Chapter 7
Chaotic Systems Reconstruction

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Abstract. This chapter deals with the multiple model approach based chaotic systems reconstruction. The approach is based on the design of unknown inputs multiple observers using Linear Matrix Inequalities ($\mathcal{LMI}$) formulation. The objective is to estimate state variables of a multiple model subject to unknown inputs affecting both states and outputs of the system. Uncertainties affecting state matrices of the system are also considered for both continuous-time and discrete-time multiple models. In order to improve the performances of the observer, poles placement in an $\mathcal{LMI}$ region is also studied. Numerical examples are given to illustrate the effectiveness the given results. Application dealing with chaotic synchronization and message decoding are also given by considering chaotic multiple model subject to hidden message. The proposed approach can be also used in a chaotic cryptosystem procedure where the plaintext (message) is encrypted using chaotic signals at the drive system side. The resulting ciphertext is embedded to the output and/or state of the drive system and is sent via public channel to the response system. The plaintext is retrieved via the synthesis approach, i.e. the designed unknown input multiple observer.

7.1 Introduction

In last two decades, many studies concerning stability analysis and design of controllers and observers for a class of systems described by multiple model approach [20] are carried out. Such representation results from the interpolation of M local

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LTI (linear time invariant) models throughout convex functions. These functions can be viewed as a weighted sum of local LTI models and quantify the relative contribution of each local model to the global model. The choice of the number of local models may be intuitively chosen by considering some operating regimes. Each LTI model can be obtained by using a direct linearization of an a priori nonlinear model around operating points, or alternatively by using an identification procedure [20, 5]. From a practical point of view, LTI model describes the system's local behavior around the i^{th}, $i:1...M$ regime. This approach includes Takagi-Sugeno fuzzy models [23] and PLDI representation [4]. Based on the Lyapunov method and Linear Matrix Inequalities ($\mathcal{LMI}$) formulation, sufficient conditions have been derived for stability analysis, controllers and observers design (see among others [7, 15, 16, 26, 24]). Recently, systems subject to unknown inputs are extensively considered in the literature. Unknown inputs can result either from model uncertainty, faults or due to the presence of unknown external excitation. This problem, usually referred as the unknown input observer design, has been considered actively for linear systems [13, 12, 27, 10, 22], for descriptor and nonlinear systems (see among other [19, 17, 14] and for multiple model approach (see for example [7, 6] and references therein). Based on unknown inputs observer design, many works have been carried out on secure communication and chaotic system reconstruction problem. The increasing need of secure communications leads to the development of many techniques which make difficult the detecting of transmitted message (see for example [18, 8, 3, 2, 1]. Indeed, the problem we are faced with consists of transmitting some coded message with a signal broadcasted by a communication channel. At the receiver side, the hidden signal is recovered by a decoding system. In this chapter, our goal is to show how to get chaotic multiple model from chaotic nonlinear system and how to design the proposed structure of observer for chaotic system reconstruction.

This chapter is organized as follows. In section 2, a considered unknown inputs multiple model in continuous-time case and his corresponding observer are given. Synthesis conditions for the proposed observer are given in $\mathcal{LMI}$ terms. Two cases are considered. The case of output signal not depending on the unknown inputs and the case when both state and output signal are affected by unknown inputs. To improve the performances of the proposed observer, the pole assignment in a $\mathcal{LMI}$ region is also studied. Unknown input estimation is given in section 3. Then these design conditions are extended to unknown inputs discrete-time multiple model in section 5. To illustrate the given synthesis $\mathcal{LMI}$ conditions, numerical examples and applications dealing with the chaotic system reconstruction for both continuous-time and discrete-time multiple model are proposed.

Throughout this chapter, $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. Superscript "$^\mathsf{T}$" denotes matrix transposition and the notation $X>Y$ where X and Y are symmetric matrices, means that $X-Y$ is positive definite. $\otimes$ is the Kronecker product, $\mathbf{I}$ is the identity matrix with compatible dimensions, the symbol $(*)$ denotes the transpose elements in the symmetric positions and $I_M=\{1,2,\cdots,M\}$.

7.2 Unknown Inputs Multiple Observer Design

Consider a continuous-time multiple model with unknown inputs defined as follows

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(\widehat{A}_i x(t) + B_i u(t) + D_i + R_i \bar{u}(t)) \\ y(t) = Cx(t) + F\bar{u}(t) \end{cases} \tag{7.1}$$

with

$$\mu_i(\xi(t)) \geq 0, \sum_{i=1}^{M} \mu_i(\xi(t)) = 1 \tag{7.2}$$

and

$$\widehat{A}_i = A_i + \Delta A_i(t) \tag{7.3}$$

M being the number of local LTI models, $x(t) \in \mathbf{R}^n$ the state vector, $u(t) \in \mathbf{R}^m$ the input vector, $\bar{u}(t) \in \mathbf{R}^q$, the unknown input and $y \in \mathbf{R}^p$ the measured outputs. $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^{n\times m}$, $D_i \in \mathbf{R}^n$ and $C \in \mathbf{R}^{p\times n}$ define the i^{th} local LTI model. Matrices $R_i \in \mathbf{R}^{n\times q}$ and $F \in \mathbf{R}^{p\times q}$ represent the influence of the unknown inputs. We assume that $q < p$ and, without loss of generality, that

Assumption 1: $rank(F) = q$ and $rank(R_i) = q$, i.e. F and R_i are full column ranks.

Assumption 2: $rank(C) = p$, i.e. C is full row rank.

$\Delta A_i(t)$ are time-varying matrices representing parametric uncertainties. These uncertainties are admissibly norm-bounded, structured and satisfy: $\Delta A_i = D_{A_i} F_{A_i} E_{A_i}$ with D_{A_i} and E_{A_i} are known real matrices with appropriate dimensions and F_{A_i} satisfies $F_{A_i}^{\top} F_{A_i} \leq \mathbf{I}$, $\forall\, i \in I_M$. The *activation functions* $\mu_i(.)$ depend on the so-called decision vector $\xi(t)$ assumed to depend on measurable variables.

In this section, we are concerned by the reconstruction of state variable $x(t)$ of multiple model (7.1) subject to unknown inputs, using only the available information, that is known input $u(t)$ and measured output $y(t)$.

The following lemma will be used in the rest of the paper.

Lemma 1. [28] : Let H and E be given matrices with appropriate dimensions and F satisfying $F^{\top} F \leq \mathbf{I}$. Then, we have for any $\varepsilon > 0$,

$$HFE + E^{\top} F^{\top} H^{\top} \leq \varepsilon H H^{\top} + \frac{1}{\varepsilon} E^{\top} E. \tag{7.4}$$

The considered unknown input multiple observer, for the unknown input multiple model (7.1), has the following structure

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) \Big(N_i z(t) + G_{i1} u(t) + G_{i2} + L_i y(t) \Big) \\ \hat{x}(t) = z(t) - Ey(t) \end{cases} \tag{7.5}$$

The considered observer only uses known variables ($u(t)$ and $y(t)$) and the same functions $\mu_i(.)$ as used for the multiple model (7.1). The unknown inputs $\bar{u}(t)$ are considered non available.

In order to estimate the state of the unknown input multiple model (7.1), the variables $N_i \in \mathbf{R}^{n\times n}$, $G_{i1} \in \mathbf{R}^{n\times m}$, $G_{i2} \in \mathbf{R}^{n}$, $L_i \in \mathbf{R}^{n\times p}$ and $E \in \mathbf{R}^{n\times p}$ must be be determined such that the state estimation error:

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{7.6}$$

satisfies $\tilde{x}(t) \to 0$ when $t \to \infty$. Multiple model subject to unknown inputs which affect state and outputs variables of the system are then studied. Poles placement in $\mathcal{LMI}$ region for the designed multiple observer is also considered.

7.2.1 Unknown Inputs Observer Design

This section addresses the case when only states are affected by unknown inputs $\bar{u}(t)$, i.e. the multiple model (7.1) with $F = 0$:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(\widehat{A}_i x(t) + B_i u(t) + D_i + R_i \bar{u}(t)), \\ y(t) = Cx(t) \end{cases} \tag{7.7}$$

The following result gives sufficient $\mathcal{LMI}$ conditions guaranteeing the global asymptotic convergence of the state estimation error (7.6).

Theorem 7.1. *The state estimation error between multiple observer (7.5) and unknown input multiple model (7.7) converges globally asymptotically towards zero, if there exists matrices $X > 0$, S and W_i and scalars ε_i such that the following conditions hold $\forall\, i \in I_M$:*

$$\begin{bmatrix} A_i^\top X + XA_i + A_i^\top C^\top S^\top + SCA_i - W_i C - C^\top W_i^\top + \varepsilon_i E_{A_i}^\top E_{A_i} & (X+SC)D_{A_i} \\ (*) & -\varepsilon_i \mathbf{I} \end{bmatrix} < 0 \tag{7.8a}$$

$$(X+SC)R_i = 0 \tag{7.8b}$$

Then multiple observer (7.5) is completely defined by:

$$E = X^{-1}S \tag{7.9a}$$

$$G_{i1} = (\mathbf{I} + X^{-1}SC)B_i \tag{7.9b}$$

$$G_{i2} = (\mathbf{I} + X^{-1}SC)D_i \tag{7.9c}$$

$$N_i = (\mathbf{I} + X^{-1}SC)A_i - X^{-1}W_i C \tag{7.9d}$$

$$L_i = X^{-1}W_i - N_i E \tag{7.9e}$$

Proof. From estimation error (7.6), the expression of $\hat{x}(t)$ given by multiple observer (7.5) and $x(t)$ given by (7.7), we get

$$\tilde{x}(t) = (\mathbf{I} + EC)x(t) - z(t) \tag{7.10}$$

The dynamic of state estimation error (7.10), taking account the expressions of $y(t)$ and $z(t)$ given in (7.7) and (7.5), is given by

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi) \Big(N_i\tilde{x}(t) + \big(T\widehat{A}_i - K_iC - N_i\big)x(t) + (TB_i - G_{i1})\,u(t) + (TD_i - G_{i2}) + TR_i\bar{u}(t) \Big) \tag{7.11}$$

with

$$K_i = N_iE + L_i, \;\; T = \mathbf{I} + EC \tag{7.12}$$

The following change of variables

$$W_i = XK_i \tag{7.13a}$$
$$S = XE \tag{7.13b}$$

with $X > 0$ and (7.12) lead to the following expression

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi) \Big(N_i\tilde{x}(t) + \big(T\widehat{A}_i - K_iC - N_i\big)x(t) + \big((\mathbf{I} + X^{-1}SC)B_i - G_{i1}\big)u(t) + \big((\mathbf{I} + X^{-1}SC)D_i - G_{i2}\big) + X^{-1}(X + SC)R_i\bar{u}(t) \Big) \tag{7.14}$$

Taking account (7.8b) and (7.9b-c), we get

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi) N_i\tilde{x}(t) \tag{7.15}$$

with

$$N_i = T\widehat{A}_i - K_iC \tag{7.16}$$

Then, the state estimation error (7.15) converges asymptotically to zero if there exist $X > 0$ such that $\forall\, i \in I_M$:

$$XN_i + N_i^{\top}X < 0 \tag{7.17}$$

With the same variable change (7.13), inequalities (7.17) are equivalent to

$$(X + SC)\widehat{A}_i - W_iC + ((X + SC)\widehat{A}_i - W_iC)^{\top} < 0 \tag{7.18}$$

By applying Lemma 1, with $\widehat{A}_i = A_i + \Delta A_i$ and $\Delta A_i = D_{A_i} F_{A_i} E_{A_i}$, the constraint (7.18) is equivalent to the existence of scalars $\varepsilon_i > 0$ such that

$$(X+SC)A_i + A_i^\top (X+SC)^\top - W_i C - C^\top W_i^\top + \varepsilon_i E_{A_i}^\top E_{A_i} + \varepsilon_i^{-1}(X+SC)D_{A_i}D_{A_i}^\top (X+SC)^\top < 0 \tag{7.19}$$

which is only the Schur complement of (7.8a). This completes the proof. □

For multiple model (7.7) without uncertainties, i.e. $\widehat{A}_i = A_i$, the following corollary gives sufficient $\mathcal{LMI}$ conditions for global asymptotic convergence of the state estimation error (7.6).

Corollary 1: The state estimation error between multiple observer (7.5) and unknown input multiple model (7.7) with $\Delta A_i = 0$ converges globally asymptotically towards zero, if there exists matrices $X > 0$, S and W_i such that the following conditions hold $\forall\, i \in I_M$:

$$(X+SC)A_i + A_i^\top (X+SC)^\top - W_i C - C^\top W_i^\top < 0 \tag{7.20a}$$

$$(X+SC)R_i = 0 \tag{7.20b}$$

Then multiple observer (7.5) is defined by (7.9).

Proof. It suffices to substitute $\widehat{A}_i$ by A_i in (7.18) to get (7.20a). The equalities constraints are not modified. □

It is important to note that equalities (7.8b)/(7.20b), with $X > 0$ and the change of variable (7.13), are equivalent to $(\mathbf{I}+EC)R_i = 0$, that is $ECR_i = -R_i\ \forall\, i \in I_M$. Lets notice that this condition contains the one for linear systems ($R_i = R$) where an solution E exits if and only if the rank constraint $rank(CR) = rank(R)$ holds (see for example [27, 10]). However, in contrast to linear systems, it is important to note that the condition on the rank is only a necessary condition for multiple model. Moreover, inequalities (7.20a) with $X > 0$ are equivalent to $X\big((\mathbf{I}+EC)A_i - K_iC\big) + \big((\mathbf{I}+EC)A_i - K_iC\big)^\top X < 0$. It is easy to note that these conditions contain the observability (detectability) conditions of $\big((\mathbf{I}+EC)A, C\big)$ for linear systems. Then, in order to assist the designer, the following procedure proposes to check two necessary conditions before solving conditions (7.8) or (7.20):

Procedure 1:

i) Check if $rank(CR_i) = rank(R_i)\ \forall\, i \in I_M$.

ii) Compute for each $i \in I_M$, a solution $E^{(i)} = -R_i(CR_i)^+$ and check the local observability of each pair $\big((\mathbf{I}+E^{(i)}C)A_i, C\big)$. Σ^+ denotes any generalized inverse of matrix Σ with $\Sigma\Sigma^+\Sigma = \Sigma$ [21].

If *(i)*-*(ii)* hold, then the designer can solve the sufficient $\mathcal{LMI}$ conditions (7.8)/(7.20) to design multiple observer (7.5).

7.2.2 $\mathcal{LMI}$ Design Conditions

This section considers the general structure of unknown inputs multiple model (7.1), that is when both the state and the output signal are affected by unknown inputs $\bar{u}(t)$. The following result gives sufficient $\mathcal{LMI}$ conditions guaranteeing the global asymptotic convergence of state estimation error (7.6).

Theorem 7.2. *The state estimation error between multiple observer (7.5) and unknown input multiple model (7.1) converges globally asymptotically towards zero, if there exists matrices $X > 0$, S and W_i and scalars ε_i such that the following conditions hold $\forall\, i \in I_M$:*

$$\begin{bmatrix} A_i^\top X + XA_i + A_i^\top C^\top S^\top + SCA_i - W_iC - C^\top W_i^\top + \varepsilon_i E_{A_i}^\top E_{A_i} & (X+SC)D_{A_i} \\ (*) & -\varepsilon_i \mathbf{I} \end{bmatrix} < 0 \tag{7.21a}$$

$$(X+SC)R_i = W_iF \tag{7.21b}$$

$$SF = 0 \tag{7.21c}$$

Then multiple observer (7.5) is completely defined by (7.9).

Proof. From estimation error (7.6) with the expression of $\hat{x}(t)$ given by observer (7.5) and multiple model (7.1), we obtain the following expression:

$$\tilde{x}(t) = (\mathbf{I} + EC)x(t) - z(t) + EF\bar{u}(t) \tag{7.22}$$

The dynamic of state estimation error is then given by

$$\begin{aligned}\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) \Big(T\big(\widehat{A}_i x(t) + B_i u(t) + R_i\bar{u}(t) + D_i\big) - N_i z(t) - \\ G_{i1}u(t) - G_{i2} - L_i y(t)\Big) + EF\dot{\bar{u}}(t)\end{aligned} \tag{7.23}$$

where T is defined in (7.12). With the expressions of $y(t)$, $z(t)$ given in (7.1) and (7.5), we obtain

$$\begin{aligned}\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi) \Big(N_i\tilde{x}(t) + \big(T\widehat{A}_i - K_iC - N_i\big)x(t) + (TB_i - G_{i1})u(t) + \\ (TD_i - G_{i2}) + \big(TR_i - K_iF\big)\bar{u}(t)\Big) + EF\dot{\bar{u}}(t)\end{aligned} \tag{7.24}$$

with K_i defined in (7.12). Thus, using the same change of variable (7.13) with (7.9b-c) and (7.21b-c), we get

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{M} \mu_i(\xi) N_i \tilde{x}(t) \tag{7.25}$$

where N_i is defined in (7.16). The rest of the proof is similar to the one of the theorem 1. This completes the proof. □

The following corollary obtained directly from theorem 2, gives sufficient linear conditions to design a multiple observer for multiple model (7.1) without uncertainties.

Corollary 2: The state estimation error between multiple observer (7.5) and unknown input multiple model (7.1) with $\Delta A_i = 0$ converges globally asymptotically towards zero, if there exists matrices $X > 0$, S and W_i such that the following conditions hold $\forall\, i \in I_M$:

$$(X+SC)A_i + A_i^\top (X+SC)^\top - W_i C - C^\top W_i^\top < 0 \tag{7.26a}$$

$$(X+SC)R_i = W_i F \tag{7.26b}$$

$$SF = 0 \tag{7.26c}$$

Then multiple observer (7.5) is defined by (7.9).

Remark 1. Classical numerical tools as the LMITOOL [25] may be used to solve the linear problem (7.8) on variables $X > 0$, S, W_i and scalars $\varepsilon_{i.}$. Examples are given in section 4 to illustrate the derived stability conditions.

Remark 2. Only uncertainties on matrices A_i are considered. Uncertainties on the other matrices lead to equalities constraints impossible to satisfy and not considered in this chapter.

Remark 3. The case of different multiple output matrices $C_i = C, \forall i \in I_M$ is not considered because it leads to non convex constraints not easy to resolve with existing numerical tools.

7.2.3 Pole Placement

In this part, we investigate how to improve the performances of the proposed observer (7.5) for multiple model (7.1). In order to achieve a desired transient performance, a pole placement should be considered. For many problems, exact pole assignment may not be necessary; it suffices to locate the pole in a sub-region of the complex left half plane [9]. This section discusses a pole assignment in $\mathcal{LMI}$ regions $S(\alpha,\beta)$.

Theorem 7.3. *A matrix $A \in \mathbf{R}^{n\times n}$ is D-stable if and only if there exists a symmetric positive definite matrix $X > 0$ such that*

$$M_D(A,D) = \alpha \otimes X + \beta \otimes (AX) + \beta^\top \otimes (AX)^\top < 0 \tag{7.27}$$

where $\alpha \in \mathbf{R}^{n\times n}$ and $\beta \in \mathbf{R}^{n\times n}$.

Since prescribed $\mathcal{LMI}$ region (7.27) will be added as supplementary constraint to these of theorem 1 or theorem 2, it should be noted that it only suffices to locate the poles of matrix $\sum_{i=1}^{M} \mu_i(\xi(t))N_i$ in prescribed $\mathcal{LMI}$ regions. Indeed, estimation error (7.25) is D-stable if there exists a matrix $X > 0$ such that

$$M_D(N_i, D) = \alpha \otimes X + \beta \otimes (N_i X) + \beta^\top \otimes (N_i X)^\top < 0 \tag{7.28}$$

With the same changes of variables (7.13) applied to inequalities (7.28), we obtain the following result.

Corollary 3: If there exit matrices $X > 0$, S and W_i such that the following conditions hold $\forall\, i \in I_M$:

$$\alpha \otimes X + \beta \otimes (X\widehat{A}_i + SC\widehat{A}_i - W_i C) + \beta^\top \otimes (X\widehat{A}_i + SC\widehat{A}_i - W_i C)^\top < 0 \tag{7.29a}$$

$$(X + SC)R_i = W_i F \tag{7.29b}$$

$$SF = 0 \tag{7.29c}$$

Then, multiple observer (7.5) is globally asymptotically convergent with the performance defined by complex region $S(\alpha, \beta)$. The multiple observer gains are as defined by (7.9).

For example, to ensure a given performance of the state estimation error, we define region $S_r(\alpha, \beta)$ as the intersection between a circle, of center $(0,0)$ and of radius β, and the left half plane limited by a vertical straight line of x-coordinate equal to $-\alpha < 0$. The corresponding $\mathcal{LMI}$ formulation of the corollary 3 is given by the following corollary.

Corollary 4: If there exit matrices $X > 0$, S, W_i and scalars ε_{i1} and ε_{i2} such that the following $\mathcal{LMI}$ conditions hold $\forall\, i \in I_M$:

$$\begin{bmatrix} -\beta X & XA_i + SCA_i - W_i C & (X+SC)D_{A_i} \\ (*) & -\beta X + \varepsilon_{i1} E_{A_i}^\top E_{A_i} & 0 \\ (*) & (*) & -\varepsilon_{i1}\mathbf{I} \end{bmatrix} < 0 \tag{7.30a}$$

$$\begin{bmatrix} A_i^\top X + XA_i + A_i^\top C^\top S^\top + SCA_i - W_i C - C^\top W_i^\top + 2\alpha X + \varepsilon_{i2} E_{A_i}^\top E_{A_i} & (X+SC)D_{A_i} \\ (*) & -\varepsilon_{i2}\mathbf{I} \end{bmatrix} < 0 \tag{7.30b}$$

$$(X + SC)R_i = W_i F \tag{7.30c}$$

$$SF = 0 \tag{7.30d}$$

Then, multiple observer (7.5) is globally asymptotically convergent with the performance defined by complex region $S_r(\alpha, \beta)$. The multiple observer gains are as defined by (7.9).

Proof. For the defined region $S_r(\alpha,\beta)$, constraints (7.29a) are equivalent to

$$\begin{bmatrix} -\beta X & X\widehat{A}_i + SC\widehat{A}_i - W_iC \\ (*) & -\beta X \end{bmatrix} < 0 \tag{7.31a}$$

$$X\widehat{A}_i + SC\widehat{A}_i - W_iC + (X\widehat{A}_i + SC\widehat{A}_i - W_iC)^\top + 2\alpha X < 0 \tag{7.31b}$$

Inequalities (7.31a) can be rewritten as follows

$$\begin{bmatrix} -\beta X & XA_i + SCA_i - W_iC \\ (*) & -\beta X \end{bmatrix} + \begin{bmatrix} (X+SC)D_{A_i} \\ 0 \end{bmatrix} F_{A_i} \begin{bmatrix} 0 & E_{A_i} \end{bmatrix} + \begin{bmatrix} 0 \\ E_{A_i}^\top \end{bmatrix} F_{A_i}^\top \begin{bmatrix} D_{A_i}^\top (X+SC)^\top & 0 \end{bmatrix} < 0 \tag{7.32}$$

Applying Lemma 1 to (7.32) and Schur complement to the result, we get $\mathcal{LMI}$ conditions (7.30a). $\mathcal{LMI}$ conditions (7.30b) are also obtained from (7.31b) using lemma 1. This completes the proof. □

Note that for the certain case, i.e. $\widehat{A}_i = A_i$, corollary 4 is rewritten as follows

$$\begin{bmatrix} -\beta X & XA_i + SCA_i - W_iC \\ (*) & -\beta X \end{bmatrix} < 0 \tag{7.33a}$$

$$XA_i + SCA_i - W_iC + (XA_i + SCA_i - W_iC)^\top + 2\alpha X < 0 \tag{7.33b}$$

$$(X+SC)R_i = W_iF \tag{7.33c}$$

$$SF = 0 \tag{7.33d}$$

7.3 Unknown Inputs Estimation

A lot of works have been considered for the unknown input estimation problem (see for example [7, 6, 11]). For example in [11], authors are proposed methods for detecting and reconstructing sensor faults using sliding mode observers whereas in [6] a method to simultaneously estimate unknown inputs and states for T-S fuzzy models is proposed.

The method proposed in this chapter is based on the hypothesis of the good estimation of the state variables [7]. Indeed, when the state estimation error is equal to zero; by replacing x by $\hat{x}$ in the equation (7.1) we obtain the following approximation:

$$\hat{y} = C\hat{x} + F\hat{\hat{u}} \tag{7.34}$$

Since the assumption 1 holds, i.e. the matrix F is of full column rank, an estimation of unknown inputs can be carried out in a simpler way by

$$\hat{\hat{u}} = (F^\top F)^{-1} F^\top (y - \hat{y}) \tag{7.35}$$

7.4 Simulation Examples

To illustrate the validness of the proposed results, two examples will be proposed. The first one illustrates the good estimation of both states and unknown input affecting multiple model in different case (without poles placement, with poles placement and uncertainties). The second one in section 4.2, deals with the chaotic system reconstruction. First by building chaotic multiple model. Then the design of a multiple observer for such chaotic multiple model is given. Simulation shows the good chaotic system reconstruction with the proposed design.

7.4.1 Academic Example

Now, consider the multiple model (7.1), where both the state and the output signal are affected by unknown inputs, with $M = 2$ and the following data:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -6 \end{bmatrix} \quad A_2 = \begin{bmatrix} -3 & 2 & 2 \\ 5 & -8 & 0 \\ 0.5 & 0.5 & -4 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.5 \\ 1 \\ -0.25 \end{bmatrix} \quad F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 \\ 0.5 \\ -2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the functions

$$\begin{cases} \xi(t) = y_1(t) \\ \mu_1(\xi(t)) = \frac{1}{2}(1 - \tanh(\xi(t))) \\ \mu_2(\xi(t)) = 1 - \mu_1(\xi(t)) \end{cases}$$

The following subsections are dedicated to design a multiple observer of the form (7.5), firstly without pole placement in subsection 4.1.1 and then with pole placement in subsection 4.1.2. Uncertainties on state matrices are also studied in section 4.1.3.

7.4.1.1 Observer Design without Pole Placement

The resolution of conditions (7.26) lead to the following result:

$$\mathrm{X} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0.22 & -0.10 \\ 0.33 & -0.37 \\ 0.30 & -0.15 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0.49 & -0.40 \\ 0.17 & -0.15 \\ 0.49 & -0.72 \end{bmatrix}$$

From (7.9), we define completely multiple observer (7.5) as follows:

$$E = \begin{bmatrix} -0.18 & 0.18 \\ -0.66 & 0.66 \\ -0.57 & 0.57 \end{bmatrix} \quad L_1 = \begin{bmatrix} 1.04 & 0.14 \\ 0.76 & -1.09 \\ -1.55 & 3.13 \end{bmatrix} \quad L_2 = \begin{bmatrix} 3.70 & -2.8 \\ -1.03 & 1.19 \\ 4.05 & -6.34 \end{bmatrix}$$

$$G_{11} = \begin{bmatrix} 1.09 \\ -0.16 \\ -0.21 \end{bmatrix} \quad G_{21} = \begin{bmatrix} -0.68 \\ 0.33 \\ -0.82 \end{bmatrix}$$

With known input $u(t)$ and unknown input $\bar{u}(t)$ given in figures $1-2$ respectively and initial conditions $x_0 = (1, 0.5, 0)^\top$ and $z_0 = (-2, 2, -3)^\top$, we obtain the simulation results given in figures $3-5$. As shown, the dynamic of the estimated state tends globally asymptotically to the model sate in spite of the presence of unknown input $\bar{u}(t)$. This allows to illustrate the effectiveness of the derived synthesis conditions.

Fig. 7.1 Known input $u(t)$

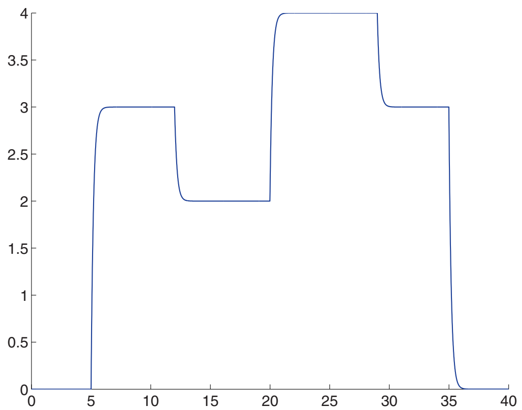
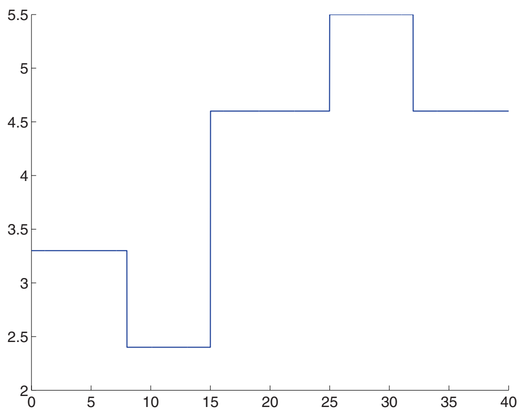


Fig. 7.2 Unknown input $\bar{u}(t)$



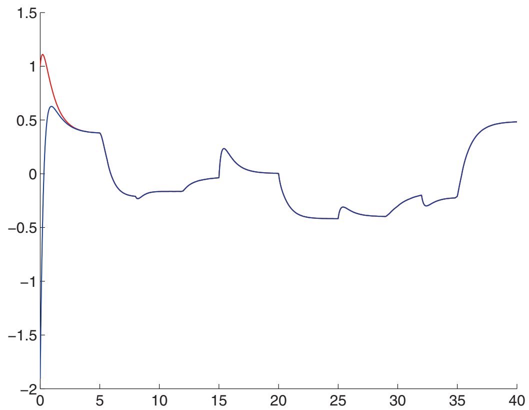


Fig. 7.3 State estimation without pole placement: $x_1(t)$ and $\hat{x}_1(t)$

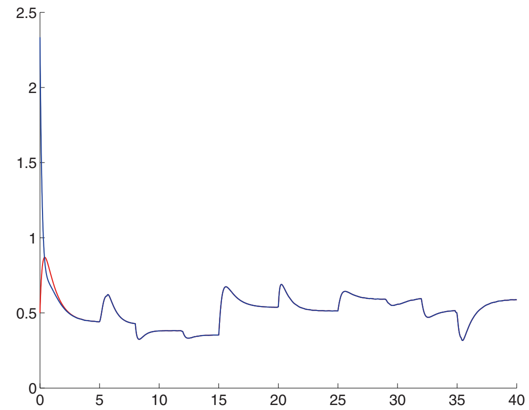


Fig. 7.4 State estimation without pole placement: $x_2(t)$ and $\hat{x}_2(t)$

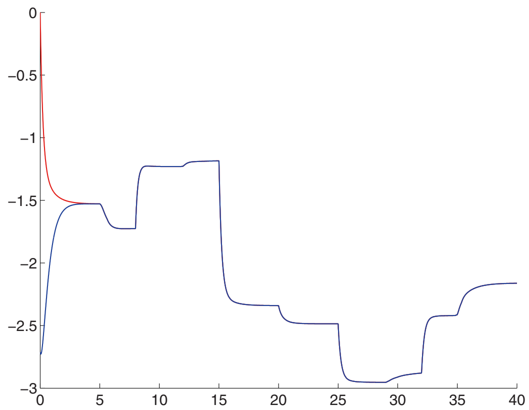


Fig. 7.5 State estimation without pole placement: $x_3(t)$ and $\hat{x}_3(t)$

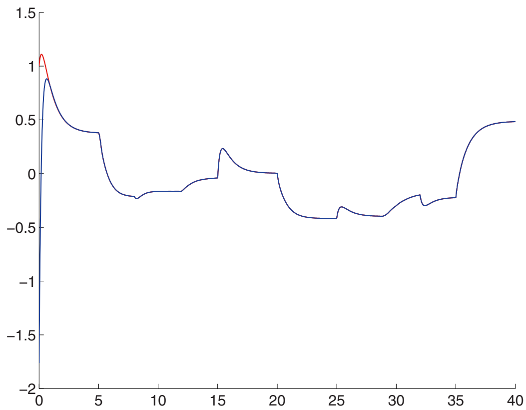


Fig. 7.6 State estimation with pole placement: $x_1(t)$ and $\hat{x}_1(t)$

7.4.1.2 Observer Design with Pole Placement

In order to improve performances of the above designed observer, the region $S_r(\alpha,\beta)$ defined in (7.33) is used. The considered region is an intersection between a circle, of center $(0,0)$ and of radius 10, and the left half plane limited by a vertical line at -2. The resolution of conditions (7.33) corresponding to the region $S_r(2,10)$ give:

$$\mathrm{X} = \begin{bmatrix} 4.15 & 0 & -1.75 \\ 0 & 2.97 & 0 \\ -1.75 & 0 & 1.94 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0.88 & 1.01 \\ 13.41 & -13.41 \\ 5.94 & -3.84 \end{bmatrix} \quad W_2 = \begin{bmatrix} 3.38 & 4.53 \\ 13.41 & -13.41 \\ 12.82 & -19.41 \end{bmatrix}$$

From (7.9), the corresponding multiple observer is given by

$$E = \begin{bmatrix} -0.47 & 0.47 \\ -1 & 1 \\ -1.42 & 1.42 \end{bmatrix} \quad L_1 = \begin{bmatrix} -0.13 & 1.6 \\ 0 & 0 \\ -7.5 & 10 \end{bmatrix} \quad L_2 = \begin{bmatrix} 4.64 & -3.88 \\ 0 & 0 \\ 8.17 & -10.88 \end{bmatrix}$$

$$G_{11} = \begin{bmatrix} 1.24 \\ 0 \\ 0.21 \end{bmatrix} \quad G_{21} = \begin{bmatrix} -0.97 \\ 0 \\ -1.67 \end{bmatrix}$$

With the same initial conditions, the known and unknown inputs given in figures $1-2$, we obtain the simulation result given in figures $6-8$. To show clearly the performance improvements of the designed multiple observer, the simulation of state estimation errors $\tilde{x}(t) = x(t) - \hat{x}(t)$ with and without pole assignment are presented in figures $9-11$.

Fig. 7.7 State estimation with pole placement: $x_2(t)$ and $\hat{x}_2(t)$

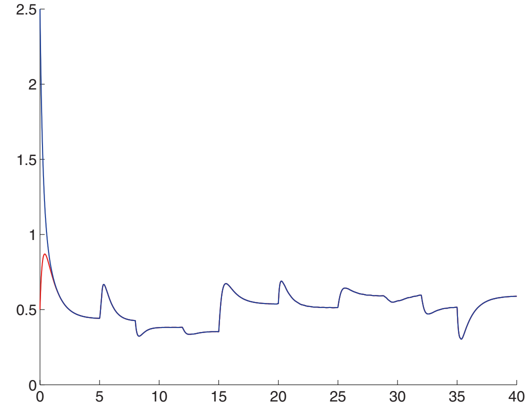


Fig. 7.8 State estimation with pole placement: $x_3(t)$ and $\hat{x}_3(t)$

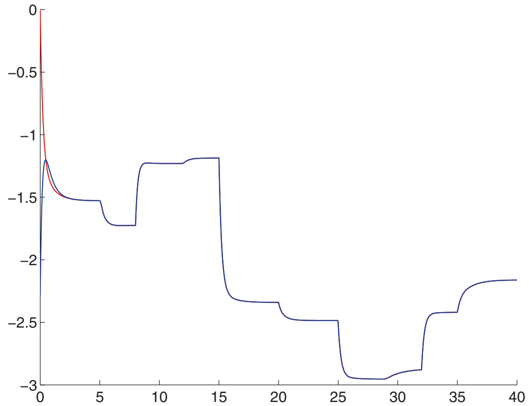
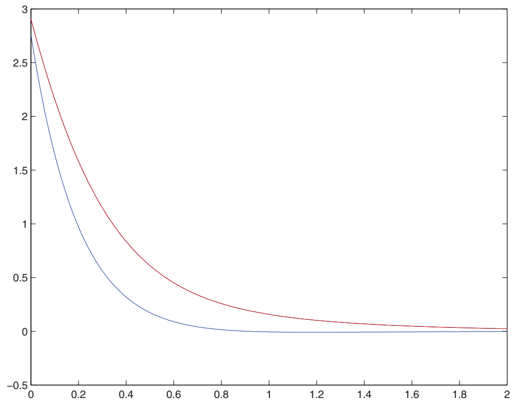


Fig. 7.9 Estimation errors with and without (red line) pole placement:$\tilde{x}_1(t) = x_1(t) - \hat{x}_1(t)$



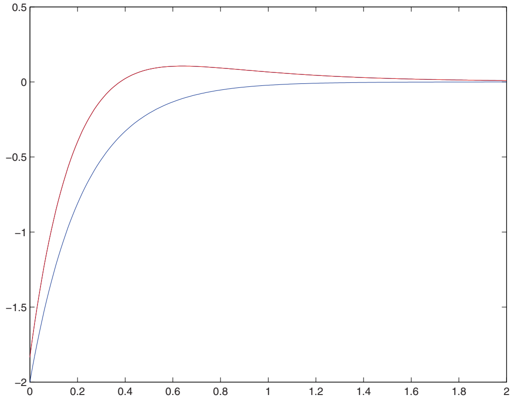


Fig. 7.10 Estimation errors with and without (red line) pole placement:$\tilde{x}_2(t) = x_2(t) - \hat{x}_2(t)$

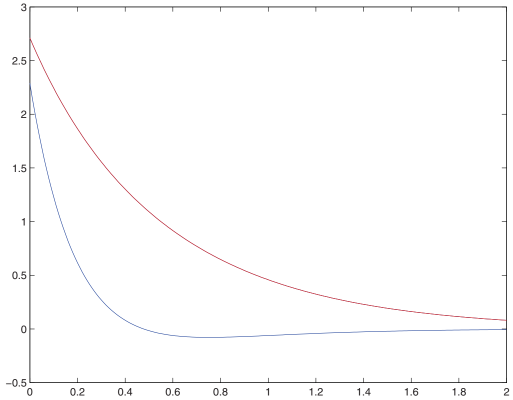


Fig. 7.11 Estimation errors with and without (red line) pole placement:$\tilde{x}_3(t) = x_3(t) - \hat{x}_3(t)$

7.4.1.3 Uncertainties on State Matrices

Now, to show the robustness of the proposed observer, consider the same example with uncertainties on state matrices as follows

$$D_{A_1} = D_{A_2} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad E_{A_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & -2 \end{bmatrix}, E_{A_2} = \begin{bmatrix} -2 & 2 & 2 \\ 5 & -4 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

The resolution of conditions (7.30) of corollary 4 corresponding to the same region $S_r(2,10)$ leads to feasible problem and gives:

$$X = \begin{bmatrix} 26.9004 & -0.4639 & -8.7213 \\ -0.4639 & 0.4138 & 0.0776 \\ -8.7213 & 0.0776 & 6.9058 \end{bmatrix}, S = \begin{bmatrix} -6.2098 & 6.2098 \\ -0.1912 & 0.1912 \\ -3.8747 & 3.8747 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 38.0992 & -13.2463 \\ 1.4271 & -2.0360 \\ 9.3190 & -7.3374 \end{bmatrix}, \; W_2 = \begin{bmatrix} 101.1735 & -60.1674 \\ 0.1294 & -0.6374 \\ 16.3630 & -40.7944 \end{bmatrix}$$

$$\varepsilon_{11} = 0.9609, \; \varepsilon_{21} = 0.1024, \; \varepsilon_{12} = 1.2122, \; \varepsilon_{22} = 0.1808$$

Then from (7.9), multiple observer (7.5) is defined with the following parameters

$$E = \begin{bmatrix} -0.7219 & 0.7219 \\ -0.9972 & 0.9972 \\ -1.4615 & 1.4615 \end{bmatrix}, \; G_{11} = \begin{bmatrix} 1.3609 \\ -0.0014 \\ 0.2308 \end{bmatrix}, \; G_{21} = \begin{bmatrix} -1.2219 \\ 0.0028 \\ -1.7115 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -4.4438 & -0.1141 & -0.7219 \\ 0.0057 & -6.1171 & 0.0028 \\ -1.9230 & -0.0381 & -8.4615 \end{bmatrix}, \; N_2 = \begin{bmatrix} -7.2484 & -0.0510 & 1.3609 \\ 0.0128 & -6.8255 & -0.0014 \\ -4.0768 & 0.0158 & -1.2692 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} -1.0969 & 2.8188 \\ 0.0171 & -0.0200 \\ -8.3702 & 10.8317 \end{bmatrix}, \; L_2 = \begin{bmatrix} 4.5317 & -3.8926 \\ 0.0038 & -0.0024 \\ 7.3941 & -10.1249 \end{bmatrix}$$

It is important to note that the derived results can tolerate some level of uncertainties on the state matrices of the multiple model. The designed unknown input observer is proved to be robust against state matrices uncertainties.

7.4.2 Application to Chaotic System Reconstruction

Results developed in section 7.2 can be applied to reconstruct states of chaotic system in multiple model representation and also for a secure communication system. Indeed, the problem we are faced with consists of transmitting some coded message with a signal broadcasted by a communication channel. At the receiver side, the hidden signal is recovered by a decoding system. The increasing need of secure communications leads to the development of many techniques which make difficult the detecting of transmitted message (se for example [18, 8, 3, 2, 1]). In this section, our goal to show how the designed observer could be used in chaotic system reconstruction and in a secure communication scheme. For this purpose we use the nonlinear Lorenz model as chaotic systems represented by his equivalent chaotic multiple model. Consider the non linear Lorenz equation [1]:

$$\begin{cases} \dot{x}_1(t) = -ax_1(t) + ax_2(t) \\ \dot{x}_2(t) = cx_1(t) - x_2(t) - x_1(t)x_3(t) \\ \dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t) \end{cases} \tag{7.36}$$

Which can be rewritten as follows

$$\dot{x}(t) = A(x(t))x(t) \tag{7.37}$$

with:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A(x(t)) = \begin{bmatrix} -a & a & 0 \\ c & -1 & -x_1(t) \\ 0 & x_1(t) & -b \end{bmatrix}$$

and a, b, and c are constants. Assume that $x_1(t) \in [-d, d]$ with $d > 0$. Then, we can write $x_1(t) = -d.\mu_1(x_1(t)) + d.\mu_2(x_1(t))$ with $\mu_1(x_1(t)) + \mu_2(x_1(t)) = 1$ which leads to the following multiple model:

$$\dot{x}(t) = (\mu_1(x_1(t))A_1 + \mu_2(x_1(t))A_2)\,x(t) \tag{7.38}$$

where

$$A_1 = \begin{bmatrix} -a & a & 0 \\ c & -1 & -d \\ 0 & d & -b \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & a & 0 \\ c & -1 & d \\ 0 & -d & -b \end{bmatrix}$$

and

$$\mu_1(x_1(t)) = \frac{1}{2}\left(1 + \frac{x_1(t)}{d}\right) \quad \mu_2(x_1(t)) = \frac{1}{2}\left(1 - \frac{x_1(t)}{d}\right)$$

Note that the obtained multiple model exactly represents the nonlinear Lorenz model under $x_1(t) \in [-d, d]$.

In the following, we consider the chaotic multiple model (7.38) with $a = 10$, $b = 8/3$, $c = 28$ and $d = 30$ in his general form:

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} \mu_i(y_1)\left(A_i x + R_i \bar{u}\right) \\ y = Cx + F\bar{u} \end{cases} \tag{7.39}$$

with:

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -30 \\ 0 & 30 & -^8/_3 \end{bmatrix} \quad A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 30 \\ 0 & -30 & -^8/_3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix} \quad F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The simulation of multiple model (7.39) without the unknown input $\bar{u}$ and with the initial value $x_0 = (1\ 1\ 1)^\top$ shows the chaotic behavior of the example plotted in the phase plan of the system (see figure 7.12).

The message to be encoded constitutes the so-called unknown input of the multiple model which plays the role of the encoder. The output of this model is transmitted using a public channel. On the receiving side, an unknown input multiple observer serves as a decoder in order to re-build the message. Clearly, the choice of LTI local models, their number, as well as the nature of the function $\mu_i(\xi(t))$ are key elements for an external person to be able to decode the embodied crypted message from only the signal $y(t)$. The goal of the proposed example is only to show

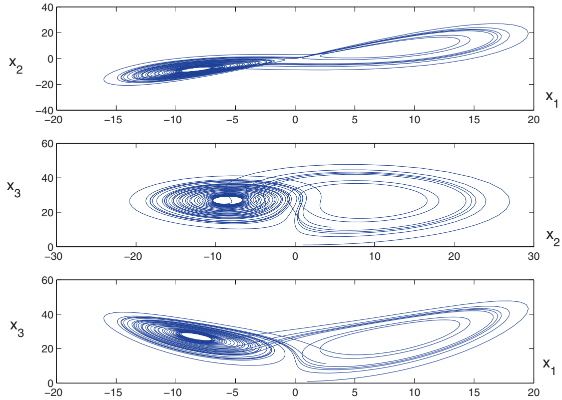


Fig. 7.12 Phase plan of the chaotic multiple model (7.38)

the feasibility of the proposed design in chaotic system reconstruction and in secure communication procedure.

Indeed, the unknown input can represent the hidden message to be transmitted. Thus the transmitted signal y is embedded with the hidden message $\bar{u}$ of the figure 7.14 .

The considered multiple observer for this application is

$$\begin{cases} \dot{z} = \sum_{i=1}^{2} \mu_i(y_1)\left(N_i z + L_i y\right) \\ \hat{x} = z - Ey \end{cases} \tag{7.40}$$

The resolution of conditions (7.26) with $B_1 = B_2 = (0,0,0)^\top$ lead to the following result:

$$X = \begin{bmatrix} 1.750 & 1.650 & -0.003 \\ 1.650 & 1.750 & -0.003 \\ -0.003 & -0.003 & 0.195 \end{bmatrix} \quad E = \begin{bmatrix} -3.05 & 3.05 \\ 3.99 & -3.99 \\ -0.004 & 0.004 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 33.66 & 47.89 & -91.72 \\ -36.18 & -45.78 & 89.96 \\ 61.12 & -32 & -2.79 \end{bmatrix} \quad L_1 = \begin{bmatrix} -16.44 & 17.44 \\ -14.94 & 15.94 \\ 253.9 & -252.9 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 35.06 & 49.54 & 91.72 \\ -37.82 & -47.14 & -89.96 \\ -62.08 & 31.20 & -2.53 \end{bmatrix} \quad L_2 = \begin{bmatrix} -19.38 & 17.32 \\ -13.67 & 17.67 \\ -252.38 & 253.37 \end{bmatrix}$$

Figure 7.13 represent the state estimation error with the initial conditions $x_0 = (1\ 1\ 1)^\top$ and $\hat{x}_0 = (0\ 0\ 0)^\top$. It shows the good reconstruction of chaotic system state. Figure 7.14 displays the hidden transmitted message and its estimate. Excepted around the time origin, the unknown input (transmitted message) is perfectly estimated.

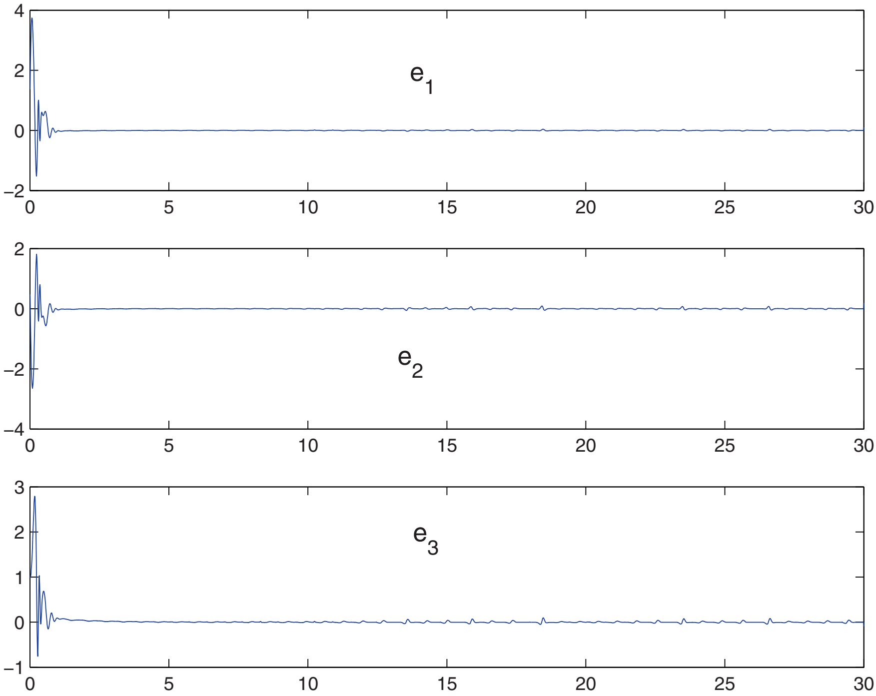


Fig. 7.13 Estimation errors $e_i = x_i - \hat{x}_i$, $i \in \{1, 2, 3\}$

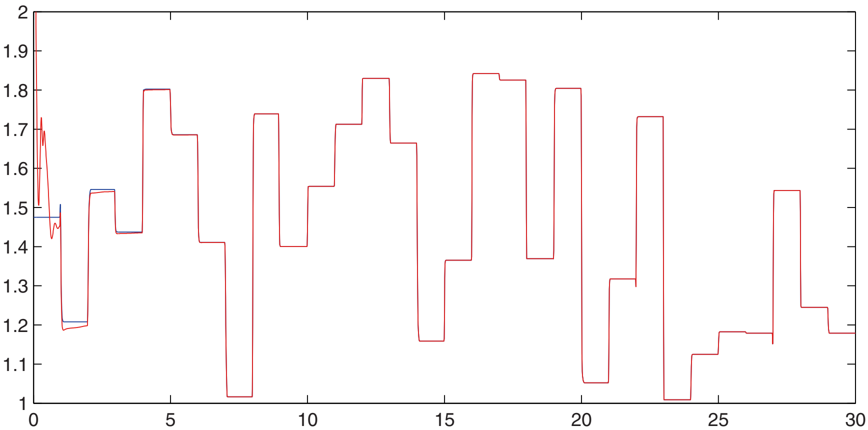


Fig. 7.14 Hidden message $\bar{u}$ and its estimate

7.5 Extension to Discrete-Time Multiple Model

Consider the class of a nonlinear discrete-time system subject to unknown inputs represented by a discret-time multiple model:

$$\begin{cases} x(t+1) = \sum_{i=1}^{M} \mu_i(\xi(t))(A_i x(t) + B_i u(t) + R_i \bar{u}(t) + D_i) \\ y(t) = Cx(t) + F\bar{u}(t) \end{cases} \tag{7.41}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ the input vector, $\bar{u}(t) \in \mathbf{R}^q$, $q < n$, contains the unknown inputs and $y \in \mathbf{R}^p$ the measured outputs. Matrices $A_i \in \mathbf{R}^{n\times n}$ and $B_i \in \mathbf{R}^{n\times m}$. Matrices $R_i \in \mathbf{R}^{n\times q}$ and $F \in \mathbf{R}^{p\times q}$ are assumed to satisfy assumptions 1 and 2. The matrices $D_i \in \mathbf{R}^n$ are introduced to take into account the operating point of the system and $C \in \mathbf{R}^{p\times n}$ is the output matrix. Functions $\mu_i(\xi(t))$ are as defined in (7.2).

The considered structure of the multiple observer is

$$\begin{cases} z(t+1) = \sum_{i=1}^{M} \mu_i(\xi(t))\big(N_i z(t) + G_{i1} u(t) + G_{i2} + L_i y(t) \\ \hat{x}(t) = z(t) - Ey(t) \end{cases} \tag{7.42}$$

where $N_i \in \mathbf{R}^{n\times n}$, $G_{i1} \in \mathbf{R}^{n\times m}$, $G_{i2} \in \mathbf{R}^n$, $E \in \mathbf{R}^{n\times p}$, $L_i \in \mathbf{R}^{n\times p}$ are the observer gains to be determined. The considered problem concerns both the reconstruction of state variable $x(t)$ and unknown input $\overline{u}(t)$, using only the available signal, that is known input $u(t)$ and measured output $y(t)$.

The following result gives sufficient conditions for the global asymptotic convergence of the state estimation error (7.6).

Theorem 7.4. *The state estimation error between multiple model (7.41) and unknown input multiple observer (7.42) converges globally asymptotically towards zero if there exists matrices $X > 0$, S and W_i such that the following conditions hold $\forall\, i \in I_M$:*

$$\begin{bmatrix} X & * \\ XA_i + SCA_i - W_i C & X \end{bmatrix} > 0 \tag{7.43a}$$

$$(X + SC)R_i = W_i F \tag{7.43b}$$

$$SF = 0 \tag{7.43c}$$

Multiple observer (7.42) is then completely defined by (7.9).

Proof. The prrof is obvious by using the same arguments used for proving theorem 1 and theorem 2. □

7.5.1 Pole Assignment

To improve performances of the multiple observer for better estimation of system state, dynamics of the multiple observer are constrained to be faster than that of the multiple model. As stated in section 2.3, it is possible to assign the poles to a specific sub-region is the complex plane [9]. For example if the prescribed region $S_r(\sigma, r)$ is a disk centered at $(\sigma, 0)$ and radius r, the $\mathcal{LMI}$ formulation of the previous problem is expressed by the following corollary.

Corollary 5: If there exist matrices X, S and W_i such that the following conditions hold $\forall\ i \in I_M$:

$$\begin{bmatrix} rX & * \\ X(A_i - \sigma\mathbf{I}) + SC(A_i - \sigma\mathbf{I}) - W_iC & rX \end{bmatrix} > 0 \tag{7.44a}$$

$$(X + SC)R_i = W_iF \tag{7.44b}$$

$$SF = 0 \tag{7.44c}$$

then the multiple observer (7.42) is globally asymptotically convergent with the performance defined by the complex region $S(\sigma, r)$. The observer parameters are as defined by (7.9).

Remark 4. Note that the $\mathcal{LMI}$ constraints (7.44) can be obtained from (7.43) by simply replacing the matrices A_i by $(A_i - \sigma\mathbf{I})/r$. Moreover if we are interested by the region $S_0(0, \alpha)$ it suffices to chose $\sigma = 0$ and $r = \alpha$.

Note that the $\mathcal{LMI}$ conditions (7.44) can be extended to incertain case, i.e. $\widehat{A}_i = A_i + \Delta A_i(t)$, as follows

Corollary 6: If there exit matrices $X > 0$, S, W_i and scalars ε_{i1} and ε_{i2} such that the following $\mathcal{LMI}$ conditions hold $\forall\ i \in I_M$:

$$\begin{bmatrix} rX & -XA_i - SCA_i + W_iC & -(X + SC)D_{A_i} \\ (*) & rX - \varepsilon_{i1}E_{A_i}^{\top}E_{A_i} & 0 \\ (*) & (*) & \varepsilon_{i1}\mathbf{I} \end{bmatrix} > 0 \tag{7.45a}$$

$$(X + SC)R_i = W_iF \tag{7.45b}$$

$$SF = 0 \tag{7.45c}$$

Then, multiple observer (7.42) is globally asymptotically convergent with the performance defined by complex region $S_r(\sigma, r)$. The multiple observer gains are as defined by (7.9).

Summarizing the estimation procedure, the design of multiple observer and the estimation of unknown inputs can be implemented as follows:

Procedure 2:

i) Solve the linear constraints (7.43) (or (7.45) for pole placement with uncertainties) with numerical tools such as the LMITOOL software [25],

ii) Deduce the observer parameters N_i, G_{i1}, G_{i2}, L_i and E of the multiple observer (7.42) using the equations (7.9).

iii) Under the assumption 1, estimate unknown input estimation using equation (7.35).

7.6 Application to Chaotic System Reconstruction

In this section, the proposed multiple observer is used to reconstruct states of chaotic systems and can be exploited in secure communication scheme. The message to be encoded is the unknown input of the chaotic multiple model.

Consider a chaotic discrete-time multiple model that results from the interpolation of two local models:

$$\begin{cases} x(t+1) = \sum_{i=1}^{2} \mu_i(\xi(t))\big(A_i x(t) + R_i \bar{u}(t)\big) \\ y(t) = Cx(t) + F\bar{u}(t) \end{cases} \tag{7.46}$$

The functions $\mu_i(.)$ depend on the multiple model output, $\xi(t) = y(t)$, and expressed by

$$\begin{cases} \xi(t) = y(t) \\ \mu_1(\xi(t)) = \frac{1}{2}(1 - \tanh(\xi(t))) \\ \mu_2(\xi(t)) = 1 - \mu_1(\xi(t)) \end{cases} \tag{7.47}$$

The numerical values of matrices are as follows:

$$A_1 = \begin{bmatrix} -1.1 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & -0.1 \\ 1 & 1.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad F = 5$$

From the structure of multiple model (7.46), we can deduce the following values:

$$S = 0, \quad E = 0, \quad G_{i1} = 0, \quad G_{i2} = 0$$

For this example, since the encoding system (7.46) can be conceived at the same time as the decoding system (observer), the computation matrices R_i is then free. Thus, $\mathcal{LMI}$ (7.43a) can be solved without taking into account equalities (7.43b-c). The resolution of $\mathcal{LMI}$ (7.43a) gives

$$X = \begin{bmatrix} 1.6718 & -2.0563 \\ -2.0563 & 7.7169 \end{bmatrix} \quad W_1 = \begin{bmatrix} -3.9158 \\ 9.0362 \end{bmatrix} \quad W_2 = \begin{bmatrix} -2.3610 \\ 13.5810 \end{bmatrix}$$

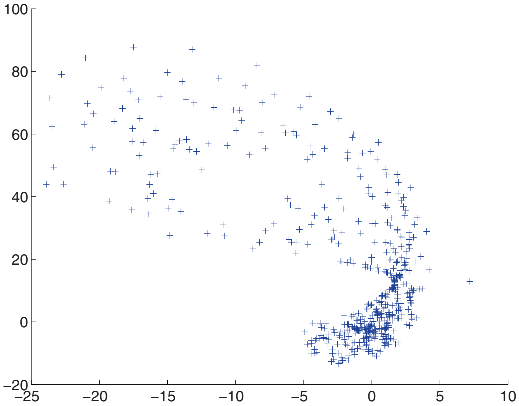


Fig. 7.15 Phase plan of the system

Using equations (7.9d-e), we obtain

$$L_1 = \begin{bmatrix} -1.3418 \\ 0.8134 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1.1192 \\ 2.0581 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -0.4291 & 1.1709 \\ -0.1067 & 0.2933 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0.2404 & -0.6596 \\ -0.0291 & 0.0709 \end{bmatrix}$$

from (7.46b) with W_i and X, we compute the values of R_1 and R_2 as follows

$$R_1 = \begin{bmatrix} -6.7090 \\ 4.0671 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 5.5959 \\ 10.2906 \end{bmatrix}$$

The designed unknown multiple observer can be applied in chaotic system reconstruction and also in a secure communication procedure. In this context, the problem consists of transmitting a resulting ciphertext embedded to the output by a communication channel. At the receiver side, the hidden signal (plaintext) is retrieved via the synthesis approach, i.e. the designed unknown input multiple. Concerning the transmission of a crypted message on a public channel of communication, one can wonder about the possibility of detecting and retrieving the message from the transmitted signal. In literature, some answers are given and one of them is satisfied here with a simple ”visual appreciation” (see for example [18, 8, 2]). So, figure 7.15, plotted in the phase plan of the system, does not show any particular behavior of periodic type or with commutation. Obviously, these observations can not establish security on the inviolability of the transmitted signal. For simulation example, consider the message to be transmitted given by figure 7.16 and the resulting output (the encoded message) of the chaotic multiple model in figure 7.17. Figures 7.18 and 7.19 show the state variable of the chaotic system and its estimation which are perfectly superposed. Finally, figure 7.20 presents the estimated unknown input (message estimate) where the message is perfectly estimated except around the time origin.

Fig. 7.16 Message $\bar{u}(t)$

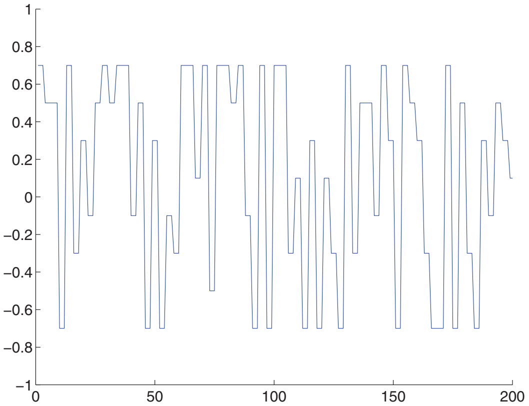


Fig. 7.17 Output $y(t)$

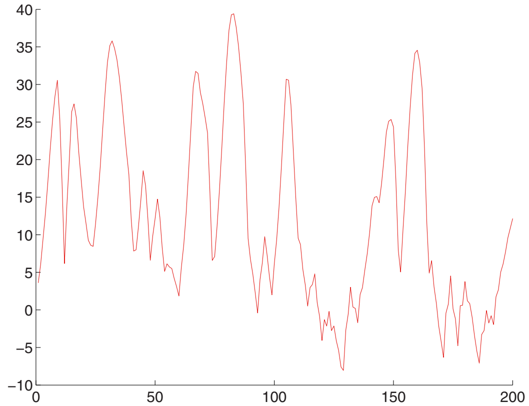
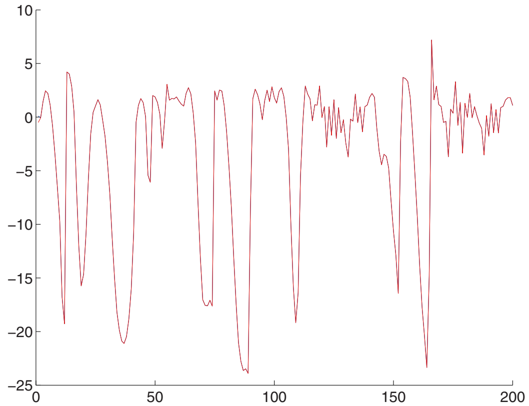


Fig. 7.18 $x_1(t)$ of chaotic multiple model and its estimate



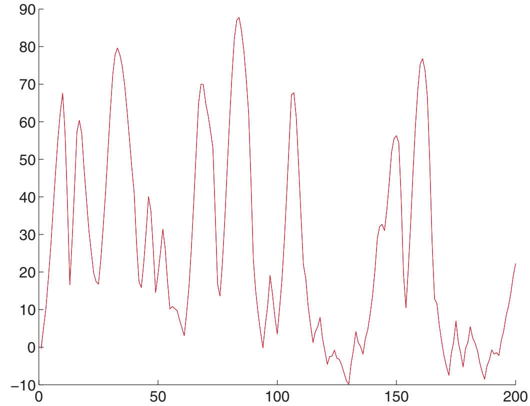


Fig. 7.19 $x_2(t)$ of chaotic multiple model and its estimate

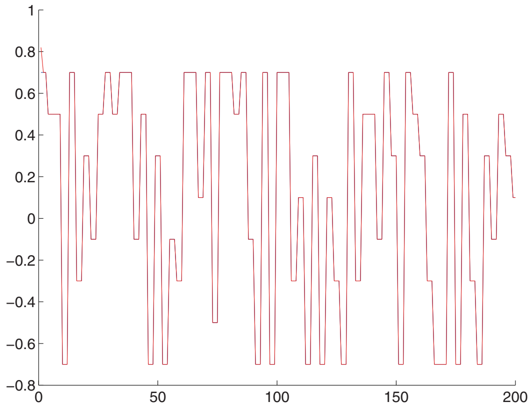


Fig. 7.20 Estimated message $\hat{\hat{u}}(t)$

7.7 Conclusion

In this chapter we have shown how to use multiple model approach, multiple observer and $\mathcal{LMI}$ formulation for chaotic system reconstruction. Indeed, an approach to design an observer for multiple models with unknown inputs affecting both the state and the output of the system is proposed. Uncertainties on state matrices are also considered. Sufficient conditions to design the proposed structure of observer are given in $\mathcal{LMI}$ terms under linear equality constraints. To improve the performances of the proposed unknown inputs multiple observer, poles assignment in $\mathcal{LMI}$ regions is also addressed for continuous-time and discrete-time. It is shown that this approach can be used in chaotic communications in the sense of signal masking and encryption. Moreover, the hidden message may be embedded in the state/output of the drive (chaotic) system which enhances the design flexibility. The designed unknown inputs multiple observer is shown to be satisfactory for message (unknown input) estimation and for chaotic system reconstruction.

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