1.3.1)$$

Proof: If we have already known $\alpha \in [0,1]$, A_α are all convex sets, $\forall x_1, x_2 \in X$ might as well suppose $\mu_{\tilde{A}}(x_2) \geqslant \mu_{\tilde{A}}(x_1) = \alpha_0$, then

$$\mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2) = \alpha_0.$$

Because A_{α_0} is a convex set, $\forall x_1, x_2 \in A_{\alpha_0}$, and $\forall \lambda \in [0,1]$, we have

$$\lambda x_1 + (1-\lambda)x_2 \in A_{\alpha_0},$$

hence
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \alpha_0.$$

Therefore
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \mu_{\tilde{A}}(x_1) \bigwedge \mu_{\tilde{A}}(x_2).$$

Conversely, if we have already known $\forall x_1, x_2 \in X, \alpha \in [0,1]$, there exist

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2),$$

then, if $\alpha \in [0,1], x_1, x_2 \in A_\alpha$, hence $\mu_{\tilde{A}}(x_1) \geqslant \alpha,\ \mu_{\tilde{A}}(x_2) \geqslant \alpha$,

such that
$$\mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2) \geqslant \alpha,$$

so
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2) \geqslant \alpha,$$

hence
$$\lambda x_1 + (1-\lambda)x_2 \in A_\alpha.$$

Therefore, A_α is a convex set.

Definition 1.3.3. Suppose $X = \mathscr{R}^n$ to be n-dimensional Euclidean space, $\tilde{A}$ is a fuzzy set in X. If $\forall \alpha \in [0,1], A_\alpha$ are all convex sets, calling fuzzy set $\tilde{A}$ a convex fuzzy set.

From Theorem 1.3.1 we know that $\tilde{A}$ is a convex set if and only if $\forall \lambda \in [0,1], x_1, x_2 \in X$, there is

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2).$$

Theorem 1.3.2. *If $\tilde{A}$ and $\tilde{B}$ are convex sets, so is $\tilde{A} \cap \tilde{B}$.*

Proof: $\forall x_1, x_2 \in X,\ \forall \lambda \in [0,1]$,

$$\begin{aligned}\mu_{(\tilde{A}\cap\tilde{B})}\big(\lambda x_1 + (1-\lambda)x_2\big) &= \mu_{\tilde{A}}\big(\lambda x_1 + (1-\lambda)x_2\big) \wedge \mu_{\tilde{B}}\big(\lambda x_1 + (1-\lambda)x_2\big) \\ &\geqslant \big(\mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2)\big) \wedge \big(\mu_{\tilde{B}}(x_1) \wedge \mu_{\tilde{B}}(x_2)\big) \\ &= \big(\mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{B}}(x_1)\big) \wedge \big(\mu_{\tilde{A}}(x_2) \wedge \mu_{\tilde{B}}(x_2)\big) \\ &= \mu_{\tilde{A}\cap\tilde{B}}(x_1) \wedge \mu_{\tilde{A}\cap\tilde{B}}(x_2).\end{aligned}$$

Therefore, $\tilde{A} \cap \tilde{B}$ denotes a convex fuzzy set.

Definition 1.3.4. Let $\tilde{A}, \tilde{B}, \tilde{\lambda} \in \mathscr{F}(X)$. Then a convex combination with respect to $\tilde{\lambda}$ of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $(\tilde{A}, \tilde{B}; \tilde{\lambda})$, with its membership function being

$$\mu_{(\tilde{A},\tilde{B};\tilde{\lambda})}(x) = \tilde{\lambda}(x)\mu_{\tilde{A}}(x) + \big(1 - \tilde{\lambda}(x)\big)\mu_{\tilde{B}}(x), \quad \forall x \in X.$$

Generally, if $\tilde{A}_i, \tilde{\lambda}_i \in \mathscr{F}(X)(1 \leqslant i \leqslant m)$ and $\sum\limits_{i=1}^{m} \tilde{\lambda}_i(x) = 1(\forall x \in X)$, then a convex combination with respect to $\tilde{\lambda}_i$ of $\tilde{A}_i$ is written as

$$\mu_{(\tilde{A}_1,\tilde{A}_2,\cdots,\tilde{A}_m;\tilde{\lambda}_1,\tilde{\lambda}_2,\cdots,\tilde{\lambda}_m)}(x) = \sum_{i=1}^{m} \tilde{\lambda}_i(x)\mu_{\tilde{A}_i}(x), \quad \forall x \in X.$$

Definition 1.3.5. Suppose $\tilde{A} \in \mathscr{F}(X)$, if $\forall \alpha \in [0,1], A_\alpha$ to be all bounded sets in X, then $\tilde{A}$ is called a bounded fuzzy set in X.

Theorem 1.3.3. *Both union and intersection of two bounded fuzzy sets are bounded fuzzy sets, respectively.*

It is easy to prove it by property and Definition 1.3.5 of α-cut sets.

1.4 Fuzzy Relativity and Operator

1.4.1 Fuzzy Relations

Definition 1.4.1. Suppose $X \times Y$ to be a Cartesian product in X and $Y, \tilde{R}$ is a fuzzy set of $X \times Y$, its membership function $\mu_{\tilde{R}}(x,y)(x \in X, y \in Y)$ determines a fuzzy relation $\tilde{R}$ in X and Y (still use the same mark).

Example 1.4.1: Suppose X=$\{x_1, x_2, x_3, x_4\}$ denotes the set of four factories, Y=$\{$electricity, coal, petroleum$\}$ denotes three kinds of energy resource set, Table 1.4.1 denotes fuzzy relations $\tilde{R}$ between factories and each energy resource, $\tilde{R}_{ij}$ denotes dependence degree from the factory i to energy resource j.

Table 1.4.1. Fuzzy Relations between Factories and Energy Resource

	Electricity	Coal	Petroleum
Factory 1	$\tilde{R}_{11}$	$\tilde{R}_{12}$	$\tilde{R}_{13}$
Factory 2	$\tilde{R}_{21}$	$\tilde{R}_{22}$	$\tilde{R}_{23}$
Factory 3	$\tilde{R}_{31}$	$\tilde{R}_{32}$	$\tilde{R}_{33}$
Factory 4	$\tilde{R}_{41}$	$\tilde{R}_{42}$	$\tilde{R}_{43}$

Example 1.4.2: Suppose $X = Y$ is a real number set, Cartesian product $X \times Y$ is the whole plane. R: " $x > y$" is an ordinary relation, that is, set R in plane. But we consider the relation as follows:

"$x \gg y$", that is "x is much greater than y", which is a fuzzy relation; write $\tilde{R}$, and we define its membership function as

$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0, & x \leqslant y, \\ \left[1 + \dfrac{100}{(x-y)^2}\right]^{-1}, & x > y. \end{cases}$$

From here we can know the following:

1^0 Fuzzy relation $\tilde{R}$ from X to Y is a fuzzy set in Cartesian product $X \times Y$. Because of Cartesian product with order relevant, i.e., $X \times Y \neq Y \times X$, $\tilde{R}$ is also with order relevant.

2^0 If two values {0,1} is taken from the membership function $\tilde{R}(x,y)$ in fuzzy relation only, then $\tilde{R}$ confirms an ordinary set in $X \times Y$, so the fuzzy relation is extended to an ordinary relation.

In Example 1.4.2, $\tilde{R}$ is a fuzzy relation between the same universe. Under the condition of $X = Y$, we call $\tilde{R}$ fuzzy relation in X.

Example 1.4.3: Suppose X={x_1,x_2,x_3} denotes three persons' sets, $\tilde{R}$ denotes fuzzy relation in three persons' trust each other, i.e.,

$$\tilde{R} = \frac{1}{(x_1,x_1)} + \frac{0.6}{(x_1,x_2)} + \frac{0.9}{(x_1,x_3)} + \frac{0.1}{(x_2,x_1)} + \frac{1}{(x_2,x_2)} + \frac{0.7}{(x_2,x_3)} + \frac{0.5}{(x_3,x_1)} + \frac{0.8}{(x_3,x_2)} + \frac{1}{(x_3,x_3)}.$$

$\mu_{\tilde{R}}(x_i,x_i) = 1$ expresses that everybody trusts most himself. $\mu_{\tilde{R}}(x_2,x_1) = 0.1$ indicates that x_2 to x_1 "distrust basically".

Definition 1.4.1 can be expanded into fuzzy relations between finite, even an infinite universe.

Since fuzzy relation $\tilde{R}$ is given through set $\tilde{R}$ in Cartesian product set $X \times X$, then some operations and properties of fuzzy relations are all those of fuzzy sets.

In addition, the fuzzy relations still have the following special operations.

Definition 1.4.2. Suppose $\tilde{R}_1$ to be a fuzzy relation from X to Y, $\tilde{R}_2$ is a fuzzy relation from Y to Z, then synthesis $\tilde{R}_1 \circ \tilde{R}_2$ of $\tilde{R}_1$ and $\tilde{R}_2$ is a fuzzy relation from X to Z; its membership function confirms as follows:

$\forall (x,z) \in X \times Z$,

$$\mu_{(\tilde{R}_1 \circ \tilde{R}_2)}(x,z) = \bigvee_{y \in Y} [\mu_{\tilde{R}_1}(x,y) \wedge \mu_{\tilde{R}_2}(y,z)], \tag{1.4.1}$$

where $x \in X, z \in Z$. If R_1, R_2 are two ordinary relations, according to method in ordinary set, its synthesis denotes

$$R_1 \circ R_2 = \{(x,z) | (x,z) \in X \times Z, \exists y \in Y, s.t.\ (x,y) \in R_1, (y,z) \in R_2\}. \tag{1.4.2}$$

From here, as an ordinary relation R_1 with R_2, its synthesis (1.4.1) and (1.4.2) should be accordant. In fact, at this time, synthesis (1.4.1) of R_1 and R_2 also can take only two values {0,1}. It is easy to prove that (1.4.1) is equivalent to (1.4.2).

Example 1.4.4: Suppose $\tilde{R}_1$ to be a fuzzy relation in X and Y, its membership function is $\mu_{\tilde{R}_1}(x,y) = e^{-k(x-y)^2}$ and $\tilde{R}_2$ is a fuzzy relation in Y and Z, its membership function is $\mu_{\tilde{R}_2}(y,z) = e^{-k(y-z)^2}(k \geqslant 1$, constant), then its synthesis $\tilde{R}_1 \circ \tilde{R}_2$ is a fuzzy relation in X and Z, its membership function is

$$\mu_{(\tilde{R}_1 \circ \tilde{R}_2)}(x,z) = \bigvee_{y \in Y} [e^{-k(x-y)^2} \bigwedge e^{-k(y-z)^2}]$$
$$= e^{-k\left(x - \frac{x+z}{2}\right)^2} = e^{-k\left(\frac{x-z}{2}\right)^2}.$$

A few special fuzzy relations, and suppose $\tilde{R}$ to be fuzzy relation in X.

(1) Inverse fuzzy relation.

Inverse fuzzy relation of fuzzy relation $\tilde{R}$ denotes $\tilde{R}^{-1}$, its membership function being

$$\mu_{\tilde{R}^{-1}}(x,y) = \mu_{\tilde{R}}(y,x),\ \forall x,y \in X.$$

Example 1.4.5: In Example 1.4.3, inverse relation of $\tilde{R}$ is

$$\tilde{R}^{-1} = \frac{1}{(x_1,x_1)} + \frac{0.1}{(x_1,x_2)} + \frac{0.5}{(x_1,x_3)} + \frac{0.6}{(x_2,x_1)} + \frac{1}{(x_2,x_2)} + \frac{0.8}{(x_2,x_3)}$$
$$+ \frac{0.9}{(x_3,x_1)} + \frac{0.7}{(x_3,x_2)} + \frac{1}{(x_3,x_3)}.$$

(2) Symmetric relation.

If fuzzy relation $\tilde{R}$ satisfies

$$\mu_{\tilde{R}^{-1}}(x,y) = \mu_{\tilde{R}}(x,y),\ \forall x,y \in X,$$

then $\tilde{R}$ is called symmetry.

Example 1.4.6: The "friend relation" is symmetric, while "paternity relation" and "consequence relation" are not symmetric.

(3) Identical relation.

Fuzzy relation $\tilde{I}$ on X called identical relation means that $\tilde{I}$ represents an ordinary relation with its membership function being

$$\mu_{\tilde{I}}(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases} \ \forall x,y \in X.$$

(4) Zero relation $\tilde{O}$ and the whole relation $\tilde{X}$ are

$$\mu_{\tilde{O}}(x,y) = 0,\ \mu_{\tilde{X}}(x,y) = 1,\ \forall x,y \in X.$$

1.4.2 The Operation Properties of the Fuzzy Relation

Proposition 1.4.1. *Synthesis of fuzzy relation satisfies combination law*

$$(\tilde{R}_1 \circ \tilde{R}_2) \circ \tilde{R}_3 = \tilde{R}_1 \circ (\tilde{R}_2 \circ \tilde{R}_3). \tag{1.4.3}$$

Proof: Because

$$\begin{aligned}
\mu_{(\tilde{R}_1\circ\tilde{R}_2)\circ\tilde{R}_3}(x,w) &= \bigvee_{z\in X}[\mu_{(\tilde{R}_1\circ\tilde{R}_2)}(x,z) \bigwedge \mu_{\tilde{R}_3}(z,w)] \\
&= \bigvee_{z\in X}\{\bigvee_{y\in X}[\mu_{\tilde{R}_1}(x,y) \bigwedge \mu_{\tilde{R}_2}(y,z)] \bigwedge \mu_{\tilde{R}_3}(z,w)\} \\
&= \bigvee_{y\in X}[\bigvee_{z\in X}(\mu_{\tilde{R}_1}(x,y) \bigwedge \mu_{\tilde{R}_2}(y,z) \bigwedge \mu_{\tilde{R}_3}(z,w))] \\
&= \bigvee_{y\in X}\{\mu_{\tilde{R}_1}(x,y) \bigwedge [\bigvee_{z\in X}(\mu_{\tilde{R}_2}(y,z) \bigwedge \mu_{\tilde{R}_3}(z,w))]\} \\
&= \bigvee_{y\in X}[\mu_{\tilde{R}_1}(x,y) \bigwedge \mu_{(\tilde{R}_2\circ\tilde{R}_3)}(y,w)] \\
&= \mu_{\tilde{R}_1\circ(\tilde{R}_2\circ\tilde{R}_3)}(x,w),
\end{aligned}$$

consequently (1.4.3) holds.

If $\tilde{R}$ is a fuzzy relation in X with X, then we stipulate

$$\underbrace{\tilde{R} \circ \tilde{R} \circ \cdots \circ \tilde{R}}_{k} = \tilde{R}^k.$$

Proposition 1.4.2. *For arbitrarily fuzzy relation $\tilde{R}$, we have*

$$\tilde{I} \circ \tilde{R} = \tilde{R} \circ \tilde{I} = \tilde{R}, \quad \tilde{O} \circ \tilde{R} = \tilde{R} \circ \tilde{O} = \tilde{O}.$$

Proposition 1.4.3. *If $\tilde{S} \subseteq \tilde{T}$, then $\tilde{R} \circ \tilde{S} \subseteq \tilde{R} \circ \tilde{T}$, $\tilde{S} \circ \tilde{R} \subseteq \tilde{T} \circ \tilde{R}$.*

Proposition 1.4.4. *For arbitrarily a tuft fuzzy relation $\{\tilde{R}_i\}_{i\in I}$ and fuzzy relation $\tilde{R}$, we have*

$$(1)\ \ \tilde{R} \circ (\bigcup_{i\in I} \tilde{R}_i) = \bigcup_{i\in I} \tilde{R} \circ \tilde{R}_i; \qquad (2)\ \ (\bigcup_{i\in I} \tilde{R}_i) \circ \tilde{R} = \bigcup_{i\in I} \tilde{R}_i \circ \tilde{R}.$$

Proof: Only prove (1). $\forall (x,z) \in X \times X$,

$$\begin{aligned}
&\mu_{\tilde{R}\circ(\bigcup_{i\in I}\mu_{\tilde{R}_i})}(x,z) = \bigvee_{y\in X}\{\mu_{\tilde{R}}(x,y) \bigwedge [\bigcup_{i\in I}\mu_{\tilde{R}_i}(y,z)]\} \\
&= \bigvee_{y\in X}\{\mu_{\tilde{R}}(x,y) \bigwedge [\bigvee_{i\in I}\mu_{\tilde{R}_i}(y,z)]\} = \bigvee_{i\in I}\{\bigvee_{y\in X}[\mu_{\tilde{R}}(x,y) \bigwedge \mu_{\tilde{R}_i}(y,z)]\} \\
&= \bigvee_{i\in I}\mu_{(\tilde{R}\circ\tilde{R}_i)}(x,z) = \bigcup_{i\in I}\mu_{(\tilde{R}\circ\tilde{R}_i)}(x,z).
\end{aligned}$$

Therefore (1) holds.

Proposition 1.4.5. (1) $\tilde{R} \circ (\bigcap_{i\in I} \tilde{R}_i) \subseteq \bigcap_{i\in I} \tilde{R} \circ \tilde{R}_i$;

(2) $(\bigcap_{i\in I} \tilde{R}_i) \circ \tilde{R} \subseteq \bigcap_{i\in I} \tilde{R}_i \circ \tilde{R}$.

Proof: Only prove (1).

$\forall i \in I$, $\bigcap_{i\in I} \tilde{R}_i \subseteq \tilde{R}_i$, hence $\forall (x,z) \in X \times X, \forall i \in I$, from Proposition 1.4.3, then

$$\mu_{[\tilde{R}\circ(\bigcap_{i\in I}\tilde{R}_i)]}(x,z) \leqslant \mu_{(\tilde{R}\circ\tilde{R}_i)}(x,z),$$

hence

$$\mu_{[\tilde{R}\circ(\bigcap\limits_{i\in I}\tilde{R}_i)]}(x,z) \leqslant \bigwedge \mu_{(\tilde{R}\circ\tilde{R}_i)}(x,z).$$

Therefore (1) holds.

Proposition 1.4.6. $(\tilde{R}_1 \circ \tilde{R}_2)^{-1} = \tilde{R}_2^{-1} \circ \tilde{R}_1^{-1}$.

Proof: $\forall (x,z) \in X \times X$, we have

$$\begin{aligned}\mu_{(\tilde{R}_1\circ\tilde{R}_2)^{-1}}(x,z) = \mu_{(\tilde{R}_1\circ\tilde{R}_2)}(z,x) &= \bigvee_{y\in X}[\mu_{\tilde{R}_1}(z,y) \wedge \mu_{\tilde{R}_2}(y,x)] \\ &= \bigvee_{y\in X}[\mu_{\tilde{R}_1^{-1}}(y,z) \wedge \mu_{\tilde{R}_2^{-1}}(x,y)] \\ &= \bigvee_{y\in X}[\mu_{\tilde{R}_2^{-1}}(x,y) \wedge \mu_{\tilde{R}_1^{-1}}(y,z)] \\ &= \mu_{(\tilde{R}_2^{-1}\circ\tilde{R}_1^{-1})}(x,z).\end{aligned}$$

Hence $(\tilde{R}_1 \circ \tilde{R}_2)^{-1} = \tilde{R}_2^{-1} \circ \tilde{R}_1^{-1}$.

Proposition 1.4.7. (1) $(\bigcup\limits_{i\in I}\tilde{R}_i)^{-1} = \bigcup\limits_{i\in I}\tilde{R}_i^{-1}$; (2) $(\bigcap\limits_{i\in I}\tilde{R}_i)^{-1} = \bigcap\limits_{i\in I}\tilde{R}_i^{-1}$.

Proof: Only prove (1).

$\forall (x,y) \in X \times X$,

$$\mu_{(\bigcup\limits_{i\in I}\tilde{R}_i)^{-1}}(x,y) = \mu_{(\bigcup\limits_{i\in I}\tilde{R}_i)}(y,x) = \mu_{\bigvee\limits_{i\in I}\tilde{R}_i}(y,x)$$
$$= \mu_{\bigvee\limits_{i\in I}\tilde{R}_i^{-1}}(x,y) = \mu_{(\bigcup\limits_{i\in I}\tilde{R}_i^{-1})}(x,y).$$

Therefore (1) holds.

Proposition 1.4.8. $(\tilde{R}^{-1})^{-1} = \tilde{R}$.

Definition 1.4.3. Suppose $\tilde{R}$ to be a fuzzy relation in X. If $\tilde{R}$ satisfies $\tilde{R}\circ\tilde{R} \subseteq \tilde{R}$, then $\tilde{R}$ is called a transitivity fuzzy relation.

Notice, if R is an ordinary relation on X; R is transitive if and only if $(x,y)\in R$ and $(y,z) \in R$, then $(x,z)\in R$. It is easy to understand transitivity in the Definition 1.4.3, when $\tilde{R}$ degenerates into the ordinary relation, with the ordinary transitivity being the same.

Proposition 1.4.9. *The union and intersection of symmetric fuzzy relation also are still symmetric.*

Proposition 1.4.10. *The intersection of transitive fuzzy relation is transitive.*

Proposition 1.4.11. *To arbitrarily fuzzy relation $\tilde{R}$, we have the following:*

(1) Existence of inclusive $\tilde{R}$ is the least symmetric fuzzy relation, that is the symmetric closure of $\tilde{R}$, recorded as $S(\tilde{R})$.

(2) Existence of the least transitive fuzzy relation contains $\tilde{R}$, that is the transitive closure of $\tilde{R}$, recorded as $T(\tilde{R})$.

Proof: Now, we only prove (1).

Use $\tilde{Q}$ to denote all sets of containments symmetric fuzzy relation $\tilde{R}$, because the whole relation $\tilde{X}$ is symmetric on X, i.e., $\tilde{X} \in \tilde{Q}$, as a result is not empty. Let $\tilde{S}_0 = \bigcap\{\tilde{S}|\tilde{S} \in \tilde{Q}\}$ from Proposition 1.4.9. Then $\tilde{S}_0$ is the least symmetric relation containing $\tilde{R}$.

Proposition 1.4.12. *Suppose $\tilde{R}_1$ and $\tilde{R}_2$ to be a symmetric fuzzy relation, then $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric $\Longleftrightarrow \tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$.*

Proof: "$\Longrightarrow$" Because $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric, then

$$\tilde{R}_1 \circ \tilde{R}_2 = (\tilde{R}_1 \circ \tilde{R}_2)^{-1} = \tilde{R}_2^{-1} \circ \tilde{R}_1^{-1} = \tilde{R}_2 \circ \tilde{R}_1.$$

"$\Longleftarrow$" If $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then

$$(\tilde{R}_1 \circ \tilde{R}_2)^{-1} = \tilde{R}_2^{-1} \circ \tilde{R}_1^{-1} = \tilde{R}_2 \circ \tilde{R}_1 = \tilde{R}_1 \circ \tilde{R}_2.$$

Therefore $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric.

Proposition 1.4.13. *If $\tilde{R}$ is transitive, then $\tilde{R}^{-1}$ is transitive.*

Proof: Because $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$, then from Proposition 1.4.6, $\forall (x, y) \in X \times X$, hence

$$\mu_{(\tilde{R}^{-1} \circ \tilde{R}^{-1})}(x, y) = \mu_{(\tilde{R} \circ \tilde{R})^{-1}}(x, y) = \mu_{(\tilde{R} \circ \tilde{R})}(y, x)$$
$$\leqslant \mu_{\tilde{R}}(y, x) = \mu_{\tilde{R}^{-1}}(x, y),$$

that is, $\tilde{R}^{-1}$ is transitive.

As for a series of propositions concerning fuzzy relation, the above is considered all for X fuzzy relations. We can throw away this restraint actually, that is, above-mentioned proposition holds as long as the synthesis exists.

1.4.3 Special Fuzzy Operators

Definition 1.4.4. For $\tilde{A},\ \tilde{B} \in \mathscr{F}$(X), its general form of union and intersection operation is defined as

$$\mu_{(\tilde{A} \bigcup \tilde{B})}(x) \triangleq \mu_{\tilde{A}}(x) \bigvee{}^{*} \mu_{\tilde{B}}(x),\ \ \mu_{(\tilde{A} \bigcap \tilde{B})}(x) \triangleq \mu_{\tilde{A}}(x) \bigwedge{}^{*} \mu_{\tilde{B}}(x).$$

Here $\bigvee^{*}$, $\bigwedge^{*}$ is binary operation in [0, 1], and is briefly called a fuzzy operator.

We take them as follows.

I. Max-product operator $(\bigvee, \cdot)$

$\mu_{\tilde{A}}(x) \bigvee \mu_{\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\};\ \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$ denotes an ordinary real number product method.

II. Boundary sum and product operator $(\oplus, \odot)$

$\mu_{\tilde{A}}(x) \oplus \mu_{\tilde{B}}(x) \triangleq \min\{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x),\ 1\},\quad \tilde{A} \odot \tilde{B} \triangleq \max\{0,\ \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\}$.

III. Probability sum and product operator $(\widehat{+}, \cdot)$

$$\mu_{\tilde{A}}(x)\widehat{+}\mu_{\tilde{B}}(x) \triangleq \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x).$$

It can be verified by using an elementary calculation that Operator I satisfies Operator $(\bigvee, \bigwedge)$ in accordance with a calculation law, but Operator II and III dissatisfy idempotent, absorptive and distributive laws.

1.5 Fuzzy Functions

A fuzzy function is one of the most important conceptions in a fuzzy optimum problem. Its discussion is divided into two parts [DPr80]. Besides, kinds of constraint functions repeatedly used in the book are introduced.

1.5.1 Fuzzy Function from Universe X to Another One Y

Definition 1.5.1. Let $\mathscr{F}(X)$ and $\mathscr{F}(Y)$ represent all fuzzy sets on universe X and Y, respectively. If there exists an ordinary mapping $f : \mathscr{F}(X) \to \mathscr{F}(Y)$, then we call f a fuzzy-valued function from X to Y, writing $\tilde{f} : X \sim\to Y$.

Definition 1.5.2. Let $\tilde{f} : X \sim\to Y, \tilde{g} : Y \sim\to Z$ be two fuzzy-valued functions. Then

$$\tilde{g} \circ \tilde{f} : \mathscr{F}(X) \to \mathscr{F}(Z),$$

i.e.,

$$\forall \tilde{A} \in \mathscr{F}(X), (\tilde{g} \circ \tilde{f})(\tilde{A}) \in \mathscr{F}(Z)$$

is called a compound fuzzy function of $\tilde{f}$ and $\tilde{g}$.

Proposition 1.5.1. *If $f : X \to Y, g : Y \to Z$ denote two ordinary mappings, two fuzzy functions $\tilde{f} : X \sim\to Y$ and $\tilde{g} : Y \sim\to Z$ can be obtained by means of the extension principle. Their compound under Definition 1.5.2 coincides with fuzzy functions in compound $g \circ f : X \to Z$ from ordinary mappings $f : X \to Y$ and $g : Y \to Z$ by means of the extension principle.*

Proof: As for $\forall \tilde{A} \in \mathscr{F}(X)$, the image

$$\mu_{\tilde{f}(\tilde{A})}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & f^{-1}(y) \neq \phi \\ 0, & f^{-1}(y) = \phi \end{cases}$$

obtained with fuzzy function $\tilde{f} : X \sim\to Y$ is extended from $f : X \to Y$ to arbitrary $y \in Y$. $\forall \tilde{B} \in \mathscr{F}(Y)$, the image

$$\mu_{\tilde{g}(\tilde{B})}(z) = \begin{cases} \sup\limits_{y \in g^{-1}(z)} \mu_{\tilde{B}}(y), & g^{-1}(z) \neq \phi \\ 0, & g^{-1}(z) = \phi \end{cases}$$

is achieved by a fuzzy function $\tilde{g} : Y \sim\rightarrow Z$ for $\forall z \in Z$, such that their compound denotes $\tilde{g} \circ \tilde{f} : X \sim\rightarrow Z$. From Definition 1.5.2, $\forall \tilde{A} \in \mathscr{F}(X), \forall z \in Z$, there exists

$$\begin{aligned}
\mu_{\tilde{g}\circ\tilde{f}(\tilde{A})}(z) &= \mu_{\tilde{g}(\tilde{f}(\tilde{A}))}(z) \\
&= \begin{cases} \sup\limits_{y\in g^{-1}(z)} \mu_{f(\tilde{A})}(y), & g^{-1}(z) \neq \phi \\ 0, & g^{-1}(z) = \phi \end{cases} \\
&= \begin{cases} \sup\limits_{y\in g^{-1}(z)} \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x), & f^{-1}(y) \neq \phi, \\ 0, & f^{-1}(y) = \phi, \end{cases} & g^{-1}(z) \neq \phi \\ 0, & g^{-1}(z) = \phi \end{cases} \\
&= \begin{cases} \sup\limits_{y\in g^{-1}(z)} \sup\limits_{x\in f^{-1}(y)} \mu_{\tilde{A}}(x), & f^{-1}(y) \neq \phi, g^{-1}(z) \neq \phi, \\ 0, & f^{-1}(y) = \phi \text{ or } g(z) = \phi; \end{cases}
\end{aligned}$$

therefore, $\forall z \in Z$,

$$\begin{aligned}
\mu_{\tilde{g}\circ\tilde{f}(\tilde{A})}(z) &= \begin{cases} \sup\limits_{x\in f^{-1}(g^{-1}(z))} \mu_{\tilde{A}}(x), & f^{-1}(g^{-1}(z)) \neq \phi \\ 0, & f^{-1}(g^{-1}(z)) = \phi \end{cases} \\
&= \begin{cases} \sup\limits_{x\in (y\circ f)^{-1}(z)} \mu_{\tilde{A}(x)}, & (g\circ f)^{-1}(z) \neq \phi, \\ 0, & (g\circ f)^{-1}(z) = \phi. \end{cases}
\end{aligned} \tag{1.5.1}$$

On the right of Formula (1.5.1) is a fuzzy function gained from the ordinary compound mapping $g \circ f$ by means of an extension principle.

1.5.2 Fuzzy Functions from Fuzzy Set $\tilde{A}$ to Another One $\tilde{B}$

Definition 1.5.3. Let $f : X \rightarrow Y$ be an ordinary mapping. If fuzzy sets $\tilde{A}$ and $\tilde{B}$ are defined on X and Y, respectively, we have $\tilde{B}\big(f(x)\big) = \mu_{\tilde{A}}(x)$ for $\forall x \in X$, then we call $\tilde{f}$ a fuzzy-valued function from fuzzy set $\tilde{A}$ to $\tilde{B}$, writing

$$\tilde{f} : \tilde{A} \sim\rightarrow \tilde{B}.$$

Let $\tilde{A} \in \mathscr{F}(X), \tilde{B} \in \mathscr{F}(Y), \tilde{C} \in \mathscr{F}(Z)$, and let $f : \tilde{A} \sim\rightarrow \tilde{B}$ and $g : \tilde{B} \sim\rightarrow \tilde{C}$. Then a composite mapping $g \circ f : X \rightarrow Z$ is a fuzzy function from $\tilde{A}$ to $\tilde{C}$, i.e, $\tilde{g} \circ \tilde{f} : \tilde{A} \sim\rightarrow \tilde{C}$.

In fact, $\forall x \in X, \mu_{(\tilde{g}\circ\tilde{f})}(x) = \tilde{g}\big(\tilde{f}(x)\big)$, hence,

$$\begin{aligned}
\mu_{(\tilde{C}\circ(\tilde{g}\circ\tilde{f}))}(x) &= \tilde{C}((\tilde{g}\circ\tilde{f})(x)) = \tilde{C}(\tilde{g}(\tilde{f}(x))) \\
&= (\tilde{C}\circ\tilde{g})(\tilde{f}(x)) = \tilde{B}(\tilde{f}(x)) = \mu_{\tilde{A}}(x),
\end{aligned}$$

i.e.,

$$\tilde{C} \circ (\tilde{g} \circ \tilde{f}) = (\tilde{C} \circ \tilde{f}) \circ \tilde{f} = \tilde{B} \circ \tilde{f} = \tilde{A}.$$

1.5.3 Fuzzy Constrained Function

We introduce some fuzzy constrained functions constantly used for the sake of discussion [Cao93a][Cao94b][Cao07][DPr80].

Definition 1.5.4. $\forall x \in X$, $g(x)$ is a real bounded function defined on X, and its infimum and supremum are written as $\inf(g)$ and $\sup(g)$, respectively, such that we define

$$\mu_{\tilde{M}}(x) = \left[\frac{g(x) - \inf(g)}{\sup(g) - \inf(g)} \right]^n, \tag{1.5.2}$$

calling $\tilde{M} : X \to [0, 1]$ a maximal set of g, where $\mu_{\tilde{M}}(x) \neq 0$, n is a natural number.

Definition 1.5.5. If c_i^1, c_i^2 are left and right endpoints of an interval, then, for $\tilde{c}_i$ freely fixed in a closed value interval $[c_i^1, c_i^2]$, its degree of accomplishment is determined by

$$\mu_{\tilde{\phi}_i}(\tilde{c}_i) = \begin{cases} 0, & \text{if} \quad c_i \leqslant c_i^1, \\ \left(\dfrac{c_i - c_i^1}{c_i^2 - c_i^1} \right)^n, & \text{if} \quad c_i^1 < c_i \leqslant c_i^2, \\ 1, & \text{if} \quad c_i > c_i^2, \end{cases} \tag{1.5.3}$$

where n denotes a natural number.

For fuzzy constraint sets and fuzzy objective sets, we have the following.

Definition 1.5.6. If $\tilde{A}_i = \{x \in \mathscr{R}^m | g_i(x) \lesssim 1\}$ $(1 \leqslant i \leqslant p)$ is a fuzzy constraint set corresponding to fuzzy constraint inequations $g_i(x) \lesssim 1$, then the membership functions of $\tilde{A}_i$ are

$$\mu_{\tilde{A}_i}(x) = \begin{cases} 0, & \text{if} \quad g_i(x) \geqslant 1 + d_i, \\ \left(1 - t_i/d_i\right)^n, & \text{if} \quad g_i(x) = 1 + t_i, 0 \leqslant t_i \leqslant d_i, \\ 1, & \text{if} \quad g_i(x) \leqslant 1, \end{cases} \tag{1.5.4}$$

where $d_i \in \mathscr{R}$ (a real number set) denotes a maximum flexible index of $g_i(x)$.

Definition 1.5.7. Regard $\tilde{A}_0 = \{x \in \mathscr{R}^m | g_0(x) \gtrsim z_0\}$ as a fuzzy objective set and assume a membership function of $\tilde{A}_0$ as follows:

$$\mu_{\tilde{A}_0}(x) = \begin{cases} 0, & \text{if} \quad g_0(x) \geqslant z_0 - d_0, \\ \left(1 - t_0/d_0\right)^n, & \text{if} \quad g_0(x) = z_0 - t_0, 0 \leqslant t_0 \leqslant d_0, \\ 1, & \text{if} \quad g_0(x) \leqslant z_0, \end{cases} \tag{1.5.5}$$

where $d_0 \geqslant 0$ is a maximum flexible index of $g_0(x)$ and z_0 an objective value.

We define symbol " $\lesssim$ " as a flexible version of $\leqslant$ at a 'certain degree' [Ver84][LL01], or approximately less than or equal to.

Definition 1.5.8. Let fuzzy sets $\tilde{A}_i(1 \leqslant i \leqslant p)$ be

$$\tilde{A}_i = \{x \in \mathscr{R}^m | g_i(x) \lesssim 1\} \ (0 \leqslant i \leqslant p')$$

and

$$\tilde{A}_i = \{x \in \mathscr{R}^m | g_i(x) \gtrsim 1\} \ (p' + 1 \leqslant i \leqslant p).$$

Then their membership functions are defined as

$$\mu_{\tilde{A}_i}(x) = \begin{cases} 1, & g_i(x) \leqslant 1 \\ e^{-\frac{1}{d_i}(g_i(x)-1)}, & 1 < g_i(x) \leqslant 1 + d_i \end{cases} \tag{1.5.6}$$

for $0 \leqslant i \leqslant p'$, and

$$\mu_{\tilde{A}_i}(x) = \begin{cases} 0, & g_i(x) \leqslant 1 \\ 1 - e^{-\frac{1}{d_i}(g_i(x)-1)}, & 1 < g_i(x) \leqslant 1 + d_i \end{cases} \tag{1.5.7}$$

for $p' + 1 \leqslant i \leqslant p$, where $d_i \geqslant 0$ is a maximum flexible index of $i-$th function $g_i(x)$.

We introduce the possibility grade of dominance of $\tilde{1}$ over $\tilde{g}_i(x)$, a concept introduced by Dubois and Prade in 1980 which represents the fuzzy extension for $g_i(x) \leqslant 1$ [DPr80].

Definition 1.5.9. The degree of possibility of $\tilde{g}_i(x) \leqslant \tilde{1}$ is defined as

$$v(\tilde{g}_i(x) \leqslant \tilde{1}) = \sup_{x,y:x \geqslant y} \min(\mu_{\tilde{1}}(x), \mu_{\tilde{g}_i(x)}(y)).$$

This formula is an extension of the inequality $x \geqslant y$ according to the extension principle. When pair (x, y) exists, such that $x \geqslant y$ and $\mu_{\tilde{1}}(x) = \mu_{\tilde{g}_i(x)}(y) = 1$, then $v(\tilde{g}_i(x) \leqslant \tilde{1}) = 1$.

When $\tilde{g}_i(x)$ and $\tilde{1}$ are convex fuzzy numbers, we have

$$v(\tilde{g}_i(x) \leqslant \tilde{1}) = 1, \text{ if and only if } \ g_i(x) \leqslant 1,$$
$$v(\tilde{g}_i(x) \leqslant \tilde{1}) = \text{ hgt } (\tilde{g}_i(x) \bigcap \tilde{1}) = \mu_{\tilde{1}}(d),$$

where d is an ordinate of the highest intersection point between $\mu_{\tilde{1}}(x)$ and $\mu_{\tilde{g}_i(x)}(y)$.

1.6 Three Mainstream Theorems in Fuzzy Mathematics

1.6.1 Decomposition Theorem

Definition 1.6.1. If $\alpha \in [0, 1], \tilde{A} \in \mathscr{F}(\mathrm{X})$, then product of number α with fuzzy set $\tilde{A}$ is defined as

$$\mu_{(\alpha \tilde{A})}(x) = \alpha \bigwedge \mu_{\tilde{A}}(x).$$

Theorem 1.6.1. (Decomposition Theorem I) *For an arbitrary* $\tilde{A} \in \mathscr{F}(X)$, *we have*

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha, \tag{1.6.1}$$

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha A_{\underset{\cdot}{\alpha}}. \tag{1.6.2}$$

Proof: Because

$$\mu_{A_\alpha}(x) = \begin{cases} 1, & x \in A_\alpha \\ 0, & x \notin A_\alpha, \end{cases}$$

then

$$\begin{aligned} \mu_{(\bigcup\limits_{\alpha \in [0,1]} \alpha A_\alpha)}(x) &= \sup_{0 \leqslant \alpha \leqslant 1} \alpha \mu_{A_\alpha}(x) = \sup_{x \in A_\alpha} \alpha \\ &= \sup_{\alpha \leqslant \mu_{\tilde{A}}(x)} \alpha = \mu_{\tilde{A}}(x). \end{aligned}$$

Therefore, (1.6.1) is proved.

Similarly, we can prove Formula (1.6.2).

Example 1.6.1: Suppose universe to be $X = \{2, 1, 7, 6, 9\}$, try to decompose the fuzzy set

$$\tilde{A} = \frac{0.1}{2} + \frac{0.3}{1} + \frac{0.5}{7} + \frac{0.9}{6} + \frac{1}{9}$$

by applying Decomposition Theorem.

Solution: The relevant cut-sets in fuzzy sets are

$$\begin{array}{ll} A_{0.1} = X, & 0 < \alpha \leqslant 0.1, \\ A_{0.3} = \{1,\ 7,\ 6,\ 9\}, & 0.1 < \alpha \leqslant 0.3, \\ A_{0.5} = \{7,\ 6,\ 9\}, & 0.3 < \alpha \leqslant 0.5, \\ A_{0.9} = \{6,\ 9\}, & 0.5 < \alpha \leqslant 0.9, \\ A_1\ = \{9\}, & 0.9 < \alpha \leqslant 1, \end{array}$$

$$\begin{aligned} \tilde{A} &= \bigcup_{\alpha \in [0,1]} \alpha A_\alpha \\ &= \bigcup_{0<\alpha \leqslant 0.1} \alpha \left(\frac{1}{2} + \frac{1}{1} + \frac{1}{7} + \frac{1}{6} + \frac{1}{9} \right) \bigcup_{0.1<\alpha \leqslant 0.3} \alpha \left(\frac{1}{1} + \frac{1}{7} + \frac{1}{6} + \frac{1}{9} \right) \\ &\quad \bigcup_{0.3<\alpha \leqslant 0.5} \alpha \left(\frac{1}{7} + \frac{1}{6} + \frac{1}{9} \right) \bigcup_{0.5<\alpha \leqslant 0.9} \alpha \left(\frac{1}{6} + \frac{1}{9} \right) \bigcup_{0.9<\alpha \leqslant 1} \alpha \left(\frac{1}{9} \right) \\ &= 0.1 A_{0.1} \bigcup 0.3 A_{0.3} \bigcup 0.5 A_{0.5} \bigcup 0.9 A_{0.9} \bigcup 1 A_1. \end{aligned}$$

Definition 1.6.2. Suppose X to be universe of discourse. If a fuzzy set-valued mapping H: [0,1]$\rightarrow P(X), \alpha \mapsto H(\alpha)$ satisfies

$$\alpha_1 < \alpha_2 \Rightarrow H(\alpha_1) \supseteq H(\alpha_2), \forall\ \alpha_1, \alpha_2 \in [0,1],$$

then H is called a collection sleeve on X, written as $\mathbf{H}(X)$.

We can get more general decomposition theorem by using Definition 1.6.2.

Theorem 1.6.2. (Decomposition Theorem II) *Let $\tilde{A} \in \mathscr{F}(X)$. If there exists a set value mapping*

$$\begin{aligned} H : [0,1] &\Longrightarrow \mathscr{F}(X), \\ \alpha &\longmapsto H(\alpha), \end{aligned}$$

such that $\forall \alpha \in [0,1], A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha$, *then*

$$\begin{aligned} &(1) \quad \tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha H(\alpha). \\ &(2) \quad \alpha_1 < \alpha_2 \Longrightarrow H(\alpha_1) \supset H(\alpha_2). \\ &(3) \quad A_\alpha = \bigcap_{\lambda < \alpha} H(\lambda), \alpha \neq 0, \quad A_{\underset{\cdot}{\alpha}} = \bigcup_{\lambda > \alpha} H(\lambda), \alpha \neq 1. \end{aligned}$$

Proof:

(1) Because $A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha$, then

$$\begin{aligned} &\alpha A_{\underset{\cdot}{\alpha}} \subseteq \alpha H(\alpha) \subseteq \alpha A_\alpha \\ \Longrightarrow\ &\tilde{A} = \bigcup_{\alpha \in [0,1)} \alpha A_{\underset{\cdot}{\alpha}} \subseteq \bigcup_{\alpha \in [0,1]} \alpha H(\alpha) \subseteq \bigcup_{\alpha \in [0,1]} \alpha A_\alpha = \tilde{A} \\ \Longrightarrow\ &\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha H(\alpha). \end{aligned}$$

Therefore, (1) is proved.

(2) Because $\forall x \in X$, we have

$$x \in A_{\alpha_2} \Longrightarrow \mu_{\tilde{A}}(x) \geqslant \alpha_2 > \alpha_1 \Longrightarrow x \in A_{\underset{\cdot}{\alpha_1}},$$

therefore

$$\alpha_1 < \alpha_2 \Longrightarrow H(\alpha_1) \supseteq A_{\underset{\cdot}{\alpha_1}} \supseteq A_{\alpha_2} \supseteq H(\alpha_2).$$

(3) Because $\forall \lambda < \alpha, H(\lambda) \supseteq A_{\underset{\cdot}{\lambda}} \supseteq A_\alpha \Longrightarrow \bigcap_{\lambda < \alpha} H(\lambda) \supseteq A_\alpha, \quad \alpha \neq 0.$

Again $$\bigcap_{\lambda < \alpha} H(\lambda) \subseteq \bigcap_{\lambda < \alpha} A_\lambda = A_{(\bigvee_{\lambda < \alpha} \lambda)} = A_\alpha, \quad \alpha \neq 0,$$

hence we have $$A_\alpha = \bigcap_{\lambda < \alpha} H(\lambda).$$

The second formula can be proved similarly.

From the knowledge of Decomposition Theorem II, fuzzy sets $\tilde{A}$ can be vividly represented by a collection of set $H(A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha)$, so $H(\alpha)$ has more extensive application actually.

1.6.2 Extension Principle

Theorem 1.6.3. (Extension Principle I, Lotfi A. Zadeh)[Zad65a,b] *Let $f : X \to Y$ be an ordinary point function and $\tilde{A} \in \mathscr{F}(X)$. Two mappings can be induced by f and f^{-1} as follows:*

$$f:\ \mathscr{F}(X)\to\mathscr{F}(Y), \qquad\qquad f^{-1}:\mathscr{F}(Y)\to\mathscr{F}(X),$$
$$\tilde{A}\mapsto f(\tilde{A})\in\mathscr{F}(Y), \qquad\qquad \tilde{B}\mapsto f^{-1}(\tilde{B})\in\mathscr{F}(X),$$

whose membership functions are denoted by

$$\mu_{f(\tilde{A})}(y)\triangleq\begin{cases}\bigvee\limits_{f(x)=y}\mu_{\tilde{A}}(x), & f^{-1}(y)\neq\phi,\\ 0, & f^{-1}(y)=\phi,\end{cases}$$
$$\mu_{f^{-1}(\tilde{B})}(x)\triangleq\tilde{B}(f(x)),\ y=f(x),$$

respectively. $f(\tilde{A})$ *is called an image of* $\tilde{A}$ *under* f *and* $f^{-1}(\tilde{B})$ *an inverse image of* $\tilde{B}$.

The representation of an α-cut set in extension principle.

Theorem 1.6.4. (Extension Principle II) *Let mapping* $f : X \to Y$ *be extended as mapping* $f : \mathscr{F}(X) \to \mathscr{F}(Y)$ *and* $f^{-1} : \mathscr{F}(Y) \to \mathscr{F}(X)$*. Then* $\forall\alpha\in[0,1], \tilde{A}\in\mathscr{F}(X), \tilde{B}\in\mathscr{F}(Y)$*, we have*

1^0 $f(\tilde{A})_{\underset{\cdot}{\alpha}}=f(A_{\underset{\cdot}{\alpha}})$.
2^0 $f^{-1}(\tilde{B})_{\underset{\cdot}{\alpha}}=f^{-1}(B_{\underset{\cdot}{\alpha}})$.
3^0 $f^{-1}(\tilde{B})_{\alpha}=f^{-1}(B_{\alpha})$.

Here, $f(\tilde{A})_\alpha$ *is a simplification of* $(f(\tilde{A}))_\alpha$.

Proof: Only 1^0 is proved and the others can be verified in a similar way.
Because

$$\begin{aligned} y\in f(\tilde{A})_{\underset{\cdot}{\alpha}} &\Longleftrightarrow \mu_{f(\tilde{A})}(y)>\alpha\\ &\Longleftrightarrow \bigvee_{f(x)=y}\mu_{\tilde{A}}(x)>\alpha\\ &\Longleftrightarrow \exists x\in X,\ \text{satisfying } f(x)=y,\ \text{such that } \mu_{\tilde{A}}(x)>\alpha\\ &\Longleftrightarrow \exists x\in X,\ \text{satisfying } f(x)=y,\ \text{such that } x\in A_{\underset{\cdot}{\alpha}}\\ &\Longleftrightarrow y\in f(A_{\underset{\cdot}{\alpha}}),\end{aligned}$$

consequently, 1^0 is proved.

Corollary 1.6.1. (Extension Principle III) *Let* $f : X \to Y, \tilde{A}\in\mathscr{F}(X), \tilde{B}\in\mathscr{F}(Y)$ *and* f^{-1}*. Then*

$$f(\tilde{A})=\bigcup_{\alpha\in[0,1)}\alpha f(A_{\underset{\cdot}{\alpha}}),$$
$$f^{-1}(\tilde{B})=\bigcup_{\alpha\in[0,1]}\alpha f^{-1}(B_{\alpha}),$$
$$f^{-1}(\tilde{B})=\bigcup_{\alpha\in[0,1)}\alpha f^{-1}(B_{\underset{\cdot}{\alpha}}).$$

Algebraically, a representation method is given to an ordinary set of fuzzy set; it is related to the extension of a fuzzy set by a classic set from another view, especially representation theorem is advanced. A more visual "collection sleeves" method is shown as follows.

1.6.3 Representation Theorem

Definition 1.6.3. Given $H_1, H_2 \in \mathbf{H}(X)$, if $\forall \alpha \in [0,1]$, we have $\bigcap\limits_{\lambda<\alpha} H_1(\lambda) = \bigcap\limits_{\lambda<\alpha} H_2(\lambda)$, then call H_1 and H_2 equivalence, written as $H_1 \sim H_2$.

Obviously, relation " $\sim$ " satisfies

(1) $H \sim H, \forall H \in \mathbf{H}(X)$ (reflection);
(2) $H_1 \sim H_2 \Rightarrow H_2 \sim H_1$ (symmetry);
(3) $H_1 \sim H_2$ and again $H_2 \sim H_3 \Rightarrow H_1 \sim H_3$ (transitivity);

which are written as $\mathbf{H}'(X) = \{|H| | H \in \mathbf{H}(X)\} = \mathbf{H}(X)/\sim$, where the class is $|H| = \{H' | H' \sim H\}$.

Definition 1.6.4. Suppose $H, H_1, H_2 \in \mathbf{H}(X)$, and $H_t \in \mathbf{H}(X) (\forall t \in T)$ with $\forall \alpha \in [0,1]$, then

(1) Contain $\subseteq$: $H_1 \subseteq H_2 \Leftrightarrow H_1(\alpha) \subseteq H_2(\alpha)$;
(2) Union $\bigcup\limits_{t\in T} H_t$: $(\bigcup\limits_{t\in T} H_t)(\alpha) \triangleq \bigcup\limits_{t\in T} H_t(\alpha)$;
(3) Intersection $\bigcap\limits_{t\in T} H_t$: $(\bigcap\limits_{t\in T} H_t)(\alpha) \triangleq \bigcap\limits_{t\in T} H_t(\alpha)$;
(4) Complement H^c : $H^c(\alpha) \triangleq (H(1-\alpha))^c$.

Theorem 1.6.5. (Representation Theorem) *Let $H \in \mathbf{H}(X)$. Then $\bigcup\limits_{\alpha\in[0,1]} \alpha H(\alpha)$ is a fuzzy set on X, writing $\tilde{A}$, and $\forall \lambda, \alpha \in [0,1]$:*

(1) $A_\alpha = \bigcap\limits_{\lambda<\alpha} H(\alpha), \quad \alpha \neq 0.$
(2) $A_{\underset{\cdot}{\alpha}} = \bigcup\limits_{\lambda>\alpha} H(\alpha), \quad \alpha \neq 1.$

Proof: According to a number-and-set-product definition, $\forall \alpha \in [0,1], H(\alpha) \in P(X)$, then $\alpha H(\alpha) \in \mathbf{H}(X)$, such that

$$\bigcup_{\alpha\in[0,1]} \alpha H(\alpha) \in \mathbf{H}(X),$$

written as $\tilde{A} = \bigcup\limits_{\alpha\in[0,1])} \alpha H(\alpha)$.

According to Decomposition Theorem II, if condition $A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha$ is satisfied, then (1) and (2) can be got. Prove that this condition holds below.

$$\begin{aligned}
\forall \alpha \in [0,1] : x \in A_{\underset{\cdot}{\alpha}} &\Longrightarrow \mu_{\tilde{A}}(x) > \alpha \\
&\Longrightarrow [\bigcup_{\lambda\in[0,1]} \lambda H(\lambda)](x) > \alpha \\
&\Longrightarrow \bigvee_{\lambda\in[0,1]} [\lambda \bigwedge H(\lambda)(x)] > \alpha \\
&\Longrightarrow \exists \alpha_0 \in [0,1], \text{such that } \alpha_0 \bigwedge H(\alpha_0)(x) > \alpha \\
&\Longrightarrow \alpha_0 > \alpha \text{ and } H(\alpha_0)(x) = 1 \\
&\Longrightarrow x \in H(\alpha_0) \subseteq H(\alpha), \quad (\alpha \neq 1);
\end{aligned}$$

$$\begin{aligned} x \in H(\alpha) &\Longrightarrow H(\alpha)(x) = 1 \\ &\Longrightarrow \bigvee_{\lambda \in [0,1]} [\lambda \bigwedge H(\lambda)(x)] \geqslant \alpha \bigwedge H(\alpha)(x) = \alpha \\ &\Longrightarrow \mu_{\tilde{A}}(x) \geqslant \alpha \Longrightarrow x \in A_\alpha. \end{aligned}$$

Hence, condition $A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha$ holds.

Corollary 1.6.2. *Suppose $H \in \mathbf{H}(X)$, if we write $\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha H(\alpha)$, then*

(1) $\forall \alpha \in [0,1], A_{\underset{\cdot}{\alpha}} \subseteq H(\alpha) \subseteq A_\alpha$,
(2) $\mu_{\tilde{A}}(x) = \bigvee_{x \in H(\alpha)} \alpha$.

It is easer to obtain them by Theorem 1.6.5.

Example 1.6.2: Suppose $X = [-1, 1]$, a collection of sets in X is

$$H(\alpha) = [\alpha^2 - 1, 1 - \alpha^2], \quad \alpha \in [0,1],$$

compute membership function of a fuzzy set $\tilde{A}$ determined by H.

Solution: Because

$$\mu_{\tilde{A}}(x) = \bigvee_{x \in H(\alpha)} \alpha, \quad \alpha \in [0,1],$$

when $-1 \leqslant x \leqslant 0$, i.e., $x = \alpha^2 - 1$,

$$\mu_{\tilde{A}}(x) = \bigvee_{\alpha = \sqrt{1+x}} \alpha = \sqrt{1+x};$$

when $0 < x \leqslant 1$, i.e., $x = 1 - \alpha^2$,

$$\mu_{\tilde{A}}(x) = \bigvee_{\alpha = \sqrt{1-x}} \alpha = \sqrt{1-x}.$$

Therefore

$$\mu_{\tilde{A}}(x) = \begin{cases} \sqrt{1+x}, -1 \leqslant x \leqslant 0, \\ \sqrt{1-x}, 0 < x \leqslant 1. \end{cases}$$

Its figure is shown as Figure 1.6.1:

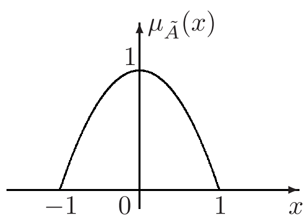

Figure 1.6.1 $\mu_{\tilde{A}}(x)$

1.7 Five-Type Fuzzy Numbers

In this section, we shall discuss properties of five types of fuzzy numbers including interval-type, $(\cdot, c)$-type, T-type, L-R-type and flat-type fuzzy numbers.

Because number 0 is an especial example in interval number $\bar{0}$ and fuzzy number $\tilde{0}$, in this book, 0 denotes them by adopting all the same mark.

1.7.1 Interval and Fuzzy Numbers

Definition 1.7.1. Let $\mathscr{R}$ denote a real number set. We call c, d interval numbers, written as $c, d \in I_{\mathscr{R}}$, where $I_{\mathscr{R}} = \{[c_i, d_i] | c_i < d_i, c_i, d_i \in \mathscr{R}, (i = 1, 2)\}$ is a set consisting of all interval numbers.

If $c = [c_1, c_2], d = [d_1, d_2]$, the operation of defined interval numbers is as follows:

$$\begin{aligned}
&c + d = [c_1 + d_1, c_2 + d_2],\\
&c - d = [(c_1 - d_1) \wedge (c_2 - d_2), (c_1 - d_1) \vee (c_2 - d_2)],\\
&c \cdot d = [c_1 d_1 \wedge c_1 d_2 \wedge c_2 d_1 \wedge c_2 d_2, c_1 d_1 \vee c_1 d_2 \vee c_2 d_1 \vee c_2 d_2],\\
&c \div d = [\frac{c_1}{d_1} \wedge \frac{c_1}{d_2} \wedge \frac{c_2}{d_1} \wedge \frac{c_2}{d_2}, \frac{c_1}{d_1} \vee \frac{c_1}{d_2} \vee \frac{c_2}{d_1} \vee \frac{c_2}{d_2}],\\
&c \vee d = [c_1 \vee d_1, c_2 \vee d_2],\\
&c \wedge d = [c_1 \wedge d_1, c_2 \wedge d_2].
\end{aligned}$$

Theorem 1.7.1. *Given $c, d \in I_{\mathscr{R}}$, then $c * d \in I_{\mathscr{R}}$, where "$*$" denotes algebra operations $\{+, -, \cdot, \div, \vee, \wedge\}$ on $\mathscr{R}$.*

Fuzzy numbers are obtained by applying the extension principle. From now on, $\mathscr{F}(\mathscr{R})$ represents the set of real fuzzy numbers.

Definition 1.7.2. Given that $\tilde{c}, \tilde{d}$ denote fuzzy numbers, written as $\tilde{c}, \tilde{d} \in \mathscr{F}(\mathscr{R})$, with α-cut of $\tilde{c}$ and $\tilde{d}$ being

$$\bar{c}_\alpha = [c_1(\alpha), c_2(\alpha)],\ \bar{d}_\alpha = [d_1(\alpha), d_2(\alpha)], \alpha \in [0, 1],$$

respectively, so that the operations of a fuzzy number are defined as follows:

$$\begin{aligned}
\tilde{c} + \tilde{d} &= \bigcup_{\alpha \in [0,1]} \alpha(\bar{c}_\alpha + \bar{d}_\alpha) = \bigcup_{\alpha \in [0,1]} \alpha[c_1(\alpha) + d_1(\alpha), c_2(\alpha) + d_2(\alpha)];\\
\tilde{c} - \tilde{d} &= \bigcup_{\alpha \in [0,1]} \alpha(\bar{c}_\alpha - \bar{d}_\alpha)\\
&= \bigcup_{\alpha \in [0,1]} \alpha[(c_1(\alpha) - d_1(\alpha)) \wedge (c_2(\alpha) - d_2(\alpha)),\\
&\qquad (c_1(\alpha) - d_1(\alpha)) \vee (c_2(\alpha) - d_2(\alpha))];\\
\tilde{c} \cdot \tilde{d} &= \bigcup_{\alpha \in [0,1]} \alpha(\bar{c}_\alpha \cdot \bar{d}_\alpha)\\
&= \bigcup_{\alpha \in [0,1]} \alpha[c_1(\alpha) d_1(\alpha) \wedge c_1(\alpha) d_2(\alpha) \wedge c_2(\alpha) d_1(\alpha) \wedge c_2(\alpha) d_2(\alpha),\\
&\qquad c_1(\alpha) d_1(\alpha) \vee c_1(\alpha) d_2(\alpha) \vee c_2(\alpha) d_1(\alpha) \vee c_2(\alpha) d_2(\alpha)];\\
\tilde{c} \div \tilde{d} &= \bigcup_{\alpha \in [0,1]} \alpha(\bar{c}_\alpha \div \bar{d}_\alpha)\\
&= \bigcup_{\alpha \in [0,1]} \alpha[\frac{c_1(\alpha)}{d_1(\alpha)} \wedge \frac{c_1(\alpha)}{d_2(\alpha)} \wedge \frac{c_2(\alpha)}{d_1(\alpha)} \wedge \frac{c_2(\alpha)}{d_2(\alpha)},
\end{aligned}$$

$$\frac{c_1(\alpha)}{d_1(\alpha)} \vee \frac{c_1(\alpha)}{d_2(\alpha)} \vee \frac{c_2(\alpha)}{d_1(\alpha)} \vee \frac{c_2(\alpha)}{d_2(\alpha)}];$$
$$\tilde{c} \vee \tilde{d} = \bigcup_{\alpha\in[0,1]} \alpha(\bar{c}_\alpha \vee \bar{d}_\alpha) = \bigcup_{\alpha\in[0,1]} \alpha[c_1(\alpha) \vee d_1(\alpha), c_2(\alpha) \vee d_2(\alpha)];$$
$$\tilde{c} \wedge \tilde{d} = \bigcup_{\alpha\in[0,1]} \alpha(\bar{c}_\alpha \wedge \bar{d}_\alpha) = \bigcup_{\alpha\in[0,1]} \alpha[c_1(\alpha) \wedge d_1(\alpha), c_2(\alpha) \wedge d_2(\alpha)].$$

Theorem 1.7.2. *Let* $\tilde{c}, \tilde{d} \in \mathscr{F}(\mathscr{R})$. *Then* $\tilde{c} * \tilde{d} \in \mathscr{F}(\mathscr{R})$.

It is easy to prove the two theorems above similar to the corresponding theorems in Ref. [LW92], [Luo84a] and [Luo84b].

Definition 1.7.3. $\tilde{c} \in \mathscr{F}(\mathscr{R})$ is called a fuzzy number, where $\mathscr{R}$ represents the set of whole real numbers, if

i) $\tilde{c}$ is normal, i.e., $x_0 \in \mathscr{R}$ exists, such that $\mu_{\tilde{c}}(x_0) = 1$.

ii) $\forall\alpha \in (0,1], \bar{c}_\alpha$ is a closed interval.

Theorem 1.7.3. *Let* $\tilde{c} \in \mathscr{F}(\mathscr{R})$ *be a fuzzy number. Then*

i) $\tilde{c}$ *is fuzzy convex.*

ii) If $\mu_{\tilde{c}}(x_0) = 1$, *then* $\mu_{\tilde{c}}(x)$ *is nondecreasing for* $x \leqslant x_0$ *and* $\mu_{\tilde{c}}(x)$ *nonincreasing for* $x \geqslant x_0$.

Proof: Because $\bar{c}_\alpha(\alpha \in (0,1])$ is the closed interval, $c_0 = \mathscr{R}$, i.e., $\forall\alpha \in [0,1], \bar{c}_\alpha$ is a convex set. $\tilde{c}$ can be proved to be fuzzy convex according to Theorem 1.3.1.

Now, take $x_1 < x_2 \leqslant x_0$ and let $\alpha = \mu_{\tilde{c}}(x_1)$. Since $\mu_{\tilde{c}}(x_0) = 1$, then $[x_1, x_0] \subset \bar{c}_\alpha$, hence $x_2 \in \bar{c}_\alpha$, such that $\mu_{\tilde{c}}(x) \geqslant \alpha$, i.e., $\mu_{\tilde{c}}(x_1) \leqslant \mu_{\tilde{c}}(x_2)$.

Similarly, $\mu_{\tilde{c}}(x_2) \leqslant \mu_{\tilde{c}}(x_1)$ can be proved if $x_0 \leqslant x_1 < x_2$.

Overall, the theorem holds.

Theorem 1.7.4. *Let* $\tilde{c} \in \mathscr{F}(R)$ *and* $\sup \tilde{c}$ *be bounded. Then* $\tilde{c}$ *is a fuzzy number* $\Leftrightarrow$ *there exists interval* $[c_1, c_2]$, *such that*

$$\mu_{\tilde{c}}(x) = \begin{cases} 1, & x \in [c_1, c_2] \neq \phi, \\ L(x), & x < c_1, \\ R(x), & x > c_2, \end{cases} \tag{1.7.1}$$

where $L(x)$ *represents an increasing function of right continuance* $(0 \leqslant L(x) < 1)$; $R(x)$ *represents a decreasing one of left continuance* $(0 \leqslant R(x) < 1)$.

Proof: *Necessity.* Let $\tilde{c} \in \mathscr{F}(\mathscr{R})$. Then

(1) Because $\tilde{c}$ is a closed convex set, $\tilde{c} = [c_1, c_2]$ and $\mu_{\tilde{c}}(x) = 1$ on $[c_1, c_2]$. It is obvious that $\mu_{\tilde{c}}(x) < 1$ for $x \notin [c_1, c_2]$.

(2) Because $\tilde{c} \in \mathscr{F}(\mathscr{R}), \forall\alpha \in [0,1]$, $\bar{c}_\alpha$ is a closed interval, we assume $\bar{c}_\alpha = [c_{1\alpha}, c_{2\alpha}] \subset [0,1]$, then

$$\tilde{c} = \bigcup_{\alpha\in(0,1]} \alpha\bar{c}_\alpha = \bigcup_{\alpha\in(0,1]} \alpha[c_{1\alpha}, c_{2\alpha}].$$

As for $x < c_1$,

$$L(x) = \mu_{\tilde{c}}(x) = \bigvee_{\alpha \in (0,1]} \alpha \wedge \chi_{[c_{1\alpha}, c_{2\alpha}]}(x)$$
$$= \bigvee_{\alpha \in (0,1]} \{\alpha | x \in [c_{1\alpha}, c_{2\alpha})\} = \bigvee_{\alpha \in (0,1]} \{\alpha | c_{1\alpha} \leqslant x < c_{2\alpha}\},$$

where χ represents a characteristic function. Therefore, $0 \leqslant L(x) < 1$.

If $x_1 < x_2 \leqslant c_1$, then $\mu_{\tilde{c}}(x_1) \leqslant \mu_{\tilde{c}}(x_2)$, otherwise, $\mu_{\tilde{c}}(x_1) > \mu_{\tilde{c}}(x_2)$. Again, $x_1 < x_2 \leqslant c_1 \Rightarrow x_2 \in (x_1, c_1 + \varepsilon) \Rightarrow \exists \lambda \in [0, 1]$, such that

$$x_2 = \lambda x_1 + (1 - \lambda)(c_1 + \varepsilon),\ c_1 + \varepsilon \in (c_1, c_2).$$

Since $\tilde{c}$ represents a convex fuzzy set on $[c_1, c_2]$,

$$\mu_{\tilde{c}}(x_2) = \mu_{\tilde{c}}(\lambda x_1 + (1 - \lambda)(c_1 + \varepsilon)) \geqslant \mu_{\tilde{c}}(x_1) \wedge \mu_{\tilde{c}}(c_1 + \varepsilon),$$
$$\geqslant \mu_{\tilde{c}}(x_1) > \mu_{\tilde{c}}(x_2),$$

which is a contradiction. Therefore, $L(x)$ is an increasing function.

(3) $L(x)$ continues on the right, otherwise there exists $x < c_1, x_n \hookrightarrow x$, then

$$\lim_{x_n \to x^*} L(x_n) = \alpha > L(x).$$

Since $x_n \in \bar{c}_\alpha$ and $\bar{c}_\alpha$ is closed, then $x \in \bar{c}_\alpha$, such that $\mu_{\tilde{c}}(x) = L(x) \geqslant \alpha$. Therefore, contradiction.

For the same reason, $\mu_{\tilde{c}}(x) = R(x)$ is a continuously decreasing function on the left for $x > c_2$, with $0 \leqslant R(x) < 1$.

Sufficiency. Let $\tilde{c}$ satisfy the condition in the theorem. Then

(1) $\tilde{c}$ is obviously normal.
(2) Prove $\bar{c}_\alpha = [c_{1\alpha}, c_{2\alpha}], \forall \alpha \in (0, 1]$.

$\mu_{\tilde{c}}(x) = L(x)$ for $x < c_1$, so we select $c_{1\alpha} = \min\{x | L(x) \geqslant \alpha\}$ and $\mu_{\tilde{c}}(x) = R(x)$ for $x > c_2$, such that we select $c_{2\alpha} = \max\{x | R(x) \geqslant \alpha\}$.

Obviously, $\bar{c}_\alpha \subset [c_{1\alpha}, c_{2\alpha}]$. Now, prove $[c_{1\alpha}, c_{2\alpha}] \subset \bar{c}_\alpha$, and we prove only $[c_{1\alpha}, c_1) \subset \bar{c}_\alpha$ (because we can prove $(c_2, c_{2\alpha}] \subset \bar{c}_\alpha$ for the same reason). Again, we prove only $c_{1\alpha} \in \bar{c}_\alpha$ due to the monotonicity of $L(x)$.

Select $x_n \hookrightarrow c_{1\alpha}$, then $L(c_{1\alpha}) = \lim\limits_{x_n \to c_{1\alpha}} L(x_n) \geqslant \alpha$, such that $c_{1\alpha} \in \bar{c}_\alpha$.

1.7.2 Type $(\cdot, c), T, L - R$ and Flat Fuzzy Numbers

Definition 1.7.4. $\tilde{c} = (\alpha, c)$ is defined as a $(\cdot, c)$ fuzzy number on a product space $\alpha_1 \times \alpha_2 \times \cdots \times \alpha_J$; its membership function is

$$\mu_{\tilde{c}}(a) = \min_j [\mu_{\tilde{c}_j}(a_j)],$$

$$\mu_{\tilde{c}}(a_j) = \begin{cases} 1 - \dfrac{|\alpha_j - a_j|}{c_j}, & \alpha_j - c_j \leqslant a_j \leqslant \alpha_j + c_j, \\ 0, & \text{otherwise}, \end{cases} \tag{1.7.2}$$

where $\alpha = (a_1, a_2, \cdots, a_J)^{\mathrm{T}}, c = (c_1, c_2, \cdots, c_J)^{\mathrm{T}}$; α denotes the center of $\tilde{c}$, c the extension of $\tilde{c}$, with $c_j > 0$.

Coming next are special cases.

Definition 1.7.5. *L is called a reference function of fuzzy numbers if L satisfies*

(i) $L(x) = L(-x)$;
(ii) $L(0) = 1$;
(iii) $L(x)$ is a nonincreasing and piecewise continuous function at $[0, +\infty)$.

Definition 1.7.6. Let L, R be reference functions of a fuzzy number $\tilde{c}$, called a L-R fuzzy number. If

$$\mu_{\tilde{c}}(x) = \begin{cases} L\left(\dfrac{c-x}{\underline{c}}\right), & x \leqslant c, \underline{c} > 0, \\ R\left(\dfrac{x-c}{\overline{c}}\right), & x \geqslant c, \overline{c} > 0, \end{cases} \tag{1.7.3}$$

we write $\tilde{c} = (c, \underline{c}, \overline{c})_{LR}$, where c is a mean value; $\underline{c}$ and $\overline{c}$ are called the left and the right spreads of $\tilde{c}$, respectively. L is called a left reference and R a right reference. If take $\tilde{c}$ to be variable $\tilde{x}$, then $\tilde{x} = (x, \underline{\xi}, \overline{\xi})_{LR}$ represents T-fuzzy variable.

Definition 1.7.7. If L and R are functions satisfying

$$T(x) = \begin{cases} 1 - |x|, & \text{if} \quad -1 \leqslant x \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.7.4}$$

then we call $\tilde{c} = (c, \underline{c}, \overline{c})_T$ T-fuzzy numbers, $T(\mathscr{R})$ representing T-fuzzy number sets. If take $\tilde{c}$ to be variable $\tilde{x}$, then $\tilde{x} = (x, \underline{\xi}, \overline{\xi})_T$ represents T-fuzzy variables.

Definition 1.7.8. Let L, R be reference functions and the quadruple $\tilde{c} = (c^-, c^+, \sigma_c^-, \sigma_c^+)_{LR}$ is called a L-R flat fuzzy number. Then

$$\mu_{\tilde{c}}(x) = \begin{cases} L\left(\dfrac{c^- - x}{\sigma_c^-}\right), & x \leqslant c^-, \sigma_c^- > 0 \\ R\left(\dfrac{x - c^+}{\sigma_c^+}\right), & x \geqslant c^+, \sigma_c^+ > 0 \\ 1, & \text{otherwise} \end{cases} \tag{1.7.5}$$

satisfying $\exists (c^-, c^+) \in \mathscr{R}, c^- < c^+$, with $\mu_{\tilde{c}}(x) = 1$.

Especially, $\tilde{c} = (c^-, c^+, \sigma_c^-, \sigma_c^+)$ is said to be a flat fuzzy number, where

$$\mu_{\tilde{c}}(x) = \begin{cases} 1 - \dfrac{c^- - x}{\sigma_c^-}, & \text{if } c^- - \sigma_c^- \leqslant x \leqslant c^-, \\ 1, & \text{if } c^- < x < c^+, \\ 1 - \dfrac{x - c^+}{\sigma_c^+}, & \text{if } c^+ \leqslant x \leqslant c^+ + \sigma_c^+, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7.6}$$

If we take interval (c^-, c^+) to be (x^-, x^+), then $\tilde{x} = (x^-, x^+, \underline{\xi}, \overline{\xi})_{LR}$ and $\tilde{x} = (x^-, x^+, \underline{\xi}, \overline{\xi})$ represent an L-R fuzzy variable and a flat fuzzy one, respectively.

Definition 1.7.9. Suppose that "$*$" represents an arbitrary ordinary binary operation in $\mathscr{R}$, such that $\forall \tilde{c}, \tilde{d} \in \mathscr{F}(\mathscr{R})$ and we define

$$\tilde{c} * \tilde{d} = \int_{x,y \in \mathscr{R}} \frac{\mu_{\tilde{c}}(x) \wedge \mu_{\tilde{d}}(y)}{x * y},$$

i.e., $\forall z \in \mathscr{R}$,

$$\mu_{\tilde{c}*\tilde{d}}(z) = \bigvee_{x*y=z} (\mu_{\tilde{c}}(x) \wedge \tilde{d}(y)),$$

where "$*$" represents arithmetic operations $+, -, \cdot, \div$.

Accordingly, we can define the operations of Type L-R, T and flat fuzzy numbers.

A. Operations properties in L-R fuzzy number

Let $\tilde{c} = (c, \underline{c}, \overline{c})_{LR}, \tilde{d} = (d, \underline{d}, \overline{d})_{LR}$ and $\tilde{p} = (p, \underline{p}, \overline{p})_{RL}$ be an L-R fuzzy number. Then

1) $\tilde{c} + \tilde{d} = (c + d, \underline{c} + \underline{d}, \overline{c} + \overline{d})_{LR}$.

2) $k \cdot \tilde{c} = \begin{cases} (kc, k\underline{c}, k\overline{c})_{LR}, & \text{when } k \geqslant 0 \\ (kc, -k\overline{c}, -k\underline{c})_{RL}, & \text{when } k < 0 \end{cases} \quad (k \in \mathscr{R}).$

Let $(-1)\tilde{c} = -\tilde{c}$ for $k = -1$. Then $-\tilde{c} = (-c, \overline{c}, \underline{c})_{RL}$.

3) $\tilde{c} - \tilde{p} = (c - p, \underline{c} + \overline{p}, \overline{c} + \underline{p})_{LR}$ for $L = R$.

4) $\tilde{c} \cdot \tilde{d} \approx (cd, c\underline{d} + d\underline{c}, c\overline{d} + d\overline{c})_{LR}$.

5) $\tilde{c} \div \tilde{p} \approx (\dfrac{c}{p}, \dfrac{\overline{p}c + \underline{c}p}{p^2}, \dfrac{\underline{p}c + \overline{c}p}{p^2})_{LR}$, $p \neq 0$, $\tilde{c}$ and $\tilde{p}$ can not be divided for $L \neq R$.

6) $\widetilde{\max}(\tilde{c}, \tilde{d}) \approx (c \vee d, \underline{c} \wedge \underline{d}, \overline{c} \vee \overline{d})_{LR}$, $\widetilde{\min}(\tilde{c}, \tilde{d}) \approx (c \wedge d, \underline{c} \vee \underline{d}, \overline{c} \wedge \overline{d})_{LR}$.

7) $\tilde{c} \leqslant \tilde{d} \Longleftrightarrow c \leqslant d, \underline{c} \geqslant \underline{d}, \overline{c} \leqslant \overline{d}$; $\tilde{c} \subseteq \tilde{d} \Longleftrightarrow c + \overline{c} \leqslant d - \overline{d}$, or $\tilde{c} = \tilde{d}$.

B. Operations properties in T-fuzzy numbers

If $\tilde{c}_1 = (c_1, \underline{c}_1, \overline{c}_1)_T, \tilde{c}_2 = (c_2, \underline{c}_2, \overline{c}_2)_T$, then

(1) $\tilde{c}_1 + \tilde{c}_2 = (c_1 + c_2, \underline{c}_1 + \underline{c}_2, \overline{c}_1 + \overline{c}_2)_T$;

(2) $\tilde{c}_1 - \tilde{c}_2 = (c_1 - c_2, \underline{c}_1 + \overline{c}_2, \overline{c}_1 + \underline{c}_2)_T$;

$$(3)\ \lambda\tilde{c} = \lambda(c, \underline{c}, \overline{c})_T = \begin{cases} (\lambda c, \lambda\underline{c}, \lambda\overline{c})_T, & \forall\lambda > 0, \\ (\lambda c, -\lambda\overline{c}, -\lambda\underline{c})_T, & \forall\lambda < 0. \end{cases}$$

$$(4)\ \tilde{c}^{-1} = (c, \underline{c}, \overline{c})_T^{-1} \approx (\frac{1}{c}, \overline{c}c^{-2}, \underline{c}c^{-2})_T.$$

C. Operation properties in flat fuzzy numbers

Let $\tilde{c} = (c^-, c^+, \sigma_c^-, \sigma_c^+)$ and $\tilde{d} = (d^-, d^+, \sigma_d^-, \sigma_d^+)$ be flat fuzzy numbers. Then

$$1)\ \ \tilde{c} + \tilde{d} = (c^- + d^-, c^+ + d^+, \sigma_c^- + \sigma_d^-, \sigma_c^+ + \sigma_d^+).$$

$$2)\ \ k \cdot \tilde{c} = \begin{cases} (kc^-, kc^+, k\sigma_c^-, k\sigma_c^+), & \text{for } k > 0, \\ (kc^+, kc^-, -k\sigma_c^-, -k\sigma_c^+), & \text{for } k \leqslant 0. \end{cases}$$

By the definition of Type L-R, T or flat fuzzy numbers, it is easy to prove their operation properties [Dia87][DPr80].

We can deduce operation properties of $(\cdot, c)$ fuzzy numbers since it is extended over flat fuzzy ones.