

Chapter 5

Superzeta Functions: an Overview

In view of the *central symmetry* of the set of Riemann zeros, $\rho \leftrightarrow 1 - \rho$, which crucially reflects the functional equation (3.26), there is no clear reason to pick either the ρ themselves (with $\rho \equiv \frac{1}{2} \pm i\tau_k$, $\text{Re } \tau_k > 0$) or the $\rho(1 - \rho) = \frac{1}{4} + \tau_k^2$ as the basic set upon which to build zeta functions over the Riemann zeros. This is a nonlinear remapping freedom as discussed in Sect. 1.1, which leads to generalized zeta functions of several kinds. This chapter gives an introductory overview of the various possibilities. In decreasing order of analytical tractability, and in our older notation to be soon discarded, we may define:

- Functions of first kind: $\sum (x - \rho)^{-s}, \quad \text{Re } s > 1;$
- Functions of second kind: $\sum_{k=1}^{\infty} (\tau_k^2 + v)^{-\sigma}, \quad \text{Re } \sigma > \frac{1}{2};$
- Functions of third kind: $\sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \text{Re } s > 1.$

In all three kinds the *argument* (in exponent) is the principal variable, the other one a shift parameter. Each parametric family starts as an analytic function of its argument in its indicated half-plane of convergence, due to (4.7) or (4.26), but it will extend to a *meromorphic function* in the entire complex plane (of its argument), with a *computable singular part*. The domain of each shift parameter is left as large as possible: simply, none of the quantities to be raised to a power should lie on the cut \mathbb{R}_- ; this will hardly matter with the fairly small parameter values that we will emphasize.

The first two families will moreover display computable *special values*, i.e., values at *integer* arguments, with explicit relationships between the two families at those arguments. The third family forcibly selects the zeros in only one half-plane, thus breaking the central symmetry for any parameter $\tau \neq 0$; then it will show a more singular analytic structure than the other two, and no known special values; it will be discussed more briefly.

We now define each superzeta family more precisely with its immediate properties, introducing *new notation* henceforth for greater coherence.

5.1 First Kind (\mathcal{Z})

$$\mathcal{Z}(s|t) = \sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-s} \equiv \sum_{\rho} \left(\rho + t - \frac{1}{2}\right)^{-s}, \quad \operatorname{Re} s > 1 \quad (5.1)$$

is the simplest generalized zeta function over the Riemann zeros: the sum runs over all zeros symmetrically, and t is just a shift parameter varying in the complex plane of the function $t \mapsto \zeta(\frac{1}{2} + t)$. With the function $z \mapsto z^{-s}$ defined in the cut plane $z \in \mathbb{C} \setminus \mathbb{R}_-$, the t -domain for (5.1) is then (Fig. 5.1)

$$\Omega_1 = \{t \in \mathbb{C} \mid t \pm i\tau_k \notin \mathbb{R}_- \ (\forall k)\}. \quad (5.2)$$

The equality between the two sums in (5.1) expresses the *central symmetry* (of Riemann's Functional Equation). Were the function $z \mapsto z^{-s}$ single-valued in the *uncut* z -plane, then that equality would translate into the functional equation " $\mathcal{Z}(s|t) \equiv (-1)^s \mathcal{Z}(s|-t)$," and this works indeed at (positive) *integers* s , see Table 7.2. For non-integer s , on the other hand, when the choice of a branch for $z \mapsto z^{-s}$ breaks the central symmetry, our closest related identity will be (5.11) which needs a *new* function \mathfrak{Z} (of third kind). We then see no finite closed functional relation between $\mathcal{Z}(s|t)$ and $\mathcal{Z}(s|-t)$, nor any other full translation of Riemann's Functional Equation. We will only find some curious further signs of the central symmetry: e.g., countably many sum rules like (7.46) below will constrain the special values $\mathcal{Z}(2n+1|t)$, $n \in \mathbb{N}$, now all *at the same* t . So in brief, for this first family \mathcal{Z} :

To see or not to see (the central symmetry fully), that is the question.

As the parameter t "lives" in the complex plane of $\zeta(\frac{1}{2} + t)$, we expect privileged locations to be $t = 0$ (the center of symmetry), and $t = \frac{1}{2}$ (the pole); thus, the simple sum $\sum_{\rho} \rho^{-n}$ is $\mathcal{Z}(n|t = \frac{1}{2})$. Indeed, the special values reduce further at those parameter values. We then add two shorter names for convenience:

$$\mathcal{Z}_0(s) \stackrel{\text{def}}{=} \mathcal{Z}(s|0), \quad \mathcal{Z}_*(s) \stackrel{\text{def}}{=} \mathcal{Z}(s|\frac{1}{2}). \quad (5.3)$$

(As for $t = -\frac{1}{2}$, its special values can easily be drawn from the case $t = +\frac{1}{2}$ by using the central symmetry, see later Table 7.2.)

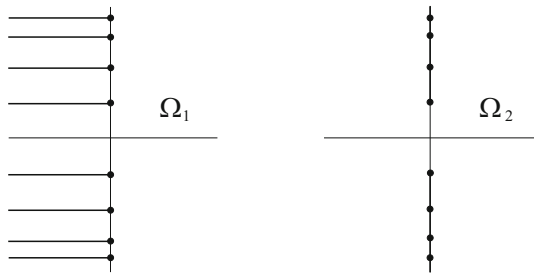
The family of first kind \mathcal{Z} is covered in detail in Chap. 7.

5.2 Second Kind (\mathcal{Z})

$$\mathcal{Z}(\sigma|t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}, \quad \operatorname{Re} \sigma > \frac{1}{2}, \quad (5.4)$$

is now defined in the natural t -domain (Fig. 5.1)

Fig. 5.1 Domains of definition in the t -plane (schematic views) for \mathcal{Z} (left) and \mathcal{Z} (right)



$$\Omega_2 = \{t \in \mathbb{C} \mid t \pm i\tau_k \notin \pm i\mathbb{R}_- \ (\forall k)\}. \quad (5.5)$$

In view of the factorization $\tau_k^2 + t^2 = (t + i\tau_k)(t - i\tau_k)$, this parameter t identifies indeed with the previous one in $\mathcal{Z}(s|t)$.

This family \mathcal{Z} has one major asset: it fully embodies the central symmetry $\tau_k \leftrightarrow -\tau_k$ (or Riemann's Functional Equation), through the identity

$$\mathcal{Z}(\sigma|t) \equiv \mathcal{Z}(\sigma|-t). \quad (5.6)$$

But otherwise it proves more singular than the first family \mathcal{Z} , and also harder to analyze (just as, say, the function $\sum_k (k^2 + a^2)^{-\sigma}$ vs $\sum_k (k+a)^{-s} = \zeta(s, a)$). We cannot reduce it to the first family (see below), and it is better suited for some purposes (e.g., in Chap. 11), so that both families are worth studying. Our results for \mathcal{Z} are now fairly on par with those for the first family, but this required multiple angles of attack which we did not find all at once. This operational difficulty has been our main problem with the second family \mathcal{Z} .

Since the parameter t has preserved its meaning, we add two more shorthand names to match (5.3),

$$\mathcal{Z}_0(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma|0), \quad \mathcal{Z}_*(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma|\frac{1}{2}). \quad (5.7)$$

Just as between $\mathcal{Z}(s|-t)$ and $\mathcal{Z}(s|t)$ before, we see no obvious finite functional relation between $\mathcal{Z}(\sigma|t)$ and $\mathcal{Z}(s|t)$, with two exceptions:

- The central parameter location $t = 0$, which is the (only) *confluence point* of the two families \mathcal{Z} and \mathcal{Z} , according to

$$\mathcal{Z}_0(\sigma) \equiv (2 \cos \pi\sigma)^{-1} \mathcal{Z}_0(2\sigma); \quad (5.8)$$

- Integer arguments: The sets of special values $\mathcal{Z}(m|t)$ and $\mathcal{Z}(n|t)$ ($m, n = 1, 2, \dots$) will display mutual linear relationships (Sect. 8.5). Here the computable special values of $\mathcal{Z}(\sigma|t)$ do refer to all integers σ just as those of $\mathcal{Z}(s|t)$ refer to all integers s . Otherwise, just as we immediately did with the parameter t , we could have tied both \mathcal{Z} and \mathcal{Z} to a common argument: but

the only meaningful identification, $s \equiv 2\sigma$ by (5.8), does not preserve integers. Hence the issue of special values prompts us to keep distinct arguments, s for \mathcal{Z} , σ for \mathcal{Z} .

The family of second kind \mathcal{Z} is covered in detail in Chap. 8.

5.3 Third Kind (3)

$$\mathfrak{Z}(s|\tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \operatorname{Re} s > 1, \quad \tau + \tau_k \notin \mathbb{R}_- \quad (\forall k). \quad (5.9)$$

Although this family explicitly breaks the central symmetry, it is actually expressible in terms of the family \mathcal{Z} of the first kind and vice-versa: for $|t|, |\tau| < \tau_1$ (to avoid overlapping cuts),

$$\mathcal{Z}(s|t) = e^{i\pi s/2} \mathfrak{Z}(s|it) + e^{-i\pi s/2} \mathfrak{Z}(s|-it), \quad (5.10)$$

which inverts as

$$\mathfrak{Z}(s|\tau) = \frac{1}{2i \sin \pi s} \left[e^{i\pi s/2} \mathcal{Z}(s|-i\tau) - e^{-i\pi s/2} \mathcal{Z}(s|+i\tau) \right]. \quad (5.11)$$

This family shares the confluence point of the other two:

$$\mathfrak{Z}(s|0) \equiv \mathcal{Z}_0(\tfrac{1}{2}s) \equiv (2 \cos \tfrac{1}{2}\pi s)^{-1} \mathcal{Z}_0(s) = \sum_{k=1}^{\infty} \tau_k^{-s}. \quad (5.12)$$

Otherwise (for $\tau \neq 0$), \mathfrak{Z} proves more singular than \mathcal{Z} or even \mathcal{Z} , hence we will only view it as subordinate to \mathcal{Z} through (5.11), see Chap. 9.

5.4 Further Generalizations (Lerch, Cramér, ...)

A Lerch-like generalization (cf. (3.38)) is

$$Z(s; x) = \sum_{\rho} x^{\rho} \rho^{-s} \quad (\text{e.g., } x > 0) \quad (5.13)$$

with numerous possible variants (allow complex x ; replace ρ^{-s} by $(\rho + a)^{-s}$, $|\rho|^{-s}$, ...; select $\operatorname{Im} \rho \geq 0$ or not). At $s = 0$, the family also captures another interesting symmetric function of the zeros, Cramér's V -function [25, Chap. I]:

$$V(z) = \sum_{\operatorname{Im} \rho > 0} e^{\rho z} \quad \text{for } \operatorname{Im} z > 0. \quad (5.14)$$

Concerning $V(z)$, it is mainly known [25, 29, 44, 54, 59] that the function $V(z) - (2\pi i)^{-1} \frac{\log z}{1 - e^{-z}}$ is meromorphic in all of \mathbb{C} , with a functional equation, and with computable: poles (all simple), residues, and finite part (but nothing more) at $z = 0$. Thus, $V(z)$ itself has a rather irregular, partially known (and hard to get) structure at $z = 0$; so we do not advocate it as a basic tool. (It is actually nicer for zeros of Selberg zeta functions! [12, 15] [16, Sect. 5] [105, Sect. 4])

Formally, the broader family $Z(s; x)$ recovers our superzeta functions at $x = 1$; however, these form a singular case. Hence the general- x family is better seen as a vast subject of its own, which we will not address here either: see, e.g., [40, 68, 83], [25, Chap. II.I], [61, Chap. VI].

5.5 Other Studies on Superzeta Functions

The present monograph grew up from articles written by us around 2001–2005. It thus makes sense here to place a dividing line between “older” and “current” literature around 2000 (a pivotal year, anyway).

We found it difficult to trace the older literature on superzeta functions. To wit, the past publications touching those functions have been very sporadic (we laboriously collated but a dozen articles, spanning almost one century) and disconnected (the works largely ignore one another); even when superzeta functions were addressed, it was never systematically and often briefly: e.g., amidst various examples. So overall, occurrences of our subject have tended to be rare and inconspicuous. We then offer the following survey truly “to the best of our knowledge”. We had to change most of the original notation to make it globally consistent here: any translation between an earlier notation “ A ” and our present notation “ B ” is provided below as $A \equiv B$.

The most ancient occurrences of this sort of symmetric functions over the Riemann zeros concern the more general functions $Z(s; x)$ of (5.13) in works of Landau [68], Mellin [83] and Cramér [25]. Out of those works, only [83] truly touches “superzeta” functions (i.e., cases for which $x = 1$), featuring:

- Two instances of our “functions of the 1st kind,”

$$Z(s) \equiv \mathcal{Z}(s | \tfrac{1}{2}), \quad \overline{Z}(s) \equiv \mathcal{Z}(s | -\tfrac{1}{2}); \quad (5.15)$$

- Two instances of our “functions of the 3rd kind,”

$$Z_1(s) \equiv e^{i\pi s/2} \mathfrak{Z}(s | \tfrac{1}{2}i), \quad Z_2(s) \equiv e^{-i\pi s/2} \mathfrak{Z}(s | -\tfrac{1}{2}i); \quad (5.16)$$

- Meromorphic continuation formulae for those four cases, which exemplify some of our later equations: successively, (7.7) at $t = +\frac{1}{2}$, (7.4) at $t = -\frac{1}{2}$, (9.4) at $\tau = -\frac{1}{2}i$ and at $\tau = +\frac{1}{2}i$;

- Explicit formulae for the polar parts of $Z(s)$ and $\bar{Z}(s)$, plus the rational special values at $s = -n$, $n \in \mathbb{N}$, but the latter [83, (12)–(13)] are affected by obvious computational mistakes and contradicted by our own results (upper parts of Tables 7.1 and 7.4, from [108]).

Overall, Mellin’s article ([83], 1917) is quite pioneering and fundamental, yet it is hardly ever referred to and never for “superzeta” functions; to make it more accessible, we provide its English translation here as Appendix D.

The next early occurrences of a superzeta function all concern the *confluent* case $t = 0$ as in (5.12). For the corresponding function $\sum_k \tau_k^{-s}$, Guinand [43, Sect. 4(A)] gave a restricted form of the functional relation (8.7), moreover assuming the Riemann Hypothesis, an assumption later removed by Chakravarty [17]; while Delsarte [27, Sect. 7] proved that same function (called by him $\phi(s)$) to be meromorphic in the whole s -plane.

Much later, cases with $t = \frac{1}{2}$ drew some attention, namely those we call $\mathcal{Z}_*(s)$ (first kind) and $\mathcal{Z}_*(\sigma)$ (second kind).

- For \mathcal{Z}_* , only the special values $\sigma_n \equiv \mathcal{Z}_*(n)$ ($= \sum_\rho \rho^{-n}$, $n = 1, 2, \dots$) got considered: the value $\mathcal{Z}_*(1)$, as in (4.14), is the sole *classic* “superzeta” occurrence (i.e., visible in some textbooks); results for $n > 1$ were given much later, by Matsuoka [82], Lehmer [72], Keiper [64], Zhang and Williams [114].
- As for \mathcal{Z} : the function $\zeta(s, \Delta_F) \equiv \mathcal{Z}_*(\sigma)$ for the more general Dedekind zeta functions (cf. Chap. 10) was proved by Kurokawa [66] to be meromorphic in the whole σ -plane while Matiyasevich [80] studied the special values $\theta_m \equiv 2 \mathcal{Z}_*(m)$ ($m = 1, 2, \dots$); $\mathcal{Z}_*(1)$ and $\mathcal{Z}_*(2)$ resurfaced lately in studies on the distribution of primes [93, pp. 191–193][71, (4.12)–(4.13)].

A full-fledged family arose still later, specifically \mathcal{Z} (first kind). (Earlier it came up only in exercises posed by Patterson [89, Ex. 3.5–7].) Deninger [28, Theorem 3.3] for $\text{Re } z > 1$, followed by Schröter and Soulé [96] for general z , described the meromorphic continuation in s of $\xi(s, z) \equiv (2\pi)^s \mathcal{Z}(s | z - \frac{1}{2})$, but with focus on the parametric special value $z \mapsto \xi'(0, z)$.

We found no results on the full second family \mathcal{Z} prior to our work [106].

Then quite lately, the third family \mathfrak{Z} was proved a meromorphic function of its argument, independently by Hirano, Kurokawa and Wakayama [51, Sect. 3] (as $Z(w, x) \equiv \mathfrak{Z}(w | -x)$) and by us [106, Sect. 6.2].

As for *extended* superzeta functions, i.e., over zeros of more general zeta- and L -functions than Riemann’s, their even smaller bibliography will be given in Chap. 10, which is devoted to this extension.

So, our topic of “superzeta functions” has benefited from some seminal contributions. Yet, these are scarce (a dozen, spread over three families and nearly one century), and almost each of them treats an isolated case under a single aspect: typically meromorphic continuation (often just qualitatively), or particular instances of special values. We thus feel that those early works, even taken together, do not do full justice to the subject.

We therefore embarked on a more systematic and dedicated coverage, but this emerged only gradually through three papers. In [106] we described the full families \mathcal{Z} and \mathfrak{Z} and only sketched the family \mathcal{Z} , which became the main

focus of [108]. In [107] we treated all three families more efficiently but in general notation, to accommodate other zeta and L -functions. Overall, these three articles do not readily fuse into a neat coherent whole; their notation and methods can also be improved. The core of the present work aims at giving an updated synthesis (including a few new results) of the current state of the subject.