On the Crossing Numbers of Cartesian Products of Stars and Graphs on Five Vertices*, **

Marián Klešč

Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University, 042 00 Košice, Slovak Republic
marian.klesc@tuke.sk

Abstract. There are known crossing numbers of Cartesian products of stars with all graphs of order at most four. In this paper we are dealing with the Cartesian products of stars with graphs on five vertices. We give the exact values of crossing numbers for some of these graphs and we summarise all known results concerning crossing numbers of these graphs. In addition, we give the crossing number of the join product of star and the cycle C_5 with one additional edge.

Keywords: graph, drawing, crossing number, star, Cartesian product.

1 Introduction

Let G be a simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *crossing number* $cr(G)$ of the graph G is the minimum number of pairwise intersections of edges in all drawings of G in the plane. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. The investigation on crossing numbers of graphs is a classical and however very difficult problem. Garey and Johnson [4] have proved that this problem is NP–complete. According to their special structure, Cartesian products of special graphs are one of few graph classes for which the exact values of crossing numbers were obtained. (For a definition of Cartesian product, see [1].)

Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in the drawing D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of the graph G. We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D. It is easy to see that for any three edge-disjoint subgraphs G_i, G_j, and G_k of the graph G the following equations hold:

$$cr_D(G_i \cup G_j) = cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j)\,,$$

* The research was supported by the Slovak VEGA grant No. 1/0636/08.

** This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0073-07.

J. Fiala, J. Kratochvíl, and M. Miller (Eds.): IWOCA 2009, LNCS 5874, pp. 324–333, 2009.

$$cr_D(G_i \cup G_j, G_k) = cr_D(G_i, G_k) + cr_D(G_j, G_k)\,. \tag{1}$$

Let P_n and C_n be the *path* and the *cycle* of length n, respectively, and the *star* S_n be the complete bipartite graph $K_{1,n}$. In [1] Beineke and Ringeisen asked on the crossing numbers of Cartesian products of small graphs with paths, cycles and stars. For the path, the crossing numbers of $G \times P_n$ are known for all graphs G of order at most five, see [8,10,11,14], and for the cycle, the crossing numbers of $G \times C_n$ are given for all graphs G with at most four vertices [1,10,11,16]. In [15], the known crossing numbers of $G \times C_n$ are summarised for graphs on five vertices. The crossing numbers of stars and all graphs of order three or four are given in [1,8,10,11]. For some graphs of order five, the crossing numbers of Cartesian products with stars are given in [13]. We extend these results and we establish the crossing numbers for Cartesian products of stars with several graphs of order five.

In this paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, in [9] he proved that

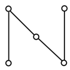
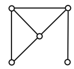
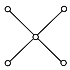
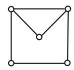
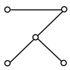
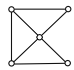
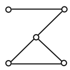
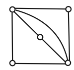
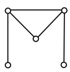
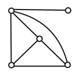
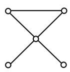
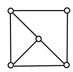
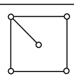
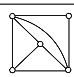
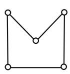
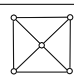
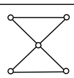
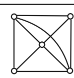
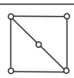
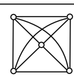
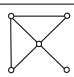
$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6\,. \tag{2}$$

For convenience, the number $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m,n)$ in our paper. In the proofs of this paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

2 Cartesian Products of Stars and Graphs of Order Five

In the Table 1, one can find all 21 connected graphs on five vertices. In [2] Bokal proved the conjecture given by Jendroľ and Ščerbová [8] that $cr(K_{1,n} \times P_m) = (m-1)\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ for the path P_m of length m. Hence, $cr(G_1 \times S_n) = cr(P_4 \times S_n) = 3\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. The same author in [3] proved that for every tree T with n_2 vertices of degree two and n_3 vertices of degree three $cr(T \times S_n) = \lfloor \frac{n}{2} \rfloor ((n_2 + 2n_3) \lfloor \frac{n-1}{2} \rfloor + 1)$. Thus, $cr(G_3 \times S_n) = 3\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. As there is a drawing of the graph $G_4 \times S_n$ with $3\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings and the graph $G_4 \times S_n$ contains $G_3 \times S_n$ as a subgraph, $cr(G_4 \times S_n) = 3\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. In [13], the crossing numbers of the graphs $G_{11} \times S_n$ and $G_{14} \times S_n$ are given. Huang and Zhao proved in [7] that the crossing number of the complete tripartite graph $K_{1,4,n}$ is $n(n-1)$. As the graph $S_4 \times S_n$ is a subdivision of the graph $K_{1,4,n}$, $cr(S_4 \times S_n) = cr(G_2 \times S_n) = n(n-1)$. Both graphs G_6 and G_9 contain the graph G_2 as a subgraph and therefore, $cr(G_6 \times S_n) \geq cr(G_2 \times S_n)$ and $cr(G_9 \times S_n) \geq cr(G_2 \times S_n)$. It is not difficult to find drawings of both graphs $G_6 \times S_n$ and $G_9 \times S_n$ with exactly $n(n-1)$ crossings. This implies that $cr(G_6 \times S_n) = cr(G_9 \times S_n) = n(n-1)$. The crossing number of the graph $G_{10} \times S_n$ is $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$, see [12]. The graph $G_{17} \times S_n$ contains $G_{10} \times S_n$ as a subgraph and hence, $cr(G_{17} \times S_n) \geq cr(G_{10} \times S_n)$. As there is a drawing of the graph $G_{17} \times S_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$ crossings, the crossing number of

Table 1. The known crossing numbers for Cartesian products of stars and graphs of order five

G_i	$cr(G_i \times S_n)$	G_i	$cr(G_i \times S_n)$
G_1 	$3\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ [2]	G_{12} 	
G_2 	n(n-1) [7]	G_{13} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$
G_3 	$3\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$ [3]	G_{14} 	n(n-1) [13]
G_4 	$3\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$	G_{15} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n+\lfloor\frac{n}{2}\rfloor$
G_5 		G_{16} 	
G_6 	n(n-1)	G_{17} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n$
G_7 		G_{18} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n+\lfloor\frac{n}{2}\rfloor$
G_8 		G_{19} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n+\lfloor\frac{n}{2}\rfloor$ [5]
G_9 	n(n-1)	G_{20} 	
G_{10} 	$4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n$ [12]	G_{21} 	
G_{11} 	n(n-1) [13]		

the graph $G_{17} \times S_n$ is $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2n$ also. He and Huang [5] proved that the crossing number of the Cartesian product $G_{19} \times S_n$ is $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2n + \lfloor\frac{n}{2}\rfloor$. In the next sections we give the exact values of crossing numbers of the graphs $G_i \times S_n$ for $i = 13, 15$, and 18.

3 The Crossing Number of $G_{13} + nK_1$

Our aim is to establish the crossing number of the graph $G_{13} \times S_n$. To prove that $cr(G_{13} \times S_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor$, we need to know the crossing number of the join of the graph G_{13} with n isolated vertices. The join product of two

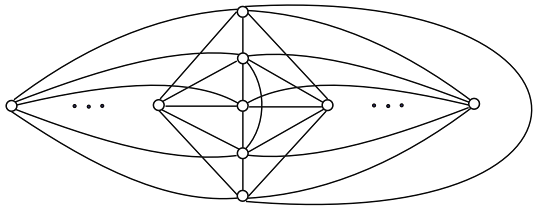


Fig. 1. The graph $G_{13} + nK_1$

graphs G_i and G_j, denoted by $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. In this section, we denote the graph G_{13} by H. The graph H consists of one 5-cycle, denoted by $C_5(H)$ in the paper, and of one additional edge. The graph $H + nK_1$ consists of one copy of the graph H and n vertices $t_1, t_2, \ldots, t_n$, where every vertex $t_i,\ i = 1, 2, \ldots, n$, is adjacent to five vertices of H. Let for $i = 1, 2, \ldots, n,\ T^i$ denote the subgraph induced by five edges incident with the vertex t_i and let $F^i = H \cup T^i$. For the simpler labelling, let H_n denote the graph $H + nK_1$, in this paper. In Figure 1 one can easily see that

$$H + nK_1 = H_n = H \cup K_{5,n} = H \cup \left(\bigcup_{i=1}^{n} T^i \right) . \tag{3}$$

Theorem 1. $cr(H + nK_1) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Proof. The drawing in Figure 1 shows that $cr(H+nK_1) \leq Z(5, n)+\lfloor \frac{n}{2} \rfloor$ and that the theorem is true if equality holds. We prove the reverse inequality by induction on n. As the graph $H+K_1$ is planar, the case $n = 1$ is trivial. The graph $H+2K_1$ contains a subgraph homeomorphic to K_5 and hence, $cr(H+2K_1) \geq 1$. A suitable drawing of the graph $H + 2K_1$ with one crossing shows that $cr(H + 2K_1) \leq 1$, and the case $n = 2$ is also true. Suppose now that for $n \geq 3$

$$cr(H_{n-2}) \geq Z(5, n - 2) + \left\lfloor \frac{n-2}{2} \right\rfloor \tag{4}$$

and consider such a drawing D of H_n that

$$cr_D(H_n) < Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor . \tag{5}$$

Assume that there are two different subgraphs T^i and T^j that do not cross each other in D. Without loss of generality, let $cr_D(T^{n-1}, T^n) = 0$. The subdrawing of $T^{n-1} \cup T^n$ induced from D divides the plane in such a way that no three vertices of degree two (the vertices of H) are placed on the boundary of some region. If the edges of $T^{n-1} \cup T^n$ cross in D the cycle $C_5(H)$, then $cr_D(H, T^{n-1} \cup T^n) \geq 1$.

Otherwise the vertices t_{n-1} and t_n are placed in D in different regions in the view of the subdrawing of $C_5(H)$. In this case the edge of H not belonging to $C_5(H)$ crosses some edge of $T^{n-1} \cup T^n$ and $cr_D(H, T^{n-1} \cup T^n) \geq 1$ again. Moreover, as $cr(K_{5,3}) = 4$, in D every subgraph T^i, $i = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least four times. Since $H_n = H + nK_1 = H_{n-2} \cup (T^{n-1} \cup T^n)$ and $H_{n-2} = K_{5,n-2} \cup H$, using (1) we have

$$cr_D(H_n) = cr_D(H_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(K_{5,n-2}, T^{n-1} \cup T^n)$$
$$+ cr_D(H, T^{n-1} \cup T^n) \geq Z(5, n-2) + \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-2) + 1$$
$$\geq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor.$$

This contradicts (5).

Hence, $cr_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$. Moreover, using (1) and (3) together with $cr(K_{5,n}) = Z(5, n)$ we have

$$cr_D(H_n) = cr_D(K_{5,n}) + cr_D(H) + cr_D(K_{5,n}, H)$$
$$\geq Z(5, n) + cr_D(H) + cr_D(K_{5,n}, H).$$

This, together with the assumption (5), implies that

$$cr_D(H) + cr_D(K_{5,n}, H) < \left\lfloor \frac{n}{2} \right\rfloor \qquad (6)$$

and hence, in D there is at least one subgraph T^i which does not cross H.

Without loss of generality, let $cr_D(H, T^n) = 0$ and let F^n be the subgraph $H \cup T^n$ of the graph H_n. In the drawing D there is at least one subgraph T^i, $i \in \{1, 2, \ldots, n-1\}$, for which $cr_D(F^n, T^i) \leq 2$, otherwise, as $H_n = K_{5,n-1} \cup F^n$, we have

$$cr_D(H_n) = cr_D(K_{5,n-1}) + cr_D(F^n) + cr_D(K_{5,n-1}, F^n)$$
$$\geq Z(5, n-1) + 3(n-1) \geq Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor.$$

This contradicts (5). Consider now the subdrawing D^* of F^n induced by D. Our next analysis depends on whether or not the 5-cycle $C_5(H)$ has an internal crossing in D^*.

Assume first, that the edges of $C_5(H)$ do not cross each other. Since the edges of T^n do not cross the edges of H, all edges of T^n are placed in D^* in one of two regions, say outside, in the view of the subdrawing of $C_5(H)$ and the edge of H not belonging to $C_5(H)$ is placed inside the cycle $C_5(H)$. The unique such drawing D^* is shown in Figure 2. It is easy to see that if, in D, some vertex t_i, $i \in \{1, 2, \ldots, n-1\}$, is placed in the region β_1, then H is crossed by at least two edges joining t_i with the vertices of H. Moreover, as T^i crosses T^n, $cr_D(F^n, T^i) \geq 3$. Consider now the region β_2. One vertex of H does not appear

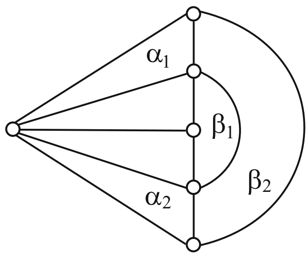


Fig. 2. The subdrawing of $F^n = H \cup T^n$

on the boundary of this region. So, if t_i is placed in D in the region β_2, then $cr_D(H, T^i) \geq 1$ and $cr_D(T^n, T^i) \geq 1$. On the boundary of the region α_1 there are only two vertices of H, and therefore, if the vertex t_i is placed in D in the region α_1, $cr_D(F^n, T^i) \geq 3$. For $cr_D(F^n, T^i) = 3$, the necessary condition is that one edge of T^i joining t_i with the vertex of H on the boundaries of the regions α_2 and β_1 crosses the edge of H on the boundary of α_1. If no edge of T^i crosses this edge of H, then $cr_D(F^n, T^i) \geq 4$. Regarding to the symmetry of D^*, the same holds for the region α_2. For the remaining three regions of D^*, three vertices of H do not appear on its boundary and one of them does not appear on the boundaries of the neighbouring regions. So, if t_i is placed in D in some of the mentioned three regions, then the edges of T^i cross the edges of F^n at least four times.

Let r be the number of vertices t_i, $i \in \{1, 2, \ldots, n-1\}$, which are placed in D in the region β_2. In the drawing D, every such subgraph T^i crosses F^n at least two times and at least one of these crossings appears on the edges of H. Let s be the number of vertices t_i placed in D in the region β_1 and such vertices t_i placed in the regions α_1 and α_2 for which T^i crosses F only three times. Every such subgraph T^i crosses also H. Since $cr_D(F^n, T^j) \leq 2$ for some $j \in \{1, 2, \ldots, n-1\}$, we have $r \geq 1$, and it follows from (6) that $r + s < \lfloor \frac{n}{2} \rfloor$. Now

$$\begin{aligned} cr_D(H_n) &= cr_D(K_{5,n-1}) + cr_D(F^n) + cr_D(K_{5,n-1}, F^n) \\ &\geq Z(5, n-1) + 2r + 3s + 4(n - r - s - 1) \\ &= Z(5, n) - 4\left\lfloor \frac{n-1}{2} \right\rfloor + 4n - 4 - 2r - s\,. \end{aligned}$$

This, together with the assumption (5), gives

$$2r + s > 4n - 4 - 4\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = 3\left\lfloor \frac{n}{2} \right\rfloor .$$

On the other hand, $r + s < \lfloor \frac{n}{2} \rfloor$ and the inequality

$$2r + s > 3\left\lfloor \frac{n}{2} \right\rfloor > 3(r + s)$$

implies that

$$r + 2s < 0\,.$$

This contradiction with $r > 0$ and $s \geq 0$ confirms that there is no drawing of the graph H_n with fewer than $Z(5,n) + \lfloor \frac{n}{2} \rfloor$ crossings in which the edges of the 5-cycle $C_5(H)$ do not cross each other.

Assume now that the edges of $C_5(H)$ cross each other in D. We remark that in D there is at least one subgraph T^j, $j \in \{1, 2, \ldots, n-1\}$, for which $cr_D(F^n, T^j) \leq 2$. As $cr_D(T^n, T^i) \neq 0$ for all $i = 1, 2, \ldots, n-1$, the inequality $cr_D(F^n, T^j) \leq 2$ implies that $cr_D(C_5(H), T^j) \leq 1$. Since $cr_D(C_5(H), T^n) = 0$, the vertex t_n of T^n lies in D^* in the region with all five vertices of $C_5(H)$ on its boundary, and the condition $cr_D(C_5(H), T^j) \leq 1$ enforces that in the subdrawing of $C_5(H) \cup T^n$ there is a region with at least four vertices of $C_5(H)$ on its boundary. This is possible only in the case when two edges incident with a common vertex of $C_5(H)$ cross, a contradiction with the requirement that the drawing D is good. This completes the proof. □

4 The Crossing Number of $G_i \times S_n$ for $i = 13, 15$, and 18

Let K be a connected graph on five vertices. Consider a graph G_K obtained by joining all vertices of K to five vertices of a connected graph G such that every vertex of K be adjacent to exactly one vertex of G. Let G_K^* be the graph obtained from G_K by contracting the edges of K.

Lemma 1. *If G is a connected graph and $K = G_{13}$, then $cr(G_K^*) \leq cr(G_K)$.*

Proof. Assume an optimal drawing of G_K. Let $x_1, x_2, \ldots, x_6$ denote the numbers of crossings on the edges of the graph K in this drawing as shown in Figure 3(a). The drawing in Figure 3(b) shows that if $x_4 \leq x_2 + x_5$, then K can be contracted such that the resulting drawing does not have more crossings than the original. Due to symmetry of the graph G_{13}, the same holds if $x_5 \leq x_1 + x_4$.

Assume that the statement of Lemma 1 is not true. Then there is a good drawing of the graph G_K in which $x_4 > x_2 + x_5$ and $x_5 > x_1 + x_4$. Combining these inequalities we have the inequality

$$x_4 > x_2 + x_5 > x_2 + x_1 + x_4\,,$$

which implies that

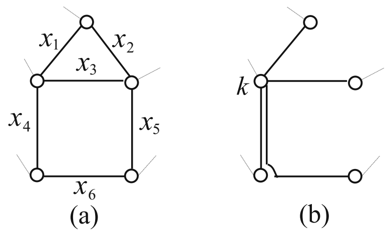


Fig. 3. The contraction of G_{13}

$$x_1 + x_2 < 0\,.$$

This contradiction with $x_i \geq 0$ for $i = 1, 2, \ldots, 5$ completes the proof. □

Lemma 2. *If G is a connected graph and $K = G_{15}$, then $cr(G^*_K) \leq cr(G_K) - 1$.*

Proof. As the graph $K = G_{15}$ contains $K_{2,3}$ as a subgraph and the graph $K_{2,3}$ is not outer-planar, in every good drawing of the graph G_K there is at least one crossing on the edges of the $K_{2,3}$-subgraph of K. In the graph G_K, the subgraph K contains two vertices of degree five and three vertices of degree three. Let us denote by v_1 and v_2 the vertices of degree five and let e_1, e_2, e_3 and f_1, f_2, f_3 be the edges of the $K_{2,3}$-subgraph of K incident with the vertices v_1 and v_2, respectively. Assume an optimal drawing of the graph G_K. If some of the edges e_1, e_2, and e_3 is crossed, then the subdrawing obtained by deleting the edges e_1, e_2, and e_3 has at least one crossing less than the original. If none of the edges e_1, e_2, and e_3 is crossed, then there is at least one crossing on the edges f_1, f_2, and f_3. In this case, the subdrawing of the subgraph $G_K - \{f_1, f_2, f_3\}$ has at least one crossing less than the original. Both subgraphs $G_K - \{e_1, e_2, e_3\}$ and $G_K - \{f_1, f_2, f_3\}$ are homeomorphic to the graph G^*_K, and therefore $cr(G^*_K) \leq cr(G_K) - 1$. □

Theorem 2. $cr(G_{13} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Proof. In Figure 4 there is the drawing of the graph $G_{13} \times S_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings. Hence, $cr(G_{13} \times S_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. To prove the reverse inequality we assume that there is a drawing of the graph $G_{13} \times S_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings. As contracting the edges of every non–central copy of G_{13} in the graph $G_{13} \times S_n$ results in a graph isomorphic to the graph $G_{13} + nK_1$, in accordance with Lemma 1 we have $cr(G_{13} + nK_1) < 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. This contradiction with Theorem 1 completes the proof. □

Theorem 3. $cr(G_{15} \times S_n) = cr(G_{18} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

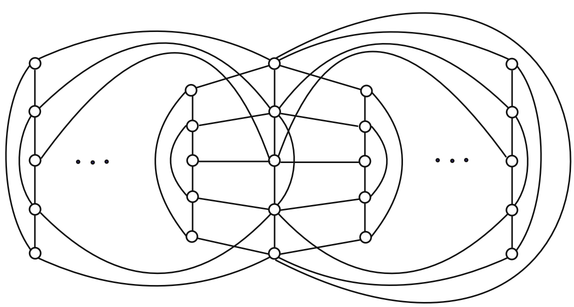


Fig. 4. The graph $G_{13} \times S_n$

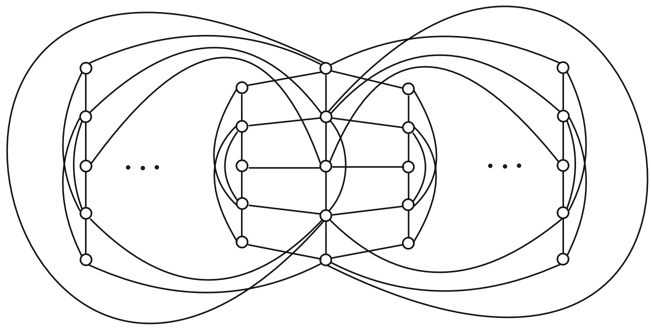


Fig. 5. The graph $G_{15} \times S_n$

Proof. In Figure 5 there is the drawing of the graph $G_{15} \times S_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$ crossings. Hence, $cr(G_{15} \times S_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$. To prove the reverse inequality we assume that there is a drawing of the graph $G_{15} \times S_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$ crossings. As contracting the edges of every non–central copy of G_{15} in the graph $G_{15} \times S_n$ results in a graph isomorphic to the graph $K_{1,1,3,n}$, in accordance with Lemma 2 we have $cr(K_{1,1,3,n}) < 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$. This is in contradiction with the result of Ho Pak Tunk [6] which states that $cr(K_{1,1,3,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ for all $n \geq 1$. Hence, $cr(G_{15} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$.

In Figure 5 it is possible to draw $n+1$ edges (one edge in every copy of G_{15}) without increasing the number of crossings and obtain the drawing of the graph $G_{18} \times S_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$ crossings. Thus, we have the upper bound. On the other hand, the graph $G_{18} \times S_n$ contains $G_{15} \times S_n$ as a subgraph and this implies that $cr(G_{18} \times S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

Acknowledgement

The author is indebted to the referees for their useful comments.

References

1. Beineke, L.W., Ringeisen, R.D.: On the crossing numbers of products of cycles and graphs of order four. J. Graph Theory 4, 145–155 (1980)
2. Bokal, D.: On the crossing numbers of Cartesian products with paths. J. Combin. Theory B 97, 381–384 (2007)
3. Bokal, D.: On the crossing numbers of Cartesian products with trees. J. Graph Theory 56, 287–300 (2007)
4. Garey, M.R., Johnson, D.S.: Crossing number is NP-complete. SIAM J. Algebraic and Discrete Methods 4, 312–316 (1983)
5. He, P., Huang, Y.: The crossing number of $W_4 \times S_n$. Journal of Zhengzhou University (Natural Science Edition) 39, 14–18 (2007)

6. Ho, P.T.: The crossing number of $K_{1,1,3,n}$. Ars Combinatoria (to appear)
7. Huang, Y., Zhao, T.: The crossing number of $K_{1,4,n}$. Discrete Math. 308, 1634–1638 (2008)
8. Jendrol, S., Ščerbová, M.: On the crossing numbers of $S_m \times P_n$ and $S_m \times C_n$. Časopis pro pěstování matematiky 107, 225–230 (1982)
9. Kleitman, D.J.: The crossing number of $K_{5,n}$. J. Combin. Theory 9, 315–323 (1970)
10. Klešč, M.: On the crossing numbers of Cartesian products of stars and paths or cycles. Mathematica Slovaca 41, 113–120 (1991)
11. Klešč, M.: The crossing numbers of products of paths and stars with 4-vertex graphs. J. Graph Theory 18, 605–614 (1994)
12. Klešč, M.: The crossing number of $K_{2,3} \times P_n$ and $K_{2,3} \times S_n$. Tatra Mountains Math. Publ. 9, 51–56 (1996)
13. Klešč, M.: On the crossing numbers of products of stars and graphs of order five. Graphs and Combinatorics 17, 289–294 (2001)
14. Klešč, M.: The crossing numbers of Cartesian products of paths with 5–vertex graphs. Discrete Math. 233, 353–359 (2001)
15. Klešč, M.: Some crossing numbers of products of cycles. Discuss. Math. Graph Theory 25, 197–210 (2005)
16. Ringeisen, R.D., Beineke, L.W.: The crossing number of $C_3 \times C_n$. J. Combin. Theory 24, 134–136 (1978)