Chapter 4
Riemann Zeros and Factorizations of the Zeta Function

This is the second half of our review on the basic properties of $\zeta(x)$.

4.1 Growth Properties of $\zeta(x)$ and $\Xi(x)$

The theory of entire functions of *finite order* (cf. Sect. 2.1) applies to Riemann's Ξ function in a classic way [26, Sect. 12] [89, Appendix 5].

We first bound $\zeta(x)$ and the trivial factor $\mathbf{G}^{-1}(x)\,(x-1)$ separately in the half-plane $\{\mathrm{Re}\ x \geq \frac{1}{2}\}$.

Applying the Euler–Maclaurin formula (1.14) to $f(u) = u^{-x}$ with Re $x > 1$ and $K = 1$, $K' = +\infty$, yields (with $\{u\} \stackrel{\mathrm{def}}{=}$ the fractional part of u here)

$$\zeta(x) = \frac{1}{x-1} + \tfrac{1}{2} - x \int_1^\infty B_1\big(\{u\}\big)\, u^{-x-1} \mathrm{d}u; \tag{4.1}$$

but as the right-hand side converges and defines an analytic function for Re $x > 0$, it analytically continues $\zeta(x)$ to this half-plane. The integral is bounded by $\int_1^\infty \frac{1}{2} u^{-\mathrm{Re}\ x - 1} \mathrm{d}u = (2\,\mathrm{Re}\ x)^{-1} \leq 1$ if Re $x \geq \frac{1}{2}$, hence as $x \to \infty$ in the latter half-plane, the bound $\zeta(x) = \mathrm{O}(|x|)$ holds.

On the other hand, exponentiating Stirling's formula (3.51), we see that $|\mathbf{G}^{-1}(x)\,(x-1)| < \mathrm{e}^{C|x| \log |x|}$ for any $C > 1$.

Hence in the half-plane $\{\mathrm{Re}\ x \geq \frac{1}{2}\}$,

$$|\Xi(x)| < \mathrm{e}^{C|x| \log |x|} \qquad (\forall C > 1); \tag{4.2}$$

but this final bound, which concerns Ξ, therefore extends to the whole x-plane thanks to the Functional Equation (3.26). At the same time,

$$\log \Xi(x) \sim \tfrac{1}{2} x \log x \qquad \text{for } x \to +\infty \tag{4.3}$$

(using Stirling's formula again, plus $\zeta(x) \sim 1$ for $x \to +\infty$). All that fixes the order of the entire function $\Xi(x)$ to be precisely $\mu_0 = 1$ in the variable x.

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By the Functional Equation, Ξ is also an entire function in the variable $u = x(x-1)$; its order (relative to this new variable) is then $\frac{1}{2}$, and this seemingly trivial rephrasing proves quite useful for certain issues.

4.2 The Riemann Zeros (Basic Features)

By the general theory of entire functions of finite order [10, Chap. 2], a function like Ξ which has a *non-integer* order ($\frac{1}{2}$, in the variable $u = x(x-1)$) must have *infinitely many zeros* (a statement invariant under this change of variables $x \leftrightarrow u$). The zeros of $\Xi(x)$ are then *nontrivial zeros* for $\zeta(x)$ (besides the trivial set $\{-2k\}$): they are the *Riemann zeros*, usually labeled ρ.

The Euler product (3.5), convergent for Re $x > 1$, implies $\zeta(x) \neq 0$ there. This plus the central symmetry restrict the ρ to lie in the strip $0 \leq \mathrm{Re}\, x \leq 1$.

In fact, $\zeta(x)$ does not vanish on the line Re $x = 1$ either. While this is a key step toward the *prime number theorem*, a major arithmetical result, both the fact itself and its derivation seem to lie off our main analytical track; so, we merely quote the result (proved in all books). This further restricts the Riemann zeros to the open *critical strip* $0 < \mathrm{Re}\, x < 1$.

No zeros can lie on the real segment (0,1) either: indeed,

$$(1 - 2^{1-x})\,\zeta(x) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-x} \qquad \text{converges for Re } x > 0,$$

and for all $x > 0$ this series is alternating and shows a *strictly* positive sum.

By central symmetry, the Riemann zeros occur in pairs $(\rho, 1-\rho)$. To make this symmetry explicit, we will rather enumerate them in pairs as

$$\{\rho = \tfrac{1}{2} \pm \mathrm{i}\tau_k\}_{k=1,2,\ldots}, \qquad \mathrm{Re}\,\tau_k > 0 \text{ and non-decreasing,} \tag{4.4}$$

and also parametrize each such pair by the single number

$$u_k = \rho(1-\rho) = \tfrac{1}{4} + {\tau_k}^2. \tag{4.5}$$

The *Riemann Hypothesis* (1859, still open) states that [92]

$$\text{(RH)} \qquad \mathrm{Re}\,\rho = \tfrac{1}{2} \quad (\forall \rho) \iff (\forall k) \quad \tau_k \in \mathbb{R}_+ \iff u_k - \tfrac{1}{4} \in \mathbb{R}_+; \tag{4.6}$$

all our arguments will be *unconditional* (independent of RH) up to Chap. 11.

Another unproved conjecture is that all zeros are *simple*, but we adopt the standard convention of counting all objects with their multiplicities if any.

The first zeros' approximate ordinates are numerically found as

$$\{\tau_k \approx 14.1347251,\ 21.0220396,\ 25.0108576,\ 30.4248761,\ 32.9350616, \ldots\}$$

4.3 Hadamard Products for $\Xi(x)$

By the general theory of entire functions of finite order as referred to in Sect. 2.1, in the case of $\Xi(x)$ of order 1, the zeros ρ satisfy $\sum_\rho |\rho|^{-1-\varepsilon} < \infty$ if $\varepsilon > 0$, and $\Xi(x)$ admits the Hadamard factorization

$$\Xi(x) = \Xi(0)\, \mathrm{e}^{[\log \Xi]'(0)\, x} \prod_\rho \Big(1 - \frac{x}{\rho}\Big)\, \mathrm{e}^{x/\rho} \qquad (\forall x \in \mathbb{C}).$$

Were $\sum |\rho|^{-1}$ to converge, this and the general bound $|1-z| \le \mathrm{e}^{|z|}$ would imply $\Xi(x) = \mathrm{O}(\mathrm{e}^{C|x|})$ for some C and all x, which is contradicted by (4.3). Hence

$$\sum_\rho |\rho|^{-1-\varepsilon} \quad \text{converges for } \varepsilon > 0, \text{ diverges for } \varepsilon = 0. \tag{4.7}$$

This growth estimate for the zeros (*note*: $\sum |\rho|^{-1} = \infty$ is a *tangible* proof of their infinite number) will be refined to the *Riemann–von Mangoldt formula* soon (Sect. 4.5), but it is needed meanwhile.

Here, using (3.24) and the special values $\Xi(0) = 1$ and (3.21), the above Hadamard formula boils down to

$$\Xi(x) = \mathrm{e}^{Bx} \prod_\rho \Big(1 - \frac{x}{\rho}\Big)\, \mathrm{e}^{x/\rho}, \qquad B \stackrel{\text{def}}{=} \log 2\sqrt{\pi} - 1 - \tfrac{1}{2}\gamma \tag{4.8}$$

$$\text{or} \quad \zeta(x) = \frac{1}{x-1}\, \mathbf{G}(x)\Xi(x) \equiv \frac{\mathrm{e}^{(\log 2\pi - 1 - \gamma/2)x}}{2(x-1)\Gamma(1+\frac{1}{2}x)} \prod_\rho \Big(1 - \frac{x}{\rho}\Big)\, \mathrm{e}^{x/\rho}. \tag{4.9}$$

The logarithmic derivative of the latter will also serve,

$$\frac{\zeta'}{\zeta}(x) \equiv (\log 2\pi - 1 - \tfrac{1}{2}\gamma) - \frac{1}{x-1} - \tfrac{1}{2}\psi(1 + \tfrac{1}{2}x) + \sum_\rho \Big[\frac{1}{x-\rho} + \frac{1}{\rho}\Big]. \tag{4.10}$$

Those are the classic forms given everywhere, for which the ordering of the factors is immaterial (they converge absolutely for all $x \in \mathbb{C}$). However, we will critically need more symmetrical forms, which require the zeros to be taken *in pairs* $(\rho, 1-\rho)$. So from now on, all symbols $\prod_\rho$ for products or $\sum_\rho$ for summations crucially *imply this grouping of zeros in pairs*.

Thereupon, following, e.g., [31, Sects. 1.10 and 2.8]: first,

$$\sum_\rho \frac{1}{\rho} = \sum_k \Big[\frac{1}{\frac{1}{2} + \mathrm{i}\tau_k} + \frac{1}{\frac{1}{2} - \mathrm{i}\tau_k}\Big] = \sum_k \frac{1}{\frac{1}{4} + \tau_k^2} = \sum_k \frac{1}{u_k} \tag{4.11}$$

(the grouping restores absolute convergence). This at once motivates the introduction of *two kinds* of zeta functions over the Riemann zeros; anticipating our future notation (Chap. 5), we set $\mathscr{Z}_*(s) \stackrel{\text{def}}{=} \sum_\rho \rho^{-s}$ and

$\mathscr{Z}_*(\sigma) \stackrel{\text{def}}{=} \sum_k u_k^{-\sigma}$, which translates (4.11) to $\mathscr{Z}_*(1) = \mathscr{Z}_*(1)$. Then, this other pair evaluation:

$$\Big(1 - \frac{x}{\rho}\Big)\Big(1 - \frac{x}{1-\rho}\Big) \equiv 1 - \frac{x(1-x)}{\frac{1}{4} + \tau^2}, \tag{4.12}$$

substituted into the initial Hadamard product (4.8) for $\Xi(x)$, turns it into

$$\Xi(x) = \mathrm{e}^{\left[B + \sum_\rho \frac{1}{\rho}\right] x} \prod_\rho \Big(1 - \frac{x}{\rho}\Big) = \mathrm{e}^{[B + \mathscr{Z}_*(1)]\, x} \prod_k \Big[1 - \frac{x(1-x)}{u_k}\Big]. \tag{4.13}$$

Now, the symmetry of the Functional Equation $\Xi(x) = \Xi(1-x)$ requires the exponents in (4.13) to vanish, or $\mathscr{Z}_*(1) = -B$. All that yields the identities

$$\mathscr{Z}_*(1) = \mathscr{Z}_*(1) = -B \equiv -[\log \Xi]'(0) = 1 + \tfrac{1}{2}\gamma - \tfrac{1}{2}\log 4\pi \tag{4.14}$$
$$(\approx 0.0230957090).$$

This result appears in *some* textbooks [26, Sect. 12, (10)–(11)] [31, 3.8(4)] [89, Sect. 3.1], but in isolation; we will see that it actually initiates *two infinite sequences* of such formulae for all the special values $\mathscr{Z}_*(k)$ and $\mathscr{Z}_*(k)$ ($k = 1, 2, \ldots$), which admit parametric generalizations as well (Chaps. 7–8).

Finally, (4.13) reduces to two manifestly symmetric product forms, less common than (4.8) [31, Sect. 2.8] [89, Sect. 3.1] but more useful for us later,

$$\Xi(x) = \prod_\rho \Big(1 - \frac{x}{\rho}\Big) \qquad \text{with zeros grouped in pairs} \tag{4.15}$$

$$\equiv \prod_{k=1}^{\infty} \Big[1 + \frac{u}{u_k}\Big], \qquad u \stackrel{\text{def}}{=} x(x-1) \tag{4.16}$$

(the latter is just a Hadamard product for Ξ as a function of u of order $\frac{1}{2}$).

4.4 Basic Bounds on ζ'/ζ

An important class of results to come, like the Riemann–von Mangoldt formula (Sect. 4.5) and the Explicit Formulae (Sect. 6.2) (plus our basic analytical continuation formula for a zeta function over the Riemann zeros in Chap. 7), require some control over the growth of ζ'/ζ in the complex plane. The problematic region is clearly about the critical strip, where the unknown Riemann zeros lie; if $\mathrm{Re}\, x > x_0$ for some $x_0 > 1$ then ζ'/ζ will be trivially bounded (and the symmetrical half-plane $\{\mathrm{Re}\, x < 1 - x_0\}$ is thereupon controlled by applying the Functional Equation to ζ'/ζ: we will skip this).

Without loss of generality, we may now fix $x_0 = 2$ (the customary choice), and only consider the upper half-plane (by conjugation symmetry).

We follow the standard exposition, as condensed in [57, Sect. 1.4] for instance; usually, however, the bounds for $[\zeta'/\zeta](x)$ are intended to hold everywhere, whereas we focus on $\mathrm{Im}\ x \gg 1$, which simplifies some equations. The logic goes as follows: (1) bound ζ'/ζ for $\mathrm{Re}\ x \geq 2$, and specially at the edge; (2) using this, bound the fluctuations in the distribution of the zeros' ordinates $\{\mathrm{Im}\ \rho\}$; (3) in turn, use the latter results to bound $[\zeta'/\zeta](x)$ when $-1 < \mathrm{Re}\ x < 2$. We then develop those successive points in (partial) detail.

(1) The Dirichlet series (3.7) for $[\zeta'/\zeta](x)$ immediately implies

$$|\zeta'/\zeta|(x) \leq |\zeta'/\zeta|(2) = \mathrm{O}(1) \qquad \text{for all } x \text{ with } \mathrm{Re}\ x \geq 2. \tag{4.17}$$

(2) This in turn constrains the distribution of the zeros' ordinates to obey the following two bounds:

$$\sum_\rho \frac{1}{4 + (T - \mathrm{Im}\ \rho)^2} = \mathrm{O}(\log T), \tag{4.18}$$

$$\#\{\rho \mid |\mathrm{Im}\ \rho - T| < 1\} < C_1 \log T \qquad \text{for some } C_1 > 0. \tag{4.19}$$

Proof. Select $x_T = 2 + \mathrm{i}T$ (with $T \gg 1$). These inequalities obeyed by every zero ρ : $\mathrm{Re}\ \frac{1}{\rho} > 0$ (due to $|\arg \rho| < \frac{1}{2}\pi$), and

$$\mathrm{Re}\ \frac{1}{x_T - \rho} = \frac{2 - \mathrm{Re}\ \rho}{(2 - \mathrm{Re}\ \rho)^2 + (T - \mathrm{Im}\ \rho)^2} > \frac{1}{4 + (T - \mathrm{Im}\ \rho)^2}$$

(due to $0 < \mathrm{Re}\ \rho < 1$), entail

$$\sum_\rho \frac{1}{4 + (T - \mathrm{Im}\ \rho)^2} < \mathrm{Re} \sum_\rho \Big[\frac{1}{x_T - \rho} + \frac{1}{\rho}\Big];$$

but if we combine the former bound (4.17) and the Stirling formula (3.52) for ψ within the series expansion (4.10) for $[\zeta'/\zeta](x_T)$, we can extract

$$\sum_\rho \Big[\frac{1}{x_T - \rho} + \frac{1}{\rho}\Big] = \mathrm{O}(\log T) \qquad \text{for } x_T = 2 + \mathrm{i}T,$$

and (4.18) follows.

Now this bound secretly means that the ordinates of zeros *cannot cluster too much*: indeed, in conjunction with the obvious inequality

$$\sum_{|\mathrm{Im}\ \rho - T| < 1} \frac{1}{4 + (T - \mathrm{Im}\ \rho)^2} > \#\{\rho \mid |\mathrm{Im}\ \rho - T| < 1\} \times \frac{1}{5},$$

it furnishes the more concrete bound (4.19) (the argument can more generally yield $\#\{\rho \mid |\mathrm{Im}\ \rho - T| < h\} < C_h \log T$ for any fixed $h > 0$: the density of the zeros' ordinates gets bounded on any fixed scale). □

(3) Then the following two bounds hold uniformly for x on the segment $[-1 + \mathrm{i}T, 2 + \mathrm{i}T]$:
(a) Provided $T \gg 1$ is not the ordinate of a zero,

$$\frac{\zeta'}{\zeta}(x) = \sum_{|\mathrm{Im}\ \rho - T| < 1} \frac{1}{x - \rho} + \mathrm{O}(\log T); \tag{4.20}$$

(b) Within unit distance from any $T' \gg 1$ there exist "good" ordinates T that are "far enough" from all zeros' ordinates, in the sense that for some absolute constant $c > 0$,

$$|T - \mathrm{Im}\ \rho| > \frac{c}{\log T'} \qquad \text{for all zeros } \rho, \tag{4.21}$$

and then, for some absolute constant C,

$$\left|\frac{\zeta'}{\zeta}\right|(x) < C \log^2 T \quad \text{uniformly for } x \in [-1 + \mathrm{i}T, 2 + \mathrm{i}T]. \tag{4.22}$$

Proof. (a) Here we may slightly streamline the usual argument by making it *intrinsic* (in the sense of Sect. 2.1). We write, for $x_T = 2 + \mathrm{i}T$ as in (2),

$$\frac{\zeta'}{\zeta}(x) = \int_{x_T}^{x} \left[\frac{\zeta'}{\zeta}\right]'(y)\,\mathrm{d}y + \frac{\zeta'}{\zeta}(x_T),$$

where the second term is O(1) by (4.17), so it will not count here. Now (4.10), differentiated once more, refers to the *intrinsic* function $[\zeta'/\zeta]'$, implying

$$-\left[\frac{\zeta'}{\zeta}\right]'(y) = \sum_{\rho} \frac{1}{(y - \rho)^2} + \mathrm{O}\Big(\frac{1}{T}\Big); \tag{4.23}$$

but, uniformly in y on the line $\{\mathrm{Im}\ y = T\}$,

$$\Bigg|\sum_{|\mathrm{Im}\ \rho - T| \geq 1} \frac{1}{(y - \rho)^2}\Bigg| \leq \sum_{|\mathrm{Im}\ \rho - T| \geq 1} \frac{1}{(T - \mathrm{Im}\ \rho)^2} < \sum_{|\mathrm{Im}\ \rho - T| \geq 1} \frac{5}{4 + (T - \mathrm{Im}\ \rho)^2}$$

which is $\mathrm{O}(\log T)$ by (4.18). All that implies

$$-\left[\frac{\zeta'}{\zeta}\right]'(y) = \sum_{|\mathrm{Im}\ \rho - T| < 1} \frac{1}{(y - \rho)^2} + \mathrm{O}(\log T). \tag{4.24}$$

Then, upon integration over the segment $[x, x_T]$ of bounded length,

$$\frac{\zeta'}{\zeta}(x) + \mathrm{O}(\log T) = \sum_{|\,\mathrm{Im}\ \rho - T| < 1} \frac{1}{x - \rho} \quad - \sum_{|\,\mathrm{Im}\ \rho - T| < 1} \frac{1}{x_T - \rho} \ + \ \mathrm{O}(\log T),$$

and the second sum has $\mathrm{O}(\log T)$ terms by (4.19), all bounded by 1: it can be absorbed into the $\mathrm{O}(\log T)$ remainder, resulting in the estimate (4.20).

(b) The nonexistence of T satisfying (4.21) would immediately violate the bound (4.19) written for T'. Then, for such "good" T and with x on the segment $[-1 + \mathrm{i}T, 2 + \mathrm{i}T]$, each summand in (4.20) is uniformly $\mathrm{O}(\log T')$, while the number of summands is $\mathrm{O}(\log T)$ by (4.19). This (plus $T' \sim T$) yields the final bound (4.22). □

The proof of the Explicit Formula (Sect. 6.2) will actually need a *sequence* $\{T_n\} \to +\infty$ of "good" ordinates fulfilling (4.22): for this, it suffices to apply (b) with $T' = n$.

4.5 The (Asymptotic) Riemann–von Mangoldt Formula

The previous results actually allow us to describe the growth of the Riemann zeros' ordinates more finely.

The classic counting function for the zeros' ordinates can be defined by

$$N(T) \stackrel{\mathrm{def}}{=} \#\{\tau_k \mid \mathrm{Re}\ \tau_k \le T\}, \tag{4.25}$$

and it obeys the fundamental estimate (the *Riemann–von Mangoldt formula*):

$$N(T) = \overline{N}_0(T) + \mathrm{O}(\log T)_{T \to +\infty}\,, \qquad \overline{N}_0(T) \stackrel{\mathrm{def}}{=} \frac{T}{2\pi}\Big(\log\frac{T}{2\pi} - 1\Big) \tag{4.26}$$

(thus the asymptotic density of zeros at the ordinate T is $\approx (2\pi)^{-1} \log(T/\,2\pi)$).

Proof. (As in all books, simply we focus here on $T \gg 1$.)

Let R_T be the positively oriented closed rectangular path with vertices $\frac{1}{2} \pm \frac{3}{2} \pm \mathrm{i}T$, where T is not the ordinate of any zero, and R'_T be its restriction to the first quadrant of vertex $x = \frac{1}{2}$. Then by the general Rouché theorem, $N(T) = (4\pi\mathrm{i})^{-1} \oint_{R_T} [\Xi'/\Xi](x)\,\mathrm{d}x$; but the symmetries (reality and the Functional Equation) at once reduce the integration to the quarter path R'_T:

$$N(T) = \pi^{-1}\,\mathrm{Im} \int_{R'_T} \left[\frac{\Xi'}{\Xi}\right](x)\,\mathrm{d}x = \pi^{-1}\,\mathrm{Im}\,\big[\log \Xi(x)\big]_2^{\frac{1}{2}+\mathrm{i}T}, \tag{4.27}$$

where log means the complex logarithm followed continuously along the path R'_T, starting from $x = 2$. We then invoke the factorization (3.24) of $\Xi(x)$ to separate the contributions from $\zeta(x)$ and the remaining trivial factor:

$$N(T) = \overline{N}(T) + S(T), \tag{4.28}$$

$$\overline{N}(T) \stackrel{\text{def}}{=} \pi^{-1} \,\mathrm{Im}\, \log\big[(\mathrm{i}T - \tfrac{1}{2})\mathbf{G}^{-1}(\tfrac{1}{2} + \mathrm{i}T)\big] \tag{4.29}$$

$$S(T) \stackrel{\text{def}}{=} \pi^{-1} \,\mathrm{Im} \int_{R'_T} \left[\frac{\zeta'}{\zeta}\right](x)\, \mathrm{d}x. \tag{4.30}$$

Now, $\overline{N}(T)$ lends itself to an estimation by the Stirling formula (3.51):

$$\overline{N}(T) = \frac{T}{2\pi}\Big(\log \frac{T}{2\pi} - 1\Big) + \frac{7}{8} + \mathrm{O}\Big(\frac{1}{T}\Big). \tag{4.31}$$

In contrast, the other term $S(T)$ proves very irregular (fluctuating), but can be bounded by $\mathrm{O}(\log T)$ – this is the key argument. Indeed, on the long segment $[2, 2+\mathrm{i}T]$ of R'_T, the Dirichlet series (3.6) holds everywhere, implying $|\log \zeta(2 + \mathrm{i}T)| \le \log \zeta(2) = \mathrm{O}(1)$. All the action then takes place on the other segment $[\frac{1}{2} + \mathrm{i}T, 2 + \mathrm{i}T]$, of fixed length, where the use of (4.20) entails

$$\mathrm{Im} \int_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} \left[\frac{\zeta'}{\zeta}\right](x)\, \mathrm{d}x = \sum_{\{|\,\mathrm{Im}\ \rho - T| < 1\}} \big[\arg(x - \rho)\big]_{\frac{1}{2}+\mathrm{i}T}^{2+\mathrm{i}T} + \mathrm{O}(\log T);$$

but since the variation of each $\arg(x - \rho)$ is less than π and the number of summands is $\mathrm{O}(\log T)$ by (4.19), it follows that the left-hand side and therefore $S(T)$ are themselves $\mathrm{O}(\log T)$, and the proof is complete. □