Chapter 2
Hybrid Systems with Time-Dependent Switching

This chapter considers a broad class of HS whose switchings are activated according to time functions, i.e., a switching occurs at a certain time instant. These switching instants can be prescribed *a priori* and fixed, or designed arbitrarily by engineers. The motivation of researching HS appears from many practical systems e.g., circuit system, and also the switching control ideas. In this chapter, several FTC methods are presented for such HS. Two natural ideas follow: One way is to design FTC law in each faulty mode such that it is stable (Lyapunov stable, asymptotical stable or input-to-state stable) or the output regulation problem of each mode is solvable, then apply the standard stability results on HS (see sections 2.1-2.3). Another way is to research directly the stability of HS without reconfiguring the controller in each unstable mode (see sections 2.4-2.5). These two ideas will be developed in this chapter. The switching control techniques as developed in Chapter 6 also have their roots in this chapter.

2.1 Output-Input Stability Technique

In this section, we apply the output-input stability concept proposed in [70, 71] to the FTC design of HS with continuous faults.

The concept of *output−input stability* (OIS) [70, 71] is a robust variant of the minimum-phase property for general smooth nonlinear control systems. Its definition requires the state and the input of the system to be bounded by a suitable function of the output and derivatives of the output. Our objective is to provide a fault tolerant strategy for a class of hybrid nonlinear systems, in which each mode is output−input stable in the healthy situation and without full state measurements. The main ideas are that:

1 An observer-based FTC method is proposed for each output−input stable mode to make each mode asymptotically stable whenever faults occur during its dwell period;
2 A set of switching laws based on this FTC method are designed to guarantee the asymptotic stability of the overall HS.

H. Yang et al.: Fault Tolerant Control Design for Hybrid Systems, LNCIS 397, pp. 11–58.
springerlink.com

To make this section more readable, we first discuss the FTC for nonlinear systems in the following two subsections 2.1.1 and 2.1.2, then extend the obtained results to hybrid case in subsection 2.1.3.

2.1.1 State Feedback Control for Nonlinear System

Consider the following affine nonlinear system with faults

$$\begin{aligned} \dot{x} &= f(x) + G(x)u + E(x)f_a \\ y &= h(x) \end{aligned} \tag{2.1}$$

where $x \in \Re^n$ is the non measured state, $u \in \Re^m$ is the input, $y \in \Re^p$ is the output, and only the case $m \leq p$ is considered. Functions $f(\cdot)$, $G(\cdot)$, $E(\cdot)$ and $h(\cdot)$ are smooth, and it is assumed that $u \in \mathscr{C}^k$, the set of k times continuously differentiable functions $u : [0;\infty) \rightarrow \Re^m$, with $k \geq 1$. For all $u \in \mathscr{C}^k$, derivatives $\dot{y}, \ddot{y}, \ldots, y^{(k+1)}$ are assumed to exist and to be continuous.

The fault effect is modelled by a "fault pattern", described by the distribution matrix $E(x)$ and a "fault parameter" $f_a \in \Re^d$, which can be time varying, and is supposed to be norm bounded, i.e., $\exists f_1 : |f_a| < f_1$. The fault pattern describes the family of faults that are investigated [152], as identified e.g. through standard methods like failure modes and effect analysis (FMEA) [10]. The fault parameter describes the size of the fault, and its time evolution. It is assumed that the distribution matrix $E(x)$ satisfies the so-called matching condition

$$E(x) = G(x) \cdot W(x) \tag{2.2}$$

i.e. it can be factorized as (2.2) for some $m \times d$ continuous matrix $W(x)$. The interpretation of the matching condition is that the effect of faults can be described by a deviation of the control signal. This model covers actuator faults and a large number of system faults.

Definition 2.1. *[70] System (2.1) with $f_a = \mathbf{0}$ is called* output-input stable *if there exist a positive integer N, a function β of class $\mathscr{K}\mathscr{L}$, and a function γ of class $\mathscr{K}_\infty$ such that for every initial state $x(0)$ and every input $u \in \mathscr{C}^{N-1}$ its solution $x(t)$ satisfies*

$$\left| \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right| \leq \beta(|x(0)|, t) + \gamma\left(\left\| \underline{y}_N \right\|_{[0,t]} \right) \tag{2.3}$$

for all t, where $\underline{y}_k \triangleq (y^\top, \dot{y}^\top, \ldots, y^{(k)\top})^\top$.

Note that (2.3) implies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\left\| \underline{y}_N \right\|_{[0,t]} \right) \tag{2.4}$$

According to [70], the system is said to be *weakly uniformly 0-detectable of order* N if inequality (2.4) holds, or just *weakly uniformly 0-detectable* when an order is not specified.

The weak uniform 0-detectability is independent on any input, which implies that even when the faulty system is not output-input stable any more, it is still weakly uniformly 0-detectable if faults satisfy the matching condition (2.2). This property is very useful for FTC.

The following structure algorithm will be helpful to construct the feedback controller later. Due to the structure of the fault distribution matrix (2.2), the term $G(x)u+E(x)f_a$ is written as $G(x)\bar{u}$ where $\bar{u}=u+W(x)f_a$.

Algorithm 2.1. *nonlinear structure algorithm*

Step 1: *Define* $\tilde{h}_1(x)\triangleq L_f h(x)$*,* $\tilde{J}_1(x)\triangleq L_G h(x)$*. Differentiating* y *with respect to time along the trajectories of (2.1) gives*

$$\dot{y}=\tilde{h}_1(x)+\tilde{J}_1(x)\bar{u} \tag{2.5}$$

Assume that matrix $\tilde{J}_1(x)$ *has constant rank* r_1 *and a fixed set of* r_1 *rows that are linearly independent for all* x*, these rows are taken as the first* r_1 *rows of* $\tilde{J}_1(x)$*.*

Denote $\check{h}_1(x)$ *and* $\hat{h}_1(x)$ *as respectively the first* r_1 *and the last* $p-r_1$ *components of* $\tilde{h}_1(x)$*, then Eq.(2.5) is divided into two parts as*

$$\dot{y}_{1\dots r_1}=\check{h}_1(x)+J_1(x)\bar{u}$$

and

$$\dot{y}_{r_1+1\dots p}=\hat{h}_1(x)+\hat{J}_1(x)\bar{u} \tag{2.6}$$

where $(\cdot)_{1\dots k}$ *denotes the first* k *elements of the signal.* $J_1(x)$ *is a matrix of full row rank, and* $\hat{J}_1(x)=f_1(x)J_1(x)$ *for some* $(p-r_1)\times r_1$ *matrix* $f_1(x)$*.*

Define $\bar{h}_1(x,\dot{y}_{1\dots r_1})\triangleq\hat{h}_1(x)+f_1(x)(\dot{y}_{1\dots r_1}-\check{h}_1(x))$*. Eq.(2.6) can be rewritten as*

$$\dot{y}_{r_1+1\dots p}=\bar{h}_1(x,\dot{y}_{1\dots r_1}) \tag{2.7}$$

Step 2: *Similar to Step 1, define*

$$\tilde{h}_2(x,\dot{y}_{1\dots r_1},\ddot{y}_{1\dots r_1})\triangleq L_f\bar{h}_1(x)+\sum_{i=1}^{r_1}\frac{\partial\bar{h}_1}{\partial\dot{y}_i}(x,\dot{y}_{1\dots r_1})\ddot{y}_i$$

$$\tilde{J}_2(x,\dot{y}_{1\dots r_1})\triangleq L_G\bar{h}_1(x)$$

Differentiating (2.7) leads to

$$\ddot{y}_{r_1+1\dots p}=\tilde{h}_2(x,\dot{y}_{1\dots r_1},\ddot{y}_{1\dots r_1})+\tilde{J}_2(x,\dot{y}_{1\dots r_1})\bar{u} \tag{2.8}$$

The termination condition of the structure algorithm at Step 2, denoted as C 1, is as follows:

C 1: *The matrix* $\begin{bmatrix} J_1(x) \\ \tilde{J}_2(x, \dot{y}_{1\ldots r_1}) \end{bmatrix}$ *is continuous and has constant rank m and there is a fixed set of* $m - r_1$ *rows of* $\tilde{J}_2(x, \dot{y}_{1\ldots r_1})$ *which together with the rows of* $J_1(x)$ *form a linearly independent set for all x and* $\dot{y}_{1\ldots r_1}$. *These rows are taken as the first* $m - r_1$ *rows of* $\tilde{J}_2(x, \dot{y}_{1\ldots r_1})$.

Denote $\check{h}_2(x)$ and $\hat{h}_2(x)$ as respectively the first $m - r_1$ and the last $p - m$ components of $\tilde{h}_2(x)$. Under C 1, since $m \le p$, Eq.(2.8) can be written similarly to Step 1 as

$$\ddot{y}_{r_1+1\ldots m} = \check{h}_2(x, \dot{y}_{1\ldots r_1}, \ddot{y}_{1\ldots r_1}) + J_2(x, \dot{y}_{1\ldots r_1})\bar{u}$$

and

$$\ddot{y}_{m+1\ldots p} = \hat{h}_2(x, \dot{y}_{1\ldots r_1}, \ddot{y}_{1\ldots r_1}) + \hat{J}_2(x, \dot{y}_{1\ldots r_1})\bar{u} \tag{2.9}$$

The following Lemma is a special case of Theorem 1 in [71], therefore its proof is omitted. It gives a necessary and sufficient OIS condition.

Lemma 2.1. *Under the termination condition C 1, the system (2.1) with* $f_a = \mathbf{0}$ *is output-input stable if and only if it is weakly uniformly 0-detectable.*

Based on Algorithm 2.1, a state feedback controller is now designed for the healthy system, $m = p$ is considered, the extension to $m \le p$ is straightforward. Two assumptions are imposed.

Assumption 2.1. *The vector* $\dot{y}_{r_1+1,\ldots,m}$ *is not affected directly by input signals, which results, for an output-input stable system (2.1) with* $f_a = \mathbf{0}$, *in the fact that* $f_1(x) = \mathbf{0}$.

Assumption 2.2. *Let* $\chi \in \Re^{2m-r_1} \triangleq (y_{1\ldots r_1}^\top, y_{r_1+1\ldots m}^\top, \dot{y}_{r_1+1\ldots m}^\top)^\top$. *When* $f_a = \mathbf{0}$, *there exists an invertible map* $T : \Re^n \to \Re^{2m-r_1}$, *such that* $\chi = T(x)$.

Since $m = p$, Eq.(2.9) is removed. Under C 1 and assumptions 2.1-2.2, the algorithm 2.1 leads to

$$\begin{bmatrix} \dot{y}_{1\ldots r_1} \\ \ddot{y}_{r_1+1\ldots m} \end{bmatrix} = \begin{bmatrix} \check{h}_1(x) \\ \check{h}_2(x) \end{bmatrix} + \begin{bmatrix} J_1(x) \\ J_2(x) \end{bmatrix} \bar{u} \tag{2.10}$$

where $\check{h}_2 = \tilde{h}_2$, $J_2 = \tilde{J}_2$.

The state feedback control design consists of the following three steps:

Step 1: Choose a Hurwitz matrix A_{10}, which gives $\dot{y}_{1\ldots r_1} = A_{10} y_{1\ldots r_1}$ provided that $J_1(x)\bar{u} = \vartheta_1(x)$ with

$$\vartheta_1(x) \triangleq A_{10} y_{1\ldots r_1} - \check{h}_1(x)$$

Step 2: Choose two $(m - r_1) \times (m - r_1)$ matrices A_{21} and A_{20} such that

$$\ddot{y}_{r_1+1\ldots m} = A_{21}\dot{y}_{r_1+1\ldots m} + A_{20} y_{r_1+1\ldots m}$$

The matrix $\begin{bmatrix} \mathbf{0} & I_{(m-r_1)\times(m-r_1)} \\ A_{20} & A_{21} \end{bmatrix}$ is Hurwitz provided that $J_2(x)\bar{u} = \vartheta_2(x)$ and

$$\vartheta_2(x) \triangleq A_{21}\dot{y}_{r_1+1\ldots m} + A_{20}y_{r_1+1\ldots m} - \check{h}_2(x)$$

Step 3: Design the state feedback controller $u_n(x)$ as

$$u_n(x) \triangleq \begin{bmatrix} J_1(x) \\ J_2(x) \end{bmatrix}^{-1} \begin{bmatrix} \vartheta_1(x) \\ \vartheta_2(x) \end{bmatrix} \tag{2.11}$$

Define

$$h_\chi(x) \triangleq \begin{bmatrix} \check{h}_1(x) \\ \mathbf{0} \\ \check{h}_2(x) \end{bmatrix}, \quad J_\chi(x) \triangleq \begin{bmatrix} J_1(x) \\ \mathbf{0} \\ J_2(x) \end{bmatrix}$$

$$\bar{A} \triangleq \begin{bmatrix} A_{10} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{(m-r_1)\times(m-r_1)} \\ \mathbf{0} & A_{20} & A_{21} \end{bmatrix}$$

Then under the control $u_n(x)$, the system (2.10) is augmented as

$$\dot{\chi} = h_\chi(x) + J_\chi(x)u_n = \bar{A}\chi \tag{2.12}$$

Therefore, $u_n(x)$ in (2.11) asymptotically stabilizes system (2.12) if A_{10},A_{20}, and A_{21} are chosen such that $\bar{A}$ is Hurwitz. An "optimized" choice of $\bar{A}$ can be refered to [61]. The weak uniform 0-detectability implies that the closed-loop system is stabilized.

2.1.2 Observer-Based FTC for Nonlinear System

Now we provide an observer-based method to stabilize system (2.1) under both healthy and faulty conditions.

The FD scheme in [56] is first applied to provide rapid and accurate estimation of states and faults. Denote $\hat{x}$ and $\hat{f}_a$ as the estimates states and faults respectively. Using the differential geometry theory, we can obtain (see [56] for details) a global diffeomorphism $z = N(x)$ with $N(0) = 0$ and $z \in \Re^n$ that satisfies

$$|\tilde{z}| \leq \mu(\lambda^*)|\tilde{z}(0)|\exp(-\lambda^* t) \tag{2.13}$$

where $\tilde{z} \triangleq z - \hat{z}$, $\lambda^* > 0$, $\mu(\lambda^*) > 0$ is polynomial in λ^*. We can also get from [56] that $f_a(t) - \hat{f}_a(t) \to 0$ when $z(t) - \hat{z}(t) = 0$. This means that rapid and accurate fault estimates can always be obtained when faults occur.

The following two lemmas provide the control strategy for the healthy case and faulty case respectively.

Lemma 2.2. *Suppose that the output-input stable system (2.1) with $f_a = \mathbf{0}$ and $m = p$ satisfies C 1 and assumptions 2.1-2.2. Given an initial $x(0)$, there exists a constant $\varepsilon_1 > 0$ such that if $|\tilde{\bar{z}}(0)| \leq \varepsilon_1$, then the control $u(\hat{x}) = u_n(\hat{x})$ makes the origin of the closed-loop system asymptotically stable.*

Proof : In the healthy case, system (2.12) controlled by $u_n(\hat{x})$ is rewritten as

$$\dot{\chi} = \bar{A}\chi + J_\chi(x)(u(\hat{x}) - u(x)) \tag{2.14}$$

Let P be the symmetric positive definite solution of the Lyapunov equation $\bar{A}^\top P + P\bar{A} = -Q$ with a given matrix $Q > 0$. Consider the Lyapunov function $V = \chi^\top P\chi$, its time derivative with respect to (2.14) is

$$\begin{aligned}\dot{V} &= -\chi^\top Q\chi + 2\chi^\top PJ_\chi(x)(u(\hat{x}) - u(x)) \\ &\leq -\lambda_{\min}(Q)|\chi|^2 + 2|\chi|\cdot|P|\cdot|J_\chi(x)|\cdot|u(\hat{x}) - u(x)|\end{aligned} \tag{2.15}$$

Consider the given initial $x(0)$, and define $\Omega \triangleq \{\chi : V(\chi) \leq \chi(0)^\top P\chi(0)\}$, which are the level sets of V with respect to χ (see Chapter 4 in [62]).

Note that $|u(\hat{x}) - u(x)|$ is continuous within the region Ω, and vanishes when $\hat{x} - x = \mathbf{0}$, i.e., $\tilde{z} = \mathbf{0}$. There exists two constants $\bar{\varepsilon}_1 > 0$ and $\kappa_1 > 0$, such that $|\bar{\tilde{z}}_2| \leq \bar{\varepsilon}_1 \Longrightarrow |u(\hat{x}) - u(x)| \leq \kappa_1|\tilde{z}|$. From inequality (2.15) it follows

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(Q)|\chi|^2 + 2\kappa_1|\chi|\cdot|P|\cdot|\tilde{z}|\sqrt{\left(\lambda_{\max}(J_\chi^\top(x)J_\chi(x))\right)_{(\chi\in\Omega)}} \\ &\leq -(1-r)\lambda_{\min}(Q)|\chi|^2\end{aligned} \tag{2.16}$$

$$\forall|\chi| \geq \sqrt{\frac{2\kappa_1|P|\cdot|\bar{\tilde{z}}_2|\sqrt{\left(\lambda_{\max}(J_\chi^\top(x)J_\chi(x))\right)_{(\chi\in\Omega)}}}{r\lambda_{\min}(Q)}} \triangleq \bar{\gamma}(|\bar{\tilde{z}}_2|), 0 < r \leq 1 \tag{2.17}$$

where $\bar{\gamma}(\cdot)$ is a class $\mathcal{K}$ function. There exists a constant $\bar{\varepsilon}_2$ such that $|\tilde{z}| \leq \bar{\varepsilon}_2$ satisfies (2.17). Based on [62], the choice of $|\tilde{z}(0)| \leq \varepsilon_1$ where $\varepsilon_1 = \min(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$, clearly results in χ being input-to-state stable with respect to $\tilde{z}$. Note that $\lim_{t\to\infty}\tilde{z}(t) = 0$. Hence the origin of the system (2.14) is asymptotically stable. On the other hand, the map $T(x)$ is invertible and not affected by the observer, and system (2.1) is weakly uniformly 0-detectable, which leads to the asymptotic stability of the origin of the system. □

Lemma 2.3. *Consider the output-input stable system (2.1) with $f_a = \mathbf{0}$ and $m = p$ satisfying C 1 and assumptions 2.1-2.2. Let a fault occur at $t = 0$. Given an initial $x(0)$, there exists a constant $\varepsilon_2 > 0$ such that for all $|\bar{\tilde{z}}_2(0)| \leq \varepsilon_2$, the control $u(\hat{x}) = u_n(\hat{x}) - W(\hat{x})\hat{f}_a$ makes the origin of the closed-loop faulty system (2.1) asymptotically stable.*

Proof: In the faulty case, the system (2.10) controlled by $u_n(\hat{x}) - W(\hat{x})\hat{f}_a$ is rewritten as

$$\dot{\chi} = \bar{A}\chi + J_\chi(x)\Big(u_n(\hat{x}) - u_n(x)\Big) + J_\chi(x)W(\hat{x})(f_a - \hat{f}_a) + J_\chi(x)\Big(W(x) - W(\hat{x})\Big)f_a \tag{2.18}$$

The time derivative of V along (2.18) is

$$\begin{aligned}\dot{V} = -\chi^\top Q\chi + 2\chi^\top PJ_\chi(x)\Big[&\Big(u_n(\hat{x}) - u_n(x)\Big) \\ &+ W(x)(f_a - \hat{f}_a) + \Big(W(x) - W(\hat{x})\Big)f_a\Big]\end{aligned} \tag{2.19}$$

There exist two constants $\bar{\varepsilon}_3 > 0$ and $\kappa_2 > 0$, such that $|\tilde{z}| \leq \bar{\varepsilon}_3 \Longrightarrow |N(t) - \hat{N}(t)| \leq \kappa_2|\tilde{z}|$ within Ω. Similarly, there exist two constants $\bar{\varepsilon}_4 > 0$ and $\kappa_3 > 0$, such that $|\tilde{z}| \leq \bar{\varepsilon}_4 \Longrightarrow |W(x) - W(\hat{x})| \leq \kappa_3|\tilde{\bar{z}}_2|$. Following (2.19), appropriate selection of $\hat{\bar{z}}_2$ leads to

$$\dot{V} \leq -\lambda_{\min}(Q)|\chi|^2 + \Xi \tag{2.20}$$

$$\begin{aligned} \Xi \triangleq\ & 2|\chi| \cdot |P| \cdot |\tilde{z}| \cdot \sqrt{\left(\lambda_{\max}(J_\chi^\top J_\chi)\right)_{(\chi\in\Omega)}} \\ & \cdot \left[\kappa_1 + \kappa_2\sqrt{\left((\lambda_{\max}(\eta^\top\eta) \cdot \lambda_{\max}(W^\top W)\right)_{(\chi\in\Omega)}} + \kappa_3 f_1\right] \end{aligned} \tag{2.21}$$

where η is defined as in [56]. Given a physical bound of control signals and f_1, the value of $\lambda_{\max}[\eta^\top\eta]$ within Ω can be estimated. As in Lemma 2.2, there exists a constant $\varepsilon_2 > 0$ such that $|\tilde{z}(0)| \leq \varepsilon_2$ makes the origin of system (2.14) asymptotically stable. On the other hand, from the structure of faults in (2.2) and Assumption 2.2, $T(x)$ exists and is still invertible, the faulty system (2.1) is still weakly uniformly 0-detectable, which leads to the asymptotic stability of the origin of the closed-loop system. □

The following theorem provides a reconfiguration strategy based on the previous analysis.

Theorem 2.1. *Assume the output-input stable system (2.1) with $f_a = \mathbf{0}$ and $m = p$ satisfies C 1, assumptions 2.1-2.3. Faults are assumed to occur at $t = t_f$. Given a $x(0)$, there exists a constant $\omega = \min(\varepsilon_1, \varepsilon_2)$ such that for all $|\tilde{z}(0)| \leq \omega$, the following control*

$$u_s(\hat{x}, t_{fd}) \triangleq \begin{cases} u_n(\hat{x}), & t \in [0, t_{fd}) \\ u_n(\hat{x}) - W(\hat{x})\hat{f}_a, & t \in [t_{fd}, \infty) \end{cases} \tag{2.22}$$

makes the origin of the closed-loop system asymptotically stable, where t_{fd} is the time instant when the fault has been estimated.

Proof: From Lemma 2.2, under the control $u_n(\hat{x})$ with the initial $|\tilde{z}(0)| \leq \omega$, one has $\dot{V} < 0, \forall t \in [0, t_f)$, and $\chi(t_f) \in \bar{\Omega}$, where $\bar{\Omega} \subset \Omega$. Eq.(2.13) implies $|\tilde{z}(t_f)| \leq |\tilde{z}(0)|$. On the other hand, the fault can be detected at $t_{fd} = t_f$ if $|\tilde{z}(0)| \leq \omega$ (see [56] and [142]), which means the faults are detected rapidly. Therefore, after $t = t_{fd}$, inequality (2.20) holds under the control $u_n(\hat{x}) - W(\hat{x})\hat{f}_a$. The result of Lemma 2.3 is then applied to complete the proof. □

Remark 2.1. *Theorem 2.1 provides a flexible control architecture which guarantees that $\dot{V} < 0\ \forall t \in [0, \infty)$ whenever the faults occur, this property is very suitable for HS [142]. The proposed strategy treats the healthy system and the faulty system with different controllers, which leads to good system performance in the sense of FTC.*

Example 2.1: [142] A DC motor example is employed to illustrate a potential application field of this approach. A series DC motor is a DC motor where the field circuit is connected in series with the armature circuit [19]. Under the hypothesis that there is no magnetic saturation, the modified model of this system is expressed as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-k_1x_1x_2 - \frac{R}{L}x_1 + u_1 + Lf_a\\ -k_2x_2 + \frac{k_1}{JL}x_1^2 - \frac{x_3}{J}\\ u_2 + 2k_1x_1f_a\end{bmatrix} \tag{2.23}$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

where $x_1 = \phi_f$ denotes the flux, $x_2 = \omega_f$ denotes the speed, $x_3 = T_L$ denotes time varying load torque, u_1 and u_2 are the voltage inputs. the speed and the flux are measured.

Let us first consider the healthy case ($f_a = 0$). Since $x_1 = y_1$, $x_2 = y_2$, and $|x_3| = J|\dot{y}_2^2 + k_2y_2 - \frac{k_1}{JL}y_1^2| \le J|\dot{y}_2|^2 + Jk_2|y_2| + \frac{k_1}{L}|y_1|^2$, it is seen that the healthy system is weakly uniformly 0-detectable of order 1. The output derivatives are

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}-k_1x_1x_2 - \frac{R}{L}x_1\\ -k_2x_2 + \frac{k_1}{JL}x_1^2 - \frac{x_3}{J}\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

so $r_1 = 1$, differentiating the equality of $\dot{y}_2$ leads to

$$\ddot{y}_2 = k_2^2x_2 - \frac{k_1k_2}{JL}x_1^2 + \frac{k_2}{J}x_3 - \frac{2k_1^2}{JL}x_1^2x_2 - \frac{2k_1R}{JL^2}x_1^2 + \left[\,\frac{2k_1}{JL}x_1 \;\; -\frac{1}{J}\,\right]\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

The matrix $\begin{bmatrix}1 & 0\\ \frac{2k_1}{JL}x_1 & -\frac{1}{J}\end{bmatrix}$ is always nonsingular. The map $T : x \longrightarrow \chi$ is also invertible and not affected by the observer. C 1 and assumptions 2.1-2.2 are satisfied. From (2.11), u_n can be designed as

$$u_n = \begin{bmatrix}k_1x_1x_2 + (\frac{R}{L} - 1)x_1\\ (Jk_2^2 - \frac{k_1}{JL})x_1^2 - (\frac{2k_1}{L} + \frac{k_1k_2}{L})x_1^2 + (k_2 + \frac{1}{J})x_3\end{bmatrix}$$

which makes $\bar{A}$ Hurwitz.

Now consider the faulty case. It is clear that $W(x) = (L,\, 2k_1x_1)^\top$, f_a is an actuator fault that affects both control channels. The invertible transformation $z_1 = x_2$, $z_2 = -\frac{x_3}{J} + \frac{k_1}{JL}x_1^2$, $z_3 = x_1$ puts system (2.23) into the form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}z_2 - k_2y_1\\ -2\frac{k_1}{JL}y_2(k_1y_1y_2 + \frac{R}{L}y_2 - u_1) - \frac{u_2}{J}\end{bmatrix} \tag{2.24}$$
$$y_2 = z_1 \tag{2.25}$$
$$\dot{z}_3 = -k_1y_1y_2 - \frac{R}{L}y_1 + u_1 + Lf_a \tag{2.26}$$
$$y_1 = z_3$$

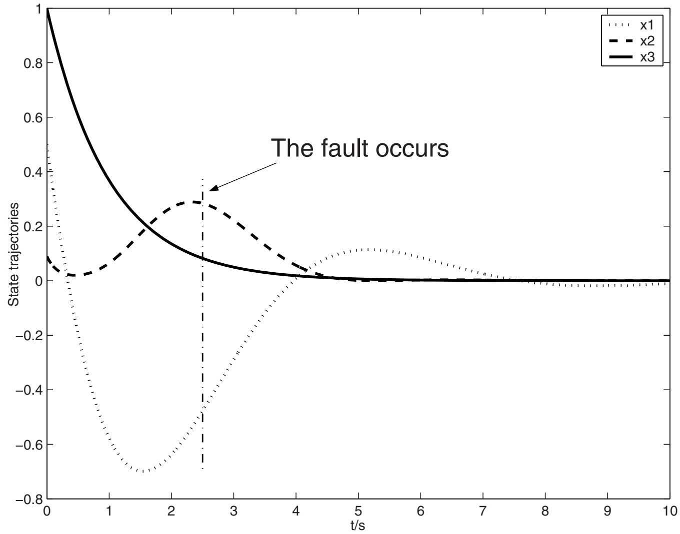


Fig. 2.1 State trajectories

Eq.(2.26) does not involve the estimation of z_1 and z_2, which implies that fault estimates are obtained without any estimation error. So the fault can be detected and compensated immediately after the fault occurs. Under control $u_n(x)$, one has $\dot{\chi} = \bar{A}\chi$, where $\chi = (y_1, y_2, \dot{y}_2)^\top$.

In the simulation, the parameters are [19]: $R = 0.0247$, $L = 0.06$, $J = 30.1$, $k_1 = 0.04329$, $k_2 = 0.0033$. The initial $x(0) = (0.5, 0.1, 1)^\top$. $\hat{x}(3) = 0.85$. The fault is considered as

$$f_a = \begin{cases} 0, & 0s \le t < 2.5s \\ 0.5 + 0.2\sin(5t), & 2.5s \le t < 10s \end{cases} \tag{2.27}$$

Fig.2.1 shows state trajectories, the origin of the closed-loop system is asymptotically stable in spite of faults.

2.1.3 FTC for Hybrid Systems

The above FTC solution is now extended to a class of switched systems taking the form

$$\begin{aligned} \dot{x} &= f_\sigma(x) + G_\sigma(x)u_\sigma + E_\sigma(x)f_{a\sigma} \\ y &= h(x) \end{aligned} \tag{2.28}$$

where each mode satisfies all the conditions in Theorem 2.1. $\sigma(t) : [t_0, \infty) \to Q = \{1, 2, \ldots, N\}$ is a switching signal, which is assumed to be a piecewise constant function continuous from the right.

The switching property is considered as in [29]: (a) the switching sequence is fixed, (b) there is a series of dwell periods Δt_{kj} for mode k when it is activated for the jth time and mode k switches to mode $(k+1)$ for the jth time at $t = t_{kj}$ when $\Delta t_{(k+1)j}$ is elapsed, (c) the states do not jump at the switching instants.

The observer-based method in Section 2.1.2 is modified for the HS as follows:

- The observer and the fault estimates scheme are switched according to the current mode at each switching time.
- The initial states of the current observer are chosen as the final states of the previous observer. The fault estimates are set to zero at each switching instant.

We also need to impose a condition on the above switching law such that the weak uniform 0-detectability of the overall HS can be guaranteed.

Assumption 2.3. *$\Delta t_{kj} (k = 1,2,\ldots,N)$ are large enough such that for any $s \in \Re+$, we have*

$$\beta_{k+1}(2\beta_k(2s,\Delta t_{kj}),\Delta t_{(k+1)j}) \leq \bar{\lambda} s < s \quad \forall k \in Q \tag{2.29}$$

where $0 < \bar{\lambda} < 1$ and $\beta_k (k \in Q)$ satisfies (2.4) for mode k.

Lemma 2.4. *Consider the HS (2.28) satisfying Assumption 2.3 in the healthy case. Then, the overall HS is weakly uniformly 0-detectable.*

Proof: Lemma 2.4 is an extension of Theorem 1 in [129] to the weak uniform 0-detectability case, its proof is omitted. □

Let V_k, $u_{sk}(\hat{x},t_{fdk})$, ω_k be respectively V, $u_s(\hat{x},t_{fd})$, ω for mode k. The FTC problem for the system (2.28) with unfixed dwell periods and fixed dwell periods will be discussed respectively.

Theorem 2.2. *Under Assumption 2.3, consider the HS (2.28) under a family of control laws $u_k(\hat{x},t_{fdk})$. There exists a constant ω_k such that $|\bar{\tilde{z}}_2(0)| \leq \omega_k$ with a given $x(0)$. If, at any time instant $\bar{t}$, the following conditions hold:*

$$|\tilde{z}(\bar{t})| \leq \omega_{k+1} \tag{2.30}$$

$$V_{k+1}(\chi(\bar{t})) < V_{k+1}(\chi(t_{(k+1)(j-1)})), \quad j > 0 \tag{2.31}$$

then, choosing $\Delta t_{kj} \geq \bar{t} - t_{kj}$, which satisfies (2.29), and setting $\sigma(t) = k+1$ at $t = t_{kj} + \Delta t_{kj}$ guarantee that the origin of the overall HS is asymptotically stable.

Proof: If the initial $|\tilde{z}(0)| \leq \omega_k$ for some $k \in Q$, it follows from Theorem 2.1 that $\dot{V}_k < 0$ as long as mode k remains active. If at some time instant $\bar{t}$ one has $|\tilde{z}(\bar{t})| \leq \omega_{k+1}$, and $\sigma(t) = k+1$ is set on, then for all $t \in [\bar{t}, t_{kj} + \Delta t_{kj})$, $\dot{V}_{k+1} < 0$ as long as $\sigma(t) = k+1$. It is concluded that if the k^{th} mode is activated only when $|\tilde{z}(t)| \leq \omega_k$, then

$$\dot{V}_\sigma(t) < 0, \;\; \forall \sigma(t) = k \tag{2.32}$$

Moreover, from (2.31), for any admissible switching time t_{kj} one has

$$V_{k+1}(\chi(t_{(k+1)j})) < V_{k+1}(\chi(t_{(k+1)(j-1)})) \tag{2.33}$$

Since the k^{th} faulty mode is still weakly uniformly 0-detectable, and T always exists, the Multiple Lyapunov function method [22] can be applied to conclude that the origin of the hybrid system is Lyapunov stable. On the other hand, for each switching time t_{kj}, $j = 1,2,\ldots$ such that $\sigma(t_{kj}^{+}) = k$, the sequence $V_{\sigma(t_{kj})}$ is decreasing and positive, and therefore has a limit $\zeta \geq 0$. One has

$$\lim_{j\to\infty}\left[V_{k+1}(\chi(t_{(k+1)(j+1)})) - V_{k+1}(\chi(t_{(k+1)j}))\right] = \zeta - \zeta = 0$$

Note that there exists a class $\mathscr{K}$ function α such that

$$\begin{aligned} 0 = &\lim_{j\to\infty}\left[V_{k+1}(\chi(t_{(k+1)(j+1)})) - V_{k+1}(\chi(t_{(k+1)j}))\right] \\ &\leq \lim_{j\to\infty}[-\alpha(\|\chi(t_{(k+1)j}\|)] \leq 0 \end{aligned} \tag{2.34}$$

Inequality (2.34) together with Lemma 2.4 implies that $x(t)$ converges to the origin, which combined with Lyapunov stability, leads to the asymptotic stability of the origin of the HS. This completes the proof. □

Remark 2.2. *Inequality (2.31) is used only when the target $k+1^{th}$ mode has been previously activated. Actually, when only a finite number of switchings is considered over the infinite time-interval, Inequality (2.31) can be relaxed to allow for finite increases in V_{k+1}, (see [28] and [29] for some analysis). In this case, inequality (2.30) alone is sufficient to enforce the asymptotic stability of the origin.*

Many real systems work under a series of prescribed dwell periods, i.e., Δt_{kj} is fixed. In this case, the goal of FTC must be achieved before each switching time whenever the faults occur. This is possible because the decay rate of V_k can be estimated. We have the following corollary.

Corollary 2.1. *Consider the HS (2.28) under a family of control laws $u_k(\hat{x}, t_{fdk})$ with fixed Δt_{kj} $k \in Q$ which satisfies (2.29). If each faulty mode satisfies (iv), T exists and is still invertible, and there exists a constant ω_k such that $|\tilde{\tilde{z}}_2(0)| \leq \omega_k$, then the origin of the overall hybrid system is asymptotically stable.*

Proof: It is clear from (2.13) that appropriate selection of λ makes (2.30) hold at a given $t_{(k+1)j}$. On the other hand, inequality (2.20) in Lemma 2.3 leads to

$$\dot{V}_k \leq -\lambda_{\min}(Q_k)|\chi|^2 + \Xi_k \leq -\iota_k V_k + \Xi_k, \quad \iota_k \triangleq \frac{\lambda_{\min}(Q_k)}{\lambda_{\max}(P_k)} \tag{2.35}$$

Note that Ξ_k is bounded within a known region and converges to zero, so the trajectory of V_k can be estimated by (2.35). The results of Theorem 2.2 can be applied to guarantee the asymptotic stability of the origin of the HS. □

2.2 Overall Fault Tolerant Regulation

This section extends the classical output regulation theories to hybrid nonlinear systems and analyzes its fault tolerance in the presence of continuous faults modeled by the exosignals.

2.2.1 Fault Tolerant Regulation for Nonlinear Systems

The considered system takes the general nonlinear form

$$\dot{x}(t) = G(x(t),u(t),f(t)) \tag{2.36}$$
$$y(t) = H(x(t),f(t)) \tag{2.37}$$
$$\dot{f}(t) = S(f(t)) \ \ \forall t \geq t_f, \text{ with } f(t) = 0 \ \ \forall t \in [0,t_f) \tag{2.38}$$
$$e(t) = y(t) - y_r(x(t)) \tag{2.39}$$

with measurable state $x \in \mathfrak{R}^n$, input $u \in \mathfrak{R}^p$, output $y \in \mathfrak{R}^m$. The regulated error e denotes the output tracking error between y and the continuous reference signal $y_r(x) : \mathfrak{R}^n \to \mathfrak{R}^m$. The vector fields G, H are assumed to be smooth and known.

Once the fault occurs, the fault signal $f \in \mathscr{F} \subset \mathfrak{R}^q$ is generated by the *neurally stable* exosystem (2.38), i.e., $\partial S(0)/\partial f$ has all its eigenvalues on the imaginary axis, which means that f is always bounded. The function S is also assumed to be smooth and known. Such model effectively describes process, actuator and sensor faults.

The following assumption is a basic requirement for the state feedback output regulation design [55].

Assumption 2.4. *There exist some $u = \alpha(x,f)$ with $f = 0$ such that $x = 0$ of healthy system (2.36) $\dot{x} = G(x,\alpha(x,0),0)$ is asymptotically stable.*

Definition 2.2. Fault tolerant regulation problem *(FTRP) for system (2.36)-(2.39) is to find a FTC law $u = \alpha(x,f)$ such that $\forall x(0) \in \mathscr{X}$ with $\mathscr{X} \subset \mathfrak{R}^n$ a neighborhood of* 0 *and $\forall f \in \mathscr{F}$, the trajectory of the closed-loop system (2.36) $\dot{x} = G(x,\alpha(x,f),f)$ is bounded $\forall t \geq 0$ and $\lim_{t\to\infty} e(t) = 0$.*

Theorem 2.3. *Suppose that the fault f can be detected/approximated accurately, and there exists a $u = \alpha(x,f)$ satisfying Assumption 2.4. The FTRP for system (2.36)-(2.39) is solvable if and only if there exists a $\mathscr{C}^k$ mapping $x = \pi(f)$ with $\pi(0) = 0$ defined for $(x,f) \in \mathscr{X} \times \mathscr{F}$ satisfying*

$$\frac{\partial \pi}{\partial f} S(f) = G(\pi(f),\alpha(\pi(f),f),f) \tag{2.40}$$
$$0 = H(\pi(f),f) - y_r(\pi(f)) \tag{2.41}$$

Proof: The proof follows the same way as that of Theorem 8.3.2 in [55], which is thus omitted. □

Remark 2.3. *It can be seen that FTRP is similar to the general output regulation problem with disturbances. Theorem 2.3 provides necessary and sufficient conditions to solve FTRP in the classical faulty case. The existence and the design of $\pi(f)$ and $\alpha(x,f)$ have been deeply investigated in many literatures, e.g. [55], [52], which are not focused on here.*

2.2.2 Overall Fault Tolerant Regulation

Now we consider the hybrid case. The system is

$$\dot{x}(t) = G_{\sigma(t)}(x(t), u_{\sigma(t)}(t), f_{\sigma(t)}(t)) \tag{2.42}$$
$$y(t) = H(x(t), f_{\sigma(t)}(t)) \tag{2.43}$$
$$\dot{f}_{\sigma(t)}(t) = S_{\sigma(t)}(f_{\sigma(t)}(t)) \ \ \forall t \geq t_f, \text{ with } f_{\sigma(t)}(t) = 0 \ \ \forall t \in [0, t_f) \tag{2.44}$$

where $\sigma(t) : [0, \infty) \to Q$ also denotes a piecewise constant switching function.

Assumption 2.5. *There exists a family of controllers $u_i = \alpha_i(x, f_i)$ for $i \in Q$ solving the FTRP for system (2.39) and (2.42)-(2.44) with $\sigma(t) = i$.*

Assumption 2.5 means that the FTRP of each mode is solvable individually. The following definition is an extension of FTRP to the successional faulty case.

Definition 2.3. Overall fault tolerant regulation problem *(OFTRP) for system (2.39) and (2.42)-(2.44) is to find a switching scheme among $u_i = \alpha_i(x, f_i)$, $i \in Q$ such that $\forall x(0) \in \mathscr{X}$ and $\forall f_i \in \mathscr{F}$, the trajectory of the closed-loop system (2.42) is bounded $\forall t \geq 0$ and $\lim_{t\to\infty} e(t) = 0$.*

Before solving the OFTRP, we give an important concept as follows

Definition 2.4. *[49]: Let $N_\sigma(T, t)$ denote the number of switchings of σ over the interval (t, T), if there exists a positive number τ_a such that*

$$N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_a}, \quad \forall T \geq t \geq 0 \tag{2.45}$$

where $N_0 > 0$ denotes the chattering bound, then the positive constant τ_a is called average dwell time *(ADT) of σ over (t, T).*

Definition 2.4 means that there may exist some switchings separated by less than τ_a, but the average dwell period among switchings of modes is not less than τ_a.

The following theorem establishes the sufficient conditions to solve OFTRP.

Theorem 2.4. *Consider a system (2.39) and (2.42)-(2.44) satisfying Assumption 2.5. Suppose that each fault can be diagnosed without delay, and each FTC law u_i is applied once a fault f_i occurs. The OFTRP is solvable if*

C1) $\tau_a > \frac{\ln B}{a}$, where $B \triangleq \max_{i \in Q} B_i$, $a \triangleq \min_{i \in Q} a_i$.
and either C2) or C3) holds for $k = 1, 2, \ldots$
C2) $\pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_k)) = \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t_k))$.
C3) $-(a - \frac{\ln B}{\tau_a})(t - t_k) + \ln k < -a^* t$, for $t \geq t_k$ and $a^* > 0$.

Remark 2.4. *Before proving Theorem 2.4, we provide some insight into the conditions C1)-C3): C1) requires that the switching of modes is slow averagely, i.e., the frequency of switching is not too much. C2) imposes a condition on the mapping π_i and the fault value f_i. It can be seen that if there is a common mapping $x = \pi(f_i)$ for all modes, and $f_{\sigma(t_{k-1})}(t_k)) = 0$, then C2) holds. Generally, C2) is hard to satisfy even in the linear case [76]. In the absence of C2), C3) requires that the dwell period of each mode is long enough. C3) can be verified by checking whether $\ln k + (a - \frac{\ln B}{\tau_a})t_k < (a - \frac{\ln B}{\tau_a} - a^*)t$ holds or not for $t \in [t_k, t_{k+1})$.*

Proof of Theorem 2.4: Since mode $\sigma(t_k)$ in the time interval $[t_k, t_{k+1})$ is controlled by $u_{\sigma(t_k)}$, thus its FTRP is solved from Assumption 2.5. According to Theorem 8.3.2 in [55], a center manifold $x = \pi_{\sigma(t_k)}(f_{\sigma(t_k)})$ of mode $\sigma(t_k)$ is locally attractive, i.e.,

$$|x(t) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t))| \leq Be^{-a(t-t_k)}|x(t_k) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t_k))|, \ t_k \leq t < t_{k+1} \quad (2.46)$$

Similarly, in $[t_{k-1}, t_k)$ one has

$$|x(t_k^-) - \pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_k^-))| \leq Be^{-a(t_k^- - t_{k-1})}|x(t_{k-1}) - \pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_{k-1}))| \quad (2.47)$$

Combining (2.46) with (2.47) yields

$$\begin{aligned}|x(t) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t))| &\leq Be^{-a(t-t_k)}\Big|x(t_k) - \pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_k)) \\ &\quad + \pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_k)) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t_k))\Big| \\ &\leq B^2 e^{-a(t-t_{k-1})}|x(t_{k-1}) - \pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_{k-1}))| \\ &\quad + Be^{-a(t-t_k)}|\pi_{\sigma(t_{k-1})}(f_{\sigma(t_{k-1})}(t_k)) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t_k))| \end{aligned} \quad (2.48)$$

By induction, we obtain

$$\begin{aligned}|x(t) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t))| &\leq B^{k+1}e^{-at}|x(0) - \pi_{\sigma(0)}(f_{\sigma(0)}(0))| \\ &\quad + \sum_{s=1}^{k}\Big(B^s e^{-a(t-t_{k-s+1})}|\pi_{\sigma(t_{k-s})}(f_{\sigma(t_{k-s})}(t_{k-s+1})) \\ &\quad - \pi_{\sigma(t_{k-s+1})}(f_{\sigma(t_{k-s+1})}(t_{k-s+1}))|\Big) \end{aligned} \quad (2.49)$$

From C1), we can pick $\lambda = a - \frac{\ln B}{\tau_a}$, we have $\tau_a = \frac{\ln B}{(a-\lambda)}$. Based on (2.45), we have

$$B^{k+1}e^{-at} \leq B^{N_0+1}e^{\frac{t}{\tau_a}\ln B - at} < B^{N_0+1}e^{-\lambda t} \quad (2.50)$$

If C2) holds, each term of the sum in (2.49) is zero. Substituting (2.50) into (2.49), we further have

$$|x(t) - \pi_{\sigma(t_k)}(f_{\sigma(t_k)}(t))| \leq B^{N_0+1}e^{-\lambda t}|x(0) - \pi_{\sigma(0)}(f_{\sigma(0)}(0))| \quad (2.51)$$

Inequality (2.51) means that $x - \pi_{\sigma(t_k)}(f_{\sigma(t_k)})$ still converges to zero $\forall t \geq t_k$, $\forall x(0) \in \mathscr{X}$ and $\forall f_i \in \mathscr{F}$. By continuity of H and y_r in each $[t_{k-1}, t_k)$, it follows that $\lim_{t\to 0} e(t) = 0$.

If C2) does not hold, one has from C1) and (2.45) that

$$\begin{aligned}B^s e^{-a(t-t_{k-s+1})} &\leq B^{N_0 + \frac{t-t_{k-s+1}}{\tau_a}} e^{-a(t-t_{k-s+1})} \\ &\leq B^{N_0} e^{\frac{t-t_{k-s+1}}{\tau_a}\ln B - a(t-t_{k-s+1})} \\ &\leq B^{N_0} e^{-\lambda(t-t_{k-s+1})} \end{aligned} \quad (2.52)$$

Since each f_i is bounded due to the neurally stable exosystems, there exists a constant $\xi > 0$ such that $\forall k = 1, 2, ...$, and $1 \le s \le k$

$$\left|\pi_{\sigma(t_{k-s})}(f_{\sigma(t_{k-s})}(t_{k-s+1})) - \pi_{\sigma(t_{k-s+1})}(f_{\sigma(t_{k-s+1})}(t_{k-s+1}))\right| \le \xi \tag{2.53}$$

It follows from (2.53) and C3) that

$$\begin{aligned}
&\sum_{s=1}^{k} \left(B^s e^{-a(t-t_{k-s+1})} |\pi_{\sigma(t_{k-s})}(f_{\sigma(t_{k-s})}(t_{k-s+1})) - \pi_{\sigma(t_{k-s+1})}(f_{\sigma(t_{k-s+1})}(t_{k-s+1}))|\right) \\
&\quad \le \xi B^{N_0} \sum_{s=1}^{k} e^{-\lambda(t-t_{k-s+1})} \\
&\quad \le \xi B^{N_0} e^{\ln k - \lambda(t-t_k)} \\
&\quad \le \xi B^{N_0} e^{-a^* t}
\end{aligned} \tag{2.54}$$

By substituting (2.50) and (2.54) into (2.49), we conclude that $x - \pi_{\sigma(t_k)}(f_{\sigma(t_k)})$ converges to zero $\forall t \ge t_k$, $\forall x(0) \in \mathscr{X}$ and $\forall f_i \in \mathscr{F}$. The result follows. □

2.3 Multiple Observers Method

2.3.1 Problem Formulation

Differently from sections 2.1-2.2, we address a class of HS with both continuous faults and discrete faults in this section. The system takes the form

$$\dot{x}(t) = A_\sigma x(t) + g_\sigma(x(t), t) + B_\sigma u_\sigma(t) + E_\sigma f_\sigma^c(t) \tag{2.55}$$

$$y(t) = Cx(t) \tag{2.56}$$

where $x(t) \in \Re^n$ is the non measured state, $y(t) \in \Re^p$ is the output, $u_\sigma(t) \in \Re^m$ is the control. A_σ, B_σ, E_σ and C are real constant matrices of appropriate dimensions. (A_σ, B_σ) is controllable. $g_\sigma(x(t), t)$ is a continuous Lipschitz function, i.e., $|g_\sigma(x_1, t) - g_\sigma(x_2, t)| \le L_\sigma |x_1 - x_2|$, where $L_\sigma > 0$ is called the Lipschitz constant. Moreover, $g_\sigma(0, t) = 0$.

The *continuous actuator fault* is modelled by a "fault pattern" as in Chapter 2.1. Suppose that there exists two constants f_σ^0 and f_σ^1 such that $|f_\sigma^c| \le f_\sigma^0$, $|\dot{f}_\sigma^c| \le f_\sigma^1$. Such fault model covers all faults that result in a deviation of the control signal from normal.

Define $Q = \{1, 2, \ldots, N\}$, where N is the number of modes. $\sigma(t) : [t_0, \infty) \to Q$ denotes the *switching function* as in sections 2.1-2.2. Denote t_j as the jth switching instant of the system (2.55)(2.56). At t_j, the system switches to mode k, where $k \in Q$, $j = 1, 2, \ldots$.

The switching property is considered as in [29]: a) the switching sequence is fixed. b) there is a series of prescribed dwell periods between each switching. We also assume that the states do not jump at the switching instants.

The *discrete fault* is represented by the faulty switching function $\sigma_f(t)$, that forces the system to switch to a mode which is not the prescribed successor at the switching instant. Similarly, $\sigma_H(t)$ denotes the healthy switching function. If $\sigma(t) = \sigma_H(t)$, then there is no discrete fault in the current mode.

The FTC problem in this section can be described as: *Keep the states of system (2.55)-(2.56) always bounded and make them converge to a small closed set in spite of continuous and discrete faults.*

Different from sections 2.1-2.2, the FTC of discrete faults must be taken into account as in [132] and [145]. Since the current mode after each switching time may be unknown due to discrete faults, some identifying work must be applied for a short period. Some related work can be seen in [129], [68], [48] and [20]. Whatever method used, the necessary time period in which mode is identified (due to computation time, decision time) may cause instability. How to overcome this finite delay is a problem to be addressed.

The main idea is as follows: 1) For the continuous faults in each mode, an adaptive observer technique is proposed to provide the rapid fault estimation, based on which the FTC law is designed. 2) For the discrete faults, a novel model-free sliding mode observer is designed, which together with a series of observers related to system modes, can identify the current mode quickly while guaranteeing the stability of the system during each transition period. 3) The above two FTC strategies are combined with the average dwell time scheme such that the states of the overall hybrid system are always bounded and converge to a small closed set.

2.3.2 FTC for Continuous Faults

In this subsection, only $f^c_\sigma(t)$ is addressed. We introduce the *input-to-state practical stability* and a lemma that will be used later.

Definition 2.5. *[113] A system $\dot{x} = f(x,u)$ is said to be* input-to-state practically stable *(ISpS) over $[0,t)$ w.r.t. u if there exist functions $\beta \in \mathscr{KL}$, $\alpha, \gamma \in \mathscr{K}_\infty$, and a constant $\varsigma > 0$, such that for any bounded input u and any initial condition $x(0)$, we have*

$$\alpha(|x(t)|) \le \beta(|x(0)|,t) + \gamma(\|u\|_{[0,t)}) + \varsigma, \quad \forall t \ge 0$$

Note that when $\varsigma = 0$, ISpS becomes input-to-state stability (ISS) [114] (see also Definition 4.1 in Chapter 4).

It has been proven in Section VI of [113] that the following property holds.

Lemma 2.5. *If there exist α_1, α_2, α_3, $\gamma_1 \in \mathscr{K}_\infty$, $\varsigma_1 > 0$ and a smooth function $V : \Re^n \to \Re_{\ge 0}$ such that*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{2.57}$$

$$\dot{V}(x) \le -\alpha_3(|x|) + \gamma_1(|u|) + \varsigma_1 \tag{2.58}$$

Then the system $\dot{x} = f(x,u)$ is ISpS over $[0,t)$ w.r.t. u.

If $\varsigma_1 = 0$, then V is called *ISS Lyapunov function*[114], and the system is ISS under (2.57) and (2.58) with the state x and the input u (see Lemma 2.14 in [114]).

Now let us consider the system (2.55)(2.56) with $\sigma(t) = k$ for some $k \in Q$ starting from $t = t_j$

$$\dot{x}(t) = A_k x(t) + g_k(x(t),t) + B_k u_k(t) + E_k f_k^c(t) \tag{2.59}$$

$$y(t) = Cx(t) \tag{2.60}$$

Assumption 2.6. *There exists a matrix K_k such that $G_k(s) = C[sI - (A_k - K_k C)]^{-1} E_k$, is strictly positive real (SPR) :*

$$\forall \omega > 0 : Re(G_k(j\omega)) > 0 \tag{2.61}$$

Moreover

$$\min_{\omega \in R^+} \sigma_{\min}(A_k - K_k C - j\omega I) > L_k \tag{2.62}$$

where $\sigma_{\min}(M)$ is the smallest singular value of M.

Remark 2.5. *Assumption 2.6 is a restriction on the triple (A_k, C, E_k) in terms of the fault to residual transfer of the observer-based residual generator associated with the linear part of the system. A known necessary condition for $G_k(s)$ to be SPR is that (A_k, C) is observable and CE_k is of full column rank. It should be noted that CE_k being of full column rank is a standard assumption in fault isolation problem [10].*

Under Assumption 2.6, it has been proven in [104] that for any given matrix $Q_k \in \Re^{n\times n} > 0$ and scalar $\varepsilon > 0$, there exist two matrices $P_k \in \Re^{n\times n} > 0$ and $R_k \in \Re^{r\times q}$ such that

$$P_k E_k = C^\top R_k \tag{2.63}$$

and

$$(A_k - K_k C)^\top P_k + P_k(A_k - K_k C) + \varepsilon L_k^2 I_n + \frac{P_k^2}{\varepsilon} + Q_k \leq 0 \tag{2.64}$$

The FD scheme for mode k is designed as

$$\dot{\hat{x}} = A_k \hat{x} + g_k(\hat{x},t) + B_k u_k + E_k \hat{f}_k^c + K_k(y - \hat{y}) \tag{2.65}$$

$$\dot{\hat{f}}_k^c = \Gamma_k R_k^\top (y - \hat{y}) - \vartheta_k \Gamma_k \hat{f}_k^c \tag{2.66}$$

$$\hat{y} = C\hat{x} \tag{2.67}$$

where $\hat{x}(t), \hat{f}_k^c(t), \hat{y}(t)$ are the estimates of $x(t), f_k^c(t), y(t)$. The weighting matrix $\Gamma_k = \Gamma_k^\top > 0$, and the constant $\vartheta_k > 0$ are chosen such that $\vartheta_k - \lambda_{\max}(\Gamma_k^{-1}) > 0$.

Remark 2.6. *The diagnostic scheme (2.65)-(2.67) plays an important role to diagnose the f_k^c. Our goal is to stabilize the system, we neither care about when the fault occurs nor design a so-called* detection observer *as in [58] to detect the fault. The diagnostic scheme (2.65)-(2.67) always works no matter the mode k is faulty or not (i.e., the normal condition can be treated as a special faulty case where $f_k^c = 0$).*

Denote $e_x(t) = x(t) - \hat{x}(t)$, $e_y(t) = y(t) - \hat{y}(t)$, $e_f(t) = f_k^c(t) - \hat{f}_k^c(t)$, we have the following lemma:

Lemma 2.6. *[57] Define a set S_k as*

$$S_k \triangleq \left\{ (e_x, e_f) \,\middle|\, \lambda_{\min}(P_k)|e_x|^2 + \lambda_{\min}(\Gamma_k^{-1})|e_f|^2 \le \frac{\beta_k}{\alpha_k} \right\}$$

where

$$\beta_k \triangleq \lambda_{\max}(\Gamma_k^{-1})(f_k^1)^2 + \sigma_k (f_k^0)^2, \quad \alpha_k \triangleq \frac{\min(c_{k1},\ c_{k2})}{\max[\lambda_{\max}(P_k),\ \lambda_{\max}(\Gamma_k^{-1})]}$$
$$c_{k1} \triangleq \lambda_{\min}(Q_k) > 0, \quad c_{k2} \triangleq \vartheta_k - \lambda_{\max}(\Gamma_k^{-1}) > 0 \tag{2.68}$$

Then under Assumption 2.6, the fault diagnostic scheme (2.65)-(2.67) guarantees that (e_x, e_f) of mode k converges to S_k exponentially at a rate greater than $e^{-\alpha_k t}$.

The following lemma gives a relation between e_x and e_f.

Lemma 2.7. *Under Assumption 2.6, the fault diagnostic scheme (2.65)-(2.67) guarantees that e_x is ISS w.r.t. e_f, i.e., there exist $\beta_{ek} \in \mathcal{KL}$, $\alpha_{ek}, \gamma_{ek} \in \mathcal{K}_\infty$ such that*

$$\alpha_{ek}(|e_x(t))| \le \beta_{ek}(|e_x(t_j)|, t) + \gamma_{ek}(\|e_f\|_{[t_j,t)}), \quad \forall t \ge t_j \tag{2.69}$$

Proof: From (2.59), (2.60), (2.65) and (2.67), we have

$$\dot{e}_x = (A_k - K_k C)e_x + g_k(x,t) - g_k(\hat{x},t) + E_k e_f \tag{2.70}$$

Choose a Lyapunov candidate $\Theta_k = e_x^\top P_k e_x$, its derivative w.r.t. time along (2.70) is

$$\dot{\Theta}_k = e_x^\top [P_k(A_k - K_k C) + (A_k - K_k C)^\top P_k] e_x + 2e_x^\top P_k(g_k(x,t) - g_k(\hat{x},t)) + 2e_x^\top P_k E_k e_f$$

Note that, for two vectors $\mathbf{a_1}$, $\mathbf{a_2}$, it holds that $2\mathbf{a_1}^\top \mathbf{a_2} \le \frac{1}{\varepsilon}\mathbf{a_1}^\top \mathbf{a_1} + \varepsilon \mathbf{a_2}^\top \mathbf{a_2}$ for $\varepsilon > 0$. Similarly, we can show that

$$2e_x^\top P_k(g_k(x,t) - g_k(\hat{x},t)) \le e_x^\top \frac{P_k^2}{\varepsilon}\, e_x + \varepsilon L_k^2\, e_x^\top e_x \tag{2.71}$$

From (2.64), we have

$$\begin{aligned} \dot{\Theta}_k &\le -e_x^\top Q_k e_x + 2e_x^\top P_k E_k e_f \\ &\le (-\lambda_{\min}(Q_k) + \varepsilon_1)|e_x|^2 + \frac{|P_k E_k|^2}{\varepsilon_1}|e_f|^2 \end{aligned} \tag{2.72}$$

where $\varepsilon_1 > 0$ is chosen such that $-\lambda_{\min}(Q_k) + \varepsilon_1 < 0$. Inequality (2.72) implies that Θ_k is an ISS-Lyapunov function with the state e_x and the input e_f. From Lemma 2.5, the result follows.

Moreover, we have

$$\dot{\Theta}_k \le \frac{-\lambda_{\min}(Q_k)+\varepsilon_1}{\lambda_{\max}(P_k)}\Theta_k + \frac{|P_k E_k|^2}{\varepsilon_1}|e_f|^2 \triangleq \iota_1\Theta_k + \iota_2|e_f|^2$$

Using the differential inequality theory (see Chapter 2 in [84]), we can obtain

$$\begin{aligned}\Theta_k &\le e^{\iota_1(t-t_j)}\Theta_k(t_j) + \int_{t_j}^{t} e^{\iota_1(t-\tau)}\iota_2|e_f(\tau)|^2 d\tau \\ &\le e^{\iota_1(t-t_j)}\Theta_k(t_j) + \sup_{\tau\in[t_j,t)}\{\iota_2|e_f(\tau)|^2\}\int_{t_j}^{t} e^{\iota_1(t-\tau)}d\tau \\ &\le \underbrace{e^{\iota_1(t-t_j)}\lambda_{\max}(P_k)|e_x(t_j)|^2}_{\beta_{ek}(|e_x(t_j)|,t)} + \underbrace{\frac{1}{-\iota_1}\sup_{\tau\in[t_j,t)}\{\iota_2|e_f(\tau)|^2\}}_{\gamma_{ek}(\|e_f\|_{[t_j,t)})} \end{aligned} \tag{2.73}$$

Define $\alpha_{ek}(\cdot) = \lambda_{\min}(P_k)(\cdot)^2$, which, together with β_{ek}, γ_{ek} in (2.73), leads to (2.69). This completes the proof. □

Supposed that $e_f(t)$ is norm bounded in each $[t_j, t_{j+1})$. Inequality (2.69) means that given an initial $|e_x(t_j)|$ (or a bound of $|e_x(t_j)|$, the value of $|e_x|$ can be estimated. Define

$$e_x(t)_{est} \triangleq \alpha_{ek''}^{-1}\circ\beta_{ek''}(|e_x(t_j)|,t) + \alpha_{ek''}^{-1}\circ\gamma_{ek''}(\|e_f(t)\|_{[t_j,t)}), \quad t_j \le t \le t_{j+1} \tag{2.74}$$

$e_x(t)_{est}$ is the estimates of $|e_x(t)|$. It follows that $|e_x(t)| \le e_x(t)_{est}$.

Now we are ready to design the FTC law. Since (A_k, B_k) is controllable, let $W_k = W_k^\top > 0$ be associated with a given symmetric positive definite matrix H_k by the Riccati equation

$$A_k^\top H_k + H_k A_k - 2H_k B_k B_k^\top H_k + W_k = 0 \tag{2.75}$$

The design of the proposed fault-tolerant controller makes use of the two following assumptions.

Assumption 2.7. *Given a solution H_k of (2.75), there exists a bounded function $\eta_k(x,t) > 0$ such that*

$$|x^\top H_k g_k(x,t)| \le \eta_k(x,t)|x^\top H_k B_k| \tag{2.76}$$

Assumption 2.8. *$rank(B_k, E_k) = rank(B_k)$.*

Remark 2.7. *Inequality (2.76) is not restrictive. Since $g_k(0,t) = 0$, from the Lipschitz condition, one has $|g_k(x,t)| \le L_k|x|$ and $\left|x^\top H_k g_k(x,t)\right| \le L_k\left|x^\top H_k\right||x|$. Since (A_k, B_k) is controllable, the ratio $\left|x^\top H_k\right| / \left|x^\top H_k B_k\right|$ is homogeneous and its maximal value is found by solving $\max(|x^\top H_k|)$ under the constraint $|x^\top H_k B_k| = 1$ providing some bounded solution x^*. Assumption 2.8 is naturally satisfied for the actuator faulty case. Indeed, $rank(B_k) = rank(B_k, E_k) \Leftrightarrow Im(E_k) \subseteq Im(B_k)$ which is equivalent to the existence of B_k^* such that $(I - B_k B_k^*)E_k = 0$.*

The fault-tolerant controller is constructed as

$$u_k(\hat{x}) = u_{k1}(\hat{x}) + u_{k2}(\hat{x}) \tag{2.77}$$

where

$$u_{k1}(\hat{x}) \triangleq -B_k^\top H_k \hat{x} - B_k^* E_k \hat{f}_k^c, \tag{2.78}$$

$$u_{k2}(\hat{x}) \triangleq -\frac{\eta_k(\hat{x},t)}{|\phi_k(\hat{x})| + \varepsilon/2}\phi_k(\hat{x}), \quad \phi_k(\hat{x}) \triangleq \eta_k(\hat{x},t) B_k^\top H_k \hat{x} \tag{2.79}$$

with ε an arbitrarily small positive scalar.

Lemma 2.8. *Suppose that assumptions 2.6-2.8 are satisfied, under the feedback control (2.77)-(2.79), mode k in (2.59)(2.60) is ISpS over* $[t_j, t)$ *w.r.t.* e_x, e_f *and a constant* $\varsigma_k > 0$.

Proof: Applying the control (2.77) to (2.59) results in the closed-loop dynamics

$$\dot{x} = (A_k - B_k B_k^\top H_k)x + B_k B_k^\top H_k e_x + E_k e_f + g_k(x,t) + B_k u_{k2}(\hat{x}) \tag{2.80}$$

Consider a Lyapunov candidate $V_k(x) = x^\top H_k x$, where $H_k > 0$ is defined by (2.75). Its derivative along the system is

$$\begin{aligned}\dot{V}_k \leq & -\lambda_{\min}(W_k)|x|^2 + 2|H_k B_k B_k^\top H_k| \cdot |x| \cdot |e_x| \\ & +2|H_k E_k| \cdot |x| \cdot |e_f| + 2x^\top H_k [B_k u_{k2}(\hat{x}) + g_k(x,t)]\end{aligned} \tag{2.81}$$

From (2.79), one has

$$\begin{aligned}& 2x^\top H_k [B_k u_{k2}(x) + g_k(x,t)] \\ = & \frac{-2\eta_k^2(x,t)|x^\top H_k B_k|^2 + 2x^\top H_k g_k(x,t)\eta_k(x,t)|x^\top H_k B_k| + \varepsilon x^\top H_k g_k(x,t)}{\eta_k(x,t)|x^\top H_k B_k| + \varepsilon/2}\end{aligned} \tag{2.82}$$

Substituting (2.76) into (2.82) yields

$$2x^\top H_k [B_k u_{k2}(x) + g_k(x,t)] \leq \frac{\varepsilon |x^\top H_k g_k(x,t)|}{\eta_k(x,t)|x^\top H_k B_k| + \varepsilon/2} \leq \varepsilon \tag{2.83}$$

Assumption 2.7 guarantees that the control $u_{k2}(x)$ is continuous and locally bounded. There always exists a number $\delta_k > 0$ such that $|u_{k2}(\hat{x}) - u_{k2}(x)| \leq \delta_k |e_x|$ for a small $|e_x|$. Due to the convergence of the estimation in Lemma 2.6, it follows that

$$2x^\top H_k [B_k (u_{k2}(\hat{x}) - u_{k2}(x)] \leq 2|H_k B_k| \cdot \delta_k |e_x| \tag{2.84}$$

where $\delta_k > 0$. It also holds that

$$2|H_k B_k B_k^\top H_k| \cdot |x| \cdot |e_x| \leq \varepsilon_2 |x|^2 + \frac{|H_k B_k B_k^\top H_k|^2}{\varepsilon_2}|e_x|^2$$

$$2|H_kE_k|\cdot|x|\cdot|e_f| \leq \varepsilon_3|x|^2+\frac{|H_kE_k|^2}{\varepsilon_3}|e_f|^2$$

where $\varepsilon_2, \varepsilon_3 > 0$ are chosen such that $-\lambda_{\min}(W_k)+\varepsilon_2+\varepsilon_3 < 0$. Substituting two inequalities above and (2.83), (2.84) into (2.81), one can further obtain

$$\begin{aligned}\dot{V}_k \leq & (-\lambda_{\min}(W_k)+\varepsilon_2+\varepsilon_3)|x|^2 \\ & +\frac{|H_kB_kB_k^\top H_k|^2}{\varepsilon_2}|e_x|^2+2|H_kB_k|\cdot\delta_k|e_x|+\frac{|H_kE_k|^2}{\varepsilon_3}|e_f|^2+\varepsilon\end{aligned}$$

From Lemma 2.5, the result follows. □

Based on previous analysis for single mode, now we consider the HS (2.55)(2.56). It can be obtained from Lemma 2.8 that there exist continuously differentiable functions $V_k : \Re^n \to \Re_{\geq 0}$, $k \in Q$ and $\bar{\gamma}_1(\cdot)$, $\bar{\gamma}_2(\cdot) \in \mathscr{K}_\infty$, such that $\forall p, q \in Q$

$$\bar{\alpha}_1|x|^2 \leq V_p(x) \leq \bar{\alpha}_2|x|^2 \tag{2.85}$$

$$\dot{V}_p(x) \leq -\lambda_0 V_p(x)+\bar{\gamma}_1(|e_x|)+\bar{\gamma}_2(|e_f|)+\varsigma_0 \tag{2.86}$$

$$V_p(x) \leq \mu V_q(x) \tag{2.87}$$

where constants $\bar{\alpha}_1$, $\bar{\alpha}_2$, λ_0, $\varsigma_0 > 0$, $\mu \geq 1$. The existence of μ is automatically guaranteed for the quadratic Lyapunov functions, e.g., $\mu = \bar{\alpha}_2/\bar{\alpha}_1$.

Since no discrete fault is considered, the system follows the prescribed switching sequence at each switching instant. The observer is modified for the overall system as follows:

- The fault diagnostic scheme is switched according to the current mode at each switching instant.
- The initial states $\hat{x}$ of the current observer are chosen as the final states of the previous observer. The fault estimates $\hat{f}_k^c$ are set to zero at each switching instant.

The following theorem provides a FTC strategy for the overall system with continuous faults.

Theorem 2.5. *Consider the HS (2.55)(2.56) with an initial $x(0)$, each mode satisfies assumptions 2.6-2.8. Let the switching function σ have an ADT τ_a. If $\tau_a > \frac{\ln\mu}{\lambda_0}$, where μ and λ_0 are chosen from (2.86)-(2.87), and $e_x(t^j(k+1))_{est} < e_x(t^j(k))_{est}$ where $t^j(k)$ denotes the time instant that mode j is activated for the kth time, then under the diagnostic scheme (2.65)-(2.67) and controller (2.77)-(2.79), the states of the overall switched system are always bounded and converge to a small closed set.*

Proof: Define $G_a^b(\lambda) = \int_a^b e^{\lambda s}\Phi ds$, where $\Phi \triangleq \bar{\gamma}_1(|e_x|)+\bar{\gamma}_2(|e_f|)+\varsigma_0$. Let $T > 0$ be an arbitrary time. Denote by $t_1, \ldots, t_{N_\sigma(T,0)}$ the switching instants on the interval $(0,T)$, where $N_\sigma(T,0)$ is defined in (2.45). Similar to [125], consider the function

$$W(s) \triangleq e^{\lambda_0 s}V_{\sigma(s)}(x(s)) \tag{2.88}$$

Since $\sigma(s)$ is constant on each interval $s \in [t_j, t_{j+1})$, from (2.86), we have $\dot{W}(s) \le e^{\lambda_0 s}\Phi, \forall s \in [t_j, t_{j+1})$. Integrating both sides of the foregoing inequality from t_j to t_{j+1}^- and from (2.87), we obtain $W(t_{j+1}) \le \mu(W(t_j) + G_{t_j}^{t_{j+1}}(\lambda_0))$. Iterating the foregoing inequality from 0 to $N_\sigma(T,0)$, we get

$$W(T^-) \le \mu^{N_\sigma(T,0)}\Big(W(0) + \sum_{j=0}^{N_\sigma(T,0)} \mu^{-j} G_{t_j}^{t_{j+1}}(\lambda_0)\Big) \tag{2.89}$$

where T^- denotes the time instant just before T.

Pick $\lambda \in (0, \lambda_0 - \frac{\ln\mu}{\tau_a})$, we have $\tau_a \ge \frac{\ln\mu}{(\lambda_0-\lambda)}$. Based on (2.45), we have

$$\begin{aligned}\mu^{N_\sigma(T,0)-j} \le& \mu^{N_0 + \frac{T}{\tau_a} - j + 1 - 1}\\ \le& \mu^{1+N_0} e^{\tau_a(\lambda_0-\lambda)(\frac{T}{\tau_a}-1-j)} \le \mu^{1+N_0} e^{(\lambda_0-\lambda)(T-t_{j+1})}\end{aligned} \tag{2.90}$$

and

$$G_{t_j}^{t_{j+1}}(\lambda_0) = \int_{t_j}^{t_{j+1}} e^{\lambda_0 s}\Phi ds \le e^{(\lambda_0-\lambda)t_{j+1}} G_{t_j}^{t_{j+1}}(\lambda) \tag{2.91}$$

Substituting (2.90), (2.91) into (2.89) yields

$$\begin{aligned}W(T^-) \le& \mu^{N_\sigma(T,0)} W(0) + \sum_{j=0}^{N_\sigma(T,0)} \mu^{1+N_0} e^{(\lambda_0-\lambda)T} G_{t_j}^{t_{j+1}}(\lambda)\\ \le& \mu^{1+N_0} e^{-\lambda T}\Big(e^{\lambda_0 T - (\lambda_0-\lambda)\tau_a} W(0) + \sum_{j=0}^{N_\sigma(T,0)} e^{\lambda_0 T} G_{t_j}^{t_{j+1}}(\lambda)\Big)\\ \le& \mu^{1+N_0} e^{-\lambda T} e^{\lambda_0 T}\Big(W(0) + G_0^{\top}(\lambda)\Big)\end{aligned}$$

It follows that

$$\begin{aligned}\bar{\alpha}_1 |x(T)|^2 \le& \mu^{1+N_0} e^{-\lambda T}(\bar{\alpha}_2 |x(0)|^2 + G_0^{\top}(\lambda))\\ \le& \mu^{1+N_0} e^{-\lambda T} \bar{\alpha}_2 |x(0)|^2 + \mu^{1+N_0}\frac{1}{\lambda}\Big(\bar{\gamma}_1(\|e_x\|_{[0,T)}) + \bar{\gamma}_2(\|e_f\|_{[0,T)})\Big) + \bar{\varsigma}\end{aligned}$$

where $\bar{\varsigma} \triangleq (\mu^{1+N_0} \cdot \varsigma_0)/\lambda$.

This implies that the HS is ISpS w.r.t. e_x, e_f and a constant $\bar{\varsigma} > 0$. On the other hand, the inequality $e_x(t^j(k+1))_{est} < e_x(t^j(k))_{est}$ guarantees the global convergence of e_x, which together with the boundness of e_f leads to convergence of the states of the overall HS to a small closed set. This completes the proof. □

Roughly speaking, Theorem 2.5 shows that, if the average dwell time is large enough, then the overall HS is stable and the states are bounded whenever the continuous actuator faults occur in each dwell period.

2.3.3 FTC for Discrete Faults

Since the discrete faults violate the prescribed switching sequence, one would naturally try to first identify the current mode at the beginning of each time interval $[t_j, t_{j+1})$ using a short time period $\Delta t_j \ll t_{j+1} - t_j$, and then control the identified mode in the rest of the time interval.

In this section, a model-free sliding mode observer is proposed to estimate the states of current unknown mode, which together with a series of observers according to system modes, can identify the current mode quickly while guaranteeing the stability of the system in each Δt_j.

In each Δt_j, the control signal is set to zero, thus no continuous fault signal appears in Δt_j.

The system (2.59)-(2.60) without input can be written as

$$\dot{x}(t) = A_{k'}x(t) + g_{k'}(x(t),t), \quad y(t) = Cx(t) \tag{2.92}$$

where $k' \in Q$ is unknown. The system (2.92) is rewritten as

$$\dot{x}(t) = \bar{A}x(t) + F_{k'}(x(t),t), \quad y(t) = Cx(t) \tag{2.93}$$

where $F_{k'}(x,t) \triangleq A_{k'}x + g_{k'}(x,t) - \bar{A}x$, $\bar{A}$ is a matrix such that the pair $(\bar{A}, C)$ is observable. There exists a matrix $\bar{L}$ such that $\bar{A} - \bar{L}C$ is Hurwitz stable. Denote $\bar{P}$ as the symmetric positive definite solution of the Lyapunov equation $(\bar{A} - \bar{L}C)^\top \bar{P} + \bar{P}(\bar{A} - \bar{L}C) = -\bar{Q}$ with a given symmetric positive definite matrix $\bar{Q}$.

A model-free sliding mode observer is designed as

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + S(\bar{e}_x(t),\rho_j) + L(y(t) - \bar{y}(t)), \quad \bar{y}(t) = C\bar{x}(t) \tag{2.94}$$

where $\bar{e}_x \triangleq x - \bar{x}$, and

$$S(\bar{e}_x(t),\rho_j) \triangleq \frac{\bar{P}^{-1}C^\top C\bar{e}_x(t)}{|C\bar{e}_x(t)|}\rho_j$$

with a constant $\rho_j > 0$ which will be designed later.

From (2.93) and (2.94), we have

$$\dot{\bar{e}}_x(t) = (\bar{A} - \bar{L}C)\bar{e}_x(t) - S(\bar{e}_x(t),\rho_j) + F_{k'}(x(t),t) \tag{2.95}$$

Assumption 2.9. *There exists a bounded function $h_{k'}(x,t)$, $|h_{k'}(x,t)| < \rho|x|$ for $\rho > 0$ such that*

$$F_{k'}(x,t) = -\bar{P}^{-1}C^\top h_{k'}(x,t) \tag{2.96}$$

Remark 2.8. *Eq.(2.96) is not hard to be satisfied if $F_{k'}(x,t)$ is bounded. It is clear that there exists a constant $\bar{F} > 0$ such that $|F_{k'}(x,t)| \leq \bar{F}|x|$. If x is bounded in Δt_j (which will be shown later), then $|F_{k'}(x,t)|$ is naturally bounded.*

Lemma 2.9. *Under Assumption 2.9, there exists a $\rho_j > 0$ such that, if the states in each Δt_j are bounded, then the origin of the system (2.95) is asymptotically stable.*

Proof: Consider a Lyapunov function candidate $\bar{V}(\bar{e}_x) = \bar{e}_x^\top \bar{P} \bar{e}_x$. Its derivative along the system (2.95) is

$$\begin{aligned} \dot{\bar{V}} &= -\bar{e}_x^\top \bar{Q} \bar{e}_x + 2\bar{e}_x^\top \bar{P} F_{k'}(x,t) - 2|C\bar{e}_x|\rho_j \\ &\leq -\bar{e}_x^\top \bar{Q} \bar{e}_x + 2|C\bar{e}_x| \cdot |x|\rho - 2|C\bar{e}_x|\rho_j \end{aligned} \tag{2.97}$$

If $|x|$ is always bounded in Δt_j, then we can choose a ρ_j large enough such that $\dot{\bar{V}} < -\bar{e}_x^\top \bar{Q} \bar{e}_x$ in Δt_j. This completes the proof. □

In order to identify the current mode, a series of following observers are also needed

$$\text{observer } i: \ \dot{\hat{x}}_i = A_i \hat{x}_i + g_i(\hat{x}_i, t) + K_i(y - \hat{y}_i), \ \hat{y}_i = C\hat{x}_i, \ i \in Q \tag{2.98}$$

which are the same as (2.65)-(2.67) without u_i and $\hat{f}_i^c$. e_{xi} denotes the state estimation error using observer i.

The sliding mode observer in (2.94) and all observers in (2.98) are invoked to estimate the current mode simultaneously in Δt_j. Set the initial states of observers to $\hat{x}(t_j^-)$ at $t = t_j$. It is supposed that all modes are *discernable* [20], i.e., for mode i without input, $|e_{xi}|$ converges faster than $|e_{xj}|, \forall j \in Q, j \neq i$. This is a quite general condition for switching control problem as for instance in [20],[129] and [68]. Roughly speaking, it means that all the modes are not overlapping.

The identifiability is analyzed in the following lemma.

Lemma 2.10. *The current mode k' can be identified at time instants $t_j + \Delta t_j$, where Δt_j can be made arbitrarily small.*

Proof: It is evident that $|e_{xk'}| - |\bar{e}_x| \leq |\bar{x} - \hat{x}_{k'}| \leq |e_{xk'}| + |\bar{e}_x|$, one has

$$|\bar{x} - \hat{x}_i| - |\bar{x} - \hat{x}_{k'}| \geq \chi, \ \ \forall i \in Q, i \neq k'$$

where $\chi \triangleq |e_{xi}| - 2|\bar{e}_x| - |e_{xk'}|$. All observers share the same initial states at $t = t_j$, so $\chi(t_j) < 0$. From Lemmas 2.7, 2.8, and (2.98), it follows that if the current mode is mode k', then $|e_{xk'}|$ converges to zero at a given rate depending on $K_{k'}$ and $Q_{k'}$. Lemma 2.9 ensures $|\bar{e}_x|$ also converges to zero at a given rate. Note that all modes are discernable, there always exist $K_{k'}$, $Q_{k'}$, $\bar{L}$, $\bar{Q}$ and ρ_j such that $\chi(t) > 0 \, \forall t \geq t_j + \Delta t_j$ with arbitrarily small Δt_j. It follows that $|\bar{x} - \hat{x}_{k'}|$ is minimal $\forall t \geq t_j + \Delta t_j$. This implies that mode k' can be identified. □

The work of identifier is to find $\hat{x}_{k'}$ that is most similar to $\bar{x}$. Although Δt_j can be made arbitrarily small as in Lemma 2.10, a small delay is necessary to overcome the possible overshoot of the state trajectories. Since $\hat{x}_i$, $\hat{x}_{k'}$ and $\bar{x}$ are all continuous and measurable, in the real implementation of the identifier, high order time derivatives of the signals can help to find the similarity (as using 1-order time derivative of signals in the simulation).

The following assumption is imposed to avoid that the system states escape into infinity or a large region before a proper controller is invoked.

Assumption 2.10. *The Δt_j determined by Lemma 2.10 is always within the following set*

$$\Omega_{\Delta t_j} \triangleq \{\Delta t_j | \Delta t_j < t_{j+1} - t_j \text{ and } |\bar{x}(t_j + \Delta t_j)| \leq \xi |\bar{x}(t_j)|\} \tag{2.99}$$

where $\xi \geq 1$, $\forall k' \in Q$, $j = 1, 2 \ldots$.

Remark 2.9. *The selection of ξ depends on system dynamics. Assumption 2.10 is not hard to be satisfied, since Δt_j can be made arbitrary small (due to Lemma 2.10). If the system without control is still stable or divergent slowly (this is the ideal case), then it is also possible that $|\bar{x}(t_j + \Delta t_j)| < \xi |\bar{x}(t_j)|$ when the current mode is detected at $t + \Delta t_j$.*

From (2.99), lemmas 2.7 and 2.9, we have

$$\begin{aligned} |x(t_j + \Delta t_j)| &\leq |\bar{x}(t_j + \Delta t_j)| + |\bar{e}_x(t_j + \Delta t_j)| \\ &\leq |\bar{x}(t_j + \Delta t_j)| + \sqrt{\bar{\alpha}_3 e^{\bar{\alpha}_4 \Delta t_j}} |\bar{e}_x(t_j)| \\ &\leq |\bar{x}(t_j + \Delta t_j)| + \sqrt{\bar{\alpha}_3} e_x(t_j)_{est} \\ &\leq \xi |\bar{x}(t_j)| + \varepsilon_j \end{aligned} \tag{2.100}$$

where $\bar{\alpha}_3 > 0$, $\bar{\alpha}_4 < 0$ are determined by $\bar{P}, \bar{Q}$. k'' denotes the mode activated in $[t_{j-1}, t_j)$. Note that $\varepsilon_j > 0$ can be calculated from the estimates $e_x(t_j)_{est}$ in (2.74). The main contribution of inequality (2.100) is that it provides a bound of $|x(t)|$ in Δt_j, which can be used to design ρ_j in (2.94).

The proposed identifier in this section has three good properties:

- It can provide accurate state estimates after each Δt_j .
- It is not affected by continuous actuator faults since no control signal are applied in Δt_j.
- It avoids the large transient overshoot of states in Δt_j.

2.3.4 FTC Framework

Based on the analysis in sections 2.3.2-2.3.3, the FTC problem for both continuous and discrete faults is discussed in this section. Fig.2.2 shows the block diagram of the framework, where the plant is connected with three parts: a series of observers and controllers, a model-free observer, and an identifier. The *fault tolerant control framework* works as the following procedure:

1) *At switching instant t_j, stop the fault diagnostic scheme (2.65)-(2.67), set control signals and fault estimates to zero.*
2) *Invoke the model free observer (2.94), a series of observers (2.98), initialize all observers at t_j with the same states $\hat{x}(t_j^-)$.*

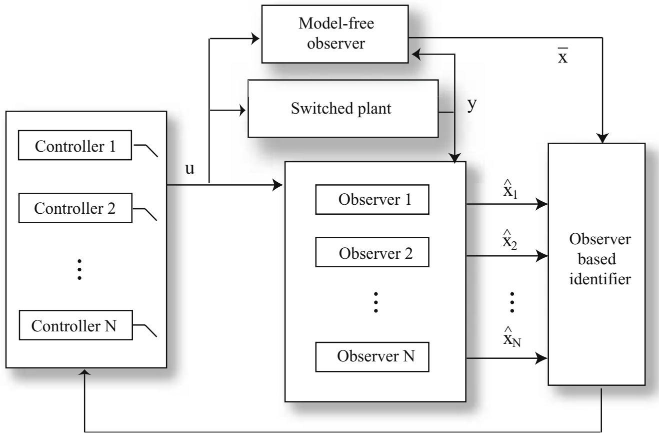


Fig. 2.2 The FTC framework

3) Choose ρ_j *by (2.97) and (2.100), invoke the identifying scheme in Lemma 2.10 into the system.*
4) Determine Δt_j *based on Lemma 2.10.*
5) At $t_j + \Delta t_j$*, stop the identifier, apply the fault diagnostic scheme (2.65)-(2.67) and controller (2.77)-(2.79) into the system according to the current mode.*
6) At switching instant t_{j+1}*, go to* 1).

The following theorem is given to guarantee the stability of overall system with both continuous and discrete faults.

Theorem 2.6. *Consider the HS (2.55)(2.56) with an initial* $x(0)$ *satisfying assumptions 2.9, 2.10, with each mode satisfying assumptions 2.6-2.8. Let the switching function* σ *have an ADT* τ_a. *If* $\tau_a > \frac{\ln \mu}{\lambda_0}$, *and* $e_x(t^j(k+1))_{est} < e_x(t^j(k))_{est}$, *then the proposed FTC framework guarantees that the states of the HS are always bounded and converge to a small closed set.*

Proof: Following the result of Theorem 2.5, we have

$$W(t_{j+1}) \le \mu(W(t_j + \Delta t_j) + G_{t_j+\Delta t_j}^{t_{j+1}}(\lambda_0)) \tag{2.101}$$

If the current mode is mode k', then

$$W(t_j + \Delta t_j) = e^{\lambda_0(t_j+\Delta t_j)} V_{k'}(x(t_j + \Delta t_j)) \tag{2.102}$$

From Lemma 2.9 and (2.99), we have

$$\begin{aligned} |x(t_j + \Delta t_j)| &\le |\bar{e}_x(t_j + \Delta t_j)| + |\bar{x}(t_j + \Delta t_j)| \\ &\le \sqrt{\bar{\alpha}_3 e^{\bar{\alpha}_4 \Delta t_j}} |\bar{e}_x(t_j)| + \xi |\bar{x}(t_j) - x(t_j) + x(t_j)| \end{aligned}$$

$$\leq (\sqrt{\bar{\alpha}_3 e^{\bar{\alpha}_4 \Delta t_j}} + \xi)|\bar{e}_x(t_j)| + \xi|x(t_j)| \tag{2.103}$$

From (2.126), we further have

$$\begin{aligned} V_{k'}(x(t_j+\Delta t_j)) &\leq \bar{\alpha}_2 |x(t_j+\Delta t_j)|^2 \\ &\leq 2\bar{\alpha}_2(\sqrt{\bar{\alpha}_3 e^{\bar{\alpha}_4 \Delta t_j}} + \xi)|\bar{e}_x(t_j)|^2 + \frac{2\bar{\alpha}_2 \xi^2}{\bar{\alpha}_1} V_{k'}(x(t_j)) \end{aligned} \tag{2.104}$$

Define $\psi(t_j) \triangleq 2\bar{\alpha}_2(\sqrt{\bar{\alpha}_3 e^{\bar{\alpha}_4 \Delta t_j}} + \xi)|\bar{e}_x(t_j)|^2 e^{\lambda_0 \Delta t_j}$, $\Delta_j \triangleq \frac{(2\bar{\alpha}_2 \xi^2)}{\bar{\alpha}_1} e^{\lambda_0 \Delta t_j}$. In each period Δt_j, there are no input and continuous actuator fault, so $e_f(t) = 0$ $\forall t \in [t_j, t_j + \Delta t_j)$, and it is natural that $G_{t_j+\Delta t_j}^{t_{j+1}}(\lambda_0) \leq G_{t_j}^{t_{j+1}}(\lambda_0)$. Iterating the inequality (2.101) from 0 to N_σ together with (2.104), where N_σ denotes $N_\sigma(T,0)$, we get

$$\begin{aligned} W(T^-) \leq & \Big(\mu^{N_\sigma} \prod_{s=0}^{N_\sigma - 1} \Delta_s\Big) W(0) + \sum_{i=1}^{N_\sigma - 1} \Big(\mu^{N_\sigma - i+1} e^{\lambda_0 t_i} \psi(t_{i-1}) \prod_{\bar{s}=i}^{N_\sigma - 1} \Delta_{\bar{s}}\Big) \\ & + \mu e^{\lambda_0 T} \psi(t_{N_\sigma - 1}) + \sum_{j=1}^{N_\sigma - 1} \Big(\mu^{N_\sigma - j+1} G_{t_{j-1}}^{t_j}(\lambda_0) \prod_{l=j}^{N_\sigma - 1} \Delta_l\Big) + \mu G_{t_{N_\sigma - 1}}^{\top}(\lambda_0) \end{aligned}$$

Since Δt_j is a bounded small time period, there exists a constant $\bar{\Delta} > 0$ such that $\prod_{s=i}^{N_\sigma - 1} \Delta_s \leq \bar{\Delta}$ $\forall i \in \{1, 2, \ldots, N_\sigma - 1\}$. Note that $e^{\lambda_0 t_i} \leq e^{\lambda_0 T}$, one has

$$\begin{aligned} W(T^-) \leq & \mu^{N_\sigma} \bar{\Delta} W(0) + e^{\lambda_0 T} \bar{\Delta} \sum_{i=1}^{N_\sigma - 1} (\mu^{N_\sigma - i+1} \psi(t_{i-1})) + \mu e^{\lambda_0 T} \psi(t_{N_\sigma - 1}) \\ & + \bar{\Delta} \sum_{j=1}^{N_\sigma - 1} (\mu^{N_\sigma - j+1} G_{t_{j-1}}^{t_j}(\lambda_0)) + \mu G_{t_{N_\sigma - 1}}^{\top}(\lambda_0) \end{aligned} \tag{2.105}$$

From (2.90) and (2.91), we get $\mu^{N_\sigma - j+1} G_{t_{j-1}}^{t_j}(\lambda_0) \leq \mu^{1+N_0} e^{(\lambda_0 - \lambda)T} G_{t_{j-1}}^{t_j}(\lambda)$, for $0 < \lambda < \lambda_0$. Taking the forgoing inequality into (2.105), and following the same way as in Theorem 2.5, we can finally obtain

$$\begin{aligned} \bar{\alpha}_1 |x(T)|^2 \leq & \beta_a(|x(0)|, t) + \gamma_{\bar{e}}(\|\bar{e}_x(t_j)\|_{[0,T)}) \\ & + \gamma_{ex}(\|e_x\|_{[0,T)}) + \gamma_{ef}(\|e_f\|_{[0,T)}) + \bar{\varsigma}_2 \end{aligned} \tag{2.106}$$

where $\beta_a \in \mathcal{KL}$, $\gamma_{\bar{e}}$, γ_{ex}, $\gamma_{ef} \in \mathcal{K}_\infty$, $\bar{\varsigma}_2 \geq 0$ are determined from (2.105).

The inequality (2.106) implies the ISpS of HS w.r.t. $e_x(t)$, $e_f(t)$, $\bar{e}_x(t_j)$ and a constant $\bar{\varsigma}_2 > 0$, where $j = 1, 2 \ldots$ which, together with $e_x(t^j(k+1))_{est} < e_x(t^j(k))_{est}$ and the boundness of e_f guarantees the global convergence of the states of the system to a small closed set. □

Remark 2.10. *Note that $\bar{e}_x(t_j)$ is a discrete vector, since its value is captured only at each switching instant. Moreover, it has been shown that $|\bar{e}_x(t_j)|$ $\forall k \in Q$ is bounded. Theorem 2.6 also implies that the value of $\Delta t_j \in \Omega_{\Delta t_j}$ does not change the system's*

ISpS property. Appropriate selection of Δt_j can reduce the bound of x in the sense of ISpS in (2.106).

Remark 2.11. *Switching the input between the nominal control strategy and zero value has been shown to be an efficient way for performance-based FTC [103]. It is natural for HS that, at each t_j, the controller is switched on according to the next mode. Setting the input to zero during a short period after each switching is reasonable.*

Example 2.2: [132] A $\bar{m}$-phase switched reluctance motor (SRM) system is employed to illustrate a potential application field of the approach. $x = [\theta_m,\ \omega_m]^\top$ is the state, where θ_m, ω_m denote the angular position and velocity of the motors.

The simplified system model is expressed as follows:

$$\begin{aligned}\dot{\theta}_m &= \omega_m \\ \dot{\omega}_m &= -\frac{\kappa_e}{J_m}\sin(\theta_m) - \frac{b_i}{J_m}\omega_m + \frac{c_i}{J_m}u_i, \quad i = 1,2,\ldots,\bar{m}\end{aligned}$$

where J_m denotes the inertia of the motor. $\kappa_e > 0$ is the elasticity constant. u_i is the voltage applied to the motor of phase i, with b_i and c_i being the related viscous friction and the amplifier gain. In the simulation, $\bar{m} = 3$ is considered. The parameters are $J_m = 0.935\, kgm^2$, $\kappa_e = 0.311\, Nm/rad$, $b_1 = 1.17\, Nms/rad$, $b_2 = 2.23\, Nms/rad$, $b_3 = 0.54\, Nms/rad$, $c_1 = 20.196\, Nm/V$, $c_2 = 35.31\, Nm/V$, $c_3 = 12.44\, Nm/V$. We further describe the model by the general form (2.55)-(2.56) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1.2513 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2.385 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -0.5775 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 21.6 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 37.765 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 13.305 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ -0.333\sin x_2 \end{bmatrix}$$

The position of the motor phase can be measured via the shaft position sensor, while the motor velocity is often estimated by timing the interval between phase commutations of SRM. A coupled output signal of the angular position and velocity is obtained shared by all phases, the output matrix $C = [1\ 2]$.

The *continuous actuator fault* is considered only in mode 1 with $E_1 = [0\ \ -12.5]^\top$. The matrix K_1 and Q_1 are chosen as

$$K_1 = \begin{bmatrix} 3 \\ -1.8 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.1105 & -0.0007 \\ -0.0007 & 0.0986 \end{bmatrix}$$

Solving Eqs.(2.63)-(2.64), we obtain $R_1 = 0.3225$ and

$$P_1 = \begin{bmatrix} 0.0157 & 0.0258 \\ 0.0258 & 0.0516 \end{bmatrix}$$

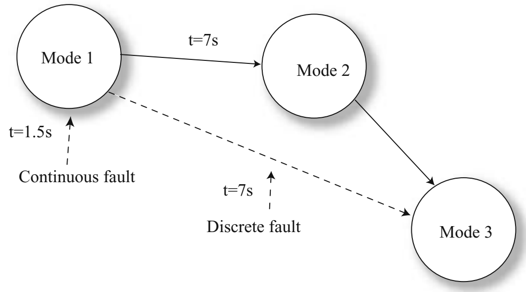


Fig. 2.3 An illustration of system's behavior

On the other hand, by choosing $W_1 = I_{2\times 2}$, we obtain the matrix H_1 from (2.75) as

$$H_1 = \begin{bmatrix} 1.0330 & 0.0327 \\ 0.0327 & 0.0325 \end{bmatrix}$$

The bounded function $\eta_1(x,t)$ is selected from (2.76) as

$$\eta_1(x,t) = \frac{0.333|0.0151x_1 - 0.0377x_2|}{|0.3266x_1 - 0.8150x_2|}$$

Take $\Gamma_1 = 20$, $\vartheta_1 = 8$, $\varepsilon = 0.01$. The related parameters of modes 2 and 3 can be obtained following the same way as for mode 1, which is omitted.

The considered switching sequence is: mode 1$\rightarrow$ mode 2$\rightarrow$ mode 3 as shown in Fig. 2.3. $N_0 = 0$. From (2.126)-(2.87), choose $\mu = 35$, $\lambda_0 = 0.8$. The switching instants are prescribed as $t_1 = 7s$, $t_2 = 14s$, which satisfy the ADT scheme in theorems 2.3 and 2.4. The system is initialized in mode 1 with $x(0) = [0.05 \;\; 0.2]^\top$.

f_1^c is assumed to occur at $t = 1.5s$ as

$$f_1^c(t) = \begin{cases} 0, & 0s \le t < 1.5s \\ 0.5 + 0.3\sin(4\pi t), & 1.5s \le t < 7s \end{cases}$$

which corresponds to an increase in the friction of the motor, that makes the voltage deviates from normal situation. Fig. 2.4 shows the fault estimation performance, from which we can see that $\hat{f}_1^c$ follows f_1^c rapidly with a very small overshoot.

The *discrete fault* occurs at $t = t_1 = 7s$, which represents the abnormal switching behavior of the motor phase that makes mode 1 switch to mode 3 as in Fig. 2.3. At $t = 7s$, the identifier scheme is invoked. The parameter of the model free observer in (2.94) is designed as

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \;\; \bar{L} = \begin{bmatrix} 2.8 \\ -1.6 \end{bmatrix}, \;\; \bar{P} = \begin{bmatrix} 2.7055 & 4.5351 \\ 4.5351 & 9.0703 \end{bmatrix}$$

where $\bar{P}$ is obtained with $\bar{Q} = \begin{bmatrix} 0.6384 & 0.6540 \\ 0.6540 & 1.8141 \end{bmatrix}$. There exists a $h(x,t)$ with ρ selected as 3. The speed of the rotor can cause an increase of the current after the corresponding voltage control has been switched off. As a consequence, such residual current can have an adverse effect on torque production at each switching instant. To avoid an unexpected oscillation of rotor, we select $\xi = 2$. From (2.97) and (2.100), we can also choose $\rho_1 = 5$. A boundary layer compensator technique [150] is used with a bound number 0.02 to eliminate the chattering.

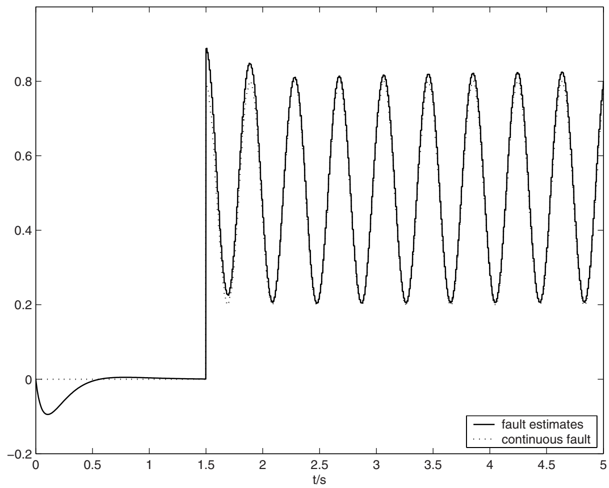
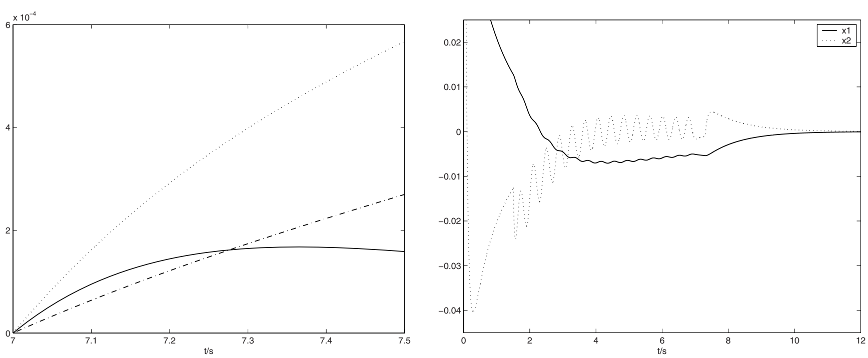


Fig. 2.4 Fault diagnosis



(a) Identifier performance ($|\bar{x}-\hat{x}_3|$: solid; $|\bar{x}-\hat{x}_2|$: dotted; $|\bar{x}-\hat{x}_1|$: dash-dotted)

(b) States trajectories

Fig. 2.5 FTC performance

Fig. 2.5(a) shows the performance of the identifier. Although $|\bar{x}-\hat{x}_1|$ is minimal at the beginning, $|\bar{x}-\hat{x}_3|$ is minimal and decreases at $7.35s$, while $|\bar{x}-\hat{x}_1|$ and $|\bar{x}-\hat{x}_2|$ still diverge. This implies that Mode 3 and consequently the discrete fault can be identified with $\Delta t_1 = 0.35$. The controller and fault diagnostic scheme for mode 3 are invoked into the system at $t = 7.35s$. The state trajectories throughout the system process is shown in Fig. 2.5(b), it can be seen that the states are always bounded.

2.4 Global Passivity

In sections 2.1-2.3, we designed FTC law in each faulty mode such that it is stable, then applied the standard stability results for HS. In the following two sections, we will research directly the stability of HS without reconfiguring the controller in each mode. We introduce, for the first time, the passivity theory into the FTC analysis of HS.

2.4.1 Passivity and Fault Diagnosis

Passivity theory, that provides a bridge between achievable system performances and energy-like considerations, has been widely used to analyze stability of nonlinear systems, where systems can not store more energy than that supplied by the environment outside [127]. Passivity concept has also been adopted for switched and HS [156], [151], where each mode is assumed to be passive.

We shall introduce the passivity theory into the FTC design for HS where each mode is passive in the healthy situation, and may be not passive due to the fault.

Consider the affine nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \Delta(x) \\ y &= h(x) \end{aligned} \tag{2.107}$$

where $x \in X \subset \Re^n$ are measurable states, $u \in U \subset \Re^m$ are inputs, $y \in Y \subset \Re^m$ are outputs. The fault is modelled by an unknown function $\Delta(x) \in \Re^n$, which effectively represents the process faults [10], and occurs at an unknown time. f, g, h and Δ are smooth functions.

Definition 2.6. *[14] A system (2.107) with $\Delta \equiv 0$ is* passive *if there exists a nonnegative function $V : X \to \Re$, which satisfies $V(0) = 0$, called the* storage function*, and a* supply rate $y^\top u$*, such that for all initial states $x(0) \in X$, $u \in U$, and $t \geq 0$*

$$\underbrace{V(x(t)) - V(x(0))}_{\textit{stored energy}} \leq \underbrace{\int_0^t y^\top(s)u(s)ds}_{\textit{supplied energy}} \tag{2.108}$$

where $x(t)$ are the states at time t.

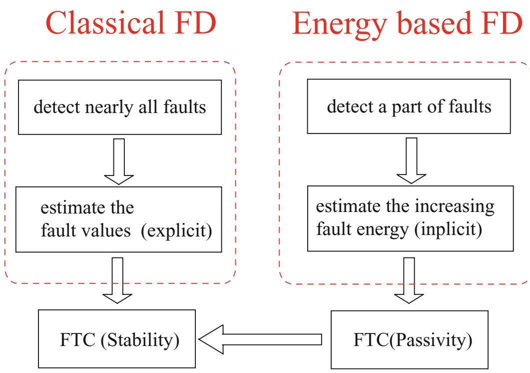


Fig. 2.6 Comparison of FD methods

The inequality (2.108) is called dissipativity inequality [127], which formalizes the property that the increase in stored energy is never greater than the amount of energy supplied by the environment. A passive system is easy to control, choosing $u = -\phi(y)$, where $\phi : U \rightarrow Y$ is a smooth function and $\phi(0) = 0$, such that $y^\top \phi(y) > 0$ for each nonzero y leads to Lyapunov stability [14].

Now we address the FD problem. As shown in Fig. 2.6, most classical methods [36, 18] are designed such that the explicit values of faults can be estimated. Here we develop a novel energy based FD technique that is concerned with the energy analysis and has its root in the passivity. Under the passivity framework, we show that only a part of faults needs to be detected and estimated implicitly.

In the following, we assume that V is a $\mathscr{C}^1$ function. The passivity property is equivalent to

$$\left[\frac{\partial V}{\partial x}(x)\right]^\top [f(x) + g(x)u] \leq y^\top u \tag{2.109}$$

Once a fault occurs, the constraint (2.108) may be violated. Adding $\Delta(x)$ into (2.109) and integrating both sides yields

$$V(x(t)) - V(x(0)) \leq \int_0^t y^\top(s)u(s)ds + \underbrace{\int_0^t \left[\frac{\partial V}{\partial x}(x)\right]^\top \Delta(x(s))ds}_{\text{fault energy } E_f} \tag{2.110}$$

As indicated in (2.110), the energy dissipativity property changes due to the fault. The fault may help to dissipate the stored energy ($E_f < 0$) or increase the stored energy ($E_f > 0$). We only care about the faults that result in $V(x(t)) - V(x(0)) > \int_0^t y^\top(s)u(s)ds$. A diagnosis threshold can be designed as

$$V(x(t)) - V(x(0)) = \int_0^t y^\top(s)u(s)ds \tag{2.111}$$

This is also called *lossless* property [14]. Note that the faults with $E_f < 0$ are not necessary to be detected since they do not change the energy dissipativity. Once the left side of (6.6) becomes larger than the right side, the fault is detected. We estimate such fault value implicitly as $V(x(t)) - V(x(0)) - \int_0^t y^\top(s)u(s)ds$. More precisely, we estimate the energy that increases due to the fault and check whether the system is still passive or not. This information will be used for fault tolerance analysis [141].

2.4.2 Fault Tolerance Analysis of Hybrid Systems

The hybrid system takes the form

$$\begin{aligned} \dot{x} &= f_\sigma(x) + g_\sigma(x)u_\sigma + \Delta_\sigma(x) \\ y &= h_\sigma(x) \end{aligned} \tag{2.112}$$

where $x \in X \subset \Re^n$ is continuous everywhere, $u_\sigma \in \Re^{m_\sigma}$, $h_\sigma \in \Re^{m_\sigma}$. All f_σ, g_σ, h_σ and Δ_σ are smooth functions. $\sigma(t) : [t_0, \infty) \to Q = \{1, 2, \ldots, N\}$ denotes the *switching function*. We denote by t_k, $k = 1, 2, \ldots$ the kth switching time. $N_{\sigma(t)}$ represents the number of switchings in $[0, t)$. t_{kj}, $k = 1, 2, \ldots$, $j \in Q$ denotes the kth switching time that mode j is activated. Suppose that there exists N non-negative storage functions $V_p(x)$, and α_1^p, $\alpha_2^p \in \mathcal{K}_\infty$, $\forall p \in Q$ that satisfy

$$\alpha_1^p(|x|) \le V_p(x) \le \alpha_2^p(|x|) \tag{2.113}$$

such that mode p is passive with $V_p(x)$ in the healthy situation.

In this work, we neither reconfigure the controller u_σ nor adjust the switching law σ. We analyze fault tolerance of the HS (2.112) under the original u_σ and σ. It will be shown that under the *global energy dissipativity*, the stability of the HS can be achieved in spite of non passive modes.

Definition 2.7. *A switched system (2.112) is* globally passive *if there exists nominal controllers u_1, u_2, ...,u_N, such that for all initial states $x(0) \in X$, and $T \ge 0$*

$$V_{\sigma(T)}(x(T)) - V_{\sigma(0)}(x(0)) - E_{tr}(x(0)) \le \int_0^T W(s)ds \tag{2.114}$$

where $W(s) \le 0$ is defined as

$$\begin{aligned} \int_0^T W(s) \triangleq \sum_{k=0}^{N_{\sigma(T)}} \int_{t_k}^{t_{k+1}} \Big(& y^\top(s)u_{\sigma(s)}(s) \\ & + \Big[\frac{\partial V_{\sigma(s)}}{\partial x}(x)\Big]^\top \Delta_{\sigma(s)}(x(s))\Big) ds \end{aligned} \tag{2.115}$$

and $E_{tr} = \sum_{k=1}^{N_{\sigma(T)}} \Big[V_{\sigma(t_k)} - V_{\sigma(t_k^-)}\Big]$ is bounded by a constant and tends to zero as $x(0)$ goes to origin.

The left side of (2.114) represents the sum of stored energies of all modes, which could also be written as $\sum_{k=0}^{N_{\sigma(T)}} \left[V_{\sigma(t_{k+1}^-)} - V_{\sigma(t_k)} \right]$ where $t_0 = 0$, $t_{N_{\sigma(T)}+1} = T$. The formulation of (2.114) is consistent with the standard passivity inequality, E_{tr} denotes the total transient energy. As shown later, E_{tr} may be eliminated under some conditions.

It is clear from (2.115) that the right hand of (2.114) denotes the total supplied energy and "fault" energy. Since $W(s) \le 0$, it follows that under the nominal controllers u_1, u_2, ...,u_N, the sum of the supplied energy during $[0,T)$ can compensate the increasing energy due to faults. This means that the total stored energies still dissipative in spite of faults.

Global passivity balances the total energy throughout the overall process, while no individual passivity of each mode is required. We shall prove that the global passivity includes the passivity property proposed in [156] as in the following proposition.

Proposition 2.1. *If each mode of a HS (2.112) is passive as in (2.108), and there exist functions $\omega_k^{k+1}(t)$, called cross supply rates such that $\omega_k^{k+1}(t) \le \phi_k^{k+1}(t)$ where $\phi_k^{k+1}(t) \in \mathscr{L}_1$ and*

$$V_q(x(t_{q(k+1)})) - V_q(x(t_{qk})) \le \int_{t_{qk}}^{t_{q(k+1)}} \omega_k^{k+1}(s)ds \tag{2.116}$$

then the system (2.112) is globally passive.

Proof: The passivity of each mode leads to the fact that each energy is non-decreasing when the related mode is activated. Suppose mode q is activated at the time T, from (2.116), we obtain

$$V_q(x(T)) - V_q(x(t_{q1})) - \Theta(x(0)) \le \int_0^T W(s)ds \tag{2.117}$$

where $W(s) \le 0$, $\Theta(x(0))$ is a constant and tends to zero as $x(0)$ goes to the origin. This constant is obtained from the fact that $\Sigma_{k=1}^{\infty} \int_{t_{qk}}^{t_{q(k+1)}} \omega_k^{k+1}(s)ds$ is bounded, since $\phi_k^{k+1}(t) \in \mathscr{L}_1$. On the other hand, for any $x(0)$, $V_{\sigma(0)}(x(0))$ is bounded, there exists a constant $\Phi \triangleq V_{\sigma(0)}(x(0)) - V_q(x(t_{q1}))$, which together with (2.117), leads to the result. □

Global passivity implies the stability as shown below.

Theorem 2.7. *If a HS (2.112) is globally passive, then the origin of the system is stable in spite of faults.*

Proof: For a given arbitrary $\varepsilon > 0$, since V_i is continuous and $V_i(0) = 0$, based on (2.113), we can choose $\varepsilon_2^i > 0$ such that $V_i < \varepsilon_2^i$ leads to $(\alpha_1^i)^{-1}(V_i) < \varepsilon$. Pick $\varepsilon_3 = \min_i[\varepsilon_2^i]$, since E_{tr} tends to zero as $x(0)$ goes to the origin, we can choose ε_4 such that $|x(0)| < \varepsilon_4$ results in $\max_i[\alpha_2^i(|x(0)|) + E_{tr}(x(0))] < \varepsilon_3$. Thus, followed by (2.114),

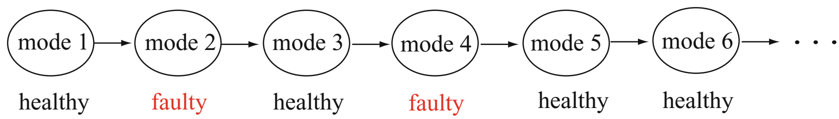


Fig. 2.7 Switching sequence

we find that if the system starts in $B(\varepsilon_4)$, we will stay within $B(\varepsilon)$. This completes the proof. □

Theorem 2.7 provides us a method to check the fault tolerance, which is equivalent to check the global passivity. However, when we use (2.114) to check the fault tolerance at any instant T, one obstacle appears since we are not sure whether there is a constant bound of the total transient energy for all $t \geq T$. This motivates the following result.

Proposition 2.2. *If a HS (2.112) is globally passive, and $V_{\sigma(t)}(x(t)) \leq V_{\sigma(t^-)}(x(t))$ at each switching instant t, then (2.114) holds with $E_{tr} = 0$.*

Proof: The result follows the fact that

$$\begin{aligned}\sum_{k=0}^{N_{\sigma(T)}} &\left[V_{\sigma(t_{k+1}^-)}(x(t_{k+1})) - V_{\sigma(t_k)}(x(t_k))\right] \\ &= V_{\sigma(T)}(x(T)) - V_{\sigma(t_{N_{\sigma(T)}})}(x(t_{N_{\sigma(T)}})) + \cdots \\ &+ V_{\sigma(t_{k+1}^-)}(x(t_{k+1})) - V_{\sigma(t_k)}(x(t_k)) + \cdots + V_{\sigma(t_1^-)}(x(t_1)) - V_{\sigma(0)}(x(0)) \\ &\geq V_{\sigma(T)}(x(T)) - V_{\sigma(0)}(x(0)) \qquad (2.118)\end{aligned}$$

Thus, from (2.115), we have $V_{\sigma(T)}(x(T)) - V_{\sigma(0)}(x(0)) \leq \int_0^T W(s)ds$. □

The condition in Proposition 2.2 guarantees that the energy in the current mode at switching time is always larger than that of the next mode. In this case, the transient energy is negative.

To further overcome the obstacle in (2.114), and allow the increase of energy at switching time, we provide a stronger version of global passivity, named "periodic fault tolerant passivity". We first define some mode sets:

- $Q_1 \subset Q$ denotes the set of healthy modes.
- $Q_2 \subset Q_1$ denotes the set of healthy modes that may be activated as the initial mode or after a healthy mode.
- $Q_3 \subset Q_1$ denotes the set of healthy modes that are activated after a faulty mode, meanwhile, are followed by a healthy mode or are the final mode.

The relation of above several sets is illustrated by Fig.2.7, from which we see that $\{\ 1, 3, 5, 6\ \} \in Q_1$. $\{\ 1, 6\ \} \in Q_2$. $5 \in Q_3$. Note that Mode 3 is activated between two faulty modes. Thus $3 \in Q_1 \setminus (Q_2 \cup Q_3)$.

Definition 2.8. *A HS (2.112) is* periodically fault tolerant passive *if there exist nominal controllers u_1, u_2, ...,u_N, such that for all initial states $x(0) \in X$, and $T \geq 0$, the following inequalities hold:*

- $\forall i \in Q_2$
$$V_i(x(t_{(k+1)i})) - V_i(x(t_{ki})) \leq 0 \tag{2.119}$$
where $0 \leq t_{(k)i} < t_{(k+1)i} \leq T$.
- $\forall i \in Q_2$, $j \in Q_3$, *such that mode* j *is the first mode of set* Q_3 *activated after mode* i. *Denote by* T_e, T_s *the end time of mode* j *and the start time of mode* i *respectively*
$$V_j(x(T_e)) - V_i(x(T_s)) \leq \int_{T_s}^{T_e} W_1(s)ds \tag{2.120}$$
where $W_1(s) \leq 0$.
- *For the case that the initial mode* i *is faulty, and there exists* $j \in Q_3$ *such that mode* j *is the first mode of set* Q_3 *activated after initial mode and is ended at* T_e
$$V_j(x(T_e)) - V_i(x(0)) \leq \int_0^{T_e} W_2(s)ds \tag{2.121}$$
where $W_2(s) \leq 0$.
- *For the case that the final mode* i *is faulty, and there exists* $j \in Q_2$ *such that mode* j *is the last mode of set* Q_2 *activated before the final mode and is started at* T_s
$$V_j(x(T)) - V_i(x(T_s)) \leq \int_{T_s}^{T} W_3(s)ds \tag{2.122}$$
where $W_3(s) \leq 0$.
- *For the case that no mode of the set* $Q_2 \cup Q_3$ *is activated*
$$V_{\sigma(T)}(x(T)) - V_{\sigma(0)}(x(0)) \leq \int_0^{T} W_4(s)ds \tag{2.123}$$
where $W_4(s) \leq 0$.

Definition 2.8 is illustrated in Fig. 2.8, from which we can see that the energy is dissipative in each small period that includes the faulty modes. Two advantages result from this property, that is 1) Inequalities (2.120)-(2.123) are not hard to justify. 2) We can check the fault tolerance in a short period after the fault occurs.

Theorem 2.8. *If a HS (2.112) is periodic fault tolerant passive, then the origin of the system is stable in spite of faults.*

Proof: We consider four cases as follows:

- Case 1: The initial and final modes are not faulty. Note that each healthy mode is passive. Inequalities (2.120)-(2.122) imply that every time when we start in the mode of the set Q_2, the energy is non-increasing until the next mode of set Q_2 is activated. The stability follows from Theorem 2.3 in [13] and Theorem 2.7.

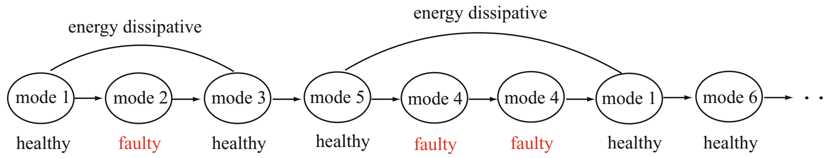


Fig. 2.8 Switching sequence

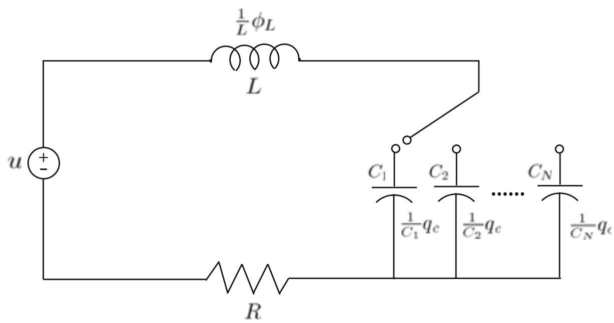


Fig. 2.9 A switched RLC circuit

- Case 2: No mode of the set $Q_2 \cup Q_3$ is activated. The stability is achieved from (2.123) and Theorem 2.7.
- Case 3: The initial mode is healthy, and the final mode is faulty. It follows from (2.122) that after the last mode of set Q_2 before final mode is activated, the energy is non-increasing. The stability is achieved from Theorem 2.3 in [13] and Theorem 2.7.
- Case 4: The initial mode is faulty, and the final mode is healthy. Similarly to Case 3, the result can be obtained from (2.121). □

Example 2.3: A switched RLC circuit that is widely employed in order to perform low-frequency signal processing in integrated circuits is taken as an example to illustrate the results. As shown in Fig. 2.9, the circuit consists of an input power source, a resistance, an inductance and N capacitors that could be switched between each other. The two measurable state variables are the charge in the capacitor and the flux in the inductance $x = [q_c, \phi_L]^\top$. The input u is the voltage.

The dynamic equations are given by

$$\begin{cases} \dot{x}_1 = \frac{1}{L}x_2 \\ \dot{x}_2 = -\frac{1}{C_i}x_1 - \frac{R}{L}x_2 + u \\ y = \frac{1}{L}x_2, \quad i = 1, 2, \ldots, N \end{cases}$$

where C_i denotes the ith capacitor. The energy function of each mode is given as

$$V_i = \frac{1}{2C_i}x_1^2 + \frac{1}{2L}x_2^2$$

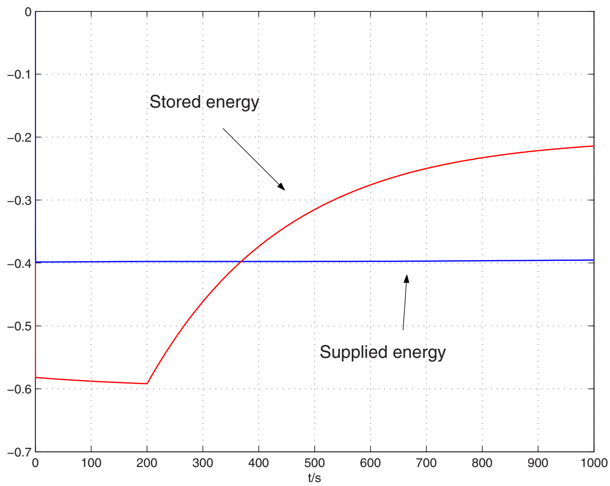


Fig. 2.10 Diagnosis performance (N=1)

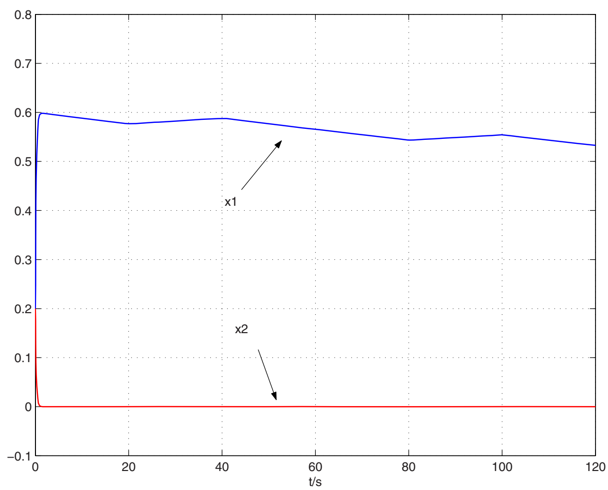


Fig. 2.11 System performance (N=3)

Let us first consider the case $N = 1$, this RLC circuit is also discussed in [91]. In the healthy situation, it can be obtained that $\dot{V} = -\frac{R}{L}x_2^2 + yu$ which satisfies the passivity. The nominal control is chosen as $u = u_n = -y$. Now we consider a leakage fault that occurs in the capacitor at $t = 200s$, the dynamic equation of $\dot{x}_2$ is changed into

$$\dot{x}_2 = -\frac{1}{C}x_1 - \frac{R}{L}x_2 + \frac{k}{C}x_1 + u \tag{2.124}$$

where $k > 0$ is an unknown faulty parameter. It follows that $\dot{V} = -\frac{R}{L}x_2^2 - \frac{k}{LC}x_1x_2 + yu$. If $-\frac{R}{L}x_2^2 \leq \frac{k}{LC}x_1x_2$, then such fault does not affect the passivity. Otherwise, the fault would be diagnosed. Set $k = -200$, $L = 0.1H$, $C = 100\mu F$, $R = 1\Omega$, the initial states are $[0.2, 0.2]^\top$. Fig. 2.10 shows the diagnosis performance, we can see that once the threshold is reached at nearly 370s, the fault is detected.

Suppose that $N = 3$, i.e., the system is switched among three capacitors. C_1 is activated in $[t_{3n}, t_{3n+1})$, C_2 is in $[t_{3n+1}, t_{3n+2})$, and C_3 is in $[t_{3n+2}, t_{3n+3})$, $n = 0, 1, \ldots$. The nominal input is $u_i = -\frac{1}{L}x_2$. The fault occurs in C_2 as (2.124) with $k = -200$, which violates the passivity of mode 2. It is clear that $1 \in Q_2$, $3 \in Q_3$. In the simulation, set $L = 0.1H$, $C_1 = 50\mu F$, $C_2 = 100\mu F$, $C_3 = 20\mu F$ and $R = 1\Omega$. Assume that the dwell period $t_{3n+3} - t_{3n+2} = 20s$, $t_{3n+2} - t_{3n+1} = 20s$, and $t_{3n+1} - t_{3n} = 20s$. We can check that each period $[t_{3n}, t_{3n+3})$ satisfies (2.120), and mode 1 satisfies (2.119). Thus the system is periodic fault tolerant passive. Fig. 2.11 shows the state trajectory, the system is still stable in spite of the fault.

2.5 General Stability Results in HS

Motivated by the fact that some modes may be unstable due to faults, in this section, we establish a new sufficient stability condition named " gain technique" for HS with unstable mode, and provide novel stabilizing switching laws such that the stability is guaranteed and each mode can be activated following any prescribed sequence whatever it is stable or not.

2.5.1 Preliminaries

The considered switched system takes the general form

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \tag{2.125}$$

where $x \in X \subset \Re^n$ are the states. f_σ is a nonlinear smooth function. Define $Q = \{1, 2, \ldots, N\}$, where N is the number of modes. $\sigma(t) : [0, \infty) \to Q$ denotes the *switching function*, which is assumed to be a piecewise constant function continuous from the right. f_i, $i \in Q$ are smooth functions with $f_i(0) = 0$, hence, the origin is an equilibrium point. We denote by t_j, $j = 1, 2, \ldots$ the jth switching instant, $t_0 = 0$. Let t_{ik}, $i \in Q$, $k = 1, 2, \ldots$ be the kth time when mode i is switched on. $N_{\sigma(t)}$ represents the number of switchings in $[0, t)$. In this work, we only consider nonZeno sequences (i.e., sequences that switch at most a finite number of times in any finite time interval). However, the developed theory allows infinite switchings in infinite time interval. We also assume that the states do not jump at the switching instants.

Specially, we define a class $\mathscr{GKL}$ function as in [135] $\gamma : [0, \infty) \times [0, \infty) \to [0, \infty)$ if $\gamma(\cdot, t)$ is of class $\mathscr{K}$ for each fixed $t \geq 0$ and $\gamma(s, t)$ increases as $t \to \infty$ for each fixed $s \geq 0$.

Denote $Q_s \subset Q$ as the set of stable modes and $Q_{us} \subset Q$ the set of unstable ones. $Q = Q_s \cup Q_{us}$, $Q_s \cap Q_{us} = \emptyset$ and $Q_s \neq \emptyset$. Suppose that there exist continuous non-negative functions $V_p : \Re^n \to \Re_{\geq 0}$, α_1^p, $\alpha_2^p \in \mathscr{K}_\infty$, $\forall p \in Q$, and $\phi_p \in \mathscr{K}\mathscr{L}$ $\forall p \in Q_s$, $\phi_p \in \mathscr{G}\mathscr{K}\mathscr{L}$ $\forall p \in Q_{us}$ that satisfy for $k = 1,2,...$

$$\alpha_1^p(|x|) \leq V_p(x) \leq \alpha_2^p(|x|), \quad \forall p \in Q \tag{2.126}$$

$$V_p(x(t)) \leq \phi_p(V_p(x(t_{pk})), t - t_{pk}),\ \forall p \in Q_s,\ \phi_p \in \mathscr{K}\mathscr{L},\ t \geq t_{pk} \tag{2.127}$$

$$V_p(x(t)) \leq \phi_p(V_p(x(t_{pk})), t - t_{pk}),\ \forall p \in Q_{us},\ \phi_p \in \mathscr{G}\mathscr{K}\mathscr{L},\ t \geq t_{pk} \tag{2.128}$$

Formulations (2.126)-(2.128) include various converging and diverging forms (e.g., the exponential decay form [47], the constant gain form [155]). For each stable mode, V_p in (2.127) is more general than a classic Lyapunov function since a bounded increase is allowed. For unstable modes, inequality (2.128) implies that V_p may increase infinitely as described by a $\mathscr{G}\mathscr{K}\mathscr{L}$ function if $t \to \infty$. $\mathscr{G}\mathscr{K}\mathscr{L}$ function is more general than the Lyapunov-like function in [148] since we do not impose an upper bound on V_p. Note that (2.127)-(2.128) are properties satisfied by functions of each mode, and do not depend on the switching sequence. V_p ($\forall p \in Q$) is not required to be differentiable.

Definition 2.9. *Given a switching function $\sigma(t)$, the origin of a switched system (2.125) is said to be* stable *under σ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x(t)| \leq \varepsilon$, $t \geq 0$, whenever $|x(0)| \leq \delta$.*

Definition 2.9 describes the stability w.r.t. a given switching function $\sigma(t)$. The objectives of this section is *to propose switching laws that stabilize the system (2.125) satisfying (2.126)-(2.128) by determining the switching instants according to any given switching sequence.*

2.5.2 Stabilization of Switched Systems

In the following, we first establish a stability condition for the considered switched systems in the finite time interval with finite numbers of switchings (Lemma 2.11). Based on such stability criterion, a stabilizing switching law will be constructed (Theorem 2.9).

Lemma 2.11. *Consider a switched system (2.125) satisfying (2.126)-(2.128). Under $\sigma(t)$, if there exists a constant $\beta > 0$ such that*

$$\sum_{k=0}^{N_{\sigma(t_s,t)}} \left(\prod_{i=k}^{N_{\sigma(t_s,t)}} \frac{\phi_{\sigma(t_i)}^{t_{i+1}-t_i}}{V_{\sigma(t_i)}^{t_i}} \right) \leq \beta, \quad t > t_s \geq 0, \quad \textit{where} \quad t_{N_{\sigma(t_s,t)}+1} \triangleq t,\ N_{\sigma(t_s,t)} \textit{ is finite} \tag{2.129}$$

Then x is bounded in $[t_s, t)$. Moreover, for any bounded $x(t_s)$, the upper bound of $|x(t)|$ can be estimated.

Remark 2.12. *Note that $\frac{\phi_{\sigma(t_i)}^{t-t_i}}{V_{\sigma(t_i)}^{t_i}}$ for $t \geq t_i$ is the bound of the gain of function $V_{\sigma(t_i)}$ when mode $\sigma(t_i)$ is activated. Condition (2.129) gives a relation among the gains of* *each activated mode and its activating period. More precisely, x is bounded in $[t_s, t)$ if the product of gains from each activated mode to the terminated mode is bounded, and the sum of these products values is also bounded. It deserves to point out that for a switched system with unstable modes, even in the finite time interval with finite switching times, x may escape to infinity under inappropriate switching law.*

Proof of Lemma 2.11: For the sake of clearness, suppose that $t_s = t_0 = 0$. Denote $N_{\sigma(t)} \triangleq N_{\sigma(0,t)}$.

Consider $t \in [0, t_1)$, we have $V^t_{\sigma(0)} \leq \frac{\phi^t_{\sigma(0)}}{V^0_{\sigma(0)}} V^0_{\sigma(0)}$. Condition (2.129) ensures that $\frac{\phi^t_{\sigma(0)}}{V^0_{\sigma(0)}} \leq \beta$. It follows from (2.126)-(2.128) that

$$|x(t_1)| \leq \underbrace{(\alpha_1^{\sigma(0)})^{-1} \circ \beta \circ \alpha_2^{\sigma(0)}}_{\vartheta_{t_1}}(|x(0)|) \tag{2.130}$$

for $\vartheta_{t_1} \in \mathscr{K}_\infty$. According to (2.126), one has

$$V^{t_1}_{\sigma(t_1)} \leq V^{t_1}_{\sigma(t_1^-)} + \alpha_2^{\sigma(t_1)}(\vartheta_{t_1}(|x(0)|)) - \alpha_1^{\sigma(t_1^-)}(\vartheta_{t_1}(|x(0)|)) \tag{2.131}$$

Define $\alpha_{t_1} = \max[\alpha_2^{\sigma(t_1)} \circ \vartheta_{t_1}, \alpha_1^{\sigma(t_1^-)} \circ \vartheta_{t_1}]$. Since $\alpha_2^{\sigma(t_1)}, \alpha_1^{\sigma(t_1^-)}, \vartheta_{t_1} \in \mathscr{K}_\infty$, it is clear that $\alpha_{t_1} \in \mathscr{K}_\infty$ and

$$\alpha_{t_1}(|x(0)|) \geq \alpha_2^{\sigma(t_1)}(\vartheta_{t_1}(|x(0)|)) - \alpha_1^{\sigma(t_1^-)}(\vartheta_{t_1}(|x(0)|)) \tag{2.132}$$

Substituting (2.132) into (2.131) results in

$$V^{t_1}_{\sigma(t_1)} \leq V^{t_1}_{\sigma(t_1^-)} + \alpha_{t_1}(|x(0)|) \tag{2.133}$$

For $t \in [t_1, t_2)$, we have

$$\begin{aligned} V^t_{\sigma(t)} \leq \frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} V^{t_1}_{\sigma(t_1)} &\leq \frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} \left[V^{t_1^-}_{\sigma(t_1^-)} + \alpha_{t_1}(|x(0)|) \right] \\ &\leq \frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} \frac{\phi^{t_1}_{\sigma(0)}}{V^0_{\sigma(0)}} V^0_{\sigma(0)} + \frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} \alpha_{t_1}(|x(0)|) \end{aligned} \tag{2.134}$$

Note that $V^0_{\sigma(0)}$ is bounded and $\alpha_{t_1} \in \mathscr{K}_\infty$. Condition (2.129) ensures that $\frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} \frac{\phi^{t_1}_{\sigma(0)}}{V^0_{\sigma(0)}} \leq \beta$ and $\frac{\phi^{t-t_1}_{\sigma(t_1)}}{V^{t_1}_{\sigma(t_1)}} \leq \beta$. It follows from (2.126)-(2.128) and (2.134) that

$$|x(t_2)| \leq \underbrace{(\alpha_1^{\sigma(0)})^{-1} \circ \beta \circ \left(\alpha_2^{\sigma(0)}(|x(0)|) + \alpha_{t_1}(|x(0)|)\right)}_{\vartheta_{t_2}(|x(0)|)} \tag{2.135}$$

for $\vartheta_{t_2} \in \mathscr{K}_\infty$. One further has

$$V_{\sigma(t_2)}^{t_2} \leq V_{\sigma(t_2^-)}^{t_2} + \alpha_2^{\sigma(t_2)}(\vartheta_{t_2}(|x(0)|)) - \alpha_1^{\sigma(t_2^-)}(\vartheta_{t_2}(|x(0)|)) \tag{2.136}$$

Define $\alpha_{t_2} = \max[\alpha_2^{\sigma(t_2)} \circ \vartheta_{t_2}, \alpha_1^{\sigma(t_2^-)} \circ \vartheta_{t_2}]$. Since $\alpha_2^{\sigma(t_2)}, \alpha_1^{\sigma(t_2^-)}, \vartheta_{t_2} \in \mathscr{K}_\infty$, it follows that $\alpha_{t_2} \in \mathscr{K}_\infty$ and

$$\alpha_{t_2}(|x(0)|) \geq \alpha_2^{\sigma(t_2)}(\vartheta_{t_2}(|x(0)|)) - \alpha_1^{\sigma(t_2^-)}(\vartheta_{t_2}(|x(0)|)) \tag{2.137}$$

Substituting (2.137) into (2.136) results in

$$V_{\sigma(t_2)}^{t_2} \leq V_{\sigma(t_2^-)}^{t_2} + \alpha_{t_2}(|x(0)|) \tag{2.138}$$

for $\alpha_{t_2} \in \mathscr{K}_\infty$.

By induction, we find that under condition (2.129) there exists a function $\alpha \in \mathscr{K}_\infty$ such that at each switching instant $t_i > 0$, $i = 1, 2, ..., N_{\sigma(t)}$

$$V_{\sigma(t_i)}(x(t_i)) \leq V_{\sigma(t_i^-)}(x(t_i)) + \alpha(|x(0)|) \tag{2.139}$$

where $\alpha(|x(0)|) \triangleq \sup_{i=1,2,...,N_{\sigma(t)}}[\alpha_{t_i}(|x(0)|)]$.

Denote $j = N_{\sigma(t)}$ for $t \geq 0$, $j \geq 0$, it follows from (2.127)-(2.128) that

$$\begin{aligned}
V_{\sigma(t)}(x(t)) &\leq \phi_{\sigma(t_j)}^{t-t_j} = \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} V_{\sigma(t_j)}^{t_j} \\
&\leq \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} \left[V_{\sigma(t_j^-)}^{t_j^-} + \alpha(|x(0)|)\right] \\
&\leq \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} \phi_{\sigma(t_{j-1})}^{t_j - t_{j-1}} + \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} \alpha(|x(0)|) \\
&\leq \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} \frac{\phi_{\sigma(t_{j-1})}^{t_j - t_{j-1}}}{V_{\sigma(t_{j-1})}^{t_{j-1}}} V_{\sigma(t_{j-1}^-)}^{t_{j-1}^-} + \left[\frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}} \frac{\phi_{\sigma(t_{j-1})}^{t_j - t_{j-1}}}{V_{\sigma(t_{j-1})}^{t_{j-1}}} + \frac{\phi_{\sigma(t_j)}^{t-t_j}}{V_{\sigma(t_j)}^{t_j}}\right] \alpha(|x(0)|) \\
&\qquad \vdots \\
&\leq \prod_{s=0}^{N_{\sigma(t)}} \frac{\phi_{\sigma(t_s)}^{t_{s+1}-t_s}}{V_{\sigma(t_s)}^{t_s}} V_{\sigma(0)}(x(0)) + \sum_{k=1}^{N_{\sigma(t)}} \left(\prod_{i=k}^{N_{\sigma(t)}} \frac{\phi_{\sigma(t_i)}^{t_{i+1}-t_i}}{V_{\sigma(t_i)}^{t_i}}\right) \alpha(|x(0)|)
\end{aligned} \tag{2.140}$$

Based on (2.126) and (2.139), since $\alpha \in \mathscr{K}_\infty$, there exists a $\mathscr{K}_\infty$ function $\bar{\alpha}$ such that

$$\bar{\alpha}(|x(0)|) = \max\left[\alpha_2^{\sigma(0)}(|x(0)|), \alpha(|x(0)|)\right] \tag{2.141}$$

Substituting (2.141) into (2.140), together with (2.129), yields

$$V_{\sigma(t)}(x(t)) \leq \sum_{k=0}^{N_{\sigma(t)}} \Big(\prod_{i=k}^{N_{\sigma(t)}} \frac{\phi_{\sigma(t_i)}^{t_{i+1}-t_i}}{V_{\sigma(t_i)}^{t_i}}\Big)\bar{\alpha}(|x(0)|) \leq \beta\bar{\alpha}(|x(0)|) \tag{2.142}$$

From (2.126), we finally obtain

$$|x(t)| \leq (\alpha_1^{\sigma(t)})^{-1}\beta\bar{\alpha}(|x(0)|) \tag{2.143}$$

Since $\beta > 0$ is a constant, $\alpha_1^{\sigma(t)}, \bar{\alpha} \in \mathscr{K}_\infty$, the stability result follows.

From above procedures, one can find that under condition (2.129), given any $x(t_s)$, β and switching sequence, each $\alpha_{t_i}(|x(t_s)|)$ can be calculated which is independent from the switching instants. Thus, for any bounded $x(t_s)$, we can find a function $\Omega(\cdot)$ such that $|x(t)| \leq \Omega(|x(t_s)|)$. This completes the proof. □

Remark 2.13. *The main contributions of Lemma 2.11 are twofold: 1) Both stable and unstable modes are allowed in the switched nonlinear system; 2) The "μ" condition is removed by introducing a difference $\alpha(|x(0)|)$ among functions V_p $\forall p \in \mathscr{M}$. However, the condition (2.129) is independent from $\alpha(|x(0)|)$. 3) The upper bound of $|x(t)|$ can be estimated without the information of switching instants in $[0,t)$. This property will be very useful in switching law design.*

Remark 2.14. *The condition (2.129) is valid since V_σ is a non-negative function and is impossible to become zero unless a stronger finite time stability [9] is achieved. For the case that finite time stability is achieved, (2.129) is available if we take j instead of $N_{\sigma(t)}$ where $V_{\sigma(t)}^t > 0$ for $t < t_{j+1}$.*

Remark 2.15. *It is often not easy to verify (2.129) on-line, which relies on the solutions of the system. However, this condition can help to construct a stabilizing switching law as shown below. The proposed stabilization scheme will automatically guarantee the validation of (2.129).*

Unlike the usual design methods that adjust both the switching sequence and switching instants [155], [130], we only redesign the switching instants such that the origin of switched system is always stable under any given switching sequence where each prescribed mode can be activated.

Assumption 2.11. *there exists a known constant $\chi \geq 1$ such that*

$$\chi = \max_{j\in\mathscr{M}, k=1,2\dots} \frac{\phi_j(V_j(x(t_{jk})),0)}{V_j(x(t_{jk}))} \tag{2.144}$$

Remark 2.16. *Assumption 1 means that the initial gain of function V_j is bounded when the corresponding mode j is just switched on at $t = t_{jk}$. In some situations, $\phi_j(V_j(x(t_{jk})),0)$ is affine w.r.t. $V_j(x(t_{jk}))$, e.g. the exponential decay form [47], the constant gain form [155]. In these cases, χ can be easily obtained* a priori.

Without loss of generality, suppose that at for a given sequence, at most m unstable modes (m is finite) are activated one by one without being interrupted by stable modes.

Choose a constant $\beta > \max[m(1+\chi)\chi^m, m(m+1)\chi^{m+1}]$, where χ is defined in (6.37). Given any required upper bound ε of $|x(t)|$ and switching sequence, the switching law is designed as:

Switching law $\mathscr{S}$ (with a given ε and a switching sequence)

1. *Let $i = 0$, choose $x(0)$ such that $(\alpha_1^{\sigma(0)})^{-1}\phi_{\sigma(0)}(V_{\sigma(0)}(x(0),0)) \leq \varepsilon$*
2. *If (**C1**) mode $\sigma(t_i)$ is stable and mode $\sigma(t_{i+1})$ is stable, then go to 3; Else, go to 5.*
3. *Choose t_{i+1} such that $(\alpha_1^{\sigma(t_{i+1})})^{-1}\phi_{\sigma(t_{i+1})}(V_{\sigma(t_{i+1})}(x(t_{i+1}),0)) \leq \varepsilon$.*
4. *Let $i = i+1$, go to 2.*
5. *If (**C2**) mode $\sigma(t_i)$ is stable and mode $\sigma(t_{i+1})$ is unstable, and there exist $h-1$ unstable modes ($h \leq m$) activated successively after mode $\sigma(t_{i+1})$, then go to 6; Else, go to 9.*
6. *Determine the bound $\Omega(|x(t_{i+1})|)$ satisfying $|x(t_{i+h+1})| \leq \Omega(|x(t_{i+1})|)$ using (2.143) in Lemma 2.11, choose t_{i+1} such that*

$$(\alpha_1^{\sigma(t_{i+h+1})})^{-1}\phi_{\sigma(t_{i+h+1})}(\alpha_2^{\sigma(t_{i+h+1})}(\Omega(|x(t_{i+1})|)),0)) \leq \varepsilon$$

   let $s = 0$.
7. *Choose t_{i+2+s} such that*

$$\sum_{k=0}^{i+1+s}\Bigg(\prod_{j=k}^{i+1+s}\frac{\phi_{\sigma(t_j)}^{t_{j+1}-t_j}}{V_{\sigma(t_j)}^{t_j}}\Bigg) \leq \frac{\beta}{(h+1-s)\chi^{h+1-s}} - 1$$

8. *Let $s = s+1$; If $s \neq h$, then go to 7; Else, let $i = i+h$, go to 2.*
9. *If (**C3**) the initial mode $\sigma(0)$ is unstable, and there exist $h-1$ unstable modes ($h \leq m$) activated successively after mode $\sigma(0)$, then go to 10.*
10. *Determine the bound $\Omega(|x(0)|)$ satisfying $|x(t_h)| \leq \Omega(|x(0)|)$ using (2.143) in Lemma 2.11, choose $x(0)$ such that*

$$(\alpha_1^{\sigma(t_h)})^{-1}\phi_{\sigma(t_h)}(\alpha_2^{\sigma(t_h)}(\Omega(|x(0)|)),0)) \leq \varepsilon$$

    let $s = 0$.
11. *Choose t_{1+s} such that $\sum_{k=0}^{s}\Big(\prod_{j=k}^{s}\frac{\phi_{\sigma(t_j)}^{t_{j+1}-t_j}}{V_{\sigma(t_j)}^{t_j}}\Big) \leq \frac{\beta}{(h+1-s)\chi^{h+1-s}} - 1$.*
12. *Let $s = s+1$; If $s \neq h$, then go to 11; Else, let $i = h$, go to 2.* ∎

The main idea behind $\mathscr{S}$ is that for current stable mode $\sigma(t_i)$, if next mode $\sigma(t_{i+1})$ is stable, we let mode $\sigma(t_i)$ be activated until t_{i+1} such that $x(t_{i+1})$ results in $|x(t)| \leq \varepsilon$ during mode $\sigma(t_{i+1})$'s working period $[t_{i+1}, t_{i+2})$ (step 3). When we predict that h unstable modes will be activated after stable mode $\sigma(t_i)$, we let mode $\sigma(t_i)$ be activated long enough until t_{i+1} such that $x(t_{i+1})$ results in $|x(t)| \leq \varepsilon$ for $t \in [t_{i+1}, t_{i+h+2})$, i.e. the total activating periods of all h unstable modes and stable mode $\sigma(t_{i+h+1})$ (step 6). This can be achieved because the upper bound $\Omega(|x(t_{i+1})|)$ can be obtained without the information of switching instants $t_{i+1}, \ldots, t_{i+h+1}$. The switching scheme among unstable modes is based on Lemma 2.11 (steps 7, 8, 11, 12). For initial stable/unstable modes, the initial states $x(0)$ are also chosen in different ways (steps 1 and 10).

Theorem 2.9. *Consider a switched system (2.125) satisfying (2.126)-(2.128) and Assumption 2.11. For any given $\varepsilon > 0$ and any switching sequence where at most m unstable modes are activated one by one, under the switching law $\mathscr{S}$, there exist an initial states $x(0)$ and a series of switching instants satisfy $0 < t_1 < t_2 < \ldots$, such that the origin is stable and $|x(t)| \leq \varepsilon\ \forall t \geq 0$.*

Proof: In the step 1 of $\mathscr{S}$, choosing $x(0)$ satisfying

$$(\alpha_1^{\sigma(0)})^{-1} \phi_{\sigma(0)}(V_{\sigma(0)}(x(0),0)) \leq \varepsilon$$

which leads to $|x(0)| \leq \varepsilon$ when mode $\sigma(0)$ is just activated. If mode $\sigma(0)$ is stable, we have from (2.126)-(2.127) that $|x(t)| \leq \varepsilon$ for $t \in [0, t_1)$. We will consider respectively three cases **C1-C3** in $\mathscr{S}$.

For **C1**, since mode $\sigma(t_i)$ is stable, it follows from (2.126)-(2.127) that there always exists a time instant $t_{i+1} > t_i$ satisfying

$$(\alpha_1^{\sigma(t_{i+1})})^{-1} \phi_{\sigma(t_{i+1})}(V_{\sigma(t_{i+1})}(x(t_{i+1}),0)) \leq \varepsilon$$

this implies that $|x(t_{i+1})| \leq \varepsilon$ when mode $\sigma(t_{i+1})$ is just activated. Since mode $\sigma(t_{i+1})$ is also stable, we have $|x(t)| \leq \varepsilon$ for $t \in [t_{i+1}, t_{i+2})$.

For **C2**, switching on mode $\sigma(t_{i+2})$ at $t = t_{i+2}$ results in

$$\frac{\phi_{\sigma(t_{i+2})}(V^{t_{i+2}}_{\sigma(t_{i+2})}, 0)}{V^{t_{i+2}}_{\sigma(t_{i+2})}} \Big(\sum_{k=0}^{i+1} \Big(\prod_{j=k}^{i+1} \frac{\phi^{t_{j+1}-t_j}_{\sigma(t_j)}}{V^{t_j}_{\sigma(t_j)}} \Big) + 1 \Big) \leq \frac{\beta}{(h+1)\chi^h}$$

Since $\beta > m(m+1)\chi^{m+1}$, $h \leq m$, we have $\frac{\beta}{(h+1)\chi^h} < \frac{\beta}{h\chi^h} - 1$. Thus we can choose $t_{i+3} > t_{i+2}$ such that

$$\frac{\phi^{t_{i+3}-t_{i+2}}_{\sigma(t_{i+2})}}{V^{t_{i+2}}_{\sigma(t_{i+3})}} \Big(\sum_{k=0}^{i+1} \Big(\prod_{j=k}^{i+1} \frac{\phi^{t_{j+1}-t_j}_{\sigma(t_j)}}{V^{t_j}_{\sigma(t_j)}} \Big) + 1 \Big) \leq \frac{\beta}{h\chi^h} - 1$$

By induction, for $s = 1,2,...,h-1$ we have $\frac{\beta}{(h+1-s)\chi^{h-s}} < \frac{\beta}{(h-s)\chi^{h-s}} - 1$. Choose t_{i+3+s} as $\mathscr{S}$, we obtain

$$\frac{\phi_{\sigma(t_{i+2+s})}^{t_{i+3+s}-t_{i+2+s}}}{V_{\sigma(t_{i+2+s})}^{t_{i+2+s}}}\Big(\sum_{k=0}^{i+1+s}\Big(\prod_{j=k}^{i+1+s}\frac{\phi_{\sigma(t_j)}^{t_{j+1}-t_j}}{V_{\sigma(t_j)}^{t_j}}\Big)+1\Big) \le \frac{\beta}{(h-s)\chi^{h-s}} - 1$$

Finally, we verify condition (2.129) with $t = t_{i+1+h}$ and $t_s = t_{i+1}$. There are finite numbers of switchings occurring in $(t_{i+1}, t_{i+1+h}]$, it follows from Lemma 2.11 that we can find a bound $\Omega(|x(t_{i+1})|)$ satisfying $|x(t_{i+h+1})| \le \Omega(|x(t_{i+1})|)$ using (2.143). Since this bound is independent from the switching instants, we can determine it before h unstable modes are switched into.

Note that mode $\sigma(t_i)$ is stable, we can find a time instant $t_{i+1} > t_i$ such that

$$(\alpha_1^{\sigma(t_{i+h+1})})^{-1}\phi_{\sigma(t_{i+h+1})}(\alpha_2^{\sigma(t_{i+h+1})}(\Omega(|x(t_{i+1})|)),0)) \le \varepsilon$$

This guarantees that $|x(t)| \le \varepsilon$ for $t \in [t_{i+1}, t_{i+h+1}]$. Mode $\sigma(t_{i+h+1})$ is also stable, we further have $|x(t)| \le \varepsilon$ for $t \in [t_{i+h+1}, t_{i+h+2})$.

For **C3**, note that $\beta > m(1+\chi)\chi^m$ and $\chi \ge 1$, which results in $\chi < \frac{\beta}{h\chi^h} - 1$. We can choose t_1 such that $\frac{\phi_{\sigma(0)}^{t_1}}{V_{\sigma(0)}^{0}} \le \frac{\beta}{h\chi^h} - 1$, the rest of the proof follows the same procedure as in **C2**, thus is omitted here. We finally obtain (2.129) with $t = t_h$ and $t_s = 0$.

Based on above analysis, one finds that for a switched system with any given switching sequence, finite or infinite numbers of switchings and both stable and unstable modes, the switching law $\mathscr{S}$ maintains the stability of the origin, and $|x(t)| \le \varepsilon$ for $t \ge 0$. This completes the proof. □

Remark 2.17. *Roughly speaking, $\mathscr{S}$ lets the activating periods of stable modes large enough and lets the activating periods of unstable modes small enough such that the state trajectory is bounded under a given switching sequence. Such idea is similar to that of dwell-time schemes in [136], [32] where an aggregated system is considered including stable modes and consequently activated unstable ones. This aggregated system would be stable if the total activating periods of stable modes are sufficient large. However, $\mathscr{S}$ provides an alternative way to approach stability in the absence of the "μ" condition.*

Example 2.4: Consider a numerical example with three modes. Let $\mathscr{M} = \{1,2,3\}$, $x = [x_1,\ x_2]^\top$, the modes take the following forms

$$f_1 = \begin{bmatrix} -x_1 + 4x_2^3 \\ -x_1 - x_2 \end{bmatrix},\quad f_2 = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_1^3 \end{bmatrix},\quad f_3 = \begin{bmatrix} x_1 - 3x_2 \\ x_1 + x_2 \end{bmatrix}$$

The prescribed switching sequence is

$$\text{mode 1} \ \rightarrow\ \text{mode 2} \ \rightarrow\ \text{mode 3} \ \rightarrow\ \text{mode 1} \ \rightarrow \cdots\cdots$$

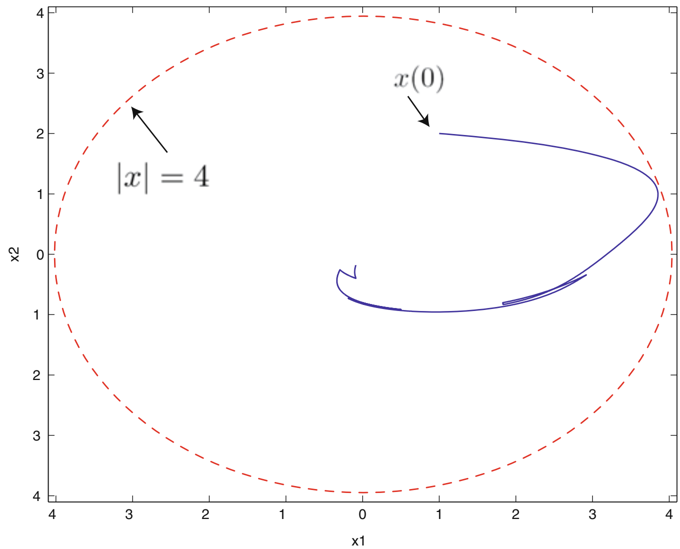


Fig. 2.12 State trajectory

For mode 1, it is not easy to find a quadratic Lyapunov function. However the origin is still stable, we choose a polynomial Lyapunov function $V_1 = x_1^2 + 2x_2^4$, this results in $V_1(x(t)) < e^{-2t}V_1(x(0))$ for $t \geq 0$. Both mode 2 and mode 3 are unstable, applying V_1 to modes 2 and 3 yields

$$\frac{dV_1(x)}{dx} f_2(x) \leq V_1^{0.5}(x) + 7V_1(x) + 4V_1^{1.5}(x) + 4V_1^3(x) \tag{2.145}$$

$$\frac{dV_1(x)}{dx} f_3(x) \leq V_1^{0.5}(x) + 11V_1(x) + 2V_1^{1.5}(x) \tag{2.146}$$

It can be seen that a common Lyapunov function is hard to impose here because inequalities (2.145)-(2.146) do not satisfy the general Lyapunov function formulation in dwell-time scheme [48]. The method in [88] is also not easy to be implemented since the right sides of (2.145) and (2.146) are polynomial forms of V_1 rather than $aV_1^m(x)$ for $a, m > 0$ in [88], and the exponents larger and smaller than 1 exist simultaneously.

We choose $V_2 = x_1^4 + 2x_2^2$, $V_3 = x_1^2 + x_2^2$. It follows that $V_2(x(t)) < e^{4t}V_2(x(0))$, $V_3(x(t)) < e^{2t}V_3(x(0))$, for $t \geq 0$. Note that MLFs techniques are difficult to be applied since the state trajectories in unstable modes are not bounded and Lyapunov-like functions are not easy to find. The "μ" condition is also hard to impose here, because V_1 and V_2 are non-quadratic.

Set $\varepsilon = 4$ which means that $|x(t)| \leq 4$ must hold for all $t \geq 0$. The prescribed switching sequence is

$$\text{mode } 1 \rightarrow \text{mode } 2 \rightarrow \text{mode } 3 \rightarrow \text{mode } 1 \rightarrow \cdots\cdots$$

Now we design the switching instants according to $\mathscr{S}$. Mode 1 is stable, choose $x(0) = [1,\ 2]^\top$ from step 1 of $\mathscr{S}$ such that $|x(t)| \leq 4$ for $t \in [0, t_1)$. Since both mode 2 and mode 3 are unstable, the switching scheme based on Lemma 1 is applied after t_1. It can be obtained from (6.37) that $\chi = 1$. $m = 2$ due to two unstable modes. Choose $\beta = 6.3 > 2(2+1)$. The activating periods of modes 2 and 3 can be calculated from step 7 of $\mathscr{S}$: $0.0059s$ for mode 2; $0.2602s$ for mode 3. Choose $t_1 = 0.9s$ from step 6 of $\mathscr{S}$ such that $|x(t)| \leq 4$ for $t \in [0, t_4)$. Consequently, choose $t_2 = 0.9059s$, $t_3 = 1.1661s$. The activating period of mode 1 is set to be $0.9s$ in the following switching process, i.e., $t_4 = 2.0661s$. Although our theory allows infinite switchings in infinite time interval, in the numerical simulation, a finite time interval $[0s,\ 4s]$ is considered. Other switching instants can be obtained straightly. Fig.2.12 shows the state trajectory, from which we can see that the stability is achieved and $|x| \leq 4$ always holds.

2.6 Conclusion

In this chapter, several FTC methods have been proposed for HS with time dependent switching. The known switching instants bring much convenience to FTC design. In sections 2.1-2.3, FTC objective has been achieved via designing the stabilizing controller in each faulty mode and a switching scheme. Sections 2.4-2.5 researched directly the stability of HS without reconfiguring the controller in each mode. It can be found that even some faulty modes are unstable, the stability of overall HS is still maintained under appropriate switching schemes.