Chapter 10
Extension to Other Zeta- and L-Functions

In this chapter, based on [107], we extend the treatment and results of the three former chapters to the setting where the Riemann zeros are replaced by the (nontrivial) zeros of a more general zeta or L-function $L(x)$, still fairly similar to $\zeta(x)$. We will use the terms: *primary function* for this function $L(x)$ which supplies the new zeros in this *extended* setting, and *Riemann case* for the former setting where $\zeta(x)$ itself was the fixed primary function.

The interest of this extension is twofold. First, it broadens the previous results in a natural way: with little work, we will accommodate three distinct kinds of superzeta functions as before, but now over the (nontrivial) zeros of numerous primary functions. Second, it sheds some further light on the results for $\zeta(x)$ itself: the origin of many final values in the previous chapters will be clarified through their more abstract specifications. For instance, various special values like $\mathrm{FP}_{s=1}\, \mathscr{Z}_0(s)$, $\mathcal{Z}_0(0), \ldots$ will now explicitly stem from the Stirling expansion (10.12) for the trivial factor $\mathbf{G}(x)$, $x \to +\infty$.

We could of course have taken this general path from the very beginning, relegating the Riemann case to the status of just a special instance. However, this would have gone against our plan to provide a most concrete and readily usable handbook. The Riemann zeta function may be a special case in a crowd, but it is important enough to deserve an autonomous presentation.

Earlier *explicit* descriptions of such extended superzeta functions, i.e., over zeros other than Riemann's, hardly exist in the literature. We set apart the case of Selberg zeta functions: their zeros correspond to eigenvalues of hyperbolic Laplacians, and zeta functions over them have been analyzed by spectral methods [15, 16, 47, 91, 100, 105]: in the cocompact case, they are indeed examples of *spectral* (Minakshisundaram–Pleijel or generalized) zeta functions; we revisit them in Appendix B. Otherwise, only Dedekind zeta functions got some mention as primary functions [47, 51, 52, 66]; already if we turn to L-series, then only the Cramér functions (5.14) over their zeros were ever considered [29, 54, 59, 62], without relating them to any superzeta functions (apart from one short note [94] on an Explicit Formula like (6.22) but actually for $\mathscr{Z}(s\,|\,\frac{3}{2})$ over the zeros of specific Dirichlet L-functions).

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Most of the notation can be taken over from the former chapters; simply, all objects will now be understood to depend on the chosen primary function L, implicitly as a rule. At the same time, the Riemann zeta function as such will continue to appear in some results.

10.1 Admissible Primary Functions $L(x)$

For the sake of definiteness, we choose here to stay fairly close to the Riemann case ($L(x) = \zeta(x)$): basically, we want to retain a reflexive functional equation $\Xi(x) = \Xi(1-x)$ for a completed function built similarly to (3.24). We therefore accept primary functions $L(x)$ such that:

(a) $L(x)$ is real, and meromorphic in $\mathbb{C}$ with at most one simple pole, $x = 1$:

$$\text{if } q \stackrel{\text{def}}{=} \text{the order of the pole } x = 1, \text{ then } q = 0 \text{ or } 1; \tag{10.1}$$

(b) $L(x) \neq 0$ in $\{\operatorname{Re} x > 1\}$, and $L(x) \to 1$ for Re $x \to +\infty$ with

$$(\log L)^{(n)}(x) = \mathrm{o}(x^{-N}) \qquad (\forall n,\ N \in \mathbb{N}); \tag{10.2}$$

(c) a completed L-function and a functional equation exist, similar to the Riemann case:

$$\Xi(x) \equiv \Xi(1-x), \qquad \Xi(x) \stackrel{\text{def}}{=} \mathbf{G}^{-1}(x)(x-1)^q L(x), \tag{10.3}$$

where both $\Xi(x)$ and $\mathbf{G}(x)$ are *real entire functions of order* $\mu_0 = 1$, and

(c_1) $\mathbf{G}(x)$, the "trivial factor," is an explicitly known finite product of inverse-Gamma (and simpler) factors, with all its zeros x_k located on the negative real axis $\{x \le 0\}$ (they form the "trivial zeros" of $L(x)$);

(c_2) the zeros of $\Xi(x)$ lie in the strip $\{0 < \operatorname{Re} x < 1\}$; since they come in symmetrical pairs as in the Riemann case, we still label them

$$\{\rho = \tfrac{1}{2} \pm \mathrm{i}\tau_k\}_{k=1,2,\dots}, \qquad \text{with } \operatorname{Re}\tau_k > 0 \text{ and non-decreasing;}$$

they are the "nontrivial zeros" of $L(x)$; *we exclude the exceptional occurrence of any of these on the real line*, for simplicity. (But see Appendix B for such a case.)

Note: all zeros, x_k or ρ, are counted with multiplicities if any.

(d)

$$\lim_{x=1} (x-1)^q L(x) \ (= \mathbf{G}(1)\Xi(1)) > 0 \tag{10.4}$$

(but dropping the normalization $\Xi(1) = \Xi(0) = 1$, too awkward to implement in general); then, Stieltjes cumulants can extend from (3.15) in the Riemann case, according to the general definition

$$\log\left[(x-1)^q L(x)\right] \equiv \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!}\, g_n^{\mathrm{c}}\,(x-1)^n : \qquad (10.5)$$

now $g_0^{\mathrm{c}} \neq 0$ may occur, while g_1^{c} extends Euler's constant γ from (3.16).

As already said, the symbols from the previous chapters (Ξ, $\mathbf{G}$, ρ, $g_n^{\mathrm{c}}, \ldots$) are consistent with their former uses in the Riemann case, but they now designate objects attached to the changeable primary function L.

Conditions (a)–(d) above are tailored to fit two basic classes which are immediate extensions from the Riemann case, and will be described as final examples: *L-functions of real primitive Dirichlet characters*, and *Dedekind zeta functions* (with $\zeta(x)$ as a special case of the latter). Our assumptions somewhat resemble the axioms of the *Selberg class* [98] but are more restrictive on some concrete details; on the other hand these can undoubtedly be refitted to different needs. For instance, zeta functions over zeros of Selberg zeta functions for compact hyperbolic surfaces have yielded results comparable to the Riemann case earlier [15, 16, 66, 91, 100], while they correspond to $\mu_0 = 2$ ($\mathbf{G}$ contains a Barnes G-function), and $q = -1$ (those Selberg zeta functions have a simple *zero* at $x = 1$, and possibly others on $(0, 1)$). To illustrate the flexibility of our approach, and since results of this class have interested physicists as well, we treat this *Selberg case* in Appendix B. Other extensions are equally conceivable (e.g., to Hecke L-functions, as achieved upon their Cramér functions [54]).

We now begin with the general results for the superzeta families, attainable for unspecified admissible primary functions L.

10.2 The Three Superzeta Families

We can then define the same three parametric zeta functions over the nontrivial zeros $\{\rho\}$ of a general primary function L satisfying the above conditions (a)–(d), just as in the Riemann case to which we refer for details (Chap. 5):

$$\mathscr{Z}(s\,|\,t) = \sum_{\rho} (\tfrac{1}{2} + t - \rho)^{-s} \equiv \sum_{\rho} (\rho + t - \tfrac{1}{2})^{-s}, \qquad \mathrm{Re}\, s > 1, \quad (10.6)$$

$$\mathcal{Z}(\sigma\,|\,t) = \sum_{k=1}^{\infty} ({\tau_k}^2 + t^2)^{-\sigma}, \qquad \mathrm{Re}\, \sigma > \tfrac{1}{2}, \quad (10.7)$$

$$\mathfrak{Z}(s\,|\,\tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \qquad \mathrm{Re}\, s > 1; \quad (10.8)$$

we also keep the shorthand names for the two points t of special interest:

$$\mathscr{Z}_0(s) = \mathscr{Z}(s\,|\,0), \qquad \mathscr{Z}_*(s) = \mathscr{Z}(s\,|\,\tfrac{1}{2}); \quad (10.9)$$
$$\mathcal{Z}_0(\sigma) = \mathcal{Z}(\sigma\,|\,0) \equiv (2\cos\pi\sigma)^{-1}\,\mathscr{Z}_0(2\sigma), \qquad \mathcal{Z}_*(\sigma) = \mathcal{Z}(\sigma\,|\,\tfrac{1}{2}); \quad (10.10)$$

more generally, all other considerations of Chap. 5 remain valid here.

We now describe the explicit results in greater detail family by family. The logic exactly follows that of the previous chapters, so we will mainly restate the formulae that have a significantly different abstract form.

10.3 The First Family $\{\mathscr{Z}\}$

This will extend the treatment of Chap. 7 from the Riemann case.

10.3.1 The Zeta Function $\mathbf{Z}(s\,|\,t)$ over the Trivial Zeros

A key role is played by the zeta function wholly analogous to $\mathscr{Z}(s\,|\,t)$ but built on the trivial zeros of $L(x)$ (which we call the shadow zeta function of $\mathscr{Z}(s\,|\,t)$):

$$\mathbf{Z}(s\,|\,t) \stackrel{\text{def}}{=} \sum_k (\tfrac{1}{2} + t - x_k)^{-s} \qquad (\operatorname{Re} s > 1). \qquad (10.11)$$

Here this function and its properties should be taken as completely known, just like the factor $\mathbf{G}$ and the trivial zeros themselves. In our later explicit examples, $\mathbf{Z}(s\,|\,t)$ will be expressible in terms of the Hurwitz zeta function (3.33).

We now specialize the results of Chap. 2 first to the trivial factor $\mathbf{G}$. By assumption (c_1), the Stirling formula (3.51) will produce a large-x expansion for $\log \mathbf{G}(x)$ with $\mu_0 = 1$, which we treat as known and reorganize according to allowed/banned terms, as

$$-\log \mathbf{G}(\tfrac{1}{2} + t) \sim \tilde{\mathbf{a}}_1 t(\log t - 1) + \mathbf{b}_1 t + \tilde{\mathbf{a}}_0 \log t + \mathbf{b}_0 + \sum_{n=1}^{\infty} \mathbf{a}_{-n} t^{-n}; \quad (10.12)$$

this expansion also governs $[\log \Xi(\tfrac{1}{2} + t) - q\log(t - \tfrac{1}{2})]$, by (10.2) and (10.3). Equation (2.50) then yields the zeta-regularized forms for $\mathbf{G}$ and Ξ as

$$\mathbf{D}(\tfrac{1}{2} + t) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathbf{Z}'(0\,|\,t)} \equiv \mathrm{e}^{+\mathbf{b}_1 t + \mathbf{b}_0}\, \mathbf{G}(\tfrac{1}{2} + t), \qquad (10.13)$$
$$\mathscr{D}(\tfrac{1}{2} + t) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathscr{Z}'(0\,|\,t)} \equiv \mathrm{e}^{-\mathbf{b}_1 t - \mathbf{b}_0}\, \Xi(\tfrac{1}{2} + t), \qquad (10.14)$$

which in turn entail this zeta-regularized decomposition of $L(x)$:

$$(x-1)^q L(x) \equiv \mathbf{D}(x)\mathscr{D}(x). \tag{10.15}$$

Then, using (10.13), the specific translation of (2.53)–(2.56) with $\mu_0 = 1$ is

$$\mathrm{FP}_{s=1}\,\mathbf{Z}(s\,|\,t) \equiv (\log \mathbf{G})'(\tfrac{1}{2}+t) + \mathbf{b}_1, \tag{10.16}$$

$$\mathbf{Z}(m\,|\,t) \equiv \frac{(-1)^{m-1}}{(m-1)!}(\log \mathbf{G})^{(m)}(\tfrac{1}{2}+t) \quad \text{for } m = 2,3,\ldots. \tag{10.17}$$

The shifted large-y expansion of $-\log \mathbf{D}(\frac{1}{2}+t+y)$ is then deduced as in Sect. 2.6.1, in the form $\tilde{\mathbf{a}}_1(t)\,y(\log y - 1) + \tilde{\mathbf{a}}_0(t)\log y + \sum_{n=1}^{\infty} \mathbf{a}_{-n}(t)\,y^{-n}$: all coefficients are computable polynomials in t and encode algebraic properties of $\mathbf{Z}(s\,|\,t)$, as explained in Sect. 2.4. In more explicit terms:

- $\mathbf{Z}(s\,|\,t)$ extends to a meromorphic function in the whole s-plane, with

$$\text{the single pole } s=1, \text{ simple, of residue } \tilde{\mathbf{a}}_1 \qquad (\text{independent of } t); \tag{10.18}$$

- The values $\mathbf{Z}(-n\,|\,t)$, $n \in \mathbb{N}$ are given by *closed polynomial formulae*,

$$\mathbf{Z}(-n\,|\,t) = -\frac{\tilde{\mathbf{a}}_1}{n+1}\,t^{n+1} - \tilde{\mathbf{a}}_0 t^n + n\sum_{j=1}^{n}(-1)^j \binom{n-1}{j-1}\mathbf{a}_{-j}t^{n-j}, \quad \text{e.g.,} \quad \mathbf{Z}(0\,|\,t) = -\tilde{\mathbf{a}}_0(t) = -\tilde{\mathbf{a}}_1 t - \tilde{\mathbf{a}}_0\,. \tag{10.19}$$

More interestingly, $\mathbf{D}$, $\mathbf{Z}$, $\mathbf{G}$ can be replaced by the (less explicit) $\mathscr{D}$, $\mathscr{Z}$, Ξ respectively, and similar results will arise, as described next.

10.3.2 The Basic Analytical Continuation Formula for $\mathscr{Z}$

As an extension from (7.4)–(7.5) (Riemann case), $\mathscr{Z}(s\,|\,t)$ admits the following integral representation, valid in the half-plane $\{\mathrm{Re}\, s < 1\}$ and for any eligible value of the parameter t avoiding the cut $(-\infty, +\frac{1}{2}]$:

$$\mathscr{Z}(s\,|\,t) \;=\; -\mathbf{Z}(s\,|\,t) + q\,(t-\tfrac{1}{2})^{-s} + \frac{\sin \pi s}{\pi}\mathscr{J}(s\,|\,t), \tag{10.20}$$

$$\mathscr{J}(s\,|\,t) \stackrel{\text{def}}{=} \int_0^\infty \frac{L'}{L}(\tfrac{1}{2}+t+y)\,y^{-s}\,\mathrm{d}y \qquad (\mathrm{Re}\ s < 1). \tag{10.21}$$

The real forms (7.6)–(7.7) and the description of the poles of $\mathscr{J}$ extend likewise, just replacing ζ by L and $\dfrac{1}{t-\frac12+y}$ by $\dfrac{q}{t-\frac12+y}$ everywhere.

As in Sect. 7.3 for the Riemann case:

• It follows that $\mathscr{Z}(s\,|\,t)$ is meromorphic in the whole s-plane with the same polar structure as $-\,\mathbf{Z}(s\,|\,t)$, which now means that

$$\mathscr{Z}(s\,|\,t) \text{ has the single pole } s=1, \text{ simple, of residue } -\tilde{\mathbf{a}}_1; \tag{10.22}$$

• If $L(x)$ admits an Euler product (like our concrete examples (10.49) and (10.72)), then the substitution of its logarithmic derivative into (10.21), followed by integration term by term, yields an *asymptotic* ($s\to-\infty$) *expansion* for $\mathscr{J}(s\,|\,t)$, and thereby for $\mathscr{Z}(s\,|\,t)$ (cf. (7.27)–(7.28) for the Riemann case).

10.3.3 Special Values of $\mathscr{Z}(s\,|\,t)$ for General t

As in Sect. 7.4 for the Riemann case:

• Almost all the *special values* of $\mathscr{Z}(s\,|\,t)$ (at integer s) are explicitly readable off (10.20): as in (7.29)–(7.38) before, we get rational values for $s\in-\mathbb{N}$, transcendental ones for $s=2,3,\ldots$, plus

$$\begin{aligned}\mathscr{Z}'(0\,|\,t) &= -\,\mathbf{Z}'(0\,|\,t) - q\log(t-\tfrac12) + \mathscr{J}(0\,|\,t)\\ &= \mathbf{b}_1 t + \mathbf{b}_0 + \log\mathbf{G}(\tfrac12+t) - \log\left[(t-\tfrac12)^q L(\tfrac12+t)\right] \end{aligned}\tag{10.23}$$

$$\begin{aligned}\mathrm{FP}_{s=1}\,\mathscr{Z}(s\,|\,t) &= -\mathrm{FP}_{s=1}\,\mathbf{Z}(s\,|\,t) + \frac{q}{t-\frac12} - \mathrm{Res}_{s=1}\mathscr{J}(s\,|\,t),\\ &= -\mathbf{b}_1 - (\log\mathbf{G})'(\tfrac12+t) + \left[\frac{q}{t-\frac12} + \frac{L'}{L}(\tfrac12+t)\right]. \end{aligned}\tag{10.24}$$

• However, the values $\mathscr{Z}(n\,|\,t)$ for $n\in\mathbb{N}^*$, now including $n=1$, emerge more directly by proceeding in full analogy with (7.39) (Riemann case):

$$\mathscr{Z}(n\,|\,t) = \frac{(-1)^{n-1}}{(n-1)!}(\log\varXi)^{(n)}(\tfrac12+t) \qquad (n=1,2,\ldots) \tag{10.25}$$

$$= -\,\mathbf{Z}(n\,|\,t) + \left[\frac{q}{(t-\frac12)^n} + \frac{(-1)^{n-1}}{(n-1)!}(\log|L|)^{(n)}(\tfrac12+t)\right] \qquad (n=2,3,\ldots). \tag{10.26}$$

Since $\mathbf{Z}(1\,|\,t)$ is infinite, (10.26) cannot hold for $n=1$ but a substitute formula exists: i.e., the subtraction of (10.25) at $n=1$ from (10.24) yields a t-independent *anomaly*, or discrepancy formula extending (7.40):

$$\mathrm{FP}_{s=1}\,\mathscr{Z}(s\,|\,t) - \mathscr{Z}(1\,|\,t) = \big(\log[\mathscr{D}/\varXi]\big)'(\tfrac12+t) = -\mathbf{b}_1 \quad \text{(constant)}. \tag{10.27}$$

Table 10.1 [107] recapitulates the special values obtained for $\mathscr{Z}(s\,|\,t)$ at general t, extending Table 7.1 from the Riemann case.

Remark. All identities stemming purely from the central symmetry of the zeros $\rho \longleftrightarrow (1-\rho)$, like $\mathscr{Z}(n\,|\,t) = (-1)^n\, \mathscr{Z}(n\,|\,-t)$ $(n = 1, 2, \ldots)$ and the sum rules (7.46) (or (7.43)), carry over unchanged. However, the rest of Table 7.2 needs rewriting (exercise!).

10.3.4 Special Values of $\mathscr{Z}(s\,|\,t)$ at $t = 0$ and $\frac{1}{2}$

Under our assumption $\{\rho\} \cap \mathbb{R} = \emptyset$, $\mathscr{Z}(s\,|\,t)$ is regular on the real t-axis; then, further simplifications occur at the particular parameter locations $t = 0$ and $\frac{1}{2}$.

- For $t = 0$, (10.25) reduces to

$$\mathscr{Z}_0(n) \equiv 0 \qquad\qquad \text{for all } n \geq 1 \text{ odd}; \tag{10.28}$$

in combination with (10.26) and (10.24), that amounts to explicit formulae for the primary function L itself:

$$(\log|L|)^{(n)}(\tfrac{1}{2}) = (\log \mathbf{G})^{(n)}(\tfrac{1}{2}) + 2^n q\,(n-1)! \qquad \text{for all } n \geq 1 \text{ odd} \tag{10.29}$$

(also directly implied by the functional equation (10.3); they extend (7.48) from the Riemann case). By (10.27) at $t = 0$, the case $n = 1$ moreover implies

$$\mathrm{FP}_{s=1}\, \mathscr{Z}_0(s) = -\mathbf{b}_1. \tag{10.30}$$

- For $t = \frac{1}{2}$, the formulae (10.23)–(10.26) bring in the Taylor series (10.5), to yield

$$\begin{aligned} \mathscr{Z}_*'(0) &= -\,\mathbf{Z}'(0\,|\,\tfrac{1}{2}) + g_0^{\mathrm{c}}; \\ \mathscr{Z}_*(1) &= -(\log \mathbf{G})'(1) + g_1^{\mathrm{c}}, \\ \mathscr{Z}_*(n) &= -\,\mathbf{Z}(n\,|\,\tfrac{1}{2}) + g_n^{\mathrm{c}}/(n-1)! \qquad (n = 2, 3, \ldots). \end{aligned} \tag{10.31}$$

The purely combinatorial relations (7.56) between the $\mathscr{Z}_*(n)$ and Keiper–Li coefficients λ_j remain unchanged; the latter now refer to the zeros of the primary function L, like the rest.

It is hard to achieve further progress while keeping the primary function L and its "accessories" ($\mathbf{G}$, $\mathbf{Z}$, etc.) completely unspecified. Case by case, on the other hand, $\mathbf{Z}(s\,|\,t)$ can be made more explicit at $t = 0$ or $\frac{1}{2}$, just as in the Riemann case. Thus, in our later examples for $L(x)$ (L-functions of real primitive Dirichlet characters; Dedekind zeta functions), both $\mathbf{Z}(s\,|\,0)$ and

$\mathbf{Z}(s\,|\,\frac{1}{2})$ will reduce to combinations of the two fixed Dirichlet series $\zeta(s)$ and $\beta(s)$. The resulting fully reduced special values of $\mathscr{Z}_0(s)$ and $\mathscr{Z}_*(s)$ will be displayed in Tables 10.3–10.6 [107], which conclude the chapter.

10.4 The Second Family $\{\mathcal{Z}\}$

This will extend the treatment of Chap. 8 from the Riemann case.

10.4.1 The Confluent Case $\mathcal{Z}(\sigma\,|\,t=0) \equiv \mathcal{Z}_0(\sigma)$

The confluence identity is unchanged from (8.2): $\mathcal{Z}_0(\sigma) \equiv (2\cos\pi\sigma)^{-1}\,\mathscr{Z}_0(2\sigma)$. Therefore:

- The function $\mathcal{Z}_0$ retains the same abstract singular structure as in Sect. 8.1; the explicit principal part formulae (8.4)–(8.5) simply extend to

$$\mathcal{Z}_0(\tfrac{1}{2}+\varepsilon) = \frac{\tilde{\mathbf{a}}_1}{4\pi}\,\varepsilon^{-2} + \frac{\mathbf{b}_1}{2\pi}\,\varepsilon^{-1} \;+ \mathrm{O}(1)_{\varepsilon\to 0}\;; \tag{10.32}$$

$$\mathcal{Z}_0(\tfrac{1}{2}-m+\varepsilon) = \mathcal{R}_m\,\varepsilon^{-1} \;+ \mathrm{O}(1)_{\varepsilon\to 0} \qquad \text{for } m=1,2,\ldots, \tag{10.33}$$

$$\mathcal{R}_m = \begin{cases} -\dfrac{1}{2\pi}\mathrm{FP}_{s=1}\,\mathscr{Z}_0(s) = \dfrac{\mathbf{b}_1}{2\pi}, & m=0,\\[2mm] \dfrac{(-1)^m}{2\pi}\big[\mathbf{Z}(1-2m\,|\,0) + q\,2^{1-2m}\big], & m=1,2,\ldots; \end{cases} \tag{10.34}$$

- The special values of $\mathcal{Z}_0$ obey the same relations (8.8)–(8.9) as before:

$$\mathcal{Z}_0(m) = \tfrac{1}{2}(-1)^m\,\mathscr{Z}_0(2m) \quad (\forall m\in\mathbb{Z}), \qquad \mathcal{Z}_0'(0) = \mathscr{Z}_0'(0). \tag{10.35}$$

  Both (10.34) and (10.35) now refer to the $\mathscr{Z}$-values of Table 10.1 (at $t=0$).

10.4.2 Algebraic Results for $\mathcal{Z}(\sigma\,|\,t)$ at General t

For the algebraically computable formulae, all the abstract results from Sect. 8.3 carry over unchanged. Concretely, the principal parts (8.21) extend to

$$\mathcal{Z}(\tfrac{1}{2}-m+\varepsilon\,|\,t) = \frac{\tilde{\mathbf{a}}_1}{4\pi}\,\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2})\,m!}\,t^{2m}\,\varepsilon^{-2} + \mathcal{R}_m(t)\,\varepsilon^{-1} \;+ \mathrm{O}(1)_{\varepsilon\to 0}\;, \tag{10.36}$$

$$\mathcal{R}_m(t) = \Gamma(\tfrac{1}{2}+m)\left[-\frac{\tilde{\mathbf{a}}_1 \sum_{j=1}^{m} \frac{1}{2j-1}}{2\pi\, \Gamma(\frac{1}{2})\, m!}\, t^{2m} + \sum_{j=0}^{m} \frac{1}{\Gamma(\frac{1}{2}+j)\,(m-j)!}\mathcal{R}_j\, t^{2(m-j)}\right] \tag{10.37}$$

where the $t = 0$ residues $\mathcal{R}_j$ are now read from (10.34). And the rational special values still obey (8.23), or equivalently

$$\mathcal{Z}(-m\,|\,t) \equiv \tfrac{1}{2}\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m}{\ell}\,\mathscr{Z}_0\big(2(-m+\ell)\big)\,t^{2\ell} \qquad (m \in \mathbb{N}), \tag{10.38}$$

with the values $\mathscr{Z}_0\big(2(-m+\ell)\big)$ now taken from Table 10.1 (at $t = 0$). So, all the polar terms of $\mathcal{Z}(\sigma\,|\,t)$, still of order 2, and the special values $\mathcal{Z}(-m\,|\,t)$ $(m \in \mathbb{N})$ are *computable polynomials* in t^2, as in the Riemann case.

The principal part at the leading pole $\sigma = \frac{1}{2}$, given by (10.32), and the leading special value $\mathcal{Z}(0\,|\,t) = \frac{1}{2}(q + \tilde{\mathbf{a}}_0)$, remain *independent of* t as before. Those two invariants amount to three constants $\tilde{\mathbf{a}}_1$, $\mathbf{b}_1$, and $\frac{1}{2}(q+\tilde{\mathbf{a}}_0)$, which also have *another embodiment*. As with (4.29) for the Riemann case, let

$$\overline{N}(T) \stackrel{\text{def}}{=} \pi^{-1}\,\mathrm{Im}\,\log\big[(x-1)^q\mathbf{G}^{-1}(x)\big]_{x=\frac{1}{2}+\mathrm{i}T} \tag{10.39}$$

be the contribution from the trivial factors of Ξ to the counting function $N(T)$ (for its zeros $\{\rho\}$). Then the generalized Stirling formula (10.12) readily yields

$$\overline{N}(T) = \frac{\tilde{\mathbf{a}}_1}{\pi}\,T(\log T - 1) + \frac{\mathbf{b}_1}{\pi}\,T + \frac{1}{2}(q+\tilde{\mathbf{a}}_0) + O\Big(\frac{1}{T}\Big), \tag{10.40}$$

which is built from those *same three constants*, cf. (4.31) for the Riemann case. We may recall that $\mathbf{b}_1$ also governs the discrepancy at $n = 1$ in (10.27).

Remark: in both our later concrete examples, the full counting function itself obeys $N(T) = \overline{N}(T) + \mathrm{O}(\log T)$, extending the Riemann–von Mangoldt formula (4.26): this is proved, e.g., in [26, Sect. 16] for Dirichlet L-functions and in [69, Satz 173 p. 89] for Dedekind zeta functions.

10.4.3 Transcendental Values of $\mathcal{Z}(\sigma\,|\,t)$ at General t

As in the Riemann case (Sect. 8.4), these values most readily emerge from a variant of the factorization (10.15), using the alternative zeta-regularized factor

$$\mathcal{D}(t^2) \stackrel{\text{def}}{=} \mathrm{e}^{-\,\mathcal{Z}'(0\,|\,t)} \tag{10.41}$$

instead of $\mathscr{D}(x)$. The main point here is the replacement of $x = \frac{1}{2} + t$ by $v = t^2$ as basic variable: this zeta-regularization of $\Xi(x)$ *preserves the central symmetry* $(x \longleftrightarrow 1 - x)$.

Rewritten in the variable $v \to +\infty$, the generalized Stirling expansion (10.12) for $[\log \Xi(x) - q \log(x-1)]$ becomes

$$\log \Xi(\sqrt{v}+\tfrac{1}{2}) \sim \tfrac{1}{2}\tilde{\mathbf{a}}_1 v^{\frac{1}{2}} \log v + (\mathbf{b}_1 - \tilde{\mathbf{a}}_1) v^{\frac{1}{2}} + \tfrac{1}{2}(\tilde{\mathbf{a}}_0 + q) \log v + \mathbf{b}_0 \; [+\mathrm{O}(v^{-\frac{1}{2}})], \tag{10.42}$$

with $\mu_0 = \frac{1}{2}$: the only "banned" terms (cf. (2.47)) are now *constants*, implying

$$\mathcal{D}(t^2) \equiv \mathrm{e}^{-\mathbf{b}_0}\, \Xi(\tfrac{1}{2}+t) \equiv \mathrm{e}^{\mathbf{b}_1 t}\, \mathscr{D}(\tfrac{1}{2}+t) \tag{10.43}$$

and the modified decomposition (cf. (8.18) for the Riemann case)

$$(t-\tfrac{1}{2})^q L(\tfrac{1}{2}+t) \equiv \mathrm{e}^{-\mathbf{b}_1 t}\, \mathbf{D}(\tfrac{1}{2}+t) \mathcal{D}(t^2). \tag{10.44}$$

All transcendental special values of $\mathcal{Z}(\sigma \,|\, t)$ immediately follow: first, $\mathcal{Z}'(0 \,|\, t) \equiv -\log \mathcal{D}(t^2) = \mathbf{b}_0 - \log \Xi(\frac{1}{2}+t)$ which also expresses in terms of $\log |L|(\frac{1}{2} \pm t)$; then (2.56), now applied with $v = t^2$ as variable and $\mu_0 = \frac{1}{2}$, yields the same form (8.26) for $\mathcal{Z}(m \,|\, t)$ as before.

The overall resulting special values of $\mathcal{Z}$ form Table 10.2, extending Table 8.1 from the Riemann case. The particular parameter locations $t = 0$ and $\frac{1}{2}$ can be covered by Sect. 10.4.1 and Table 10.2 respectively (mimicking Sect. 8.6.2 from the Riemann case), without need for further Tables.

The various combinatorial linear identities relating special values of $\mathcal{Z}$ at positive integers to those of $\mathscr{Z}$ (Sect. 8.5) or to the Keiper–Li coefficients λ_j ((8.34)–(8.36)) persist identically, with all objects now in their extended meaning (linked to the function L).

10.5 The Third Family $\{\mathfrak{Z}\}$

The abstract treatment of Chap. 9 for the Riemann case extends unchanged. So, the function $\mathfrak{Z}$ keeps the same qualitative meromorphic structure; we just rewrite its polar expansions (9.5)–(9.8) in their abstract extended form:

$$(\text{about } s=1) \qquad \mathfrak{Z}(1+\varepsilon \,|\, \tau) = \frac{\tilde{\mathbf{a}}_1}{\pi}\, \varepsilon^{-2} + \frac{\mathbf{b}_1}{\pi}\, \varepsilon^{-1} + \mathrm{O}(1)_{\varepsilon \to 0} \tag{10.45}$$

($s = 1$ is a double pole with the same principal part as $\mathcal{Z}(\frac{1}{2}s) \equiv \mathfrak{Z}(s \,|\, 0)$, fixed for all τ); and, for $n = 1, 2, \ldots$ (now using (10.34) for $\mathcal{R}_j$):

$$\mathfrak{Z}(1-n+\varepsilon \,|\, \tau) = \left[-\frac{\tilde{\mathbf{a}}_1}{\pi n}\, \tau^n + 2 \sum_{0<2m\le n} \binom{n-1}{2m-1} \mathcal{R}_m\, \tau^{n-2m} \right] \varepsilon^{-1} + \mathrm{O}(1)_{\varepsilon \to 0}\,; \tag{10.46}$$

for $n = 1$, the next term is also attained (i.e., the finite part at $s = 0$):

$$\mathfrak{Z}(\varepsilon \,|\, \tau) = -\frac{\tilde{\mathbf{a}}_1}{\pi}\,\tau\,\varepsilon^{-1} + \left(\tfrac{1}{2}(\tilde{\mathbf{a}}_0 + q) - \frac{\mathbf{b}_1}{\pi}\,\tau\right) + \mathrm{o}(1)_{\varepsilon\to 0}\,. \tag{10.47}$$

10.6 Special Concrete Examples

We finally illustrate the preceding results upon the two classes of primary zeta functions announced in Sect. 10.1. As a rule, it suffices to specialize the general formulae as indicated below. To enhance the practical side of this work, we display the ultimate results, namely the special values of $\mathscr{Z}(s\,|\,0) \equiv \mathscr{Z}_0(s)$ and $\mathscr{Z}(s\,|\,\frac{1}{2}) \equiv \mathscr{Z}_*(s)$, in Tables 10.3–10.6 (corresponding to Tables 7.3–7.4 for $L(x) = \zeta(x)$, which fits better here as a special case of Dedekind zeta function). Then, any further results can be readily derived from the general formalism, as in the Riemann case.

At the particular parameter locations $t = 0$ and $\frac{1}{2}$, the relevant values of $\mathbf{Z}(s\,|\,t)$ become more explicit, using

$$\begin{aligned} &\zeta(s,1) \equiv \zeta(s); \qquad \zeta(s,\tfrac{1}{2}) \equiv (2^s - 1)\,\zeta(s);\\ &2^{-2s}\zeta(s,\tfrac{1}{2} \mp \tfrac{1}{4}) \equiv \tfrac{1}{2}[(1-2^{-s})\,\zeta(s) \pm \beta(s)] \end{aligned} \tag{10.48}$$

(cf. (3.27)). Due to this, the two fixed Dirichlet series $\zeta(s)$ and $\beta(s)$ (itself a particular Dirichlet L-function, reviewed in Sect. 3.5) will continue to occur as such in the $t = 0$ and $\frac{1}{2}$ special values, concurrently with the variable primary function $L(x)$ itself.

We now describe those two classes in turn. At this more technical stage, we abandon the idea of being self-contained, and instead, we will selectively quote (without proof) the prerequisites we really need, which are classic but often quite scattered in lengthy treatises. The most explicit properties of the resulting functions $\mathscr{Z}_0(s)$ and $\mathscr{Z}_*(s)$ are tabulated at the end.

A somewhat different third class, where the primary function $L(x)$ is chosen to be a *Selberg zeta function*, is moreover described in Appendix B.

10.6.1 L-Functions of Real Primitive Dirichlet Characters

A *Dirichlet L-function* is associated with a *character* χ of a multiplicative group of integers mod d ($d \in \mathbb{N}^*$ is called the *modulus* or *conductor*), as [14, 26, 33]

$$L_\chi(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \chi(k)\, k^{-x} \equiv \prod_{\{p\}} \big(1-\chi(p)\, p^{-x}\big)^{-1} \qquad (\mathrm{Re}\, x > 1) \tag{10.49}$$

$$\equiv d^{-x} \sum_{n=1}^{d} \chi(n)\, \zeta(x, n/d). \tag{10.50}$$

Such a character is necessarily even or odd; its parity "bit" a is defined by

$$a = 0 \text{ or } 1, \quad \text{according to} \quad \chi(-1) = (-1)^a. \tag{10.51}$$

Their Admissibility

Dirichlet L-functions satisfy conditions (a)–(b) of Sect. 10.1 without further ado (reality apart).

We now restrict to *primitive* characters [26, Chap. 5], with $d > 1$ to exclude the case $\chi \equiv 1$ (for which $L_\chi(x) \equiv \zeta(x)$, a case which will better fit the other class below). Then, $L_\chi(x)$ is *entire* (i.e., $q = 0$), and has a functional equation [26, Chap. 9]:

$$\Xi_\chi(x) \equiv W_\chi\, \Xi_{\overline{\chi}}(1-x), \tag{10.52}$$

with

$$\Xi_\chi(x) \stackrel{\text{def}}{=} (2\sqrt{\pi})^{-a} (d/\pi)^{x/2}\, \Gamma\big(\tfrac{1}{2}(x+a)\big)\, L_\chi(x), \tag{10.53}$$

$$W_\chi \stackrel{\text{def}}{=} (-\mathrm{i})^a d^{-1/2} \sum_{n \bmod d} \chi(n)\ \mathrm{e}^{2\pi \mathrm{i} n/d}; \tag{10.54}$$

the latter sum, called the *Gaussian sum* for χ, has *modulus* $d^{1/2}$ [26, Chap. 9, (5)], implying $|W_\chi| = 1$. (*Note*: in (10.53) we chose to insert an entirely optional prefactor $(2\sqrt{\pi})^{-a}$, only to streamline a part of Sect. 10.6.2 later.)

We finally keep the *real* ($\overline{\chi} = \chi$) primitive characters only. These consist of the *Kronecker symbols* for the *quadratic number fields* K with discriminant $d_K \equiv (-1)^a d$ [26, Chap. 5], so these characters are best labeled $\chi_{(-1)^a d}$ (see also Sect. 10.6.2). Their Gaussian sums are *fully* known ($= \mathrm{i}^a d^{1/2}$) [49, Sect. 58, Theorem 164], now implying $W_\chi \equiv +1$; it follows that their L-functions (now real) satisfy condition (c) of Sect. 10.1 as well, with

$$q \equiv 0, \qquad \mathbf{G}(x) \equiv (2\sqrt{\pi})^a \frac{(\pi/d)^{x/2}}{\Gamma\big(\frac{1}{2}(x+a)\big)}, \qquad a = \begin{cases} 1 & \text{for } \chi \text{ odd} \\ 0 & \text{for } \chi \text{ even.} \end{cases} \tag{10.55}$$

Finally, condition (d) with $q = 0$, or $L_\chi(1) > 0$, is ensured by *Dirichlet's class number formula* giving $L_\chi(1)$ in terms of *invariants of the quadratic field* K and chiefly its *class number* $h > 0$, as [115][26, Sect. 6]

$$L_\chi(1) = \begin{cases} 2\pi h \,/\, W\sqrt{d} & (W = \#\{\textit{roots of unity}\} < \infty) \quad \text{if } a = 1 \\ 2h \log \varepsilon / \sqrt{d} & (\varepsilon = \textit{fundamental unit} > 1) \quad \text{if } a = 0. \end{cases} \tag{10.56}$$

So in the end, L-functions of real primitive Dirichlet characters are admissible primary functions for us. Any choice of such an L-function L_χ as primary function will henceforth be abbreviated "a Dirichlet-L case."

Results for Superzeta Functions in Dirichlet-*L* Cases

Based on (10.55), we can specify the other quantities needed in Table 10.1:

- The shadow zeta function (10.11) becomes

$$\mathbf{Z}(s \,|\, t) = 2^{-s}\zeta\big(s, \tfrac{1}{2}(\tfrac{1}{2} + a + t)\big) \tag{10.57}$$

- The leading coefficients in the Stirling formula (10.12) are

$$\begin{aligned} \tilde{\mathbf{a}}_1 &= \tfrac{1}{2}\,, & \tilde{\mathbf{a}}_0 &= \tfrac{1}{2}(a - \tfrac{1}{2}), \\ \mathbf{b}_1 &= \tfrac{1}{2}\log\big[d/(2\pi)\big], & \mathbf{b}_0 &= \tfrac{1}{4}\log\big[(8\pi)^{1-2a} d\big]. \end{aligned} \tag{10.58}$$

This completes the tool kit needed to handle Dirichlet-L cases for general t.

We then list further results only for the particular cases $t = 0$ and $\frac{1}{2}$, in Tables 10.3 and 10.4 respectively and in the following comments.

• $\boldsymbol{t = 0}$**:** the identities (10.29), resulting from $\mathscr{Z}_0(n) \equiv 0$ for odd $n \geq 1$, yield more explicit formulae for the Dirichlet L-function L_χ itself:

$$\begin{aligned} (\log L_\chi)^{(n)}(\tfrac{1}{2}) &\equiv -2^{-n}\psi^{(n-1)}(\tfrac{1}{4} + \tfrac{1}{2}a) + \delta_{n,1}\,\tfrac{1}{2}\log\frac{\pi}{d} \qquad \text{for } n \geq 1 \text{ odd} \\ &\equiv \begin{cases} \frac{1}{2}(n-1)!\,\big[(2^n - 1)\,\zeta(n) + (1-2a)\,2^n\beta(n)\big], & n > 1, \\ \frac{1}{2}\gamma + \frac{1}{4}(1 - 2a)\,\pi + \frac{1}{2}\log(8\pi/d), & n = 1, \end{cases} \end{aligned} \tag{10.59}$$

in which, n being odd, $\frac{1}{2}(n-1)!\,2^n\beta(n)$ reduces to $\frac{1}{4}\pi^n|E_{n-1}|$ by (3.32) while $\zeta(n)$ remains elusive.

Note: for odd $n > 1$, $(\log L_\chi)^{(n)}(\frac{1}{2}) = L_n^a$ depends on χ solely through its parity bit a, with

$$L_n^0 = (\log|\zeta|)^{(n)}(\tfrac{1}{2}) \quad \text{and} \quad L_n^1 = L_n^0 - \tfrac{1}{2}\pi^n|E_{n-1}| \qquad \text{for odd } n > 1, \tag{10.60}$$

cf. (7.49); for more such identities bypassing $\zeta(n)$, see Appendix C.

• $\boldsymbol{t = \frac{1}{2}}$**:** by (10.5) with $q = 0$, the lowest generalized Stieltjes cumulants are

$$g_0^{\mathrm{c}}[\chi] \equiv -\log L_\chi(1), \qquad g_1^{\mathrm{c}}[\chi] \equiv \frac{L'_\chi}{L_\chi}(1) \tag{10.61}$$

(we restate their χ-dependence from now on, to clarify later formulae). Then, we can evaluate $g_0^{\rm c}[\chi]$ *always*, but $g_1^{\rm c}[\chi]$ *only when* $a=1$ *(odd case)*, as follows.

First, the general formula (10.50), together with the special values (3.37), (3.40), (3.41) of the Hurwitz zeta function, plus $\chi(d)=0$ and $\sum_{n=1}^{d}\chi(n)=0$, yield these special values for $L_\chi(x)$:

$$L_\chi(0) = -\frac{1}{d}\sum_{n=1}^{d-1}\chi(n)\,n \qquad \text{(rational)}, \tag{10.62}$$

$$L'_\chi(0) = -L_\chi(0)\log d + \sum_{n=1}^{d-1}\chi(n)\log\Gamma(n/d) \qquad \text{(transcendental)}, \tag{10.63}$$

$$L_\chi(1) = -\frac{1}{d}\sum_{n=1}^{d-1}\chi(n)\,\psi(n/d) \qquad \text{(transcendental)}. \tag{10.64}$$

Next, the functional equation (10.3) using (10.55) implies the following.

– When $a=1$ (the odd-χ case):

$$L_\chi(1) = \pi d^{-1/2}L_\chi(0), \qquad \frac{L'_\chi}{L_\chi}(1) = \gamma + \log\frac{2\pi}{d} - \frac{L'_\chi}{L_\chi}(0), \tag{10.65}$$

which, together with (10.62), (10.63) yield an *algebraic* explicit formula for $\pi^{-1}L_\chi(1)$ [26, Chap. 6, (17)] (superseding (10.64)) plus a *transcendental* one for $[L'_\chi/L_\chi](1)$ in terms of Gamma values, overall giving

$$\left.\begin{aligned} g_0^{\rm c}[\chi] &= -\log L_\chi(1), \qquad L_\chi(1) = -\frac{\pi}{d^{3/2}}\sum_{n=1}^{d-1}\chi(n)\,n, \\ g_1^{\rm c}[\chi] &= \frac{L'_\chi}{L_\chi}(1) = \gamma + \log 2\pi + \frac{\sum_{n=1}^{d-1}\chi(n)\log\Gamma(n/d)}{\frac{1}{d}\sum_{n=1}^{d-1}\chi(n)\,n} \end{aligned}\right\} \quad \text{if } a=1. \tag{10.66}$$

Example: For each of $d=3$ and 4 (the lowest possible values of d), the real primitive character is unique and odd, given by: $\chi_{-d}(\pm 1 \bmod d) = \pm 1$, else $\chi_{-d}(n)=0$ (thus, $L_{\chi_{-4}}(x)\equiv\beta(x)$ as in (3.27)); then (10.66) yields

$$\begin{aligned} g_0^{\rm c}[\chi_{-3}] &= -\log(\pi/3^{3/2}), \quad g_1^{\rm c}[\chi_{-3}] = \log[(2\pi)^4/3^{3/2}] + \gamma - 6\log\Gamma(\tfrac{1}{3}); \\ g_0^{\rm c}[\chi_{-4}] &= -\log(\pi/4), \qquad g_1^{\rm c}[\chi_{-4}] = \log(4\pi^3) + \gamma - 4\log\Gamma(\tfrac{1}{4}). \end{aligned} \tag{10.67}$$

– When $a = 0$ (the even-χ case):

$$L_\chi(0) \equiv 0 \qquad \text{(the first trivial zero: } \mathbf{G}(0) = 0\text{)}, \tag{10.68}$$
$$L_\chi(1) = 2d^{-1/2} L'_\chi(0), \tag{10.69}$$

and $L'_\chi(0)$, starting from (10.63), simplifies further through (10.68) and the reflection formula for Γ; we thus obtain a *transcendental* explicit formula for $L_\chi(1)$ [26, Chap. 6, (18)] (still more elementary than (10.64)), giving

$$g_0^{\mathrm{c}}[\chi] = -\log L_\chi(1), \quad L_\chi(1) = -\frac{1}{d^{1/2}} \sum_{n=1}^{d-1} \chi(n) \log \sin \frac{\pi n}{d} \qquad \text{if } a = 0. \tag{10.70}$$

On the other hand, $g_1^{\mathrm{c}}[\chi]$ stays *unspecified* because the functional equation only relates $L'_\chi(1)$ to $L''_\chi(0)$ when χ is even, leaving us no wiser: thus for an even character, $g_1^{\mathrm{c}}[\chi]$ seems to generalize Euler's constant in a nontrivial way (actually related to the *Euler–Kronecker invariant* of the field K, see (10.83) in Sect. 10.6.2).

Example: The lowest modulus for an *even* real primitive Dirichlet character is $d = 5$, with $\chi_{+5}(\pm 1 \bmod 5) = +1$, $\chi_{+5}(\pm 2 \bmod 5) = -1$, else $\chi_{+5}(n) = 0$; then (10.70) only yields

$$g_0^{\mathrm{c}}[\chi_{+5}] = -\log\Big(\frac{2}{\sqrt{5}} \log\Big[2\cos\frac{\pi}{5}\Big]\Big) = -\log\Big(\frac{2}{\sqrt{5}} \log \frac{\sqrt{5}+1}{2}\Big). \tag{10.71}$$

Remark. The $g_n^{\mathrm{c}}[\chi]$ for general n also relate, through (10.50), to the Laurent coefficients $\gamma_m(w)$ of the Hurwitz zeta function $\zeta(x, w)$ around $x = 1$ [5, 65, 114]; however, these generalized Stieltjes constants $\gamma_m(w)$ are even more elusive than the original ones $\gamma_m = \gamma_m(1)$ from (3.9).

The overall resulting special values of $\mathscr{Z}(s \mid 0)$ and $\mathscr{Z}(s \mid \frac{1}{2})$ in Dirichlet-L cases are presented in Tables 10.3 and 10.4 respectively, drawn from [107].

10.6.2 Dedekind Zeta Functions

For any algebraic number field K, its *Dedekind zeta function* is defined as

$$\zeta_K(x) \stackrel{\text{def}}{=} \sum_{\mathfrak{a}} N(\mathfrak{a})^{-x} \equiv \prod_{\mathfrak{p}} \big(1 - N(\mathfrak{p})^{-x}\big)^{-1} \qquad (\operatorname{Re} x > 1), \tag{10.72}$$

where $\mathfrak{a}$ (resp. $\mathfrak{p}$) runs over all *integral* (resp. *prime*) ideals of K, and $N(\mathfrak{a})$ is the *norm* of $\mathfrak{a}$ [49, Sect. 42][23, Sect. 10.5].

Their Admissibility

Dedekind zeta functions at once satisfy conditions (a)–(c) of Sect. 10.1 with

$$q \equiv 1, \qquad \mathbf{G}(x) \equiv \frac{\left(4^{r_2}\pi^{n_K}/|d_K|\right)^{x/2}}{x\Gamma(x/2)^{r_1}\Gamma(x)^{r_2}}\,, \tag{10.73}$$

where r_1 (resp. $2r_2$) is the number of real (resp. complex) *conjugate fields* of K, $n_K \equiv r_1 + 2r_2$ is the *degree* of K, and d_K ($\gtrless 0$) its *discriminant* [49, Sect. 42]. With Laurent series at $x = 1$ having the form

$$\zeta_K(x) = \frac{\mathfrak{R}_K}{x-1} + \mathfrak{C}_K + \cdots\,, \tag{10.74}$$

condition (d) asks for $\mathfrak{R}_K > 0$; this is now ensured by *Dedekind's class number formula* for this residue, involving further *positive invariants* of the field K [49, Theorems 121, 124][23, Theorem 10.5.1]:

$$\mathfrak{R}_K = \frac{2^{r_1+r_2}\pi^{r_2}hR}{W\sqrt{|d_K|}}, \tag{10.75}$$

$h =$ the *class number*, $R =$ the *regulator*, $W =$ the *number of roots of unity*.

So, Dedekind zeta functions $\zeta_K(x)$ are admissible primary functions for us: any such choice will be denoted "a Dedekind-ζ case" here. For $K = \mathbb{Q}$, which has $r_1 = 1$, $r_2 = 0$, and $d_K = 1$, one recovers the *Riemann case*: $\zeta_K(x) \equiv \zeta(x)$, with the trivial factor $\mathbf{G}(x) \equiv \pi^{x/2}\left[x\Gamma(x/2)\right]^{-1}$ as in (3.24).

Results for Superzeta Functions in Dedekind-ζ Cases

Based on (10.73), we can specify the other quantities needed in Table 10.1:

- The shadow zeta function (10.11), counting all zeros of $\mathbf{G}(x)$ with their multiplicities, becomes

$$\mathbf{Z}(s\,|\,t) = r_1 2^{-s}\zeta(s, \tfrac14 + \tfrac12 t) + r_2\,\zeta(s, \tfrac12 + t) - (\tfrac12 + t)^{-s} \tag{10.76}$$

- The leading coefficients in the Stirling formula (10.12) are

$$\begin{aligned} \tilde{\mathbf{a}}_1 &= \tfrac12 n_K, & \tilde{\mathbf{a}}_0 &= 1 - \tfrac14 r_1, \\ \mathbf{b}_1 &= \tfrac12 \log\left[\,|d_K|/(2\pi)^{n_K}\right], & \mathbf{b}_0 &= \tfrac14 \log\left[(8\pi)^{r_1}|d_K|\,\right]. \end{aligned} \tag{10.77}$$

This completes the tool kit needed to handle Dedekind-ζ cases for general t.

We then list further results only for the particular cases $t = 0$ and $\frac12$, in Tables 10.5 and 10.6 respectively and in the following comments.

• $\boldsymbol{t=0}$**:** the identities (10.29), resulting from $\mathscr{Z}_0(n)\equiv 0$ for odd $n\ge 1$, yield more explicit identities for the Dedekind zeta function ζ_K itself:

$$(\log|\zeta_K|)^{(n)}(\tfrac12)\equiv -2^{-n}r_1\psi^{(n-1)}(\tfrac14)-r_2\psi^{(n-1)}(\tfrac12)+\delta_{n,1}\,\tfrac12\log\frac{4^{r_2}\pi^{n_K}}{|d_K|} \qquad \text{for } n\ge 1 \text{ odd}$$

$$\equiv\begin{cases}\tfrac12(n-1)!\,\big[n_K(2^n-1)\,\zeta(n)+r_1 2^n\beta(n)\big], & n>1,\\ \tfrac12\,n_K\,\gamma+\tfrac14\,r_1\,\pi+\tfrac12\log\big[(8\pi)^{n_K}/|d_K|\big], & n=1,\end{cases} \tag{10.78}$$

in which, n being odd, $\frac12(n-1)!\,2^n\beta(n)$ reduces to $\frac14\pi^n|E_{n-1}|$ by (3.32) while $\zeta(n)$ remains elusive. (For related identities bypassing $\zeta(n)$, see Appendix C.)

• $\boldsymbol{t=\frac12}$**:** by (10.5) with $q=1$ and (10.74), the lowest generalized Stieltjes cumulants are

$$g_0^{\rm c}\{K\}\equiv-\log\mathfrak{R}_K,\qquad g_1^{\rm c}\{K\}\equiv\mathfrak{C}_K/\mathfrak{R}_K \tag{10.79}$$

(we restate their K-dependence to clarify the next formulae). Then, somewhat similarly to the previous example: $g_0^{\rm c}\{K\}$ is *always* expressible – now through Dedekind's class number formula (10.75); and $g_1^{\rm c}\{K\}$, or equivalently the term $\mathfrak{C}_K$, is *sometimes* expressible – through certain *Kronecker limit formulae*; $g_1^{\rm c}\{K\}$ (also known as the *Euler–Kronecker invariant* γ_K [52]) links to $\mathscr{Z}_*(1)=\sum\rho^{-1}$ [47, Theorem B(2)][52, (1.4.1)], as Table 10.6 shows.

– We first consider $K=\mathbb{Q}$: then $\zeta_K(x)\equiv\zeta(x)$, and by (3.15),

$$g_0^{\rm c}\{\mathbb{Q}\}=0,\qquad g_1^{\rm c}\{\mathbb{Q}\}=\gamma. \tag{10.80}$$

– Next, if K is a *quadratic* number field: letting $\chi=\chi_{d_K}$, the real primitive character of modulus $|d_K|$ given by the Kronecker symbol for the discriminant d_K (cf. Sect. 10.6.1), then [49, Sect. 49]

$$\zeta_K(x)\equiv\zeta(x)\,L_\chi(x). \tag{10.81}$$

(*Note*: this factorization nicely extends to their trivial factors, resp. completed functions, given the factor $(2\sqrt{\pi})^a$ we inserted in (10.55)). Now as a rule, zeta functions over zeros (and their linear invariants) obviously *add up* when their primary functions are *multiplied*. Thus, for their Stieltjes cumulants,

$$g_n^{\rm c}\{K\}=g_n^{\rm c}\{\mathbb{Q}\}+g_n^{\rm c}[\chi]\qquad(\forall n), \tag{10.82}$$

and here in particular,

$$g_0^{\rm c}\{K\}=g_0^{\rm c}[\chi],\qquad g_1^{\rm c}\{K\}=\gamma+g_1^{\rm c}[\chi]. \tag{10.83}$$

Following a well-known practice, we can thereby *pass results* from the real primitive Dirichlet characters χ to the quadratic number fields K, and vice-versa. Results on the χ side were supplied before (Sect. 10.6.1). Now on the K side, we can add the class number formula for $g_0^{\rm c}\{K\}$ – which for quadratic K reduces to Dirichlet's (10.56), and Kronecker limit formulae for $g_1^{\rm c}\{K\}$, which come in two types – one simpler form handles *imaginary* quadratic K ($d_K < 0$) using the *Dedekind η-function* at specific points [19, Sect. 6][113, Sect. 2], while much more involved forms hold for *real* quadratic fields ($d_K > 0$) [113, Sect. 3]. All in all, we find the more reduced formulae to be still *those found on the χ side* when they exist:

- For $g_0^{\rm c}$, either (10.66) or (10.70) (according to parity) is a more explicit expression of $L_\chi(1)$ than the class number formula for K (here, (10.56))
- For $g_1^{\rm c}$ in the *odd-χ* case ($d_K < 0$), (10.66) is more elementary than the simpler Kronecker limit formula (which calls Dedekind η-function values)
- Inversely, for $g_1^{\rm c}$ in the *even-χ* case ($d_K > 0$), only the K side provides *something new*, namely Kronecker limit formulae for $g_1^{\rm c}\{K\}$ that are now exceedingly involved, but then Sect. 10.6.1 had left us without any closed expression for the corresponding $g_1^{\rm c}[\chi]$.

Any further description of those aspects would carry us too far here, so we simply refer the reader to the above literature.

Example: Two basic quadratic number fields, both imaginary (i.e., with $r_1 = 0$ and $r_2 = 1$), are: $K = \mathbb{Q}(\mathrm{i})$ (for which $d_K = -4$, $\chi = \chi_{-4} : L_\chi(x) \equiv \beta(x)$ as in (3.27)), and: $K = \mathbb{Q}(\sqrt{-3})$ (for which $d_K = -3$, $\chi = \chi_{-3}$) [14]; hence their specific cumulants $g_0^{\rm c}\{K\}$ and $g_1^{\rm c}\{K\}$ most readily follow from (10.67) and (10.83).

– More generally, $g_1^{\rm c}\{K\}$ will also be expressible: if K is a quadratic extension of another field F, in terms of $g_1^{\rm c}\{F\}$ [45]; if ζ_K factorizes in Dirichlet L-functions (e.g., for cyclotomic fields and their subfields [23, Sect. 10.5.4]).

The overall resulting special values of $\mathscr{Z}(s\,|\,0)$ and $\mathscr{Z}(s\,|\,\frac{1}{2})$ in Dedekind-ζ cases are presented in Tables 10.5 and 10.6 extending Tables 7.3 and 7.4 from the Riemann case, respectively, as drawn from [107].

10.7 Tables of Formulae for the Special Values

Note. The superzeta functions of all three kinds also have explicitly computable polar decompositions, displayed in the main text.

10.7.1 For General Primary Functions $L(x)$ at General t

Table 10.1 Special values of the function of first kind $\mathscr{Z}(s\,|\,t)$ over the zeros $\{\rho\}$ of a general primary zeta function $L(x)$, admissible in the sense of Sect. 10.1, and having a pole of order $q \le 1$ at $x = 1$. Notation: see (10.3) for $\varXi(x)$, (10.11) and (10.19) for $\mathbf{Z}(-n\,|\,t)$, (10.12) for $\tilde{\mathbf{a}}_j$, $\mathbf{b}_j$; n is an integer. For $L(x) = \zeta(x)$ (the Riemann case), cf. Table 7.1

s	$\mathscr{Z}(s\,|\,t) = \sum_\rho (\frac{1}{2} + t - \rho)^{-s}$
$-n \le 0$	$-\mathbf{Z}(-n\,|\,t) + q(t - \frac{1}{2})^n$ †
0	$\tilde{\mathbf{a}}_1 t + \tilde{\mathbf{a}}_0 + q$
0 *(s-derivative)*	$\boxed{\mathscr{Z}'(0\,|\,t) = \mathbf{b}_1 t + \mathbf{b}_0 - \log \varXi(\frac{1}{2} + t)}$
$+1$ *(finite part)*	$\boxed{\mathrm{FP}_{s=1}\, \mathscr{Z}(s\,|\,t) = -\mathbf{b}_1 + (\log \varXi)'(\frac{1}{2} + t)}$
$+n \ge 1$	$\dfrac{(-1)^{n-1}}{(n-1)!} (\log \varXi)^{(n)}(\frac{1}{2} + t)$

† With $(t - \frac{1}{2})^0 \stackrel{\text{def}}{=} 1$ for $t = \frac{1}{2}$ (continuity in t is imperative)

Table 10.2 As Table 10.1, but for the function of second kind $\mathcal{Z}(\sigma\,|\,t)$; m is an integer. For $L(x) = \zeta(x)$ (the Riemann case), cf. Table 8.1

σ	$\mathcal{Z}(\sigma\,|\,t) = \sum_{k=1}^{\infty} ({\tau_k}^2 + t^2)^{-\sigma}$
$-m \le 0$	$\frac{1}{2}\Big[q(t^2 - \frac{1}{4})^m - \sum_{j=0}^{m} \binom{m}{j} (-1)^j\, \mathbf{Z}(-2j\,|\,0)\, t^{2(m-j)}\Big]$ †
0	$\frac{1}{2}(q + \tilde{\mathbf{a}}_0)$
0 *(σ-derivative)*	$\boxed{\mathcal{Z}'(0\,|\,t) = \mathbf{b}_0 - \log \varXi(\frac{1}{2} \pm t)}$
$+m \ge 1$	$\dfrac{(-1)^{m-1}}{(m-1)!} \dfrac{\mathrm{d}^m}{\mathrm{d}(t^2)^m} \log \varXi(\frac{1}{2} \pm t)$

† With $(t - t_0)^0 \stackrel{\text{def}}{=} 1$ for $t = t_0$ (continuity in t is imperative)

10.7.2 Dirichlet-L Cases, Functions of First Kind at $t = 0$ and $\frac{1}{2}$

Table 10.3 Special values of the zeta function $\mathscr{Z}(s\,|\,t)$ at $t = 0$ over the zeros $\{\rho\}$ of an L-function L_χ, for a real primitive Dirichlet character χ with modulus $d > 1$ and parity bit $a = 0$ or 1. Notation: see (1.4)–(1.7), (10.49)–(10.51); n is an integer

s		$\mathscr{Z}_0(s) \equiv \sum_\rho (\rho - \frac{1}{2})^{-s}$ $[t = 0]$
$-n \le 0$	even	$2^{-n-1}(a-\frac{1}{2})E_n$
	odd	$-\frac{1}{2}(1-2^{-n})\frac{B_{n+1}}{n+1}$
0		$\frac{1}{2}(a-\frac{1}{2})$
0 *(derivative)*		$\boxed{\mathscr{Z}_0'(0) = (\frac{3}{4}-a)\log 2 + (a-\frac{1}{2})\log\bigl[\Gamma(\frac{1}{4})^2/\pi\bigr] - \log L_\chi(\frac{1}{2})}$
+1 *(finite part)*		$\boxed{\mathrm{FP}_{s=1}\,\mathscr{Z}_0(s) = \frac{1}{2}\bigl[\log 2\pi - \log d\bigr]}$
$+n \ge 1$	odd	$0\,^*$
	even	$-\frac{1}{2}\bigl[(2^n-1)\,\zeta(n) + (1-2a)\,2^n\beta(n)\bigr] - \frac{(\log L_\chi)^{(n)}(\frac{1}{2})}{(n-1)!}\ ^\dagger$

[*] This amounts to the formulae (10.59) yielding $(\log L_\chi)^{(n)}(\frac{1}{2})$ for n odd

[†] Here $\zeta(n) \equiv (2\pi)^n|B_n|/(2\,n!)$, while $\beta(n)$ (Sect. 3.5) and $(\log L_\chi)^{(n)}(\frac{1}{2})$ remain elusive

Table 10.4 As Table 10.3, but at $t = \frac{1}{2}$. For the generalized Stieltjes cumulants $g_n^{\mathrm{c}}[\chi]$, see (10.5) with $q = 0$, (10.66) and (10.70)

s	$\mathscr{Z}_*(s) \equiv \sum_\rho \rho^{-s}$ $[t = \frac{1}{2}]$
$-n < 0$	$\bigl[(a-1)(2^n-1) + a\,2^n\bigr]\frac{B_{n+1}}{n+1}$
0	$\frac{1}{2}a$
0 *(derivative)*	$\boxed{\mathscr{Z}_*'(0) = \frac{1}{2}\bigl[(1-a)\log 2 + a\log\pi\bigr] + g_0^{\mathrm{c}}[\chi]}$
+1 *(finite part)*	$\boxed{\mathrm{FP}_{s=1}\,\mathscr{Z}_*(s) = (a-\frac{1}{2})\log 2 - \frac{1}{2}\gamma + g_1^{\mathrm{c}}[\chi]}$
+1	$(a-1)\log 2 - \frac{1}{2}\log(\pi/d) - \frac{1}{2}\gamma + g_1^{\mathrm{c}}[\chi]$
$+n > 1$	$\bigl[(a-1)(1-2^{-n}) - a\,2^{-n}\bigr]\zeta(n) + \frac{g_n^{\mathrm{c}}[\chi]}{(n-1)!}\ ^\dagger$

[†] For n even, $\zeta(n) \equiv (2\pi)^n|B_n|/(2\,n!)$

10.7.3 Dedekind-ζ Cases, Functions of First Kind at $t = 0$ and $\frac{1}{2}$

Table 10.5 Special values of the zeta function $\mathscr{Z}(s\,|\,t)$ at $t = 0$ over the zeros $\{\rho\}$ of a Dedekind zeta function ζ_K, for an algebraic number field K. Notation: see (1.4)–(1.7), (10.72) and (10.73); n is an integer. For $K = \mathbb{Q}$ (the Riemann case), cf. Table 7.3

s		$\mathscr{Z}_0(s) \equiv \sum_\rho (\rho - \frac{1}{2})^{-s}$ $\quad [t = 0]$
$-n \le 0$	even	$2^{-n+1}(1 - \frac{1}{8} r_1 E_n)$
	odd	$-\frac{1}{2} n_K (1-2^{-n}) \dfrac{B_{n+1}}{n+1}$
0		$2 - \frac{1}{4} r_1$
0 *(derivative)*		$\boxed{\mathscr{Z}_0'(0) = (2+\frac{3}{4}r_1+\frac{1}{2}r_2)\log 2 - \frac{1}{2} r_1 \log\big[\Gamma(\frac{1}{4})^2/\pi\big] - \log\lvert\zeta_K\rvert(\frac{1}{2})}$
$+1$ *(finite part)*		$\boxed{\mathrm{FP}_{s=1}\, \mathscr{Z}_0(s) = \frac{1}{2}\big[n_K \log 2\pi - \log\lvert d_K\rvert\big]}$
$+n \ge 1$	odd	$0\,^*$
	even	$2^{n+1} - \frac{1}{2}\big[n_K (2^n - 1)\,\zeta(n) + r_1\, 2^n \beta(n)\big] - \dfrac{(\log\lvert\zeta_K\rvert)^{(n)}(\frac{1}{2})}{(n-1)!}\,^\dagger$

*This amounts to the formulae (10.78) yielding $(\log|\zeta_K|)^{(n)}(\frac{1}{2})$ for n odd

†Here $\zeta(n) \equiv (2\pi)^n |B_n|/(2\,n!)$, while $\beta(n)$ (Sect. 3.5) and $(\log|\zeta_K|)^{(n)}(\frac{1}{2})$ remain elusive

Table 10.6 As Table 10.5, but at $t = \frac{1}{2}$. For the generalized Stieltjes cumulants $g_n^{\mathrm{c}}\{K\}$, see (10.5) with $q = 1$, (10.79)–(10.83). For $K = \mathbb{Q}$ (the Riemann case), cf. Table 7.4

s	$\mathscr{Z}_*(s) \equiv \sum_\rho \rho^{-s}$ $\quad [t = \frac{1}{2}]$
$-n < 0$	$\big[-r_1(2^n - 1) + r_2\big] \dfrac{B_{n+1}}{n+1} + 1$
0	$\frac{1}{2} r_2 + 2$
0 *(derivative)*	$\boxed{\mathscr{Z}_*'(0) = \frac{1}{2}\big[(r_1+r_2)\log 2 + r_2 \log\pi\big] + g_0^{\mathrm{c}}\{K\}}$
$+1$ *(finite part)*	$\boxed{\mathrm{FP}_{s=1}\, \mathscr{Z}_*(s) = 1 - \frac{1}{2} r_1 \log 2 - \frac{1}{2} n_K \gamma + g_1^{\mathrm{c}}\{K\}}$
$+1$	$1 + \frac{1}{2}\log\lvert d_K\rvert - (r_1+r_2)\log 2 - \frac{1}{2} n_K \log\pi - \frac{1}{2} n_K \gamma + g_1^{\mathrm{c}}\{K\}$
$+n > 1$	$1 - [r_1(1-2^{-n}) + r_2]\,\zeta(n) + \dfrac{g_n^{\mathrm{c}}\{K\}}{(n-1)!}\,^\dagger$

†For n even, $\zeta(n) \equiv (2\pi)^n |B_n|/(2\,n!)$