Trivially-Perfect Width

Ling-Ju Hung[1], Ton Kloks*, and Chuan Min Lee[2,**]

[1] Department of Computer Science and Information Engineering
National Chung Cheng University
Min-Hsiung, Chia-Yi 621, Taiwan
hunglc@cs.ccu.edu.tw

[2] Department of Computer and Communication Engineering
Ming Chuan University
5 De Ming Rd., Guishan District, Taoyuan County 333, Taiwan
joneslee@mail.mcu.edu.tw

Abstract. The $\mathcal{G}$-width of a class of graphs $\mathcal{G}$ is defined as follows. A graph G has $\mathcal{G}$-width k if there are k independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ in G such that G can be embedded into a graph $H \in \mathcal{G}$ with the property that for every edge e in H which is not an edge in G, there exists an i such that both endpoints of e are in $\mathbb{N}_i$. For the class $\mathfrak{TP}$ of trivially-perfect graphs we show that $\mathfrak{TP}$-width is NP-complete and we present fixed-parameter algorithms.

1 Introduction

The recognition problem of probe interval graphs was introduced by Zhang *et al.* [8,14]. This problem stems from the physical mapping of chromosomal DNA of humans and other species. Since then probe graphs of many other graph classes have been investigated by various authors. We generalize the concept to the graph-class-width parameters.

Definition 1. *Let $\mathcal{G}$ be a class of graphs which contains all cliques. The $\mathcal{G}$-*width *of a graph G is the minimum number k of independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ in G such that there exists an embedding $H \in \mathcal{G}$ of G with the property that for every edge $e = (x, y)$ in H which is not an edge of G, there exists an i with $x, y \in \mathbb{N}_i$.*

In this paper we investigate the width-parameter for the class $\mathfrak{TP}$ of trivially-perfect graphs, henceforth called the *trivially-perfect width*, or $\mathfrak{TP}$-width. If a graph G has $\mathfrak{TP}$-width k then we call G also a *k-probe trivially-perfect graph*. This paper deals with the recognition problem of k-probe trivially-perfect graphs. We refer to the *partitioned case* of the problem when a collection of, possibly overlapping, independent sets $\mathbb{N}_i$, $i = 1, \ldots, k$ is a part of the input. We call such a collection of independent sets a *witness*.

* This author is supported by the National Science Council of Taiwan under grant NSC 97–2811–E–194–001.

** The author is partially supported by the National Science Council of Taiwan under grant NSC 97–2218–E–130–002–MY2.

J. Fiala, J. Kratochvíl, and M. Miller (Eds.): IWOCA 2009, LNCS 5874, pp. 301–311, 2009.

Trivially-perfect graphs were first studied by Wolk [13]. However, Golumbic gave the class its name [4].

Definition 2 ([4]). *A graph is* trivially perfect *if for each induced subgraph the independence number is equal to the number of maximal cliques.*

We use Wolk's characterization of the class.

Theorem 1 ([13]). *A graph is trivially perfect if and only if every connected induced subgraph has a universal vertex.*[1]

We denote the class of trivially-perfect graphs by $\mathfrak{TP}$. The class can also be characterized by excluding the C_4 and P_4 as induced subgraphs; thus the class of trivially-perfect graphs is exactly the class of *chordal cographs*. In this paper we study the $\mathfrak{TP}$-width of graphs.

It follows from Theorem 1 that a connected graph $G = (V, E)$ is trivially perfect if and only if there exists a rooted tree T, with node-set V such that two vertices x and y are adjacent in G if and only if one lies on the path from the root to the other. Thus the set of vertices of any path from the root to a leaf in T induces a maximal clique in G, and these are all the maximal cliques in G.

It can be seen that the class of partitioned k-probe trivially-perfect graphs can be characterized by a finite set of forbidden induced, partitioned subgraphs, see, *e.g.*, [6,11,12]. We think that a similar statement holds also for the unpartitioned case, but we have no proof of this yet.

In the following, we write some of our notational customs. For two sets A and B we write $A + B$ and $A - B$ instead of $A \cup B$ and $A \setminus B$. We write $A \subseteq B$ if A is a subset of B with possible equality and we write $A \subset B$ if A is a subset of B and $A \neq B$. For a set A and an element x we write $A + x$ instead of $A + \{x\}$ and $A - x$ instead of $A - \{x\}$. It will be clear from the context when x is an element instead of a set.

A graph G is a pair $G = (V, E)$ where V is a *finite* set, of which the elements are called the vertices of G, and where E is a set of two-element subsets of V, of which the elements are called the edges of G. We denote edges of a graph as (x, y) and we call x and y the endvertices of the edge. For a vertex x we write $N(x)$ for its set of neighbors and we write $N[x] = N(x) + x$ for the closed neighborhood of x. For a subset $W \subseteq V$ we write $N(W) = \bigcup_{x \in W} N(x) - W$ for its neighborhood and we write $N[W] = N(W) + W$ for its closed neighborhood. Usually we use $n = |V|$ to denote the number of vertices of G and we use $m = |E|$ to denote the number of edges of G.

For a graph $G = (V, E)$ and a subset $S \subseteq V$ of vertices we write $G[S]$ for the subgraph *induced* by S, that is, the graph with S as its set of vertices and with those edges of E that have both endvertices in S. For a subset $W \subseteq V$ we write $G - W$ for the graph $G[V - W]$. For a vertex x we write $G - x$ rather than $G - \{x\}$. We usually denote graph classes by calligraphic capitals.

[1] A vertex is *universal* if it is adjacent to all other vertices.

The paper is organized as follows: in Section 2, we show that the rankwidth of k-probe trivially-perfect graphs is bounded and for constant k, k-probe trivially-perfect graphs can be recognized in $O(n^3)$ by formulating the problem in C_2MS-logic. In Section 3, we give a linear time algorithm for the recognition of partitioned k-probe trivially-perfect graphs. In Section 4, we show that $\mathfrak{TP}$-width is NP-complete. In Section 5, we give a fixed-parameter algorithm to check if a given graph is a k-probe trivially-perfect graph.

Since the results are comparable to those of rankwidth, it seems worthwhile to examine $\mathcal{G}$-width for specific classes of graphs. We started with one of the 'easiest' in this paper. Although for this class rankwidth gives an alternative solution, a closer examination of the method seems not a bad idea anyway. Next in line are the distance-hereditary graphs; for those no monadic-order formulation is available at the moment. We hope that, with this paper, we obtained a sufficient understanding of the methods to solve this, and other classes in future research.

2 Trivially-Perfect Width Is Fixed-Parameter Tractable

In this section we show that for constant k, k-probe trivially-perfect graphs can be recognized in $O(n^3)$ time.

The following Theorem 2 is a characterization which can be formulated in C_2MS-logic [3]. We will prove that k-probe trivially-perfect graphs have *rankwidth* at most 2^k shortly. It is known that problems which can be formulated in C_2MS-logic can be solved in $O(n^3)$ time on graphs with bounded rankwidth [3].

Definition 3. *Let $(G, \mathcal{N})$ be a partitioned graph with a witness*

$$\mathcal{N} = \{\mathbb{N}_i \mid i = 1, \dots, k\}$$

of k, possibly overlapping independent sets in G. A vertex ω is probe universal *if for every vertex $x \neq \omega$ either*

(i) *$(x, \omega) \in E$, or*
(ii) *there exists $i \in \{1, \dots, k\}$ with $\{x, \omega\} \subseteq \mathbb{N}_i$.*

Theorem 2. *A graph $G = (V, E)$ is a k-probe trivially-perfect graph if and only if there exist independent sets $\mathbb{N}_i$, $i = 1, \dots, k$, such that every connected induced subgraph has a probe universal vertex.*

Proof. Assume that G is a k-probe trivially-perfect graph. Let $\mathcal{N}$ be a witness of k independent sets $\mathbb{N}_i$ and let H be an embedding, obtained by adding edges in these independent sets. Let $C \subseteq V$ be a subset of vertices such that $G[C]$ is connected. Then $H[C]$ is also connected. By Theorem 1 $H[C]$ has a universal vertex ω. Let x be another vertex of $G[C]$. Since H is an embedding of $(G, \mathcal{N})$, either x and ω are adjacent in G, or there exists an i such that $x, \omega \in \mathbb{N}_i$.

Assume that there exists a witness $\mathcal{N}$ of k independent sets $\mathbb{N}_i$ such that every connected induced subgraph of G has a probe universal vertex. Let C be a component of G. We show that $G[C]$ can be embedded into a trivially perfect graph. Let ω be a probe universal vertex in $G[C]$. By induction we may assume that $G[C] - \omega$ with the induced witness has a trivially-perfect embedding H'. We obtain an embedding H of $G[C]$ by adding ω as a universal vertex to H'. □

Definition 4 ([10]). *A* rank-decomposition *of a graph $G = (V, E)$ is a pair (T, τ) where T is a ternary tree and τ a bijection from the leaves of T to the vertices of G. Let e be an edge in T and consider the two sets A and B of leaves of the two subtrees of $T - e$. Let M_e be the submatrix of the adjacency matrix of G with rows indexed by the vertices of A and columns indexed by the vertices of B. The* width of e *is the rank over $GF(2)$ of M_e. The* width of (T, τ) *is the maximum width over all edges e in T and the* rankwidth *of G is the minimum width over all rank-decompositions of G.*

The class of graphs with rankwidth at most 1 is exactly the class of distance-hereditary graphs [2,5,10]. Note that every trivially-perfect graph is distance hereditary [1], since every induced path has length 1 or 2 by Theorem 1.

Theorem 3. *k-Probe trivially-perfect graphs have rankwidth at most 2^k.*

Proof. Consider a rank-decomposition (T, τ) with width 1 for an embedding H of a k-probe trivially-perfect graph G. Consider an edge e in T and assume that M_e is an all 1s-matrix. Each independent set $\mathbb{N}_i$ creates a 0-submatrix in M_e. If $k = 1$ this proves that the rankwidth of G is at most 2. In general, for $k \geq 0$, note that there are at most 2^k different neighborhoods from one leaf-set of $T - e$ to the other. It follows that the rank of M_e is at most 2^k. □

Theorem 4. *For each $k \geq 0$ there exists an $O(n^3)$ algorithm which checks whether a graph G with n vertices is a k-probe trivially-perfect graph; that is, $\mathfrak{TP}$-width is in FPT.*

Proof. By Theorem 3 k-probe trivially-perfect graphs have bounded rankwidth. It is well-known that C_2MS-problems can be solved in $O(n^3)$ time for graphs of bounded rankwidth (see [3], and follow pointers from there). By Theorem 2, the recognition of k-probe trivially-perfect graphs is such a problem. □

3 Partitioned *k*-Probe Trivially-Perfect Graphs

Obviously, the result of the previous section holds as well when the collection of independent sets $\mathbb{N}_1, \ldots, \mathbb{N}_k$ is a part of the input. Thus for each k there is an $O(n^3)$ algorithm that checks whether a graph G with a witness of k independent sets $\mathbb{N}_i$, can be embedded into a trivially-perfect graph. However, there are a few drawbacks to this solution. First of all, Theorem 4 only shows the *existence* of an $O(n^3)$ recognition algorithm. In any case, *a priori*, it is unclear how to obtain the algorithm explicitly. Furthermore, the constants involved in the algorithm make the solution impractical. Already there is an exponential blow-up when one moves from $\mathfrak{TP}$-width to rankwidth.

In this section we show that there exists an easy algorithm for the recognition of partitioned k-probe trivially-perfect graphs by recursively eliminating a probe universal vertex.

Proposition 1. *Let* $(G, \mathcal{N})$ *be a partitioned graph with a witness*

$$\mathcal{N} = \{\mathbb{N}_i \mid i = 1, \ldots, k\}$$

of k*, possibly overlapping independent sets. Then* $(G, \mathcal{N})$ *is a partitioned* k*-probe trivially-perfect graph if and only if every component of* G*, with the induced witness, is a partitioned* k*-probe trivially-perfect graph.*

Theorem 5. *There exists a linear-time algorithm to check whether a partitioned graph* $(G, \mathcal{N})$ *with a witness* $\mathcal{N}$ *of* k *independent sets, is a partitioned* k*-probe trivially-perfect graph.*

Proof. If G is disconnected then, by Proposition 1, we can check each component individually. Assume G is connected. It is easy to see that we can compute an elimination ordering by probe universal vertices in linear time, by keeping a list of vertices ordered by degree. □

Remark 1. Note that the algorithm described in Theorem 5 is *fully* polynomial. The algorithm can be used to compute an embedding in $O(n^2)$ time.

4 $\mathfrak{TP}$-Width Is NP-Complete

Let $\mathfrak{T}$ be the class of complete graphs (cliques). We first show that $\mathfrak{T}$-width is NP-complete.

Theorem 6. $\mathfrak{T}$*-Width is NP-complete.*

Proof. Let $(G, \mathcal{N})$ with witness $\mathcal{N} = \{\mathbb{N}_i \mid i = 1, \ldots, k\}$ be a partitioned k-probe complete graph. Thus every nonedge of G has its endvertices in one of the independent sets $\mathbb{N}_i$. That is, $\mathcal{N}$ forms a clique-cover of the edges of $\bar{G}$. This proves that a graph G has $\mathfrak{T}$-width at most k if and only if the edges of $\bar{G}$ can be covered with k cliques. The problem to cover the edges of a graph by a minimum number of cliques is NP-complete [9]. □

Theorem 7. $\mathfrak{TP}$*-Width is NP-complete.*

Proof. Let $G = (V, E)$ be a graph with n vertices and m edges. Add a vertex ω and make ω adjacent to all vertices of G. Additionally, add a clique C of n^2 vertices and make every vertex of C adjacent to every vertex of G. Let G' be the graph constructed in this way. Note that, when we add edges between nonadjacent vertices of V we obtain an embedding of G' into a trivially-perfect graph. We show that this is the only feasible embedding.

For each nonedge $\{x, y\}$ in G we now have a collection of C_4's using x, y, the vertices of the clique C and ω. Assume that there is an embedding of G'

into a trivially-perfect graph with x and y not adjacent. Then each vertex of C is adjacent to ω. Thus each vertex of C must be in one of the independent sets $\mathbb{N}_s$, $s = 1, \ldots, k$, and no two are in the same independent set since C is a clique. Then $k \geq n^2$ which is a contradiction, since making a clique of G creates a trivially-perfect embedding, and this needs at most $\binom{n}{2} - m$ independent sets. Thus the only feasible embedding makes a clique of G. That is, the $\mathfrak{TP}$-width of G' is the same as the $\mathfrak{T}$-width of G, and by Theorem 6 this is hard to compute. □

5 A Fixed-Parameter Algorithm to Compute $\mathfrak{TP}$-Width

In this section we show that there exists for each k an $O(n^3)$ algorithm which checks if a graph G is a k-probe trivially-perfect graph.

Let $(G, \mathcal{N})$ be a partitioned graph with a witness

$$\mathcal{N} = \{\mathbb{N}_i \mid i = 1, \ldots, k\}$$

of k independent sets. We call the vertices of the independent sets *nonprobes* and we call the vertices which are not in any independent set *probes*. The *k-label* $\alpha(x)$ of a vertex x is the 0/1-vector of length k with the i^{th} entry $\alpha_i(x)$ equal to 1 if and only if $x \in \mathbb{N}_i$. We write $\alpha(x) \leq \alpha(y)$ if $\alpha_i(x) \leq \alpha_i(y)$ for all $i = 1, \ldots, k$. We write $\alpha(x) \perp \alpha(y)$ if there is no i with $\alpha_i(x) = \alpha_i(y) = 1$.

We use (G, α) to denote a labeled graph. If X is a subset of vertices then we write $\alpha(X)$ for the restriction of the labeling α to the vertices of X. For a labeled subset X we write (X, α), instead of $(G[X], \alpha)$ and instead of $(G[X], \alpha(X))$.

Consider the equivalence relation $\equiv$ defined by $x \equiv y$ if $N(x) = N(y)$. Denote the equivalence class of a vertex x by (x). Define the partial order $\preceq$ by:

$$(x) \preceq (y) \quad \text{if} \quad N(x) \subseteq N(y)$$

Likewise, we consider the equivalence relation $\equiv'$ defined by $x \equiv' y$ if $N[x] = N[y]$. The equivalence class of a vertex x under this relation is denoted by $[x]$. We consider the partial order defined by:

$$[x] \preceq [y] \quad \text{if} \quad N[x] \subseteq N[y]$$

Lemma 1. *A graph G is trivially perfect if and only if for every pair of adjacent vertices x and y, either $[x] \preceq [y]$ or $[y] \preceq [x]$.*

Proof. Note that a graph G has two adjacent vertices with incomparable closed neighborhoods, if and only if G contains an induced P_4 or C_4. □

Definition 5 ([7]). *A module M in a graph $G = (V, E)$ is a set of vertices such that for every vertex $y \notin M$ either*

1. *$N(y) \cap M = \varnothing$, or*
2. *$M \subseteq N(y)$.*

A module M is trivial *if $|M| \leq 1$ or if $M = V$.*

Remark 2. Assume $G = (V, E)$ is connected and trivially perfect. Let T be a rooted tree with node-set V such that two vertices are adjacent in G if and only if in T, one lies on the path from the root to the other. We refer to T as the *tree-model* of G. Note that for each node x in T the vertices in the subtree rooted at this node form a module X in G. The neighborhood $N_G(X)$ of this module is the path in T from the root to the parent of x.

Definition 6. *A* probe module *is a labeled set (X, α) which induces a partitioned k-probe trivially-perfect graph with the additional property that for each vertex $y \notin X$ there exists a y-*extension*, which is a label $\alpha(y)$ such that either*

(a) *$N(y) \cap X = \varnothing$, or*
(b) *For each $x \in X$ $[x \in N(y) \Leftrightarrow \alpha(y) \perp \alpha(x)]$.*

Definition 7. *Two disjoint probe modules (X, α) and (Y, β) are* twins *if $(X + Y, \gamma)$ is a probe module with the inherited labeling $\gamma(X) = \alpha$ and $\gamma(Y) = \beta$, such that either*

(i) *no vertex of X is adjacent to any vertex of Y, or*
(ii) *one of X and Y is a probe clique, and for every pair of vertices $x \in X$ and $y \in Y$, x and y are adjacent if and only if $\gamma(x) \perp \gamma(y)$.*

Definition 8. *Let (X, α) be a probe module in a connected graph G. Then (X, α)* embeds *if the labeling α extends such that (G, α) has an embedding H which has $H[X]$ as a module. The graph H is an* embedding of X*.*

Definition 9. *Let (X, α) be a probe module. A* label-set *of (X, α) is a maximal subset of vertices of X with the same label. The* characteristic *$\chi(X)$ is the set of labels for which the label-set is nonempty.*

Remark 3. For ease of description we describe a trivially-perfect graph G also by its *cotree*. This representation is a binary tree where the leaves are labeled by the vertices of G and the internal nodes labeled by the join-operator $\otimes$ or the union-operator $\oplus$. In case of a join-operator, the set of leaves in at least one of the two subtrees must induce a clique, since G is chordal.

Lemma 2 (The Telescope Lemma). *Let (X, α) and (Y, α) be twin probe-modules. Assume that $\chi(X) \supseteq \chi(Y)$. Then $(X + Y, \alpha)$ embeds if and only if (X, α) embeds.*

Proof. Let H be an embedding of $(X + Y, \alpha)$. Consider the cotree of H. Since $H[X + Y]$ is a module in H, we may assume that $(X + Y)$ forms a subtree. Let H' be the trivially-perfect graph obtained from H by replacing $H[X + Y]$ by the union or join of $H[X]$ and $H[Y]$, whichever is appropriate. Then H' is an embedding of (X, α); we obtain a cotree by making X and Y twin-branches.

Now let H be an embedding of (X, α). Assume that a vertex $z \notin (X + Y)$ is adjacent in G to a vertex $y \in Y$. Then $\alpha(z) \perp \alpha(y)$ in any z-extension of

$(X+Y,\alpha)$. There exists a vertex $x \in X$ such that $\alpha(x) = \alpha(y)$, which implies that z is also adjacent to x in G.

Consider a cotree of $H - Y$ such that X forms a subtree. Add $H[Y]$ to $H - Y$ as a twin-branch of $H[X]$ and let H' be the graph that results. We prove that we can add edges between Y and $N_H(X) - Y$, such that $X + Y$ becomes a module in H'.

Let $z \notin (X+Y)$. If z is not adjacent to any vertex of X in H then z is also not adjacent to any vertex in Y. Assume z is adjacent to X in H. Let $y \in Y$, and assume that z is not adjacent to y in G. We prove that $\alpha_i(z) = \alpha_i(y) = 1$ for some entry i. There exists a vertex $x \in X$ such that $\alpha(x) = \alpha(y)$. If z is not adjacent to x in G, $\alpha_i(y) = \alpha_i(x) = \alpha_i(z) = 1$ for some entry i. Assume z is adjacent to x in G. Then $\alpha(z) \perp \alpha(x)$ in any z-extension of $(X+Y,\alpha)$. Since $(X+Y,\alpha)$ is a probe module and $\alpha(x) = \alpha(y) \perp \alpha(z)$, z is adjacent to y, which is a contradiction. □

Definition 10. *A* true – *or* false twinset *is a set of vertices such that every pair is a true – or false twin, respectively.*[2] *A k-*twinset *is either a false twinset with at least 3 vertices or a true twinset with at least $k+2$ vertices.*

Lemma 3 (The Twinset Lemma). *Let S be k-twinset. Then G has $\mathfrak{TP}$-width at most k if and only if $G - x$ has $\mathfrak{TP}$-width at most k for any $x \in S$.*

Proof. Assume that G has a false twinset $\{x, y, z\}$. Assume that $G - x$ has an embedding H. If one of y and z is a nonprobe in H, then we can make a copy for x as a true twin. Note that creating a true twin does not introduce a P_4 or C_4 so the new embedding is also trivially perfect. Now assume that both y and z are probes. Then their neighborhood in H must induce a clique. We may now add x as a false twin of y and z in H. Note that also in this case no P_4 or C_4 is introduced.

Assume that G has a true twinset S with $k+2$ vertices. Let $x \in S$ and let H be an embedding of $G - x$. Since $S - x$ is a clique there exists an ordering of the vertices of $S - x$ such that for every pair $a, b \in S - x$, $N_H[a] \subseteq N_H[b]$ or $N_H[b] \subseteq N_H[a]$. Let y be the smallest vertex in this ordering. If y has a neighbor in H which is not a neighbor in G, then this is a new neighbor of all the vertices in $S - x$. This is a contradiction, since $S - x$ is a clique with $k+1$ vertices, and creating a common neighbor for $S - x$ would require $k+1$ independent sets. □

Definition 11. *A k-witness $\mathcal{N}$ is* well-linked *if for every $\mathbb{N}_i \in \mathcal{N}$, every vertex $x \notin \mathbb{N}_i$ has a neighbor in $\mathbb{N}_i$.*

Lemma 4. *Every k-probe trivially-perfect graph has a witness with k independent sets which is well-linked.*

Proof. Starting with any witness, repeatedly add a vertex x to an independent set $\mathbb{N}_i$ if it has no neighbor in that set. □

[2] A *true twin* is a pair of vertices x and y with $N[x] = N[y]$. A *false twin* is a pair of vertices x and y with $N(x) = N(y)$.

Lemma 5 (The Well-Linkedness Lemma). *Let $(G, \mathcal{N})$ be a k-probe trivially-perfect graph with a well-linked witness $\mathcal{N}$ and corresponding labeling α. Let H be an embedding. For every nonadjacent pair x and y in G with $N_H(x) \subseteq N_H[y]$,*

$$(x) \preceq (y) \quad \Leftrightarrow \quad \alpha(x) \geq \alpha(y)$$

Proof. Assume $\alpha(x) \geq \alpha(y)$. Let $z \in N_G(x)$. Then $z \in N_H[y]$. Since x and y are not adjacent, $z \neq y$. Thus $z \in N_H(y)$. If $z \notin N_G(y)$, then there exists an i with $\{z, y\} \subseteq \mathbb{N}_i$. Now $\alpha(x) \geq \alpha(y)$ implies that also $x \in \mathbb{N}_i$, which contradicts that z is adjacent to x. Hence $(x) \preceq (y)$.

Assume $(x) \preceq (y)$, that is, $N_G(x) \subseteq N_G(y)$. *A fortiori*, x and y are not adjacent. Assume $\neg(\alpha(x) \geq \alpha(y))$. Then there exists an i with $y \in \mathbb{N}_i$ and $x \notin \mathbb{N}_i$. Since $\mathcal{N}$ is well-linked, there exists a vertex $z \in N_G(x) \cap \mathbb{N}_i$. Since $(x) \preceq (y)$, $z \in N_G(y)$, contradicting that z and y are both in $\mathbb{N}_i$. □

Definition 12. *Let (X, α) be a probe module. A vertex $\gamma \in N(X)$ is X-*minimal *if there exists no $y \in N(X)$ with $(y) \neq (\gamma)$ and $(y) \preceq (\gamma)$ and also no $z \in N(X)$ with $[z] \neq [\gamma]$ and $[z] \preceq [\gamma]$.*

Remark 4. Notice that X-minimality of a vertex is independent of the actual labeling of the probe module (X, α).

Lemma 6. *Assume G has no k-twinset. Assume that (X, α) embeds as a branch in the tree-model of a well-linked embedding. Let Υ be the set of X-minimal vertices. Then $|\Upsilon| \leq 2^{k+1} + k - 1$.*

Proof. Consider a well-linked embedding H. Let T be a tree-model of H. Consider the path M from the root to the ancestor of X in T and let $M_0, M_1, \ldots$ be a partition of M into modules. By minimality of the embedding we may assume that each vertex of M_i has a neighbor in every subtree of M_i. Assume they are ordered such that $N_H[x_i] \subset N_H[x_{i+1}]$ for each $x_i \in M_i$ and $x_{i+1} \in M_{i+1}$, for $i = 0, 1, \ldots$.

Notice that each label-set of each M_s is a module in G. Since there is no k-twinset, each label-set of nonprobes has at most 2 vertices and each label-set of probes has at most $k + 1$ vertices. Thus

$$|M_s| \leq 2(2^k - 1) + (k + 1) = 2^{k+1} + k - 1$$

By the Well-Linkedness Lemma, a vertex $x \in M_s$ is minimal if it has a label $\alpha(x)$ such that all other label-sets $\alpha' \geq \alpha(x)$ in $M_0, \ldots, M_s$ are empty. It follows that there are at most $\sum_{i=0}^{k} \binom{k}{i} = 2^k$ label-sets of minimal vertices, at most $2^k - 1$ of minimal nonprobes, each containing at most 2 elements, and at most one label-set of minimal probes, containing at most $k + 1$ elements. Thus the number of minimal elements is bounded by $2^{k+1} + k - 1$. □

Lemma 7. *Assume G is a connected k-probe trivially-perfect graph without k-twinset. Let (X, α) be a probe module that embeds as a branch into a well-linked embedding H. Let T be a tree-model of H and let M_0 be the lowest set of ancestors of X in T that forms a module in H. There exists a set Ω, of size $|\Omega| \leq 2^{2(k+1)}$ such that $M_0 \subseteq \Omega$. This set Ω can be computed in linear time.*

Proof. Start with $\Omega = \varnothing$. Repeatedly compute the set of X-minimal vertices in G, add them to Ω, and delete them from the graph. After at most 2^k repetitions, each label-set of M_0 is contained in Ω. Since each set of maximal elements has at most $2^{k+1} + k - 1$ vertices,

$$|\Omega| \leq 2^k(2^{k+1} + k - 1) \leq 2^{2k+1} + 2^{2k} \leq 2^{2(k+1)}$$ □

Definition 13. *A* pattern *is a cotree of a k-labeled trivially perfect graph such that for every internal node the characteristics of the two subtrees are incomparable.*

Remark 5. Consider a cotree of an embedding of a labeled graph (G, α). By Lemma 2, we may repeatedly prune branches for which the characteristic is contained in the characteristic of the other branch. The result is a pattern.

Lemma 8. *There are $O(2^{(k+3)2^{2^k}})$ non-isomorphic patterns.*

Proof. The characteristic of every internal node is the union of the characteristics of its children. This union is larger than the two constituent sets since those are incomparable. A binary tree with depth at most 2^k has at most $2^{2^k} - 1$ internal nodes. The number of binary trees with t internal nodes can be bounded by the Catalan number $C_t = \binom{2t}{t}\frac{1}{t+1}$. Thus the number of cotrees with $t + 1$ leaves is bounded by $2^t C_t \sim \frac{2^{3t}}{t^{3/2}\sqrt{\pi}}$. There are at most $2^{k(t+1)}$ labelings for the leaves. Thus the number of patterns is bounded by $c2^{kc}\frac{2^{3c}}{c^{3/2}\sqrt{\pi}}$, where $c = 2^{2^k}$. □

Remark 6. Similar to Lemma 6 it can be shown that the number of feasible, incomparable induced patterns of branches incident with an internal node of the cotree is bounded by a constant. Actually, this proves a well-quasi-ordering, which implies a finite set of forbidden induced subgraphs for $\mathfrak{TP}$-width $\leq k$. We elaborate on this in the full version of this paper.

Theorem 8. *For each k, there exists an $O(n^3)$-time algorithm for the recognition of k-probe trivially-perfect graphs.*

Proof. Consider a partition $(M_0, M_1, \ldots, M_s)$ of the vertices into probe modules. Initially, each module consists of a single vertex. For each probe module we keep the possible embeddings, either as a clique-module or as a branch, as a list of patterns. The algorithm tries to merge modules into new modules. By Lemma 7 there are only a constant number of possible extensions for each module. Assume that a probe module (X, α) unions with some other probe modules. If there is a module (Y, β) with $\chi(Y) \subseteq \chi(X)$ then (Y, β) merges together with (X, α) for those labelings. There are at most 2^k module extensions in which the characteristic enlarges. A suitable merge of two probe modules can be found in $O(n^2)$ time. □

6 Conclusion

So far, we have limited our research to classes of graphs that have bounded rankwidth. For classes such as threshold graphs and cographs we were able to show that the width parameter is fixed-parameter tractable. One of the classes for which this is still open is the class of distance-hereditary graphs.

Acknowledgement

Ton Kloks is currently a guest of the Department of Computer Science and Information Engineering of National Chung Cheng University. He gratefully acknowledges the funding for this research by the National Science Council of Taiwan and the hospitality of the department. This research is supported by the National Science Council of Taiwan under grant NSC 97–2221–E–194–055.

References

1. Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph classes: A survey. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (1999)
2. Chang, M.-S., Hsieh, S.-Y., Chen, G.-H.: Dynamic programming on distance-hereditary graphs. In: Leong, H.-V., Jain, S., Imai, H. (eds.) ISAAC 1997. LNCS, vol. 1350, pp. 344–353. Springer, Heidelberg (1997)
3. Courcelle, B., Oum, S.: Vertex minors, monadic second-order logic, and a conjecture by Seese. Journal of Combinatorial Theory, Series B 97, 91–126 (2007)
4. Golumbic, M.C.: Trivially perfect graphs. Discrete Mathematics 24, 105–107 (1978)
5. Howorka, E.: A characterization of distance-hereditary graphs. Quarterly Journal of Mathematics 28, 417–420 (1977)
6. Kruskal, J.: Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture. Transactions of the American Mathematical Society 95, 210–225 (1960)
7. Möhring, R., Radermacher, F.: Substitution decomposition for discrete structures and connections with combinatorial optimization. In: Burkard, R., Cuninghame-Green, R., Zimmermann, U. (eds.) Algebraic and Combinatorial Methods in Operations Research. Annals of Discrete Mathematics, vol. 19. Mathematics studies, vol. 95, pp. 257–355. Elsevier/North-Holland, Amsterdam (1984)
8. McMorris, F.R., Wang, C., Zhang, P.: On probe interval graphs. Discrete Applied Mathematics 88, 315–324 (1998)
9. Orlin, J.: Contentment in graph theory: covering graphs with cliques. Indagationes Mathematicae 39, 406–424 (1977)
10. Oum, S.: Graphs of bounded rank-width, PhD Thesis, Princeton University (2005)
11. Pouzet, M.: Applications of well-quasi-ordering and better-quasi-ordering. In: Rival, I. (ed.) Graphs and Order, pp. 503–519. D. Reidel Publishing Company (1985)
12. Thomassé, T.: On better-quasi-ordering countable series-parallel orders. Transactions of the American Mathematical Society 352, 2491–2505 (2000)
13. Wolk, E.S.: A note on The comparability graph of a tree. Proceedings American Mathematical Society 16, 17–20 (1965)
14. Zhang, P., Schon, E.A., Fischer, S.G., Cayanis, E., Weiss, J., Kistler, S., Bourne, P.E.: An algorithm based on graph theory for the assembly of contigs in physical mapping of DNA. In: CABIOS, vol. 10, pp. 309–317 (1994)