A Structural Lemma in 2-Dimensional Packing, and Its Implications on Approximability*

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Abstract. We present a new lemma stating that, given an arbitrary packing of a set of rectangles into a larger rectangle, a "structured" packing of nearly the same set of rectangles exists. In this paper, we use it to show the existence of a polynomial-time approximation scheme for 2-dimensional geometric knapsack in the case where the range of the profit to area ratio of the rectangles is bounded by a constant. As a corollary, we get an approximation scheme for the problem of packing rectangles into a larger rectangle to occupy the maximum area. Moreover, we show that our approximation scheme can be used to find a $(1+\varepsilon)$-approximate solution to 2-dimensional fractional bin packing, the LP relaxation of the popular set covering formulation of 2-dimensional bin packing, which is the key to the practical solution of the problem.

1 Introduction

Due to their practical relevance, 2-dimensional (geometric) packing problems always received considerable attention in the combinatorial optimization literature. Given that the structure of their solutions can be extremely complicated, after some early approximability results proved in the early 1980s [1,2,12,13], the study of these problems was limited for a long time to the design of heuristic algorithms that could be useful in practice, without any proof of approximation guarantee. Moreover, in the last few years, some progress was made towards the solution of some instances to proven optimality by enumerative methods. Only in the last decade it was observed that the tools used

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in the late 1970s and early 1980s [15,21,29,32] to settle the approximability status of the main 1-dimensional packing problems could in fact be used with the same purpose also for their 2-dimensional counterparts. For two of the three basic 2-dimensional packing problems, namely 2-Dim Strip Packing and 2-Dim Bin Packing, the picture of approximability is now pretty clear, due to a series of recent relevant results listed in the state-of-the-art review below. For the third basic problem, namely 2-Dim Geometric Knapsack, the main question, concerning the existence of a *polynomial-time approximation scheme* (PTAS), remains open. In this problem, we are given a collection of two dimensional rectangular items with profits and a bin. The goal is to find the maximum profit subset of items that can be packed feasibly in the bin. In this paper we present the first, to the best of our knowledge, nontrivial PTAS for a variant (in fact, restriction) of 2-Dim Geometric Knapsack in which items are rectangles of arbitrary size and the bin cannot be enlarged. The restriction requires that the range of profit to area ratios of the items to be bounded from above by a constant. As a special case, this implies a PTAS for case when the profit of an item is equal to its area. This result has several applications in 2-Dim Bin Packing and 2-Dim Strip Packing. For example, it has already been used independently by Harren and van Stee [20] and by Jansen et al. [23] to derive an approximation algorithm for 2-Dim Bin Packing with absolute approximation ratio 2 (which is best possible unless $P = NP$). For the 2-Dim Strip Packing the PTAS has been used to achieve approximation algorithms with absolute ratios $1.939\ldots$ by Harren and van Stee [20] and $1.75 + \varepsilon$ by Jansen and Prädel [22], respectively.

The main result leading to our PTAS for 2-Dim Geometric Knapsack is a new lemma about the structure of the packings of the items in a bin. Very roughly, it says that given any complicated packing of items in a bin, there is a simpler packing with almost the same value of items. We also show that the PTAS above can be used to solve to near-optimality the column generation problem for 2-Dim Fractional Bin Packing, which is the LP relaxation of the natural (exponentially-large) Set Covering formulation of 2-Dim Bin Packing and plays a key role in the state-of-the-art practical solution break approaches to the problem (see e.g., [11]). By the well known connection between approximate separation and optimization [18,19,31], this leads to an *asymptotic polynomial-time approximation scheme* (APTAS) for 2-Dim Fractional Bin Packing itself.

Basic notions: In the 2-dimensional packing problems considered in this paper we are given a set I of *items*, the i-th corresponding to a *rectangle* having *width* (or *basis*) w_i, *height* h_i, and *profit* p_i, to be packed into *bins*, corresponding to *unit squares*. We will let $a_i := w_i \cdot h_i$ denote the *area* of item i. For a subset $S \subseteq I$, we will use the notations $b(S) := \sum_{i \in S} w_i$, $h(S) := \sum_{i \in S} h_i$, $p(S) := \sum_{i \in S} p_i$, $a(S) := \sum_{i \in S} a_i$.

A set C of items can be packed into a bin if the items can be placed into the bin without any two overlapping with each other. We only consider the orthogonal packing case, where the items must be placed so that their edges are parallel to the edges of the bin. We address both the classical version without rotations, in which the edges associated with the item heights have to be parallel to each other, and the version with rotations, in which this restriction is not imposed. In the latter case, we will assume w.l.o.g. $w_i \geq h_i$ for $i \in I$.

In 2-Dim Geometric Knapsack, only one bin is available and the objective is to pack a subset of the items having maximum profit into the bin. In 2-Dim Bin Packing, an unlimited number of bins is available and the objective is to pack all the items in I into the minimum number of bins. 2-Dim Fractional Bin Packing is the variant in which bins can be assigned a real value in $[0, 1]$, and the objective is to assign values to bins and pack the items into these bins so that, for each item, the sum of the values assigned to the bins containing the item is at least 1, and the sum of the values assigned to the bins is minimized (for those familiar, this is just a solution to the configuration LP for 2-Dim Bin Packing).

Given an instance I of a minimization problem, we let $\text{opt}(I)$ denote the value of the optimal solution of the problem for I. Given an algorithm for the problem, we say that it has *asymptotic approximation guarantee* ρ if there exists a constant δ such that the value of the solution found by the algorithm is at most $\rho\,\text{opt}(I) + \delta$ for each instance I. If $\delta = 0$, then the algorithm has *(absolute) approximation guarantee* ρ. An APTAS is a family of polynomial-time algorithms such that, for each $\varepsilon > 0$, there is a member of the family with asymptotic approximation guarantee $1 + \varepsilon$. If $\delta = 0$ for every ε, then this is a PTAS. In case, the running time of the algorithm is polynomial in $|I|$ and $1/\varepsilon$ we obtain an *asymptotic fully polynomial time approximation scheme*(AFPTAS). The definitions for a maximization problem are analogous, replacing "at most $\rho\,\text{opt}(I) + \delta$" by "at least $\rho\,\text{opt}(I) - \delta$" and "$1+\varepsilon$" by "$1-\varepsilon$". In the paper we will let $\text{opt}_{2\text{KP}}(I)$ denote the optimal solution value of 2-Dim Geometric Knapsack for the given instance I.

State-of-the-art: For 2-Dim Geometric Knapsack, a basic result of Steinberg [33] easily leads to an approximation guarantee arbitrarily close to 3 [10]. The best known approximation algorithm for the problem, due to Jansen and Zhang [27], has an approximation guarantee of $2 + \varepsilon$, for any $\varepsilon > 0$. On the other hand, no inapproximability result is known. PTASs are known only with resource augmentation, i.e. the algorithm can use a bin with both sides slightly enlarged [17], or even with only one side slightly enlarged [24] (but the optimum does not have this privilege). Without resource augmentation, a PTAS is also known in case all items are much smaller than the bin [16] or when all items are squares [25].

As to the other two relevant 2-dimensional packing problems, for 2-Dim Strip Packing the result in [33] yields a polynomial-time algorithm with (absolute) approximation guarantee 2, and Kenyon and Rémila [28] showed the existence of an AFPTAS. This was recently extended by Jansen and van Stee [26] to the case in which the items can be rotated. Furthermore there is an APTAS by Jansen and Solis-Oba with additive constant 1 [24].

For 2-Dim Bin Packing, Bansal et al. [3] showed that it does not admit an APTAS unless P=NP. For the case without rotations, Caprara [8] presented a polynomial-time algorithm with asymptotic approximation guarantee arbitrarily close to Π_∞, where $\Pi_\infty = 1.691\ldots$ is the so-called harmonic constant in the context of bin-packing [30]. For the case with rotations, an asymptotic approximation guarantee arbitrarily close to 2 follows from the result of [26]. APTASs are known for the 2-Stage and the Guillotine 2-Dim Bin Packing [6,9], in which the items must be packed in a certain structured way, as well as for the cases in which one or two sides of the bins can be slightly enlarged [7,3,14]. Building upon the results of [8,28], Bansal et al. derived in [4,5] a randomized approximation algorithm for 2-Dim Bin Packing, with and without rotations, with asymptotic approximation guarantee arbitrarily close to $1 + \ln \Pi_\infty = 1.525\ldots$. This latter algorithm runs in polynomial time if there exists an APTAS for 2-Dim Fractional Bin Packing, a question that was open before this paper.

Our results: The main result of this paper is a technical lemma on the structure of packings of items into a bin. Roughly speaking, the lemma concerns a packing into a bin of a set of items that can be partitioned into three subsets, namely "fat and tall", "fat and low", and "thin and tall", and for which the number of widths of the "fat and low" items as well as the number of heights of the "thin and tall" items is bounded by a constant. The (fairly complex) formal statement is:

Lemma 1 (Structural lemma). *Consider a set of items (rectangles) that fits into a bin (unit square) of the form* $L \cup O \cup V$*, where* $w_i \geq \varepsilon$ *for* $i \in L \cup O$*;* $h_i \geq \varepsilon$ *for* $i \in L \cup V$*; and the number of distinct widths of the items in* O *and heights of the items in* V *is at most* d*, where* ε *and* d *are given constants. Let* $\overline{w}_1, \overline{w}_2 \ldots$ *be the distinct widths of the items in* O*;* $h(\overline{w}_j)$ *be the total height of the items having width* $\overline{w}_j$*;* $\overline{h}_1, \overline{h}_2 \ldots$ *be the distinct heights of the items in* V*; and* $b(\overline{h}_j)$ *be the total width of the items having height* $\overline{h}_j$*. Then, there exists a constant* $f(d, \varepsilon)$ *such that, for any* $\delta > 0$*, the following set of rectangles fits into a unit square: the items in* L *plus, for* $j = 1, 2, \ldots$*, a set of rectangles of width* $\overline{w}_j$ *and height* δ *for a total height at least* $h(\overline{w}_j) - \delta f(d, \varepsilon)$*, and a set of rectangles of height* $\overline{h}_j$ *and width* δ *for a total width at least* $b(\overline{h}_j) - \delta f(d, \varepsilon)$*.*

By using this lemma, we are able to prove the following theorem, that shows that 2-Dim Geometric Knapsack has a PTAS if the range of the profit/area ratios, namely $\max_{i \in I}(p_i/a_i)/\min_{i \in I}(p_i/a_i)$, is bounded from above by a constant. Note that, by possibly scaling the profits, this is equivalent to saying that there exists a constant r such that $p_i/a_i \in [1, r]$ for $i \in I$.

Theorem 1. *For any fixed* $r \geq 1$*, there exists a PTAS for* 2*-Dim Geometric Knapsack with and without rotations restricted to instances* I *for which* $p_i/a_i \in [1, r]$ *for* $i \in I$*.*

As a corollary, we get a PTAS for the problem of maximizing the area occupied in a bin, whose existence was open so far.

Corollary 1. *There exists a PTAS for the special case of* 2*-Dim Geometric Knapsack with and without rotations in which* $p_i = w_i \cdot h_i$ *for* $i \in I$*.*

Although the straightforward column generation (or dual separation) problem for the customary LP formulation of 2-Dim Fractional Bin Packing is a general 2-Dim Geometric Knapsack, to which Theorem 1 does not apply, we show that the column generation problem for a closely-related variant can be solved near-optimally. By the well known connection between approximate separation and optimization [18,19,31], this implies:

Theorem 2. *There exists an APTAS for* 2*-Dim Fractional Bin Packing with and without rotations.*

As mentioned above, the results in [4,5] along with Theorem 2 imply:

Corollary 2 ([4,5]). *For any fixed* $\varepsilon > 0$*, there exists a polynomial-time approximation algorithm for* 2*-Dim Bin Packing without rotations with approximation guarantee arbitrarily close to* $1 + \ln \Pi_\infty = 1.525 \ldots$*.*

For the case without rotations, the bins can be assumed to be unit squares without loss of generality. For the case with rotations, our results hold also for the case in which the bins can be arbitrary rectangles, and we address the case of unit squares only for simplicity of presentation. For the sake of readability, in the coming sections we present the above results in reverse order, which corresponds to increasing technical difficulty. For a full discription of the proof of the structural lemma we refer to the full version.

Next-fit decreasing height: Throughout the paper, we will extensively use the *next-fit decreasing height* (NFDH) procedure introduced by [13]:

Observation 1 ([13]) *Consider a set of items I and its packing into bins by NFDH, letting m be the number of these bins and, for $j = 1, \dots, m$, S_j be the subset of items packed into the j-th bin. The following hold:*

(i) if $m > 1$ and $w_i \geq h_i$ for $i \in I$, then the area $a(S_1) \geq 1/4$;
(ii) if $m > 2$, then $\max\{a(S_1), a(S_2)\} \geq 1/4$;
(iii) if $m > 1$, then $a(S_1) \geq (1 - \max_{i \in S_1} w_i) \cdot (1 - \max_{i \in S_1} h_i)$.

2 An APTAS for 2-Dim Fractional Bin Packing

In this section we prove Theorem 2. It is well known that 2-Dim Bin Packing can be formulated as the Set Covering problem in which the set I of items has to be covered by *configurations* from the collection $\mathcal{C} \subseteq 2^I$, where each configuration $C \in \mathcal{C}$ corresponds to a set of items that can be packed into a bin. The associated 2-Dim Fractional Bin Packing is the continuous relaxation of this Set Covering problem:

$$\min\{\sum_{C \in \mathcal{C}} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), \; x_C \geq 0 \; (C \in \mathcal{C})\}. \tag{1}$$

The dual of this LP is given by:

$$\max\{\sum_{i \in I} \pi_i : \sum_{i \in C} \pi_i \leq 1 \; (C \in \mathcal{C}), \; \pi_i \geq 0 \; (i \in I)\}. \tag{2}$$

The well known connection between approximate separation and optimization for (1) reads:

Theorem 3 ([18,19,31]). *There exists a PTAS for (1) if, for any $\varepsilon > 0$, there exists a polynomial-time algorithm that, given $(\pi_i^*) \in \mathbb{R}_+^{|I|}$ such that $\max_{C \in \mathcal{C}} \sum_{i \in C} \pi_i^* \geq 1 + \varepsilon$, finds a configuration $C^* \in \mathcal{C}$ such that $\sum_{i \in C^*} \pi_i^* > 1$.*

Note that a PTAS for the 2-Dim Geometric Knapsack associated with the items in I in which the item profits correspond to the dual values π_i^* would suffice in Theorem 3. Since the existence of such a PTAS remains open, we now introduce a variant of (1) that, on the one hand, is almost equivalent to the original problem and, on the other, has a dual separation problem that fulfils the requirements of Theorem 1. The definition of this variant and its properties is the novelty of this section. The variant is simply obtained by imposing a bound of $4a_i$ on each dual variable π_i:

$$\max\{\sum_{i \in I} \pi_i : \sum_{i \in C} \pi_i \leq 1 \; (C \in \mathcal{C}), \; 0 \leq \pi_i \leq 4a_i \; (i \in I)\}, \tag{3}$$

which corresponds to the primal problem with the additional variables y_i:

$$\min\{\sum_{C\in\mathcal{C}} x_C + \sum_{i\in I} 4a_i y_i : \sum_{C\ni i} x_C + y_i \geq 1\ (i\in I),\ x_C, y_i \geq 0\ (C\in\mathcal{C}, i\in I)\}. \quad (4)$$

Lemma 2. *Given any solution of (4) of value z^*, one can obtain in polynomial time a solution of (1) of value at most z^*+1 for the case with rotations and z^*+2 for the case without rotations.*

Proof. Consider a solution (x_C^*, y_i^*) of (4). Let $I^* := \{i \in I : y_i^* > 0\}$ be the set of items associated with a positive y-component in the solution. We pack the items in I^* into bins by NFDH. If at least one of these bins contains a subset of items $S^* \subseteq I^*$ such that $a(S^*) \geq 1/4$, we do the following. We let $\alpha := \min_{i\in S^*} y_i^*$, and define the new solution of (4) in which y_i^* is decreased by α for $i \in S^*$ and x_{S^*} is increased by α. It is immediate to verify that the new solution is feasible, and not worse than the previous one since $\sum_{i\in S^*} 4a_i\alpha = 4\alpha a(S^*) \geq \alpha$.
We repeat the procedure above until no bin packed by NFDH with the items in I^* has area occupied at least $1/4$. In this case, by Observation 1(ii), for the case without rotations we have that NFDH packs the items in I^* into at most two bins, associated with, say, subsets S_1^* and S_2^*. At this point, we define the new solution of (4) in which y_i^* is set to 0 for $i \in S_1^* \cup S_2^*$ and $x_{S_1^*}, x_{S_2^*}$ are increased to value 1. This solution is feasible also for (1) (by neglecting the y variables) and has a value which is larger than the previous one by at most 2. On the other hand, for the case with rotations, by Observation 1(i) NFDH packs the items in I^* into one bin, and the reasoning is analogous. Note that the number of iterations of the above procedure is at most $|I|$ as, in each iteration, at least one y_i^* is decreased from a positive value to 0.

Lemma 3. *There exists a PTAS for (4) with and without item rotations.*

Proof. By the counterpart of Theorem 3 for (4), for any $\varepsilon > 0$ we need a polynomial-time algorithm that, given $(\pi_i^*) \in [0, 4a_i]^{|I|}$ such that $\max_{C\in\mathcal{C}} \sum_{i\in C} \pi_i^* \geq 1+\varepsilon$, finds a configuration $C^* \in \mathcal{C}$ such that $\sum_{i\in C^*} \pi_i^* > 1$. In other words, if the 2-Dim Geometric Knapsack associated with the items in I having profits π_i^* satisfies $\text{opt}_{\text{2KP}}(I) \geq 1+\varepsilon$, we want a solution of the problem of value > 1. Letting $\sigma := \varepsilon/3$, we first remove all the items $i \in I$ such that $\pi_i^* \leq \sigma a_i$, whose overall contribution to $\text{opt}_{\text{2KP}}(I)$ is at most σ. For the items left, the range of the profit/area ratios is $[\sigma, 4]$, i.e., it becomes $[1, 4/\sigma]$ after scaling. Then, we apply the PTAS of Theorem 1 with internal precision σ where now $r = 4/\sigma$. The solution found by this PTAS, after scaling profits back to their original values, has value at least $(1-\sigma)\ (\text{opt}_{\text{2KP}}(I) - \sigma) \geq (1-\sigma)(1+\varepsilon-\sigma) > 1$.

3 A PTAS for 2-Dim Geometric Knapsack

In this section we prove Theorem 1. Recall that we are assuming $p_i/a_i \in [1, r]$ for $i \in I$, where r is a constant. For simplicity, we will assume that r is integer. By Observation 1(ii), items for a total area at least $\min\{a(I), 1\}/4$ can be packed into the bin. Together with $p_i/a_i \geq 1$ for $i \in I$ this implies $\text{opt}_{\text{2KP}}(I) \geq \min\{a(I), 1\}/4$.

Let $\overline{\varepsilon} < 1/2$ denote the accuracy required. Letting $\delta < \overline{\varepsilon}^2$ be a suitable constant threshold specified below, we distinguish the case in which $a(I) \geq \delta$, for which we apply the algorithm described below, from the case in which $a(I) < \delta$. In this second case, if rotations are allowed all the items are packed into the bin by NFDH, by Observation 1(i). On the other hand, for the case without rotations handling instances in which the overall area $a(I)$ of the items is very small may be tricky. In fact, for the case in which $a(I) < \delta$, we adopt a completely different method illustrated in the full version.

Description of the main algorithm: We first illustrate the case without rotations, as it is more complex. Let $\varepsilon > 0$ denote an internal accuracy parameter, assuming for simplicity that $1/\varepsilon$ is integer. We will show how to find in polynomial time a solution of value at least $(1-\alpha\varepsilon)\ \text{opt}_{2\text{KP}}(I) - \varepsilon$, where $\alpha > 2$ is a suitable constant (independent of ε). Note that this yields a PTAS for the case in which $a(I) > \delta$, implying $\text{opt}_{2\text{KP}}(I) > \delta$ (recalling $\delta < \overline{\varepsilon}^2 < 1/4$), by setting for instance $\varepsilon := (\overline{\varepsilon}\delta)/\alpha$.

Size classification: Let $\varepsilon_0 := 1$ and, for $j = 0, \ldots, 2/\varepsilon$, ε_{j+1} be a suitable constant, depending on $\varepsilon, r, \varepsilon_j$, to be specified later, such that $\varepsilon_{j+1} < \varepsilon^2 \varepsilon_j$. Let $I_j \subseteq I$ denote the subset of items that have width or height in the interval $(\varepsilon_{j+1}, \varepsilon_j]$. We apply the method that follows for all values $m = 0, \ldots, 2/\varepsilon$, and take the best solution produced. We neglect the items in I_m (i.e., we find a solution in which none of these items is packed) and partition the rectangles in $I \setminus I_m$ as follows: Let L (large) denote the rectangles having both height and width $> \varepsilon_m$; O (horizontal) denote the rectangles having width $> \varepsilon_m$ and height $\leq \varepsilon_{m+1}$; V (vertical) denote the rectangles having height $> \varepsilon_m$ and width $\leq \varepsilon_{m+1}$; S (small) denote the rectangles having both height and width $\leq \varepsilon_{m+1}$.

Rounding the items in O and V: In order to apply Lemma 1, we modify the widths of the items in O (resp., the heights of the items in V) so that there are only a constant number of distinct widths (resp., heights). In this phase we allow the items in O to be sliced horizontally (resp., the items in V to be sliced vertically) so as to be able to form subsets whose total height (resp., width) is exactly a given value. At the end of the algorithm, we will pack the items in O and V with their original sizes and without slicing them.

We partition the items in O into groups O_{jk} for which the width and the profit/area ratio is approximately the same, as follows:

$$O_{jk} := \{i \in O : w_i \in ((1-\varepsilon)^j, (1-\varepsilon)^{j-1}],\ p_i/a_i \in (r(1-\varepsilon)^k, r(1-\varepsilon)^{k-1}]\}.$$

Note that we have to consider $j = 1, \ldots, \lceil (\log \varepsilon_m)/(\log(1-\varepsilon)) \rceil$, as $w_i \in (\varepsilon_m, 1]$, and $k = 1, \ldots, \lceil (\log 1/r)/(\log(1-\varepsilon)) \rceil + 1$, as $p_i/a_i \in [1, r]$. This implies that the total number of groups is at most

$$g_m := \lceil (\log \varepsilon_m)/(\log(1-\varepsilon)) \rceil \cdot (\lceil (\log 1/r)/(\log(1-\varepsilon)) \rceil + 1) \tag{5}$$

For simplicity, we redefine (decrease) the profits of the items in each group O_{jk} so that their profit/*height* ratio is equal to $r(1-\varepsilon)^{j+k}$, i.e., the profit of any (slice of) item in O_{jk} having height h is given by $r(1-\varepsilon)^{j+k} \cdot h$. Given that items in O_{jk} can be sliced, this implies that it is better to pack the items in O_{jk} with smallest width. Analogously, we redefine the profits of the items in each group V_{jk} so that their profit/*width* ratio is equal to $r(1-\varepsilon)^{j+k}$.

For each group O_{jk}, if $h(O_{jk}) > 1/(1-\varepsilon)^j$, we keep only the items with the smallest width for a total height equal to $1/(1-\varepsilon)^j$. Accordingly, in the reminder of this section we will assume $h(O_{jk}) \leq 1/(1-\varepsilon)^j$. Then, we consider the items in increasing order of widths, and define rg_m/ε subgroups $O_{jk1}, O_{jk2}, \ldots$ of consecutive items, so that the total height of the items in each subgroup $O_{jk\ell}$ is $h(O_{jk})\varepsilon/(rg_m)$. Note that the overall number of subgroups of items in O is at most rg_m^2/ε.

For each subgroup $O_{jk\ell}$, we define the *increased width* $\overline{w}_{jk\ell}$ of all items as the largest original width of an item in the subgroup. Finally, for each subgroup $O_{jk\ell}$ we further slice the items into $\lfloor h(O_{jk})\varepsilon/(rg_m\delta_m)\rfloor$ identical slices of width $\overline{w}_{jk\ell}$ and height δ_m, where $\delta_m \in [(\varepsilon_{m+1}/\varepsilon), \varepsilon_m]$ is a suitable constant, depending on $\varepsilon, r, \varepsilon_m$, to be specified later. The possible residual slice of height $< \delta_m$ is neglected.

The rounding procedure for the items in V is analogous, leading to at most rg_m^2/ε distinct *increased heights* and, for each increased height $\overline{h}_{jk\ell}$, to identical slices of height $\overline{h}_{jk\ell}$ and width δ_m. After having defined the slices as above, we consider these slices as single items that cannot be sliced further. Overall, this leaves us with a modified instance I' with the items in L and the items corresponding to slices from O and V. Note that $|I'| \leq |I|$ as $\delta_m > \varepsilon_{m+1}$.

Enumeration of the solutions for I': We enumerate all 2-Dim Geometric Knapsack solutions associated with I' as these are polynomially many. Specifically, since the area of each item in I' is at least $\delta_m\varepsilon_m$, only the $O(|I|^{1/(\delta_m\varepsilon_m)})$ subsets with at most $1/(\delta_m\varepsilon_m)$ items may be fit into the bin. Moreover, we can test in constant time if each of these subsets indeed fits into the bin, since we can assume that the bottom left corner of each item is placed into the bin at some (x, y) position which is an integer linear combination of the widths and heights of the items in the subset, and therefore we have $O(2^{2/(\delta_m\varepsilon_m)})$ possible positions for each item. For each solution for I', and the associated packing into the bin, we pack the small items in S and the original items in O and V by the greedy procedure of the next section. Among the solutions defined in this way, we keep the best one.

Converting the solution for I' into one for I: We use the empty spaces left in the bin by the items in I' to pack the items in S, and the space occupied by the slices of items in O and V to pack the original items in O and V. All (original) items in O, V and S are *unpacked* at the beginning of this phase. In order to pack the items in S, we draw horizontal and vertical lines through the coordinates of each corner of the items in I', and let the *cells* be the rectangles that are empty among those defined by these lines. We consider the cells one by one (in an arbitrary order) and, for each cell C, having area a_C, we consider the unpacked items in S in decreasing order of profit/area ratios and define a subset R by selecting the first items until condition $a(R) \geq a_C - 2\varepsilon_{m+1}$ is satisfied. We pack all the items in R by NFDH into the cell, given that they fit as we now show. Indeed, by Observation 1(iii), letting w_C and h_C be the width and height of the cell, after scaling all small item widths by $1/w_C$ and all item heights by $1/h_C$, we have that the area of any subset of items in S packed by NFDH in the cell, in case some items are unpacked, is at least $(1 - \varepsilon_{m+1}/w_C) \cdot (1 - \varepsilon_{m+1}/h_C) \cdot (w_C \cdot h_C) \geq w_C \cdot h_C - 2\varepsilon_{m+1} = a_C - 2\varepsilon_{m+1}$.

As to the items in O, for each group O_{jk}, we consider the slices of width $\overline{w}_{jk\ell}$ and height δ_m in increasing order of widths (i.e., by increasing ℓ). For each such slice, we consider the unpacked (original) items in O_{jk} in increasing order of widths, and define a subset R by selecting the first items until condition $h(R) \geq \delta_m - \varepsilon_{m+1}$ is satisfied. We pack all items in R in the slice (noting that they clearly fit). Note that the order in which we consider slices and items guarantees that we never run out of items. We do the same for the items in V.

The case with rotations and proof of approximation guarantee: For a full description of the case with rotations and the following lemma we refer to the full version.

Lemma 4. *By defining* $\varepsilon_0 := 1$ *and, for each* $m = 0, \ldots, 2/\varepsilon$, $\delta_m := \varepsilon^2/(2r^2 g_m^2 f(rg_m^2/\varepsilon, \varepsilon_m))$ *and* $\varepsilon_{m+1} := \varepsilon/(2r(2/(\delta_m \varepsilon_m) + 1)^2)$, *the value of the 2-Dim Geometric Knapsack solution produced by the algorithm above is at least* $(1 - 13\varepsilon)\,\mathrm{opt}_{2\mathrm{KP}}(I) - \varepsilon$, *where* g_m *is defined by (5) and* $f(\cdot, \cdot)$ *is the constant in the statement of Lemma 1.*

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