Chapter 1
Introduction

1.1 Symmetric Functions

The non-real zeros of the Riemann zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-s} \qquad (\text{Re } s > 1), \tag{1.1}$$

called the *Riemann zeros* and usually denoted ρ, are most elusive quantities. Thus, no individual Riemann zero is analytically known; and the Riemann Hypothesis (RH): Re $\rho = \frac{1}{2}$ $(\forall \rho)$, has stayed unresolved since 1859 [92].

For analogous finite or infinite sets of numbers $\{v_k\}$, like the roots of a polynomial, the eigenvalues of a matrix, or the discrete spectrum of a linear operator, the *symmetric functions* of $\{v_k\}$ tend to be much more accessible. Some common *types* of additive symmetric functions, to be denoted Theta, Zeta and (log Delta) here, are formally given by

$$\begin{aligned}
\text{Theta}(z) &\stackrel{\text{def}}{=} \textstyle\sum_k \mathrm{e}^{-z x_k}, \\
\text{Zeta}(s) &\stackrel{\text{def}}{=} \textstyle\sum_k x_k^{-s}, \\
\text{Delta}(a) &\stackrel{\text{def}}{=} \textstyle\prod_k (x_k + a) \qquad \text{or, if this diverges,} \\
(\log \text{Delta})^{(m)}(a) &\stackrel{\text{def}}{=} (-1)^{m-1}(m-1)! \textstyle\sum_k (x_k + a)^{-m} \quad \text{for some } m \ge 1,
\end{aligned}$$

where $x_k = v_k$ or some other function $f(v_k)$ (such that no $x_k = 0$ and, e.g., Re $x_k \to +\infty$). It is useful to allow at least the a-shift in this remapping, thereby obtaining a two-variable or generalized zeta function, in analogy with the Hurwitz function $\zeta(s,a) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (k+a)^{-s}$:

$$\text{Zeta}(s,a) \stackrel{\text{def}}{=} \textstyle\sum_k (x_k + a)^{-s}. \tag{1.2}$$

Here we think of s as the main argument, the variable in which analyticity properties and special values are studied, and of a as an (auxiliary) shift

A. Voros, *Zeta Functions over Zeros of Zeta Functions*, Lecture Notes of the Unione Matematica Italiana 8, DOI 10.1007/978-3-642-05203-3_1,

parameter which adds flexibility; i.e., we view $\mathrm{Zeta}(s,a)$ as a parametric family in the type $\mathrm{Zeta}(s)$ (accordingly denoting $\mathrm{Zeta}'(s,a) \stackrel{\mathrm{def}}{=} \partial_s \mathrm{Zeta}(s,a)$).

The gain with $\mathrm{Zeta}(s,a)$ is that it encompasses the last three types above:

$$\mathrm{Zeta}(s) = \mathrm{Zeta}(s,0),$$

$$\mathrm{Delta}(a) \stackrel{\mathrm{def}}{=} \exp[-\mathrm{Zeta}'(0,a)]$$

(when Delta is an *infinite* product, this is zeta-regularization, see Chap. 2),

$$(\log \mathrm{Delta})^{(m)}(a) = (-1)^{m-1}(m-1)!\, \mathrm{Zeta}(m,a);$$

while the Zeta type is simply a Mellin transform of the Theta type, as

$$\mathrm{Zeta}(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Theta}(z)\, \mathrm{e}^{-za}\, z^{s-1} \mathrm{d}z.$$

So, formally, all those types of symmetric functions look interchangeable and their properties convertible from one to the other. However, experience (especially from spectral theory) tells that zeta functions are those which display the most explicit properties, reaching to computable special values (values at integers) as in the case of $\zeta(s)$ itself.

Again from spectral techniques we borrow the idea that, besides the above shift operation, *nonlinear* remappings $x_k = f(v_k)$ may prove suitable before building the symmetric functions. For instance, if $\{v_k\}$ is the spectrum of a Laplacian on a manifold, both choices $x_k = v_k$ and $x_k = \sqrt{v_k}$ have their own merits: on the circle, with the spectrum $\{n^2\}_{n\in\mathbb{Z}}$, the resulting Theta-type functions are, respectively, a Jacobi θ function and $\coth z/2$, a generating function for the Poisson summation formula as in (1.13); whereas on a compact hyperbolic surface (normalized to curvature -1), an even better choice than $\sqrt{v_k}$ is $x_k = (v_k - \frac{1}{4})^{1/2}$, as the Selberg trace formula shows. (This formula expresses additive symmetric functions of precisely *the latter* x_k as dual sums carried by the periodic geodesics of the surface, see Sect. 6.3.1.)

It is then very natural to study symmetric functions of the Riemann zeros in a similar manner, and this has happened. Indeed,

- Some zeta functions built over the Riemann zeros have appeared in a few works, as early as 1917
- A universal tool exists to evaluate fairly general additive symmetric functions of the Riemann zeros: the Guinand–Weil "Explicit Formulae."

Still, we feel that our subject (zeta functions over the Riemann zeros) remains far from exhausted. For one thing, the existing studies are surprisingly few over a long stretch of time; they are neither systematic nor error-free, are often unaware of one another, and none has made it to the classic textbooks on $\zeta(s)$; consequently there is no comprehensive, easily accessible treatment of zeta functions over the Riemann zeros. Calculations in this field continue to appear on a case-by-case basis.

Neither do the classic Explicit Formulae settle the issue of these zeta functions as mere special instances: on the contrary, the most interesting particular zeta functions over the Riemann zeros lie outside the standard range of validity (i.e., of convergence) of those formulae.

In contrast, a dedicated study of these zeta functions uncovers a wealth of explicit results, many of which were not given or even suspected in the literature. The question of the importance or usefulness of those results will not be addressed: the answer may lie in an undefined future.

There is no standardized terminology for zeta functions over zeros of zeta functions. Chakravarty [17] used the name "secondary zeta functions," but to denote several Dirichlet series apart from $\zeta(x)$ itself which is the "primary" zeta function (that which supplies its zeros). Here, to have a short and specific name, we choose to call "superzeta" functions all second-generation zeta functions built over zeros of other, "primary", zeta functions.

We continue this introduction with some essential notation, then we will recall the most basic tools that will often serve later.

1.2 Essential Basic Notation

As a rule, we refer to [1, 33].

Bernoulli polynomials (definition by generating function):

$$\frac{z\,\mathrm{e}^{zy}}{\mathrm{e}^{z}-1} \equiv \sum_{n=0}^{\infty} B_n(y)\,\frac{z^n}{n!} \qquad (B_0(y)=1,\ B_1(y)=y-\tfrac{1}{2},\ldots). \tag{1.3}$$

Bernoulli numbers: $\quad B_n \equiv B_n(0) \quad$ or $\quad \dfrac{z}{\mathrm{e}^{z}-1} \equiv \displaystyle\sum_{n=0}^{\infty} B_n\,\frac{z^n}{n!}$

$$(B_0=1,\ B_1=-\tfrac{1}{2},\ B_2=\tfrac{1}{6},\ldots;\ B_{2m+1}=0 \text{ for } m=1,2,\ldots). \tag{1.4}$$

Euler numbers: $\quad \dfrac{1}{\cosh z} \equiv \displaystyle\sum_{n=0}^{\infty} E_n\,\frac{z^n}{n!}$

$$(E_0=1,\ E_2=-1,\ E_4=5,\ldots;\ E_{2m+1}=0 \text{ for } m=0,1,\ldots). \tag{1.5}$$

Digamma function $\psi(x)$ and Euler's constant γ:

$$\psi(x) \stackrel{\mathrm{def}}{=} [\Gamma'/\Gamma]\,(x); \qquad \gamma=-\psi(1)\approx 0.5772156649\,. \tag{1.6}$$

The *finite part* of a meromorphic function $z \mapsto f(z)$ at a pole a is

$$\mathrm{FP}_{z=a} f(z) \stackrel{\text{def}}{=} \text{the constant term in the Laurent series of } f \text{ at } a. \tag{1.7}$$

For the complex functions $z \mapsto z^{-s}$, we use the principal determination

$$z^{-s} \stackrel{\text{def}}{=} |z|^{-s}\, \mathrm{e}^{-\mathrm{i}s\, \arg z} \qquad (-\pi < \arg z < +\pi) \tag{1.8}$$

in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

In any generalized zeta function,

$$\mathrm{Zeta}'(s,a) \stackrel{\text{def}}{=} \partial_s \mathrm{Zeta}(s,a). \tag{1.9}$$

1.3 The Poisson Summation Formula

We mention this formula here only for later reference and comparison purposes. In its simplest form, it involves a dual pair of functions $(h,\, \hat{h}) : \mathbb{R} \mapsto \mathbb{C}$ which are Fourier transforms of each other:

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\tau)\, \mathrm{e}^{-\mathrm{i}\tau u}\, \mathrm{d}\tau, \qquad h(\tau) = \int_{-\infty}^{\infty} \hat{h}(u)\, \mathrm{e}^{\mathrm{i}\tau u}\, \mathrm{d}u. \tag{1.10}$$

Under rather mild conditions, they satisfy the *Poisson summation formula*,

$$\sum_{k \in \mathbb{Z}} h(k) \equiv 2\pi \sum_{r \in \mathbb{Z}} \hat{h}(2\pi r). \tag{1.11}$$

Sufficient conditions for (1.11) are, e.g., that for some $\delta,\ \varepsilon > 0$,

(i) $\hat{h}(u) = \mathrm{O}(|u|^{-1-\delta})$ for $u \to \pm\infty$
(ii) $h(\tau) = \mathrm{O}(|\tau|^{-1-\varepsilon})$ for $\tau \to \pm\infty$
(iii) (Without loss of generality) the function h (hence $\hat{h}$) is even.

Proof. (Sketched). The function $H(\tau) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} h(\tau + k)$ is periodic of period 1, and has the Fourier series $\tilde{H}(\tau) = \sum_{r \in \mathbb{Z}} h_r\, \mathrm{e}^{2\pi \mathrm{i} r \tau}$, where $h_r = \int_0^1 H(\tau)\, \mathrm{e}^{-2\pi \mathrm{i} r \tau}\, \mathrm{d}\tau \equiv 2\pi\, \hat{h}(2\pi r)$; both series above converge uniformly by (ii), resp. (i), hence both their sums $H(\tau)$ and $\tilde{H}(\tau)$ are continuous functions, implying $\tilde{H}(\tau) = H(\tau)$ pointwise [89, Appendix 1]. Then (1.11) just says $H(0) \equiv \tilde{H}(0)$ (while (iii) simply endorses the even symmetry of the summations in (1.11)). $\square$

A basic function pair (in which z is an implied parameter) is

$$h(\tau) \stackrel{\text{def}}{=} \tfrac{1}{2}\,\mathrm{e}^{-z\,|\tau|}, \qquad \hat{h}(u) \equiv \frac{1}{2\pi}\,\frac{z}{z^2+u^2} \qquad (\mathrm{Re}\ z > 0). \tag{1.12}$$

In this case, the summation formula (1.11) expresses the partial fraction decomposition

$$\frac{1}{\mathrm{e}^z - 1} + \frac{1}{2}\left(= \frac{1}{2\tanh\, z/2}\right) \equiv \lim_{R\to+\infty} \sum_{r=-R}^{+R} \frac{1}{z + 2\pi \mathrm{i} r}. \tag{1.13}$$

Conversely, (1.13) gives back the general Poisson summation formula (1.11) simply through integration against suitable test functions [16, Sect. 1].

1.4 Euler–Maclaurin Summation Formulae

These formulae serve to approximate sums by integrals. We will use a very basic version but in several specific forms, which we then sketch.

For a differentiable function f of a real variable u, integration by parts against the first Bernoulli polynomial $B_1(u) = u - \frac{1}{2}$ yields $\frac{1}{2}\big[f(0)+f(1)\big] = \int_0^1 f(u)\,\mathrm{d}u + \int_0^1 B_1(u) f'(u)\,\mathrm{d}u$. Now let $K \in \mathbb{N}$ and $K' \in \mathbb{N} \cup \{+\infty\}$ with $K' > K$, and just here, $\{u\} \stackrel{\text{def}}{=}$ the fractional part of u. Then a simple summation of the preceding formula over successive unit intervals gives

$$\sum_{k=K}^{K'} f(k) - \tfrac{1}{2}\big[f(K) + f(K')\big] - \int_K^{K'} f(u)\,\mathrm{d}u = \int_K^{K'} B_1\big(\{u\}\big) f'(u)\,\mathrm{d}u \stackrel{\text{def}}{=} R, \tag{1.14}$$

only assuming f and f' to be *integrable* over the interval under use. If moreover f is a *monotonic* function, then the difference or remainder R is easily bounded, resulting in

$$\sum_{k=K}^{K'} f(k) - \tfrac{1}{2}\big[f(K) + f(K')\big] = \int_K^{K'} f(u)\,\mathrm{d}u + R, \qquad |R| \le \tfrac{1}{2}|f(K) - f(K')|.$$

In particular if $K' = +\infty$, then $f(+\infty) = 0$, and

$$\sum_{k=K}^{+\infty} f(k) = \int_K^{+\infty} f(u)\,\mathrm{d}u + \mathrm{O}(f(K)) \qquad \text{for } K \to +\infty. \tag{1.15}$$

1.5 Meromorphic Properties of Mellin Transforms

The representation of meromorphic functions as Mellin transforms, the resulting control over their principal parts and effective meromorphic continuation, constitute the technical tool we will constantly invoke throughout. Therefore, although the subject is classic [36, Chap. VII.6] [58] [112, Chap. III], we give a detailed review of the results we will use.

For a locally integrable function $f(x)$, $x \in (0, +\infty)$, its *Mellin transform* can be formally defined as a function of $s \in \mathbb{C}$, as

$$\mathrm{M}f(s) = \int_0^\infty f(x)\, x^{-s-1} \mathrm{d}x. \tag{1.16}$$

We list some increasingly detailed analytic features in s for this transform.

Stage 1 (root). If for some $-\infty \le \mu < \nu \le +\infty$,

$$f(x) = \mathrm{O}(x^\nu) \quad (x \to 0^+) \qquad \text{and} \qquad f(x) = \mathrm{O}(x^\mu) \quad (x \to +\infty), \tag{1.17}$$

then $\mathrm{M}f$ is defined and holomorphic in the strip $\{\mu < \mathrm{Re}\, s < \nu\}$ (simply by the convergence properties of the integral (1.16) in that strip). *Note:* $\mathrm{M}f(s) \to 0$ for $s \to \infty$ in the strip, by the Riemann–Lebesgue lemma.

Stage 2. If moreover, for some *strictly descending* sequence $\mu_0 > \mu_1 > \cdots > \mu_N$ with $\mu_0 < \nu$ and some sequence of polynomials $\{p_n(y)\}$, the function f obeys a *large-x asymptotic estimate*

$$f(x) \sim \sum_{n=0}^{N} p_n(\log x)\, x^{\mu_n} + \mathrm{O}(x^{\mu'}) \quad \text{for } x \to +\infty \quad \text{with } \mu' < \mu_N,$$

then $\mathrm{M}f$ continues to a meromorphic function of s in the wider strip $\{\mu' < \mathrm{Re}\, s < \nu\}$, where its only singularities are *poles at $s = \mu_n$, with principal parts* $p_n\bigl(-\frac{\mathrm{d}}{\mathrm{d}s}\bigr)\, \dfrac{1}{s - \mu_n}$ for $n = 0, 1, \ldots, N$.

Proof. Let $g_N(x) = f(x) - \sum\limits_{n=0}^{N} p_n(\log x)\, x^{\mu_n}\big|_{\{x>1\}}$. Then g_N satisfies all assumptions of Stage 1 with μ' in place of μ, hence $\mathrm{M}g_N$ is holomorphic in the wider strip; while $\int_1^\infty \bigl[p_n(\log x)\, x^{\mu_n}\bigr]\, x^{-s-1} \mathrm{d}x$ precisely yields the stated principal part at $s = \mu_n$ by direct evaluation. □

Stage 2'. If moreover the above sequences are *infinite*, $\mu_n \downarrow -\infty$, and the function f admits a *complete* large-x asymptotic expansion

$$f(x) \sim \sum_{n=0}^{\infty} p_n(\log x)\, x^{\mu_n} \qquad \text{for } x \to +\infty, \tag{1.18}$$

then all results of Stage 2 extend to $N = \infty$ (simply by letting $\mu' \to -\infty$ in the above): $\mathrm{M}f$ is then meromorphic in the half-plane $\{\mathrm{Re}\ s < \nu\}$, and its only singularities are

$$\textit{poles at } s = \mu_n, \textit{ with principal parts } p_n\Big(-\frac{\mathrm{d}}{\mathrm{d}s}\Big)\frac{1}{s-\mu_n} \quad \text{for } n \in \mathbb{N}. \tag{1.19}$$

In particular, in the absence of logarithmic terms (p_n constant), the corresponding pole at μ_n is *simple, with residue* p_n.

Stage 3'. Independently of Stages 2, 2', if now for some strictly *ascending* sequence $\nu_0 < \nu_1 < \cdots$ with $\nu_0 > \mu$ and $\nu_n \uparrow +\infty$, the function f admits a complete *small-x* asymptotic expansion (here without logarithmic terms for simplicity)

$$f(x) \sim \sum_{n=0}^{\infty} f_n\, x^{\nu_n} \qquad \text{for } x \to 0^+, \tag{1.20}$$

then $\mathrm{M}f$ continues to a meromorphic function in the half-plane $\{\mathrm{Re}\ s > \mu\}$, where its only singularities are

$$\textit{simple poles at } s = \nu_n, \textit{ with residues } (-f_n) \qquad \text{for } n \in \mathbb{N} \tag{1.21}$$

(arguing just as above, but with the bounds $x = 0$ and $+\infty$ interchanged under $s \mapsto -s$; hence the residues change sign).

Note that we could more generally have allowed, e.g., suitable complex exponents μ_n and ν_n, logarithmic terms also in the expansion (1.20) at $x = 0$, etc. Here we aim at economy, and refer to [58, 76] for more general settings.

Effective continuation formulae. In Stages 2–3, now assuming all asymptotic conditions to be *differentiable*, the Mellin representations can be explicitly modified to perform an *effective* meromorphic continuation. The trick consists of successive *integrations by parts*, watching the growth conditions at the integration bound where divergence sets in.

We illustrate the idea in the case of Stage 2 (or 2') assuming a simple pole at $s = \mu_0$ (i.e., $p_0 = \text{const.}$). We then have, in principle for $\mu_0 < \mathrm{Re}\ s < \nu$,

$$\mathrm{M}f(s) = \int_0^\infty [f(x)\, x^{-\mu_0}]\, x^{\mu_0 - s - 1}\mathrm{d}x = \frac{1}{s-\mu_0}\int_0^\infty [f(x)\, x^{-\mu_0}]'\, x^{\mu_0 - s}\mathrm{d}x, \tag{1.22}$$

but the latter integral satisfies all conditions of Stage 1 with μ_1 in place of μ, so it defines an analytic function in the *wider* strip $\{\mu_1 < \mathrm{Re}\ s < \nu\}$; the right-hand side thus continues $\mathrm{M}f$ meromorphically in this strip (note how the pole at $s = \mu_0$ and its residue $[f(x)\, x^{-\mu_0}]_0^\infty = p_0$ become explicit).

Now in the newly accessed strip $\{\mu_1 < \mathrm{Re}\, s < \mu_0\}$, the integration by parts can be reversed, but the growth conditions produce a new result,

$$\mathrm{M}f(s) = \int_0^\infty [f(x)\, x^{-\mu_0} - p_0]\, x^{\mu_0 - s - 1} \mathrm{d}x = \int_0^\infty [f(x) - p_0\, x^{\mu_0}]\, x^{-s-1} \mathrm{d}x, \tag{1.23}$$

upon which the whole process can then be restarted to pass the next pole μ_1.

A multiple pole can likewise be passed by further iterated integrations by parts; as we will not explicitly need to do this, we simply refer to [58] [106, Appendix A].