Kernel in Oriented Circulant Graphs

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Abstract. A kernel in a directed graph $D(V,E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S , such that (u,v) is an arc of D. The problem of existence of a kernel is NP-complete for a general digraph. In this paper we introduce the strong kernel problem of an undirected graph G and solve it in polynomial time for circulant graphs.

Keywords: oriented graph, kernel, strong kernel number, NP-complete, strongly connected.

1 Introduction

The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, game theory, combinatorics and coding theory [3], [4]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating-set-based routing to networks with unidirectional links is proposed in [1], [9]. A few years ago a new interest for these studies arose due to their applications in finite model theory. Indeed variants of kernel are the best properties to provide counter examples of $0-1$ laws in fragments of monadic second order logic [8].

A *kernel* [6] in a directed graph $D(V,E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u,v) is an arc of D. The concept of kernels in digraphs was introduced in different ways [10], [14]. Von Neumann and Morgenstern [14] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. All odd length directed cycles and most tournaments have no kernels [3], [4]. If D is finite, the decision problem of the existence of a kernel is NP-complete for a general digraph [5], [13], and for a planar digraph with indegrees ≤ 2, outdegrees ≤ 2 and degrees ≤ 3 [7]. It is further known that a finite digraph all of whose cycles have even length has a kernel [11], and that the question of the number of kernels is NP-complete even for this restricted class of digraphs [12].

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In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph G and its applications are extensively studied. Our aim in this paper is to investigate all strong orientations of G and to determine the strong kernel number of G. This number is different from the independent domination number γ_i for undirected graphs where γ_i is the cardinality of a minimum independent dominating set [2].

2 Kernel in Oriented Graphs

An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. There are $2^{|E|}$ orientations for G. Let $O_x(G)$ denote the set of all orientations of G. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G.

An orientation O of an undirected graph G is said to be an *acyclic orientation* if it contains no directed cycles. Let $O_a(G)$ denote the set of all acyclic orientations of G.

An orientation O of an undirected graph G is said to be *strong* if for any two vertices x, y of $G(O)$, there are both (x, y)-path and (y, x)-path in $G(O)$. Let $O_s(G)$ denote the set of all strong orientations of G.

An orientation O of an undirected graph G is said to be *weak* if for any two vertices x, y of $G(O)$, there is either a (x, y)-path or a (y, x)-path in $G(O)$. Let $O_w(G)$ denote the set of all weak orientations of G.

So far the only known problem on this topic is the kernel problem which investigates the existence of some kernel in a digraph. We introduce kernel number (κ_x), acyclic kernel number (κ_a), strong kernel number (κ_s), and weak kernel number (κ_w) for various orientations of a given graph.

Notation 1. *Let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G. The kernel number of $G(O)$ to be defined below is denoted by $\kappa(G(O))$. For convenience we write $\kappa(G(O))$ as $\kappa(O)$.*

Definition 1. *The kernel number κ_x of G is defined as*

$$\kappa_x(G) = \min \{\kappa(O) : O \in O_x(G)\}$$

Definition 2. *The acyclic kernel number κ_a of G is defined as*

$$\kappa_a(G) = \min \{\kappa(O) : O \in O_a(G)\}$$

Definition 3. *The strong kernel number κ_s of G is defined as*

$$\kappa_s(G) = \min \{\kappa(O) : O \in O_s(G)\}$$

Definition 4. *The weak kernel number κ_w of G is defined as*

$$\kappa_w(G) = \min\{\kappa(O) : O \in O_w(G)\}$$

The *acyclic (strong or weak) kernel problem* of an undirected graph G is to find a kernel K of $G(O)$ for some acyclic (strong or weak) orientation O of G such that $|K| = \kappa_x(\kappa_s \text{ or } \kappa_w)$.

In this section we exhibit the relationship among the parameters $\gamma_i, \kappa_x, \kappa_s$ and κ_w.

Theorem 1. *Let G be a graph. Then $\gamma_i = \kappa_x$, where γ_i is the independent domination number.*

Proof. It is always true that $\gamma_i \leq k_x$, since a kernel of a directed graph is an independent dominating set of the corresponding underlying undirected graph. We identify a minimum independent dominating set and orient all the edges incident at each of these vertices as incoming edges. This implies $k_x \leq \gamma_i$. Hence $\gamma_i = \kappa_x$. □

Illustration 1. For the graph G in Figure 1(a), $\Gamma = \{3, 4\}$ is an independent dominating set. Thus $\gamma_i = 2$. Figure 1(b) shows an arbitrary orientation of G with the edges incident at 3, 4 as incoming edges. Hence $\kappa_x = 2$.

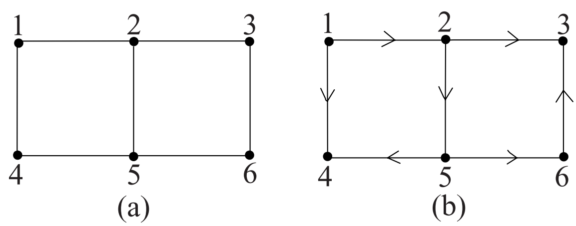

Fig. 1. (a): Independent domination number = 2; (b): Kernel number = 2

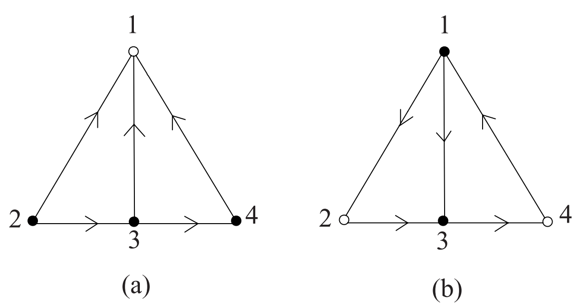

Fig. 2. (a): Weak kernel number = 1; (b): Strong kernel number = 2

Illustration 2. In Figure 2(a), $K = \{1\}$ is a weak kernel. Thus $\kappa_w = 1$. On the other hand, in Figure 2(b) $K = \{2, 4\}$ is a strong kernel. Thus $\kappa_s = 2$.

Lemma 1. *For any graph G, O_a and O_s are disjoint.*

Proof. Let O be some orientation in O_s and $G(O)$, the corresponding oriented graph. Then there exist (u, v)-path and (v, u)-path, for all $u, v \in V(G)$. This shows that there exists at least one cycle in graph $G(O)$. Hence $O \notin O_a$. Thus O_a and O_s are disjoint. □

The following result is trivial as every strong orientation of a graph G is also a weak orientation.

Lemma 2. *For any graph G, O_s is a subset of O_w.*

Lemma 3. *Given any hamiltonian graph $G, O_a \cap O_w \neq \phi$.*

Proof. It is enough to find an orientation which is both acyclic and weakly oriented. Let $C = v_1v_2...v_nv_1$ be a hamiltonian cycle in G. Orient the path $v_1v_2...v_{n-1}v_n$ in the clockwise direction and the edge v_nv_1 in the anticlockwise direction. Orient every other edge v_iv_j of G from v_i to v_j whenever $i < j$. The resulting orientation O on G is clearly weak, as there is always either a (v_i, v_j)-path or (v_j, v_i)-path along the orientation of C. Next we claim that $G(O)$ is acyclic. Suppose there is a cycle $v_{i_1}v_{i_2}...v_{i_k}v_{i_1}$. By definition $i_1 < i_2 < ... < i_k$ and hence $i_k < i_1$ is not possible. Thus $v_{i_1}v_{i_2}...v_{i_k}v_{i_1}$ is not a directed cycle. This proves that O is an acyclic orientation. □

Remark 1. *If $O_a \cap O_w \neq \phi$, then it is not necessary that G is hamiltonian. See Figure 3.*

Remark 2. *For an arbitrary graph $G, O_a \cap O_w$ may be empty. The graph G in Figure 4(a), has an acyclic orientation O and $G(O)$ is not weakly connected. Thus $O_a \cap O_w = \phi$, whereas for the graph in Figure 4(b), $O_a \cap O_w \neq \phi$.*

Lemma 4. *For any noncomplete graph G on 3 or more vertices, O_w is a proper subset of O_x.*

Proof. Since G is not complete, there exist two independent vertices u and v. Consider an orientation O such that all the edges incident at u and v are incoming edges. Then neither there is a (u, v)-path nor there is a (v, u)-path. This

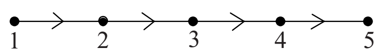

Fig. 3. $O \in O_a \cap O_w$ but $G(O)$ is not hamiltonian

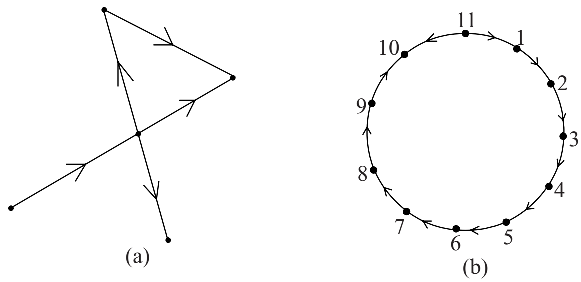

Fig. 4. (a): an acyclic orientation G; (b): $O_a \cap O_w \neq \phi$

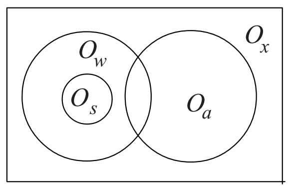

Fig. 5. Relation between different orientations for hamiltonian Graphs

implies that $O \notin O_w$. Since O_w is a subset of O_x and there exist an orientation $O \in O_x \setminus O_w, O_w$ is a proper subset of O_x. □

Figure 5 depicts the set theoretical relationship among O_x, O_a, O_s, O_w, for hamiltonian graphs.

Next we exhibit the relation among the parameters such as kernel number, acyclic kernel number, strong kernel number and weak kernel number.

The proof follows from Lemma 1 and Lemma 2.

Theorem 2. *Let G be an undirected graph. Then* (i) $\kappa_x \leq \kappa_w \leq \kappa_s$ (ii) $\kappa_x \leq \kappa_a$.

The salient feature of this paper is the following theorem where we obtain a lower bound for the strong kernel number of regular graphs.

Theorem 3. *Let G be an r-regular graph on n vertices. Then* $\kappa_s \geq \lceil n/r \rceil$.

Proof. Let $O \in O_s$ and K be some kernel of $G(O)$. For any vertex $v \in K$ there are at most $r - 1$ incoming edges. Thus $|K| \geq \lceil n/r \rceil$. □

3 Kernel in Oriented Circulant Graphs

In this section we obtain the strong kernel number for oriented circulant graphs, proving that the strong kernel problem is polynomially solvable.

We begin with the definition of circulant digraph [15].

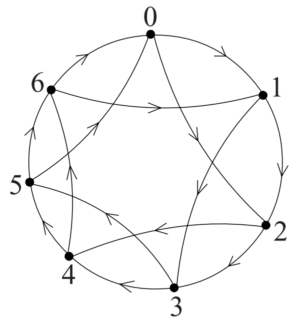

Fig. 6. Circulant digraph(DC_7(1,2))

Definition 5. *A circulant digraph $D(C_n(S))$, where $S \subseteq \{1, 2, ..., n-1\}$, $n \geq 2$, is defined as a digraph with vertex set $V = \{0, 1, 2, ..., n-1\}$ and the edge set $E = \{(i, j)\}$: there is $s \in S$ such that $j - i \equiv s \pmod n)\}$. See Figure 6.*

If $S = \{1, 2, ..., j\}$ we write $D(C_n(1, 2, ..., j))$ instead of $D(C_n(S))$. We observe that kernel does not exist for $D(C_7(1, 2))$ in Figure 6. Thus it is interesting to note that the directed circulant graphs do not possess kernels in certain cases. This motivates us to consider oriented circulant graphs.

Definition 6. [15] *A circulant undirected graph $C_n(S)$, where $S \subseteq \{1, 2, ..., \lfloor n/2 \rfloor\}$ $n \geq 3$, is defined as an undirected graph with vertex set $V = \{0, 1, 2, ..., n-1\}$ and the edge set $E = \{(i, j)$: there is $s \in S$ such that $\mid j - i \mid \equiv s \pmod n)\}$.*

Clearly $C_n(S)$ is $2\,|S|$-regular.

Lemma 5. *Let G be $C_n(1, 2, ..., s)$, $1 \leq s \leq \lfloor n/2 \rfloor$. Then G has at least 2^{m-n} strong orientations where m denotes the number of edges in G.*

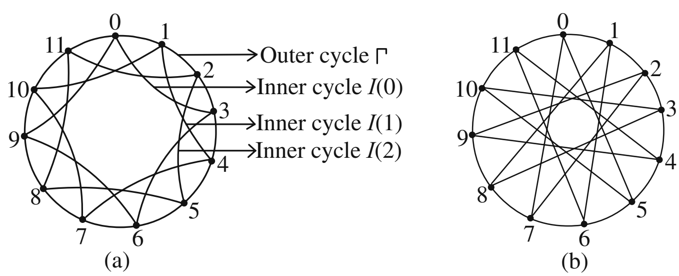

Fig. 7. (a) Cycles marked in $C_{12}(1, 3)$; (b) $C_{12}(1, 5)$ has one hamiltonian cycle Γ and exactly one inner hamiltonian cycle $I(0)$

Proof. The circulant graph G contains the cycle $C_n(1)$ on n vertices as a subgraph. Let it be oriented in clockwise direction. Clearly, this orientation induces a strong orientation of G. Now the remaining $m-n$ edges can be oriented in 2^{m-n} ways. Hence G has at least 2^{m-n} strong orientations. □

We proceed to prove that the strong kernel problem is polynomially solvable for $C_n\left(1,2,\ldots,s\right), 1 \leq s \leq \lfloor n/2 \rfloor$.

Lemma 6. *Let G be $C_n(1,s), 2 \leq s \leq \lfloor n/2 \rfloor$. The edge set E of $C_n(1,s)$ is partitioned into the following cycles:*

1. *The outer hamiltonian cycle $\Gamma = (0,1,2,\ldots,0)$.*
2. *l number of edge-disjoint inner cycles $I(k) = (k, k+s, k+2s, \ldots, k+(\frac{n}{l}-1)s, k), 0 \leq k \leq l-1$, each of length $\frac{n}{l}$ where $l = g.c.d(n,s)$ and vertex labels taken modulo n.*

Proof. **Case 1: (s divides n)**
Since s divides n, we have $l = s$. In addition to the outer hamiltonian cycle Γ, $C_n(1,s)$ has the following inner cycles: $I(0) = (0, s, \ldots, 0); I(1) = (1, s+1, \ldots, 1); I(2) = (2, s+2, \ldots, 2); \ldots I(s-1) = (s-1, 2s-1, \ldots, s-1)$. See Figure 7(a). For $i \neq j, V(I(i)) \cap V(I(j)) = \phi, 1 \leq i,j \leq l$ and $\cup V(I(k)) = \{0,1,2,3,\ldots,n-1\}$. Hence the cycles $I(0), I(1), I(2), \ldots, I(s-1)$ are edge-disjoint and these cycles together with Γ partition the set E.

Case 2: (s does not divide n)
If s does not divide n, then $l = 1$. In this case, in addition to the outer hamiltonian cycle Γ, there is one more hamiltonian cycle which is $I(0) = 0, s, 2s, \ldots, 0$. See Figure 7(b). □

Notation 2. *Let $C_n^{sub}(1,s)$ be a subgraph of $C_n(1,s)$ where the edge set of $C_n^{sub}(1,s)$ consists of : (i) the edges of the outer cycle Γ (ii) the edges of the inner cycle $I(0)$.*

Notation 3. *A segment $\Pi^+(x)$ of $C_n^{sub}(1,s)$ is a path of consecutive vertices $x, x+1, \ldots, x+l$ of length l. Similarly a segment $\Pi^-(x)$ is a path of consecutive vertices $x, x-1, \ldots, x-l$ of length l, where $l = g.c.d(n,s)$. See Figure 8.*

The following Lemma gives a strong orientation of $C_n\left(1,2,...,s\right), 2 \leq s \leq \lfloor n/2 \rfloor$.

Lemma 7. *Let $C_n^{sub}(1,s)$ have an orientation O satisfying the following conditions.*

1. *Inner cycle $I(0)$ is oriented clockwise*
2. *$\Pi^+(il)$ is oriented either in clockwise or in anticlockwise direction where $i = 0, 1, \ldots, \lceil n/l \rceil - 1$. Then O is a strong orientation.*

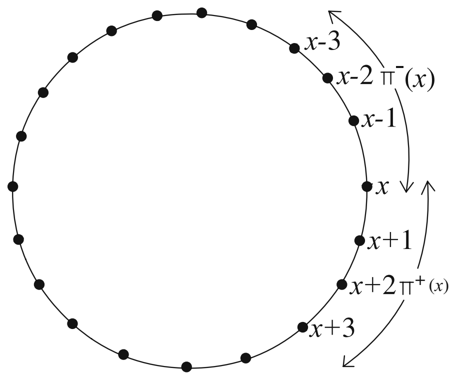

Fig. 8. $l = 3, \Pi^+(x) = x, x+1, x+2, x+3; \Pi^-(x) = x, x-1, x-2, x-3$

Proof. Let u, v be any two vertices of $C_n^{sub}(1, s)$ and let u be in $\Pi^+(il)$ such that $u = il + t$ and v be in $\Pi^+(jl)$ such that $v = jl + h$. Both $\Pi^+(il)$ and $\Pi^+(jl)$ are oriented either clockwise or anticlockwise. Thus there are four possible cases.

Case 1: ($\Pi^+(il)$ and $\Pi^+(jl)$ are Oriented Clockwise)
Here is a directed path from u to v : $(il + t, il + t + 1, \ldots, (i + 1)l, (i + 1)l + s, \ldots, jl, jl + 1, \ldots, jl + h)$ *mod* n. In the same way we trace out a directed path from v to u : $(jl+h, jl+h+1, \ldots, (j+1)l, (j+1)l+s, \ldots, il, il+1, il+2, \ldots, il+t)$ *mod* n. See Figure 9.

Case 2: ($\Pi^+(il)$ is Oriented Clockwise and $\Pi^+(jl)$ is Oriented Anticlockwise)
There is a directed path from u to v: $(il + t, il + t + 1, \ldots, (i + 1)l, (i + 1)l + s, \ldots, (j+1)l, (j+1)l-1, \ldots, jl+h)$ *mod* n and there exist a directed path from v to u : $(jl + h, jl + h - 1, \ldots, jl, jl + s, \ldots, il, il + 1, \ldots, il + t)$ *mod* n.

Case 3: ($\Pi^+(il)$ is oriented anticlockwise and $\Pi^+(jl)$ is oriented clockwise)
Here is a directed path from u to v : $(il + t, il + t - 1, \ldots, il, il + s, \ldots, jl, jl + 1, \ldots, jl + h)$ *mod* n. In the same way we exhibit a directed path from v to u : $(jl + h, jl + h + 1, \ldots, (j + 1)l, (j + 1)l + s, \ldots, (i + 1)l, (i + 1)l - 1, \ldots, il + t)$ *mod* n.

Case 4: ($\Pi^+(il)$ and $\Pi^+(jl)$ are oriented anticlockwise)
Here is a directed path from u to v : $(il+t, il+t-1, \ldots, il, il+s, \ldots, (j+1)l, (j+1)l-1, \ldots, jl+h)$ *mod* n. In the same way we trace out a directed path from v to u : $(jl+h, jl+h-1, \ldots, jl, jl+s, \ldots, (i+1)l, (i+1)l-1, \ldots, il+t)$ *mod* n. □

Theorem 4. *Let G be $C_n(1, 2, \ldots, s), 1 \leq s \leq \lfloor n/2 \rfloor$. Then $\kappa_s = \lceil n/2s \rceil$.*

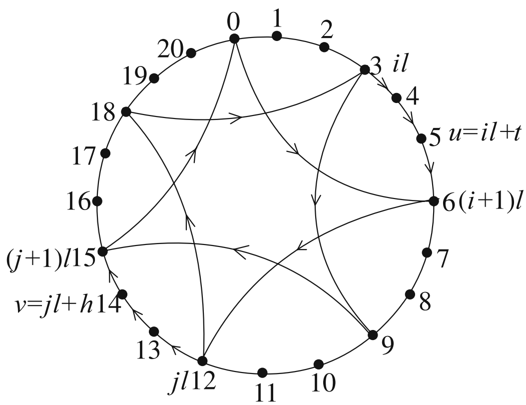

Fig. 9. $C_{21}(1,6); \Pi^+(il)$ and $\Pi^+(jl)$ are oriented clockwise

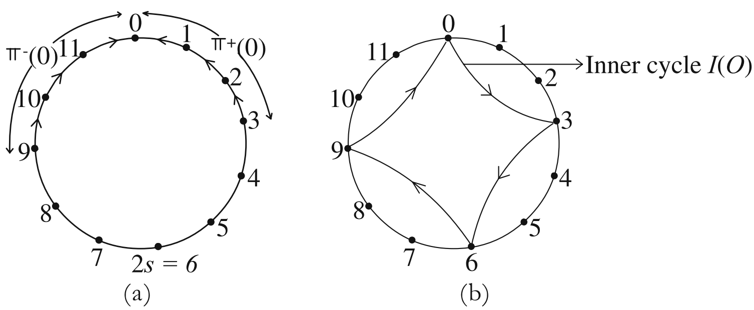

Fig. 10. (a) $\Pi^-(0)$ is oriented in clockwise direction and $\Pi^+(0)$ is oriented in anticlockwise direction in $C_{12}(1,2,3)$; (b) Inner cycle $I(0)$ is oriented clockwise in $C_{12}(1,2,3)$

Proof. By Theorem 3, $\kappa_s \geq \lceil n/2s \rceil$. We claim $\kappa_s \leq \lceil n/2s \rceil$. Two cases arise.

Case 1: (s divides n)
Subcase 1: (n/s is even)
We claim that $K = \{0, 2s, 4s, \ldots, (n/2s-1)2s\}$ is a kernel of $C_n(1,2,\ldots,s)$.

Step 1: For $0 \leq k \leq n/2s$, $\Pi^+(2ks)$ is oriented anticlockwise and $\Pi^-(2ks)$ is oriented clockwise. See Figure 10(a).

Step 2: Orient the inner cycle $I(0)$ as clockwise. See Figure 10(b).

Step 3: The remaining edges at $0, 2s, 4s, \ldots, (n-2s)$ are oriented as incoming edges into the respective vertices. See Figure 11.

Let $u \in V \setminus K$ such that $2k_0s < u < 2(k_0+1)s$ for some k_0. Then either u is adjacent to $2k_0s$ or adjacent to $2(k_0+1)s$ in $C_n(1,2,\ldots,s)$. By the construction of strong orientation of $C_n(1,2,\ldots,s)$, either $\overleftarrow{(u, 2k_0s)}$ or $\overleftarrow{(u, 2(k_0+1)s)}$ is an arc. Also K is an independent set of $C_n(1,2,\ldots,s)$. Clearly $|K| = \lceil n/2s \rceil$.

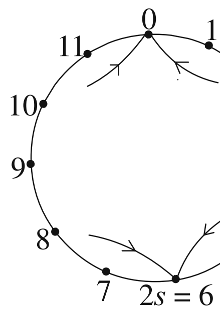

Fig. 11. Edges at $0, 2s, 4s, \ldots, n-2s$ are oriented as incoming edges in C_{12}(1,2,3)

Subcase 2: (n/s is odd)

We claim that $K = \{0, 2s, 4s, \ldots, (\lfloor n/2s \rfloor - 1)2s, n-s-1\}$ is a kernel of $C_n(1, 2, \ldots, s)$.

Step 1: For $0 \leq k < \lfloor n/2s \rfloor$, $\Pi^{+}(2ks)$ is oriented anticlockwise and $\Pi^{-}(2ks)$ is oriented clockwise.

Step 2: All other segments $\Pi(i)$ are oriented in the clockwise direction.

Step 3: Orient the inner cycle $I(0)$ as clockwise.

Step 4: The remaining edges at $0, 2s, 4s, \ldots, (\lfloor n/2s \rfloor - 1)2s, n-s-1$ are oriented as incoming edges into the respective vertices.

Step 5: The remaining unoriented edges of $C_n(1, 2, \ldots, s)$ are oriented arbitrarily.

When $k = \lfloor n/2s \rfloor$, the vertex $2s \lfloor n/2s \rfloor$ is adjacent to 0. Hence we choose the vertex $n-s-1$ which is not adjacent to 0 and $n-s-1 \in (n-2s, n-s)$.

Let $u \in V \setminus K$ such that $2k_0 s < u < 2(k_0+1)s$ for some k_0. Then either u is adjacent to $2k_0 s$ or adjacent to $2(k_0+1)s$ in $C_n(1, 2, \ldots, s)$. By the construction of strong orientation of $C_n(1, 2, \ldots, s)$, either $\overleftarrow{(u, 2k_0 s)}$ or $\overleftarrow{(u, 2(k_0+1)s)}$ is an arc. Also K is an independent set of C_n(1,2,...,s). Clearly $|K| = \lceil n/2s \rceil$.

Case 2: (s does not divide n)

Subcase 1: ($\lfloor n/2s \rfloor 2s < n - s$)

We claim that $K = \{0, 2s, \ldots, 2s \lfloor n/2s \rfloor\}$ is a kernel of $C_n(1, 2, \ldots, s)$.

Step 1: For $0 \leq k \leq \lfloor n/2s \rfloor$, $\Pi^{+}(2ks)$, is oriented anticlockwise and $\Pi^{-}(2ks)$ is oriented clockwise.

Step 2: All other segments $\Pi(i)$ are oriented in the clockwise direction.

Step 3: Orient the inner cycle $I(0)$ as clockwise.

Step 4: The remaining edges at $0, 2s, \ldots, 2s \lfloor n/2s \rfloor$ are oriented as incoming edges into the respective vertices.

Step 5: The remaining unoriented edges of $C_n(1, 2, \ldots, s)$ are oriented arbitrarily.

Let $u \in V \setminus K$ such that $2k_0s < u < 2(k_0+1)s$ for some k_0. Then either u is adjacent to $2k_0s$ or adjacent to $2(k_0+1)s$ are adjacent in $C_n(1,2,\ldots,s)$. By the construction of strong orientation of $C_n(1,2,\ldots,s)$, either $\overleftarrow{(u,2k_0s)}$ or $\overleftarrow{(u,2(k_0+1)s)}$ is an arc. Also K is an independent set of $C_n(1,2,\ldots,s)$. Clearly $|K| = \lceil n/2s \rceil$.

Subcase 2: $(n - s < \lfloor n/2s \rfloor 2s \leq n-1)$
Subcase 2(a): $(n - s < \lfloor n/2s \rfloor 2s < n-1)$
We claim that $K = \{0, 2s, 4s, \ldots, (\lfloor n/2s \rfloor - 1)2s, n-s-1\}$ is a kernel of $C_n(1,2,\ldots,s)$.

Step 1: For $0 \leq k < \lfloor n/2s \rfloor$, $\Pi^+(2ks)$ is oriented anticlockwise and $\Pi^-(2ks)$ is oriented clockwise.
Step 2: All other segments $\Pi(i)$ are oriented in the clockwise direction.
Step 3: Orient the inner cycle $I(0)$ as clockwise.
Step 4: The remaining edges at $0, 2s, 4s, \ldots, (\lfloor n/2s \rfloor - 1)2s, n-s-1$ are oriented as incoming edges into the respective vertices.
Step 5: The remaining unoriented edges of $C_n(1,2,\ldots,s)$ are oriented arbitrarily.

The proof is similar to that of subcase 1 of case 2.

Subcase 2(b): $(\lfloor n/2s \rfloor 2s = n-1)$
We claim that $l = g.c.d\ (n,s) = 1$. If possible let $l > 1$. This implies that $n = k_1 l$ and $s = k_2 l$ for some integers k_1 and k_2. But $n - 1 = ks$, where $k = 2\lfloor n/2s \rfloor$. Therefore $n - ks = 1$. Hence $g.c.d(n,s) = 1$.
Therefore $\lfloor n/2s \rfloor 2s = n - 1 \Longrightarrow l(k_1 - 2k_2 \lfloor k_1/2k_2 \rfloor) = 1$.
Since $l > 1$, $(k_1 - 2k_2 \lfloor k_1/2k_2 \rfloor) = \frac{1}{l}$ is not an integer, a contradiction.
We next claim that the vertex $(\lfloor n/2s \rfloor - 1)2s$ is adjacent to the vertex $n-s-1$.
Consider

$$\begin{aligned}(n-s-1) - (\lfloor n/2s \rfloor - 1)2s &= n - s - 1 - \lfloor n/2s \rfloor 2s + 2s \\ &= n + s - 1 - \lfloor n/2s \rfloor 2s = s\end{aligned}$$

Since the distance between the two vertices $(\lfloor n/2s \rfloor - 1)2s$ and $n-s-1$ is s, they are adjacent to each other. Hence we choose the vertices $n-s-1$ and $(\lfloor n/2s \rfloor - 1)2s - 1$ in K.
We claim that $K = \{0, 2s-1, 4s-1, \ldots, (\lfloor n/2s \rfloor - 1)2s - 1, n-s-1\}$.
Step 1: Mark the vertices $0, 2ks-1, k = 1,2,\ldots,\lfloor n/2s \rfloor - 1$ and $n-s-1$.
Step 2: $\Pi^+(0)$, $\Pi^+(2ks-1)$ and $\Pi^+(n-s-1)$ are oriented anticlockwise and $\Pi^-(0)$, $\Pi^-(2ks-1)$ and $\Pi^-(n-s-1)$ are oriented clockwise.
Step 3: All other segments $\Pi(i)$ are oriented in the clockwise direction.
Step 4: Orient the inner cycle $I(0)$ as clockwise.
Step 5: The remaining edges at $0, 2s-1, 4s-1, \ldots, (\lfloor n/2s \rfloor - 1)2s - 1, n-s-1$ are oriented as incoming edges into the respective vertices.
Step 6: The remaining unoriented edges of $C_n(1,2,\ldots,s)$ are oriented arbitrarily.

The proof is similar to that of subcase 1 of case2. □
Thus we have the following Theorem.

Theorem 5. *The strong kernel problem for* $C_n(1, 2, \ldots, s), 1 \leq s \leq \lfloor n/2 \rfloor$ *is polynomially solvable.*

4 Conclusion

We have introduced variations of kernel number for oriented graphs and have exhibited relations among them. We have estimated the lower bound for the strong kernel number for regular graphs. We have also proved that the strong kernel problem is polynomially solvable for circulant graphs. Further the various parameters introduced in Section 2 have opened new avenues for further research in this field.

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