Integrality Properties of Certain Special Balanceable Families

Nicola Apollonio[1] and Massimiliano Caramia[2]

[1] Istituto per le Applicazioni del Calcolo, M. Picone, Via G. Amendola, 122/D 70126 Bari, Italy
nicola.apollonio@uniroma1.it
[2] Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133 Roma, Italy
caramia@disp.uniroma2.it

Abstract. Balanceable clutters are clutters whose bipartite representation contains no odd wheel and no odd 3-path configuration as induced subgraph (this is Truemper's characterization of balanceable matrices). In this paper we study a proper subclass of balanceable clutters called quasi-graphical defined by forbidding one-sided even wheels and one-sided even 3-path configurations. We characterize Mengerian quasi-graphical clutters and, as a consequence, we show that a recent conjecture in [5] is true for quasi-graphical clutters.

Keywords: Wheels, 3-path configurations, Mengerian Clutters.

1 Introduction

An *unbalanced hole submatrix* of a $\{-1, 0, 1\}$ matrix A is a square submatrix of A having exactly two nonzero entries per row and per columns whose sum of the entries is not divisible by four and minimal with this property. A $\{-1, 0, 1\}$ matrix A is *balanced* if it does not contain any unbalanced hole submatrix. A *binary* matrix is a matrix with $0, 1$ entries. A binary matrix is *balanceable* if it can be *signed* to become balanced, where *signing* a binary matrix A consists of multiplying some of its entries by -1. A finite family of subsets of a finite ground set is *balanceable* if so is its *incidence* matrix, i.e., the binary matrix whose columns are the incidence vectors of the members of the family (taken over the ground set). The *bipartite graph* of a finite family $\mathcal{C} = (L_j \mid j \in P)$ of subsets of V is the bipartite graph $B(\mathcal{C})$ with color classes V and P in which $v \in V$ and $j \in P$ are connected by an edge if $v \in L_j$. Truemper characterized balanceable families as those finite families whose bipartite graph contains neither *odd wheels* nor odd *3-path-configuration* as induced subgraphs (see e.g., [6]). Recall that a (bipartite) *uv-3-path configuration* ($3\text{PC}(u, v)$) is a bipartite graph consisting of three internally vertex-disjoint uv-paths P_1, P_2 and P_3 such that $V(P_i) \cup V(P_j)$, $i \neq j$, induces a chordless cycle and u and v are not adjacent. A 3-*path configuration* (3PC) is a $3\text{PC}(u, v)$ for some u and v. Since $3\text{PC}(u, v)$ is a bipartite graph, the length of each of the three uv-paths

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is odd or even accordingly to whether u and v belong to different color classes or to the same color class, respectively. In the former case each path has length at least three and the 3PC is said to be *odd*. In the latter case, if each path has length at least four, we say that the 3PC is *even*[1]. A (bipartite) *wheel* is a bipartite graph (C, v) consisting of a chordless cycle C and a vertex $v \notin V(C)$ that has at least three neighbors on C; C and v are referred to as the *rim* and the *center* of the wheel, respectively. Each edge of the wheel incident to the center is called a *spoke*. The wheel is *odd* if it has an odd number of spokes. It is *even* otherwise. A k-wheel is a bipartite wheel with k spokes. In this paper we study a proper subclass of balanceable matrices, namely, the class of quasi-graphical families defined as follows: a finite family $\mathcal{C} = (L_j \mid j \in P)$ of subsets of V is quasi-graphical if it is balanceable and its bipartite graph contain neither even $\mathcal{C}$-wheel nor even $\mathcal{C}$-3PC as induced subgraph where an even $\mathcal{C}$-wheel is an even wheel whose center is in the color class P and an even $\mathcal{C}$-3PC is an even 3PC whose vertices of degree three are both in the color class P. The choice of the term quasi-graphical is explained in Remark 2. With any finite family $\mathcal{C}$ of subsets of a common ground set V and a function $w \in \mathbb{Z}_+^V$ we can associate the following pair of dual linear programs:

$$\begin{array}{c} \text{minimize } wy \text{ subject to} \\ \sum (y(v) : v \in L) \geq 1 \qquad \forall L \in \mathcal{C} \\ y \in \mathbb{R}_+^V, \end{array} \tag{1}$$

$$\begin{array}{c} \text{maximize } \mathbf{1}x \text{ subject to} \\ \sum (x(L) : v \in L \in \mathcal{C}) \leq w(v) \qquad \forall v \in V \\ x \in \mathbb{R}_+^{\mathcal{C}}, \end{array} \tag{2}$$

The main aim of this paper is to characterize Mengerian quasi-graphical families, namely, those quasi-graphical families $\mathcal{C}$ for which problem (2) has an integral optimal solution for any $w \in \mathbb{Z}_+^V$, i.e, the defining system of (1) is Totally Dual Integral. Our result relies on the notion of pie introduced by Golumbic and Jamison in [8] in the context of Edge-Path-Tree graphs and closely follow a similar characterization for Edge-Path-Tree families given in [1] to which it specializes. Odd pies can be viewed as natural generalizations of odd circuits in graphs. A pie is a collection of subsets of a common ground set whose members can be cyclically ordered so that each member intersects exactly its two neighbors in the order and each element of the ground set occurs in at most two members of the collection. We base the characterizations of Mengerian quasi-graphical families on Lovász's 2-matching characterization of Mengerianity (see Theorem 1) and the additional observation (see Theorem 2) that if a quasi-graphical family $\mathcal{C}$ does not contain any odd pie as minor either it contains the Q_6 clutter as minor or the members of certain 2-matchings in $\mathcal{C}$ can be chosen "as uncrossing as possible". Recall that the Q_6 clutter is the clutter whose members are the

[1] We stress here that if H is a 3PC(u, v), with u and v belonging to the same color class, but u and v are linked by a path of length two, then H must not be considered an even wheel.

edge sets of the four triangles of the complete graph on four vertices. A Venn representation of the Q_6 is given in Figure 3 (a).

Terminology. Throughout the rest of the paper $\mathcal{C} = (L_j \mid j \in P)$ denotes a finite family of subsets of a finite ground set V. We also denote $\cup(L \mid L \in \mathcal{C})$ by $V(\mathcal{C})$ and we say that $\mathcal{C}$ is a family on V if $V = V(\mathcal{C})$. We use the term *collection* for families with no repeated members. A *clutter* is a collection whose elements are inclusionwise incomparable. For $X, Y \subseteq V$ and $X \cap Y = \emptyset$ the family of the (inclusionwise) minimal members in $\{L - Y \mid L \cap X = \emptyset, L \in \mathcal{C}\}$ is denoted by $\mathcal{C} \setminus X/Y$ and is referred to as a *minor* of $\mathcal{C}$. If $\mathcal{C}$ is a clutter so is $\mathcal{C} \setminus X/Y$. It is well known that $\mathcal{C} \setminus X/Y = \mathcal{C}/Y \setminus X$. When $X = \emptyset$ or $Y = \emptyset$ the notation will be abridged to $\mathcal{C} \setminus X$ (*deletion minor*) and $\mathcal{C}/X$ (*contraction minor*), respectively. In a graph, a chordless cycle on four or more vertices is called a *hole*. In a bipartite graph, a hole is *odd* if its length is not divisible by four. Throughout the rest of the paper we use the following concrete coloring for the bipartite graph $B(\mathcal{C})$ of $\mathcal{C} = (L_j \mid j \in P)$: the vertices in the color class P are represented by solid circles; those in the color class V are represented by empty circles.

In a graph every odd cycle contains an odd circuit, i.e., a subgraph where each vertex occurs in two edges. The natural generalization to families of the notion of circuit in a graph, is the notion of pie introduced in [8]. A *pie* is a collection $\mathcal{P} = (L_j \mid j \in N)$ on some finite ground set V such that $n := |N| \geq 3$ and

- for some permutation $(j_1, \ldots, j_n)$ of N one has $L_{j_i} \cap L_{j_{i+1}} \neq \emptyset$ and $L_{j_h} \cap L_{j_i} = \emptyset$ if $|i - h| \notin \{1, n-1\}$, (addition over indices is modulo n);
- if $n = 3$ then $\cap_{j \in N} L_j = \emptyset$.

Two members L_h and L_i of a pie are *consecutive* if $L_h \cap L_i \neq \emptyset$. The number n is the *size* of the pie; a pie of size n is a *n-pie*. The pie is *odd* if n is odd and *even* otherwise. We set $B_{j_i} = L_{j_i} \cap L_{j_{i+1}}$, $i = 1, \ldots, n$ (addition over the indices is taken modulo n) and we call B_{j_i}, the i-th *branch* of the pie. Observe that by the definition of pie one has $B_i \cap B_j = \emptyset$, for $i \neq j$, $i, j \in N$. If $\mathcal{P}$ is a pie in $\mathcal{C}$ we say that $\mathcal{C}$ *contains* a pie. Notice that a family might contain odd pies without containing odd pies as minor: in a Q_6 the collection formed by any three of its members is a 3-pie though no minor of the Q_6 is an odd pie.

Organization. The rest of the paper goes as follows. In the next section we give the characterization of Mengerian quasi-graphical families and discuss some consequences—mainly the fact that the Conjecture in [5] asserting that every minimal non-packing clutter has a transversal of size 2 holds true within quasi-graphical families–. The characterization uses Theorem 2 which is technical and hence proved in Section 2.1.

2 Mengerian Quasi-graphical Families

In this section we characterize Mengerian quasi-graphical clutters. The characterization closely follows the characterization of Mengerian Edge-Path-Tree families given in [1]. We need the following two results. Recall that a *w-matching* x

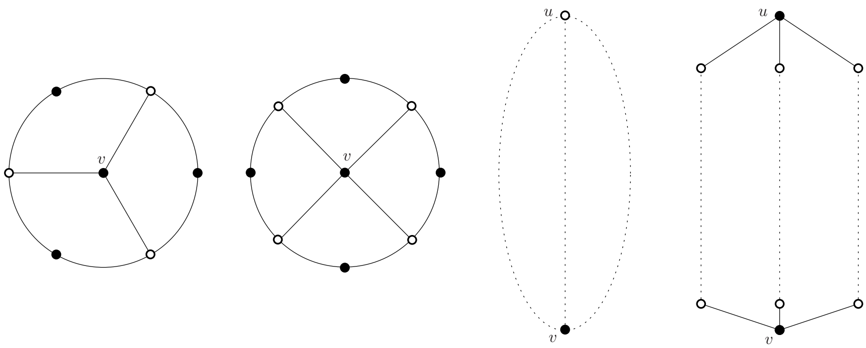


Fig. 1. An odd wheel, an even $\mathcal{C}$-wheel, an odd 3PC(u, v) and an even $\mathcal{C}$-3PC(u, v). Solid lines represent edges and dotted lines represent paths.

of $\mathcal{C}$ is any integral point in the polyhedron of (2). The number $\sum(x(L) \mid L \in \mathcal{C})$ is called the *size* of x and the maximum size of a w-matching of $\mathcal{C}$ is denoted by $\nu_w(\mathcal{C})$.

Theorem 1 (Lovász). *A family $\mathcal{C}$ of subsets of a given ground set V is Mengerian if and only if $\nu_{2w}(\mathcal{C}) = 2\nu_w(\mathcal{C})$ for each $w \in \mathbb{Z}_+^V$.*

Theorem 2. *Let $\mathcal{C}$ be a quasi-graphical family. Assume that $\mathcal{C}$ is a clutter without any odd pie as minor and let $\mathcal{P}$ be an odd pie in $\mathcal{C}$. Then either $\mathcal{C}$ contains the Q_6 clutter as minor or $F_0 \cup F_1 \subseteq L \Delta L'$ for some two members L and L' of $\mathcal{C}$ which are consecutive in $\mathcal{P}$ and some two disjoint members F_0 and F_1 of $\mathcal{C}$.*

Theorem 1 is Lovász's 2-matching characterization of Mengerian families. Theorem 2 is technical and its proof will be postponed after the characterization. Recall that a family $\mathcal{C}$ is *balanced* if $B(\mathcal{C})$ does not contain any induced odd hole. Balanced families are Mengerian (see, e.g., [9]).

Theorem 3. *Let $\mathcal{P}$ be a quasi-graphical family. Then $\mathcal{P}$ is Mengerian if and only if $\mathcal{P}$ contains neither odd pies nor the Q_6 clutter as minors.*

Proof. Neither odd pies nor the Q_6 clutter are Mengerian—in particular odd pies are not even ideal: if $\mathcal{P}' = \{L'_1, \ldots, L'_k\}$ is an odd pie minor of $\mathcal{P}$, then $\mathcal{P}'$ can be contracted to the edge set of an odd polygon—hence necessity follows. To prove sufficiency we need the following fact whose proof can be found in [3].

Claim. Let $\mathcal{E} = (E_j \mid j \in P)$ be a quasi-graphical family on V. If $\mathcal{E}$ does not contain any odd pie then $\mathcal{E}$ is balanced.

Without loss of generality $\mathcal{C}$ is a clutter on V. Suppose that $\mathcal{C}$ contains neither odd pies nor the Q_6 clutter as minors but it is not Mengerian. By Theorem 1 one has $\nu_{2w}(\mathcal{C}) > 2\nu_w(\mathcal{C})$ for some $w \in \mathbb{Z}_+^V$. Let w be chosen so as to minimize $\sum_{v \in V} w(v)$ and let $V^* := \{v \in E \mid w(v) \geq 1\}$ be its support. Therefore, for

$v \in V^*$, $\nu_{2(w-\chi_v)}(\mathcal{C}) = 2\nu_{w-\chi_v}(\mathcal{C})$, $\chi_v \in \mathbb{Z}_+^V$, being the incidence vector of edge v over V. Let $x \in \mathbb{Z}_+^{\mathcal{C}}$ be a $2w$-matching of size $\nu_{2w}(\mathcal{C})$ and let $\mathcal{M} = \{L \in \mathcal{C} \mid x(L) \geq 1\}$ be its support. The clutter $\mathcal{M}$ must contain some odd pie otherwise, by Claim 2, we would have $\nu_{2w}(\mathcal{C}) = \nu_{2w}(\mathcal{M}) = 2\nu_w(\mathcal{M}) \leq 2\nu_w(\mathcal{C})$. Let $\mathcal{P} = \{L_1, \ldots L_n\} \subseteq \mathcal{M} \subseteq \mathcal{C}$ be any odd pie in $\mathcal{M}$. Clearly $\mathcal{P}$ is a an odd pie in $\mathcal{C}$. Notice that $V(\mathcal{P}) \subseteq V^*$. Possibly after renumbering we may suppose that L_i and L_j are consecutive in $\mathcal{P}$ if and only if $|i-j| \in \{1, n-1\}$. Therefore, by Theorem 2, there are disjoint members F_0 and F_1 of $\mathcal{C}$ such that, for some $j = 1, \ldots, n$, one has $F_i \subseteq L_j \Delta L_{j+1}$, $i = 0, 1$. Define $\overline{x}$ as follows:

$$\overline{x}(L) = \begin{cases} x(L) - 1 \text{ if } L \in \{L_j, L_{j+1}\} \\ x(L) + 1 \text{ if } L \in \{F_0, F_1\} \\ x(L) \qquad \text{otherwise.} \end{cases}$$

By construction,

$$\sum_{L \ni v} \overline{x}(L) = \begin{cases} \sum_{L \ni v} x(L) - 1 \text{ if } v \in (L_j \Delta L_{j+1}) - (F_0 \cup F_1) \\ \sum_{L \ni v} x(L) - 2 \text{ if } v \in L_j \cap L_{j+1} \\ \sum_{L \ni v} x(L) \qquad \text{otherwise.} \end{cases}$$

Let $v_j \in P_j \cap L_{j+1}$. It follows that $\overline{x}$ is a $2(w - \chi_{v_j})$-matching of size

$$\sum_{L \in \mathcal{M} \cup \{F_0, F_1\}} \overline{x}(L) = \sum_{L \in \mathcal{M}} x(L),$$

contradicting the minimality of w. □

Corollary 1. *An ideal quasi-graphical family is Mengerian if and only if it does not contain the Q_6 clutter as minor.*

Remark 1. A clutter has the packing property if Problem (1) has an integral optimal dual solution for all $w \in \{0, 1, +\infty\}^V$ (see [6]). In [5] the authors conjecture that every minimally non packing clutter has a transversal of size 2. In the same paper the authors show that the conjecture implies the replication conjecture of Conforti and Cornuejols for packing clutters which is in turn equivalent to the conjecture that a clutter is Mengerian if and only if it is packing [6]. By corollary 1 all these conjectures hold true for quasi-graphical families.

Corollary 1 can be specialized to *Edge-Path-Tree* families. A family $\mathcal{E} = (E_i \mid i \in P)$ is an *Edge-Path-Tree* family if there exists a tree $T = (V, E)$ such that E_i is the edge set of some path in T. To see this we need to recall some well known preliminary notion. With every binary matrix A with m rows one can associate the binary matroid $M(A)$ generated by the columns of $[I_m, A]$, I_m being the identity matrix of order m. Such a matroid is defined as the matroid whose circuits are the minimal supports of the vectors in the nullspace of $[I_m, A]$, $[I_m, A]$ being a viewed as a matrix over $GF(2)$. Two binary matrices are $GF(2)$-*equivalent* if one arises from the other by a sequence of $GF(2)$-pivoting[2]. Any binary matrix A is $GF(2)$-equivalent to itself. $GF(2)$-equivalent matrices generate the same binary matroid and, conversely, if A and A' have the same order and $M(A) = M(A')$ then A and A' are $GF(2)$-equivalent. A *minor* in $M(A)$ is a matroid of the form $M(C)$ where C is a submatrix of some matrix A' which is $GF(2)$-equivalent to A. The operation of $GF(2)$-pivoting a matrix $A \in \{0,1\}^{I \times J}$ can be described in terms of the bipartite graph $B(A)$ as follows: if A' is the result of $GF(2)$-pivoting on a nonzero entry $a_{i,j}$ of A, then $B(A')$ results from $B(A)$ by complementing the edges between $N(i) - \{j\}$ and $N(j) - \{i\}$, where, for a vertex h of $B(A)$, $N(h)$ denotes the set of neighbors of h (see [4,6]). Let W_4 denote the wheel with four spokes and whose rim has eight vertices. The proof of the following lemma, which is similar to an analogous result in [4] about odd wheels and odd 3PC's, can be found in [3].

Lemma 1. *Let G be a bipartite graph such that G is either an even 3-path configuration $3PC(u,v)$ or an even wheel (C,v). Then G can be pivoted into a bipartite graph containing a W_4 whose center is in the same color class of u and v, if $G = 3PC(u,v)$ and in the same color class of v if $G = (C,v)$.*

Corollary 2 ([1]). *An Edge-Path-Tree family is Mengerian if and only if it does not contain any odd pie as minor. Consequently, every ideal Edge-Path-Tree family is Mengerian.*

Proof. Let $\mathcal{E} = (E_i \mid i \in P)$ be an edge-path-tree family on E and let $A(\mathcal{E})$ be its incidence matrix. Let $|E| = m$. Recall that a binary matrix is *regular* if it can be signed to become totally unimodular (see e.g. [6,9]). Since $\mathcal{E}$ is an Edge-Path-Family it follows that $A(\mathcal{E})$ is the unsigned pattern of a *network matrix* and any such matrix is a totally unimodular matrix [9]. Therefore $A(\mathcal{E})$ is regular and, consequently, $\mathcal{E}$ is balanceable, [6]. In [7], Fournier observed that $\mathcal{E}$ is an Edge-Path-Tree family if and only if the binary matroid $M(\mathcal{E})$ generated by the columns of $[I_m, A(\mathcal{E})]$ is a graphic matroid. By Tutte's deep characterizations of regular and graphic matroids (see e.g., [9]) $M(\mathcal{E})$ is graphic if and only if $A(\mathcal{E})$ is regular and $M(\mathcal{E})$ contains neither $M^*(K_{3,3})$ nor $M^*(K_5)$ as matroid-minors (the co-graphic matroid of the $K_{3,3}$ and K_5, respectively). Let $\mathcal{W}_4$ be a family on F with five members such that $B(\mathcal{W}_4) \cong W_4$. We show that $M(\mathcal{W}_4) = M^*(K_{3,3})$; let $\mathcal{W}_4^*$ be the dual of $\mathcal{W}_4$, i.e., the family $(\{L \in \mathcal{W}_4 \mid f \in L\} \mid f \in F)$; $\mathcal{W}_4^*$ is an Edge-Path-Tree family obtained as follows: let G be a copy of the $K_{3,3}$ and T let be spanning tree of G whose degree sequence is $(1,1,1,1,3,3)$. Let $F = E(G) - E(T)$ and for each f in F let $P(f)$ be the unique path of T connecting the endpoints of f. Thus $\mathcal{W}_4^* = (P(f) \mid f \in F)$ and $M(\mathcal{W}_4^*) = M(K_{3,3})$

[2] Recall that pivoting A over $GF(2)$ on a nonzero entry (the pivot element) means replacing

$$A = \begin{pmatrix} 1 & a \\ b & D \end{pmatrix} \quad \text{by} \quad \tilde{A} = \begin{pmatrix} 1 & a \\ b & D + ba \end{pmatrix}$$

where the rows and columns of A have been permutated so that the pivot element is $a_{1,1}$ ([6], p. 69, [9], p. 280).

hence $M(\mathcal{W}_4) = M^*(K_{3,3})$. Therefore $A(\mathcal{E})$ contains neither even $\mathcal{E}$-wheels nor even $\mathcal{E}$-3PC's because, by Lemma 1, these graphs could be pivoted into graphs containing a W_4 as induced subgraph. Accordingly, $M(\mathcal{E})$ would contain an $M^*(K_{3,3})$ minor contradicting that $M(\mathcal{E})$ is a graphic matroid. Therefore every Edge-Path-Tree family is a quasi-graphical family. Moreover, no Edge-Path-Tree family can contain the Q_6 clutter as minor. Indeed every minor of an Edge-Path-Tree family is an Edge-Path-Tree family but the binary matroid generated by $[I_6, A_{Q_6}]$, A_{Q_6} being the incidence matrix of the Q_6 clutter, is the co-graphic matroid of the K_5. □

Remark 2. In view of the proof of Corollary 2 the term quasi-graphical is due to the fact that quasi graphical families contain families which generate regular matroids with no $M^*(K_{3,3})$ minor, that is almost graphic matroids.

2.1 Proof of Theorem 2

Throughout the rest of the section we set $N = \{1, \ldots, n\}$ and $P = \{1, \ldots, p\}$; moreover, $\mathcal{P} = \{L_1, \ldots, L_n\}$ is an odd pie in $\mathcal{C} = \{L_1, \ldots, L_n, L_{n+1}, \ldots, L_p\}$, $\mathcal{C}$ being a clutter on $V(\mathcal{C}) = V$. Possibly after renumbering, L_i and L_j are consecutive in $\mathcal{P}$ if and only if $|i - j| \in \{1, n-1\}$. For $i \in N$ we denote by S_i the set of elements of $V(\mathcal{P})$ occurring in L_i and in no other member of $\mathcal{P}$. By the definition of branch it follows that

- $S_i \cap S_j = \emptyset$, $i \neq j$, $i, j \in N$ and $S_i \cap B_j = \emptyset$, $i, j \in N$;
- $\cup_{j\in N} L_j = (\cup_{j\in J} S_j) \cup (\cup_{j\in N} B_j)$.
- $S_j \cup S_{j+1} \subseteq L_j \Delta L_{j+1}$ (addition over indices is modulo n) and $\cup_{j\in N} S_j = \Delta_{j\in N} L_j$,

where B_j is the j-th branch of $\mathcal{P}$, $j \in N$. We also observe explicitly that if $v \in B_i$ for some $i \in N$ then $v \in P_j$ for some $j \neq i$ if and only if $|i - j| \in \{1, n-1\}$. We denote by $[\mathcal{P}]$ the clutter $\mathcal{P} \setminus (V - V(\mathcal{P}))$ and by N^* the set of indices of $[\mathcal{P}]$. Thus $N \subseteq N^* \subseteq P$, $[\mathcal{P}] = \{L_j \mid j \in N^*\}$ and $[\mathcal{P}]$ is the set of members of $\mathcal{C}$ contained in $V(\mathcal{P})$. Finally, for $s \in N^*$ let

$$\kappa_N(s) = |\{j \in N \mid P_t \cap B_j \neq \emptyset\}|.$$

Lemma 2. *If $\mathcal{C}$ is a quasi-graphical clutter then $\kappa_N(s)$ is either zero or two for each $s \in N^*$.*

Proof. For $j = 1, \ldots n$, if $L_s \cap B_j \neq \emptyset$ let v_j be an element in $L_s \cap B_j$, otherwise let v_j be an element arbitrarily chosen in B_j. Thus $\{v_1, \ldots v_n\} \cup N$ induces an odd hole C in $B(\mathcal{C})$. Now $\kappa_N(s) \leq 2$ otherwise $V(C) \cup \{s\}$ would induce a $\mathcal{C}$-wheel in $B(\mathcal{C})$ with at least three spokes contradicting the assumption that $\mathcal{C}$ is quasi-graphical. Thus $\kappa_N(s) \leq 2$. Suppose $\kappa_N(s) = 1$; hence for some $h \in N$, $v_h \in L_s \cap B_h \neq \emptyset$ and $v_j \notin L_s$, for $j \neq h$. As $[\mathcal{P}]$ is a clutter L_s intersects $\cup_{j\in N} S_j$. Suppose first that there is $v \in L_s \cap S_l$ for some $l \notin \{h, h+1\}$ (addition is taken modulo n). Thus $V(C) \cup \{v\} \cup \{s\}$ induces a 3PC(v_h, l), contradicting that $\mathcal{C}$ is quasi-graphical. Hence $L_s \cap S_l = \emptyset$ for every $l \notin \{h, h+1\}$. Necessarily there are $u \in L_s \cap S_h$ and $z \in L_s \cap S_{h+1}$ (for if not L_s would be included either in L_h or in L_{h+1}). Possibly after renumbering, we may suppose that $h = 1$. Let us consider the graph G induced by $V(C) \cup \{s\} \cup \{u, z\}$ (see Figure 2 (a)). the set $(V(C) - \{v_1\}) \cup \{s\} \cup \{u, z\}$ induces a hole C' in G and the neighbors of v_1 on C' are 1, 2 and s. Therefore $V(C') \cup \{v_1\}$ induces the odd wheel (C', v_1) in $B(\mathcal{C})$ contradicting that $\mathcal{C}$ is quasi-graphical. We conclude that $\kappa_N(s) \neq 1$ and hence $\kappa_N(s) \in \{0, 2\}$ as stated. □

Let $N_0^* = \{s \in N^* \mid \kappa_N(s) = 0 \,\text{and}\, L_s \cap L_j \neq \emptyset \,\forall j \in N\}$.

Lemma 3. *Let $\mathcal{C}$ be a quasi-graphical clutter. If $\kappa_N(t) = 0$ for some $t \in N^*$ then either $L_t \subseteq L_j \Delta L_{j+1}$ for some $j \in N$, or $|N| = 3$ and $N_0^* = \{t\}$.*

Proof. Since $\kappa_N(t) = 0$, L_t does not intersect any branch of $\mathcal{P}$. Hence $L_t \subseteq \cup_{j \in N} S_j = \Delta_{j \in N} L_j$. Suppose that L_t intersects two nonconsecutive members of the pie. Then $|N| = n \geq 5$. Let L_j and L_l be such members and let C be the hole induced in $B(\mathcal{C})$ by $\{v_1, \ldots, v_n\} \cup N$ where $v_i \in B_i$ $(i = 1, \ldots, n)$. Pick $u_j \in L_t \cap L_j$ and $u_l \in L_t \cap L_l$. Notice that $u_j, u_l \notin V(C)$ because $u_j \in S_j$ and $u_l \in S_l$. Moreover, j and l are at distance at least four on C. Thus $V(C) \cup \{t\} \cup \{u_j, u_l\}$ induces an even 3PC(j, l), that is, an even $\mathcal{C}$-3PC. This contradicts that $\mathcal{C}$ is quasi-graphical. Therefore for $n \geq 5$, if $\kappa_N(t) = 0$ then $L_t \subseteq S_j \cup S_{j+1} \subseteq L_j \Delta L_{j+1}$ for some $j \in N$. It follows that, if $\kappa_N(t) = 0$ but $L_t \not\subseteq L_j \Delta L_{j+1}$ for all $j \in N$, then necessarily $|N| = 3$ and $L_t \cap S_j \neq \emptyset$ for all $j \in N$. Hence $t \in N_0^*$. To prove the rest of the lemma we need the following fact whose proof can be found in [2]. A chain is a family of inclusionwise nested members.

Claim. Let $\mathcal{C} = (L_j \mid j \in P)$ be a quasi-graphical family (not necessarily a clutter) and let $\mathcal{P} = \{L_1, L_2, L_3\}$ be a 3-pie in $\mathcal{C}$. Moreover, let $N_0(\mathcal{C}, \mathcal{P}) \subseteq P$ be the set of indices of those members of $\mathcal{C}$ which intersect each member of $\mathcal{P}$ but no branch of $\mathcal{P}$. Then the family $(L_j \mid j \in N_0(\mathcal{C}, \mathcal{P}))$ is a chain.

Since in a clutter nonempty members of nontrivial chains are singletons the last part of the lemma is a straightforward consequence of the claim after noticing that $N_0(\mathcal{C}, \mathcal{P}) = N_0^*$. □

Let $\mathcal{P}$ be a 3-pie in $\mathcal{C}$. For $s \in N^*$ we say that L_s *wraps* L_1 if $L_s \cap B_1$ and $L_s \cap B_3$ are both nonempty. Similarly, L_s *wraps* L_2 if $L_s \cap B_1$ and $L_s \cap B_2$ are both nonempty and L_s *wraps* L_3 if $L_s \cap B_2$ and $L_s \cap B_3$ are both nonempty.

Lemma 4. *Let $\mathcal{C}$ be a quasi-graphical clutter and $\mathcal{P}$ be a 3-pie in $\mathcal{C}$ with $N_0^* = \{t\}$. If for $s \in N^* - \{t\}$ and $i \in N$, L_s wraps L_i and meets L_t then $L_s \cap L_t \cap L_{i+1}$ and $L_s \cap L_t \cap L_{i+2}$ are both empty, addition over indices being modulo 3.*

Proof. Since $s \neq t$ it follows that $\kappa_N(s) = 2$ (by Lemma 2). Possibly after renumbering, L_s wraps L_1. Hence L_s intersects B_1 and B_3. Let $v_1 \in L_s \cap B_1$, $v_3 \in L_s \cap B_3$. Moreover, let $u \in L_t \cap L_2$ if $L_s \cap L_t \cap L_3 = \emptyset$ and $u \in L_s \cap L_t \cap L_2$ otherwise (see Figure 3 (b)). Analogously, let $v \in L_t \cap L_3$ if $L_s \cap L_t \cap L_3 = \emptyset$ and

$v \in L_s \cap L_t \cap L_3$ otherwise. Observe that $|L_j \cap \{v_1, v_3, u, v\}| = 2$ for $j \in N \cup \{t\}$ and that $|L_s \cap \{v_1, v_3, u, v\}| = 2$ if and only if $L_s \cap L_t \cap L_2$ and $L_s \cap L_t \cap L_3$ are both empty. Therefore if at least one among $L_s \cap L_t \cap L_2$ and $L_s \cap L_t \cap L_3$ is nonempty then $|L_s \cap \{v_1, v_3, u, v\}| \geq 3$ and $N \cup \{s, t\} \cup \{v_1, v_3, u, v\}$ induces a $\mathcal{C}$-wheel with at least three spokes in $B(\mathcal{C})$ contradicting that $\mathcal{C}$ is quasi-graphical. □

Lemma 5. *Let $\mathcal{C}$ be a quasi-graphical clutter and let $\mathcal{P}$ be a pie in $\mathcal{C}$. For $j \in N$ and $v \in B_j$ let $\delta(v) = \{t \in N^* \mid v \in L_t\}$. Then $(\delta(v) \mid v \in B_j)$ is a chain for each $j \in N$.*

Proof. Suppose not. Hence, for some $i \in N$ there are $u(r), u(s) \in B_i$ and $r, s \in N^*$ such that $L_r \in \delta(u(r)) - \delta(u(s))$ and $L_s \in \delta(u(s)) - \delta(u(r))$. By Lemma 2, $\kappa_N(r)$, $\kappa_N(s) \in \{0, 2\}$. Hence there are (not necessarily distinct) indices $h, j \in N - \{i\}$ such that $L_r \cap B_h \neq \emptyset$ and $L_s \cap B_j \neq \emptyset$. For $l \notin \{h, i, j\}$, let $v_l \in B_l$. Since $\kappa_N(r) = \kappa_N(s) = 2$ and the branches of a pie are pairwise disjoint it follows that $v_l \notin L_r$ and $v_l \notin L_s$ for $l \in N - \{h, i, j\}$. Also observe that by the definition of branch $u(r)$, $u(s) \in L_{i+1}$. Possibly after renumbering, we may suppose that $h = 1$ and $i \neq n$. Hence $i + 1 \neq 1$ (modulo n). Let us distinguish three cases.

Case (a): $h = j = 1$ and there is some $z \in L_r \cap L_s \cap B_1$. The assumptions on the indices guarantee that $B_1 \neq B_i$ and $B_1 \neq B_{i+1}$. Thus $\{1, z, r, u(r), i+1, v_{i+1}, \ldots, n, v_n\}$ induces a hole C in $B(\mathcal{C})$. The unique neighbor of s on C is z; the unique neighbor of $u(s)$ on C is $i+1$; and since $u(s) \in L_s$, $u(s)$ and s induce an edge in $B(\mathcal{C})$. Thus $V(C) \cup \{u(s), s\}$ induces a 3PC$(z, i+1)$ in $B(\mathcal{C})$, contradicting that $\mathcal{C}$ is quasi-graphical.

Case (b): $h = j = 1$ and $L_r \cap L_s \cap B_1 = \emptyset$. Let $z(r) \in L_r \cap B_1$ and $z(s) \in L_s \cap B_1$. Possibly after renumbering, we may suppose that $\pi := |\{1, i+1, i+2, \ldots, n\}| \geq |\{2, 3, \ldots, i\}|$. Thus if $n = 3$ then $\pi = 2$ and if $n \geq 5$ then $\pi \geq 3$. Let C be the hole induced in $B(\mathcal{C})$ by $\{1, z(r), r, u(r), i+1, u(s), s, z(s)\}$ and let Q be the path induced in $B(\mathcal{C})$ by $\{1, v_n, n, v_{n-1}, \ldots, v_{i+1}, i+1\}$. The length of Q is $2(\pi - 1)$. Thus if $n \geq 5$ then $2(\pi - 1) \geq 4$ and $V(C) \cup V(Q)$ induces an even 3PC$(1, i+1)$ in $B(\mathcal{C})$. If $n = 3$ then $i = 2$. Hence $B_1 \cup B_2 \subseteq L_2$ and $\{u(r), u(s), z(r), z(s)\} \subseteq L_2$. It follows that $V(C) \cup \{2\}$ induces a $\mathcal{C}$-wheel with four spokes in $B(\mathcal{C})$. In either case the fact that $\mathcal{C}$ is quasi-graphical is contradicted.

Case (c): $h = 1 \neq j$. Let $z(r) \in L_r \cap B_1$ and $z(s) \in L_s \cap B_j$. As $i \neq j$ and $i \neq n$, possibly after renumbering, we may suppose that $i < j$. Let (see Figure 2(b))

$$D_1 = \{z(s), j+1, v_{j+1}, \ldots, v_n, 1, z(r)\},\ D_2 = \{2, v_2, \ldots, v_{i-1}, i\},$$

$$D_3 = \{i+1, v_{i+1}, \ldots, v_{j-1}, j\}.$$

We claim that $j = i+1$. For, if not, $D_1 \cup \{r, u(r)\} \cup D_3$ induces a hole C in $B(\mathcal{C})$ and $V(C) \cup \{s, u(s)\}$ is a 3PC$(i+1, z(s))$, contradicting that $\mathcal{C}$ is quasi-graphical. Hence $D_3 = \{i+1\}$. Next we claim that $i = 2$. For, if not, $z(r) \notin L_i$ because $z(r) \in B_1$ and L_1 and L_i are not consecutive. Hence $z(r)$ and i are not adjacent in $B(\mathcal{C})$. Thus $D_1 \cup \{r, u(r), i, u(s), s\}$ induces a hole C in $B(\mathcal{C})$ and $i+1$ has exactly three neighbors on C, namely, $u(r)$, $u(s)$ and $z(s)$ (Figure 2(c)). Therefore $(C, i+1)$ is an odd wheel in $\mathcal{C}$ contradicting that $\mathcal{C}$ is quasi-graphical. We conclude that

$i = 2$ and, consequently, that $D_2 = \{2\}$ and $D_3 = \{3\}$. Let now C be the hole induced by $\{1, z(r), 2, u(r), 3, z(s), \ldots, n, v_n\}$ (Figure 2(d)). Since L_r intersects B_1 and B_2 it follows that L_r must intersects some S_j with $j \neq 2$. For, if not, L_r would be included in L_2 and $[\mathcal{P}]$ would not be a clutter contradicting that $[\mathcal{P}]$ is a deletion minor of the clutter $\mathcal{C}$. Thus there is some $q \in L_r \cap S_j$ for some $j \in N - \{2\}$. Suppose first that $j \neq 3$ and let $D = \{z(s), \ldots j, q, r, u(r), 2, u(s), s\}$. Clearly D induces a hole C' in $B(\mathcal{C})$. Since 3 has exactly three neighbors on C', namely, $z(s)$, $u(s)$ and $u(r)$ it follows that $(C', 3)$ is an induced odd wheel in $B(\mathcal{C})$ (Figure 2(e)). Thus $j = 3$. Let now $D = D_1 \cup \{r, q, 3\}$. The graph C' induced by D is still a hole. Moreover, 2 and $u(s)$ have each exactly one neighbor on C, namely, $z(r)$ and 3, respectively. Since $u(s)$ and 2 are adjacent in $B(\mathcal{C})$ it follows that $V(C') \cup \{2, s\}$ induces a 3PC$(z(s), 3)$ in $B(\mathcal{C})$ (Figure 2(f)). We conclude that case (c) cannot occur and this completes the proof of the lemma. □

Proof of the Theorem. Let $\mathcal{P} = (L_j\ j \in N)$ be a pie in the quasi-graphical clutter $\mathcal{C} = (L_J \mid j \in P)$ where, $N = \{1 \ldots, n\} \subseteq P = \{1, \ldots, p\}$ and, possibly after renumbering, L_i and L_j are consecutive in $\mathcal{P}$ if and only if $|i - j| \in \{1, n - 1\}$. Observe first that $\kappa_N(t)$ must be zero for some $t \in N^*$ otherwise $\mathcal{C}$ would contain an odd pie as minor contradicting the hypotheses of the Theorem. To

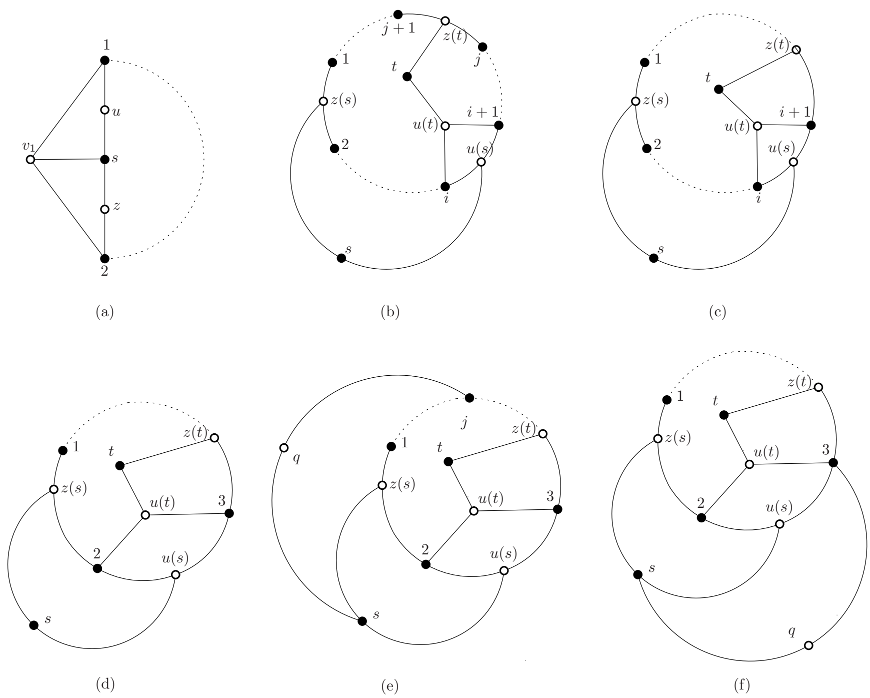
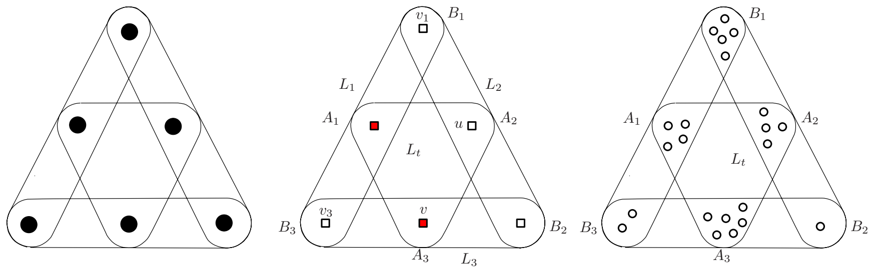


Fig. 2. (a): the odd wheel in Lemma 2; (b)÷(f): the various cases occurring in part (c) of Lemma 5. Solid lines represent edges and dotted lines represent paths.



(a) (b) (c)

Fig. 3. (a): a Venn-representation of the Q_6 clutter; (b): Lemma 4, (red) boxes are (possibly) elements of L_s; (c): the factorization on the r.h.s of (3)

see this let us argue as follows. By Lemma 2 if $\kappa_N(s) > 0$ for each $s \in N^*$ then $\kappa_N(s) = 2$ for each $s \in N^*$. For $j = 1, \ldots, n$ let v_j be such that $\delta(v) \subseteq \delta(v_j)$ for each $v \in B_j$. Thus $|L_s \cap \{v_1, \ldots, v_k\}| = 2$ for each $s \in N^*$. It follows that $[\mathcal{P}]/(V(\mathcal{P}) - \{v_1, \ldots, v_n\})$ is an n-pie. A contradiction. Hence $\kappa_N(t) = 0$, for some $t \in N^*$. By Lemma 3 either

–case (a) $L_t \subseteq L_j \Delta L_{j+1}$ for some $t \in N^*$ and some $j \in N$; or
–case (b) $n = 3$, $N_0^* = \{t\}$ and $\kappa_N(s) = 2$ for each $s \in N^* - \{t\}$.

In case (a), since $L_t \subseteq L_j \Delta L_{j+1} \subseteq L_j \cup L_{j+1}$ and $[\mathcal{P}]$ is a clutter, it follows that L_t meets both L_j and L_{j+1}. Thus $\mathcal{Q} = \{L_j, L_{j+1}, L_t\}$ is a 3-pie with index set $\{j, j+1, t\}$. Possibly after relabelling the members of $\mathcal{C}$ we may suppose that $j = 1$, and $t = 3$. Hence $\mathcal{Q}$ is a 3-pie in $\mathcal{C}$ whose index set is $M = \{1, 2, 3\}$ and where M^* is the index set of $[\mathcal{Q}]$. By Lemma 2 one has $\kappa_M(s) = 0$ for some $s \in M^*$ otherwise, as above, $[\mathcal{Q}]$ (and hence $\mathcal{C}$) would contain an odd pie minor. Therefore $L_s \subseteq L_1 \Delta L_2 \Delta L_3$ for some $s \in M^*$. Thus $L_s \subseteq (L_1 \Delta L_2) - L_3$ (because $L_t \subseteq L_j \Delta L_{j+1}$). Hence $L_s \cap L_t = \emptyset$ and the theorem is proved in case (a) with $F_0 = L_s$ and $F_1 = L_t$.

Suppose we are in case (b). We show that $[\mathcal{P}]$ contains the Q_6 clutter as minor. Let $A_1 = L_t \cap L_1$, $A_2 = L_t \cap L_2$ and $A_3 = L_t \cap L_3$ (see Figure 3 (c)). We claim that

$$L_s \subseteq A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \text{ for every } s \in N^* - \{t\}. \tag{3}$$

To see (3) suppose that it does not hold. Thus there exists some v in $L_i \cap L_s - (L_{i+1} \cup L_{i+2} \cup L_t)$ for some $i \in N$ and some $s \in N^* - \{t\}$. Observe that L_{i+1}, L_{i+2} and L_t are members of $[\mathcal{P}] \setminus v$. Since $N_0^* = \{t\}$, it follows that $\mathcal{Q} = \{L_{i+1}, L_{i+2}, L_t\}$ is a 3-pie in $[\mathcal{P}] \setminus v$ and hence in $[\mathcal{P}]$ and $\mathcal{C}$. Let $M = \{i+1, i+2, t\}$, be the index set of $\mathcal{Q}$ where $i+1$ and $i+2$ are modulo 3 and let M^* be the index set of $[\mathcal{Q}]$. Remark that $[\mathcal{Q}]$ is a minor of $[\mathcal{P}]$. For no $r \in P$ member L_r can be contained in $L_{i+1} \Delta L_{i+2} \Delta L_t$ otherwise any such member either would be contained in one among $S_1 \cup S_2$, $S_1 \cup S_3$ and $S_2 \cup S_3$ contradicting that we are in case (b) or $r \in N_0^*$ and $r \neq t$ contradicting Lemma 3.

Hence $\kappa_M(s) = 2$ for each $s \in M^*$ and, as above, $[\mathcal{Q}]$ contains an odd pie as minor contradicting the hypotheses of the theorem. Thus we conclude that (3) holds. Next we claim that

$$N^* = N \cup \{t\}. \tag{4}$$

To prove (4) let us argue as follows. Since $\kappa_N(s) = 2$ for all $s \in N^* - \{t\}$ it follows that for each $s \in N^* - \{t\}$ there is $i \in N$ such that L_s wraps i and meets L_t (by (3)). By Lemma 4, $L_s \cap A_{i+1}$ and $L_s \cap A_{i+2}$ are both empty ($i+1$ and $i+2$ are modulo 3). Therefore (still by (3)) $L_s \subseteq A_i \cup B_i \cup B_{i+2} \subseteq L_i$. Thus $s = i$ (because $[\mathcal{P}]$ is a clutter) and $N^* = \{1, 2, 3, t\}$. Hence (4) holds and $[\mathcal{P}] = \{L_1, L_2, L_3, L_t\}$.

Let now a_i and b_i be arbitrarily chosen elements in A_i and B_i, respectively ($i = 1, 2, 3$). Thus, for $j \in N \cup \{t\}$, $L_j \cap \{a_1, a_2, a_3, b_1, b_2, b_3\}$ is one of the four members of the Q_6 clutter. Hence $[\mathcal{P}]/(V(\mathcal{P}) - \{a_1, a_2, a_3, b_1, b_2, b_3\})$ is the Q_6 clutter and the proof of part (b) is completed. □

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