Chapter 2
Infinite Products and Zeta-Regularization

Our purpose here is to review some symmetric-function techniques suitable for certain *divergent* sequences $\{x_k\}_{k=1,2,...}$, i.e., infinite (real or complex) sequences with $|x_k| \uparrow \infty$. (The x_k are counted with multiplicities if any.) We specifically wish to control their symmetric functions of the Zeta and Delta types as sketched in Sect. 1.1. We will elaborate on a scheme initiated for positive eigenvalue spectra in [104], where Theta-type functions provided a natural and most convenient base. For sequences such as the Riemann zeros, however, Theta-type functions exhibit less accessible and rather intricate properties, whereas Delta-type functions are openly present (as the "trivial" Gamma factors) and thus provide a privileged gateway; at the same time, a setting which requires positive sequences is inadequate. We must then adapt [104] to a broader perspective better adjusted to the idiosyncrasies of sequences like the Riemann zeros, and favoring conditions placed on Delta-type functions. The latter option, however, lengthens some intermediate calculations, so we emphasize that these are totally elementary (nineteenth-century mathematics!), all the more that a parameter μ_0, the order of the Delta function, remains low; and only $\mu_0 = 1$ and $\frac{1}{2}$ will serve for the Riemann zeros.

Our goal here is to output a toolbox of basic special-value formulae that are general enough and systematic yet economical, being tailored to our current final needs. Alternatively, we refer to [32, 53, 60, 61, 90] for very powerful and general, but more elaborate, frameworks. For greater convenience, Sect. 2.6 groups the main practical results to be exported for later use. Thus, the bulk of this chapter may be skipped on first reading. (Inversely, it can serve as a tutorial for zeta-regularization alone.)

2.1 Informal Discussion

We begin with a heuristic description of our basic targets for symmetric functions of infinite sequences $\{x_k\}$ when $|x_k| \uparrow \infty$.

First, we want such a sequence to admit a *zeta function* $Z(s \,|\, \{x_k\}) \stackrel{\text{def}}{=} \sum_k x_k^{-s}$ (shortened to $Z(s)$ except when the sequence may vary). We will need

A. Voros, *Zeta Functions over Zeros of Zeta Functions*, Lecture Notes of the Unione Matematica Italiana 8, DOI 10.1007/978-3-642-05203-3_2,

this function $Z(s)$ to be *analytic in a complex neighborhood of* $s = 0$, but its defining series $\sum_k x_k^{-s}$ will converge at best in some half-plane $\{\mathrm{Re}\ s > \mu_0\}$ with $0 < \mu_0 < +\infty$; we then ask $Z(s)$ to admit a *meromorphic* extension to the whole s-plane, with *computable poles* (to make way for some *explicit* analytical continuation methods). We at once assume that $\sum_k |x_k|^{-\mu_0}$ *diverges* because it will always be the case for us, and otherwise extra subtleties arise.

Next, we want to have a *Delta-type* function: formally, $\Delta(x \mid \{x_k\}) = \Delta(x) = \prod_k (x + x_k)$ like a characteristic polynomial, but this nice intrinsic product is well defined as it stands for finite sequences only. In the case of a diverging sequence as above, only a *modified* infinite product converges (everywhere), the *Weierstrass* product

$$\Delta_0(x) \stackrel{\text{def}}{=} \prod_{k=1}^{\infty} \Big(1 + \frac{x}{x_k}\Big) \exp\Bigg\{ \sum_{m=1}^{[\mu_0]} \frac{1}{m} \Big(-\frac{x}{x_k}\Big)^m \Bigg\} \qquad (\forall x \in \mathbb{C}), \tag{2.1}$$

with the usual notation $[\mu] \stackrel{\text{def}}{=}$ the integer part of μ. Thus Δ_0 is an entire function, having $\{-x_k\}$ as its set of zeros; but conversely, the latter features are too general to select $\Delta(x) = \Delta_0(x)$ uniquely.

To narrow the ambiguity on Δ we note that $\Delta_0(x)$ is an entire function of *finite order* μ_0 [10, Theorem 2.6.5]. This discussion will then largely borrow from the theory of such functions and their parametrization [10, Chap. 2] [26, Sect. 11] [89, Appendix 5]. We recall that in full generality, the order of an entire function $f(x)$ is $\mu_0 \stackrel{\text{def}}{=} \inf\{\mu \in \mathbb{R} \mid f(x) = \mathrm{O}(\mathrm{e}^{|x|^\mu})\}$, with $\mu_0 > 0$ for all but polynomial f; thus here for $\Delta_0(x)$, $0 < \mu_0 < \infty$.

A central result is that any entire function of order μ_0 with this prescribed set of zeros $\{-x_k\}$ has the *Hadamard product* form:

$$\Delta(x) = \mathrm{e}^{P(x)} \prod_{k=1}^{\infty} \Big(1 + \frac{x}{x_k}\Big) \exp\Bigg\{ \sum_{m=1}^{[\mu_0]} \frac{1}{m} \Big(-\frac{x}{x_k}\Big)^m \Bigg\} \qquad (\forall x \in \mathbb{C}) \tag{2.2}$$

where $P(x)$ is some polynomial P of degree $\le [\mu_0]$. From now on, $\Delta(x)$ will then be the *general* function (2.2) of order μ_0, with prescribed zeros $\{-x_k\}$ but P unspecified.

Spaces of entire functions have a basic symmetry, *translations* in $\mathbb{C}$. Now, finite products $\Delta_{\mathrm{f}}(x \mid \{x_k\}) = \prod_1^K (x + x_k)$ are manifestly covariant:

$$\Delta_{\mathrm{f}}(x \mid \{x_k + y\}) \equiv \Delta_{\mathrm{f}}(x + y \mid \{x_k\}) \qquad (\forall y \in \mathbb{C}),$$

but *not the Weierstrass product inside (2.2)*. Indeed, $\Delta_0(x)$ as in (2.1) is also specified by a particular *Taylor series at* $x = 0$ for its logarithm,

$$-\log \Delta_0(x) = \sum_{m=m_0}^{\infty} \frac{Z(m)}{m} (-x)^m \qquad (\text{convergent for } |x| < \inf_k |x_k|), \tag{2.3}$$

$$\text{where } m_0 \stackrel{\text{def}}{=} [\mu_0] + 1 \ (= \text{the least integer} > \mu_0); \quad \text{i.e.,} \tag{2.4}$$

$$(\log \Delta_0)^{(m)}(0) = (-1)^{m-1}(m-1)!\, Z(m) \qquad \text{for } m > \mu_0 \tag{2.5}$$

$$= 0 \qquad \text{for } m = 0, 1, \ldots, [\mu_0]. \tag{2.6}$$

More generally, the polynomial P in (2.2) identifies with the Taylor expansion to order $[\mu_0]$ of $\log \Delta$ at $x = 0$. The vanishing conditions (2.6) mean $P \equiv 0$, thus specifying Δ_0 in particular, but they are distinguishing *one base point in* $\mathbb{C}$, here $x = 0$, and this breaks translation invariance. It is thus important to *recover this symmetry* in (2.2).

Thus, the following related formula regains convergence and covariance:

$$(\log \Delta)^{(m_0)}(x) = (-1)^{m_0-1}(m_0-1)! \sum_k (x + x_k)^{-m_0}; \tag{2.7}$$

inversely, $(\log \Delta)^{(m_0)}$ can be seen as the basic *intrinsic* function, yielding $\log \Delta$ as its *general* m_0*-th primitive* (only a function mod P); whereas the particular primitive

$$(\log \Delta_0)(x) = \left[\int_0^x\right]^{m_0} (\log \Delta)^{(m_0)}, \tag{2.8}$$

cannot be fully covariant since it depends on a lower bound at each integration step. Still, if $\Delta_0(x)$ continues here to mean $\Delta_0(x \,|\, \{x_k\})$, then (2.7) implies

$$\begin{aligned} (\log \Delta_0)^{(m_0)}(x \,|\, \{x_k + y\}) &= (\log \Delta_0)^{(m_0)}(x + y), \\ \text{i.e.,} \quad \Delta_0(x \,|\, \{x_k + y\}) &= \mathrm{e}^{P_y(x)}\, \Delta_0(x + y) \\ \text{with} \quad P_y(x) &= \text{a polynomial in } x \text{ of degree} \le [\mu_0]. \end{aligned} \tag{2.9}$$

Translation covariance is thus achieved, but through a multiplier e^{P_y}; the conditions (2.6) then fix the polynomial $P_y(x)$, as

$$\log \frac{\Delta_0(x \,|\, \{x_k + y\})}{\Delta_0(x + y)} = P_y(x) = -\sum_{m=0}^{m_0} \frac{1}{m!} (\log \Delta_0)^{(m)}(y)\, x^m. \tag{2.10}$$

Thus, not only $P_y \not\equiv 0$, but its coefficients look like (and turn out to be) *transcendental* functions in the shift parameter y.

Further conditions are now required to gain some effective control over the meromorphic properties of $Z(s)$. At this level, details may have to be fine-tuned to each context. While there exist very powerful general formalisms as already mentioned, here we choose to stay fairly close to the setting of the Riemann zeros, so as to quickly reach our final applications.

2.2 A Class of Eligible Sequences $\{x_k\}$

So, we consider infinite complex sequences $\{x_k\}_{k=1,2,\dots}$ such that $x_k \neq 0$ $(\forall k)$, $|x_k| \uparrow \infty$, and $|\arg x_k| < \pi - \phi_0$ for some $\phi_0 > 0$. We now deem such a sequence *eligible of order* μ_0 for some $\mu_0 \in (0, +\infty)$ if it satisfies two (non-independent) conditions.

(1) As in Sect. 2.1, the series

$$\sum_{k=1}^{\infty} |x_k|^{-s} \textit{ converges if (and only if) } \operatorname{Re} s > \mu_0, \tag{2.11}$$

allowing us to define the *zeta function of the sequence*, by the Dirichlet series

$$Z(s \,|\, \{x_k\}) \equiv Z(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} x_k^{-s} \qquad \text{for } \operatorname{Re} s > \mu_0. \tag{2.12}$$

(2) Moreover, for some real sequence $\mu_0 > \mu_1 > \cdots > \mu_n \downarrow -\infty$ (with μ_0 as above) and some angle $\phi_1 \in (0, \phi_0)$, the function $\log \Delta_0(x)$ defined through (2.1) obeys a *complete large-x asymptotic expansion* of the form [106]

$$\log \Delta_0(x) \sim \sum_{n=0}^{\infty} (\tilde{a}_{\mu_n} \log x + a_{\mu_n})\, x^{\mu_n}, \qquad x \to \infty, \ |\arg x| < \phi_1\,, \tag{2.13}$$

uniformly in $\arg x$ (to make (2.13) repeatedly differentiable [35, Sect. 1.6]).

Equation (2.13) can be called a *(generalized) Stirling expansion*: in the prototypical case of the integer sequence $\{x_k = k\}$ of order $\mu_0 = 1$, (2.13) amounts to the classic Stirling formula for $-\log \Gamma(1+x)$, as reviewed in Sect. 3.6. (Higher powers of $\log x$ could be allowed, as in Sect. 1.5, but they would unnecessarily complicate both the treatment and the results.)

We will systematically extend the range of all coefficients like $\tilde{a}_{\mu_n}$, a_{μ_n} having general (real-valued) indices by the convention

$$\tilde{a}_\mu,\, a_\mu \stackrel{\text{def}}{=} 0 \qquad \text{for arbitrary } \mu \in (-\infty, \mu_0) \setminus \{\mu_n\}. \tag{2.14}$$

2.3 Meromorphic Continuation of the Zeta Function

We can now begin to use the results of Sect. 1.5.

At the root, the estimates (2.6) and (2.13) together make the following Mellin transform of the type (1.16) well defined: for $m_0 \stackrel{\text{def}}{=} [\mu_0]+1$ as in (2.4),

$$I(s) \stackrel{\text{def}}{=} \int_0^{\infty} \log \Delta_0(y)\, y^{-s-1} \mathrm{d}y \qquad\qquad (\mu_0 < \operatorname{Re} s < m_0). \tag{2.15}$$

Then on the one hand, exploiting Sect. 1.5 thoroughly, $I(s)$ continues to a meromorphic function in all of $\mathbb{C}$ where its only singularities are:

- Due to (2.13), at most double poles at $s = \mu_n$, with the principal parts

$$I(\mu_n + \varepsilon) = \tilde{a}_{\mu_n}\,\varepsilon^{-2} + a_{\mu_n}\,\varepsilon^{-1} + \mathrm{O}(1)_{\varepsilon\to 0} \qquad (2.16)$$

- And due to (2.3), at most simple poles at the integers $s = m \geq m_0$, with residues $(-1)^m Z(m)/m$.

On the other hand, repeated integrations by parts upon (2.15) lead to

$$I(s) = \frac{(-1)^{m_0}\Gamma(-s)}{\Gamma(m_0 - s)} \int_0^\infty (\log \Delta_0)^{(m_0)}(y)\, y^{m_0 - s - 1}\mathrm{d}y \qquad (\mu_0 < \mathrm{Re}\ s < m_0); \qquad (2.17)$$

and if here we substitute (2.7) for $(\log \Delta_0)^{(m_0)}(y)$ and integrate term by term relying on $\int_0^\infty (1+y)^{-(z+w)} y^{z-1}\mathrm{d}y = \Gamma(z)\Gamma(w)/\Gamma(z+w)$, we find

$$I(s) \equiv -\frac{\Gamma(-s)\Gamma(m_0)}{\Gamma(m_0 - s)} \sum_k \int_0^\infty (y + x_k)^{-m_0} y^{m_0 - s - 1}\mathrm{d}y \equiv \cdots \equiv \frac{\pi}{s\,\sin \pi s}\, Z(s).$$

But since $I(s)$ is meromorphic in all of $\mathbb{C}$, the reverse formula

$$Z(s) \equiv \frac{s\,\sin \pi s}{\pi}\, I(s) \qquad (\mu_0 < \mathrm{Re}\ s < m_0) \qquad (2.18)$$

actually extends to the whole plane; it then continues $Z(s)$ from the half-plane $\{\mathrm{Re}\ s > \mu_0\}$ as in (2.12) to a meromorphic function in all of $\mathbb{C}$, with singularities explicitly induced by those of $I(s)$ which we just described.

As the main qualitative consequence, the zeros of the prefactor $(s\,\sin \pi s)$ counteract all the poles of $I(s)$ located at integers, implying that:

- At any *integer* $\mu_n = m$, by (2.16), $Z(s)$ has

$$\text{at most a } \textit{simple} \text{ pole, of residue } (-1)^m m\,\tilde{a}_m\,, \qquad (2.19)$$
$$\text{with } \mathrm{FP}_{s=m} Z(s) = (-1)^m(\tilde{a}_m + m\,a_m) \qquad (2.20)$$

  (and likewise for arbitrary integers $m < \mu_0$, under the convention (2.14))
- At integers $s > \mu_0$, since all the poles of $I(s)$ are simple, $Z(s)$ is devoid of poles (indeed, we know it is analytic for $\mathrm{Re}\ s > \mu_0$)
- Last but not least, $s = 0$ is a *regular point* for $Z(s)$ thanks to the *double zero* of the prefactor $(s\,\sin \pi s)/\pi$ there, with (2.16) implying

$$Z(\varepsilon) = \tilde{a}_0 + a_0\varepsilon + \mathrm{O}(\varepsilon^2)_{\varepsilon\to 0} \quad \Rightarrow \quad Z(0) = \tilde{a}_0, \quad Z'(0) = a_0 \qquad (2.21)$$

  (again with the convention (2.14) if $0 \notin \{\mu_n\}$).

Summary

All the poles of $Z(s)$ lie in the single descending sequence $\{\mu_n\}$; they are at most double in general, simple at integers, and definitely absent at $s = 0$ (a regular point); the values $Z(m)$ for integer $m \le \mu_0$, and $Z'(0)$ have closed expressions in terms of the generalized Stirling expansion coefficients.

2.4 The Generalized Zeta Function

All previous considerations about $Z(s)$ transfer to shifted sequences $\{x_k + x\}$ up to reasonable limitations on the shift parameter x (e.g., $(x + x_k) \notin \mathbb{R}_-$ $(\forall k)$).

Thus, for an eligible sequence $\{x_k\}$ as in Sect. 2.2, the *generalized zeta function*

$$Z(s,x) \stackrel{\text{def}}{=} \sum_k (x_k + x)^{-s} \qquad (\text{Re } s > \mu_0) \tag{2.22}$$

satisfies:

- The first eligibility condition (2.11), with μ_0 unchanged
- The obvious but essential functional relation

$$\partial_x Z(s,x) = -sZ(s+1,x); \tag{2.23}$$

- *Explicit covariance*, since the identity $Z(s, X|\{x_k + x\}) \equiv Z(s, X + x \,|\, \{x_k\})$, obviously obeyed for Re $s > \mu_0$, then extends to all s by meromorphic continuation.

The other eligibility condition (2.13) asks for a large-X expansion like

$$\log \Delta_0(X \,|\, \{x_k + x\}) \sim \sum_\mu \left[\tilde{a}_\mu(x) \log X + a_\mu(x)\right] X^\mu, \tag{2.24}$$

but thanks to the covariance identity (2.10) with $P_x(X)$ polynomial in X,

$$\log \Delta_0(X \,|\, \{x_k + x\}) \equiv P_x(X) + \log \Delta_0(X + x \,|\, \{x_k\}); \tag{2.25}$$

then this last term can have its large-$(X + x)$ expansion (2.13) reordered in powers of X, and the form (2.24) is obtained; this operation, which may output a modified sequence $\{\mu\} \neq \{\mu_n\}$ (but μ_0 is invariant), is algebraic at any order, therefore:

$$\begin{array}{llll} & a_\mu(x) \text{ for } \mu \notin \mathbb{N}, \text{ and all } \tilde{a}_\mu(x) & \text{are } \textit{polynomial in } x; \\ \text{but} & a_m(x) \text{ for } m = 0, 1, \ldots, [\mu_0] & \text{are } \textit{transcendental in } x \end{array} \tag{2.26}$$

as the polynomial $P_x(X)$ adds a transcendental dependence on x, by (2.10). This dichotomy will visibly affect all concrete outputs, see Sect. 2.6.3.

Then, all previous results about $Z(s) = Z(s,0)$ such as (2.19)–(2.21) extend to $Z(s,x)$, just by replacing $\tilde{a}_m,\ a_m$ by $\tilde{a}_m(x),\ a_m(x)$ respectively.

In Sect. 2.3, $Z(s)$ was derived from the function Δ_0; it is now desirable to express $Z(s,x)$ likewise, since Delta-type functions will provide our main input data in applications.

At integer s, just by inspection, (2.7) and its higher derivatives yield

$$Z(m,x) \equiv \frac{(-1)^m}{(m-1)!}(-\log \Delta)^{(m)}(x) \qquad \text{for integer } m \ge m_0 \tag{2.27}$$

valid for general Δ, which includes Δ_0 (cf. (2.5) for $x=0$).

For more general s, we combine (2.18) not with the integral formula (2.15), but with (2.17) which uses the intrinsic covariant function $\log \Delta_0^{(m_0)}$; so, we may simply shift the whole formula by x, with the result:

$$Z(s,x) = \frac{(-1)^{m_0}}{\Gamma(s)\Gamma(m_0-s)} \int_0^\infty (-\log \Delta_0)^{(m_0)}(x+y)\, y^{m_0-s-1} \mathrm{d}y \qquad (\mu_0 < \mathrm{Re}\ s < m_0), \tag{2.28}$$

again also valid with general Δ in place of Δ_0. (*Remark.* Symbolically, this amounts to extending (2.27) to derivatives of non-integer order, as $Z(s,x) \equiv \Gamma(s)^{-1}(-\mathrm{d}/\mathrm{d}x)^s(-\log \Delta_0)(x)$.)

2.5 The Zeta-Regularized Product

One can now define $\prod_k (x+x_k)$ for an infinite eligible sequence, as

$$\Delta_\infty(x) \stackrel{\text{def}}{=} \exp\left[-Z'(0,x)\right] \qquad ({}' \equiv \partial/\partial s,\ \text{as in (1.9)}). \tag{2.29}$$

Indeed this straightforwardly works for a finite sequence $\{x_k\}$, and it prescribes a finite value in any case since we forced $Z(s,x)$ to be regular at $s=0:\ \Delta_\infty(x)$ is called the *zeta-regularized* form for the products $\Delta(x)$.

However, the prescription (2.29) is an impractical one as it needs analytical continuation (from Re $s > \mu_0 > 0$ to $s=0$), which is not a constructive operation. Eligibility conditions, like here or in [104], are precisely tools to pin down $\Delta_\infty(x)$ quite concretely.

First and foremost, although this does not obviously show on its definition, $\Delta_\infty(x)$ indeed belongs to the family (2.2), i.e., it satisfies

$$\Delta_\infty(x) \equiv \mathrm{e}^{-P_\infty(x)}\Delta_0(x) \quad \text{for a polynomial } P_\infty(x) \text{ of degree } \le \mu_0. \tag{2.30}$$

Proof. The definition (2.29) and functional relation (2.23) imply, for $m \in \mathbb{N}$,

$$(-\log \varDelta_\infty)^{(m)}(x) = (-1)^m \partial_s\, s(s+1)\dots(s+m-1)Z(s+m,x)|_{s=0}. \quad (2.31)$$

Now, $\partial_s\, sf(s)|_{s=0} \equiv \mathrm{FP}_{s=0} f$ for a function $f(s)$ having at most a simple pole at $s=0$, *and the functions* $Z(s+m,x)$ *are precisely like that*, by (2.19). So, the previous formula reduces for $m>0$ to

$$(-\log \varDelta_\infty)^{(m)}(x) = (-1)^m \mathrm{FP}_{s=m}\big[(s-m+1)\dots(s-1)Z(s,x)\big], \quad (2.32)$$

and at the regular point $s = m_0$ $(=[\mu_0]+1)$, this boils down to

$$\begin{aligned}(-\log \varDelta_\infty)^{(m_0)}(x) &= (-1)^{m_0}(m_0-1)!\, Z(m_0,x) && (2.33)\\ &\equiv (-\log \varDelta)^{(m_0)}(x) \quad \text{by identification with (2.27).} && \square\end{aligned}$$

In other words, $\varDelta_\infty(x)$ is a particular $\varDelta$ function; but one which has also inherited *manifest covariance* (under translations) from $Z(s,x)$ itself, by its very definition (2.29). A serious limitation, however, is that this zeta-regularization prescription will be stable under *no other* change of x-variable than a pure translation (not even $x \mapsto \lambda x$).

Next, the yet unknown polynomial P_∞ can be characterized in two independent ways, where the *harmonic numbers* H_n will appear:

$$H_n \stackrel{\mathrm{def}}{=} \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) - \psi(1) \qquad (n \in \mathbb{N}) \qquad (H_0 = 0). \quad (2.34)$$

- Taking successive logarithmic derivatives of (2.30) at $x=0$ and enforcing (2.6), we get

$$P_\infty(x) = \sum_{m=0}^{[\mu_0]} (-\log \varDelta_\infty)^{(m)}(0)\, \frac{x^m}{m!}. \quad (2.35)$$

  If, in analogy with (2.5) for $\log \varDelta_0$, we pose

$$\begin{aligned} Z_\infty(m) &\stackrel{\mathrm{def}}{=} \frac{(-1)^m}{(m-1)!}(-\log \varDelta_\infty)^{(m)}(0) \quad \text{for } m \in \mathbb{N}^*, && (2.36)\\ (Z_\infty(m) &\equiv Z(m) \qquad \text{if } m \ge m_0), \end{aligned}$$

  then $P_\infty(x) = P_\infty(0) + \sum_{m=1}^{[\mu_0]} \frac{Z_\infty(m)}{m}(-x)^m$ gets specified via (2.31)–(2.32):

$$\begin{aligned} P_\infty(0) &= Z'(0) && (2.37)\\ Z_\infty(m) &= \frac{1}{(m-1)!}\mathrm{FP}_{s=m}\Big[\frac{\Gamma(s)}{\Gamma(s-m+1)}Z(s)\Big] \quad (m>0) \\ &= \mathrm{FP}_{s=m} Z(s) + (-1)^m m\, H_{m-1}\, \tilde a_m && (2.38)\end{aligned}$$

$$= Z(m) \quad \text{if } Z(s) \text{ is regular at } m \ (\iff \tilde{a}_m = 0). \tag{2.39}$$

$$\text{E.g.,} \quad Z_\infty(1) \equiv \mathrm{FP}_{s=1} Z(s) \qquad (\text{because } H_0 = 0). \tag{2.40}$$

Note that $P_\infty(x)$ nicely completes the Taylor series (2.3), to give

$$-\log \varDelta_\infty(x) = Z'(0) + \sum_{m=1}^{\infty} \frac{Z_\infty(m)}{m} (-x)^m \qquad (|x| < \inf_k |x_k|). \tag{2.41}$$

- A more tangible specification of P_∞ involves the large-x (generalized Stirling) expansion of $\log \varDelta_\infty(x)$.
  By substitution into (2.37) and (2.38) of the respective values (2.21) for $Z'(0)$ and (2.20) for $\mathrm{FP}_{s=m} Z(s)$, the previous results become

$$\varDelta_\infty(x) \equiv \mathrm{e}^{-P_\infty(x)} \varDelta_0(x), \qquad P_\infty(x) = \sum_{m=0}^{[\mu_0]} (H_m\, \tilde{a}_m + a_m)\, x^m. \tag{2.42}$$

  In turn, this and (2.13) imply that the large-x expansion assumes a special ("canonical") form in the case of $\log \varDelta_\infty(x)$:

$$\log \varDelta_\infty(x) \sim \sum_{m=0}^{[\mu_0]} \tilde{a}_m (\log x - H_m)\, x^m + \sum_{\mu_n \notin \mathbb{N}} (\tilde{a}_{\mu_n} \log x + a_{\mu_n})\, x^{\mu_n}, \tag{2.43}$$

  in which powers x^m for $m \in \mathbb{N}$ are *banned in free form* (i.e., when not paired with $x^m \log x$ *exactly* as shown). Inversely, given an eligible sequence $\{x_k\}$, enforcing this restriction fixes the polynomial P_∞ and thereby $\varDelta_\infty(x)$ itself.

  This result can be understood in two other ways.

- By brute force: The initial asymptotic information (2.13) about $\log \varDelta_0(x)$, put into the integral representation (2.28) of $Z(s, x)$, can generate a large-x expansion for the latter (exercise!):

$$Z(s, x) \sim -\frac{x^{-s}}{\varGamma(s)} \sum_{n=0}^{\infty} \left[\left(\tilde{a}_{\mu_n} \partial_\mu + a_{\mu_n} \right) \frac{\varGamma(s-\mu)}{\varGamma(-\mu)}\, x^\mu \right]_{\mu=\mu_n}, \tag{2.44}$$

  on which the formal evaluation of $-Z'(0, x)$ precisely restores (2.43) (here, the free powers x^μ get killed by the pole of $\varGamma(-\mu)$ when $\mu \in \mathbb{N}$).
- By *symbolic integration* [104]: In order to continue $Z(s, x)$ toward $s = 0$, one idea is to integrate $Z(m_0, x)$ m_0 times using (2.23) backwards; the integration bound is then found by reference to $m > m_0$, as

$$(\log \varDelta)^{(m-1)}(x) = \int_{+\infty}^{x} (\log \varDelta)^{(m)}(x')\, \mathrm{d}x'. \tag{2.45}$$

This integral naturally diverges as soon as $m - 1 < \mu_0$, but it can nevertheless be defined on the asymptotic terms of $(\log \Delta)^{(m)}(x)$ by, e.g., $\int_{+\infty}^{x} x'^{\,\mu} \mathrm{d}x' = x^{\mu-1}/(\mu-1)$, followed by analytical continuation and/or differentiation in μ, plus finite-part extraction if needed. It is then easily verified that in degrees $n \in \mathbb{N}$, only canonical (allowed) terms $x^n(\log x - H_n)$ emerge, e.g., $\int_{+\infty}^{x} x'^{\,-1}\mathrm{d}x' = \log x$, $\int_{+\infty}^{x} \log x' \mathrm{d}x' = x\,(\log x - 1)$, etc. Consequently, the canonical nature of the series (2.43) symbolically means [104]

$$\log \Delta_\infty(x) = \left[\int_{+\infty}^{x}\right]^{m_0} (\log \Delta)^{(m_0)}, \tag{2.46}$$

a covariant formula, to be compared to (2.8) for the Weierstrass product. Warning: such a symbolic integration does not follow the usual change-of-variable rules!

2.6 Practical Results

Our strategy is to harvest explicit properties of the zeta function $Z(s,x)$ from our assumed knowledge of a Delta function $\Delta(x)$ and its properties. Here we recapitulate the facts gathered in this chapter that serve that purpose.

2.6.1 Zeta-Regularization: a Zeta-Free Recipe

The definition (2.29) of the zeta-regularized form, $\Delta_\infty(x) \stackrel{\text{def}}{=} \exp\bigl[-Z'(0,x)\bigr]$, goes in the wrong direction for us: it precisely uses a zeta function that we aim at understanding (plus analytical continuation in totally abstract form).

Fortunately, given an *eligible* sequence $\{x_k\}$ of order μ_0 in the sense of Sect. 2.2, and an infinite Hadamard product $\Delta(x \,|\, \{x_k\})$ as in (2.2), with a known (generalized) Stirling expansion (2.13), we now basically have a *mechanical* zeta-regularization rule. It suffices to reexpand any terms of (2.13) having $\mu_n = m \in \mathbb{N}$ (in finite number since $m \le [\mu_0]$) over the basis $\{x^m(\log x - H_m),\, x^m\}$, as

$$\log \Delta(x) \sim \sum_{m=0}^{[\mu_0]} (\tilde{a}_m(\log x - H_m) + b_m)\, x^m + \sum_{\mu_n \notin \mathbb{N}} (\tilde{a}_{\mu_n} \log x + a_{\mu_n})\, x^{\mu_n} \tag{2.47}$$

(recalling that $H_m = \sum_{j=1}^{m} 1/j$ are the harmonic numbers). Now (2.43) demonstrates that in the zeta-regularized form, terms which remain in (2.47) with coefficients denoted $\tilde{a}_\mu$, a_μ are *a*llowed: namely, all terms with $\mu_n \notin \mathbb{N}$, and, for $\mu_n \in \mathbb{N}$,

$$\cdots, \quad \tilde{a}_2\, x^2(\log x - \tfrac{3}{2}), \quad \tilde{a}_1\, x(\log x - 1), \quad \tilde{a}_0 \log x\,; \tag{2.48}$$

while all terms with b_m are *b*anned, namely

$$b_m\, x^m \quad \text{for all } m \in \mathbb{N}, \quad \textit{including additive constants } (b_0). \tag{2.49}$$

Consequently, the zeta-regularized form has to be, simply (cf. (2.42)),

$$\varDelta_\infty(x) \equiv \exp\Bigg[-\sum_{m=0}^{[\mu_0]} b_m x^m\Bigg]\, \varDelta(x), \qquad b_m = H_m\, \tilde{a}_m + a_m. \tag{2.50}$$

With $\varDelta_\infty(x)$ thus determined, a shifted large-x expansion of $\log \varDelta_\infty$ yields

$$\log \varDelta_\infty(x+y) \sim \sum_{m=0}^{[\mu_0]} \tilde{a}_m(y)\,(\log x - H_m)\, x^m + \sum_{\mu \notin \mathbb{N}} \big[\tilde{a}_\mu(y) \log x + a_\mu(y)\big]\, x^\mu, \tag{2.51}$$

in which all coefficients are now *algebraically computable polynomials* of y.

Such a covariant shift performed on all the above results (2.37)–(2.41) yields our final special-value formulae in their most general and useful form:

$$-\log \varDelta_\infty(y) = Z'(0,y) \tag{2.52}$$

$$\frac{(-1)^{m-1}}{(m-1)!}\,(\log \varDelta_\infty)^{(m)}(y) = Z_\infty(m,y) \qquad (m = 1, 2, \ldots), \tag{2.53}$$

with

$$Z_\infty(1,y) \equiv \mathrm{FP}_{s=1} Z(s,y), \tag{2.54}$$

$$Z_\infty(m,y) = \mathrm{FP}_{s=m} Z(s,y) + (-1)^m m\, H_{m-1}\, \tilde{a}_m(y) \qquad (m \geq 1) \tag{2.55}$$

$$\equiv Z(m,y) \qquad \text{if } Z(s,y) \text{ is regular at } s = m. \tag{2.56}$$

Our main application, concerning the Riemann zeros, only uses $\mu_0 \leq 1$, in which case the fully general formula (2.55) can be skipped as this makes no difference. However, Appendix B (devoted to the zeros of *Selberg* zeta functions) makes a brief incursion into $\mu_0 = 2$ territory, where that omission would cause fatal errors.

2.6.2 A Subclass: "Theta-Eligible" Sequences

Some auxiliary sequences to be needed alongside the Riemann zeros are specially simple (and positive) eligible sequences $\{x_k\}$ (called "admissible" sequences in [104]): they allow easier direct control through their *Theta* functions $\Theta(z)$, under the following two assumptions:

$$(1) \qquad \Theta(z) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \mathrm{e}^{-zx_k} \qquad \text{converges for Re } z > 0; \tag{2.57}$$

(2) a *complete small-z asymptotic expansion* can be written for $\Theta(z)$,

$$\Theta(z) \sim \sum_{n=0}^{\infty} c_{\mu_n} z^{-\mu_n} \qquad \text{for } z \to 0^+,\ |\arg z| < \theta; \tag{2.58}$$

the sequence $\{\mu_n\}$ is as in Sect. 2.2, and $0 < \theta \le \frac{1}{2}\pi$. In linear spectral problems of physics, $\Theta(z)$ is the trace of the heat operator ("partition function"). Certain complex sequences $\{x_k\}$ can qualify as well [53, 60, 61, 90]. For the Riemann zeros themselves, however, the simplest $\Theta(z)$ would be $V(\mathrm{i}z)$ where V is *Cramér's function* (5.14). Now we will see in Sect. 5.4 that the singular structure of $V(z)$ about the origin is only partially and indirectly available, and much less simple than (2.58). This makes us unwilling to choose $V(z)$ as principal symmetric function over the Riemann zeros.

Sequences that are theta-eligible as above gain simpler formulae.

The zeta function is more readily accessed than in Sect. 2.3, as

$$Z(s) \equiv \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta(z)\, z^{s-1} \mathrm{d}z \qquad (\text{Re } s > \mu_0). \tag{2.59}$$

By Sect. 1.5 Stage 3' (but with $s \mapsto -s$), $Z(s)$ is meromorphic in $\mathbb{C}$, and its singularities are at most *simple* poles at the μ_n, with residues $c_{\mu_n}/\Gamma(\mu_n)$; this implies that all the negative integers including 0 are regular points which give the special values

$$Z(-n) = (-1)^n n!\, c_{-n} \quad (\forall n \in \mathbb{N}) \qquad \text{(with the convention (2.14))}. \tag{2.60}$$

Likewise,

$$Z(s,x) \equiv \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta(z)\, \mathrm{e}^{-xz}\ z^{s-1} \mathrm{d}z \qquad (\text{Re } s > \mu_0,\ |\arg x| < \tfrac{1}{2}\pi + \theta), \tag{2.61}$$

from which a direct calculation of the large-x expansion of $-Z'(0,x)$ yields [104]

$$\log \Delta_\infty(x) \sim \sum_{m=0}^{[\mu_0]} \frac{(-1)^m}{m!} c_m (\log x - H_m)\, x^m - \sum_{\mu_n \notin \mathbb{N}} \Gamma(-\mu_n)\, c_{\mu_n} x^{\mu_n}$$
$$(x \to \infty,\ |\arg x| < \tfrac{1}{2}\pi + \theta). \tag{2.62}$$

Thus, not only our previous eligibility conditions hold, but the generalized Stirling expansion of $\log \mathit{\Delta}_\infty(x)$ fully stems from the small-z expansion (2.58) of $\mathit{\Theta}(z)$.

2.6.3 Explicit Properties of the Generalized Zeta Function

Here we sum up the explicit properties of the function $Z(s,x)$ which can be gathered from the assumed knowledge of an eligible Delta function $\mathit{\Delta}(x)$ with its properties. They concern two facets of the (generalized) zeta function: its principal parts, and its values at integers.

- *The singular structure:* All s-plane poles of $Z(s,x)$ have computable locations (namely, the exponents μ in the large-X expansion (2.24)), and their principal parts involve coefficients $\tilde{a}_\mu$ and a_μ by (2.16), excepting a_m at integer points m by (2.19): such coefficients precisely give rise to *polynomials* in x by (2.26), so the principal parts at the poles of $Z(s,x)$ are *rational* objects (algebraically computable).
- *Explicit-value formulae:* They will concern all the values (or finite parts) of $Z(s,x)$ at integer points $s \in \mathbb{Z}$, plus $Z'(0,x)$ all by itself: we dub the totality of them "special values." Here an essential splitting is seen between *negative* and *positive* integer points s [106, Sect. 4].
  - The values (or finite parts) at $s = -m \in -\mathbb{N}$ are given by (2.20), using the coefficients $\tilde{a}_{-m}$ and a_{-m} except a_0, that again extend to *polynomials in* x; hence those $Z(-m,x)$, beginning with $Z(0,x)$, are *rational* objects as well (algebraically computable).
  - The values (or finite parts) at $s = +m \in \mathbb{N}$ are given: for $m > \mu_0$, by (2.53) and (2.56); and for $m \le \mu_0$, by (2.20) which involves a_m when $m > 0$, whose parametric extension $a_m(x)$ is transcendental by (2.26). At $m = 0$, as an exception, (2.52)–(2.53) show that a natural companion to the values $\{Z(m,x)\}_{m>0}$ is $Z'(0,x) = a_0(x)$ rather than $Z(0,x)$ (already put in the set $\{Z(-m,x)\}$ of rational values). We then declare our set of special values for $+m \in \mathbb{N}$ to be *not* $\{Z(m,x)\}_{m\in\mathbb{N}}$, but $\{Z'(0,x)\} \cup \{Z(m,x)\}_{m\in\mathbb{N}^*}$ instead. Manifestly, all the values of *this* set show a *transcendental* nature (e.g., as functions of x).

  Therefore, all special-value tables for zeta functions will reflect this algebraic vs transcendental splitting. A prototype for these tables follows; it is set with redundancies on purpose, to display a gamut of cases. In the $s = 0$ line of Table 2.1 we expanded the polynomial coefficient $\tilde{a}_0(x)$ explicitly; the reader is invited to do the same for the remaining polynomials $\tilde{a}_{\pm m}$ and a_{-m} (exercise!).

Table 2.1 Special values of the generalized zeta function $Z(s,x)$ for a general eligible sequence of order μ_0, in terms of the zeta-regularized product $\varDelta_\infty(x)$ given by (2.50) and its expansion polynomials $\tilde{a}_m(x)$, $a_m(x)$ given by (2.51) (upper part: rational values, lower part: transcendental values). Notation: see (1.7), (2.29), (2.34); m is an integer

s	$Z(s,x) = \sum_k (x_k + x)^{-s}$
Regular $-m < 0$	$-(-1)^m m\, a_{-m}(x)$
$-m \le 0$ *(finite part)*	$\boxed{\mathrm{FP}_{s=-m} Z(s,x) = (-1)^m \left[\tilde{a}_{-m}(x) - m\, a_{-m}(x)\right]}$
0	$\tilde{a}_0(x) \equiv \sum_{j=0}^{[\mu_0]} \tilde{a}_j x^j$
0 *(s-derivative)*	$\boxed{Z'(0,x) = -\log \varDelta_\infty(x)}$
+1 *(finite part)*	$\boxed{\mathrm{FP}_{s=1} Z(s,x) = (\log \varDelta_\infty)'(x)}$
$+1 \le m \le \mu_0$ *(finite part)*	$\boxed{\mathrm{FP}_{s=m} Z(s,x) = (-1)^{m-1} \left[\frac{1}{(m-1)!} (\log \varDelta_\infty)^{(m)}(x) + m\, H_{m-1} \tilde{a}_m(x)\right]}$
Regular $+m \ge 1$	$\frac{(-1)^{m-1}}{(m-1)!} (\log \varDelta_\infty)^{(m)}(x)$