

Lecture Notes of the Unione Matematica Italiana

André Voros

8

Zeta Functions over Zeros of Zeta Functions

 Springer



Editorial Board



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Via Ferrata 5a
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e-mail: brezzi@imati.cnr.it

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State College
PA. 16802, USA
e-mail: bressan@math.psu.edu

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Mathematisches Institut
Universitätsstraße 30
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e-mail: fabrizio.catanese@uni-bayreuth.de

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Department of Statistics
Stanford University
450 Serra Mall
Stanford, CA 94305-4065, USA
e-mail: diaconis@math.stanford.edu,
cgates@stat.stanford.edu

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Dipartimento di Matematica e Applicazioni
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e-mail n.fusco@unina.it

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5734 University Avenue
Chicago, IL 60637-1514
USA
e-mail: cek@math.uchicago.edu

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Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa, Italy
e-mail: fricci@sns.it

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Korteweg-de Vries Instituut
Universiteit van Amsterdam
Plantage Muidersgracht 24
1018 TV Amsterdam, The Netherlands
e-mail: geer@science.uva.nl

Cédric Villani
Institut Henri Poincaré
11 rue Pierre et Marie Curie
75230 Paris Cedex 05
France
e-mail: cedric.villani@upmc.fr

André Voros

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André Voros
CEA-Saclay
Institut de Physique Théorique (IPhT)
Orme des Merisiers
91191 Gif-sur-Yvette
France
andre.voros@cea.fr

Appendix D:

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To Estelle
To Magali and Sandrine

Preface

In the Riemann zeta function $\zeta(s)$, the non-real zeros or *Riemann zeros*, denoted ρ , play an essential role mainly in number theory, and thereby generate considerable interest. However, they are very elusive objects. Thus, no individual zero has an analytically known location; and the *Riemann Hypothesis*, which states that all those zeros should lie on the *critical line*, i.e., have real part $\frac{1}{2}$, has challenged mathematicians since 1859 (exactly 150 years ago).

For analogous symmetric sets of numbers $\{v_k\}$, such as the roots of a polynomial, the eigenvalues of a finite or infinite matrix, etc., it is well known that *symmetric functions* of the $\{v_k\}$ tend to have more accessible properties than the individual elements v_k . And, we find the largest wealth of explicit properties to occur in the (*generalized*) *zeta functions* of the generic form

$$\text{Zeta}(s, a) = \sum_k (v_k + a)^{-s}$$

(with the extra option of replacing v_k here by selected functions $f(v_k)$).

Not surprisingly, then, zeta functions over the Riemann zeros have been considered, some as early as 1917. What is surprising is *how small* the literature on those zeta functions has remained overall. We were able to spot them in barely a dozen research articles over the whole twentieth century and in none of the *books* featuring the Riemann zeta function. So the domain exists, but it has remained largely confidential and sporadically covered, in spite of a recent surge of interest.

Could it then be that those zeta functions have few or uninteresting properties? In actual fact, their study yields an abundance of quite explicit results. The significance or usefulness of the latter may then be questioned: at this moment, we can only answer that regarding the Riemann zeros, any explicit result, even of a collective nature, is of *potential* value. Hence we may turn over the idea that zeta functions over the Riemann zeros have stagnated because they were not so interesting: it could also be that those functions have lagged behind in their use simply because their properties never came to be fully displayed.

So, the primary aim of this monograph is to fill that very specific but definite *gap*, by offering a coherent and synthetic description of the zeta functions over the Riemann zeros (and immediate extensions thereof); we call them “superzeta” functions here for brevity. Modeled on special-function handbooks (our main reference case being the Hurwitz zeta function $\zeta(s, a)$), this book centers on delivering extensive lists of concrete explicit properties and tables of handy special-value formulae for superzeta functions, grouped in three core chapters plus Appendix B (for the variant case built over zeros of Selberg zeta functions). In that core, we mainly wish to provide readers, assuming they have specific queries about superzeta functions, with a broad panel of explicit answers. For such a purpose, the key contents of the book may be just Chap. 5 (for initial orientation) and the final results, including 20 tables of special-value formulae. The rest of the text is rather backstage material, showing justifications, perspective, and references for those end results.

For the reasons given above, we grade no individual result or formula as more or less “useful,” but place them all on an equal footing. Our main justification to date for tackling those superzeta functions is simply “Because they’re there” (like a famous mountaineer’s reply).

We now outline the contents.

Two introductory chapters review our main analytical techniques: miscellaneous notation and tools, specially the Mellin transformation (Chap. 1), and zeta-regularized products (Chap. 2). The next two chapters, still introductory, survey the Riemann zeta function itself (and close kin, the Dirichlet beta and Hurwitz zeta functions), so as to make the book reasonably self-contained and tutorial. All review sections are, however, filtered hierarchically: the aspects most central to us are exposed in detail, others more sketchily (and a few just get mentioned). We do not try to compete with the many exhaustive treatises on the Riemann zeta function; on the other hand, a shorter tutorial like ours might suit readers seeking to learn about that function from a purely *analytical*, as opposed to number-theoretical, angle.

The next two chapters begin to address the superzeta functions themselves: Chap. 5 gives an overview, and the following one introduces Explicit Formulae from number theory, which are then applied to superzeta functions (and compared to Selberg trace formulae for spectral zeta functions).

Chapters 7–10 form the core of the study: *three kinds* of superzeta functions are thoroughly described in the first three chapters, then extended to zeros of more general *L*- or zeta-functions in Chap. 10; except for Chap. 9, every core chapter (plus Appendix B for the Selberg case) culminates in detailed Tables of special-value formulae.

To close, Chap. 11 shows one application of a superzeta function: a recently obtained asymptotic criterion for the Riemann Hypothesis (based on the Keiper–Li coefficients used by *Li’s criterion*). Finally, four Appendices treat extra issues (A: some numerical aspects; B: extension to zeta functions over zeros of *Selberg* zeta functions; C: on $(\log |\zeta|)^{(2m+1)}(\frac{1}{2})$, etc.; D: an English translation of Mellin’s seminal 1917 paper in German).

As we aim to throw light on an unpublicized subject on which this is the very first book as far as we know, our text is kept concrete and expository through the first half at least, favoring elementary and economical techniques. Exercises are also proposed in the form of peripheral results left for the reader to derive. Our wish is to have built a compact reference guide, a kind of “Everything you always wanted to know about superzeta functions . . .” handbook. For the sake of improvement, we gratefully welcome any error reports from readers (and will post errata as needed).

This text is thus directed at readers interested in analytical aspects of number theory. It ought to be accessible to mathematicians from the graduate level; its main assumed background is in analysis (real and complex: series and integrals, analytic and special functions, asymptotics).

For this study, I am primarily indebted to Prof. P. Cartier who initiated our collaboration on trace formulae in the late 80s, ushering me into an area entirely new to me; I express to him my gratitude for his stimulating help and encouragement.

This book could never have been born without the moral support, the help, and guidance from colleagues in the Institut de Physique Théorique (CEA-Saclay), the Orsay area, and the Chevaleret campus (The Math. Departments of the Paris 6–7 Universities); I can only thank them collectively here, but most warmly, for their assistance.

My deepest thanks to my spouse Estelle, for her enduring patience, understanding, and support at all times but specially during the completion of the writing which seemed to stretch forever; this strain was also shared from a greater distance by our daughters Magali and Sandrine: my thanks go to them too.

Saclay, July 2009

André Voros

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List of Special Symbols

Occasional, or inversely universal, symbols are not listed. (For us, sets like \mathbb{N} , \mathbb{R}_+ include 0, while \mathbb{N}^* , \mathbb{R}_+^* , etc., exclude it.) Appendix D is not included.

\mathcal{A}_S	Area of a surface S
a	Parity bit (0 or 1) for a Dirichlet character
B	$(\log \Xi)'(0)$ for the completed Riemann zeta function $\Xi(\cdot)$
B_n	Bernoulli numbers
$B_n(\cdot)$	Bernoulli polynomials
$\mathbf{D}(x)$	Zeta-regularized form of trivial factor $\mathbf{G}(x)$
$\mathcal{D}(x)$	Zeta-regularized form of completed zeta- or L -function $\Xi(x)$
$\mathcal{D}(v)$	Zeta-regularized form of $\Xi(\frac{1}{2} + v^{1/2})$
d	Period (“modulus”) of a Dirichlet character
d_K	Discriminant of an algebraic number field K
E_n	Euler numbers
$\text{FP}_{x=x_0} f$	Finite part of a function $f(x)$ at x_0
g	Genus (for a surface)
g_n	Stieltjes constants (in our normalization), see γ_{n-1}
g_n^c	Stieltjes cumulants (in our normalization), see $\tilde{\gamma}_{n-1}$
$\mathbf{G}(\cdot)$	Trivial (entire) factor in a zeta- or L -function
H_n	Harmonic numbers
$K_\nu(\cdot)$	Modified Bessel function
$L(\cdot)$	Generic (“primary”) zeta- or L -function
$L_\chi(\cdot)$	L -function for a Dirichlet character χ
ℓ_ϖ	Length of a periodic geodesic ϖ
Mf	Mellin transform of a function f
$N(T)$	Counting function of (e.g., Riemann) zeros’ ordinates
$\bar{N}(T)$	Trivial-factor contribution to $N(T)$
$\bar{N}_0(T)$	Asymptotic form of $N(T)$ (mod $O(\log T)$)
n_K	Degree of an algebraic number field K
$\{p\}$	Set of prime numbers
RH	Riemann Hypothesis
\mathcal{R}_m	Residue of $\mathcal{Z}_0(\sigma)$ at the pole $\sigma = \frac{1}{2} - m$
$\mathcal{R}_m(t)$	Residue of $\mathcal{Z}(\sigma t)$ at the pole $\sigma = \frac{1}{2} - m$

$S(T)$	Contribution of $\zeta(x)$ to the function $N(T)$ for the Riemann zeros
$\{u_k\}$	Sequence $\{\rho(1 - \rho) \mid \text{Im } \rho > 0\}$ over the nontrivial zeros
$V(\cdot)$	Cramér's function
$Z(\cdot)$	Generic Zeta-type function
$Z(\cdot, \cdot)$	Generic generalized zeta function
$Z'(\cdot, \cdot)$	Derivative of $Z(\cdot, \cdot)$ in the <i>first</i> argument (the exponent)
$\mathbf{Z}(s \mid t)$	Generalized zeta function over the trivial zeros of a zeta function
$\mathcal{Z}(s \mid t)$	Superzeta function of first kind
$\mathcal{Z}_0(s)$	$\mathcal{Z}(s \mid t = 0)$
$\mathcal{Z}_*(s)$	$\mathcal{Z}(s \mid t = \frac{1}{2})$
$\mathcal{Z}(\sigma \mid t)$	Superzeta function of second kind
$\mathcal{Z}_0(\sigma)$	$\mathcal{Z}(\sigma \mid t = 0)$
$\mathcal{Z}_*(\sigma)$	$\mathcal{Z}(\sigma \mid t = \frac{1}{2})$
$\mathfrak{Z}(s \mid \tau)$	Superzeta function of third kind
$\beta(\cdot)$	Dirichlet beta-function
$\Gamma(\cdot)$	Euler Gamma function
γ	Euler's constant
γ_j	Stieltjes constants, <i>see</i> our modified notation g_{j+1}
$\tilde{\gamma}_j = \eta_j$	Stieltjes “cumulants”, <i>see</i> our modified notation g_{j+1}^c
$\Delta(\cdot)$	Generic Hadamard product
$\Delta_0(\cdot)$	Generic Weierstrass product
$\Delta_\infty(\cdot)$	Generic zeta-regularized product
δ_1	Discrepancy in $\mathcal{Z}(s \mid t)$ at $s = 1$
$\delta_{j,k}$	Kronecker delta
$\zeta(\cdot)$	Riemann zeta function
$\zeta(\cdot, \cdot)$	Hurwitz zeta function
$\zeta_K(\cdot)$	Dedekind zeta function for an algebraic number field K
$\zeta_S(\cdot)$	Selberg zeta function for a hyperbolic surface S
η_j	Stieltjes “cumulants”, <i>see</i> $\tilde{\gamma}_j$
$\Theta(\cdot)$	Theta-type function
κ_k	Wavenumbers ($[\text{eigenvalues} - \frac{1}{4}]^{1/2}$) for a hyperbolic 2D Laplacian
$A(n)$	von Mangoldt function
λ_n	Keiper–Li coefficients
μ_0	Order (of a sequence, of an entire function)
$\Xi(\cdot)$	Completed zeta- or L -function
$\{\varpi\}$	Set of primitive oriented periodic geodesics on a surface
$\{\rho\}$	Set of nontrivial zeros of a zeta- or L -function (e.g., Riemann)
$\{\tau_k\}$	Sequence $\{i^{-1}(\rho - \frac{1}{2}) \mid \text{Im } \rho > 0\}$ over the nontrivial zeros
χ	Generic Dirichlet character
χ_{d_K}	Kronecker symbol for the discriminant d_K
$\psi(\cdot)$	Digamma function $[\Gamma'/\Gamma](\cdot)$
Ω_1, Ω_2	t -domains for superzeta functions of first, second kind

Chapter 1

Introduction

1.1 Symmetric Functions

The non-real zeros of the Riemann zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-s} \quad (\text{Re } s > 1), \quad (1.1)$$

called the *Riemann zeros* and usually denoted ρ , are most elusive quantities. Thus, no individual Riemann zero is analytically known; and the Riemann Hypothesis (RH): $\text{Re } \rho = \frac{1}{2} (\forall \rho)$, has stayed unresolved since 1859 [92].

For analogous finite or infinite sets of numbers $\{v_k\}$, like the roots of a polynomial, the eigenvalues of a matrix, or the discrete spectrum of a linear operator, the *symmetric functions* of $\{v_k\}$ tend to be much more accessible. Some common *types* of additive symmetric functions, to be denoted Theta, Zeta and (log Delta) here, are formally given by

$$\begin{aligned} \text{Theta}(z) &\stackrel{\text{def}}{=} \sum_k e^{-zx_k}, \\ \text{Zeta}(s) &\stackrel{\text{def}}{=} \sum_k x_k^{-s}, \\ \text{Delta}(a) &\stackrel{\text{def}}{=} \prod_k (x_k + a) \quad \text{or, if this diverges,} \\ (\log \text{Delta})^{(m)}(a) &\stackrel{\text{def}}{=} (-1)^{m-1} (m-1)! \sum_k (x_k + a)^{-m} \quad \text{for some } m \geq 1, \end{aligned}$$

where $x_k = v_k$ or some other function $f(v_k)$ (such that no $x_k = 0$ and, e.g., $\text{Re } x_k \rightarrow +\infty$). It is useful to allow at least the a -shift in this remapping, thereby obtaining a two-variable or generalized zeta function, in analogy with the Hurwitz function $\zeta(s, a) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (k+a)^{-s}$:

$$\text{Zeta}(s, a) \stackrel{\text{def}}{=} \sum_k (x_k + a)^{-s}. \quad (1.2)$$

Here we think of s as the main argument, the variable in which analyticity properties and special values are studied, and of a as an (auxiliary) shift

parameter which adds flexibility; i.e., we view $Zeta(s, a)$ as a parametric family in the type $Zeta(s)$ (accordingly denoting $Zeta'(s, a) \stackrel{\text{def}}{=} \partial_s Zeta(s, a)$).

The gain with $Zeta(s, a)$ is that it encompasses the last three types above:

$$Zeta(s) = Zeta(s, 0),$$

$$Delta(a) \stackrel{\text{def}}{=} \exp[-Zeta'(0, a)]$$

(when $Delta$ is an *infinite* product, this is zeta-regularization, see Chap. 2),

$$(\log Delta)^{(m)}(a) = (-1)^{m-1} (m-1)! Zeta(m, a);$$

while the $Zeta$ type is simply a Mellin transform of the $Theta$ type, as

$$Zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty Theta(z) e^{-za} z^{s-1} dz.$$

So, formally, all those types of symmetric functions look interchangeable and their properties convertible from one to the other. However, experience (especially from spectral theory) tells that zeta functions are those which display the most explicit properties, reaching to computable special values (values at integers) as in the case of $\zeta(s)$ itself.

Again from spectral techniques we borrow the idea that, besides the above shift operation, *nonlinear* remappings $x_k = f(v_k)$ may prove suitable before building the symmetric functions. For instance, if $\{v_k\}$ is the spectrum of a Laplacian on a manifold, both choices $x_k = v_k$ and $x_k = \sqrt{v_k}$ have their own merits: on the circle, with the spectrum $\{n^2\}_{n \in \mathbb{Z}}$, the resulting $Theta$ -type functions are, respectively, a Jacobi θ function and $\coth z/2$, a generating function for the Poisson summation formula as in (1.13); whereas on a compact hyperbolic surface (normalized to curvature -1), an even better choice than $\sqrt{v_k}$ is $x_k = (v_k - \frac{1}{4})^{1/2}$, as the Selberg trace formula shows. (This formula expresses additive symmetric functions of precisely *the latter* x_k as dual sums carried by the periodic geodesics of the surface, see Sect. 6.3.1.)

It is then very natural to study symmetric functions of the Riemann zeros in a similar manner, and this has happened. Indeed,

- Some zeta functions built over the Riemann zeros have appeared in a few works, as early as 1917
- A universal tool exists to evaluate fairly general additive symmetric functions of the Riemann zeros: the Guinand–Weil “Explicit Formulae.”

Still, we feel that our subject (zeta functions over the Riemann zeros) remains far from exhausted. For one thing, the existing studies are surprisingly few over a long stretch of time; they are neither systematic nor error-free, are often unaware of one another, and none has made it to the classic textbooks

on $\zeta(s)$; consequently there is no comprehensive, easily accessible treatment of zeta functions over the Riemann zeros. Calculations in this field continue to appear on a case-by-case basis.

Neither do the classic Explicit Formulae settle the issue of these zeta functions as mere special instances: on the contrary, the most interesting particular zeta functions over the Riemann zeros lie outside the standard range of validity (i.e., of convergence) of those formulae.

In contrast, a dedicated study of these zeta functions uncovers a wealth of explicit results, many of which were not given or even suspected in the literature. The question of the importance or usefulness of those results will not be addressed: the answer may lie in an undefined future.

There is no standardized terminology for zeta functions over zeros of zeta functions. Chakravarty [17] used the name “secondary zeta functions,” but to denote several Dirichlet series apart from $\zeta(x)$ itself which is the “primary” zeta function (that which supplies its zeros). Here, to have a short and specific name, we choose to call “superzeta” functions all second-generation zeta functions built over zeros of other, “primary”, zeta functions.

We continue this introduction with some essential notation, then we will recall the most basic tools that will often serve later.

1.2 Essential Basic Notation

As a rule, we refer to [1, 33].

Bernoulli polynomials (definition by generating function):

$$\frac{z e^{zy}}{e^z - 1} \equiv \sum_{n=0}^{\infty} B_n(y) \frac{z^n}{n!} \quad (B_0(y) = 1, B_1(y) = y - \frac{1}{2}, \dots). \quad (1.3)$$

$$\text{Bernoulli numbers:} \quad B_n \equiv B_n(0) \quad \text{or} \quad \frac{z}{e^z - 1} \equiv \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (1.4)$$

$$(B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \dots; B_{2m+1} = 0 \text{ for } m = 1, 2, \dots).$$

$$\text{Euler numbers:} \quad \frac{1}{\cosh z} \equiv \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} \quad (1.5)$$

$$(E_0 = 1, E_2 = -1, E_4 = 5, \dots; E_{2m+1} = 0 \text{ for } m = 0, 1, \dots).$$

Digamma function $\psi(x)$ and Euler’s constant γ :

$$\psi(x) \stackrel{\text{def}}{=} [\Gamma'/\Gamma](x); \quad \gamma = -\psi(1) \approx 0.5772156649. \quad (1.6)$$

The *finite part* of a meromorphic function $z \mapsto f(z)$ at a pole a is

$$\text{FP}_{z=a}f(z) \stackrel{\text{def}}{=} \text{the constant term in the Laurent series of } f \text{ at } a. \quad (1.7)$$

For the complex functions $z \mapsto z^{-s}$, we use the principal determination

$$z^{-s} \stackrel{\text{def}}{=} |z|^{-s} e^{-is \arg z} \quad (-\pi < \arg z < +\pi) \quad (1.8)$$

in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

In any generalized zeta function,

$$\text{Zeta}'(s, a) \stackrel{\text{def}}{=} \partial_s \text{Zeta}(s, a). \quad (1.9)$$

1.3 The Poisson Summation Formula

We mention this formula here only for later reference and comparison purposes. In its simplest form, it involves a dual pair of functions $(h, \hat{h}) : \mathbb{R} \mapsto \mathbb{C}$ which are Fourier transforms of each other:

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\tau) e^{-i\tau u} d\tau, \quad h(\tau) = \int_{-\infty}^{\infty} \hat{h}(u) e^{i\tau u} du. \quad (1.10)$$

Under rather mild conditions, they satisfy the *Poisson summation formula*,

$$\sum_{k \in \mathbb{Z}} h(k) \equiv 2\pi \sum_{r \in \mathbb{Z}} \hat{h}(2\pi r). \quad (1.11)$$

Sufficient conditions for (1.11) are, e.g., that for some $\delta, \varepsilon > 0$,

- (i) $\hat{h}(u) = O(|u|^{-1-\delta})$ for $u \rightarrow \pm\infty$
- (ii) $h(\tau) = O(|\tau|^{-1-\varepsilon})$ for $\tau \rightarrow \pm\infty$
- (iii) (Without loss of generality) the function h (hence \hat{h}) is even.

Proof. (Sketched). The function $H(\tau) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} h(\tau + k)$ is periodic of period 1, and has the Fourier series $\tilde{H}(\tau) = \sum_{r \in \mathbb{Z}} h_r e^{2\pi i r \tau}$, where $h_r = \int_0^1 H(\tau) e^{-2\pi i r \tau} d\tau \equiv 2\pi \hat{h}(2\pi r)$; both series above converge uniformly by (ii), resp. (i), hence both their sums $H(\tau)$ and $\tilde{H}(\tau)$ are continuous functions, implying $\tilde{H}(\tau) = H(\tau)$ pointwise [89, Appendix 1]. Then (1.11) just says $H(0) \equiv \tilde{H}(0)$ (while (iii) simply endorses the even symmetry of the summations in (1.11)). \square

A basic function pair (in which z is an implied parameter) is

$$h(\tau) \stackrel{\text{def}}{=} \frac{1}{2} e^{-z|\tau|}, \quad \hat{h}(u) \equiv \frac{1}{2\pi} \frac{z}{z^2 + u^2} \quad (\text{Re } z > 0). \quad (1.12)$$

In this case, the summation formula (1.11) expresses the partial fraction decomposition

$$\frac{1}{e^z - 1} + \frac{1}{2} \left(= \frac{1}{2 \tanh z/2} \right) \equiv \lim_{R \rightarrow +\infty} \sum_{r=-R}^{+R} \frac{1}{z + 2\pi i r}. \quad (1.13)$$

Conversely, (1.13) gives back the general Poisson summation formula (1.11) simply through integration against suitable test functions [16, Sect. 1].

1.4 Euler–Maclaurin Summation Formulae

These formulae serve to approximate sums by integrals. We will use a very basic version but in several specific forms, which we then sketch.

For a differentiable function f of a real variable u , integration by parts against the first Bernoulli polynomial $B_1(u) = u - \frac{1}{2}$ yields $\frac{1}{2}[f(0) + f(1)] = \int_0^1 f(u) du + \int_0^1 B_1(u) f'(u) du$. Now let $K \in \mathbb{N}$ and $K' \in \mathbb{N} \cup \{+\infty\}$ with $K' > K$, and just here, $\{u\} \stackrel{\text{def}}{=} \text{the fractional part of } u$. Then a simple summation of the preceding formula over successive unit intervals gives

$$\sum_{k=K}^{K'} f(k) - \frac{1}{2}[f(K) + f(K')] - \int_K^{K'} f(u) du = \int_K^{K'} B_1(\{u\}) f'(u) du \stackrel{\text{def}}{=} R, \quad (1.14)$$

only assuming f and f' to be *integrable* over the interval under use. If moreover f is a *monotonic* function, then the difference or remainder R is easily bounded, resulting in

$$\sum_{k=K}^{K'} f(k) - \frac{1}{2}[f(K) + f(K')] = \int_K^{K'} f(u) du + R, \quad |R| \leq \frac{1}{2}|f(K) - f(K')|.$$

In particular if $K' = +\infty$, then $f(+\infty) = 0$, and

$$\sum_{k=K}^{+\infty} f(k) = \int_K^{+\infty} f(u) du + O(f(K)) \quad \text{for } K \rightarrow +\infty. \quad (1.15)$$

1.5 Meromorphic Properties of Mellin Transforms

The representation of meromorphic functions as Mellin transforms, the resulting control over their principal parts and effective meromorphic continuation, constitute the technical tool we will constantly invoke throughout. Therefore, although the subject is classic [36, Chap. VII.6] [58] [112, Chap. III], we give a detailed review of the results we will use.

For a locally integrable function $f(x)$, $x \in (0, +\infty)$, its *Mellin transform* can be formally defined as a function of $s \in \mathbb{C}$, as

$$Mf(s) = \int_0^\infty f(x) x^{-s-1} dx. \quad (1.16)$$

We list some increasingly detailed analytic features in s for this transform.

Stage 1 (root). If for some $-\infty \leq \mu < \nu < +\infty$,

$$f(x) = O(x^\nu) \quad (x \rightarrow 0^+) \quad \text{and} \quad f(x) = O(x^\mu) \quad (x \rightarrow +\infty), \quad (1.17)$$

then Mf is defined and holomorphic in the strip $\{\mu < \operatorname{Re} s < \nu\}$ (simply by the convergence properties of the integral (1.16) in that strip). *Note:* $Mf(s) \rightarrow 0$ for $s \rightarrow \infty$ in the strip, by the Riemann–Lebesgue lemma.

Stage 2. If moreover, for some *strictly descending* sequence $\mu_0 > \mu_1 > \dots > \mu_N$ with $\mu_0 < \nu$ and some sequence of polynomials $\{p_n(y)\}$, the function f obeys a *large- x asymptotic estimate*

$$f(x) \sim \sum_{n=0}^N p_n(\log x) x^{\mu_n} + O(x^{\mu'}) \quad \text{for } x \rightarrow +\infty \quad \text{with } \mu' < \mu_N,$$

then Mf continues to a meromorphic function of s in the wider strip $\{\mu' < \operatorname{Re} s < \nu\}$, where its only singularities are *poles at $s = \mu_n$, with principal parts $p_n(-\frac{d}{ds}) \frac{1}{s - \mu_n}$* for $n = 0, 1, \dots, N$.

Proof. Let $g_N(x) = f(x) - \sum_{n=0}^N p_n(\log x) x^{\mu_n} \Big|_{\{x>1\}}$. Then g_N satisfies all assumptions of Stage 1 with μ' in place of μ , hence Mg_N is holomorphic in the wider strip; while $\int_1^\infty [p_n(\log x) x^{\mu_n}] x^{-s-1} dx$ precisely yields the stated principal part at $s = \mu_n$ by direct evaluation. \square

Stage 2'. If moreover the above sequences are *infinite*, $\mu_n \downarrow -\infty$, and the function f admits a *complete large- x asymptotic expansion*

$$f(x) \sim \sum_{n=0}^\infty p_n(\log x) x^{\mu_n} \quad \text{for } x \rightarrow +\infty, \quad (1.18)$$

then all results of Stage 2 extend to $N = \infty$ (simply by letting $\mu' \rightarrow -\infty$ in the above): Mf is then meromorphic in the half-plane $\{\operatorname{Re} s < \nu\}$, and its only singularities are

$$\text{poles at } s = \mu_n, \text{ with principal parts } p_n \left(-\frac{d}{ds} \right) \frac{1}{s - \mu_n} \quad \text{for } n \in \mathbb{N}. \quad (1.19)$$

In particular, in the absence of logarithmic terms (p_n constant), the corresponding pole at μ_n is *simple, with residue* p_n .

Stage 3'. Independently of Stages 2, 2', if now for some strictly *ascending* sequence $\nu_0 < \nu_1 < \dots$ with $\nu_0 > \mu$ and $\nu_n \uparrow +\infty$, the function f admits a complete *small- x* asymptotic expansion (here without logarithmic terms for simplicity)

$$f(x) \sim \sum_{n=0}^{\infty} f_n x^{\nu_n} \quad \text{for } x \rightarrow 0^+, \quad (1.20)$$

then Mf continues to a meromorphic function in the half-plane $\{\operatorname{Re} s > \mu\}$, where its only singularities are

$$\text{simple poles at } s = \nu_n, \text{ with residues } (-f_n) \quad \text{for } n \in \mathbb{N} \quad (1.21)$$

(arguing just as above, but with the bounds $x = 0$ and $+\infty$ interchanged under $s \mapsto -s$; hence the residues change sign).

Note that we could more generally have allowed, e.g., suitable complex exponents μ_n and ν_n , logarithmic terms also in the expansion (1.20) at $x = 0$, etc. Here we aim at economy, and refer to [58, 76] for more general settings.

Effective continuation formulae. In Stages 2–3, now assuming all asymptotic conditions to be *differentiable*, the Mellin representations can be explicitly modified to perform an *effective* meromorphic continuation. The trick consists of successive *integrations by parts*, watching the growth conditions at the integration bound where divergence sets in.

We illustrate the idea in the case of Stage 2 (or 2') assuming a simple pole at $s = \mu_0$ (i.e., $p_0 = \text{const.}$). We then have, in principle for $\mu_0 < \operatorname{Re} s < \nu$,

$$Mf(s) = \int_0^{\infty} [f(x) x^{-\mu_0}] x^{\mu_0-s-1} dx = \frac{1}{s - \mu_0} \int_0^{\infty} [f(x) x^{-\mu_0}]' x^{\mu_0-s} dx, \quad (1.22)$$

but the latter integral satisfies all conditions of Stage 1 with μ_1 in place of μ , so it defines an analytic function in the *wider* strip $\{\mu_1 < \operatorname{Re} s < \nu\}$; the right-hand side thus continues Mf meromorphically in this strip (note how the pole at $s = \mu_0$ and its residue $[f(x) x^{-\mu_0}]_0^{\infty} = p_0$ become explicit).

Now in the newly accessed strip $\{\mu_1 < \operatorname{Re} s < \mu_0\}$, the integration by parts can be reversed, but the growth conditions produce a new result,

$$Mf(s) = \int_0^\infty [f(x)x^{-\mu_0} - p_0]x^{\mu_0-s-1}dx = \int_0^\infty [f(x) - p_0x^{\mu_0}]x^{-s-1}dx, \quad (1.23)$$

upon which the whole process can then be restarted to pass the next pole μ_1 .

A multiple pole can likewise be passed by further iterated integrations by parts; as we will not explicitly need to do this, we simply refer to [58] [106, Appendix A].

Chapter 2

Infinite Products and Zeta-Regularization

Our purpose here is to review some symmetric-function techniques suitable for certain *divergent* sequences $\{x_k\}_{k=1,2,\dots}$, i.e., infinite (real or complex) sequences with $|x_k| \uparrow \infty$. (The x_k are counted with multiplicities if any.) We specifically wish to control their symmetric functions of the Zeta and Delta types as sketched in Sect. 1.1. We will elaborate on a scheme initiated for positive eigenvalue spectra in [104], where Theta-type functions provided a natural and most convenient base. For sequences such as the Riemann zeros, however, Theta-type functions exhibit less accessible and rather intricate properties, whereas Delta-type functions are openly present (as the “trivial” Gamma factors) and thus provide a privileged gateway; at the same time, a setting which requires positive sequences is inadequate. We must then adapt [104] to a broader perspective better adjusted to the idiosyncrasies of sequences like the Riemann zeros, and favoring conditions placed on Delta-type functions. The latter option, however, lengthens some intermediate calculations, so we emphasize that these are totally elementary (nineteenth-century mathematics!), all the more that a parameter μ_0 , the order of the Delta function, remains low; and only $\mu_0 = 1$ and $\frac{1}{2}$ will serve for the Riemann zeros.

Our goal here is to output a toolbox of basic special-value formulae that are general enough and systematic yet economical, being tailored to our current final needs. Alternatively, we refer to [32, 53, 60, 61, 90] for very powerful and general, but more elaborate, frameworks. For greater convenience, Sect. 2.6 groups the main practical results to be exported for later use. Thus, the bulk of this chapter may be skipped on first reading. (Inversely, it can serve as a tutorial for zeta-regularization alone.)

2.1 Informal Discussion

We begin with a heuristic description of our basic targets for symmetric functions of infinite sequences $\{x_k\}$ when $|x_k| \uparrow \infty$.

First, we want such a sequence to admit a *zeta function* $Z(s | \{x_k\}) \stackrel{\text{def}}{=} \sum_k x_k^{-s}$ (shortened to $Z(s)$ except when the sequence may vary). We will need

this function $Z(s)$ to be *analytic in a complex neighborhood of $s = 0$* , but its defining series $\sum_k x_k^{-s}$ will converge at best in some half-plane $\{\operatorname{Re} s > \mu_0\}$ with $0 < \mu_0 < +\infty$; we then ask $Z(s)$ to admit a *meromorphic extension* to the whole s -plane, with *computable poles* (to make way for some *explicit* analytical continuation methods). We at once assume that $\sum_k |x_k|^{-\mu_0}$ *diverges* because it will always be the case for us, and otherwise extra subtleties arise.

Next, we want to have a *Delta-type* function: formally, $\Delta(x | \{x_k\}) = \Delta(x) = \prod_k (x + x_k)$ like a characteristic polynomial, but this nice intrinsic product is well defined as it stands for finite sequences only. In the case of a diverging sequence as above, only a *modified* infinite product converges (everywhere), the *Weierstrass product*

$$\Delta_0(x) \stackrel{\text{def}}{=} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) \exp \left\{ \sum_{m=1}^{[\mu_0]} \frac{1}{m} \left(-\frac{x}{x_k}\right)^m \right\} \quad (\forall x \in \mathbb{C}), \quad (2.1)$$

with the usual notation $[\mu] \stackrel{\text{def}}{=} \text{the integer part of } \mu$. Thus Δ_0 is an entire function, having $\{-x_k\}$ as its set of zeros; but conversely, the latter features are too general to select $\Delta(x) = \Delta_0(x)$ uniquely.

To narrow the ambiguity on Δ we note that $\Delta_0(x)$ is an entire function of *finite order* μ_0 [10, Theorem 2.6.5]. This discussion will then largely borrow from the theory of such functions and their parametrization [10, Chap. 2] [26, Sect. 11] [89, Appendix 5]. We recall that in full generality, the order of an entire function $f(x)$ is $\mu_0 \stackrel{\text{def}}{=} \inf\{\mu \in \mathbb{R} \mid f(x) = O(e^{|x|^\mu})\}$, with $\mu_0 > 0$ for all but polynomial f ; thus here for $\Delta_0(x)$, $0 < \mu_0 < \infty$.

A central result is that any entire function of order μ_0 with this prescribed set of zeros $\{-x_k\}$ has the *Hadamard product* form:

$$\Delta(x) = e^{P(x)} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) \exp \left\{ \sum_{m=1}^{[\mu_0]} \frac{1}{m} \left(-\frac{x}{x_k}\right)^m \right\} \quad (\forall x \in \mathbb{C}) \quad (2.2)$$

where $P(x)$ is some polynomial P of degree $\leq [\mu_0]$. From now on, $\Delta(x)$ will then be the *general* function (2.2) of order μ_0 , with prescribed zeros $\{-x_k\}$ but P unspecified.

Spaces of entire functions have a basic symmetry, *translations* in \mathbb{C} . Now, finite products $\Delta_f(x | \{x_k\}) = \prod_1^K (x + x_k)$ are manifestly covariant:

$$\Delta_f(x | \{x_k + y\}) \equiv \Delta_f(x + y | \{x_k\}) \quad (\forall y \in \mathbb{C}),$$

but *not the Weierstrass product inside (2.2)*. Indeed, $\Delta_0(x)$ as in (2.1) is also specified by a particular *Taylor series at $x = 0$* for its logarithm,

$$-\log \Delta_0(x) = \sum_{m=m_0}^{\infty} \frac{Z(m)}{m} (-x)^m \quad (\text{convergent for } |x| < \inf_k |x_k|), \quad (2.3)$$

$$\text{where } m_0 \stackrel{\text{def}}{=} [\mu_0] + 1 \text{ (= the least integer } > \mu_0); \text{ i.e.,} \quad (2.4)$$

$$(\log \Delta_0)^{(m)}(0) = (-1)^{m-1} (m-1)! Z(m) \quad \text{for } m > \mu_0 \quad (2.5)$$

$$= 0 \quad \text{for } m = 0, 1, \dots, [\mu_0]. \quad (2.6)$$

More generally, the polynomial P in (2.2) identifies with the Taylor expansion to order $[\mu_0]$ of $\log \Delta$ at $x = 0$. The vanishing conditions (2.6) mean $P \equiv 0$, thus specifying Δ_0 in particular, but they are distinguishing *one base point* in \mathbb{C} , here $x = 0$, and this breaks translation invariance. It is thus important to *recover this symmetry* in (2.2).

Thus, the following related formula regains convergence and covariance:

$$(\log \Delta)^{(m_0)}(x) = (-1)^{m_0-1} (m_0 - 1)! \sum_k (x + x_k)^{-m_0}; \quad (2.7)$$

inversely, $(\log \Delta)^{(m_0)}$ can be seen as the basic *intrinsic* function, yielding $\log \Delta$ as its *general m_0 -th primitive* (only a function mod P); whereas the particular primitive

$$(\log \Delta_0)(x) = \left[\int_0^x \right]^{m_0} (\log \Delta)^{(m_0)}, \quad (2.8)$$

cannot be fully covariant since it depends on a lower bound at each integration step. Still, if $\Delta_0(x)$ continues here to mean $\Delta_0(x | \{x_k\})$, then (2.7) implies

$$\begin{aligned} (\log \Delta_0)^{(m_0)}(x | \{x_k + y\}) &= (\log \Delta_0)^{(m_0)}(x + y), \\ \text{i.e., } \Delta_0(x | \{x_k + y\}) &= e^{P_y(x)} \Delta_0(x + y) \end{aligned} \quad (2.9)$$

with $P_y(x) = \text{a polynomial in } x \text{ of degree } \leq [\mu_0]$.

Translation covariance is thus achieved, but through a multiplier e^{P_y} ; the conditions (2.6) then fix the polynomial $P_y(x)$, as

$$\log \frac{\Delta_0(x | \{x_k + y\})}{\Delta_0(x + y)} = P_y(x) = - \sum_{m=0}^{m_0} \frac{1}{m!} (\log \Delta_0)^{(m)}(y) x^m. \quad (2.10)$$

Thus, not only $P_y \neq 0$, but its coefficients look like (and turn out to be) *transcendental* functions in the shift parameter y .

Further conditions are now required to gain some effective control over the meromorphic properties of $Z(s)$. At this level, details may have to be fine-tuned to each context. While there exist very powerful general formalisms as already mentioned, here we choose to stay fairly close to the setting of the Riemann zeros, so as to quickly reach our final applications.

2.2 A Class of Eligible Sequences $\{x_k\}$

So, we consider infinite complex sequences $\{x_k\}_{k=1,2,\dots}$ such that $x_k \neq 0$ ($\forall k$), $|x_k| \uparrow \infty$, and $|\arg x_k| < \pi - \phi_0$ for some $\phi_0 > 0$. We now deem such a sequence *eligible of order* μ_0 for some $\mu_0 \in (0, +\infty)$ if it satisfies two (non-independent) conditions.

(1) As in Sect. 2.1, the series

$$\sum_{k=1}^{\infty} |x_k|^{-s} \text{ converges if (and only if) } \operatorname{Re} s > \mu_0, \quad (2.11)$$

allowing us to define the *zeta function of the sequence*, by the Dirichlet series

$$Z(s | \{x_k\}) \equiv Z(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} x_k^{-s} \quad \text{for } \operatorname{Re} s > \mu_0. \quad (2.12)$$

(2) Moreover, for some real sequence $\mu_0 > \mu_1 > \dots > \mu_n \downarrow -\infty$ (with μ_0 as above) and some angle $\phi_1 \in (0, \phi_0)$, the function $\log \Delta_0(x)$ defined through (2.1) obeys a *complete large- x asymptotic expansion* of the form [106]

$$\log \Delta_0(x) \sim \sum_{n=0}^{\infty} (\tilde{a}_{\mu_n} \log x + a_{\mu_n}) x^{\mu_n}, \quad x \rightarrow \infty, \quad |\arg x| < \phi_1, \quad (2.13)$$

uniformly in $\arg x$ (to make (2.13) repeatedly differentiable [35, Sect. 1.6]).

Equation (2.13) can be called a (*generalized*) *Stirling expansion*: in the prototypical case of the integer sequence $\{x_k = k\}$ of order $\mu_0 = 1$, (2.13) amounts to the classic Stirling formula for $-\log \Gamma(1+x)$, as reviewed in Sect. 3.6. (Higher powers of $\log x$ could be allowed, as in Sect. 1.5, but they would unnecessarily complicate both the treatment and the results.)

We will systematically extend the range of all coefficients like \tilde{a}_{μ_n} , a_{μ_n} having general (real-valued) indices by the convention

$$\tilde{a}_{\mu}, a_{\mu} \stackrel{\text{def}}{=} 0 \quad \text{for arbitrary } \mu \in (-\infty, \mu_0) \setminus \{\mu_n\}. \quad (2.14)$$

2.3 Meromorphic Continuation of the Zeta Function

We can now begin to use the results of Sect. 1.5.

At the root, the estimates (2.6) and (2.13) together make the following Mellin transform of the type (1.16) well defined: for $m_0 \stackrel{\text{def}}{=} [\mu_0] + 1$ as in (2.4),

$$I(s) \stackrel{\text{def}}{=} \int_0^{\infty} \log \Delta_0(y) y^{-s-1} dy \quad (\mu_0 < \operatorname{Re} s < m_0). \quad (2.15)$$

Then on the one hand, exploiting Sect. 1.5 thoroughly, $I(s)$ continues to a meromorphic function in all of \mathbb{C} where its only singularities are:

- Due to (2.13), at most double poles at $s = \mu_n$, with the principal parts

$$I(\mu_n + \varepsilon) = \tilde{a}_{\mu_n} \varepsilon^{-2} + a_{\mu_n} \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0} \quad (2.16)$$

- And due to (2.3), at most simple poles at the integers $s = m \geq m_0$, with residues $(-1)^m Z(m)/m$.

On the other hand, repeated integrations by parts upon (2.15) lead to

$$I(s) = \frac{(-1)^{m_0} \Gamma(-s)}{\Gamma(m_0 - s)} \int_0^\infty (\log \Delta_0)^{(m_0)}(y) y^{m_0 - s - 1} dy \quad (\mu_0 < \operatorname{Re} s < m_0); \quad (2.17)$$

and if here we substitute (2.7) for $(\log \Delta_0)^{(m_0)}(y)$ and integrate term by term relying on $\int_0^\infty (1+y)^{-(z+w)} y^{z-1} dy = \Gamma(z)\Gamma(w)/\Gamma(z+w)$, we find

$$I(s) \equiv -\frac{\Gamma(-s)\Gamma(m_0)}{\Gamma(m_0 - s)} \sum_k \int_0^\infty (y + x_k)^{-m_0} y^{m_0 - s - 1} dy \equiv \dots \equiv \frac{\pi}{s \sin \pi s} Z(s).$$

But since $I(s)$ is meromorphic in all of \mathbb{C} , the reverse formula

$$Z(s) \equiv \frac{s \sin \pi s}{\pi} I(s) \quad (\mu_0 < \operatorname{Re} s < m_0) \quad (2.18)$$

actually extends to the whole plane; it then continues $Z(s)$ from the half-plane $\{\operatorname{Re} s > \mu_0\}$ as in (2.12) to a meromorphic function in all of \mathbb{C} , with singularities explicitly induced by those of $I(s)$ which we just described.

As the main qualitative consequence, the zeros of the prefactor $(s \sin \pi s)$ counteract all the poles of $I(s)$ located at integers, implying that:

- At any integer $\mu_n = m$, by (2.16), $Z(s)$ has

$$\text{at most a simple pole, of residue } (-1)^m m \tilde{a}_m, \quad (2.19)$$

$$\text{with } \operatorname{FP}_{s=m} Z(s) = (-1)^m (\tilde{a}_m + m a_m) \quad (2.20)$$

(and likewise for arbitrary integers $m < \mu_0$, under the convention (2.14))

- At integers $s > \mu_0$, since all the poles of $I(s)$ are simple, $Z(s)$ is devoid of poles (indeed, we know it is analytic for $\operatorname{Re} s > \mu_0$)
- Last but not least, $s = 0$ is a *regular point* for $Z(s)$ thanks to the *double zero* of the prefactor $(s \sin \pi s)/\pi$ there, with (2.16) implying

$$Z(\varepsilon) = \tilde{a}_0 + a_0 \varepsilon + O(\varepsilon^2)_{\varepsilon \rightarrow 0} \quad \Rightarrow \quad Z(0) = \tilde{a}_0, \quad Z'(0) = a_0 \quad (2.21)$$

(again with the convention (2.14) if $0 \notin \{\mu_n\}$).

Summary

All the poles of $Z(s)$ lie in the single descending sequence $\{\mu_n\}$; they are at most double in general, simple at integers, and definitely absent at $s = 0$ (a regular point); the values $Z(m)$ for integer $m \leq \mu_0$, and $Z'(0)$ have closed expressions in terms of the generalized Stirling expansion coefficients.

2.4 The Generalized Zeta Function

All previous considerations about $Z(s)$ transfer to shifted sequences $\{x_k + x\}$ up to reasonable limitations on the shift parameter x (e.g., $(x + x_k) \notin \mathbb{R}_- (\forall k)$).

Thus, for an eligible sequence $\{x_k\}$ as in Sect. 2.2, the *generalized zeta function*

$$Z(s, x) \stackrel{\text{def}}{=} \sum_k (x_k + x)^{-s} \quad (\text{Re } s > \mu_0) \quad (2.22)$$

satisfies:

- The first eligibility condition (2.11), with μ_0 unchanged
- The obvious but essential functional relation

$$\partial_x Z(s, x) = -sZ(s + 1, x); \quad (2.23)$$

- *Explicit covariance*, since the identity $Z(s, X | \{x_k + x\}) \equiv Z(s, X + x | \{x_k\})$, obviously obeyed for $\text{Re } s > \mu_0$, then extends to all s by meromorphic continuation.

The other eligibility condition (2.13) asks for a large- X expansion like

$$\log \Delta_0(X | \{x_k + x\}) \sim \sum_{\mu} [\tilde{a}_{\mu}(x) \log X + a_{\mu}(x)] X^{\mu}, \quad (2.24)$$

but thanks to the covariance identity (2.10) with $P_x(X)$ polynomial in X ,

$$\log \Delta_0(X | \{x_k + x\}) \equiv P_x(X) + \log \Delta_0(X + x | \{x_k\}); \quad (2.25)$$

then this last term can have its large- $(X + x)$ expansion (2.13) reordered in powers of X , and the form (2.24) is obtained; this operation, which may output a modified sequence $\{\mu\} \neq \{\mu_n\}$ (but μ_0 is invariant), is algebraic at any order, therefore:

$$\begin{array}{ll} a_{\mu}(x) \text{ for } \mu \notin \mathbb{N}, \text{ and all } \tilde{a}_{\mu}(x) \text{ are polynomial in } x; & (2.26) \\ \text{but } a_m(x) \text{ for } m = 0, 1, \dots, [\mu_0] \text{ are transcendental in } x & \end{array}$$

as the polynomial $P_x(X)$ adds a transcendental dependence on x , by (2.10). This dichotomy will visibly affect all concrete outputs, see Sect. 2.6.3.

Then, all previous results about $Z(s) = Z(s, 0)$ such as (2.19)–(2.21) extend to $Z(s, x)$, just by replacing \tilde{a}_m, a_m by $\tilde{a}_m(x), a_m(x)$ respectively.

In Sect. 2.3, $Z(s)$ was derived from the function Δ_0 ; it is now desirable to express $Z(s, x)$ likewise, since Delta-type functions will provide our main input data in applications.

At integer s , just by inspection, (2.7) and its higher derivatives yield

$$Z(m, x) \equiv \frac{(-1)^m}{(m-1)!} (-\log \Delta)^{(m)}(x) \quad \text{for integer } m \geq m_0 \quad (2.27)$$

valid for general Δ , which includes Δ_0 (cf. (2.5) for $x = 0$).

For more general s , we combine (2.18) not with the integral formula (2.15), but with (2.17) which uses the intrinsic covariant function $\log \Delta_0^{(m_0)}$; so, we may simply shift the whole formula by x , with the result:

$$Z(s, x) = \frac{(-1)^{m_0}}{\Gamma(s)\Gamma(m_0-s)} \int_0^\infty (-\log \Delta_0)^{(m_0)}(x+y) y^{m_0-s-1} dy \quad (\mu_0 < \operatorname{Re} s < m_0), \quad (2.28)$$

again also valid with general Δ in place of Δ_0 . (*Remark.* Symbolically, this amounts to extending (2.27) to derivatives of non-integer order, as $Z(s, x) \equiv \Gamma(s)^{-1}(-d/dx)^s(-\log \Delta_0)(x)$.)

2.5 The Zeta-Regularized Product

One can now define $\prod_k(x + x_k)$ for an infinite eligible sequence, as

$$\Delta_\infty(x) \stackrel{\text{def}}{=} \exp[-Z'(0, x)] \quad (' \equiv \partial/\partial s, \text{ as in (1.9)}). \quad (2.29)$$

Indeed this straightforwardly works for a finite sequence $\{x_k\}$, and it prescribes a finite value in any case since we forced $Z(s, x)$ to be regular at $s = 0$: $\Delta_\infty(x)$ is called the *zeta-regularized* form for the products $\Delta(x)$.

However, the prescription (2.29) is an impractical one as it needs analytical continuation (from $\operatorname{Re} s > \mu_0 > 0$ to $s = 0$), which is not a constructive operation. Eligibility conditions, like here or in [104], are precisely tools to pin down $\Delta_\infty(x)$ quite concretely.

First and foremost, although this does not obviously show on its definition, $\Delta_\infty(x)$ indeed belongs to the family (2.2), i.e., it satisfies

$$\Delta_\infty(x) \equiv e^{-P_\infty(x)} \Delta_0(x) \quad \text{for a polynomial } P_\infty(x) \text{ of degree } \leq \mu_0. \quad (2.30)$$

Proof. The definition (2.29) and functional relation (2.23) imply, for $m \in \mathbb{N}$,

$$(-\log \Delta_\infty)^{(m)}(x) = (-1)^m \partial_s s(s+1) \dots (s+m-1) Z(s+m, x)|_{s=0}. \quad (2.31)$$

Now, $\partial_s s f(s)|_{s=0} \equiv \text{FP}_{s=0} f$ for a function $f(s)$ having at most a simple pole at $s=0$, and the functions $Z(s+m, x)$ are precisely like that, by (2.19). So, the previous formula reduces for $m > 0$ to

$$(-\log \Delta_\infty)^{(m)}(x) = (-1)^m \text{FP}_{s=m} [(s-m+1) \dots (s-1) Z(s, x)], \quad (2.32)$$

and at the regular point $s = m_0 (= [\mu_0] + 1)$, this boils down to

$$\begin{aligned} (-\log \Delta_\infty)^{(m_0)}(x) &= (-1)^{m_0} (m_0 - 1)! Z(m_0, x) \\ &\equiv (-\log \Delta)^{(m_0)}(x) \quad \text{by identification with (2.27)}. \quad \square \end{aligned} \quad (2.33)$$

In other words, $\Delta_\infty(x)$ is a particular Δ function; but one which has also inherited *manifest covariance* (under translations) from $Z(s, x)$ itself, by its very definition (2.29). A serious limitation, however, is that this zeta-regularization prescription will be stable under *no other* change of x -variable than a pure translation (not even $x \mapsto \lambda x$).

Next, the yet unknown polynomial P_∞ can be characterized in two independent ways, where the *harmonic numbers* H_n will appear:

$$H_n \stackrel{\text{def}}{=} \sum_{j=1}^n \frac{1}{j} = \psi(n+1) - \psi(1) \quad (n \in \mathbb{N}) \quad (H_0 = 0). \quad (2.34)$$

- Taking successive logarithmic derivatives of (2.30) at $x=0$ and enforcing (2.6), we get

$$P_\infty(x) = \sum_{m=0}^{[\mu_0]} (-\log \Delta_\infty)^{(m)}(0) \frac{x^m}{m!}. \quad (2.35)$$

If, in analogy with (2.5) for $\log \Delta_0$, we pose

$$\begin{aligned} Z_\infty(m) &\stackrel{\text{def}}{=} \frac{(-1)^m}{(m-1)!} (-\log \Delta_\infty)^{(m)}(0) \quad \text{for } m \in \mathbb{N}^*, \\ (Z_\infty(m) &\equiv Z(m) \quad \text{if } m \geq m_0), \end{aligned} \quad (2.36)$$

then $P_\infty(x) = P_\infty(0) + \sum_{m=1}^{[\mu_0]} \frac{Z_\infty(m)}{m} (-x)^m$ gets specified via (2.31)–(2.32):

$$P_\infty(0) = Z'(0) \quad (2.37)$$

$$\begin{aligned} Z_\infty(m) &= \frac{1}{(m-1)!} \text{FP}_{s=m} \left[\frac{\Gamma(s)}{\Gamma(s-m+1)} Z(s) \right] \quad (m > 0) \\ &= \text{FP}_{s=m} Z(s) + (-1)^m m H_{m-1} \tilde{a}_m \end{aligned} \quad (2.38)$$

$$= Z(m) \quad \text{if } Z(s) \text{ is regular at } m \quad (\iff \tilde{a}_m = 0). \quad (2.39)$$

$$\text{E.g., } Z_\infty(1) \equiv \text{FP}_{s=1} Z(s) \quad (\text{because } H_0 = 0). \quad (2.40)$$

Note that $P_\infty(x)$ nicely completes the Taylor series (2.3), to give

$$-\log \Delta_\infty(x) = Z'(0) + \sum_{m=1}^{\infty} \frac{Z_\infty(m)}{m} (-x)^m \quad (|x| < \inf_k |x_k|). \quad (2.41)$$

- A more tangible specification of P_∞ involves the large- x (generalized Stirling) expansion of $\log \Delta_\infty(x)$.
By substitution into (2.37) and (2.38) of the respective values (2.21) for $Z'(0)$ and (2.20) for $\text{FP}_{s=m} Z(s)$, the previous results become

$$\Delta_\infty(x) \equiv e^{-P_\infty(x)} \Delta_0(x), \quad P_\infty(x) = \sum_{m=0}^{[\mu_0]} (H_m \tilde{a}_m + a_m) x^m. \quad (2.42)$$

In turn, this and (2.13) imply that the large- x expansion assumes a special (“canonical”) form in the case of $\log \Delta_\infty(x)$:

$$\log \Delta_\infty(x) \sim \sum_{m=0}^{[\mu_0]} \tilde{a}_m (\log x - H_m) x^m + \sum_{\mu_n \notin \mathbb{N}} (\tilde{a}_{\mu_n} \log x + a_{\mu_n}) x^{\mu_n}, \quad (2.43)$$

in which powers x^m for $m \in \mathbb{N}$ are *banned in free form* (i.e., when not paired with $x^m \log x$ exactly as shown). Inversely, given an eligible sequence $\{x_k\}$, enforcing this restriction fixes the polynomial P_∞ and thereby $\Delta_\infty(x)$ itself.

This result can be understood in two other ways.

- By brute force: The initial asymptotic information (2.13) about $\log \Delta_0(x)$, put into the integral representation (2.28) of $Z(s, x)$, can generate a large- x expansion for the latter (exercise!):

$$Z(s, x) \sim -\frac{x^{-s}}{\Gamma(s)} \sum_{n=0}^{\infty} \left[\left(\tilde{a}_{\mu_n} \partial_\mu + a_{\mu_n} \right) \frac{\Gamma(s - \mu)}{\Gamma(-\mu)} x^\mu \right]_{\mu=\mu_n}, \quad (2.44)$$

on which the formal evaluation of $-Z'(0, x)$ precisely restores (2.43) (here, the free powers x^μ get killed by the pole of $\Gamma(-\mu)$ when $\mu \in \mathbb{N}$).

- By *symbolic integration* [104]: In order to continue $Z(s, x)$ toward $s = 0$, one idea is to integrate $Z(m_0, x)$ m_0 times using (2.23) backwards; the integration bound is then found by reference to $m > m_0$, as

$$(\log \Delta)^{(m-1)}(x) = \int_{+\infty}^x (\log \Delta)^{(m)}(x') dx'. \quad (2.45)$$

This integral naturally diverges as soon as $m - 1 < \mu_0$, but it can nevertheless be defined on the asymptotic terms of $(\log \Delta)^{(m)}(x)$ by, e.g., $\int_{+\infty}^x x'^{\mu} dx' = x^{\mu-1}/(\mu - 1)$, followed by analytical continuation and/or differentiation in μ , plus finite-part extraction if needed. It is then easily verified that in degrees $n \in \mathbb{N}$, only canonical (allowed) terms $x^n(\log x - H_n)$ emerge, e.g., $\int_{+\infty}^x x'^{-1} dx' = \log x$, $\int_{+\infty}^x \log x' dx' = x(\log x - 1)$, etc. Consequently, the canonical nature of the series (2.43) symbolically means [104]

$$\log \Delta_{\infty}(x) = \left[\int_{+\infty}^x \right]^{m_0} (\log \Delta)^{(m_0)}, \quad (2.46)$$

a covariant formula, to be compared to (2.8) for the Weierstrass product. Warning: such a symbolic integration does not follow the usual change-of-variable rules!

2.6 Practical Results

Our strategy is to harvest explicit properties of the zeta function $Z(s, x)$ from our assumed knowledge of a Delta function $\Delta(x)$ and its properties. Here we recapitulate the facts gathered in this chapter that serve that purpose.

2.6.1 Zeta-Regularization: a Zeta-Free Recipe

The definition (2.29) of the zeta-regularized form, $\Delta_{\infty}(x) \stackrel{\text{def}}{=} \exp[-Z'(0, x)]$, goes in the wrong direction for us: it precisely uses a zeta function that we aim at understanding (plus analytical continuation in totally abstract form).

Fortunately, given an *eligible* sequence $\{x_k\}$ of order μ_0 in the sense of Sect. 2.2, and an infinite Hadamard product $\Delta(x | \{x_k\})$ as in (2.2), with a known (generalized) Stirling expansion (2.13), we now basically have a *mechanical* zeta-regularization rule. It suffices to reexpand any terms of (2.13) having $\mu_n = m \in \mathbb{N}$ (in finite number since $m \leq [\mu_0]$) over the basis $\{x^m(\log x - H_m), x^m\}$, as

$$\log \Delta(x) \sim \sum_{m=0}^{[\mu_0]} (\tilde{a}_m(\log x - H_m) + b_m) x^m + \sum_{\mu_n \notin \mathbb{N}} (\tilde{a}_{\mu_n} \log x + a_{\mu_n}) x^{\mu_n} \quad (2.47)$$

(recalling that $H_m = \sum_{j=1}^m 1/j$ are the harmonic numbers). Now (2.43) demonstrates that in the zeta-regularized form, terms which remain in (2.47) with coefficients denoted \tilde{a}_{μ} , a_{μ} are allowed: namely, all terms with $\mu_n \notin \mathbb{N}$, and, for $\mu_n \in \mathbb{N}$,

$$\dots, \quad \tilde{a}_2 x^2 (\log x - \frac{3}{2}), \quad \tilde{a}_1 x (\log x - 1), \quad \tilde{a}_0 \log x; \quad (2.48)$$

while all terms with b_m are *banned*, namely

$$b_m x^m \quad \text{for all } m \in \mathbb{N}, \quad \text{including additive constants } (b_0). \quad (2.49)$$

Consequently, the zeta-regularized form has to be, simply (cf. (2.42)),

$$\Delta_\infty(x) \equiv \exp \left[- \sum_{m=0}^{[\mu_0]} b_m x^m \right] \Delta(x), \quad b_m = H_m \tilde{a}_m + a_m. \quad (2.50)$$

With $\Delta_\infty(x)$ thus determined, a shifted large- x expansion of $\log \Delta_\infty$ yields

$$\log \Delta_\infty(x+y) \sim \sum_{m=0}^{[\mu_0]} \tilde{a}_m(y) (\log x - H_m) x^m + \sum_{\mu \notin \mathbb{N}} [\tilde{a}_\mu(y) \log x + a_\mu(y)] x^\mu, \quad (2.51)$$

in which all coefficients are now *algebraically computable polynomials* of y .

Such a covariant shift performed on all the above results (2.37)–(2.41) yields our final special-value formulae in their most general and useful form:

$$-\log \Delta_\infty(y) = Z'(0, y) \quad (2.52)$$

$$\frac{(-1)^{m-1}}{(m-1)!} (\log \Delta_\infty)^{(m)}(y) = Z_\infty(m, y) \quad (m = 1, 2, \dots), \quad (2.53)$$

with

$$Z_\infty(1, y) \equiv \text{FP}_{s=1} Z(s, y), \quad (2.54)$$

$$Z_\infty(m, y) = \text{FP}_{s=m} Z(s, y) + (-1)^m m H_{m-1} \tilde{a}_m(y) \quad (m \geq 1) \quad (2.55)$$

$$\equiv Z(m, y) \quad \text{if } Z(s, y) \text{ is regular at } s = m. \quad (2.56)$$

Our main application, concerning the Riemann zeros, only uses $\mu_0 \leq 1$, in which case the fully general formula (2.55) can be skipped as this makes no difference. However, Appendix B (devoted to the zeros of *Selberg* zeta functions) makes a brief incursion into $\mu_0 = 2$ territory, where that omission would cause fatal errors.

2.6.2 A Subclass: “Theta-Eligible” Sequences

Some auxiliary sequences to be needed alongside the Riemann zeros are specially simple (and positive) eligible sequences $\{x_k\}$ (called “admissible”

sequences in [104]): they allow easier direct control through their *Theta* functions $\Theta(z)$, under the following two assumptions:

$$(1) \quad \Theta(z) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} e^{-zx_k} \quad \text{converges for } \operatorname{Re} z > 0; \quad (2.57)$$

(2) a *complete small- z asymptotic expansion* can be written for $\Theta(z)$,

$$\Theta(z) \sim \sum_{n=0}^{\infty} c_{\mu_n} z^{-\mu_n} \quad \text{for } z \rightarrow 0^+, \quad |\arg z| < \theta; \quad (2.58)$$

the sequence $\{\mu_n\}$ is as in Sect. 2.2, and $0 < \theta \leq \frac{1}{2}\pi$. In linear spectral problems of physics, $\Theta(z)$ is the trace of the heat operator (“partition function”). Certain complex sequences $\{x_k\}$ can qualify as well [53, 60, 61, 90]. For the Riemann zeros themselves, however, the simplest $\Theta(z)$ would be $V(iz)$ where V is *Cramér’s function* (5.14). Now we will see in Sect. 5.4 that the singular structure of $V(z)$ about the origin is only partially and indirectly available, and much less simple than (2.58). This makes us unwilling to choose $V(z)$ as principal symmetric function over the Riemann zeros.

Sequences that are theta-eligible as above gain simpler formulae.

The zeta function is more readily accessed than in Sect. 2.3, as

$$Z(s) \equiv \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta(z) z^{s-1} dz \quad (\operatorname{Re} s > \mu_0). \quad (2.59)$$

By Sect. 1.5 Stage 3’ (but with $s \mapsto -s$), $Z(s)$ is meromorphic in \mathbb{C} , and its singularities are at most *simple* poles at the μ_n , with residues $c_{\mu_n}/\Gamma(\mu_n)$; this implies that all the negative integers including 0 are regular points which give the special values

$$Z(-n) = (-1)^n n! c_{-n} \quad (\forall n \in \mathbb{N}) \quad (\text{with the convention (2.14)}). \quad (2.60)$$

Likewise,

$$Z(s, x) \equiv \frac{1}{\Gamma(s)} \int_0^{\infty} \Theta(z) e^{-xz} z^{s-1} dz \quad (\operatorname{Re} s > \mu_0, \quad |\arg x| < \frac{1}{2}\pi + \theta), \quad (2.61)$$

from which a direct calculation of the large- x expansion of $-Z'(0, x)$ yields [104]

$$\log \Delta_{\infty}(x) \sim \sum_{m=0}^{[\mu_0]} \frac{(-1)^m}{m!} c_m (\log x - H_m) x^m - \sum_{\substack{\mu_n \notin \mathbb{N} \\ \mu_n < \mu_0}} \Gamma(-\mu_n) c_{\mu_n} x^{\mu_n} \quad (x \rightarrow \infty, \quad |\arg x| < \frac{1}{2}\pi + \theta). \quad (2.62)$$

Thus, not only our previous eligibility conditions hold, but the generalized Stirling expansion of $\log \Delta_\infty(x)$ fully stems from the small- z expansion (2.58) of $\Theta(z)$.

2.6.3 *Explicit Properties of the Generalized Zeta Function*

Here we sum up the explicit properties of the function $Z(s, x)$ which can be gathered from the assumed knowledge of an eligible Delta function $\Delta(x)$ with its properties. They concern two facets of the (generalized) zeta function: its principal parts, and its values at integers.

- *The singular structure:* All s -plane poles of $Z(s, x)$ have computable locations (namely, the exponents μ in the large- X expansion (2.24)), and their principal parts involve coefficients \tilde{a}_μ and a_μ by (2.16), excepting a_m at integer points m by (2.19): such coefficients precisely give rise to *polynomials* in x by (2.26), so the principal parts at the poles of $Z(s, x)$ are *rational* objects (algebraically computable).
- *Explicit-value formulae:* They will concern all the values (or finite parts) of $Z(s, x)$ at integer points $s \in \mathbb{Z}$, plus $Z'(0, x)$ all by itself: we dub the totality of them “special values.” Here an essential splitting is seen between *negative* and *positive* integer points s [106, Sect. 4].
 - The values (or finite parts) at $s = -m \in -\mathbb{N}$ are given by (2.20), using the coefficients \tilde{a}_{-m} and a_{-m} except a_0 , that again extend to *polynomials in x* ; hence those $Z(-m, x)$, beginning with $Z(0, x)$, are *rational* objects as well (algebraically computable).
 - The values (or finite parts) at $s = +m \in \mathbb{N}$ are given: for $m > \mu_0$, by (2.53) and (2.56); and for $m \leq \mu_0$, by (2.20) which involves a_m when $m > 0$, whose parametric extension $a_m(x)$ is transcendental by (2.26). At $m = 0$, as an exception, (2.52)–(2.53) show that a natural companion to the values $\{Z(m, x)\}_{m>0}$ is $Z'(0, x) = a_0(x)$ rather than $Z(0, x)$ (already put in the set $\{Z(-m, x)\}$ of rational values). We then declare our set of special values for $+m \in \mathbb{N}$ to be *not* $\{Z(m, x)\}_{m \in \mathbb{N}}$, but $\{Z'(0, x)\} \cup \{Z(m, x)\}_{m \in \mathbb{N}^*}$ instead. Manifestly, all the values of *this* set show a *transcendental* nature (e.g., as functions of x).

Therefore, all special-value tables for zeta functions will reflect this algebraic vs transcendental splitting. A prototype for these tables follows; it is set with redundancies on purpose, to display a gamut of cases. In the $s = 0$ line of Table 2.1 we expanded the polynomial coefficient $\tilde{a}_0(x)$ explicitly; the reader is invited to do the same for the remaining polynomials $\tilde{a}_{\pm m}$ and a_{-m} (exercise!).

Table 2.1 Special values of the generalized zeta function $Z(s, x)$ for a general eligible sequence of order μ_0 , in terms of the zeta-regularized product $\Delta_\infty(x)$ given by (2.50) and its expansion polynomials $\tilde{a}_m(x)$, $a_m(x)$ given by (2.51) (upper part: rational values, lower part: transcendental values). Notation: see (1.7), (2.29), (2.34); m is an integer

s	$Z(s, x) = \sum_k (x_k + x)^{-s}$
Regular $-m < 0$ $-m \leq 0$ (finite part)	$-(-1)^m m a_{-m}(x)$ <div style="border: 1px solid black; padding: 5px; display: inline-block;">$\text{FP}_{s=-m} Z(s, x) = (-1)^m [\tilde{a}_{-m}(x) - m a_{-m}(x)]$</div>
0	$\tilde{a}_0(x) \equiv \sum_{j=0}^{[\mu_0]} \tilde{a}_j x^j$
0 (s-derivative)	<div style="border: 1px solid black; padding: 5px; display: inline-block;">$Z'(0, x) = -\log \Delta_\infty(x)$</div>
+1 (finite part)	<div style="border: 1px solid black; padding: 5px; display: inline-block;">$\text{FP}_{s=1} Z(s, x) = (\log \Delta_\infty)'(x)$</div>
$+1 \leq m \leq \mu_0$ (finite part)	<div style="border: 1px solid black; padding: 5px; display: inline-block;">$\text{FP}_{s=m} Z(s, x) = (-1)^{m-1} \left[\frac{1}{(m-1)!} (\log \Delta_\infty)^{(m)}(x) + m H_{m-1} \tilde{a}_m(x) \right]$</div>
Regular $+m \geq 1$	$\frac{(-1)^{m-1}}{(m-1)!} (\log \Delta_\infty)^{(m)}(x)$

Chapter 3

The Riemann Zeta Function $\zeta(x)$: a Primer

This chapter and the next one recall basic properties of the *Riemann zeta* function $\zeta(x)$ in very elementary terms. We do not try to compete with the many exhaustive texts on $\zeta(x)$ to which we refer the reader, e.g., [14, 26, 31, 48, 55, 57, 63, 84, 89, 101]. We mainly survey the basic facts, arguments, and formulae to the extent that they will serve later, to make this book reasonably self-contained. We also highlight the *analytical*, as opposed to the purely *arithmetical*, features because those play a major role throughout. This part can then be a tutorial to $\zeta(x)$ from an analytical angle. The end of this chapter also reviews the *Dirichlet beta* and *Hurwitz zeta* functions.

3.1 Definition and Immediate Properties

$$\zeta(x) = \sum_{k=1}^{\infty} k^{-x} \quad (\operatorname{Re} x > 1) \quad (3.1)$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} \frac{1}{\Gamma(x)} \int_0^{\infty} e^{-kz} z^{x-1} dz = \frac{1}{\Gamma(x)} \int_0^{\infty} \sum_{k=1}^{\infty} e^{-kz} z^{x-1} dz \\ &= \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{1}{e^z - 1} z^{x-1} dz \quad (\operatorname{Re} x > 1). \end{aligned} \quad (3.2)$$

It then follows from Sect. 1.5 Stage 3' (applied with (1.4) and $x = -s$), that $\zeta(x)$ is a meromorphic function in all of \mathbb{C} having the single pole $x = 1$ which is simple and of residue 1, and that the special values at $x = -n$, $n \in \mathbb{N}$ are computable, and rational:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \quad \text{for } n \in \mathbb{N} \quad (\text{e.g., } \zeta(0) = -\tfrac{1}{2}), \quad (3.3)$$

$$\implies \zeta(-2k) = 0 \quad (k = 1, 2, \dots) : \quad \text{the trivial zeros.} \quad (3.4)$$

3.2 The Euler Infinite Product

The Euler factorization over the set of prime integers (this set to be denoted $\{p\}$ throughout) is the fundamental *arithmetical* property of $\zeta(s)$:

$$\zeta(x) = \prod_{\{p\}} (1 - p^{-x})^{-1} \quad (\text{Re } x > 1); \quad (3.5)$$

and it translates to Dirichlet series for $\log \zeta(x)$ and $[\zeta'/\zeta](x)$:

$$\log \zeta(x) = \sum_{\{p\}} \sum_{r=1}^{\infty} \frac{p^{-rx}}{r} = \sum_{n \geq 2} \frac{\Lambda(n)}{\log n} n^{-x} \quad (\text{Re } x > 1), \quad (3.6)$$

$$\frac{\zeta'}{\zeta}(x) = - \sum_{\{p\}} \sum_{r=1}^{\infty} (\log p) p^{-rx} = - \sum_{n \geq 2} \Lambda(n) n^{-x} \quad (\text{Re } x > 1), \quad (3.7)$$

where

$$\Lambda(n) \stackrel{\text{def}}{=} \begin{cases} \log p & \text{if } n = p^r, \text{ } p \text{ prime} \\ 0 & \text{otherwise} \end{cases} \quad (\text{the von Mangoldt function}). \quad (3.8)$$

3.3 The Stieltjes and Cumulant Expansions

Two Laurent expansions related to $\zeta(x)$ about $x = 1$ will have some relevance.

The first and more customary one introduces the *Stieltjes constants* γ_n (their notation fluctuates in the literature, here we follow [1, Sect. 23.2]):

$$\zeta(x) = \frac{1}{x-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (x-1)^n, \quad (3.9)$$

$$\gamma_n = \lim_{K \rightarrow \infty} \left\{ \sum_{k=1}^K \frac{(\log k)^n}{k} - \frac{(\log K)^{n+1}}{n+1} \right\} \quad (\text{e.g., } \gamma_0 = \gamma). \quad (3.10)$$

For instance, the γ_n are surveyed in [18, 21], numerically tabulated in [18, 64].

However, the truly useful expansion here will be [56, Sect. 5] [9]

$$\frac{\zeta'}{\zeta}(x) = -\frac{1}{x-1} - \sum_{n=0}^{\infty} \tilde{\gamma}_n (x-1)^n, \quad (3.11)$$

but the $\tilde{\gamma}_n$ (also called η_n) are much less studied or computed – we only saw one table for them [56] until quite recently [77] [21, Appendix D].

Remark. There also exists a direct arithmetical specification [9, p. 286] [46],

$$\tilde{\gamma}_n \equiv \eta_n = \frac{(-1)^n}{n!} \lim_{K \rightarrow \infty} \left\{ \sum_{k=1}^K \Lambda(k) \frac{(\log k)^n}{k} - \frac{(\log K)^{n+1}}{n+1} \right\}, \quad (3.12)$$

generalizing an earlier result for $\tilde{\gamma}_0 = -\gamma$ [102, p. 251].

As none of those constants is fully standardized anyway, here we advocate a newer notation which matches the relevant *degrees* of the coefficients:

$$g_n \equiv n \gamma_{n-1}, \quad \text{and} \quad g_n^c \equiv (-1)^n (n-1)! \tilde{\gamma}_{n-1}; \quad (3.13)$$

this way, we obtain one Taylor series vs its *cumulant series*,

$$(x-1)\zeta(x) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} g_n (x-1)^n, \quad (3.14)$$

$$\log[(x-1)\zeta(x)] = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} g_n^c (x-1)^n. \quad (3.15)$$

Only the g_n^c (which we dub “*Stieltjes cumulants*”) will effectively enter this work, in some special-value formulae. They relate to the Stieltjes constants rewritten as g_n by the canonical link between a sequence and its cumulant sequence: if both g_n and g_n^c are assigned the degree n , then the g_n^c are standard (up to signs) homogeneous polynomials of global degree n in the g_m , $m \leq n$ (and vice-versa):

$$\begin{aligned} g_1^c &= g_1 = \gamma, & g_2^c &= g_2 + g_1^2, & g_3^c &= g_3 + 3g_2g_1 + 2g_1^3, \dots \end{aligned} \quad (3.16)$$

(≈ 0.577215665) (≈ 0.187546233) (≈ 0.103377264).

3.4 The Functional Equation and Completed Zeta Function $\Xi(x)$

We now recall the basic Riemann’s Functional Equation for $\zeta(s)$ (capitalizing its name, to distinguish it from generic functional equations). Textbooks on Riemann’s zeta function are keen to give more than one derivation of this Functional Equation, and sometimes quite many. In contrast here, one complex-analytic proof by Riemann [31, Sect. 1.6] is definitely more relevant to our analytical approach, and we give just that one.

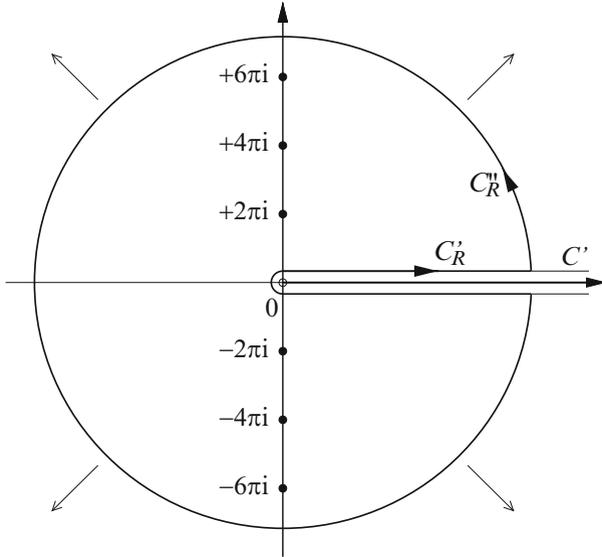


Fig. 3.1 Integration contour yielding Riemann's functional equation

First, the integral specification (3.2) is easily transformed to a Hankel loop integral,

$$\zeta(x) = \frac{\Gamma(1-x)}{2i\pi} \int_{C'} \frac{(-z)^{x-1}}{e^z - 1} dz \quad (x \notin \mathbb{N}^*), \quad (3.17)$$

where C' is an infinite path encircling the half-axis \mathbb{R}_+ clockwise at a distance $0 < \varepsilon \ll 1$ (Fig. 3.1); the right-hand side of (3.17) then analytically continues $\zeta(x)$ to all of $\mathbb{C} \setminus \mathbb{N}^*$.

Next, for $R \gg 1$, let C'_R be the truncation of C' at the radius $(2R+1)\pi$, C''_R be the positively oriented great circular arc of that same radius connecting the endpoints of C'_R , and $C_R = C'_R + C''_R$ (a closed contour). Then

$$\begin{aligned} \zeta(x) &= \frac{\Gamma(1-x)}{2i\pi} \lim_{R \rightarrow +\infty} \int_{C'_R} \frac{(-z)^{x-1}}{e^z - 1} dz \\ &= \frac{\Gamma(1-x)}{2i\pi} \lim_{R \rightarrow +\infty} \left[\int_{C_R} - \int_{C''_R} \right] \frac{(-z)^{x-1}}{e^z - 1} dz \end{aligned} \quad (3.18)$$

where R is now restricted to *integers*, which ensures that $(e^z - 1)^{-1}$ stays bounded on C''_R . Then, for $\operatorname{Re} x < 0$, $\int_{C''_R} = O(R^{\operatorname{Re} x}) \rightarrow 0$ as $R \rightarrow +\infty$, hence we are left with the contour integral \int_{C_R} alone,

$$\zeta(x) = \frac{\Gamma(1-x)}{2i\pi} \lim_{R \rightarrow +\infty} \int_{C_R} \frac{(-z)^{x-1}}{e^z - 1} dz, \quad (\operatorname{Re} x < 0) \quad (3.19)$$

which is easy to evaluate by the residue calculus: specifically, the partial fraction decomposition (1.13) of $(e^z - 1)^{-1}$ (i.e., the *Poisson summation formula* (1.11) as explained in Sect. 1.3), entails

$$\int_{C_R} \frac{(-z)^{x-1}}{e^z - 1} dz = 2i\pi \sum_{0 < |r| \leq R} (2\pi i r)^{x-1},$$

hence (3.19) yields $\zeta(x) = \Gamma(1-x) \sum_{r \in \mathbb{Z}^*} (2\pi i r)^{x-1}$, or

$$\zeta(x) \equiv \frac{\Gamma(1-x)}{(2\pi)^{1-x}} \left(2 \sin \frac{\pi x}{2} \right) \zeta(1-x), \quad (3.20)$$

which is *Riemann's Functional Equation*; obtained here for $\operatorname{Re} x < 0$, this identity extends everywhere by meromorphic continuation. \square

The Functional Equation implies further special values, now transcendental, at nonnegative integers (mainly by transfer from (3.3)):

$$\zeta'(0) = -\frac{1}{2} \log 2\pi, \quad (3.21)$$

$$\operatorname{FP}_{x=1} \zeta(x) = \gamma, \quad (3.22)$$

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!} |B_{2m}| \quad (m = 1, 2, \dots). \quad (3.23)$$

Then, the *completed zeta function*

$$\Xi(x) \stackrel{\text{def}}{=} \mathbf{G}^{-1}(x) (x-1)\zeta(x), \quad (3.24)$$

$$\text{with: } \mathbf{G}(x) \stackrel{\text{def}}{=} \frac{\pi^{x/2}}{2\Gamma(1 + \frac{1}{2}x)} \quad (\text{an explicit entire factor}), \quad (3.25)$$

is an entire function with the same zeros as $\zeta(x)$ except the trivial ones (3.4), and on which the Functional Equation (3.20) simplifies to

$$\Xi\left(\frac{1}{2} + t\right) \equiv \Xi\left(\frac{1}{2} - t\right) \quad (\text{an even function}) \quad (3.26)$$

i.e., the point $x = \frac{1}{2}$ is a *center of symmetry* for Ξ . (Note: the traditional completed function is $\xi(x) = \frac{1}{2} \Xi(x)$, but $\Xi(0) = 1$ is a nicer normalization.)

3.5 The Dirichlet Beta Function $\beta(x)$

Our study of $\zeta(x)$ will specifically need another Dirichlet series: the L -function for the odd Dirichlet character mod 4 (see Chap. 10), variously denoted $L_{\chi_{-4}}$ or Dirichlet β -function [33, Sect. 1.12 (end)] [1, Sect. 23.2] [14, Sect. 1.7]

$$\beta(x) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-x} \quad (\text{Re } x > 1) \quad (3.27)$$

$$\equiv \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{1}{2 \cosh z} z^{x-1} dz \quad (\text{Re } x > 1). \quad (3.28)$$

Its analytical treatment fully parallels that of $\zeta(x)$ except that $\beta(x)$ comes out as an *entire* function. It has the rational special values

$$\beta(-n) = \frac{1}{2} E_n \quad \text{for } n \in \mathbb{N} \quad (\text{e.g., } \beta(0) = \frac{1}{2}), \quad (3.29)$$

in particular its trivial zeros are the *odd* negative integers. It admits a completed beta function $\Xi_{\beta}(x)$ and a functional equation, according to

$$\Xi_{\beta}(x) \stackrel{\text{def}}{=} (4/\pi)^{(x+1)/2} \Gamma(\frac{1}{2}(x+1)) \beta(x) \equiv \Xi_{\beta}(1-x). \quad (3.30)$$

Further transcendental special values at nonnegative integers follow:

$$\beta'(0) = -\frac{3}{2} \log 2 - \log \pi + 2 \log \Gamma(\frac{1}{4}), \quad (3.31)$$

$$\beta(2m+1) = \frac{(\pi/2)^{2m+1}}{2(2m)!} |E_{2m}| \quad (m = 0, 1, \dots). \quad (3.32)$$

3.6 The Hurwitz Zeta Function $\zeta(x, w)$

It is a generalized zeta function [33, Sect. 1.10] [14, Sect. 1.6], defined by

$$\zeta(x, w) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (k+w)^{-x} \quad (\text{Re } x > 1, \text{ Re } w > 0) \quad (3.33)$$

$$\equiv \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{e^{(1-w)z}}{e^z - 1} z^{x-1} dz \quad (\text{Re } x > 1, \text{ Re } w > 0), \quad (3.34)$$

with the obvious functional equations

$$\zeta(x, w+1) = \zeta(x, w) - w^{-x}, \quad (3.35)$$

$$\partial_w \zeta(x, w) = -x \zeta(x+1, w). \quad (3.36)$$

Its analytical treatment initially parallels that of $\zeta(x) \equiv \zeta(x, 1)$, simply replacing the expansion (1.4) by (1.3) with $y = 1 - w$. Thus for fixed w , $\zeta(x, w)$ is meromorphic in the whole x -plane, with a single pole at $x = 1$, simple and of residue 1, and it has the rational special values

$$\zeta(-n, w) = -\frac{B_{n+1}(w)}{n+1} \quad (n \in \mathbb{N}) \quad (3.37)$$

(e.g., $\zeta(0, w) = \frac{1}{2} - w$).

Transcendental special-value formulae also arise for $x = +n \in \mathbb{N}$, see below.

The contour integration argument used to get the Functional Equation for $\zeta(x)$ (Sect. 3.4) extends to $\zeta(x, w)$ as long as $0 < \operatorname{Re} w \leq 1$ (in order for $e^{(1-w)z}(e^z - 1)^{-1}$ to remain bounded on the arc C''_R). The outcome is still a functional relation; however, it pairs the Hurwitz function no longer with itself, but with another generalization of the Riemann zeta function,

$$F(u, x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{u^n}{n^x} \quad (\text{a Lerch function or polylogarithm}), \quad (3.38)$$

according to the *Jonquière relation* [33, 1.11(16)] [101, Sect. 2.17],

$$\zeta(x, w) \equiv \frac{\Gamma(1-x)}{(2\pi)^{1-x} i} [e^{i\pi x/2} F(e^{2i\pi w}, 1-x) - e^{-i\pi x/2} F(e^{-2i\pi w}, 1-x)] \quad (3.39)$$

for $\operatorname{Re} x < 0$, $0 < \operatorname{Re} w \leq 1$ (a relation mentioned only for later reference).

For a general w -value, $\zeta(x, w)$ loses not only the central symmetry, but also the arithmetical properties, of $\zeta(x)$.

We now turn to the special-value formulae for $x = +n \in \mathbb{N}$; they involve the Digamma function $\psi(w)$ as in (1.6), which is transcendental:

$$\zeta'(0, w) = \log [\Gamma(w)/\sqrt{2\pi}] \quad (3.40)$$

$$\operatorname{FP}_{x=1} \zeta(x, w) = -\psi(w) \quad (3.41)$$

$$\zeta(n, w) = (-1)^n \psi^{(n-1)}(w)/(n-1)! \quad \text{for } n = 2, 3, \dots \quad (3.42)$$

These formulae are classic [33, Sect. 1.10] [14, Sect. 1.6] ((3.40) is known as Lerch's formula); nevertheless, we rederive them here as a first and major illustration of the zeta-regularization formalism of Chap. 2 [104, Sect. 6c] [90, Ex. 3].

Proof. Basically, $\zeta(x, 1+y)$ is just the generalized zeta function for the sequence of integers $\{x_k = 1, 2, \dots\}$, which is theta-eligible (Sect. 2.6.2) of order $\mu_0 = 1$, having the Theta function (cf. (1.4))

$$\Theta(z) = \sum_{k=1}^{\infty} e^{-kz} = \frac{1}{e^z - 1} = \sum_{n=-1}^{\infty} \frac{B_{n+1}}{n+1} z^n. \quad (3.43)$$

Then, by definition of the corresponding zeta-regularized product $\Delta_\infty(y)$,

$$\zeta'(0, 1 + y) = -\log \Delta_\infty(y), \quad (3.44)$$

and here, the general special-value formulae (2.53)–(2.56) further yield

$$\text{FP}_{x=1} \zeta(x, 1 + y) = (\log \Delta_\infty)'(y) \quad (3.45)$$

$$\zeta(n, 1 + y) = \frac{(-1)^n}{(n-1)!} (-\log \Delta_\infty)^{(n)}(y) \quad \text{for } n = 2, 3, \dots \quad (3.46)$$

If we pose $D(y) \stackrel{\text{def}}{=} [-\log \Delta_\infty(y)] - \log \Gamma(1 + y)$, it is then only a question of proving that this difference $D(y) \equiv -\frac{1}{2} \log 2\pi$. The argument for this proceeds from $n \geq 2$ to $n = 1$, and finally $n = 0$.

The last displayed formula (which is intrinsic) turns explicit first: for $n = 2$,

$$(-\log \Delta_\infty)''(y) = \zeta(2, 1 + y) = \sum_{k=1}^{\infty} (k + y)^{-2} = \psi'(1 + y) \quad (3.47)$$

by a basic formula giving ψ' [1, (6.4.10)], hence $(-\log \Delta_\infty)''(y) = (\log \Gamma)''(1 + y)$ and likewise for higher derivatives: this proves (3.42), and amounts to

$$D(y) = b_1 y + b_0. \quad (3.48)$$

(And by (2.26), constants such as b_1 and b_0 ought to be transcendental.)

Then, by applying ∂_x at $x = 0$ to the functional relation (3.35) and using (3.44), we find $-\log \Delta_\infty(y) = -\log \Delta_\infty(y - 1) + \log y$, which is the *same* functional equation as obeyed by $\log \Gamma(1 + y)$: consequently, the above difference $D(y)$ is 1-periodic, and this *symmetry* result forces $b_1 = 0$ in (3.48), or $(-\log \Delta_\infty)'(y) \equiv \psi(1 + y)$. With this, (3.45) becomes (3.41), which is thereby proved.

Last, we can identify the constant b_0 in (3.48) by setting $y = 0$ therein:

$$b_0 = D(0) = \zeta'(0, 1) - \log \Gamma(1) = \zeta'(0) \quad (3.49)$$

$= -\frac{1}{2} \log 2\pi$ by (3.21), hence $\Delta_\infty(y) \equiv \sqrt{2\pi}/\Gamma(1 + y)$, establishing the last pending special-value formula (3.40).

As an extra bonus, we can invoke the fully explicit (and rational) large- y expansion of $-\log \Delta_\infty(y)$ yielded by (2.62) for the case (3.43):

$$-\log \Delta_\infty(y) \sim B_0 y(\log y - 1) - B_1 \log y + \sum_{m=1}^{\infty} (2m - 2)! \frac{B_{2m}}{(2m)!} y^{1-2m} \quad (3.50)$$

($|\arg y| < \pi$); hence the previous results also imply

$$\log \Gamma(1 + y) \sim y(\log y - 1) + \frac{1}{2} \log y + \frac{1}{2} \log 2\pi + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} y^{1-2m},$$

($y \rightarrow \infty, |\arg y| < \pi$), (3.51)

i.e., the *complete Stirling expansion* for $\log \Gamma(1 + y)$ in one shot. □

We find this joint proof of the transcendental special-value formulae for $\zeta(s, w)$ and of the Stirling expansion for $\log \Gamma$ to have a condensed and compelling character, and it highlights an important connection between (3.51) and (3.40). We finally display the derivative of (3.51), to be often invoked:

$$\psi(1 + y) \sim \log y + \frac{1}{2} y^{-1} - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m} y^{-2m}. \tag{3.52}$$

The Hurwitz zeta function, and particularly its set of special values (3.37), (3.40)–(3.42), constitute a kind of a prototype for the methods and results that will later arise for the superzeta functions. We therefore recapitulate the special-value formulae of $\zeta(x, w)$ in Table 3.1 below, to serve as a template to forthcoming Tables and for later reference.

Table 3.1 Special values of $\zeta(x, w)$ (upper part: rational, lower part: transcendental, cf. Sect. 2.6.3 and Table 2.1). Notation: see (1.3), (1.6), (1.7); n is an integer

x	$\zeta(x, w) = \sum_{k=0}^{\infty} (k + w)^{-x}$
$-n \leq 0$	$-\frac{B_{n+1}(w)}{n+1}$
0	$\frac{1}{2} - w$
0 (<i>x-derivative</i>)	$\zeta'(0, w) = \log[\Gamma(w)/\sqrt{2\pi}]$
$+1$ (<i>finite part</i>)	$\text{FP}_{x=1}\zeta(x, w) = -\psi(w)$
$+n > 1$	$\frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(w)$

Chapter 4

Riemann Zeros and Factorizations of the Zeta Function

This is the second half of our review on the basic properties of $\zeta(x)$.

4.1 Growth Properties of $\zeta(x)$ and $\Xi(x)$

The theory of entire functions of *finite order* (cf. Sect. 2.1) applies to Riemann's Ξ function in a classic way [26, Sect. 12] [89, Appendix 5].

We first bound $\zeta(x)$ and the trivial factor $\mathbf{G}^{-1}(x)(x-1)$ separately in the half-plane $\{\operatorname{Re} x \geq \frac{1}{2}\}$.

Applying the Euler–Maclaurin formula (1.14) to $f(u) = u^{-x}$ with $\operatorname{Re} x > 1$ and $K = 1$, $K' = +\infty$, yields (with $\{u\} \stackrel{\text{def}}{=} \text{the fractional part of } u \text{ here}$)

$$\zeta(x) = \frac{1}{x-1} + \frac{1}{2} - x \int_1^\infty B_1(\{u\}) u^{-x-1} du; \quad (4.1)$$

but as the right-hand side converges and defines an analytic function for $\operatorname{Re} x > 0$, it analytically continues $\zeta(x)$ to this half-plane. The integral is bounded by $\int_1^\infty \frac{1}{2} u^{-\operatorname{Re} x-1} du = (2 \operatorname{Re} x)^{-1} \leq 1$ if $\operatorname{Re} x \geq \frac{1}{2}$, hence as $x \rightarrow \infty$ in the latter half-plane, the bound $\zeta(x) = O(|x|)$ holds.

On the other hand, exponentiating Stirling's formula (3.51), we see that $|\mathbf{G}^{-1}(x)(x-1)| < e^{C|x|\log|x|}$ for any $C > 1$.

Hence in the half-plane $\{\operatorname{Re} x \geq \frac{1}{2}\}$,

$$|\Xi(x)| < e^{C|x|\log|x|} \quad (\forall C > 1); \quad (4.2)$$

but this final bound, which concerns Ξ , therefore extends to the whole x -plane thanks to the Functional Equation (3.26). At the same time,

$$\log \Xi(x) \sim \frac{1}{2} x \log x \quad \text{for } x \rightarrow +\infty \quad (4.3)$$

(using Stirling's formula again, plus $\zeta(x) \sim 1$ for $x \rightarrow +\infty$). All that fixes the order of the entire function $\Xi(x)$ to be precisely $\mu_0 = 1$ in the variable x .

By the Functional Equation, Ξ is also an entire function in the variable $u = x(x - 1)$; its order (relative to this new variable) is then $\frac{1}{2}$, and this seemingly trivial rephrasing proves quite useful for certain issues.

4.2 The Riemann Zeros (Basic Features)

By the general theory of entire functions of finite order [10, Chap. 2], a function like Ξ which has a *non-integer* order ($\frac{1}{2}$, in the variable $u = x(x - 1)$) must have *infinitely many zeros* (a statement invariant under this change of variables $x \leftrightarrow u$). The zeros of $\Xi(x)$ are then *nontrivial zeros* for $\zeta(x)$ (besides the trivial set $\{-2k\}$): they are the *Riemann zeros*, usually labeled ρ .

The Euler product (3.5), convergent for $\operatorname{Re} x > 1$, implies $\zeta(x) \neq 0$ there. This plus the central symmetry restrict the ρ to lie in the strip $0 \leq \operatorname{Re} x \leq 1$.

In fact, $\zeta(x)$ does not vanish on the line $\operatorname{Re} x = 1$ either. While this is a key step toward the *prime number theorem*, a major arithmetical result, both the fact itself and its derivation seem to lie off our main analytical track; so, we merely quote the result (proved in all books). This further restricts the Riemann zeros to the open *critical strip* $0 < \operatorname{Re} x < 1$.

No zeros can lie on the real segment $(0,1)$ either: indeed,

$$(1 - 2^{1-x})\zeta(x) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-x} \quad \text{converges for } \operatorname{Re} x > 0,$$

and for all $x > 0$ this series is alternating and shows a *strictly* positive sum.

By central symmetry, the Riemann zeros occur in pairs $(\rho, 1 - \rho)$. To make this symmetry explicit, we will rather enumerate them in pairs as

$$\{\rho = \frac{1}{2} \pm i\tau_k\}_{k=1,2,\dots}, \quad \operatorname{Re} \tau_k > 0 \text{ and non-decreasing}, \quad (4.4)$$

and also parametrize each such pair by the single number

$$u_k = \rho(1 - \rho) = \frac{1}{4} + \tau_k^2. \quad (4.5)$$

The *Riemann Hypothesis* (1859, still open) states that [92]

$$\text{(RH)} \quad \operatorname{Re} \rho = \frac{1}{2} \quad (\forall \rho) \iff (\forall k) \quad \tau_k \in \mathbb{R}_+ \iff u_k - \frac{1}{4} \in \mathbb{R}_+; \quad (4.6)$$

all our arguments will be *unconditional* (independent of RH) up to Chap. 11.

Another unproved conjecture is that all zeros are *simple*, but we adopt the standard convention of counting all objects with their multiplicities if any.

The first zeros' approximate ordinates are numerically found as

$$\{\tau_k \approx 14.1347251, 21.0220396, 25.0108576, 30.4248761, 32.9350616, \dots\}$$

4.3 Hadamard Products for $\Xi(x)$

By the general theory of entire functions of finite order as referred to in Sect. 2.1, in the case of $\Xi(x)$ of order 1, the zeros ρ satisfy $\sum_{\rho} |\rho|^{-1-\varepsilon} < \infty$ if $\varepsilon > 0$, and $\Xi(x)$ admits the Hadamard factorization

$$\Xi(x) = \Xi(0) e^{[\log \Xi]'(0)x} \prod_{\rho} \left(1 - \frac{x}{\rho}\right) e^{x/\rho} \quad (\forall x \in \mathbb{C}).$$

Were $\sum |\rho|^{-1}$ to converge, this and the general bound $|1 - z| \leq e^{|z|}$ would imply $\Xi(x) = O(e^{C|x|})$ for some C and all x , which is contradicted by (4.3). Hence

$$\sum_{\rho} |\rho|^{-1-\varepsilon} \quad \text{converges for } \varepsilon > 0, \text{ diverges for } \varepsilon = 0. \quad (4.7)$$

This growth estimate for the zeros (*note*: $\sum |\rho|^{-1} = \infty$ is a *tangible* proof of their infinite number) will be refined to the *Riemann–von Mangoldt formula* soon (Sect. 4.5), but it is needed meanwhile.

Here, using (3.24) and the special values $\Xi(0) = 1$ and (3.21), the above Hadamard formula boils down to

$$\Xi(x) = e^{Bx} \prod_{\rho} \left(1 - \frac{x}{\rho}\right) e^{x/\rho}, \quad B \stackrel{\text{def}}{=} \log 2\sqrt{\pi} - 1 - \frac{1}{2}\gamma \quad (4.8)$$

$$\text{or } \zeta(x) = \frac{1}{x-1} \mathbf{G}(x)\Xi(x) \equiv \frac{e^{(\log 2\pi - 1 - \gamma/2)x}}{2(x-1)\Gamma(1 + \frac{1}{2}x)} \prod_{\rho} \left(1 - \frac{x}{\rho}\right) e^{x/\rho}. \quad (4.9)$$

The logarithmic derivative of the latter will also serve,

$$\frac{\zeta'}{\zeta}(x) \equiv (\log 2\pi - 1 - \frac{1}{2}\gamma) - \frac{1}{x-1} - \frac{1}{2}\psi(1 + \frac{1}{2}x) + \sum_{\rho} \left[\frac{1}{x-\rho} + \frac{1}{\rho}\right]. \quad (4.10)$$

Those are the classic forms given everywhere, for which the ordering of the factors is immaterial (they converge absolutely for all $x \in \mathbb{C}$). However, we will critically need more symmetrical forms, which require the zeros to be taken *in pairs* $(\rho, 1 - \rho)$. So from now on, all symbols \prod_{ρ} for products or \sum_{ρ} for summations crucially *imply this grouping of zeros in pairs*.

Thereupon, following, e.g., [31, Sects. 1.10 and 2.8]: first,

$$\sum_{\rho} \frac{1}{\rho} = \sum_k \left[\frac{1}{\frac{1}{2} + i\tau_k} + \frac{1}{\frac{1}{2} - i\tau_k} \right] = \sum_k \frac{1}{\frac{1}{4} + \tau_k^2} = \sum_k \frac{1}{u_k} \quad (4.11)$$

(the grouping restores absolute convergence). This at once motivates the introduction of *two kinds* of zeta functions over the Riemann zeros; anticipating our future notation (Chap. 5), we set $\mathcal{Z}_*(s) \stackrel{\text{def}}{=} \sum_{\rho} \rho^{-s}$ and

$\mathcal{Z}_*(\sigma) \stackrel{\text{def}}{=} \sum_k u_k^{-\sigma}$, which translates (4.11) to $\mathcal{Z}_*(1) = \mathcal{Z}_*(1)$. Then, this other pair evaluation:

$$\left(1 - \frac{x}{\rho}\right) \left(1 - \frac{x}{1-\rho}\right) \equiv 1 - \frac{x(1-x)}{\frac{1}{4} + \tau^2}, \quad (4.12)$$

substituted into the initial Hadamard product (4.8) for $\Xi(x)$, turns it into

$$\Xi(x) = e^{[B + \sum_\rho \frac{1}{\rho}]x} \prod_\rho \left(1 - \frac{x}{\rho}\right) = e^{[B + \mathcal{Z}_*(1)]x} \prod_k \left[1 - \frac{x(1-x)}{u_k}\right]. \quad (4.13)$$

Now, the symmetry of the Functional Equation $\Xi(x) = \Xi(1-x)$ requires the exponents in (4.13) to vanish, or $\mathcal{Z}_*(1) = -B$. All that yields the identities

$$\begin{aligned} \mathcal{Z}_*(1) = \mathcal{Z}_*(1) = -B &\equiv -[\log \Xi]'(0) = 1 + \frac{1}{2}\gamma - \frac{1}{2} \log 4\pi \quad (4.14) \\ &(\approx 0.0230957090). \end{aligned}$$

This result appears in some textbooks [26, Sect. 12, (10)–(11)] [31, 3.8(4)] [89, Sect. 3.1], but in isolation; we will see that it actually initiates *two infinite sequences* of such formulae for all the special values $\mathcal{Z}_*(k)$ and $\mathcal{Z}_*(k)$ ($k = 1, 2, \dots$), which admit parametric generalizations as well (Chaps. 7–8).

Finally, (4.13) reduces to two manifestly symmetric product forms, less common than (4.8) [31, Sect. 2.8] [89, Sect. 3.1] but more useful for us later,

$$\Xi(x) = \prod_\rho \left(1 - \frac{x}{\rho}\right) \quad \text{with zeros grouped in pairs} \quad (4.15)$$

$$\equiv \prod_{k=1}^{\infty} \left[1 + \frac{u}{u_k}\right], \quad u \stackrel{\text{def}}{=} x(x-1) \quad (4.16)$$

(the latter is just a Hadamard product for Ξ as a function of u of order $\frac{1}{2}$).

4.4 Basic Bounds on ζ'/ζ

An important class of results to come, like the Riemann–von Mangoldt formula (Sect. 4.5) and the Explicit Formulae (Sect. 6.2) (plus our basic analytical continuation formula for a zeta function over the Riemann zeros in Chap. 7), require some control over the growth of ζ'/ζ in the complex plane. The problematic region is clearly about the critical strip, where the unknown Riemann zeros lie; if $\text{Re } x > x_0$ for some $x_0 > 1$ then ζ'/ζ will be trivially bounded (and the symmetrical half-plane $\{\text{Re } x < 1 - x_0\}$ is thereupon controlled by applying the Functional Equation to ζ'/ζ ; we will skip this).

Without loss of generality, we may now fix $x_0 = 2$ (the customary choice), and only consider the upper half-plane (by conjugation symmetry).

We follow the standard exposition, as condensed in [57, Sect.1.4] for instance; usually, however, the bounds for $[\zeta'/\zeta](x)$ are intended to hold everywhere, whereas we focus on $\text{Im } x \gg 1$, which simplifies some equations. The logic goes as follows: (1) bound ζ'/ζ for $\text{Re } x \geq 2$, and specially at the edge; (2) using this, bound the fluctuations in the distribution of the zeros' ordinates $\{\text{Im } \rho\}$; (3) in turn, use the latter results to bound $[\zeta'/\zeta](x)$ when $-1 < \text{Re } x < 2$. We then develop those successive points in (partial) detail.

(1) The Dirichlet series (3.7) for $[\zeta'/\zeta](x)$ immediately implies

$$|\zeta'/\zeta|(x) \leq |\zeta'/\zeta|(2) = O(1) \quad \text{for all } x \text{ with } \text{Re } x \geq 2. \quad (4.17)$$

(2) This in turn constrains the distribution of the zeros' ordinates to obey the following two bounds:

$$\sum_{\rho} \frac{1}{4 + (T - \text{Im } \rho)^2} = O(\log T), \quad (4.18)$$

$$\#\{\rho \mid |\text{Im } \rho - T| < 1\} < C_1 \log T \quad \text{for some } C_1 > 0. \quad (4.19)$$

Proof. Select $x_T = 2 + iT$ (with $T \gg 1$). These inequalities obeyed by every zero ρ : $\text{Re } \frac{1}{x_T - \rho} > 0$ (due to $|\arg \rho| < \frac{1}{2}\pi$), and

$$\text{Re } \frac{1}{x_T - \rho} = \frac{2 - \text{Re } \rho}{(2 - \text{Re } \rho)^2 + (T - \text{Im } \rho)^2} > \frac{1}{4 + (T - \text{Im } \rho)^2}$$

(due to $0 < \text{Re } \rho < 1$), entail

$$\sum_{\rho} \frac{1}{4 + (T - \text{Im } \rho)^2} < \text{Re } \sum_{\rho} \left[\frac{1}{x_T - \rho} + \frac{1}{\rho} \right];$$

but if we combine the former bound (4.17) and the Stirling formula (3.52) for ψ within the series expansion (4.10) for $[\zeta'/\zeta](x_T)$, we can extract

$$\sum_{\rho} \left[\frac{1}{x_T - \rho} + \frac{1}{\rho} \right] = O(\log T) \quad \text{for } x_T = 2 + iT,$$

and (4.18) follows.

Now this bound secretly means that the ordinates of zeros *cannot cluster too much*: indeed, in conjunction with the obvious inequality

$$\sum_{|\text{Im } \rho - T| < 1} \frac{1}{4 + (T - \text{Im } \rho)^2} > \#\{\rho \mid |\text{Im } \rho - T| < 1\} \times \frac{1}{5},$$

it furnishes the more concrete bound (4.19) (the argument can more generally yield $\#\{\rho \mid |\operatorname{Im} \rho - T| < h\} < C_h \log T$ for any fixed $h > 0$: the density of the zeros' ordinates gets bounded on any fixed scale). \square

(3) Then the following two bounds hold uniformly for x on the segment $[-1 + iT, 2 + iT]$:

(a) Provided $T \gg 1$ is not the ordinate of a zero,

$$\frac{\zeta'}{\zeta}(x) = \sum_{|\operatorname{Im} \rho - T| < 1} \frac{1}{x - \rho} + O(\log T); \quad (4.20)$$

(b) Within unit distance from any $T' \gg 1$ there exist “good” ordinates T that are “far enough” from all zeros' ordinates, in the sense that for some absolute constant $c > 0$,

$$|T - \operatorname{Im} \rho| > \frac{c}{\log T'} \quad \text{for all zeros } \rho, \quad (4.21)$$

and then, for some absolute constant C ,

$$\left| \frac{\zeta'}{\zeta}(x) \right| < C \log^2 T \quad \text{uniformly for } x \in [-1 + iT, 2 + iT]. \quad (4.22)$$

Proof. (a) Here we may slightly streamline the usual argument by making it *intrinsic* (in the sense of Sect. 2.1). We write, for $x_T = 2 + iT$ as in (2),

$$\frac{\zeta'}{\zeta}(x) = \int_{x_T}^x \left[\frac{\zeta'}{\zeta} \right]'(y) dy + \frac{\zeta'}{\zeta}(x_T),$$

where the second term is $O(1)$ by (4.17), so it will not count here. Now (4.10), differentiated once more, refers to the *intrinsic* function $[\zeta'/\zeta]'$, implying

$$- \left[\frac{\zeta'}{\zeta} \right]'(y) = \sum_{\rho} \frac{1}{(y - \rho)^2} + O\left(\frac{1}{T}\right); \quad (4.23)$$

but, uniformly in y on the line $\{\operatorname{Im} y = T\}$,

$$\left| \sum_{|\operatorname{Im} \rho - T| \geq 1} \frac{1}{(y - \rho)^2} \right| \leq \sum_{|\operatorname{Im} \rho - T| \geq 1} \frac{1}{(T - \operatorname{Im} \rho)^2} < \sum_{|\operatorname{Im} \rho - T| \geq 1} \frac{5}{4 + (T - \operatorname{Im} \rho)^2}$$

which is $O(\log T)$ by (4.18). All that implies

$$- \left[\frac{\zeta'}{\zeta} \right]'(y) = \sum_{|\operatorname{Im} \rho - T| < 1} \frac{1}{(y - \rho)^2} + O(\log T). \quad (4.24)$$

Then, upon integration over the segment $[x, x_T]$ of bounded length,

$$\frac{\zeta'}{\zeta}(x) + O(\log T) = \sum_{|\operatorname{Im} \rho - T| < 1} \frac{1}{x - \rho} - \sum_{|\operatorname{Im} \rho - T| < 1} \frac{1}{x_T - \rho} + O(\log T),$$

and the second sum has $O(\log T)$ terms by (4.19), all bounded by 1: it can be absorbed into the $O(\log T)$ remainder, resulting in the estimate (4.20).

(b) The nonexistence of T satisfying (4.21) would immediately violate the bound (4.19) written for T' . Then, for such “good” T and with x on the segment $[-1 + iT, 2 + iT]$, each summand in (4.20) is uniformly $O(\log T')$, while the number of summands is $O(\log T)$ by (4.19). This (plus $T' \sim T$) yields the final bound (4.22). \square

The proof of the Explicit Formula (Sect. 6.2) will actually need a *sequence* $\{T_n\} \rightarrow +\infty$ of “good” ordinates fulfilling (4.22): for this, it suffices to apply (b) with $T' = n$.

4.5 The (Asymptotic) Riemann–von Mangoldt Formula

The previous results actually allow us to describe the growth of the Riemann zeros’ ordinates more finely.

The classic counting function for the zeros’ ordinates can be defined by

$$N(T) \stackrel{\text{def}}{=} \#\{\tau_k \mid \operatorname{Re} \tau_k \leq T\}, \quad (4.25)$$

and it obeys the fundamental estimate (the *Riemann–von Mangoldt formula*):

$$N(T) = \bar{N}_0(T) + O(\log T)_{T \rightarrow +\infty}, \quad \bar{N}_0(T) \stackrel{\text{def}}{=} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) \quad (4.26)$$

(thus the asymptotic density of zeros at the ordinate T is $\approx (2\pi)^{-1} \log(T/2\pi)$).

Proof. (As in all books, simply we focus here on $T \gg 1$.)

Let R_T be the positively oriented closed rectangular path with vertices $\frac{1}{2} \pm \frac{3}{2} \pm iT$, where T is not the ordinate of any zero, and R'_T be its restriction to the first quadrant of vertex $x = \frac{1}{2}$. Then by the general Rouché theorem, $N(T) = (4\pi i)^{-1} \oint_{R_T} [\Xi'/\Xi](x) dx$; but the symmetries (reality and the Functional Equation) at once reduce the integration to the quarter path R'_T :

$$N(T) = \pi^{-1} \operatorname{Im} \int_{R'_T} \left[\frac{\Xi'}{\Xi} \right](x) dx = \pi^{-1} \operatorname{Im} [\log \Xi(x)]_{\frac{1}{2} + iT}, \quad (4.27)$$

where \log means the complex logarithm followed continuously along the path R'_T , starting from $x = 2$. We then invoke the factorization (3.24) of $\Xi(x)$ to separate the contributions from $\zeta(x)$ and the remaining trivial factor:

$$N(T) = \bar{N}(T) + S(T), \quad (4.28)$$

$$\bar{N}(T) \stackrel{\text{def}}{=} \pi^{-1} \operatorname{Im} \log \left[(iT - \tfrac{1}{2}) \mathbf{G}^{-1}(\tfrac{1}{2} + iT) \right] \quad (4.29)$$

$$S(T) \stackrel{\text{def}}{=} \pi^{-1} \operatorname{Im} \int_{R'_T} \left[\frac{\zeta'}{\zeta} \right] (x) dx. \quad (4.30)$$

Now, $\bar{N}(T)$ lends itself to an estimation by the Stirling formula (3.51):

$$\bar{N}(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + \frac{7}{8} + O\left(\frac{1}{T}\right). \quad (4.31)$$

In contrast, the other term $S(T)$ proves very irregular (fluctuating), but can be bounded by $O(\log T)$ – this is the key argument. Indeed, on the long segment $[2, 2+iT]$ of R'_T , the Dirichlet series (3.6) holds everywhere, implying $|\log \zeta(2+iT)| \leq \log \zeta(2) = O(1)$. All the action then takes place on the other segment $[\frac{1}{2} + iT, 2+iT]$, of fixed length, where the use of (4.20) entails

$$\operatorname{Im} \int_{\frac{1}{2}+iT}^{2+iT} \left[\frac{\zeta'}{\zeta} \right] (x) dx = \sum_{\{|\operatorname{Im} \rho - T| < 1\}} [\arg(x - \rho)]_{\frac{1}{2}+iT}^{2+iT} + O(\log T);$$

but since the variation of each $\arg(x - \rho)$ is less than π and the number of summands is $O(\log T)$ by (4.19), it follows that the left-hand side and therefore $S(T)$ are themselves $O(\log T)$, and the proof is complete. \square

Chapter 5

Superzeta Functions: an Overview

In view of the *central symmetry* of the set of Riemann zeros, $\rho \leftrightarrow 1 - \rho$, which crucially reflects the functional equation (3.26), there is no clear reason to pick either the ρ themselves (with $\rho \equiv \frac{1}{2} \pm i\tau_k$, $\text{Re } \tau_k > 0$) or the $\rho(1 - \rho) = \frac{1}{4} + \tau_k^2$ as the basic set upon which to build zeta functions over the Riemann zeros. This is a nonlinear remapping freedom as discussed in Sect. 1.1, which leads to generalized zeta functions of several kinds. This chapter gives an introductory overview of the various possibilities. In decreasing order of analytical tractability, and in our older notation to be soon discarded, we may define:

- Functions of first kind: $\sum (x - \rho)^{-s}, \quad \text{Re } s > 1;$
- Functions of second kind: $\sum_{k=1}^{\infty} (\tau_k^2 + v)^{-\sigma}, \quad \text{Re } \sigma > \frac{1}{2};$
- Functions of third kind: $\sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \text{Re } s > 1.$

In all three kinds the *argument* (in exponent) is the principal variable, the other one a shift parameter. Each parametric family starts as an analytic function of its argument in its indicated half-plane of convergence, due to (4.7) or (4.26), but it will extend to a *meromorphic function* in the entire complex plane (of its argument), with a *computable singular part*. The domain of each shift parameter is left as large as possible: simply, none of the quantities to be raised to a power should lie on the cut \mathbb{R}_- ; this will hardly matter with the fairly small parameter values that we will emphasize.

The first two families will moreover display computable *special values*, i.e., values at *integer* arguments, with explicit relationships between the two families at those arguments. The third family forcibly selects the zeros in only one half-plane, thus breaking the central symmetry for any parameter $\tau \neq 0$; then it will show a more singular analytic structure than the other two, and no known special values; it will be discussed more briefly.

We now define each superzeta family more precisely with its immediate properties, introducing *new notation* henceforth for greater coherence.

5.1 First Kind (\mathcal{Z})

$$\mathcal{Z}(s|t) = \sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-s} \equiv \sum_{\rho} \left(\rho + t - \frac{1}{2}\right)^{-s}, \quad \operatorname{Re} s > 1 \quad (5.1)$$

is the simplest generalized zeta function over the Riemann zeros: the sum runs over all zeros symmetrically, and t is just a shift parameter varying in the complex plane of the function $t \mapsto \zeta(\frac{1}{2} + t)$. With the function $z \mapsto z^{-s}$ defined in the cut plane $z \in \mathbb{C} \setminus \mathbb{R}_-$, the t -domain for (5.1) is then (Fig. 5.1)

$$\Omega_1 = \{t \in \mathbb{C} \mid t \pm i\tau_k \notin \mathbb{R}_- \ (\forall k)\}. \quad (5.2)$$

The equality between the two sums in (5.1) expresses the *central symmetry* (of Riemann's Functional Equation). Were the function $z \mapsto z^{-s}$ single-valued in the *uncut* z -plane, then that equality would translate into the functional equation “ $\mathcal{Z}(s|t) \equiv (-1)^s \mathcal{Z}(s|-t)$,” and this works indeed at (positive) *integers* s , see Table 7.2. For non-integer s , on the other hand, when the choice of a branch for $z \mapsto z^{-s}$ breaks the central symmetry, our closest related identity will be (5.11) which needs a *new* function \mathfrak{Z} (of third kind). We then see no finite closed functional relation between $\mathcal{Z}(s|t)$ and $\mathcal{Z}(s|-t)$, nor any other full translation of Riemann's Functional Equation. We will only find some curious further signs of the central symmetry: e.g., countably many sum rules like (7.46) below will constrain the special values $\mathcal{Z}(2n+1|t)$, $n \in \mathbb{N}$, now all *at the same* t . So in brief, for this first family \mathcal{Z} :

To see or not to see (the central symmetry fully), that is the question.

As the parameter t “lives” in the complex plane of $\zeta(\frac{1}{2} + t)$, we expect privileged locations to be $t = 0$ (the center of symmetry), and $t = \frac{1}{2}$ (the pole); thus, the simple sum $\sum_{\rho} \rho^{-n}$ is $\mathcal{Z}(n|t = \frac{1}{2})$. Indeed, the special values reduce further at those parameter values. We then add two shorter names for convenience:

$$\mathcal{Z}_0(s) \stackrel{\text{def}}{=} \mathcal{Z}(s|0), \quad \mathcal{Z}_*(s) \stackrel{\text{def}}{=} \mathcal{Z}(s|\frac{1}{2}). \quad (5.3)$$

(As for $t = -\frac{1}{2}$, its special values can easily be drawn from the case $t = +\frac{1}{2}$ by using the central symmetry, see later Table 7.2.)

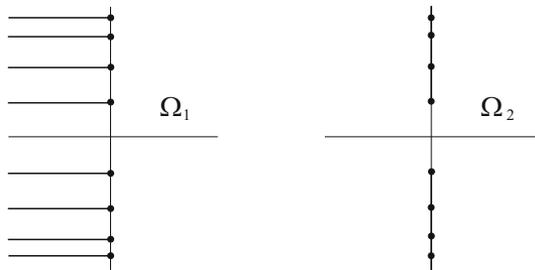
The family of first kind \mathcal{Z} is covered in detail in Chap. 7.

5.2 Second Kind (\mathcal{Z})

$$\mathcal{Z}(\sigma|t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}, \quad \operatorname{Re} \sigma > \frac{1}{2}, \quad (5.4)$$

is now defined in the natural t -domain (Fig. 5.1)

Fig. 5.1 Domains of definition in the t -plane (schematic views) for \mathcal{Z} (left) and \mathcal{Z} (right)



$$\Omega_2 = \{t \in \mathbb{C} \mid t \pm i\tau_k \notin \pm i\mathbb{R}_- \ (\forall k)\}. \quad (5.5)$$

In view of the factorization $\tau_k^2 + t^2 = (t + i\tau_k)(t - i\tau_k)$, this parameter t identifies indeed with the previous one in $\mathcal{Z}(s|t)$.

This family \mathcal{Z} has one major asset: it fully embodies the central symmetry $\tau_k \leftrightarrow -\tau_k$ (or Riemann's Functional Equation), through the identity

$$\mathcal{Z}(\sigma|t) \equiv \mathcal{Z}(\sigma|-t). \quad (5.6)$$

But otherwise it proves more singular than the first family \mathcal{Z} , and also harder to analyze (just as, say, the function $\sum_k (k^2 + a^2)^{-\sigma}$ vs $\sum_k (k+a)^{-s} = \zeta(s, a)$). We cannot reduce it to the first family (see below), and it is better suited for some purposes (e.g., in Chap. 11), so that both families are worth studying. Our results for \mathcal{Z} are now fairly on par with those for the first family, but this required multiple angles of attack which we did not find all at once. This operational difficulty has been our main problem with the second family \mathcal{Z} .

Since the parameter t has preserved its meaning, we add two more shorthand names to match (5.3),

$$\mathcal{Z}_0(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma|0), \quad \mathcal{Z}_*(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma|\frac{1}{2}). \quad (5.7)$$

Just as between $\mathcal{Z}(s|-t)$ and $\mathcal{Z}(s|t)$ before, we see no obvious finite functional relation between $\mathcal{Z}(\sigma|t)$ and $\mathcal{Z}(s|t)$, with two exceptions:

- The central parameter location $t = 0$, which is the (only) *confluence point* of the two families \mathcal{Z} and \mathcal{Z} , according to

$$\mathcal{Z}_0(\sigma) \equiv (2 \cos \pi\sigma)^{-1} \mathcal{Z}_0(2\sigma); \quad (5.8)$$

- Integer arguments: The sets of special values $\mathcal{Z}(m|t)$ and $\mathcal{Z}(n|t)$ ($m, n = 1, 2, \dots$) will display mutual linear relationships (Sect. 8.5). Here the computable special values of $\mathcal{Z}(\sigma|t)$ do refer to all integers σ just as those of $\mathcal{Z}(s|t)$ refer to all integers s . Otherwise, just as we immediately did with the parameter t , we could have tied both \mathcal{Z} and \mathcal{Z} to a common argument: but

the only meaningful identification, $s \equiv 2\sigma$ by (5.8), does not preserve integers. Hence the issue of special values prompts us to keep distinct arguments, s for \mathcal{Z} , σ for \mathcal{Z} .

The family of second kind \mathcal{Z} is covered in detail in Chap. 8.

5.3 Third Kind (3)

$$\mathfrak{Z}(s|\tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \operatorname{Re} s > 1, \quad \tau + \tau_k \notin \mathbb{R}_- \quad (\forall k). \quad (5.9)$$

Although this family explicitly breaks the central symmetry, it is actually expressible in terms of the family \mathcal{Z} of the first kind and vice-versa: for $|t|, |\tau| < \tau_1$ (to avoid overlapping cuts),

$$\mathcal{Z}(s|t) = e^{i\pi s/2} \mathfrak{Z}(s|it) + e^{-i\pi s/2} \mathfrak{Z}(s|-it), \quad (5.10)$$

which inverts as

$$\mathfrak{Z}(s|\tau) = \frac{1}{2i \sin \pi s} \left[e^{i\pi s/2} \mathcal{Z}(s|-i\tau) - e^{-i\pi s/2} \mathcal{Z}(s|+i\tau) \right]. \quad (5.11)$$

This family shares the confluence point of the other two:

$$\mathfrak{Z}(s|0) \equiv \mathcal{Z}_0(\tfrac{1}{2}s) \equiv (2 \cos \tfrac{1}{2}\pi s)^{-1} \mathcal{Z}_0(s) = \sum_{k=1}^{\infty} \tau_k^{-s}. \quad (5.12)$$

Otherwise (for $\tau \neq 0$), \mathfrak{Z} proves more singular than \mathcal{Z} or even \mathcal{Z} , hence we will only view it as subordinate to \mathcal{Z} through (5.11), see Chap. 9.

5.4 Further Generalizations (Lerch, Cramér, ...)

A Lerch-like generalization (cf. (3.38)) is

$$Z(s; x) = \sum_{\rho} x^{\rho} \rho^{-s} \quad (\text{e.g., } x > 0) \quad (5.13)$$

with numerous possible variants (allow complex x ; replace ρ^{-s} by $(\rho + a)^{-s}$, $|\rho|^{-s}$, ...; select $\operatorname{Im} \rho \geq 0$ or not). At $s = 0$, the family also captures another interesting symmetric function of the zeros, Cramér's V -function [25, Chap. I]:

$$V(z) = \sum_{\operatorname{Im} \rho > 0} e^{\rho z} \quad \text{for } \operatorname{Im} z > 0. \quad (5.14)$$

Concerning $V(z)$, it is mainly known [25, 29, 44, 54, 59] that the function $V(z) - (2\pi i)^{-1} \frac{\log z}{1 - e^{-z}}$ is meromorphic in all of \mathbb{C} , with a functional equation, and with computable: poles (all simple), residues, and finite part (but nothing more) at $z = 0$. Thus, $V(z)$ itself has a rather irregular, partially known (and hard to get) structure at $z = 0$; so we do not advocate it as a basic tool. (It is actually nicer for zeros of Selberg zeta functions! [12, 15] [16, Sect. 5] [105, Sect. 4])

Formally, the broader family $Z(s; x)$ recovers our superzeta functions at $x = 1$; however, these form a singular case. Hence the general- x family is better seen as a vast subject of its own, which we will not address here either: see, e.g., [40, 68, 83], [25, Chap. II.I], [61, Chap. VI].

5.5 Other Studies on Superzeta Functions

The present monograph grew up from articles written by us around 2001–2005. It thus makes sense here to place a dividing line between “older” and “current” literature around 2000 (a pivotal year, anyway).

We found it difficult to trace the older literature on superzeta functions. To wit, the past publications touching those functions have been very sporadic (we laboriously collated but a dozen articles, spanning almost one century) and disconnected (the works largely ignore one another); even when superzeta functions were addressed, it was never systematically and often briefly: e.g., amidst various examples. So overall, occurrences of our subject have tended to be rare and inconspicuous. We then offer the following survey truly “to the best of our knowledge”. We had to change most of the original notation to make it globally consistent here: any translation between an earlier notation “ A ” and our present notation “ B ” is provided below as $A \equiv B$.

The most ancient occurrences of this sort of symmetric functions over the Riemann zeros concern the more general functions $Z(s; x)$ of (5.13) in works of Landau [68], Mellin [83] and Cramér [25]. Out of those works, only [83] truly touches “superzeta” functions (i.e., cases for which $x = 1$), featuring:

- Two instances of our “functions of the 1st kind,”

$$Z(s) \equiv \mathcal{Z}(s | \tfrac{1}{2}), \quad \overline{Z}(s) \equiv \mathcal{Z}(s | -\tfrac{1}{2}); \quad (5.15)$$

- Two instances of our “functions of the 3rd kind,”

$$Z_1(s) \equiv e^{i\pi s/2} \mathfrak{Z}(s | \tfrac{1}{2}i), \quad Z_2(s) \equiv e^{-i\pi s/2} \mathfrak{Z}(s | -\tfrac{1}{2}i); \quad (5.16)$$

- Meromorphic continuation formulae for those four cases, which exemplify some of our later equations: successively, (7.7) at $t = +\frac{1}{2}$, (7.4) at $t = -\frac{1}{2}$, (9.4) at $\tau = -\frac{1}{2}i$ and at $\tau = +\frac{1}{2}i$;

- Explicit formulae for the polar parts of $Z(s)$ and $\bar{Z}(s)$, plus the rational special values at $s = -n$, $n \in \mathbb{N}$, but the latter [83, (12)–(13)] are affected by obvious computational mistakes and contradicted by our own results (upper parts of Tables 7.1 and 7.4, from [108]).

Overall, Mellin’s article ([83], 1917) is quite pioneering and fundamental, yet it is hardly ever referred to and never for “superzeta” functions; to make it more accessible, we provide its English translation here as Appendix D.

The next early occurrences of a superzeta function all concern the *confluent* case $t = 0$ as in (5.12). For the corresponding function $\sum_k \tau_k^{-s}$, Guinand [43, Sect. 4(A)] gave a restricted form of the functional relation (8.7), moreover assuming the Riemann Hypothesis, an assumption later removed by Chakravarty [17]; while Delsarte [27, Sect. 7] proved that same function (called by him $\phi(s)$) to be meromorphic in the whole s -plane.

Much later, cases with $t = \frac{1}{2}$ drew some attention, namely those we call $\mathcal{Z}_*(s)$ (first kind) and $\mathcal{Z}_*(\sigma)$ (second kind).

- For \mathcal{Z}_* , only the special values $\sigma_n \equiv \mathcal{Z}_*(n)$ ($= \sum_\rho \rho^{-n}$, $n = 1, 2, \dots$) got considered: the value $\mathcal{Z}_*(1)$, as in (4.14), is the sole *classic* “superzeta” occurrence (i.e., visible in some textbooks); results for $n > 1$ were given much later, by Matsuoka [82], Lehmer [72], Keiper [64], Zhang and Williams [114].
- As for \mathcal{Z} : the function $\zeta(s, \Delta_F) \equiv \mathcal{Z}_*(\sigma)$ for the more general Dedekind zeta functions (cf. Chap. 10) was proved by Kurokawa [66] to be meromorphic in the whole σ -plane while Matiyasevich [80] studied the special values $\theta_m \equiv 2 \mathcal{Z}_*(m)$ ($m = 1, 2, \dots$); $\mathcal{Z}_*(1)$ and $\mathcal{Z}_*(2)$ resurfaced lately in studies on the distribution of primes [93, pp. 191–193][71, (4.12)–(4.13)].

A full-fledged family arose still later, specifically \mathcal{Z} (first kind). (Earlier it came up only in exercises posed by Patterson [89, Ex. 3.5–7].) Deninger [28, Theorem 3.3] for $\text{Re } z > 1$, followed by Schröter and Soulé [96] for general z , described the meromorphic continuation in s of $\xi(s, z) \equiv (2\pi)^s \mathcal{Z}(s | z - \frac{1}{2})$, but with focus on the parametric special value $z \mapsto \xi'(0, z)$.

We found no results on the full second family \mathcal{Z} prior to our work [106].

Then quite lately, the third family \mathfrak{Z} was proved a meromorphic function of its argument, independently by Hirano, Kurokawa and Wakayama [51, Sect. 3] (as $Z(w, x) \equiv \mathfrak{Z}(w | -x)$) and by us [106, Sect. 6.2].

As for *extended* superzeta functions, i.e., over zeros of more general zeta- and L -functions than Riemann’s, their even smaller bibliography will be given in Chap. 10, which is devoted to this extension.

So, our topic of “superzeta functions” has benefited from some seminal contributions. Yet, these are scarce (a dozen, spread over three families and nearly one century), and almost each of them treats an isolated case under a single aspect: typically meromorphic continuation (often just qualitatively), or particular instances of special values. We thus feel that those early works, even taken together, do not do full justice to the subject.

We therefore embarked on a more systematic and dedicated coverage, but this emerged only gradually through three papers. In [106] we described the full families \mathcal{Z} and \mathfrak{Z} and only sketched the family \mathcal{Z} , which became the main

focus of [108]. In [107] we treated all three families more efficiently but in general notation, to accommodate other zeta and L -functions. Overall, these three articles do not readily fuse into a neat coherent whole; their notation and methods can also be improved. The core of the present work aims at giving an updated synthesis (including a few new results) of the current state of the subject.

Chapter 6

Explicit Formulae

This chapter discusses the approach to the superzeta functions (introduced in the previous chapter) by means of *Explicit Formulae*.

In number theory, Explicit Formulae designate certain summation formulae connecting the primes and the Riemann zeros [55, Chap. IV]. As we may also refer to explicit formulae in their generic sense of closed-form results, we will specifically use capital initials to distinguish that number-theoretical meaning. We focus on just one type of Explicit Formula, the Guinand–Weil form [42, 43, 111], which accommodates general test functions and appears closest to the Poisson summation formula [50]. It is formally capable of evaluating superzeta functions, and we will try to use it for that goal.

6.1 The Guinand–Weil Explicit Formula

As in the Poisson summation formula (Sect. 1.3), the main actors are a dual pair consisting of a fairly general test function $h(\tau)$ ($\mathbb{R} \mapsto \mathbb{C}$) and its Fourier transform $\hat{h}(u)$, namely

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\tau) e^{-i\tau u} d\tau, \quad h(\tau) = \int_{-\infty}^{\infty} \hat{h}(u) e^{i\tau u} du. \quad (6.1)$$

Now, moreover, one assumes that for some $\delta, \varepsilon > 0$,

- (i) h extends to an analytic function in a strip $S_\delta = \{|\operatorname{Im} \tau| < \frac{1}{2} + \delta\}$;
- (ii₁) $h(\tau) = O(|\tau|^{-1-\varepsilon})$ for $\tau \rightarrow \infty$ in S_δ ;
- (iii) h is an even function (hence so is \hat{h}) – to embody the central symmetry.

Such a pair (h, \hat{h}) then fulfills the *Explicit Formula*

$$\sum_k h(\tau_k) \equiv h(\pm \frac{1}{2}i) + W[h] - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(\log n), \quad (6.2)$$

where the last sum is *arithmetical* as it runs over the prime powers p^r ,

$$\sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(\log n) \equiv \sum_{\{p\}} \sum_{r=1}^{\infty} \frac{\log p}{\exp \frac{1}{2} r \log p} \hat{h}(r \log p), \quad (6.3)$$

and the linear functional W (a variant of the *Weil functional*) has two equivalent expressions $W[h] \equiv \widehat{W}[\hat{h}]$, by virtue of a generalized Parseval formula [60, Part II]:

$$W[h] = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(\tau) [\psi(\frac{1}{4} + \frac{1}{2}i\tau) - \log \pi] d\tau \quad (6.4)$$

$$\equiv -\frac{1}{2}(\log 4\pi + \gamma) \hat{h}(0) - \int_{-\infty}^{\infty} \frac{e^{|u|/2} \hat{h}(u) - \hat{h}(0)}{4 \sinh |u|} du = \widehat{W}[\hat{h}]. \quad (6.5)$$

The Explicit Formula (6.2) thus “evaluates” a general additive symmetric function of all Riemann zeros, in terms of more elementary objects indeed (the Digamma function ψ , and the von Mangoldt function Λ).

In comparison to the Poisson summation formula (Sect. 1.3): the more stringent condition (i) is natural as it conceals an exponential decay condition $\hat{h}(u) = O(e^{-t_0 u})$ for some $t_0 > \frac{1}{2}$ when $u \rightarrow +\infty$, needed to make the sum (6.3) converge; (ii₁), similar to the earlier (ii), causes the sum $\sum_k h(\tau_k)$ in (6.2), in view of (4.7), and now the integral (6.4) as well, to converge (here we mean the growth condition (ii₁) over \mathbb{R} ; imposing it over all of S_δ may be more of a technical trick); as for the parity condition (iii), it is formally unchanged but the analyticity condition (i) makes it much more restrictive here (any old $h(|\tau|)$ like (1.12) will no longer do as an even function).

6.2 Derivation of the Explicit Formula

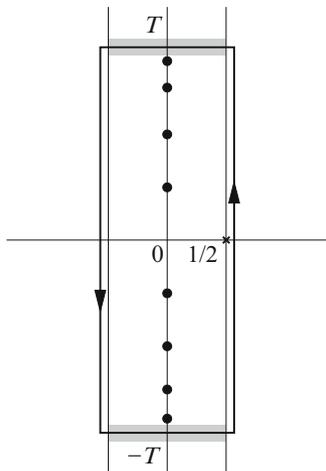
We specifically treat the form with the term W given by (6.4). It is convenient here to rewrite the completion formula (3.24) of the Riemann zeta function as

$$G(x) \stackrel{\text{def}}{=} \frac{\Xi(x)}{x(x-1)} \equiv \pi^{-x/2} \Gamma(x/2) \zeta(x); \quad (6.6)$$

G is a symmetric ($x \leftrightarrow 1-x$) function with shares the Riemann zeros plus the pole at $x=1$ with $\zeta(x)$, and thus has the symmetric pole at $x=0$ as well.

Proof. The idea is to pick a positively oriented self-symmetric rectangular contour L_T with vertices $(\pm t_0 \pm iT)$ such that $\frac{1}{2} < t_0 < \frac{1}{2} + \delta$ and T is not the ordinate of any Riemann zero (Fig. 6.1), and simply to equate two evaluations of the contour integral

Fig. 6.1 Contour of integration L_T in the t -plane, used to derive the explicit formula. The shaded areas mark the zones where the path has to keep “far enough” from the zeros of $\Xi(\frac{1}{2} + t)$ (\bullet , not placed on scale)



$$\mathcal{I}_T = \frac{1}{4\pi i} \oint_{L_T} h(-it) d \log G\left(\frac{1}{2} + t\right) \quad (6.7)$$

in a suitable $T \rightarrow +\infty$ limit.

- By the residue calculus: h being even, any two symmetric zeros and poles of $G(\frac{1}{2} + t) = \Xi(\frac{1}{2} + t)/(t^2 - \frac{1}{4})$ contribute equally to the contour integral, giving

$$\begin{aligned} \mathcal{I}_T &= \sum_{\operatorname{Re} \tau_k < T} h(\tau_k) - h(\pm \frac{1}{2} i) \\ &\rightarrow \sum_{k=1}^{\infty} h(\tau_k) - h(\pm \frac{1}{2} i) \stackrel{\text{def}}{=} \mathcal{I}_{\infty} \quad \text{for } T \rightarrow +\infty, \end{aligned} \quad (6.8)$$

where the infinite sum converges by condition (ii₁) on h and property (4.7).

- Now using the right-hand side of (6.6) for G , and again by symmetry,

$$2\pi i \mathcal{I}_T = \left[\int_{t_0 - iT}^{t_0 + iT} - \int_{-t_0 - iT}^{-t_0 + iT} \right] h(-it) \left[[\zeta'/\zeta](\frac{1}{2} + t) + \frac{1}{2} [\psi((\frac{1}{2} + t)/2) - \log \pi] \right] dt. \quad (6.9)$$

The first trick is now to send T to $+\infty$ along “good” ordinates only, i.e., satisfying the bound (4.22) for ζ'/ζ , so that the integrand is $O(h(T) \log^2 T)$ on the short side $[-t_0 + iT, t_0 + iT]$ of the rectangle, of length $O(1)$, and this integral is negligible in the $T \rightarrow +\infty$ limit again by condition (ii₁) on h ; by this same condition, the integrand on the long side $[t_0 - iT, t_0 + iT]$, which is $O(h(|t|) \log |t|)$, stays integrable in the $T \rightarrow +\infty$ limit, yielding

$$2\pi i \lim_{T \rightarrow +\infty} \mathcal{I}_T = \int_{t_0 - i\infty}^{t_0 + i\infty} h(-it) \left[[\zeta'/\zeta](\frac{1}{2} + t) + \frac{1}{2} [\psi(\frac{1}{4} + \frac{1}{2}t) - \log \pi] \right] dt. \quad (6.10)$$

The second ingredient is the Dirichlet series (3.7) for $[\zeta'/\zeta](\frac{1}{2}+t)$: it converges uniformly on the integration line, hence its substitution here yields the convergent series

$$-\sum_{n \geq 2} \int_{t_0 - i\infty}^{t_0 + i\infty} h(-it) \frac{\Lambda(n)}{\sqrt{n}} n^{-t} dt + \frac{1}{2} \int_{t_0 - i\infty}^{t_0 + i\infty} h(-it) \left[\psi\left(\frac{1}{4} + \frac{1}{2}t\right) - \log \pi \right] dt. \quad (6.11)$$

The final trick is to observe that every resulting individual integrand is now analytic and integrable in the strip $0 \leq \operatorname{Re} t < \frac{1}{2} + \delta$, due to conditions (i) and (ii₁) on h , hence each integration line can be shifted to the imaginary axis. Up to a rotation $t = i\tau$, and using (6.1), finally

$$\mathcal{I}_\infty = -\sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(\log n) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(\tau) \left[\psi\left(\frac{1}{4} + \frac{1}{2}i\tau\right) - \log \pi \right] d\tau, \quad (6.12)$$

and the identification with (6.8) results in the formulae (6.2)–(6.4). \square

We refer the reader to [8] (up to $x \equiv e^u$) for the second form (6.5) as we displayed it: it has several equivalent variants [4, 42, 43, 111][13, Sect. 1.2][70, Chap. XVII], none of which is simple or short to present, and here it will bring us no more explicit results than the first form (6.4); however, it is useful for the theory as its structure will warrant some comments below.

6.3 Pattern-Matching with the Selberg Trace Formula

We make a digression to sketch a domain of *spectral theory* which has been much studied for its own sake, and which presents formal analogies with Explicit Formulae: Selberg trace formulae. A number of spectral results and techniques developed around the latter can be transposed, and valuable inspiration obtained, for our present subject; but some significant differences also arise. We will conclude that to study superzeta functions in tight parallelism with spectral zeta functions should currently not be our best strategy.

6.3.1 The Selberg Trace Formula (Compact Surface Case)

Let S be a *hyperbolic surface*, i.e., a two-dimensional manifold endowed with a Riemannian metric of *constant negative curvature* (here normalized to -1); here for simplicity we take this surface *compact*. Then, on the *geometric* side, let ϖ label all primitive (i.e., non-repeated) oriented periodic geodesics of S , with ℓ_ϖ denoting the length of ϖ , and $\mathcal{A}_S = \operatorname{area}(S)$; and on the

spectral side, let $\{0 = v_0 < v_1 \leq v_2 \leq \dots\}$ describe the eigenvalues of the positive Laplacian on S , parametrized by the hyperbolic wavenumbers $\kappa_k \stackrel{\text{def}}{=} (v_k - \frac{1}{4})^{1/2}$ (with $\kappa_k \geq 0$ for $v_k \geq \frac{1}{4}$, else $\arg \kappa_k = \text{either } +\frac{1}{2}\pi \text{ or } -\frac{1}{2}\pi$ indifferently); this spectrum has the asymptotic density

$$\#\{v_k \leq \frac{1}{4} + \kappa^2\} \sim \frac{\mathcal{A}_S}{4\pi} \kappa^2 \quad \text{for } \kappa^2 \rightarrow +\infty \quad (\text{Weyl's law}). \quad (6.13)$$

Now, let (h, \hat{h}) be a pair of even test functions exactly as in Sect. 6.1 above, except that (ii₁) has to be tightened to:

(ii₂) $h(\tau) = O(|\tau|^{-2-\varepsilon})$ for $\tau \rightarrow \infty$ in S_δ .

Then the following summation formula (*Selberg trace formula*) holds:

$$\sum_{k=0}^{\infty} h(\kappa_k) \equiv I[h] + \sum_{\{\varpi\}} \sum_{r=1}^{\infty} \frac{\ell_\varpi}{2 \sinh \frac{1}{2} r \ell_\varpi} \hat{h}(r \ell_\varpi), \quad (6.14)$$

where the linear functional I has two expressions, equivalent by the Parseval formula for distributions:

$$\begin{aligned} I[h] &= \frac{\mathcal{A}_S}{4\pi} \int_{-\infty}^{\infty} h(\tau) \tau \tanh \pi \tau \, d\tau \\ &\equiv -\frac{\mathcal{A}_S}{4\pi} \int_{-\infty}^{\infty} \frac{d\hat{h}(u)}{\sinh \frac{1}{2} u} = \hat{I}[\hat{h}]. \end{aligned} \quad (6.15)$$

The bound (ii₂) is needed (over \mathbb{R}) to make the sum $\sum_k h(\kappa_k)$ in (6.14) converge (and the integral in $I[h]$ as well), given that the integrated spectral density (6.13) for the eigenvalues (parametrized by κ_k) grows faster than its counterpart (4.26) for the Riemann zeros (parametrized by τ_k).

For derivations of (6.14)–(6.15) and for further reading, see [50, 91, 97]. Explicit results can also be reached for the spectral zeta functions of this problem by methods parallel to those we will use for superzeta functions; those results are listed in Appendix B.

6.3.2 Comparison with the Explicit Formula

Both (6.2) and (6.14) generalize the Poisson summation formula (1.11) in similar ways. The sum in the right-hand side of (6.14), which runs over the lengths of periodic geodesics counted with repetitions, bears some resemblance to the arithmetical sum of the Explicit Formula in the form (6.3): inspiring analogies between them are drawn in, e.g., [50, 76]. However, we wish to rather emphasize *several differences*.

Since the Weyl law (6.13) is steeper than the Riemann–von Mangoldt law (4.26), the kernel $I(\tau) = \tau \tanh \pi \tau$ in (6.15) understandably grows faster

at ∞ than the kernel $W(\tau) \sim \log \frac{1}{2}|\tau|$ in (6.4). Equivalently at the Fourier-transformed level, already $\widehat{W}(u) = -\frac{e^{|u|/2}}{4 \sinh |u|} - \frac{1}{2} \log \pi \delta(u)$ fails to be an integrable *function* at $u = 0$ and has to be regularized as in (6.5) to define a proper *functional*, but $\widehat{I}(u) = \frac{\mathcal{A}_S}{4\pi} \frac{d}{du} \frac{1}{\sinh u/2}$ is even more singular there.

However, this difference at $u = 0$ between the two summation formulae has a paradoxical side, if we believe that the Explicit Formula is the more “elementary” of the two (it is “only” about the integers, after all). The main ($u = 0$) singularity in \widehat{I} may be stronger than in \widehat{W} , but it has a *much simpler* structure: the function $\widehat{I}(u)$ on \mathbb{R} has a meromorphic extension to all of \mathbb{C} (and its complex poles have a dynamical meaning [15][16, Sect. 5]), whereas $\widehat{W}(u)$ has a non-analytic singularity at $u = 0$ and a global analytic structure which hitherto admit only *quite* elaborate interpretations (see, e.g., [24, 41]). Accordingly, the nonlocal regularization needed for \widehat{W} in (6.5) at $u = 0$ is definitely more contrived [4, 8, 70, 111] than its counterpart (6.15) for \widehat{I} .

The sole culprit for all that trouble might just be the plain $(\log T)$ factor in the asymptotic law (4.26); nevertheless, the resulting complication is vicious enough to hamper the practical use of the Explicit Formula for superzeta functions, mainly those of second kind (Sect. 6.4.2).

6.4 Explicit Formulae for the Superzeta Functions

From a naive standpoint, Explicit Formulae should answer our problem at hand: formally, they evaluate additive symmetric functions of the zeros, which the superzeta functions \mathcal{Z} , \mathcal{Z} precisely are. On paper, it would then “suffice” to select a test function h that turns $\sum_k h(\tau_k)$ into the desired superzeta function, and take the resulting Explicit Formula (6.2) as ... an explicit formula for this superzeta function (picking whichever form (6.4) or (6.5) of $W[h]$ is nicer). Only the more singular family of the third kind \mathfrak{Z} seems intractable by this approach (which critically uses central symmetry); however, as we saw in Sect. 5.3, it is reducible to the family \mathcal{Z} in another way.

We now describe the concrete implementations of Explicit Formulae for each family \mathcal{Z} and \mathcal{Z} in turn, but mainly to discover their limitations.

6.4.1 The Family of the First Kind \mathcal{Z}

In order to recover $\mathcal{Z}(s|t) \equiv \sum_k [(t + i\tau_k)^{-s} + (t - i\tau_k)^{-s}]$ from the Explicit Formula (6.2), the appropriate test function pair is

$$h(\tau) = (t + i\tau)^{-s} + (t - i\tau)^{-s}, \quad \hat{h}(u) = \frac{1}{\Gamma(s)} |u|^{s-1} e^{-t|u|}, \quad (6.16)$$

where both s, t are implied parameters, with $\operatorname{Re} s > 0, \operatorname{Re} t > 0$ [34, I.3(19)].

Indeed, in the smaller domain $\{\operatorname{Re} s > 1\}$ imposed at once by (ii₁),

$$\sum_k h(\tau_k) \equiv \mathcal{Z}(s|t). \quad (6.17)$$

We next evaluate $W[h]$ by contour integration and the residue calculus:

$$W[h] = I_+ + I_-, \quad I_{\pm} \stackrel{\text{def}}{=} \frac{1}{4\pi} \int_{-\infty}^{\infty} (t \pm i\tau)^{-s} [\psi(\frac{1}{4} + \frac{1}{2}i\tau) - \log \pi] d\tau. \quad (6.18)$$

With $\operatorname{Re} s > 1$, both integrands are integrable at ∞ in all τ -plane directions, but have cuts prescribed according to (1.8): $[+it, +i\infty)$ for I_+ , and its opposite for I_- ; the function $\psi(\frac{1}{4} + \frac{1}{2}i\tau)$ has poles with residues $2i$ at all points $\tau = (2n + \frac{1}{2})i, n \in \mathbb{N}$ (Fig. 6.2). Hence for I_+ , the integrand is analytic in the whole lower half-plane, the real-axis integration path can be translated all the way as $\{\operatorname{Im} \tau = t_0\}, t_0 \downarrow -\infty$, yielding $I_+ = 0$; while for I_- the path can be translated in the opposite direction $t_0 \uparrow +\infty$, but it then picks up the contributions from all the poles of ψ , yielding

$$\begin{aligned} W[h] = I_- &= 2\pi i \frac{1}{4\pi} 2i \sum_{n=0}^{\infty} (t - i(2n + \frac{1}{2})i)^{-s} \\ &= - \sum_{n=0}^{\infty} (2n + \frac{1}{2} + t)^{-s} = -2^{-s} \zeta(s, \frac{1}{4} + \frac{1}{2}t) \end{aligned} \quad (6.19)$$

(a result which can also be reached from (6.5): exercise!); so that

$$\begin{aligned} h(\pm \frac{1}{2}i) + W[h] &= (t - \frac{1}{2})^{-s} + (t + \frac{1}{2})^{-s} - 2^{-s} \zeta(s, \frac{1}{4} + \frac{1}{2}t) \\ &= (t - \frac{1}{2})^{-s} - 2^{-s} \zeta(s, \frac{5}{4} + \frac{1}{2}t). \end{aligned} \quad (6.20)$$

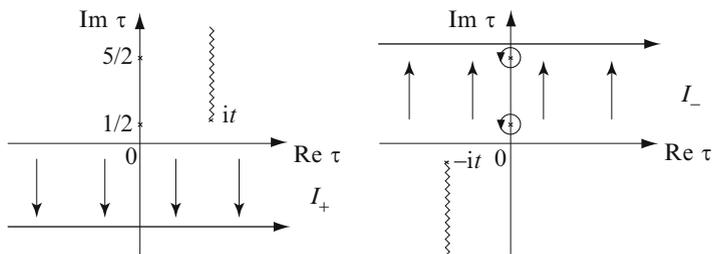


Fig. 6.2 Path deformations in the τ -planes with cuts (zigzag lines) for the evaluation of $W[h]$ by the integrals I_{\pm} of (6.18)

Now the other critical condition, (i), leads to a *major restriction on the parameter*,

$$\operatorname{Re} t > \frac{1}{2}, \quad (6.21)$$

which precisely rules out our two favorite locations, $t = \frac{1}{2}$ (a borderline case, see Remark below), and $t = 0$ (excluded outright)!

The Explicit Formula for the family of the first kind is then [28]

$$\mathcal{Z}(s|t) \equiv (t - \frac{1}{2})^{-s} - 2^{-s} \zeta(s, \frac{5}{4} + \frac{1}{2}t) - \frac{1}{\Gamma(s)} \sum_{n \geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+t}} (\log n)^{s-1} \quad \text{if } \operatorname{Re} t > \frac{1}{2}, \quad (6.22)$$

initially for $\operatorname{Re} s > 1$. Now, with $\operatorname{Re} t > \frac{1}{2}$, the argument proving the Explicit Formula shows the last sum to converge uniformly in s , hence it defines an entire function. And the Hurwitz zeta function $\zeta(s, w)$ is meromorphic in the whole s -plane (Sect. 3.6). Thus, the right-hand side of (6.22) provides a *meromorphic continuation* of $\mathcal{Z}(s|t)$ to all $s \in \mathbb{C}$ (at fixed t with $\operatorname{Re} t > \frac{1}{2}$).

In particular, $\mathcal{Z}(s|t)$ then has exactly the *singularities* of $-2^{-s} \zeta(s, \frac{5}{4} + \frac{1}{2}t)$: namely, the single pole $s = 1$, which is simple and of residue $-\frac{1}{2}$ (independent of t). The *special values* $\mathcal{Z}'(0|t)$ and $\mathcal{Z}(j|t)_{j \in \mathbb{Z}}$ can also be read from (6.22) by mere inspection (with the help of (3.6)–(3.7) on the transcendental side).

This approach is, however, curtailed throughout by the restriction $\operatorname{Re} t > \frac{1}{2}$, which seems quite essential here. An unrestricted picture will only emerge in the next chapter, thanks to a more flexible representation of $\mathcal{Z}(s|t)$ valid for general t .

Remark. A curious effect occurs in the case $\operatorname{Re} t = \frac{1}{2}$, where the analyticity condition (i) is just *marginally* violated: the arithmetical sum in (6.22) remains absolutely convergent, albeit more slowly, provided $\operatorname{Re} s < 0$ (as in the Bertrand series $\sum_n n^{-1} (\log n)^{s-1}$). So for $\operatorname{Re} t = \frac{1}{2}$, the Explicit Formula (6.22) can survive but in the half-plane $\operatorname{Re} s < 0$, while it originally held for $\operatorname{Re} s > 1$. (However, we see no vital use for this “borderline Explicit Formula,” even at $t = \frac{1}{2}$ itself, after our forthcoming extension to general t .)

6.4.2 The Family of the Second Kind \mathcal{Z}

In order to recover $\mathcal{Z}(\sigma|t) = \sum_k (\tau_k^2 + t^2)^{-\sigma}$ from the Explicit Formula (6.2), now the appropriate test function pair is

$$h(\tau) = (\tau^2 + t^2)^{-\sigma}, \quad \hat{h}(u) = \frac{\pi^{-1/2}}{\Gamma(\sigma)} \left(\frac{|u|}{2t} \right)^{\sigma - \frac{1}{2}} K_{\sigma - \frac{1}{2}}(t|u|) \quad (6.23)$$

for implied parameters σ, t such that $\operatorname{Re} \sigma > 0$, $\operatorname{Re} t > 0$, where $K_\nu(z)$ is the modified Bessel function [34, I.3(8)].

Then, in the smaller domain $\{\operatorname{Re} \sigma > \frac{1}{2}\}$ imposed at once by (ii₁),

$$\sum_k h(\tau_k) \equiv \mathcal{Z}(\sigma | t). \quad (6.24)$$

Under the further restriction $\operatorname{Re} t > \frac{1}{2}$ (as before) imposed by (i), the arithmetical sum (6.3) also converges, yielding this Explicit Formula for the family of the second kind,

$$\begin{aligned} \mathcal{Z}(\sigma | t) &\equiv (t^2 - \tfrac{1}{4})^{-\sigma} + W[h] \\ &- \frac{\pi^{-1/2}}{\Gamma(\sigma)} (2t)^{\frac{1}{2}-\sigma} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (\log n)^{\sigma-\frac{1}{2}} K_{\sigma-\frac{1}{2}}(t \log n) \quad \text{if } \operatorname{Re} t > \tfrac{1}{2}, \end{aligned} \quad (6.25)$$

initially for $\operatorname{Re} \sigma > \frac{1}{2}$; for the current function pair (6.23) however, we cannot evaluate the integral giving $W[h]$, be it (6.4) or (6.5), in contrast to the case \mathcal{Z} with the function pair (6.16).

Still, for the spectrum $\{v_k\}$ of a hyperbolic Laplacian as in Sect. 6.3.1, Randol [91] used Selberg trace formulae (6.14) similar to (6.25) to achieve the meromorphic continuation in σ of the generalized spectral zeta function $\sum_k (v_k + \varepsilon)^{-\sigma}$ ($\varepsilon \geq 0$), which our $\mathcal{Z}(\sigma | t)$ parallels if we identify $\varepsilon \equiv t^2 - \frac{1}{4}$. In fact, a trace formula precisely using the function pair (6.23) works ([91] actually focuses on $\varepsilon = 0$, see Remark below).

By analogy with [91], (6.25) should then be able to continue $\mathcal{Z}(\sigma | t)$ ($\operatorname{Re} t > \frac{1}{2}$) to a meromorphic function in the whole σ -plane, but now with greater difficulty since Weil's $W[h]$ is more complicated than Selberg's $I[h]$ for the same function h .

Another parallel is then reached with the first family, for which the Explicit Formula (6.22) achieved the meromorphic continuation in s of $\mathcal{Z}(s | t)$ ($\operatorname{Re} t > \frac{1}{2}$): but now, we lack a closed form for $W[h]$ in (6.25), to match the result (6.19) for the simpler case of \mathcal{Z} .

So on all grounds of comparison, (6.25) for the family \mathcal{Z} is an “even less Explicit Formula.”

Overall, our discussion then indicates that:

1. Analytic properties are harder to access for the family \mathcal{Z} than for \mathcal{Z}
2. In neither case should we rely too much on Explicit Formulae alone.

Remark. On the boundary $\operatorname{Re} t = \frac{1}{2}$, again as in (6.22), the right-hand side sum in (6.25) stays absolutely convergent in $\{\operatorname{Re} \sigma < 0\}$. This effect was noted in [91] upon the parallel Selberg trace formula at $\varepsilon = 0$ ($\iff t = \frac{1}{2}$), which is now an important case as it describes the *Minakshisundaram–Pleijel* spectral zeta function $\sum_{k \neq 0} v_k^{-\sigma}$ [85, Sect. 7] for the hyperbolic surface.

6.4.3 *Concluding Remarks*

Explicit Formulae can shed some light upon the zeta families of the first two kinds (\mathcal{Z} and \mathcal{Z}) but not directly on \mathfrak{Z} . Still, they are valid (i.e., converge) only over a parameter range which excludes the most interesting points $t = \frac{1}{2}$ and 0; and their potential to yield explicit results is limited, mainly by their W -functional term of an awkward singular nature.

Many variants of Explicit Formulae exist, so that better versions for superzeta functions can be envisioned (as with trace formulae for spectral zeta functions [15][16, Sect. 4]). Still at this time, we find it more economical overall to analyze our superzeta functions directly, rather than to coerce them into some Explicit Formula framework.

Chapter 7

The Family of the First Kind $\{\mathcal{Z}(s | t)\}$

We now enter the core of our study, with the detailed description of each superzeta family in turn, in the three following Chaps. 7–9. As a rule, we refer to Sect. 5.5 for bibliographical data most relevant to the superzeta functions themselves. Chapter 10 will extend the whole setting to zeros of more general zeta- and L -functions. Tables of special values conclude Chaps. 7, 8 and 10. The present chapter is then devoted to the first family \mathcal{Z} .

From Sect. 5.1, we recall the definition [28, 96, 108]

$$\mathcal{Z}(s | t) \stackrel{\text{def}}{=} \sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-s} \equiv \sum_{\rho} (\rho + t - \frac{1}{2})^{-s} \quad (\text{Re } s > 1), \quad (7.1)$$

valid for $t \in \Omega_1 \stackrel{\text{def}}{=} \{t \in \mathbb{C} | (\frac{1}{2} + t - \rho) \notin \mathbb{R}_- \ (\forall \rho)\}$, and these shorthand names for the two most interesting cases $t \in \Omega_1$,

$$\mathcal{Z}_0(s) = \mathcal{Z}(s | 0), \quad \mathcal{Z}_*(s) = \mathcal{Z}(s | \frac{1}{2}), \quad (7.2)$$

to be described last in this chapter.

7.1 The Basic Analytical Continuation Formula

Let

$$\mathbf{z}(s | t) = \sum_{k=1}^{\infty} \left(\frac{1}{2} + t + 2k\right)^{-s} \equiv 2^{-s} \zeta\left(s, \frac{5}{4} + \frac{1}{2}t\right) \quad (7.3)$$

be the “shadow” zeta function of $\mathcal{Z}(s | t)$, by which we mean the same sum as $\sum (\frac{1}{2} + t - \rho)^{-s}$ but now over the *trivial zeros* of $\zeta(x)$, a sum which is then expressible in terms of the Hurwitz zeta function.

This new sum enters an integral representation for $\mathcal{Z}(s|t)$, valid in the half-plane $\{\operatorname{Re} s < 1\}$ and for all $t \in \Omega_1 \setminus (-\infty, +\frac{1}{2}]$:

$$\mathcal{Z}(s|t) = -\mathbf{Z}(s|t) + (t - \frac{1}{2})^{-s} + \frac{\sin \pi s}{\pi} \mathcal{J}(s|t), \quad (7.4)$$

$$\mathcal{J}(s|t) \stackrel{\text{def}}{=} \int_0^\infty \frac{\zeta'}{\zeta} (\frac{1}{2} + t + y) y^{-s} dy \quad (\operatorname{Re} s < 1); \quad (7.5)$$

In fact, all discontinuities of $(t - \frac{1}{2})^{-s}$ and $-\mathbf{Z}(s|t)$ across the real cut $(-\infty, +\frac{1}{2}]$ precisely cancel against jumps of $\mathcal{J}(s|t)$, as seen by computing (7.4) for $t \pm i0$, $t \in \mathbb{R}$. So, even after this analytical continuation in s , $\mathcal{Z}(s|t)$ stays a regular function of t in all of Ω_1 , including the real axis.

The above result easily adapts to real $t < \frac{1}{2}$. One way is to take the half-sum over $t \pm i0$ (giving a principal-value integral in (7.5), as exemplified in (8.7) later). A more regular real form valid for $t > -\frac{1}{2}$ (at least) is

$$\mathcal{Z}(s|t) = -\mathbf{Z}(s|t) + \frac{\sin \pi s}{\pi} \int_0^\infty \left[\frac{\zeta'}{\zeta} (\frac{1}{2} + t + y) + \frac{1}{t - \frac{1}{2} + y} \right] y^{-s} dy \quad (0 < \operatorname{Re} s < 1); \quad (7.6)$$

this form only converges in the stated strip of the s -plane, but unlike (7.4) it remains well defined in the $t \rightarrow +\frac{1}{2}$ limit, where it gives

$$\mathcal{Z}_*(s) = -\mathbf{Z}(s|\frac{1}{2}) + \frac{\sin \pi s}{\pi} \int_0^\infty \left[\frac{\zeta'}{\zeta} (1 + y) + \frac{1}{y} \right] y^{-s} dy \quad (0 < \operatorname{Re} s < 1). \quad (7.7)$$

7.2 Derivations

We provide two distinct arguments, with pros and cons for each.

7.2.1 Derivation by Contour Integration

We start from the last term in (7.4), which we transform in parallel with the proof of the Functional Equation in Sect. 3.4 (up to $x \mapsto 1 - s$), using the same notation for the integration paths, and identifying $x \equiv \frac{1}{2} + t$ throughout:

$$\frac{\sin \pi s}{\pi} \mathcal{J}(s|t) = \frac{\sin \pi s}{\pi} \int_0^\infty \frac{\zeta'}{\zeta} (x + y) y^{-s} dy \quad (\operatorname{Re} s < 0) \quad (7.8)$$

$$= \frac{1}{2i\pi} \int_{C'} \frac{\zeta'}{\zeta}(x+y) (-y)^{-s} dy \quad (s \in \mathbb{C}) \quad (7.9)$$

$$\vdots$$

$$= \frac{1}{2i\pi} \lim_{R \rightarrow +\infty} \int_{C_R} \frac{\zeta'}{\zeta}(x+y) (-y)^{-s} dy \quad (\operatorname{Re} s > 1), \quad (7.10)$$

now true provided R is chosen to keep the contour C_R “far enough” from the zeros of $\zeta(x+y)$, up to extra small deformations over lengths $O(1)$ near the negative real axis and over the critical strip: then ζ'/ζ is $O(\log^2 R)$ on the path C_R'' by Sect. 4.4. Now we can evaluate (7.10) by the residue calculus: upon $R \rightarrow +\infty$, by construction (Fig. 7.1), the poles of ζ'/ζ at the Riemann zeros contribute $\mathcal{Z}(s|t)$, those at the trivial zeros contribute $\mathbf{Z}(s|t)$, while the pole of ζ itself contributes $-(t - \frac{1}{2})^{-s}$, whence (7.4) follows. \square

Disadvantages of this approach are that one must carefully fiddle with the integration path in three zones to make the argument rigorous (at two critical-strip and one negative-axis crossings), and rely on the nontrivial (albeit classic) bound (4.22), or $[\zeta'/\zeta](x_0 + iT) = O(\log^2 T)$ at “good” ordinates T in the critical strip; as for the negative real axis, it suffices to cross it near some *odd* integer $(x+y)$. Extra work is also needed to reach the real form (7.6). Finally, we cannot transpose this approach to the other families.

On the other hand, the foregoing argument is short and intuitive, and it displays clear parallels:

- With the proof of the Functional Equation in Sect. 3.4, and even more when $\zeta(x)$ is replaced by the Hurwitz zeta function as in Sect. 3.6 because then, both resulting relations are non-reflexive: we thus view (7.4) as a full analog for $\mathcal{Z}(s|t)$ of the Jonquière functional relation (3.39) for $\zeta(x, w)$;

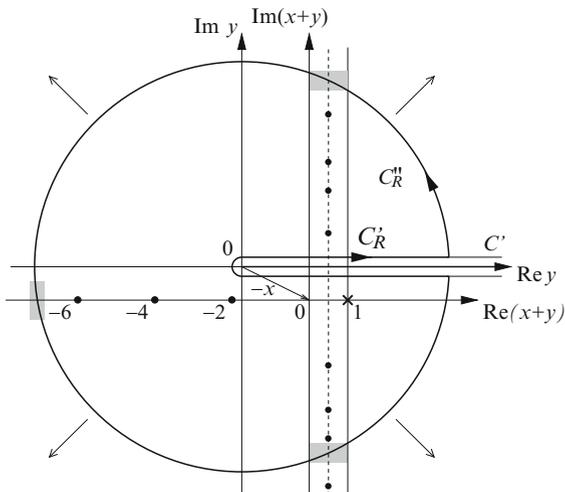


Fig. 7.1 Integration contour yielding the basic analytical continuation formula (7.4). The shaded areas mark the zones where the path has to keep far enough from the zeros of $\zeta(x+y)$ (\bullet): see text (The nontrivial zeros are not placed on scale)

- With the proof of the Explicit Formula specialized to the function pair (6.16), only with a different integration path; indeed, in Sect. 7.3 we will interpret (7.4) as a kind of extended Explicit Formula.

7.2.2 Derivation by Eligibility of the Riemann Zeros

The key input here is that the standard Hadamard factorization (4.9) of $\zeta(x)$ only involves infinite products over sequences eligible in the sense of (2.11)–(2.13), of order $\mu_0 = 1$; hence the whole formalism of Chap. 2 *mechanically* applies, with $m_0 = 2$. We then begin by expounding this aspect.

First, the Gamma factor $\Gamma(1 + \frac{1}{2}x)^{-1}$ in (4.9) is a function $\Delta(x)$ like (2.2) of order $\mu_0 = 1$ for the sequence of negated trivial zeros $\{2k\}_{k=1,2,\dots}$, whose generalized zeta function is $\mathbf{Z}(s|t)$. (Again here, $x \equiv \frac{1}{2} + t$; zeta-regularization being translation-invariant, it won't matter which variable x or t is used.) And Stirling's formula (3.51) gives

$$\log \Gamma(1 + \frac{1}{2}x) \sim \frac{1}{2} [x(\log x - 1) - \underline{(\log 2)} x + \log x + \underline{\log \pi}] \left[+ \sum_1^{\infty} a'_{-n} x^{-n} \right] \\ \text{for } x \rightarrow \infty, \quad |\arg x| < \pi, \quad (7.11)$$

a result which indeed meets the asymptotic requirement (2.13). We readily underlined the terms to be banned in the zeta-regularized form; the latter then immediately follows for the Gamma factor, by (2.50), as

$$\mathbf{D}(x) \stackrel{\text{def}}{=} \exp[-\mathbf{Z}'(0|x - \frac{1}{2})] \equiv 2^{-x/2} \pi^{1/2} / \Gamma(1 + \frac{1}{2}x). \quad (7.12)$$

We then recast the Hadamard factorization (4.9) of $\zeta(x)$ as

$$\zeta(x) = \frac{1}{x-1} \mathbf{D}(x) \mathcal{D}(x), \quad (7.13)$$

thereby defining

$$\mathcal{D}(x) \stackrel{\text{def}}{=} (x-1) 2^{x/2} \pi^{-1/2} \Gamma(1 + \frac{1}{2}x) \zeta(x) \equiv (4\pi)^{-1/2} (2\pi)^{x/2} \Xi(x); \quad (7.14)$$

this $\mathcal{D}(x)$ will prove to be the zeta-regularized form of the other factor $\Xi(x)$. But to justify such a statement, first we must check that the zeros $\{\rho\}$ of Ξ also form an eligible sequence: indeed, their zeta function $\mathcal{Z}_*(s)$ satisfies (2.11) with $\mu_0 = 1$, due to (4.7); and their Hadamard product $\Xi(x)$ given by (4.8) is a good function $\Delta(x)$, which also complies with (2.13):

$$\log \Xi(x) \sim \frac{1}{2} [x(\log x - 1) - \underline{(\log 2\pi)} x + 3 \log x + \underline{\log 4\pi}] \left[+ \sum_1^{\infty} a''_{-n} x^{-n} \right] \\ \text{for } x \rightarrow \infty, \quad |\arg x| < \frac{1}{2}\pi - \varepsilon \quad (7.15)$$

(using Stirling's formula again upon (3.24), plus $\log \zeta(x) = O(x^{-\infty})$). So, Chap. 2 fully applies to the sequence $\{\rho\}$. Thereafter, (7.14) shows that $\log \mathcal{D}(x)$ is precisely $\log \Xi(x)$ but for the banned terms suppressed, which then qualifies $\mathcal{D}(x)$ for being $\exp[-\mathcal{Z}'(0|x-\frac{1}{2})]$, the zeta-regularized product; this will ultimately be confirmed in (7.32) below.

Remark. In such conversions, *monic* polynomial factors like $(x-1)$ in (4.9) are passive spectators ($\log[x^q + \dots] = q \log x + O(1/x)$ has no banned terms). Even more crucially, $\zeta(x)$ with $\log \zeta(x) = O(x^{-\infty})$ is untouched.

After those preliminaries, the integral representation (7.4) will follow directly from the factorization (4.9) or (7.13), actually from its logarithmic derivative of order $m_0 = 2$ which is intrinsic. This can be evaluated by applying (2.27), i.e.,

$$Z(2, x) = (-\log \Delta)''(x) \quad (\text{for general } \Delta) \quad (7.16)$$

with $Z = \mathcal{Z}$ and $Z = \mathbf{Z}$ in turn, resulting in

$$[\zeta'/\zeta]'(\frac{1}{2} + t) = (t - \frac{1}{2})^{-2} - \mathcal{Z}(2|t) - \mathbf{Z}(2|t). \quad (7.17)$$

The idea is now to generalize this identity from $s = 2$ to some domain of the half-plane $\text{Re } s < 1$, so as to get a genuine analytical continuation for \mathcal{Z} . The machinery of Sect. 2.4 prompts us to use the identity (2.28), i.e.,

$$Z(s|t) = \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty Z(2|t+y) y^{1-s} dy \quad \text{when } \mu_0 = 1, \quad (7.18)$$

for $Z = \mathcal{Z}$ and \mathbf{Z} , but the latter formula only holds for $1 < \text{Re } s < 2$. Now this can be remedied using the continuation method shown in (1.23). Thus, (7.18) successively yields

$$Z(s|t) = -\frac{\sin \pi s}{\pi(1-s)^2} \int_0^\infty \frac{d}{dy} [yZ(2|t+y)] y^{1-s} dy \quad (0 < \text{Re } s < 2) \quad (7.19)$$

$$= +\frac{\sin \pi s}{\pi(1-s)} \int_0^\infty \tilde{Z}_t(2, y) y^{1-s} dy \quad (0 < \text{Re } s < 1), \quad (7.20)$$

where $\tilde{Z}_t(2, y)$ stands for $Z(2|t+y)$ minus its leading large- y behavior (drawn from (7.16) and the generalized Stirling expansion (2.13) with $\mu_0 = 1$):

$$\tilde{Z}_t(2, y) \stackrel{\text{def}}{=} Z(2|t+y) + \tilde{a}_1 y^{-1} = o(y^{-1})_{y \rightarrow +\infty}; \quad (7.21)$$

this subtraction is mandated for a valid integration by parts in (7.20).

Accordingly, we then twist the decomposition (7.17) to

$$\tilde{\mathcal{Z}}_t(2, y) \equiv -\tilde{\mathbf{Z}}_t(2, y) + (t - \frac{1}{2} + y)^{-2} - [\zeta'/\zeta]'(\frac{1}{2} + t + y), \quad (7.22)$$

$$\tilde{\mathcal{Z}}_t(2, y) \stackrel{\text{def}}{=} \mathcal{Z}(2|t+y) + \frac{1}{2y}, \quad \tilde{\mathbf{Z}}_t(2, y) \stackrel{\text{def}}{=} \mathbf{Z}(2|t+y) - \frac{1}{2y}, \quad (7.23)$$

where the last line uses the specific Stirling expansions (7.15) and (7.11). Now, on the modified decomposition (7.22), at fixed $t \in \Omega_1 \setminus (-\infty, +\frac{1}{2}]$, we can apply the Mellin transformation (7.20) term by term. Then, the left-hand side yields $\mathcal{Z}(s|t)$; as for the right-hand side, the first term yields $-\mathbf{Z}(s|t)$ by exactly the same argument, the second term trivially evaluates to $(t-\frac{1}{2})^{-s}$, and the last term can be subjected to an ultimate integration by parts now valid in the whole half-plane $\{\text{Re } t < \frac{1}{2}\}$, using

$$\mathcal{J}(s|t) \stackrel{\text{def}}{=} \frac{1}{1-s} \int_0^\infty -\left[\frac{\zeta'}{\zeta}\right]'(\tfrac{1}{2} + t + y) y^{1-s} dy = \int_0^\infty \frac{\zeta'}{\zeta}(\tfrac{1}{2} + t + y) y^{-s} dy; \quad (7.24)$$

all that yields the desired formula (7.4). If the last two terms in (7.22) are kept together instead, (7.6) can be obtained likewise. \square

This method of derivation seems more robust than the former, as it simply builds on the general eligible nature of the sequence of Riemann zeros, and it (alone) will adapt to the second family \mathcal{Z} as well. It does without the custom estimate (4.22) for ζ'/ζ in the critical strip (however, [61, p. 40] argues that such an estimate is automatically implied by the general zeta-regularization axioms therein).

On the negative side, this approach obfuscates the kinship, which was manifest in the previous derivation, between the present results for $\mathcal{Z}(s|t)$ and the Jonquière formula for $\zeta(x, w)$ on the one hand, the Explicit Formula (6.22) on the other hand.

7.3 Analytic Properties of the Family $\{\mathcal{Z}(s|t)\}$

The subsequent statements refer to s as variable, with t fixed. It helps to keep in mind that (7.4) is to $\mathcal{Z}(s|t)$ very much like the Jonquière functional relation (3.39) to $\zeta(x, w)$. Incidentally, since neither relation is reflexive (in contrast to the Riemann Functional Equation), we expect nothing remarkable about the s -plane zeros of \mathcal{Z} (so, the reader will be spared later books on zeta functions built over zeros of zeta functions built over zeros of \dots , etc.).

First of all, (7.4) gives an explicit one-step analytical continuation of $\mathcal{Z}(s|t)$ to the half-plane $\{\text{Re } s < 1\}$. It also implies its *meromorphic continuation in s* to all of \mathbb{C} , because as a Mellin transform, $\mathcal{J}(s|t)$ has a known global meromorphic structure: with $(\zeta'/\zeta)(x) = O(x^{-\infty})$ for $x \rightarrow +\infty$, the application of Sect. 1.5 shows \mathcal{J} to be meromorphic in the whole s -plane, having only simple poles at $s = +1, +2, \dots$, with residues

$$\text{Res}_{s=n} \mathcal{J}(s|t) = -(\log|\zeta|)^{(n)}(\tfrac{1}{2} + t)/(n-1)! \quad (n = 1, 2, \dots). \quad (7.25)$$

Then, $\sin \pi s \mathcal{J}(s|t)$ in (7.4) has no singularities; it follows therefrom that $\mathcal{Z}(s|t)$ is meromorphic in the whole s -plane with the same polar structure as $-\mathbf{Z}(s|t)$: i.e., by (7.3) and Sect. 3.6,

$$\mathcal{Z}(s|t) \text{ has the single pole } s = 1, \text{ simple, of residue } -\frac{1}{2}. \quad (7.26)$$

Next, the substitution of the Dirichlet series (3.7) for $\zeta'/\zeta(\frac{1}{2} + t + y)$ into (7.5), followed by y -integration term by term, formally yields

$$\mathcal{J}(s|t) = -\Gamma(1-s) \sum_{n \geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+t}} (\log n)^{s-1}. \quad (7.27)$$

This is rigorously an identity if the Dirichlet series for $\zeta'/\zeta(\frac{1}{2} + t + y)$ converges uniformly over the integration range $\{y > 0\}$, which is the case if $\operatorname{Re} t > \frac{1}{2}$; then in fact, (7.3) plus (7.4) plus (7.27) just boil down to the *Explicit Formula* (6.22) for the family \mathcal{Z} [28]. (For $\operatorname{Re} t = \frac{1}{2}$, see the Remark ending Sect. 6.4.1.)

But the above integration term by term stays valid even for general t (fixed), now in the sense of asymptotic series as $s \rightarrow -\infty$. So, the Explicit Formula (6.22) retains *some* meaning at all t : when its right-hand side diverges, it just goes into an *asymptotic expansion* for $\mathcal{Z}(s|t)$, as

$$\mathcal{Z}(s|t) \sim (t - \frac{1}{2})^{-s} - 2^{-s} \zeta(s, \frac{5}{4} + \frac{1}{2}t) - \frac{1}{\Gamma(s)} \sum_{n \geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+t}} (\log n)^{s-1} \quad (7.28)$$

for $s \rightarrow -\infty$ (or $\operatorname{Re} s \rightarrow -\infty$ in \mathbb{C}), at any fixed $t \in \Omega_1$.

In reverse, (7.4)–(7.7) qualify as extensions of the Explicit Formula (6.22) to $\{\operatorname{Re} t \leq \frac{1}{2}\}$. Only the series (7.27) buried in (6.22) is affected in this process: when this *Dirichlet* series diverges, (7.5) delivers its adequate *re-summation*; this achieves a transposition of the Borel summation method which works for divergent power series.

7.4 Special Values of $\mathcal{Z}(s|t)$ for General t

Again thanks to the factor $(\sin \pi s)$ vanishing at integers, and to the known polar structure (7.25) of $\mathcal{J}(s|t)$, (7.4) immediately outputs formulae for all the *special values* of $\mathcal{Z}(s|t)$ (in the sense of Sect. 2.6.3):

$$\mathcal{Z}(-n|t) = -\mathbf{Z}(-n|t) + (t - \frac{1}{2})^n \quad (n \in \mathbb{N}) \quad (7.29)$$

$$= -2^n \zeta(-n, \frac{5}{4} + \frac{1}{2}t) + (t - \frac{1}{2})^n \quad (n \in \mathbb{N}), \quad (7.30)$$

$$\mathcal{Z}'(0|t) = -\mathbf{Z}'(0|t) - \log \left[(t - \frac{1}{2}) \zeta(\frac{1}{2} + t) \right] \quad (7.31)$$

$$\begin{aligned}
&= -\frac{1}{2}(\log 2)t + \frac{1}{2} \log \frac{\pi}{\sqrt{2}} - \log \Gamma\left(\frac{5}{4} + \frac{1}{2}t\right) \\
&\quad - \log \left[\left(t - \frac{1}{2}\right) \zeta\left(\frac{1}{2} + t\right) \right] \\
&\equiv -\log \mathcal{D}\left(\frac{1}{2} + t\right), \tag{7.32}
\end{aligned}$$

$$\text{FP}_{s=1} \mathcal{Z}(s|t) = -\text{FP}_{s=1} \mathbf{Z}(s|t) + \left[\frac{1}{t - \frac{1}{2}} + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + t\right) \right] \tag{7.33}$$

$$= \frac{1}{2}(\log 2 + \psi\left(\frac{5}{4} + \frac{1}{2}t\right)) + \left[\frac{1}{t - \frac{1}{2}} + \frac{\zeta'}{\zeta}\left(\frac{1}{2} + t\right) \right] \tag{7.34}$$

$$\equiv (\log \mathcal{D})'\left(\frac{1}{2} + t\right), \tag{7.35}$$

and, for $n = 2, 3, \dots$,

$$\mathcal{Z}(+n|t) = -\mathbf{Z}(+n|t) + \left[\left(t - \frac{1}{2}\right)^{-n} - \frac{(-1)^n}{(n-1)!} [\log |\zeta|]^{(n)}\left(\frac{1}{2} + t\right) \right] \tag{7.36}$$

$$\begin{aligned}
&= -2^{-n} \zeta\left(n, \frac{5}{4} + \frac{1}{2}t\right) \\
&\quad + \left[\left(t - \frac{1}{2}\right)^{-n} - \frac{(-1)^n}{(n-1)!} [\log |\zeta|]^{(n)}\left(\frac{1}{2} + t\right) \right] \tag{7.37}
\end{aligned}$$

$$\equiv \frac{(-1)^{n-1}}{(n-1)!} (\log \mathcal{D})^{(n)}\left(\frac{1}{2} + t\right) \quad (n = 2, 3, \dots), \tag{7.38}$$

also using (3.40), (3.41), (7.3), (7.14). Those values are listed in Table 7.1 in still another equivalent form which clearly shows that the quantities in square brackets above, apparently singular for $t = +\frac{1}{2}$, are in fact globally regular (for the special values reached at $t = \frac{1}{2}$, see Sect. 7.6.2 and Table 7.4).

At $s = 0$, the identity (7.32) confirms that \mathcal{D} is the zeta-regularized form of Ξ in the variable x (or t). Then, (7.35) and (7.38) just exemplify the general formulae (2.53)–(2.56).

Now the point $s = 1$ ($= \mu_0$) deserves extra attention. Upon logarithmic differentiations, the symmetric Hadamard product formula (4.15), $\Xi\left(\frac{1}{2} + t\right) \equiv \prod_{\rho} \left[1 - \left(\frac{1}{2} + t\right)/\rho\right]$, directly yields

$$\frac{(-1)^{n-1}}{(n-1)!} (\log \Xi)^{(n)}\left(\frac{1}{2} + t\right) = \sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-n} \equiv \mathcal{Z}(n|t), \quad n = 1, 2, \dots \tag{7.39}$$

When $n = 1$, both product and sum (then only *semiconvergent*) must be performed with zeros grouped in symmetrical pairs, as usual: the two terms $(t \pm i\tau_k)^{-1}$ interfere sufficiently to make the grouped series converge; $\mathcal{Z}(1|t)$ thereby acquires a finite value, as exemplified earlier by (4.14) at $t = \frac{1}{2}$. But here a double paradox arises.

1. Although $\mathcal{Z}(1|t)$ is finite, $\mathcal{Z}(s|t)$ still has a pole at $s = 1$: thus, that same interference inside the (convergent) series $\sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-s}$ ($s > 1$) does not keep its sum from diverging for $s \rightarrow 1^+$.

2. A standard finite value assignment for a meromorphic function at a pole is its *finite part* (1.7); but for $\mathcal{Z}(s|t)$ at $s = 1$, this prescription differs from

(7.39) at $n = 1$; the discrepancy (or “anomaly”) is

$$\delta_1(t) \stackrel{\text{def}}{=} \text{FP}_{s=1} \mathcal{Z}(s|t) - \mathcal{Z}(1|t) \equiv [\log(\mathcal{D}/\Xi)]'(\tfrac{1}{2} + t) \equiv \tfrac{1}{2} \log 2\pi \quad (7.40)$$

according to (7.14), (7.35). Since $\text{FP}_{s=1} \mathcal{Z}(s|t)$ and $\mathcal{Z}(1|t)$ share the same t -derivative ($-\mathcal{Z}'(2|t)$), it is clear why $\delta_1(t) = \text{const.}$, but why is it not zero? This will actually relate to a *nonzero residue* in the function \mathcal{Z} of the second kind, see (8.5), and also to one term of the Riemann–von Mangoldt formula, see (10.40).

Due to this $\delta_1 \neq 0$, in every Table for the family \mathcal{Z} (this chapter, Chap. 10, and Appendix A), two lines harbor values at $s = 1$, and these *apparently* conflict.

Remark. Equation (7.39) instantly yielded all the special values $\mathcal{Z}(n|t)$ for $n = 1, 2, \dots$ as displayed in Table 7.1. However, we didn’t favor this shortcut because it all too easily suggests that those $\mathcal{Z}(n|t)$ form the *totality* of computable transcendental special values; it thus misses the first two, $\mathcal{Z}'(0|t)$ and $\text{FP}_{s=1} \mathcal{Z}(s|t)$, which match the important special values (3.40) and (3.41) of the Hurwitz zeta function. In contrast, our lengthier treatment sees and includes those latter two values as well. We add that $\mathcal{Z}'(0|t)$ itself has been computed, and its interest argued, in [28, 76, 96].

7.5 Imprints of the Central Symmetry $\rho \longleftrightarrow (1 - \rho)$

The stake here is to detect manifestations of Riemann’s Functional Equation, inasmuch as it does not simply translate into a functional equation for the family $\mathcal{Z}(s|t)$ (cf. Sect. 5.1).

7.5.1 $t \longleftrightarrow (-t)$ Symmetry at Integer t

A basic symmetry identity, obvious from (7.39), is

$$\mathcal{Z}(n|t) = (-1)^n \mathcal{Z}(n|-t) \quad \text{for } n = 1, 2, \dots, \quad (7.41)$$

and in particular, at $t = 0$,

$$\mathcal{Z}_0(n) \equiv \sum_{\rho} \left(\tfrac{1}{2} - \rho\right)^{-n} \equiv 0 \quad \text{for each odd } n \geq 1. \quad (7.42)$$

Symmetry relations can be expected for the nonpositive integers as well, but possibly modified, as these other integers are only reached under analytical continuation. Working directly from Table 7.1, Table 7.2 lists the

correct symmetry relations for all the special values; e.g., for the first line, $\mathcal{Z}(-n|t) - (-1)^n \mathcal{Z}(-n|-t) = \frac{2^n}{n+1} [B_{n+1}(\frac{1}{4} + \frac{1}{2}t) - (-1)^n B_{n+1}(\frac{1}{4} - \frac{1}{2}t)] = \frac{2^n}{n+1} [B_{n+1}(\frac{1}{4} + \frac{1}{2}t) + B_{n+1}(\frac{3}{4} + \frac{1}{2}t)] = \frac{1}{n+1} B_{n+1}(\frac{1}{2} + t)$, using the symmetry and duplication formulae for the Bernoulli polynomials [1, Sect. 23.1].

7.5.2 Sum Rules at an Arbitrary Fixed t

A less obvious effect, now for a *general* t kept at a *single* value, has the following heuristic basis: if the central symmetry made all the odd- n values $\mathcal{Z}(n|t=0)$ redundant ($\equiv 0$ through (7.42)), then some trace of this could persist for $t \neq 0$. Indeed, one extension of (7.42) to general t consists of this countable set of “sum rules”:

$$\mathcal{Z}(n|t) + \frac{1}{2} \sum_{k=n+1}^{\infty} \binom{k-1}{n-1} (2t)^{k-n} \mathcal{Z}(k|t) = 0 \quad \text{for each odd } n \geq 1. \quad (7.43)$$

Proof. In the identity

$$(t - \frac{1}{2} + \rho)^{-n} \equiv (-1)^n (\frac{1}{2} + t - \rho)^{-n} [1 - 2t/(\frac{1}{2} + t - \rho)]^{-n}, \quad (7.44)$$

do the generalized binomial expansion of the last factor (in brackets) about $t = 0$, sum over the zeros grouped in pairs, then identify \mathcal{Z} on both resulting sides thanks to central symmetry. The said expansion converges in the domain $\{t \in \mathbb{C} \mid 2|t| < |t \pm \tau_1|\}$ (intersection of two disks). \square

We know of no canonical, or maximally reduced, presentation for a set of identities based on infinite series like (7.43). Still, we may strive to make those series converge more rapidly.

For instance, if in place of (7.44) we use the identity

$$(\frac{1}{2} - \rho)^{-n} \equiv (\frac{1}{2} + t - \rho)^{-n} [1 - t/(\frac{1}{2} + t - \rho)]^{-n}, \quad (7.45)$$

then expand about $t = 0$ and sum as before, and finally invoke (7.42), we thus obtain variant sum rules which converge manifestly better than (7.43):

$$0 = \sum_{k=n}^{\infty} \binom{k-1}{n-1} t^{k-n} \mathcal{Z}(k|t) \quad \text{for each odd } n \geq 1. \quad (7.46)$$

We believe that the latter approach is the most symmetrical one and enjoys the largest possible domain of convergence (the strip $\{|\operatorname{Im} t| < \frac{1}{2}\tau_1\}$): we hence advocate these newer sum rules (7.46) as superior to (7.43).

Such sum rules then allow us to eliminate any odd- n value $\mathcal{Z}(n|t)$ as a series over all higher- k values $\mathcal{Z}(k|t)$, from which higher odd- k values can be eliminated again, up to any finite rank; in this precise sense, odd- n values are finitely redundant. If we now continue indefinitely, every odd- n value ends up formally reexpressed in a series over all higher even- n values only, as $\mathcal{Z}(2m+1|t) = \sum_{j>m} A_{m,j} \mathcal{Z}(2j|t) t^{2j-(2m+1)}$; but then (exercise!) these series have to be *divergent* for all $t \neq 0$; while at $t = 0$, only (7.42) is recovered.

7.6 Special Values of $\mathcal{Z}(s|t)$ at $t = 0$ and $\frac{1}{2}$

For integer t , the values $\zeta(\pm n, \frac{5}{4} + \frac{1}{2}t)$ which occur in (7.30) and (7.37) can be made slightly more explicit, and even simpler for half-integer t . We moreover argued in Sect. 5.1 that remarkable parameter values had to be $t = 0$ and $\frac{1}{2}$. At these points, (7.3) now yields

$$\mathbf{Z}(s|0) \equiv \frac{1}{2}[(2^s - 1)\zeta(s) + 2^s\beta(s)] - 2^s, \quad \mathbf{Z}(s|\frac{1}{2}) \equiv (1 - 2^{-s})\zeta(s) - 1; \quad (7.47)$$

then, using the special values for $\zeta(s)$ and $\beta(s)$ from Chap. 3, the special values of $\mathcal{Z}(s|0) \equiv \mathcal{Z}_0(s)$ and $\mathcal{Z}(s|\frac{1}{2}) \equiv \mathcal{Z}_*(s)$ reduce further. The results are displayed in Tables 7.3 and 7.4 respectively, with a few comments below. Moreover, some numerical data are described and plotted in Appendix A.

7.6.1 The Function $\mathcal{Z}_0(s)$ (the Confluent Case $t = 0$)

At $t = 0$, thanks to the confluence identities (5.12), $\mathcal{Z}_0(s)$ will actually subsume the superzeta functions of all three kinds.

Here, the central symmetry greatly simplifies most of the special values, notably $\mathcal{Z}_0(n) \equiv 0$ for all odd $n \geq 1$ by (7.42); equivalently, $(\log \Xi)^{(n)}(\frac{1}{2}) = 0$ by the Functional Equation, and either result amounts (e.g., by (7.37)) to a sequence of identities for $\zeta(x)$ itself:

$$(\log |\zeta|)^{(n)}(\frac{1}{2}) \equiv -2^{-n}\psi^{(n-1)}(\frac{1}{4}) + \delta_{n,1} \frac{1}{2} \log \pi \quad \text{for odd } n \geq 1, \quad (7.48)$$

$$\equiv \begin{cases} \frac{1}{2}(n-1)! [(2^n - 1)\zeta(n) + 2^n\beta(n)], & n > 1, \\ \frac{1}{2}\gamma + \frac{1}{4}\pi + \frac{1}{2}\log 8\pi, & n = 1, \end{cases} \quad (7.49)$$

in which, n being odd, $\frac{1}{2}(n-1)!2^n\beta(n)$ reduces to $\frac{1}{4}\pi^n|E_{n-1}|$ by (3.32) while $\zeta(n)$ remains elusive. App. C gives more identities relevant to $(\log |\zeta|)^{(n)}(\frac{1}{2})$.

At $n = 1$, moreover, since $\mathcal{Z}_0(1) = 0$, the evaluation of the discrepancy δ_1 by (7.40) at $t = 0$ implies

$$\text{FP}_{s=1} \mathcal{Z}_0(s) \equiv \frac{1}{2} \log 2\pi \quad (\approx 0.918938533). \quad (7.50)$$

Inversely now, at $n = 0$, $\mathcal{Z}'_0(0)$ (≈ 0.811817944) (for $t = 0$) is much less explicit than its counterpart $\mathcal{Z}'_*(0)$ for $t = \frac{1}{2}$ (see below), as it involves $\zeta(\frac{1}{2})$ about which very little is known.

All resulting special values of $\mathcal{Z}_0(s)$ are listed in Table 7.3, following [108, Table 3] (we found no earlier references).

7.6.2 The Function $\mathcal{Z}_*(s)$ (the Case $t = \frac{1}{2}$)

Here, our results (listed in Table 7.4) are the $t \rightarrow \frac{1}{2}$ limits of (7.30)–(7.38), which make the Stieltjes cumulant expansion (3.15) emerge; equivalently, they can be traced to the integral representation (7.7). Unlike the $\mathcal{Z}_0(n)$, the special values $\mathcal{Z}_*(n)$ have received some attention before.

- Rational special values $\mathcal{Z}_*(-n)$ for $n \in \mathbb{N}$ were stated in [83, (12)] but with errors (as discussed in Sect. 5.5; see also Appendix D).

By pure observation, we report these intriguing identities:

$$\mathcal{Z}_*(-n) \equiv 1 + 2^{n+1} \mathcal{Z}_0(-n) \quad \text{for odd } n, \quad \equiv 1 \quad \text{for even } n \quad (n > 0). \quad (7.51)$$

- The explicit value $\mathcal{Z}'_*(0) = \frac{1}{2} \log 2$ (≈ 0.346573590) is much simpler than $\mathcal{Z}'_0(0)$; it yields the *zeta-regularized product of all the Riemann zeros*, as

$$\text{“} \prod_{\rho} \rho \text{”} = e^{-\mathcal{Z}'_*(0)} = 2^{-1/2}. \quad (7.52)$$

- The transcendental special values $\mathcal{Z}_*(n)$ ($n = 1, 2, \dots$) have been the subject of a few studies (often under the name σ_n):

- $\mathcal{Z}_*(1) = -B = \frac{1}{2}\gamma - \frac{1}{2} \log 4\pi + 1$ (≈ 0.0230957090), as given by (4.14) [26, Sect. 12][31, Sect. 3.8][89, Sect. 3.1]

- Every higher $\mathcal{Z}_*(n)$ boils down to this *affine* function of the n -th Stieltjes *cumulant* g_n^c from (3.15) [106]

$$\mathcal{Z}_*(n) = \frac{1}{(n-1)!} g_n^c - (1 - 2^{-n}) \zeta(n) + 1 \quad (n > 1); \quad (7.53)$$

this formula subsumes earlier ones relating $\mathcal{Z}_*(n)$ to the Stieltjes *constants* instead – at the price of greater complexity [82][64, Sect. 6][114,

Theorem 5]. Another affine relationship [72, (12)], equivalent to (7.53) by the Functional Equation [2, 18], is

$$\mathcal{Z}_*(n) = -\frac{1}{(n-1)!} (\log |\zeta|)^{(n)}(0) - (-1)^n 2^{-n} \zeta(n) + 1 \quad (n > 1). \quad (7.54)$$

For numerical tables, see [72, Table 5][64, Appendix].

Those $\mathcal{Z}_*(n)$ ($n = 1, 2, \dots$) also enter two interesting countable sets of linear identities.

1. The (infinite) sum rules (7.43) or better, (7.46), specialized to $t = \frac{1}{2}$; (remark: [64, (18)] *seemingly* states more sum rules than (7.43) at $t = \frac{1}{2}$: i.e., $\mathcal{Z}_*(n) = (-1)^n \sum_{k=n}^{\infty} \binom{k-1}{n-1} \mathcal{Z}_*(k)$ for odd or even n ; but the even- n identities are redundant, reducing to finite linear combinations of higher odd- k ones).

2. (Finite, triangular) relations to the Keiper–Li coefficients

$$\lambda_j \stackrel{\text{def}}{=} \sum_{\rho} [1 - (1 - 1/\rho)^j], \quad j = 1, 2, \dots, \quad (7.55)$$

which enter *Li's criterion* for the Riemann Hypothesis, namely $\lambda_j > 0$ ($\forall j$) (see [9, 73] and Chap. 11; note: the λ_j of [73], used here, are j times the earlier λ_j of [64]). The above-mentioned relations read as [64, (27)][9, Theorem 2]

$$\lambda_j = \sum_{n=1}^j (-1)^{n+1} \binom{j}{n} \mathcal{Z}_*(n) \iff \mathcal{Z}_*(n) = \sum_{j=1}^n (-1)^{j+1} \binom{n}{j} \lambda_j. \quad (7.56)$$

The leftmost identities follow by straightforward expansion of (7.55), upon which a standard inversion procedure yields the converse identities. Then, (7.53) implies affine relations of the λ_j to the Stieltjes cumulants g_n^c with $n \leq j$; this, plus the arithmetical formula (3.12) for the cumulants, together account for the quite intricate aspect displayed by the fully developed λ_j [9, Theorem 2].

7.7 Tables of Formulae for the Special Values of \mathcal{Z}

Note. The singular structure in s of the meromorphic functions $\mathcal{Z}(s|t)$ is also explicit: their only pole, at $s = 1$, is simple and of residue $(-\frac{1}{2})$, cf. (7.26).

7.7.1 Function of First Kind for General t

Table 7.1 Special values of $\mathcal{Z}(s|t)$ (upper part: rational, lower part: transcendental, cf. Sect. 2.6.3 and Tables 2.1 and 3.1); see also (7.29)–(7.40). Notation: see (1.3), (1.7), (1.9), (3.24); n is an integer

s	$\mathcal{Z}(s t) = \sum_{\rho} (\frac{1}{2} + t - \rho)^{-s}$
$-n \leq 0$	$\frac{2^n}{n+1} B_{n+1}(\frac{1}{4} + \frac{1}{2}t) + (t + \frac{1}{2})^n + (t - \frac{1}{2})^n \dagger$
0	$\frac{1}{2}(t + \frac{7}{2})$
0 (s -derivative)	$\mathcal{Z}'(0 t) = -\frac{1}{2}(\log 2\pi)t + \frac{1}{4}(\log 8\pi) - \log \Xi(\frac{1}{2} + t)$
$+1$ (finite part)	$\text{FP}_{s=1} \mathcal{Z}(s t) = \frac{1}{2} \log 2\pi + (\log \Xi)'(\frac{1}{2} + t)$
$+n \geq 1$	$\frac{(-1)^{n-1}}{(n-1)!} (\log \Xi)^{(n)}(\frac{1}{2} + t)$

[†] With $(t - t_0)^0 \stackrel{\text{def}}{=} 1$ for $t = t_0$ (continuity in t is imperative)

Table 7.2 Special values of $\mathcal{Z}(s|-t)$ in terms of those at $(+t)$ (reflecting the central symmetry of the Functional Equation). Notation as in Table 7.1

s	$\mathcal{Z}(s -t)$
$-n \leq 0$	$(-1)^n \mathcal{Z}(-n t) + \frac{B_{n+1}(\frac{1}{2} - t)}{n+1}$
0	$\mathcal{Z}(0 t) - t$
0 (s -derivative)	$\mathcal{Z}'(0 -t) = \mathcal{Z}'(0 t) + (\log 2\pi)t$
$+1$ (finite part)	$\text{FP}_{s=1} \mathcal{Z}(s -t) = -\text{FP}_{s=1} \mathcal{Z}(s t) + \log 2\pi$
$+n \geq 1$	$(-1)^n \mathcal{Z}(+n t)$

7.7.2 Function of First Kind at $t = 0$ and $\frac{1}{2}$

Table 7.3 Special values of the function $\mathcal{Z}(s|t)$ at $t = 0$ (the confluent case, Sect. 7.6.1). Notation: see (1.4)–(1.7); n is an integer

s	$\mathcal{Z}_0(s) \equiv \sum_{\rho} (\rho - \frac{1}{2})^{-s}$	$[t = 0]$
$-n \leq 0$	$\begin{cases} \text{even} & 2^{-n+1}(1 - \frac{1}{8}E_n) \\ \text{odd} & -\frac{1}{2}(1-2^{-n})\frac{B_{n+1}}{n+1} \end{cases}$	
0	7/4	
0 (derivative)	$\mathcal{Z}'_0(0) = \log [2^{11/4}\pi^{1/2}\Gamma(\frac{1}{4})^{-1} \zeta(\frac{1}{2}) ^{-1}]$	
+1 (finite part)	$\text{FP}_{s=1} \mathcal{Z}_0(s) = \frac{1}{2} \log 2\pi$	
$+n \geq 1$	$\begin{cases} \text{odd} & 2^{n+1} - \frac{1}{2}[(2^n - 1)\zeta(n) + 2^n\beta(n)] - \frac{0^*}{(n-1)!} \\ \text{even} & \frac{(\log \zeta)^{(n)}(\frac{1}{2})^\dagger}{(n-1)!} \end{cases}$	

* This amounts to the formulae (7.49) yielding $(\log|\zeta|)^{(n)}(\frac{1}{2})$ for n odd
 † Here $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$, while $\beta(n)$ (Sect. 3.5) and $(\log|\zeta|)^{(n)}(\frac{1}{2})$ remain elusive

Table 7.4 As Table 7.3, but at $t = \frac{1}{2}$ (Sect. 7.6.2)

s	$\mathcal{Z}_*(s) \equiv \sum_{\rho} \rho^{-s}$	$[t = \frac{1}{2}]$
$-n < 0$	$1 - (2^n - 1)\frac{B_{n+1}}{n+1}$	
0	2	
0 (derivative)	$\mathcal{Z}'_*(0) = \frac{1}{2} \log 2$	
+1 (finite part)	$\text{FP}_{s=1} \mathcal{Z}_*(s) = 1 - \frac{1}{2} \log 2 + \frac{1}{2}\gamma$	
+1	$1 - \frac{1}{2} \log 4\pi + \frac{1}{2}\gamma$	
$+n > 1$	$1 - (1-2^{-n})\zeta(n) + \frac{g_n^c}{(n-1)!} \equiv 1 - (-1)^n 2^{-n}\zeta(n) - \frac{(\log \zeta)^{(n)}(0)^\dagger}{(n-1)!}$	

† For n even, $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$. For the Stieltjes cumulants g_n^c , see (3.15)

Chapter 8

The Family of the Second Kind $\{\mathcal{Z}(\sigma | t)\}$

From Sect. 5.2 (and [106], with different notation), we recall the definition

$$\mathcal{Z}(\sigma | t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}, \quad \operatorname{Re} \sigma > \frac{1}{2}, \quad (8.1)$$

valid for $t \in \Omega_2 \stackrel{\text{def}}{=} \{t \in \mathbb{C} \mid t \pm i\tau_k \notin \pm i\mathbb{R}_- \ (\forall k)\}$, and these shorthand names at the two points $t \in \Omega_2$ of special interest:

$$\mathcal{Z}_0(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma | 0) \equiv (2 \cos \pi \sigma)^{-1} \mathcal{Z}_0(2\sigma), \quad \mathcal{Z}_*(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma | \frac{1}{2}), \quad (8.2)$$

where the identity at $t = 0$ repeats the *confluence relation* (5.8).

This second family embodies the central symmetry of the zeros (i.e., the Functional Equation) at once through $\mathcal{Z}(\sigma | t) \equiv \mathcal{Z}(\sigma | -t)$, but other properties prove less readily accessible than in the previous case.

That first family \mathcal{Z} benefited from a “magic formula,” the extension (7.4) of the Explicit Formula (6.22) to general t , which sufficed to display all explicit properties of \mathcal{Z} by inspection. For sure, a formula parallel to (7.4) can be worked out for the family \mathcal{Z} (exercise!): for $\frac{1}{2} < \operatorname{Re} s < 1$, $|t| < \tau_1$, $t \notin (-\infty, \frac{1}{2}]$,

$$\begin{aligned} \mathcal{Z}(\sigma | t) &\equiv (t^2 - \frac{1}{4})^{-\sigma} + W[h] + \frac{\sin \pi \sigma}{\pi} \int_0^{\infty} \frac{\zeta'}{\zeta} (\frac{1}{2} + t + y) [y(2t + y)]^{-\sigma} dy, \\ W[h] &= \frac{1}{4\pi} \int_{-\infty}^{\infty} (\tau^2 + t^2)^{-\sigma} [\psi(\frac{1}{4} + \frac{1}{2}i\tau) - \log \pi] d\tau; \end{aligned} \quad (8.3)$$

this extends the Explicit Formula (6.25) beyond $\operatorname{Re} t > \frac{1}{2}$ all right, so it *should* yield the meromorphic continuation in σ and explicit properties for the family \mathcal{Z} unrestrictedly, by analogy with the family \mathcal{Z} ; but that task now calls for the continuation in σ of two intractable integrals in (8.3), vs. that of only $W[h]$ in (6.25), and only $\mathcal{J}(s | t)$ in (7.4) for the family \mathcal{Z} .

In contrast, for the second family \mathcal{Z} , several natural approaches gave partial or incomplete results, and only a specific combination gradually proved

satisfactory: first, describe the confluent case \mathcal{Z}_0 completely by transfer from \mathcal{Z}_0 ; next, use an expansion around $t = 0$ to get all the general- t rational formulae; and last, obtain the transcendental special values explicitly from a symmetric zeta-regularized factorization of $\Xi(\frac{1}{2} + t)$. (Thus, our initial results [106] were neither as complete, nor as fully reduced as they are now.)

8.1 The Confluent Case $\mathcal{Z}(\sigma | t = 0) \equiv \mathcal{Z}_0(\sigma)$

We then begin by using the confluence identity from (8.2), to transfer all previous results about $\mathcal{Z}_0(s)$ onto $\mathcal{Z}_0(\sigma)$ straightforwardly.

First, that identity makes $\mathcal{Z}_0(\sigma)$ *meromorphic in all of \mathbb{C} with a double pole at $\sigma = \frac{1}{2}$ and simple poles at $\sigma = \frac{1}{2} - m$ ($m = 1, 2, \dots$), with the principal parts*

$$\begin{aligned} \mathcal{Z}_0\left(\frac{1}{2} + \varepsilon\right) &= -\frac{1}{4\pi} \operatorname{Res}_{s=1} \mathcal{Z}_0(s) \varepsilon^{-2} - \frac{1}{2\pi} \operatorname{FP}_{s=1} \mathcal{Z}_0(s) \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0} \\ &= \frac{1}{8\pi} \varepsilon^{-2} - \frac{\log 2\pi}{4\pi} \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0}; \\ \mathcal{Z}_0\left(\frac{1}{2} - m + \varepsilon\right) &= \frac{(-1)^{m+1}}{2\pi} \mathcal{Z}_0(1-2m) \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0} \quad \text{for } m = 1, 2, \dots; \end{aligned} \quad (8.4)$$

((8.4) follows from (7.26) and (7.50)). So the residues are, by Table 7.3,

$$\mathcal{R}_m \stackrel{\text{def}}{=} \operatorname{Res}_{\sigma=\frac{1}{2}-m} \mathcal{Z}_0(\sigma) = \begin{cases} -\frac{1}{2\pi} \operatorname{FP}_{s=1} \mathcal{Z}_0(s) = -\frac{\log 2\pi}{4\pi}, & m = 0, \\ \frac{(-1)^m}{8\pi m} (1 - 2^{1-2m}) B_{2m}, & m = 1, 2, \dots \end{cases} \quad (8.5)$$

(all of them are negative). Concerning the residue \mathcal{R}_0 at the double pole: since $\mathcal{Z}_0(1) = 0$ by (7.42), then $-2\pi\mathcal{R}_0 = \operatorname{FP}_{s=1} \mathcal{Z}_0(s)$ also equals δ_1 , the discrepancy (7.40) affecting the family \mathcal{Z} at $s = 1$; so, $\delta_1 \neq 0$ relates to the presence of this nonzero residue \mathcal{R}_0 in the double pole $\sigma = \frac{1}{2}$ of \mathcal{Z}_0 .

Then, the confluence identity in (8.2) implies an *integral representation* for $\mathcal{Z}_0(\sigma)$ [106, (72)], just by specializing (7.4) to $t = \pm i0$:

$$\mathcal{Z}_0(\sigma) = \frac{-\mathbf{Z}(2\sigma | 0) + 2^{2\sigma} e^{\mp 2\pi i\sigma}}{2 \cos \pi\sigma} + \frac{\sin \pi\sigma}{\pi} \int_0^{+e^{\pm i\pi}} \zeta' \left(\frac{1}{2} + y\right) y^{-2\sigma} dy; \quad (8.6)$$

a real variant (among others [106, (73)–(74)]) is the half-sum of these:

$$\mathcal{Z}_0(\sigma) = \frac{-\mathbf{Z}(2\sigma | 0) + 2^{2\sigma} \cos 2\pi\sigma}{2 \cos \pi\sigma} + \frac{\sin \pi\sigma}{\pi} \operatorname{PV} \int_0^{+\infty} t^{-2\sigma} \zeta' \left(\frac{1}{2} + t\right) dt \quad (8.7)$$

(PV = principal value); both integrals converge in the half-plane $\{\operatorname{Re} \sigma < \frac{1}{2}\}$.

Finally, the confluence identity delivers all the *special values* of $\mathcal{Z}_0(\sigma)$, as

$$\mathcal{Z}_0(m) = \frac{1}{2}(-1)^m \mathcal{Z}_0(2m) \quad (\forall m \in \mathbb{Z}), \quad (8.8)$$

$$\mathcal{Z}'_0(0) = \mathcal{Z}'_0(0) \quad (\approx 0.811817944); \quad (8.9)$$

the fully explicit values then closely follow Table 7.3, see Table 8.3 below.

Remark. Although points $\sigma = \frac{1}{2} + m$, $m = 1, 2, \dots$ also map to integer arguments $s = 2\sigma$ for \mathcal{Z}_0 , these are now odd, so that in the confluence identity, poles of $(2 \cos \pi\sigma)^{-1}$ get cancelled by zeros of $\mathcal{Z}_0(2\sigma)$ at those points. Hence $\mathcal{Z}_0(\sigma)$ proves holomorphic for $\text{Re } \sigma > \frac{1}{2}$ as it should, but it gains no computable special values at $\sigma = \frac{1}{2} + m$ in the process.

Now our main challenge is to treat general values $t \neq 0$ of the parameter.

8.2 Meromorphic Continuation in σ for General t

Our key idea for the meromorphic continuation of $\mathcal{Z}(\sigma | t)$ is that its underlying sequence $\{x_k\} = \{\tau_k^2 + t^2\} = \{\rho(1 - \rho) + t^2 - \frac{1}{4}\}$ (at fixed t), of zeta function $\mathcal{Z}(\sigma | \{x_k\}) = \mathcal{Z}(\sigma | t)$, is again *eligible* in the sense of Sect. 2.2. Only the order μ_0 will change: throughout this chapter,

$$\mu_0 = \frac{1}{2}. \quad (8.10)$$

Indeed the first condition, convergence of $\mathcal{Z}(\sigma | t)$ if and only if $\text{Re } \sigma > \frac{1}{2}$, is equivalent to the property (4.7), and as such was part of the definition (8.1).

Now to get hold of an explicit Delta-type function, we specialize to $t^2 = \frac{1}{4}$, because the symmetric Hadamard product formula (4.16) for Ξ is tied to the specific sequence $\{u_k = \tau_k^2 + \frac{1}{4}\}$. Then,

- The Delta function Δ_0 is Ξ itself but as a function of $u \stackrel{\text{def}}{=} x(x-1)$, or

$$\Delta_0(u) = \prod_{k=1}^{\infty} \left[1 + \frac{u}{u_k} \right] \equiv \Xi(x), \quad u \stackrel{\text{def}}{=} x(x-1); \quad (8.11)$$

- By (3.24) and $\log \zeta(x) = O(x^{-\infty})$ ($|\arg x| < \pi/2 - \varepsilon$), the large- u expansion of $\log \Xi$ is simply the large- x expansion of $-\log[\mathbf{G}(x)/(x-1)]$,

$$\log \Xi \sim \frac{x}{2} \left(\log \frac{x}{2\pi} - 1 \right) + \log u - \frac{1}{2} \log \frac{x}{4\pi} + \sum_{m \geq 1} \frac{B_{2m} 2^{2m-1}}{2m(2m-1)} x^{1-2m} \quad (8.12)$$

but with x substituted by the positive asymptotic branch $x(u)$, namely

$$x(u) = \frac{1}{2} + \sqrt{u + \frac{1}{4}} \sim u^{1/2} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2^{-2n} \Gamma(3/2)}{n! \Gamma(-n + 3/2)} u^{1/2-n}, \quad (8.13)$$

and the result fulfills the second eligibility condition (2.13),

$$\log \Delta_0(u) \sim \sum_{n=0}^{\infty} [\tilde{a}_{\frac{1-n}{2}} \log u + a_{\frac{1-n}{2}}] u^{\frac{1-n}{2}} \quad (u \rightarrow \infty, |\arg u| < \pi - \varepsilon) \quad (8.14)$$

$$= \frac{1}{4} u^{1/2} \log u - \frac{1}{2} (\log 2\pi + 1) u^{1/2} + \frac{7}{8} \log u + \frac{1}{4} \log 8\pi + o(1). \quad (8.15)$$

The sequence of exponents is $\{\mu_n = \frac{1-n}{2}\}_{n=0,1,\dots}$, and the coefficients are algebraically computable to any order in principle. Still, a reduced general formula for a_{μ_n} looks inaccessible this way; only the \tilde{a}_{μ_n} , which all (aside from \tilde{a}_0) arise from the single substitution of (8.13) into the leading prefactor $\frac{1}{2}x$ of $\log x$ ($\sim \frac{1}{2} \log u$) in (8.12), are readily expressible, in particular

$$\tilde{a}_{-1-m} \equiv 0, \quad m = 0, 1, 2, \dots \quad (8.16)$$

Exactly the same qualitative structure arises for all shifts t^2 (except $t = 0$ which is more regular, see previous section): see [106, Sects. 3.2 and 4].

Then, it automatically follows from the general results (2.18)–(2.21) in Sect. 2.3 that $\mathcal{Z}(\sigma | t)$ is meromorphic in the whole σ -plane, with *double* poles at *all half-integers* of the form $\sigma = \frac{1}{2} - m$ ($m \in \mathbb{N}$) when $t \neq 0$; in contrast, all the *integer* σ are regular points, by (2.19) and (8.16) for $\sigma < 0$, (2.21) for $\sigma = 0$, (8.1) for $\sigma > 0$. The principal parts at the poles, and the values $\mathcal{Z}(-m | t)$ for $m \in \mathbb{N}$, have to be algebraically computable polynomials of t according to Sect. 2.6.3, but by and large they look intractable here. For this reason, we will devise another scheme for practical computations.

But before that, to make a parallel with Sect. 7.2.2 and to prepare for Sect. 8.4 below, we conclude by giving the zeta-regularized form of $\Delta_0(v - \frac{1}{4})$, which we will denote $\mathcal{D}(v)$ ($v \equiv t^2$). When $\mu_0 < 1$ like here, this calculation is even simpler than when $\mu_0 = 1$ (not to mention $\mu_0 > 1$). The only banned terms in asymptotic expansions when $\mu_0 < 1$ are the *additive constants*, hence a mere glance at the large- u expansion (8.15) of $\log \Delta_0 = \log \Xi$ imposes

$$\mathcal{D}(t^2) \stackrel{\text{def}}{=} e^{-\mathcal{Z}'(0|t)} \equiv (8\pi)^{-1/4} \Xi(\frac{1}{2} + t), \quad (8.17)$$

instead of $\mathcal{D}(\frac{1}{2} + t)$ as in (7.14). The main point here is the change of basic variable from t to $v = t^2$, which does not leave zeta-regularization invariant; unlike $\mathcal{D}(\frac{1}{2} + t)$, the new zeta-regularization of $\Xi(\frac{1}{2} + t)$ is explicitly symmetric (under $t \leftrightarrow -t$). The corresponding factorization of $\zeta(x)$ is [108]

$$\zeta(\frac{1}{2} + t) = (2\pi)^{t/2} \frac{1}{t - \frac{1}{2}} \mathbf{D}(\frac{1}{2} + t) \mathcal{D}(t^2) \quad (8.18)$$

instead of (7.13), and it is inspired by an earlier factorization for hyperbolic *Selberg zeta functions* over spectral determinants [104] (see (B.18)). The prefactor $(2\pi)^{t/2}$ specifically corrects for the discrepancy between

the zeta-regularizations with respect to t (as in \mathbf{D}) and t^2 (in \mathcal{D}). (The zeta-regularized form (8.17) is also closest to the determinant of Riemann zeros introduced by Berry and Keating for other purposes [6].)

8.3 Algebraic Results for $\mathcal{Z}(\sigma | t)$ at General t

Our best tool here is a straightforward expansion of $\mathcal{Z}(\sigma | t)$ around $t = 0$ [106], convergent for $|t| < \tau_1$ (followed by analytical continuation in t , in case we *had to* go to larger t):

$$\mathcal{Z}(\sigma | t) = \sum_{k=1}^{\infty} (\tau_k^2)^{-\sigma} \left(1 + \frac{t^2}{\tau_k^2}\right)^{-\sigma} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma-\ell)\ell!} \mathcal{Z}_0(\sigma + \ell) t^{2\ell}. \quad (8.19)$$

This series is infinite, reflecting the generally transcendental nature of the dependence on the shift parameter (Sect. 2.4), nevertheless it discloses some information in closed form through finite partial sums.

Thus, the right-hand side fully reflects the σ -plane meromorphic structure that we just found for $\mathcal{Z}(\sigma | t)$: at fixed $t \neq 0$, the poles are all the points $\sigma = \frac{1}{2} - m$ ($m \in \mathbb{N}$), as when $t = 0$, but now *all of them are double*, not just $\sigma = \frac{1}{2}$. We are not sure whether (8.19) alone could *prove* all the meromorphic properties of $\mathcal{Z}(\sigma | t)$ as we did above; but it does make algebraically computable quantities much more explicit, as follows.

- First, the principal part of $\mathcal{Z}(\sigma | t)$ at each pole only depends on a *finite* stretch of the series (8.19),

$$\mathcal{Z}\left(\frac{1}{2} - m + \varepsilon | t\right) = \sum_{\ell=0}^m \frac{\Gamma(\frac{1}{2} + m - \varepsilon)}{\Gamma(\frac{1}{2} + m - \ell - \varepsilon)\ell!} \mathcal{Z}_0\left(\frac{1}{2} - m + \ell + \varepsilon\right) t^{2\ell} + \mathcal{O}(1)_{\varepsilon \rightarrow 0}, \quad (8.20)$$

hence it can be extracted in closed form; specifically, upon importing the polar structure of $\mathcal{Z}_0(\sigma)$ from (8.4)–(8.5), we obtain

$$\mathcal{Z}\left(\frac{1}{2} - m + \varepsilon | t\right) = \frac{1}{8\pi} \frac{\Gamma(\frac{1}{2} + m)}{\Gamma(\frac{1}{2}) m!} t^{2m} \varepsilon^{-2} + \mathcal{R}_m(t) \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0}, \quad (8.21)$$

$$\mathcal{R}_m(t) = \Gamma\left(\frac{1}{2} + m\right) \left[-\frac{\sum_{j=1}^m \frac{1}{2^{j-1}}}{4\pi \Gamma(\frac{1}{2}) m!} t^{2m} + \sum_{j=0}^m \frac{1}{\Gamma(\frac{1}{2} + j) (m-j)!} \mathcal{R}_j t^{2(m-j)} \right]. \quad (8.22)$$

So, all the poles $\sigma = \frac{1}{2} - m$ of $\mathcal{Z}(\sigma | t)$ inherit their polar parts of order 2 from the single double pole $\sigma = \frac{1}{2}$ of $\mathcal{Z}_0(\sigma)$; whereas each residue $\mathcal{R}_m(t)$ gets contributions from all the residues \mathcal{R}_j of $\mathcal{Z}_0(\sigma)$ for $j \leq m$, themselves known

from (8.5). At $m = 0$, (8.21) boils down to (8.4) for all t , hence the double pole $\sigma = \frac{1}{2}$ has a principal part *wholly independent of t* , given by (8.4).

• Next, at the negative or null integers σ , the series (8.19) terminates:

$$\mathcal{Z}(-m | t) \equiv \sum_{\ell=0}^m \binom{m}{\ell} \mathcal{Z}_0(-m + \ell) t^{2\ell} \quad (m \in \mathbb{N}), \quad (8.23)$$

with the values $\mathcal{Z}_0(-m + \ell)$ known from (8.8) and Table 7.3; explicit rational special values $\mathcal{Z}(-m | t)$ thereby result, the leading one being

$$\mathcal{Z}(0 | t) = 7/8 \quad (\text{independent of } t). \quad (8.24)$$

In the end, all the principal parts of $\mathcal{Z}(\sigma | t)$, and the special values $\mathcal{Z}(-m | t)$, $m \in \mathbb{N}$ (Table 8.1, upper part), are *explicit polynomials* in t^2 . (Those which are t -independent, namely the principal part (8.4) at the leading pole $\sigma = \frac{1}{2}$ and the leading special value $\mathcal{Z}(0 | t) \equiv 7/8$, also *shape the counting function $N(T)$* described in Sect. 4.5, but this will only become clear in the more general setting of Chap. 10, (10.40).)

8.4 Transcendental Values of $\mathcal{Z}(\sigma | t)$ for General t

These special values of $\mathcal{Z}(\sigma | t)$ now very simply follow from the general formalism of Chap. 2, specialized to $\mu_0 = \frac{1}{2}$.

First, invoking the new zeta-regularized product $\mathcal{D}(t^2)$ given by (8.17),

$$\mathcal{Z}'(0 | t) \equiv -\log \mathcal{D}(t^2) = \frac{1}{4} \log 8\pi - \log \Xi\left(\frac{1}{2} \pm t\right), \quad (8.25)$$

hence $\mathcal{Z}'(0 | t)$ also expresses in terms of $\log |\zeta|(\frac{1}{2} \pm t)$.

Then, (2.27) now applied to $\Delta = \Xi$ with $v = t^2$ as variable in place of x , $Z = \mathcal{Z}$, and $\mu_0 = \frac{1}{2}$, yields

$$\mathcal{Z}(m | t) = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^m}{d(t^2)^m} \log \Xi\left(\frac{1}{2} \pm t\right), \quad m = 1, 2, \dots \quad (8.26)$$

These special values of $\mathcal{Z}(\sigma | t)$ for general t are listed in Table 8.1, lower part.

8.5 Imprints of the Central Symmetry $\rho \longleftrightarrow (1 - \rho)$

The special values $\mathcal{Z}(m | t)$, that we just evaluated, can be further reduced using the central symmetry. This is mainly useful for the transcendental values ($m = 1, 2, \dots$) since the others are fairly explicit already.

In the formula (8.26) for $\mathcal{Z}(m|t)$, by the chain rule, the right-hand side must express as a linear combination of derivatives $(\log \Xi)^{(n)}(x)$ at $x = \frac{1}{2} \pm t$, which are essentially values $\mathcal{Z}(n|t)$ by (7.39). This change of variables $t^2 \mapsto t$ can be carried out explicitly by exploiting the central symmetry, as follows.

Setting $\rho \equiv \frac{1}{2} + i\tau$ throughout here, we start from the identity (cf. (4.12))

$$\left(1 - \frac{s}{t + \frac{1}{2} - \rho}\right) \left(1 - \frac{s}{t - \frac{1}{2} + \rho}\right) \equiv 1 - \frac{s(2t - s)}{\tau^2 + t^2}, \quad (8.27)$$

then expand the logarithms of both sides around $s = 0$, and identify like powers of s to get a triangular sequence of linear identities,

$$\frac{(t + \frac{1}{2} - \rho)^{-n} + (t - \frac{1}{2} + \rho)^{-n}}{n} \equiv \sum_{n/2 \leq m \leq n} (-1)^{n-m} \binom{m}{n-m} (2t)^{2m-n} \frac{(\tau^2 + t^2)^{-m}}{m} \quad \text{for } n = 1, 2, \dots \quad (8.28)$$

Now, in one direction, summing this over (half) the Riemann zeros $\{\rho\}$ yields *the special values of the first kind in terms of those of the second kind*:

$$\frac{\mathcal{Z}(n|t)}{n} \equiv \sum_{n/2 \leq m \leq n} (-1)^{n-m} \binom{m}{n-m} (2t)^{2m-n} \frac{\mathcal{Z}(m|t)}{m} \quad \text{for } n = 1, 2, \dots; \quad (8.29)$$

this clearly extends (7.42) for n odd, and (8.8) for n even, away from $t = 0$.

The other way round, the triangular relations (8.28) necessarily invert (save at the singular point $t = 0$) into likewise triangular relations:

$$(\tau^2 + t^2)^{-m} \equiv \sum_{n=1}^m V_{m,n}(t) [(t + \frac{1}{2} - \rho)^{-n} + (t - \frac{1}{2} + \rho)^{-n}] \quad \text{for } m = 1, 2, \dots \quad (8.30)$$

First, this specifies $V_{m,n}(t)$: it has to be the coefficient of $(t + \frac{1}{2} - \rho)^{-n}$ in the Laurent series of $(\tau^2 + t^2)^{-m} = (t + \frac{1}{2} - \rho)^{-m} [2t - (t + \frac{1}{2} - \rho)]^{-m}$ in powers of $(t + \frac{1}{2} - \rho)$, whence $V_{m,n}(t) = \binom{2m-n-1}{m-1} (2t)^{-2m+n}$. Then, upon summing (8.30) over half the Riemann zeros $\{\rho\}$, we obtain the end result

$$\left. \begin{array}{l} \mathcal{Z}(m|t) \equiv \sum_{n=1}^m \binom{2m-n-1}{m-1} (2t)^{-2m+n} \mathcal{Z}(n|t) \quad \text{if } t \neq 0, \\ \text{vs} \\ \mathcal{Z}(m|0) \equiv \frac{1}{2} (-1)^m \mathcal{Z}(2m|0) \quad \text{(by (8.8) at } t = 0, \end{array} \right\} \quad \text{for } m = 1, 2, \dots \quad (8.31)$$

The pair of mutually inverse relations (8.29) and (8.31) extend to general t -values previous results written only for $t = \frac{1}{2}$ [80, 106].

Another useful approach to (8.31) is to view it as the general identity

$$(-1)^m \frac{d^m}{d(t^2)^m} = \sum_{n=1}^m \frac{(2m-n-1)!}{(m-n)!(n-1)!} (2t)^{-(2m-n)} (-1)^n \frac{d^n}{dt^n}, \quad (8.32)$$

applied to $\log \Xi(\frac{1}{2} + t)$. Then it is enough to verify (8.32) upon holomorphic functions, i.e., to write a Cauchy integral in the t^2 -plane for the left-hand side, and to evaluate it by the residue calculus in the t variable.

Finally, Table 8.2 lists all the special values of \mathcal{Z} for general t expressed in terms of those of \mathcal{Z} , which were simpler to obtain. We included such a formula for $\mathcal{Z}(-m | t)$, to be derived as an exercise! (See footnote in Table!)

8.6 Results for $\mathcal{Z}(\sigma | t)$ at $t = 0$ and $t = \frac{1}{2}$

For the family \mathcal{Z} , these parameter locations present no further special features; we review their end results here mainly for convenience and to parallel Sect. 7.6 for the family \mathcal{Z} . Some numerical data also appear in Appendix A.

8.6.1 The Function $\mathcal{Z}_0(s)$ (the Confluent Case $t = 0$)

Here this case came first, in Sect. 8.1. We repeat its special values in Table 8.3, to match Table 7.3 for \mathcal{Z}_0 . (Cf. [106]; we found no earlier occurrences.)

8.6.2 The Function $\mathcal{Z}_*(s)$ (the Case $t = \frac{1}{2}$)

This other remarkable case now simply follows by specializing the above general- t results; still, we collate the final formulae here and in Table 8.4, to match Sect. 7.6.2 and Table 7.4 for \mathcal{Z}_* .

Principal parts of $\mathcal{Z}_*(\sigma)$ at $\sigma = \frac{1}{2} - m$, $m \in \mathbb{N}$ (reducing to (8.4) at $m = 0$, just as for any t):

$$\begin{aligned} \mathcal{Z}_*(\tfrac{1}{2} - m + \varepsilon) &= \frac{2^{-2m}}{8\pi} \frac{\Gamma(\frac{1}{2} + m)}{\Gamma(\frac{1}{2}) m!} \varepsilon^{-2} + \mathcal{R}_m(\tfrac{1}{2}) \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0}, \quad (8.33) \\ \mathcal{R}_m(\tfrac{1}{2}) &= \frac{\Gamma(\frac{1}{2} + m)}{2^{2m}} \left[-\frac{\sum_{j=1}^m \frac{1}{2^{j-1}}}{4\pi \Gamma(\frac{1}{2}) m!} + \sum_{j=0}^m \frac{2^{2j}}{\Gamma(\frac{1}{2} + j) (m-j)!} \mathcal{R}_j \right], \end{aligned}$$

where \mathcal{R}_j are the residues of $\mathcal{Z}_0(\sigma)$ at $\sigma = \frac{1}{2} - j$, as given by (8.5).

The special values of $\mathcal{Z}_*(\sigma)$ are listed in Table 8.4. In particular,

- $\mathcal{Z}'_*(0) = \frac{1}{4} \log 8\pi$ (≈ 0.806042857) is simpler than $\mathcal{Z}'_0(0) = \mathcal{Z}'_*(0) - \log \Xi(\frac{1}{2})$ ($= \mathcal{Z}'_0(0) \approx 0.811817944$); cf. (8.9) and the numerical remark (A.2)
- $\mathcal{Z}_*(1) = \mathcal{Z}'_*(1) = -B$ (≈ 0.0230957090), cf. (4.14)
- More generally, $\mathcal{Z}_*(m)$ ($m \geq 1$) expresses linearly in terms of the $\mathcal{Z}'_*(n)$ with $1 \leq n \leq m$ [80]: now this result is simply (8.31), written for $t = \frac{1}{2}$. The triangular linear relations just mentioned, plus the similar ones (7.56) between the $\mathcal{Z}'_*(n)$ and the Keiper–Li coefficients λ_j , together imply that such linear relations directly connect the $\mathcal{Z}_*(m)$ and the λ_j . The latter relations are best found also directly [110], from the generating function [64, 73]

$$\frac{d}{dz} \log \Xi\left(\frac{1}{1-z}\right) \equiv \sum_{j=1}^{\infty} \lambda_j z^{j-1}. \quad (8.34)$$

Now the symmetric infinite product formula (4.16) for Ξ implies

$$\log \Xi\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty} \log \left[1 + \frac{z}{(1-z)^2 u_k} \right] = - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{z^m}{(1-z)^{2m}} \mathcal{Z}_*(m);$$

then, expanding $(1-z)^{-2m}$ by the generalized binomial formula, reordering in powers of z and substituting the output into (8.34), we get

$$\lambda_j = -j \sum_{m=1}^j \frac{(-1)^m}{m} \binom{m+j-1}{2m-1} \mathcal{Z}_*(m), \quad j = 1, 2, \dots \quad (8.35)$$

This, by the same technique we used to invert (8.29) into (8.31), admits the closed-form inverse (exercise!)

$$\mathcal{Z}_*(m) = \sum_{j=1}^m (-1)^{j+1} \binom{2m}{m-j} \lambda_j, \quad m = 1, 2, \dots \quad (8.36)$$

The other triangular linear relations of Table 8.2 (our “Bologna formula”), when specialized to $t = \frac{1}{2}$, yield the $\mathcal{Z}_*(-m)$ of Table 8.4 in terms of the $\mathcal{Z}'_*(-n)$ of Table 7.4, which means an infinite sequence of curious affine identities between the Euler and Bernoulli numbers,

$$2^{-2m} \sum_{j=0}^m \binom{m}{j} (-1)^j E_{2j} = \quad (8.37)$$

$$-2(-1)^m \sum_{\frac{m+1}{2} \leq k \leq m} \binom{m}{2k-1-m} \frac{2^{2k-1}-1}{k} B_{2k} + \frac{m!^2}{(2m+1)!} \quad \text{for } m = 1, 2, \dots$$

8.7 Tables of Formulae for the Special Values of \mathcal{Z}

Note. The singular structures in σ of the meromorphic functions $\mathcal{Z}(\sigma | t)$ (all their poles and principal parts) are also explicitly computable, see Sect. 8.3.

8.7.1 Function of Second Kind for General t

Table 8.1 Special values of $\mathcal{Z}(\sigma | t)$ (upper part: rational, lower part: transcendental, cf. Sect. 2.6.3 and Table 2.1). Notation: see (1.5), (1.9), (3.24); m is an integer. Compare with Table 7.1

σ	$\mathcal{Z}(\sigma t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}$
$-m \leq 0$	$(t^2 - \frac{1}{4})^m - 2^{-2m-3} \sum_{j=0}^m \binom{m}{j} (-1)^j E_{2j} (2t)^{2(m-j)}$ †
0	7/8
0 (σ -derivative)	$\mathcal{Z}'(0 t) = \frac{1}{4} \log 8\pi - \log \Xi(\frac{1}{2} \pm t)$
$+m \geq 1$	$\frac{(-1)^{m-1}}{(m-1)!} \frac{d^m}{d(t^2)^m} \log \Xi(\frac{1}{2} \pm t)$

† With $(t - t_0)^0 \stackrel{\text{def}}{=} 1$ for $t = t_0$ (continuity in t is imperative)

Table 8.2 Special values $\mathcal{Z}(m | t)$ expressed in terms of the special values $\mathcal{Z}(n | t)$ of first kind (given by Tables 7.1 for general t , 7.3 for $t = 0$, 7.4 for $t = \frac{1}{2}$); m, n are integers

σ	$\mathcal{Z}(\sigma t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}$
$-m \leq 0$	$\frac{1}{2} \sum_{n=m}^{2m} \binom{m}{n-m} (-1)^{n-m} (2t)^{2m-n} \mathcal{Z}(-n t) - \frac{2^{2m-2} m!^2}{(2m+1)!} t^{2m+1}$ †
0	$\frac{1}{2} \mathcal{Z}(0 t) - \frac{1}{4} t$
0 (σ -/ s -derivatives)	$\mathcal{Z}'_{(\sigma)}(0 t) = \mathcal{Z}'_{(s)}(0 t) + (\frac{1}{2} \log 2\pi) t$
$+m \geq 1$	$\begin{cases} \sum_{n=1}^m \binom{2m-n-1}{m-1} (2t)^{-2m+n} \mathcal{Z}(n t) & (t \neq 0) \\ \frac{1}{2} (-1)^m \mathcal{Z}(2m 0) & (t = 0) \end{cases}$

† “Bologna formula” (the inspiration for this last missing formula came to the author in the deeply serene atmosphere of a cloister in the Santo Stefano basilica in Bologna)

8.7.2 Function of Second Kind at $t = 0$ and $\frac{1}{2}$

Table 8.3 Special values of the function $\mathcal{Z}(\sigma | t)$ at $t = 0$ (the confluent case, Sect. 8.1; also compare with Table 7.3). Notation: see (1.5), (3.27); m is an integer

σ	$\mathcal{Z}_0(\sigma) = \sum_{k=1}^{\infty} \tau_k^{-2\sigma}$	$[t = 0]$
$-m \leq 0$	$(-1)^m 2^{-2m} (1 - \frac{1}{8} E_{2m})$	
0	$7/8$	
0 (derivative)	$\mathcal{Z}'_0(0) = \log [2^{11/4} \pi^{1/2} \Gamma(\frac{1}{4})^{-1} \zeta(\frac{1}{2}) ^{-1}]$	
$+m \geq 1$	$(-1)^m \left\{ 2^{2m} - \frac{1}{4} [(2^{2m} - 1) \zeta(2m) + 2^{2m} \beta(2m)] - \frac{(\log \zeta)^{(2m)} (\frac{1}{2})}{2(2m-1)!} \right\}^\dagger$	

$^\dagger \zeta(2m) \equiv (2\pi)^{2m} \frac{|B_{2m}|}{2(2m)!}$, while $\beta(2m)$ (Sect. 3.5) and $(\log |\zeta|)^{(2m)} (\frac{1}{2})$ remain elusive

Table 8.4 As Table 8.3, but at $t = \frac{1}{2}$ (Sect. 8.6.2)

σ	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} (\tau_k^2 + \frac{1}{4})^{-\sigma} \equiv \sum_{k=1}^{\infty} u_k^{-\sigma}$	$[t = \frac{1}{2}]$
$-m < 0$	$-2^{-2m-3} \sum_{j=0}^m \binom{m}{j} (-1)^j E_{2j}$	
0	$7/8$	
0 (derivative)	$\mathcal{Z}'_*(0) = \frac{1}{4} \log 8\pi$	
$+m \geq 1$	$\sum_{n=1}^m \binom{2m-n-1}{m-1} \mathcal{L}_*(n)^\dagger$	

† For the values $\mathcal{L}_*(n)$, see Table 7.4

Chapter 9

The Family of the Third Kind $\{\mathfrak{Z}(s | \tau)\}$

From Sect. 5.3, we recall the definition [51, 106]

$$\mathfrak{Z}(s | \tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \text{Re } s > 1. \quad (9.1)$$

For general $\tau (\neq 0)$, $\mathfrak{Z}(s | \tau)$ is built on a *one-sided* set of zeros, say $(\frac{1}{2} + i\tau_k)$ only; hence it seems harder to tackle than the other two families, and the earlier studies (just quoted) have revealed more singular features indeed.

The confluence point (5.12) is again the exception: $\mathfrak{Z}(s | 0) \equiv \mathcal{Z}_0(\frac{1}{2}s)$, so it is fully covered by Sect. 8.1. We then turn to general $\tau \neq 0$.

On the rational side, a possible approach is to expand $\mathfrak{Z}(s | \tau)$ in $\{|\tau| < \tau_1\}$, similarly to (8.19) for $\mathcal{Z}(\sigma | t)$, see also [51]:

$$\mathfrak{Z}(s | \tau) = \sum_{k=0}^{\infty} \tau_k^{-s} \left(1 + \frac{\tau}{\tau_k}\right)^{-s} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-s)}{\Gamma(1-s-\ell)} \ell! \mathcal{Z}_0\left(\frac{1}{2}(s+\ell)\right) \tau^\ell. \quad (9.2)$$

This yields poles for $\mathfrak{Z}(s | \tau)$ located at $s \in 1 - \mathbb{N}$, with explicitly computable principal parts; when $\tau \neq 0$, all the nonpositive integers are truly poles, ruling out the possibility of any special values on the rational side.

On the transcendental side, we would need to extend the analysis of Sect. 7.4, resp. 8.4 from the infinite products \mathcal{D} , resp. \mathcal{D} to products over *upper half-plane* zeros only. Now, such products are definitely more ill-behaved [51]; we lack analogs of (7.13), resp. (8.18) for them, and we thus cannot reach special-value formulae for $\mathfrak{Z}(n | \tau)$ at $n = 2, 3, \dots$ either.

However, the family \mathfrak{Z} can be described quite explicitly otherwise, simply by reducing it to the family of the first kind \mathcal{Z} built over all zeros taken symmetrically, paradoxical as this may seem. Still for $|\tau| < \tau_1$, the obvious linear relation

$$\mathcal{Z}(s | t) = e^{i\pi s/2} \mathfrak{Z}(s | it) + e^{-i\pi s/2} \mathfrak{Z}(s | -it) \quad (9.3)$$

yields the simple inversion formula [83, Sect. 8] (which we missed earlier):

$$\mathfrak{Z}(s|\tau) = \frac{1}{2i \sin \pi s} [e^{i\pi s/2} \mathcal{Z}(s|-i\tau) - e^{-i\pi s/2} \mathcal{Z}(s|+i\tau)]. \quad (9.4)$$

Thus, \mathfrak{Z} mainly extracts the “symmetry-breaking component” of \mathcal{Z} : i.e., the factor in brackets in (9.4), purely created by the multivaluedness of the function $z \mapsto z^{-s}$ for general s (giving zero for $s = 1, 2, \dots$ by Table 7.2; however, the use of (9.4) at $s = 1$ demands the same special care as before). The more singular nature of \mathfrak{Z} shows up in the extra denominator ($\sin \pi s$).

Since \mathcal{Z} is the most accessible family, \mathfrak{Z} is best described by (9.4) alone (or its analytical continuation, if we had to leave the disk $\{|\tau| < \tau_1\}$). Equation (9.4) at once implies that \mathfrak{Z} is meromorphic in the whole s -plane with poles at integers, and details readily follow just by expanding (9.4) around each integer:

- The point $s = 1$ specially comes out as a *double pole*:

$$\mathfrak{Z}(1 + \varepsilon|\tau) = \frac{1}{2\pi} \varepsilon^{-2} - \frac{\log 2\pi}{2\pi} \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0}; \quad (9.5)$$

it has the principal part of $\mathcal{Z}_0(\frac{1}{2}s) \equiv \mathfrak{Z}(s|0)$, independently of τ (cf. (8.4))

- Each point $s = 1 - n$, $n = 1, 2, \dots$ is a *simple pole* as soon as $\tau \neq 0$:

$$\mathfrak{Z}(1 - n + \varepsilon|\tau) = -\frac{(-i)^n}{2\pi} [\mathcal{Z}(1 - n|i\tau) + (-1)^n \mathcal{Z}(1 - n|-i\tau)] \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0}; \quad (9.6)$$

by Table 7.2, the residue at $s = 1 - n$ boils down to

$$-\frac{(-i)^n B_n(\frac{1}{2} + i\tau)}{2\pi n} = \frac{1}{2\pi n} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n}{2k} (1 - 2^{1-2k}) B_{2k} \tau^{n-2k} \quad (9.7)$$

- Exceptionally at $s = 0$ where \mathcal{Z}' is known in addition to \mathcal{Z} , the expansion attains the next order, namely the *finite part*:

$$\mathfrak{Z}(\varepsilon|\tau) = -\frac{\tau}{2\pi} \varepsilon^{-1} + \left(\frac{7}{8} + \frac{\log 2\pi}{2\pi} \tau\right) + o(1)_{\varepsilon \rightarrow 0} \quad (9.8)$$

- Finally, around $s = +2, +3, \dots$, the expansion yields the finite values

$$\mathfrak{Z}(n|\tau) = \frac{i^n}{2} \left[(\mp 1)^n \mathcal{Z}(n|\pm i\tau) + \frac{i}{\pi} \left(\mathcal{Z}'(n|i\tau) - (-1)^n \mathcal{Z}'(n|-i\tau) \right) \right]. \quad (9.9)$$

So indeed when $\tau \neq 0$, \mathfrak{Z} seems to lack special values as explicit as those of \mathcal{Z} and \mathcal{Z} : all the points $s = 1 - n$ ($n \in \mathbb{N}$) are singular, whereas the regular values at $n = 2, 3, \dots$ appear to involve derivative-values $\mathcal{Z}'(n|\pm i\tau)$ which

are unlikely to be computable in closed form (already the $\zeta'(n)$ aren't!). While we cannot guarantee that the specific combination of $\mathcal{Z}'(n|\pm i\tau)$ entering (9.9) isn't computable, any alternative evidence of simpler forms for $\mathfrak{Z}(n|\tau)$ is also absent. Only a sequence of binary relations can be made explicit, by specializing the identity (9.3) to $s \in \mathbb{N}^*$,

$$i^n \mathfrak{Z}(n|it) + i^{-n} \mathfrak{Z}(n|-it) \equiv \mathcal{Z}(n|t), \quad n = 1, 2, \dots, \quad (9.10)$$

straightforwardly except for $n = 1$: then, *finite parts* are to be taken *on the left-hand side only*, and $\mathcal{Z}(1|t)$ from (7.39) used on the right-hand side.

In conclusion, the family \mathfrak{Z} is more singular than the other two and seemingly devoid of computable special values (hence we provide no Tables for it); on the other hand, it can be wholly reduced to the well-understood first family \mathcal{Z} through the identity (9.4).

Chapter 10

Extension to Other Zeta- and L -Functions

In this chapter, based on [107], we extend the treatment and results of the three former chapters to the setting where the Riemann zeros are replaced by the (nontrivial) zeros of a more general zeta or L -function $L(x)$, still fairly similar to $\zeta(x)$. We will use the terms: *primary function* for this function $L(x)$ which supplies the new zeros in this *extended* setting, and *Riemann case* for the former setting where $\zeta(x)$ itself was the fixed primary function.

The interest of this extension is twofold. First, it broadens the previous results in a natural way: with little work, we will accommodate three distinct kinds of superzeta functions as before, but now over the (nontrivial) zeros of numerous primary functions. Second, it sheds some further light on the results for $\zeta(x)$ itself: the origin of many final values in the previous chapters will be clarified through their more abstract specifications. For instance, various special values like $\text{FP}_{s=1} \mathcal{Z}_0(s)$, $\mathcal{Z}_0(0), \dots$ will now explicitly stem from the Stirling expansion (10.12) for the trivial factor $\mathbf{G}(x)$, $x \rightarrow +\infty$.

We could of course have taken this general path from the very beginning, relegating the Riemann case to the status of just a special instance. However, this would have gone against our plan to provide a most concrete and readily usable handbook. The Riemann zeta function may be a special case in a crowd, but it is important enough to deserve an autonomous presentation.

Earlier *explicit* descriptions of such extended superzeta functions, i.e., over zeros other than Riemann's, hardly exist in the literature. We set apart the case of Selberg zeta functions: their zeros correspond to eigenvalues of hyperbolic Laplacians, and zeta functions over them have been analyzed by spectral methods [15, 16, 47, 91, 100, 105]: in the cocompact case, they are indeed examples of *spectral* (Minakshisundaram–Pleijel or generalized) zeta functions; we revisit them in Appendix B. Otherwise, only Dedekind zeta functions got some mention as primary functions [47, 51, 52, 66]; already if we turn to L -series, then only the Cramér functions (5.14) over their zeros were ever considered [29, 54, 59, 62], without relating them to any superzeta functions (apart from one short note [94] on an Explicit Formula like (6.22) but actually for $\mathcal{Z}(s | \frac{3}{2})$ over the zeros of specific Dirichlet L -functions).

Most of the notation can be taken over from the former chapters; simply, all objects will now be understood to depend on the chosen primary function L , implicitly as a rule. At the same time, the Riemann zeta function as such will continue to appear in some results.

10.1 Admissible Primary Functions $L(x)$

For the sake of definiteness, we choose here to stay fairly close to the Riemann case ($L(x) = \zeta(x)$): basically, we want to retain a reflexive functional equation $\Xi(x) = \Xi(1-x)$ for a completed function built similarly to (3.24). We therefore accept primary functions $L(x)$ such that:

- (a) $L(x)$ is real, and meromorphic in \mathbb{C} with at most one simple pole, $x = 1$:

$$\text{if } q \stackrel{\text{def}}{=} \text{the order of the pole } x = 1, \text{ then } q = 0 \text{ or } 1; \quad (10.1)$$

- (b) $L(x) \neq 0$ in $\{\text{Re } x > 1\}$, and $L(x) \rightarrow 1$ for $\text{Re } x \rightarrow +\infty$ with

$$(\log L)^{(n)}(x) = o(x^{-N}) \quad (\forall n, N \in \mathbb{N}); \quad (10.2)$$

- (c) a completed L -function and a functional equation exist, similar to the Riemann case:

$$\Xi(x) \equiv \Xi(1-x), \quad \Xi(x) \stackrel{\text{def}}{=} \mathbf{G}^{-1}(x)(x-1)^q L(x), \quad (10.3)$$

where both $\Xi(x)$ and $\mathbf{G}(x)$ are *real entire functions of order* $\mu_0 = 1$, and

- (c₁) $\mathbf{G}(x)$, the “trivial factor,” is an explicitly known finite product of inverse-Gamma (and simpler) factors, with all its zeros x_k located on the negative real axis $\{x \leq 0\}$ (they form the “trivial zeros” of $L(x)$);
- (c₂) the zeros of $\Xi(x)$ lie in the strip $\{0 < \text{Re } x < 1\}$; since they come in symmetrical pairs as in the Riemann case, we still label them

$$\{\rho = \frac{1}{2} \pm i\tau_k\}_{k=1,2,\dots}, \quad \text{with } \text{Re } \tau_k > 0 \text{ and non-decreasing;}$$

they are the “nontrivial zeros” of $L(x)$; we *exclude the exceptional occurrence of any of these on the real line*, for simplicity. (But see Appendix B for such a case.)

Note: all zeros, x_k or ρ , are counted with multiplicities if any.

- (d)
$$\lim_{x \rightarrow 1} (x-1)^q L(x) (= \mathbf{G}(1)\Xi(1)) > 0 \quad (10.4)$$

(but dropping the normalization $\Xi(1) = \Xi(0) = 1$, too awkward to implement in general); then, Stieltjes cumulants can extend from (3.15) in the Riemann case, according to the general definition

$$\log [(x - 1)^q L(x)] \equiv \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} g_n^c (x - 1)^n : \quad (10.5)$$

now $g_0^c \neq 0$ may occur, while g_1^c extends Euler's constant γ from (3.16).

As already said, the symbols from the previous chapters (Ξ , \mathbf{G} , ρ , g_n^c, \dots) are consistent with their former uses in the Riemann case, but they now designate objects attached to the changeable primary function L .

Conditions (a)–(d) above are tailored to fit two basic classes which are immediate extensions from the Riemann case, and will be described as final examples: *L-functions of real primitive Dirichlet characters*, and *Dedekind zeta functions* (with $\zeta(x)$ as a special case of the latter). Our assumptions somewhat resemble the axioms of the *Selberg class* [98] but are more restrictive on some concrete details; on the other hand these can undoubtedly be refitted to different needs. For instance, zeta functions over zeros of Selberg zeta functions for compact hyperbolic surfaces have yielded results comparable to the Riemann case earlier [15, 16, 66, 91, 100], while they correspond to $\mu_0 = 2$ (\mathbf{G} contains a Barnes G -function), and $q = -1$ (those Selberg zeta functions have a simple zero at $x = 1$, and possibly others on $(0, 1)$). To illustrate the flexibility of our approach, and since results of this class have interested physicists as well, we treat this *Selberg case* in Appendix B. Other extensions are equally conceivable (e.g., to Hecke L -functions, as achieved upon their Cramér functions [54]).

We now begin with the general results for the superzeta families, attainable for unspecified admissible primary functions L .

10.2 The Three Superzeta Families

We can then define the same three parametric zeta functions over the nontrivial zeros $\{\rho\}$ of a general primary function L satisfying the above conditions (a)–(d), just as in the Riemann case to which we refer for details (Chap. 5):

$$\mathcal{Z}(s | t) = \sum_{\rho} \left(\frac{1}{2} + t - \rho\right)^{-s} \equiv \sum_{\rho} \left(\rho + t - \frac{1}{2}\right)^{-s}, \quad \text{Re } s > 1, \quad (10.6)$$

$$\mathcal{Z}(\sigma | t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}, \quad \text{Re } \sigma > \frac{1}{2}, \quad (10.7)$$

$$\mathfrak{Z}(s | \tau) = \sum_{k=1}^{\infty} (\tau_k + \tau)^{-s}, \quad \text{Re } s > 1; \quad (10.8)$$

we also keep the shorthand names for the two points t of special interest:

$$\mathcal{Z}_0(s) = \mathcal{Z}(s|0), \quad \mathcal{Z}_*(s) = \mathcal{Z}(s|\frac{1}{2}); \quad (10.9)$$

$$\mathcal{Z}_0(\sigma) = \mathcal{Z}(\sigma|0) \equiv (2 \cos \pi\sigma)^{-1} \mathcal{Z}_0(2\sigma), \quad \mathcal{Z}_*(\sigma) = \mathcal{Z}(\sigma|\frac{1}{2}); \quad (10.10)$$

more generally, all other considerations of Chap. 5 remain valid here.

We now describe the explicit results in greater detail family by family. The logic exactly follows that of the previous chapters, so we will mainly restate the formulae that have a significantly different abstract form.

10.3 The First Family $\{\mathcal{Z}\}$

This will extend the treatment of Chap. 7 from the Riemann case.

10.3.1 The Zeta Function $\mathbf{Z}(s|t)$ over the Trivial Zeros

A key role is played by the zeta function wholly analogous to $\mathcal{Z}(s|t)$ but built on the trivial zeros of $L(x)$ (which we call the shadow zeta function of $\mathcal{Z}(s|t)$):

$$\mathbf{Z}(s|t) \stackrel{\text{def}}{=} \sum_k (\frac{1}{2} + t - x_k)^{-s} \quad (\text{Re } s > 1). \quad (10.11)$$

Here this function and its properties should be taken as completely known, just like the factor \mathbf{G} and the trivial zeros themselves. In our later explicit examples, $\mathbf{Z}(s|t)$ will be expressible in terms of the Hurwitz zeta function (3.33).

We now specialize the results of Chap. 2 first to the trivial factor \mathbf{G} . By assumption (c₁), the Stirling formula (3.51) will produce a large- x expansion for $\log \mathbf{G}(x)$ with $\mu_0 = 1$, which we treat as known and reorganize according to allowed/banned terms, as

$$-\log \mathbf{G}(\frac{1}{2} + t) \sim \tilde{\mathbf{a}}_1 t (\log t - 1) + \mathbf{b}_1 t + \tilde{\mathbf{a}}_0 \log t + \mathbf{b}_0 + \sum_{n=1}^{\infty} \mathbf{a}_{-n} t^{-n}; \quad (10.12)$$

this expansion also governs $[\log \Xi(\frac{1}{2} + t) - q \log(t - \frac{1}{2})]$, by (10.2) and (10.3). Equation (2.50) then yields the zeta-regularized forms for \mathbf{G} and Ξ as

$$\mathbf{D}(\frac{1}{2} + t) \stackrel{\text{def}}{=} e^{-\mathbf{Z}'(0|t)} \equiv e^{+\mathbf{b}_1 t + \mathbf{b}_0} \mathbf{G}(\frac{1}{2} + t), \quad (10.13)$$

$$\mathcal{D}(\frac{1}{2} + t) \stackrel{\text{def}}{=} e^{-\mathcal{Z}'(0|t)} \equiv e^{-\mathbf{b}_1 t - \mathbf{b}_0} \Xi(\frac{1}{2} + t), \quad (10.14)$$

which in turn entail this zeta-regularized decomposition of $L(x)$:

$$(x - 1)^q L(x) \equiv \mathbf{D}(x) \mathcal{D}(x). \tag{10.15}$$

Then, using (10.13), the specific translation of (2.53)–(2.56) with $\mu_0 = 1$ is

$$\text{FP}_{s=1} \mathbf{Z}(s | t) \equiv (\log \mathbf{G})'(\tfrac{1}{2} + t) + \mathbf{b}_1, \tag{10.16}$$

$$\mathbf{Z}(m | t) \equiv \frac{(-1)^{m-1}}{(m-1)!} (\log \mathbf{G})^{(m)}(\tfrac{1}{2} + t) \quad \text{for } m = 2, 3, \dots \tag{10.17}$$

The shifted large- y expansion of $-\log \mathbf{D}(\frac{1}{2} + t + y)$ is then deduced as in Sect. 2.6.1, in the form $\tilde{\mathbf{a}}_1(t) y(\log y - 1) + \tilde{\mathbf{a}}_0(t) \log y + \sum_{n=1}^{\infty} \mathbf{a}_{-n}(t) y^{-n}$: all coefficients are computable polynomials in t and encode algebraic properties of $\mathbf{Z}(s | t)$, as explained in Sect. 2.4. In more explicit terms:

- $\mathbf{Z}(s | t)$ extends to a meromorphic function in the whole s -plane, with

$$\text{the single pole } s = 1, \text{ simple, of residue } \tilde{\mathbf{a}}_1 \quad (\text{independent of } t); \tag{10.18}$$

- The values $\mathbf{Z}(-n | t)$, $n \in \mathbb{N}$ are given by *closed polynomial formulae*,

$$\mathbf{Z}(-n | t) = -\frac{\tilde{\mathbf{a}}_1}{n+1} t^{n+1} - \tilde{\mathbf{a}}_0 t^n + n \sum_{j=1}^n (-1)^j \binom{n-1}{j-1} \mathbf{a}_{-j} t^{n-j}, \tag{10.19}$$

e.g., $\mathbf{Z}(0 | t) = -\tilde{\mathbf{a}}_0(t) = -\tilde{\mathbf{a}}_1 t - \tilde{\mathbf{a}}_0$.

More interestingly, \mathbf{D} , \mathbf{Z} , \mathbf{G} can be replaced by the (less explicit) \mathcal{D} , \mathcal{Z} , Ξ respectively, and similar results will arise, as described next.

10.3.2 The Basic Analytical Continuation Formula for \mathcal{Z}

As an extension from (7.4)–(7.5) (Riemann case), $\mathcal{Z}(s | t)$ admits the following integral representation, valid in the half-plane $\{\text{Re } s < 1\}$ and for any eligible value of the parameter t avoiding the cut $(-\infty, +\frac{1}{2}]$:

$$\mathcal{Z}(s | t) = -\mathbf{Z}(s | t) + q(t - \tfrac{1}{2})^{-s} + \frac{\sin \pi s}{\pi} \mathcal{J}(s | t), \tag{10.20}$$

$$\mathcal{J}(s | t) \stackrel{\text{def}}{=} \int_0^{\infty} \frac{L'}{L}(\tfrac{1}{2} + t + y) y^{-s} dy \quad (\text{Re } s < 1). \tag{10.21}$$

The real forms (7.6)–(7.7) and the description of the poles of \mathcal{J} extend likewise, just replacing ζ by L and $\frac{1}{t - \frac{1}{2} + y}$ by $\frac{q}{t - \frac{1}{2} + y}$ everywhere.

As in Sect. 7.3 for the Riemann case:

- It follows that $\mathcal{Z}(s|t)$ is meromorphic in the whole s -plane with the same polar structure as $-\mathbf{Z}(s|t)$, which now means that

$$\mathcal{Z}(s|t) \text{ has the single pole } s = 1, \text{ simple, of residue } -\tilde{\mathbf{a}}_1; \quad (10.22)$$

- If $L(x)$ admits an Euler product (like our concrete examples (10.49) and (10.72)), then the substitution of its logarithmic derivative into (10.21), followed by integration term by term, yields an *asymptotic* ($s \rightarrow -\infty$) expansion for $\mathcal{J}(s|t)$, and thereby for $\mathcal{Z}(s|t)$ (cf. (7.27)–(7.28) for the Riemann case).

10.3.3 Special Values of $\mathcal{Z}(s|t)$ for General t

As in Sect. 7.4 for the Riemann case:

- Almost all the *special values* of $\mathcal{Z}(s|t)$ (at integer s) are explicitly readable off (10.20): as in (7.29)–(7.38) before, we get rational values for $s \in -\mathbb{N}$, transcendental ones for $s = 2, 3, \dots$, plus

$$\begin{aligned} \mathcal{Z}'(0|t) &= -\mathbf{Z}'(0|t) - q \log(t - \tfrac{1}{2}) + \mathcal{J}(0|t) \\ &= \mathbf{b}_1 t + \mathbf{b}_0 + \log \mathbf{G}(\tfrac{1}{2} + t) - \log [(t - \tfrac{1}{2})^q L(\tfrac{1}{2} + t)] \end{aligned} \quad (10.23)$$

$$\begin{aligned} \text{FP}_{s=1} \mathcal{Z}(s|t) &= -\text{FP}_{s=1} \mathbf{Z}(s|t) + \frac{q}{t - \frac{1}{2}} - \text{Res}_{s=1} \mathcal{J}(s|t), \\ &= -\mathbf{b}_1 - (\log \mathbf{G})'(\tfrac{1}{2} + t) + \left[\frac{q}{t - \frac{1}{2}} + \frac{L'}{L}(\tfrac{1}{2} + t) \right]. \end{aligned} \quad (10.24)$$

- However, the values $\mathcal{Z}(n|t)$ for $n \in \mathbb{N}^*$, now including $n = 1$, emerge more directly by proceeding in full analogy with (7.39) (Riemann case):

$$\mathcal{Z}(n|t) = \frac{(-1)^{n-1}}{(n-1)!} (\log \Xi)^{(n)}(\tfrac{1}{2} + t) \quad (n = 1, 2, \dots) \quad (10.25)$$

$$\begin{aligned} &= -\mathbf{Z}(n|t) + \left[\frac{q}{(t - \frac{1}{2})^n} + \frac{(-1)^{n-1}}{(n-1)!} (\log |L|)^{(n)}(\tfrac{1}{2} + t) \right] \\ &\quad (n = 2, 3, \dots). \end{aligned} \quad (10.26)$$

Since $\mathbf{Z}(1|t)$ is infinite, (10.26) cannot hold for $n = 1$ but a substitute formula exists: i.e., the subtraction of (10.25) at $n = 1$ from (10.24) yields a t -independent *anomaly*, or discrepancy formula extending (7.40):

$$\text{FP}_{s=1} \mathcal{Z}(s|t) - \mathcal{Z}(1|t) = (\log [\mathcal{D}/\Xi])'(\tfrac{1}{2} + t) = -\mathbf{b}_1 \quad (\text{constant}). \quad (10.27)$$

Table 10.1 [107] recapitulates the special values obtained for $\mathcal{Z}(s|t)$ at general t , extending Table 7.1 from the Riemann case.

Remark. All identities stemming purely from the central symmetry of the zeros $\rho \longleftrightarrow (1 - \rho)$, like $\mathcal{Z}(n|t) = (-1)^n \mathcal{Z}(n|-t)$ ($n = 1, 2, \dots$) and the sum rules (7.46) (or (7.43)), carry over unchanged. However, the rest of Table 7.2 needs rewriting (exercise!).

10.3.4 Special Values of $\mathcal{Z}(s|t)$ at $t = 0$ and $\frac{1}{2}$

Under our assumption $\{\rho\} \cap \mathbb{R} = \emptyset$, $\mathcal{Z}(s|t)$ is regular on the real t -axis; then, further simplifications occur at the particular parameter locations $t = 0$ and $\frac{1}{2}$.

- For $t = 0$, (10.25) reduces to

$$\mathcal{Z}_0(n) \equiv 0 \quad \text{for all } n \geq 1 \text{ odd}; \quad (10.28)$$

in combination with (10.26) and (10.24), that amounts to explicit formulae for the primary function L itself:

$$(\log |L|)^{(n)}\left(\frac{1}{2}\right) = (\log \mathbf{G})^{(n)}\left(\frac{1}{2}\right) + 2^n q (n-1)! \quad \text{for all } n \geq 1 \text{ odd} \quad (10.29)$$

(also directly implied by the functional equation (10.3); they extend (7.48) from the Riemann case). By (10.27) at $t = 0$, the case $n = 1$ moreover implies

$$\text{FP}_{s=1} \mathcal{Z}_0(s) = -\mathbf{b}_1. \quad (10.30)$$

- For $t = \frac{1}{2}$, the formulae (10.23)–(10.26) bring in the Taylor series (10.5), to yield

$$\begin{aligned} \mathcal{Z}'_*(0) &= -\mathbf{Z}'(0|\tfrac{1}{2}) + g_0^c; \\ \mathcal{Z}'_*(1) &= -(\log \mathbf{G})'(1) + g_1^c, \\ \mathcal{Z}'_*(n) &= -\mathbf{Z}(n|\tfrac{1}{2}) + g_n^c / (n-1)! \quad (n = 2, 3, \dots). \end{aligned} \quad (10.31)$$

The purely combinatorial relations (7.56) between the $\mathcal{Z}'_*(n)$ and Keiper–Li coefficients λ_j remain unchanged; the latter now refer to the zeros of the primary function L , like the rest.

It is hard to achieve further progress while keeping the primary function L and its “accessories” (\mathbf{G} , \mathbf{Z} , etc.) completely unspecified. Case by case, on the other hand, $\mathbf{Z}(s|t)$ can be made more explicit at $t = 0$ or $\frac{1}{2}$, just as in the Riemann case. Thus, in our later examples for $L(x)$ (L -functions of real primitive Dirichlet characters; Dedekind zeta functions), both $\mathbf{Z}(s|0)$ and

$\mathbf{Z}(s | \frac{1}{2})$ will reduce to combinations of the two fixed Dirichlet series $\zeta(s)$ and $\beta(s)$. The resulting fully reduced special values of $\mathcal{Z}_0(s)$ and $\mathcal{Z}_*(s)$ will be displayed in Tables 10.3–10.6 [107], which conclude the chapter.

10.4 The Second Family $\{\mathcal{Z}\}$

This will extend the treatment of Chap. 8 from the Riemann case.

10.4.1 The Confluent Case $\mathcal{Z}(\sigma | t = 0) \equiv \mathcal{Z}_0(\sigma)$

The confluence identity is unchanged from (8.2): $\mathcal{Z}_0(\sigma) \equiv (2 \cos \pi \sigma)^{-1} \mathcal{Z}_0(2\sigma)$. Therefore:

- The function \mathcal{Z}_0 retains the same abstract singular structure as in Sect. 8.1; the explicit principal part formulae (8.4)–(8.5) simply extend to

$$\mathcal{Z}_0(\tfrac{1}{2} + \varepsilon) = \frac{\tilde{\mathbf{a}}_1}{4\pi} \varepsilon^{-2} + \frac{\mathbf{b}_1}{2\pi} \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0}; \quad (10.32)$$

$$\mathcal{Z}_0(\tfrac{1}{2} - m + \varepsilon) = \mathcal{R}_m \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0} \quad \text{for } m = 1, 2, \dots, \quad (10.33)$$

$$\mathcal{R}_m = \begin{cases} -\frac{1}{2\pi} \text{FP}_{s=1} \mathcal{Z}_0(s) = \frac{\mathbf{b}_1}{2\pi}, & m = 0, \\ \frac{(-1)^m}{2\pi} [\mathbf{Z}(1 - 2m | 0) + q 2^{1-2m}], & m = 1, 2, \dots; \end{cases} \quad (10.34)$$

- The special values of \mathcal{Z}_0 obey the same relations (8.8)–(8.9) as before:

$$\mathcal{Z}_0(m) = \tfrac{1}{2}(-1)^m \mathcal{Z}_0(2m) \quad (\forall m \in \mathbb{Z}), \quad \mathcal{Z}'_0(0) = \mathcal{Z}'_0(0). \quad (10.35)$$

Both (10.34) and (10.35) now refer to the \mathcal{Z} -values of Table 10.1 (at $t = 0$).

10.4.2 Algebraic Results for $\mathcal{Z}(\sigma | t)$ at General t

For the algebraically computable formulae, all the abstract results from Sect. 8.3 carry over unchanged. Concretely, the principal parts (8.21) extend to

$$\mathcal{Z}(\tfrac{1}{2} - m + \varepsilon | t) = \frac{\tilde{\mathbf{a}}_1}{4\pi} \frac{\Gamma(\frac{1}{2} + m)}{\Gamma(\frac{1}{2}) m!} t^{2m} \varepsilon^{-2} + \mathcal{R}_m(t) \varepsilon^{-1} + \mathcal{O}(1)_{\varepsilon \rightarrow 0}, \quad (10.36)$$

$$\mathcal{R}_m(t) = \Gamma\left(\frac{1}{2} + m\right) \left[-\frac{\tilde{\mathbf{a}}_1 \sum_{j=1}^m \frac{1}{2j-1}}{2\pi \Gamma\left(\frac{1}{2}\right) m!} t^{2m} + \sum_{j=0}^m \frac{1}{\Gamma\left(\frac{1}{2} + j\right) (m-j)!} \mathcal{R}_j t^{2(m-j)} \right] \quad (10.37)$$

where the $t = 0$ residues \mathcal{R}_j are now read from (10.34). And the rational special values still obey (8.23), or equivalently

$$\mathcal{Z}(-m | t) \equiv \frac{1}{2} \sum_{\ell=0}^m (-1)^{m-\ell} \binom{m}{\ell} \mathcal{Z}_0(2(-m + \ell)) t^{2\ell} \quad (m \in \mathbb{N}), \quad (10.38)$$

with the values $\mathcal{Z}_0(2(-m + \ell))$ now taken from Table 10.1 (at $t = 0$). So, all the polar terms of $\mathcal{Z}(\sigma | t)$, still of order 2, and the special values $\mathcal{Z}(-m | t)$ ($m \in \mathbb{N}$) are *computable polynomials* in t^2 , as in the Riemann case.

The principal part at the leading pole $\sigma = \frac{1}{2}$, given by (10.32), and the leading special value $\mathcal{Z}(0 | t) = \frac{1}{2}(q + \tilde{\mathbf{a}}_0)$, remain *independent of t* as before. Those two invariants amount to three constants $\tilde{\mathbf{a}}_1$, \mathbf{b}_1 , and $\frac{1}{2}(q + \tilde{\mathbf{a}}_0)$, which also have *another embodiment*. As with (4.29) for the Riemann case, let

$$\overline{N}(T) \stackrel{\text{def}}{=} \pi^{-1} \operatorname{Im} \log [(x-1)^q \mathbf{G}^{-1}(x)]_{x=\frac{1}{2}+iT} \quad (10.39)$$

be the contribution from the trivial factors of Ξ to the counting function $N(T)$ (for its zeros $\{\rho\}$). Then the generalized Stirling formula (10.12) readily yields

$$\overline{N}(T) = \frac{\tilde{\mathbf{a}}_1}{\pi} T(\log T - 1) + \frac{\mathbf{b}_1}{\pi} T + \frac{1}{2}(q + \tilde{\mathbf{a}}_0) + O\left(\frac{1}{T}\right), \quad (10.40)$$

which is built from those *same three constants*, cf. (4.31) for the Riemann case. We may recall that \mathbf{b}_1 also governs the discrepancy at $n = 1$ in (10.27).

Remark: in both our later concrete examples, the full counting function itself obeys $N(T) = \overline{N}(T) + O(\log T)$, extending the Riemann–von Mangoldt formula (4.26): this is proved, e.g., in [26, Sect. 16] for Dirichlet L -functions and in [69, Satz 173 p. 89] for Dedekind zeta functions.

10.4.3 Transcendental Values of $\mathcal{Z}(\sigma | t)$ at General t

As in the Riemann case (Sect. 8.4), these values most readily emerge from a variant of the factorization (10.15), using the alternative zeta-regularized factor

$$\mathcal{D}(t^2) \stackrel{\text{def}}{=} e^{-\mathcal{Z}'(0|t)} \quad (10.41)$$

instead of $\mathcal{D}(x)$. The main point here is the replacement of $x = \frac{1}{2} + t$ by $v = t^2$ as basic variable: this zeta-regularization of $\Xi(x)$ preserves the central symmetry ($x \longleftrightarrow 1 - x$).

Rewritten in the variable $v \rightarrow +\infty$, the generalized Stirling expansion (10.12) for $[\log \Xi(x) - q \log(x - 1)]$ becomes

$$\log \Xi(\sqrt{v} + \frac{1}{2}) \sim \frac{1}{2} \tilde{\mathbf{a}}_1 v^{\frac{1}{2}} \log v + (\mathbf{b}_1 - \tilde{\mathbf{a}}_1) v^{\frac{1}{2}} + \frac{1}{2} (\tilde{\mathbf{a}}_0 + q) \log v + \mathbf{b}_0 [+ O(v^{-\frac{1}{2}})], \quad (10.42)$$

with $\mu_0 = \frac{1}{2}$: the only “banned” terms (cf. (2.47)) are now *constants*, implying

$$\mathcal{D}(t^2) \equiv e^{-\mathbf{b}_0} \Xi(\frac{1}{2} + t) \equiv e^{\mathbf{b}_1 t} \mathcal{D}(\frac{1}{2} + t) \quad (10.43)$$

and the modified decomposition (cf. (8.18) for the Riemann case)

$$(t - \frac{1}{2})^q L(\frac{1}{2} + t) \equiv e^{-\mathbf{b}_1 t} \mathbf{D}(\frac{1}{2} + t) \mathcal{D}(t^2). \quad (10.44)$$

All transcendental special values of $\mathcal{Z}(\sigma | t)$ immediately follow: first, $\mathcal{Z}'(0 | t) \equiv -\log \mathcal{D}(t^2) = \mathbf{b}_0 - \log \Xi(\frac{1}{2} + t)$ which also expresses in terms of $\log |L|(\frac{1}{2} \pm t)$; then (2.56), now applied with $v = t^2$ as variable and $\mu_0 = \frac{1}{2}$, yields the same form (8.26) for $\mathcal{Z}(m | t)$ as before.

The overall resulting special values of \mathcal{Z} form Table 10.2, extending Table 8.1 from the Riemann case. The particular parameter locations $t = 0$ and $\frac{1}{2}$ can be covered by Sect. 10.4.1 and Table 10.2 respectively (mimicking Sect. 8.6.2 from the Riemann case), without need for further Tables.

The various combinatorial linear identities relating special values of \mathcal{Z} at positive integers to those of \mathcal{Z} (Sect. 8.5) or to the Keiper–Li coefficients λ_j ((8.34)–(8.36)) persist identically, with all objects now in their extended meaning (linked to the function L).

10.5 The Third Family $\{\mathfrak{Z}\}$

The abstract treatment of Chap. 9 for the Riemann case extends unchanged. So, the function \mathfrak{Z} keeps the same qualitative meromorphic structure; we just rewrite its polar expansions (9.5)–(9.8) in their abstract extended form:

$$(\text{about } s = 1) \quad \mathfrak{Z}(1 + \varepsilon | \tau) = \frac{\tilde{\mathbf{a}}_1}{\pi} \varepsilon^{-2} + \frac{\mathbf{b}_1}{\pi} \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0} \quad (10.45)$$

($s = 1$ is a double pole with the same principal part as $\mathcal{Z}(\frac{1}{2}s) \equiv \mathfrak{Z}(s | 0)$, fixed for all τ); and, for $n = 1, 2, \dots$ (now using (10.34) for \mathcal{R}_j):

$$\mathfrak{Z}(1 - n + \varepsilon | \tau) = \left[-\frac{\tilde{\mathbf{a}}_1}{\pi n} \tau^n + 2 \sum_{0 < 2m \leq n} \binom{n-1}{2m-1} \mathcal{R}_m \tau^{n-2m} \right] \varepsilon^{-1} + O(1)_{\varepsilon \rightarrow 0}; \quad (10.46)$$

for $n = 1$, the next term is also attained (i.e., the finite part at $s = 0$):

$$3(\varepsilon | \tau) = -\frac{\tilde{\mathbf{a}}_1}{\pi} \tau \varepsilon^{-1} + \left(\frac{1}{2}(\tilde{\mathbf{a}}_0 + q) - \frac{\mathbf{b}_1}{\pi} \tau \right) + o(1)_{\varepsilon \rightarrow 0}. \quad (10.47)$$

10.6 Special Concrete Examples

We finally illustrate the preceding results upon the two classes of primary zeta functions announced in Sect. 10.1. As a rule, it suffices to specialize the general formulae as indicated below. To enhance the practical side of this work, we display the ultimate results, namely the special values of $\mathcal{Z}(s | 0) \equiv \mathcal{Z}_0(s)$ and $\mathcal{Z}(s | \frac{1}{2}) \equiv \mathcal{Z}_*(s)$, in Tables 10.3–10.6 (corresponding to Tables 7.3–7.4 for $L(x) = \zeta(x)$, which fits better here as a special case of Dedekind zeta function). Then, any further results can be readily derived from the general formalism, as in the Riemann case.

At the particular parameter locations $t = 0$ and $\frac{1}{2}$, the relevant values of $\mathbf{Z}(s | t)$ become more explicit, using

$$\begin{aligned} \zeta(s, 1) &\equiv \zeta(s); & \zeta(s, \tfrac{1}{2}) &\equiv (2^s - 1)\zeta(s); \\ 2^{-2s}\zeta(s, \tfrac{1}{2} \mp \tfrac{1}{4}) &\equiv \tfrac{1}{2}[(1 - 2^{-s})\zeta(s) \pm \beta(s)] \end{aligned} \quad (10.48)$$

(cf. (3.27)). Due to this, the two fixed Dirichlet series $\zeta(s)$ and $\beta(s)$ (itself a particular Dirichlet L -function, reviewed in Sect. 3.5) will continue to occur as such in the $t = 0$ and $\frac{1}{2}$ special values, concurrently with the variable primary function $L(x)$ itself.

We now describe those two classes in turn. At this more technical stage, we abandon the idea of being self-contained, and instead, we will selectively quote (without proof) the prerequisites we really need, which are classic but often quite scattered in lengthy treatises. The most explicit properties of the resulting functions $\mathcal{Z}_0(s)$ and $\mathcal{Z}_*(s)$ are tabulated at the end.

A somewhat different third class, where the primary function $L(x)$ is chosen to be a *Selberg zeta function*, is moreover described in Appendix B.

10.6.1 L -Functions of Real Primitive Dirichlet Characters

A *Dirichlet L -function* is associated with a *character* χ of a multiplicative group of integers mod d ($d \in \mathbb{N}^*$ is called the *modulus* or *conductor*), as [14, 26, 33]

$$L_\chi(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \chi(k) k^{-x} \equiv \prod_{\{p\}} (1 - \chi(p) p^{-x})^{-1} \quad (\operatorname{Re} x > 1) \quad (10.49)$$

$$\equiv d^{-x} \sum_{n=1}^d \chi(n) \zeta(x, n/d). \quad (10.50)$$

Such a character is necessarily even or odd; its parity “bit” a is defined by

$$a = 0 \text{ or } 1, \quad \text{according to } \chi(-1) = (-1)^a. \quad (10.51)$$

Their Admissibility

Dirichlet L -functions satisfy conditions (a)–(b) of Sect. 10.1 without further ado (reality apart).

We now restrict to *primitive* characters [26, Chap. 5], with $d > 1$ to exclude the case $\chi \equiv 1$ (for which $L_\chi(x) \equiv \zeta(x)$, a case which will better fit the other class below). Then, $L_\chi(x)$ is *entire* (i.e., $q = 0$), and has a functional equation [26, Chap. 9]:

$$\Xi_\chi(x) \equiv W_\chi \Xi_{\bar{\chi}}(1-x), \quad (10.52)$$

with

$$\Xi_\chi(x) \stackrel{\text{def}}{=} (2\sqrt{\pi})^{-a} (d/\pi)^{x/2} \Gamma(\tfrac{1}{2}(x+a)) L_\chi(x), \quad (10.53)$$

$$W_\chi \stackrel{\text{def}}{=} (-i)^a d^{-1/2} \sum_{n \bmod d} \chi(n) e^{2\pi i n/d}; \quad (10.54)$$

the latter sum, called the *Gaussian sum* for χ , has *modulus* $d^{1/2}$ [26, Chap. 9, (5)], implying $|W_\chi| = 1$. (Note: in (10.53) we chose to insert an entirely optional prefactor $(2\sqrt{\pi})^{-a}$, only to streamline a part of Sect. 10.6.2 later.)

We finally keep the *real* ($\bar{\chi} = \chi$) primitive characters only. These consist of the *Kronecker symbols* for the *quadratic number fields* K with discriminant $d_K \equiv (-1)^a d$ [26, Chap. 5], so these characters are best labeled $\chi_{(-1)^a d}$ (see also Sect. 10.6.2). Their Gaussian sums are *fully* known (= $i^a d^{1/2}$) [49, Sect. 58, Theorem 164], now implying $W_\chi \equiv +1$; it follows that their L -functions (now real) satisfy condition (c) of Sect. 10.1 as well, with

$$q \equiv 0, \quad \mathbf{G}(x) \equiv (2\sqrt{\pi})^a \frac{(\pi/d)^{x/2}}{\Gamma(\tfrac{1}{2}(x+a))}, \quad a = \begin{cases} 1 & \text{for } \chi \text{ odd} \\ 0 & \text{for } \chi \text{ even.} \end{cases} \quad (10.55)$$

Finally, condition (d) with $q = 0$, or $L_\chi(1) > 0$, is ensured by *Dirichlet’s class number formula* giving $L_\chi(1)$ in terms of *invariants of the quadratic field* K and chiefly its *class number* $h > 0$, as [115][26, Sect. 6]

$$L_\chi(1) = \begin{cases} 2\pi h / W \sqrt{d} & (W = \#\{\text{roots of unity}\} < \infty) \quad \text{if } a = 1 \\ 2h \log \varepsilon / \sqrt{d} & (\varepsilon = \text{fundamental unit} > 1) \quad \text{if } a = 0. \end{cases} \quad (10.56)$$

So in the end, L -functions of real primitive Dirichlet characters are admissible primary functions for us. Any choice of such an L -function L_χ as primary function will henceforth be abbreviated “a Dirichlet- L case.”

Results for Superzeta Functions in Dirichlet- L Cases

Based on (10.55), we can specify the other quantities needed in Table 10.1:

- The shadow zeta function (10.11) becomes

$$\mathbf{Z}(s | t) = 2^{-s} \zeta\left(s, \frac{1}{2}\left(\frac{1}{2} + a + t\right)\right) \quad (10.57)$$

- The leading coefficients in the Stirling formula (10.12) are

$$\begin{aligned} \tilde{\mathbf{a}}_1 &= \frac{1}{2}, & \tilde{\mathbf{a}}_0 &= \frac{1}{2}\left(a - \frac{1}{2}\right), \\ \mathbf{b}_1 &= \frac{1}{2} \log [d / (2\pi)], & \mathbf{b}_0 &= \frac{1}{4} \log [(8\pi)^{1-2a} d]. \end{aligned} \quad (10.58)$$

This completes the tool kit needed to handle Dirichlet- L cases for general t .

We then list further results only for the particular cases $t = 0$ and $\frac{1}{2}$, in Tables 10.3 and 10.4 respectively and in the following comments.

- $\mathbf{t} = \mathbf{0}$: the identities (10.29), resulting from $\mathcal{Z}_0(n) \equiv 0$ for odd $n \geq 1$, yield more explicit formulae for the Dirichlet L -function L_χ itself:

$$\begin{aligned} (\log L_\chi)^{(n)}\left(\frac{1}{2}\right) &\equiv -2^{-n} \psi^{(n-1)}\left(\frac{1}{4} + \frac{1}{2}a\right) + \delta_{n,1} \frac{1}{2} \log \frac{\pi}{d} && \text{for } n \geq 1 \text{ odd} \\ &\equiv \begin{cases} \frac{1}{2}(n-1)! [(2^n - 1) \zeta(n) + (1 - 2a) 2^n \beta(n)], & n > 1, \\ \frac{1}{2}\gamma + \frac{1}{4}(1 - 2a) \pi + \frac{1}{2} \log(8\pi/d), & n = 1, \end{cases} \end{aligned} \quad (10.59)$$

in which, n being odd, $\frac{1}{2}(n-1)! 2^n \beta(n)$ reduces to $\frac{1}{4} \pi^n |E_{n-1}|$ by (3.32) while $\zeta(n)$ remains elusive.

Note: for odd $n > 1$, $(\log L_\chi)^{(n)}\left(\frac{1}{2}\right) = L_n^a$ depends on χ solely through its parity bit a , with

$$L_n^0 = (\log |\zeta|)^{(n)}\left(\frac{1}{2}\right) \quad \text{and} \quad L_n^1 = L_n^0 - \frac{1}{2} \pi^n |E_{n-1}| \quad \text{for odd } n > 1, \quad (10.60)$$

cf. (7.49); for more such identities bypassing $\zeta(n)$, see Appendix C.

- $\mathbf{t} = \frac{1}{2}$: by (10.5) with $q = 0$, the lowest generalized Stieltjes cumulants are

$$g_0^c[\chi] \equiv -\log L_\chi(1), \quad g_1^c[\chi] \equiv \frac{L'_\chi}{L_\chi}(1) \quad (10.61)$$

(we restate their χ -dependence from now on, to clarify later formulae). Then, we can evaluate $g_0^c[\chi]$ *always*, but $g_1^c[\chi]$ *only when* $a = 1$ (*odd case*), as follows.

First, the general formula (10.50), together with the special values (3.37), (3.40), (3.41) of the Hurwitz zeta function, plus $\chi(d) = 0$ and $\sum_{n=1}^d \chi(n) = 0$, yield these special values for $L_\chi(x)$:

$$L_\chi(0) = -\frac{1}{d} \sum_{n=1}^{d-1} \chi(n) n \quad (\text{rational}), \quad (10.62)$$

$$L'_\chi(0) = -L_\chi(0) \log d + \sum_{n=1}^{d-1} \chi(n) \log \Gamma(n/d) \quad (\text{transcendental}), \quad (10.63)$$

$$L_\chi(1) = -\frac{1}{d} \sum_{n=1}^{d-1} \chi(n) \psi(n/d) \quad (\text{transcendental}). \quad (10.64)$$

Next, the functional equation (10.3) using (10.55) implies the following.

– When $a = 1$ (the odd- χ case):

$$L_\chi(1) = \pi d^{-1/2} L_\chi(0), \quad \frac{L'_\chi}{L_\chi}(1) = \gamma + \log \frac{2\pi}{d} - \frac{L'_\chi}{L_\chi}(0), \quad (10.65)$$

which, together with (10.62), (10.63) yield an *algebraic* explicit formula for $\pi^{-1} L_\chi(1)$ [26, Chap. 6, (17)] (superseding (10.64)) plus a *transcendental* one for $[L'_\chi/L_\chi](1)$ in terms of Gamma values, overall giving

$$\left. \begin{aligned} g_0^c[\chi] &= -\log L_\chi(1), & L_\chi(1) &= -\frac{\pi}{d^{3/2}} \sum_{n=1}^{d-1} \chi(n) n, \\ g_1^c[\chi] &= \frac{L'_\chi}{L_\chi}(1) = \gamma + \log 2\pi + \frac{\sum_{n=1}^{d-1} \chi(n) \log \Gamma(n/d)}{\frac{1}{d} \sum_{n=1}^{d-1} \chi(n) n} \end{aligned} \right\} \text{ if } a = 1. \quad (10.66)$$

Example: For each of $d = 3$ and 4 (the lowest possible values of d), the real primitive character is unique and odd, given by: $\chi_{-d}(\pm 1 \bmod d) = \pm 1$, else $\chi_{-d}(n) = 0$ (thus, $L_{\chi_{-4}}(x) \equiv \beta(x)$ as in (3.27)); then (10.66) yields

$$\begin{aligned} g_0^c[\chi_{-3}] &= -\log(\pi/3^{3/2}), & g_1^c[\chi_{-3}] &= \log[(2\pi)^4/3^{3/2}] + \gamma - 6 \log \Gamma(\tfrac{1}{3}); \\ g_0^c[\chi_{-4}] &= -\log(\pi/4), & g_1^c[\chi_{-4}] &= \log(4\pi^3) + \gamma - 4 \log \Gamma(\tfrac{1}{4}). \end{aligned} \quad (10.67)$$

– When $a = 0$ (the even- χ case):

$$L_\chi(0) \equiv 0 \quad (\text{the first trivial zero: } \mathbf{G}(0) = 0), \quad (10.68)$$

$$L_\chi(1) = 2d^{-1/2}L'_\chi(0), \quad (10.69)$$

and $L'_\chi(0)$, starting from (10.63), simplifies further through (10.68) and the reflection formula for Γ ; we thus obtain a *transcendental* explicit formula for $L_\chi(1)$ [26, Chap. 6, (18)] (still more elementary than (10.64)), giving

$$g_0^c[\chi] = -\log L_\chi(1), \quad L_\chi(1) = -\frac{1}{d^{1/2}} \sum_{n=1}^{d-1} \chi(n) \log \sin \frac{\pi n}{d} \quad \text{if } a = 0. \quad (10.70)$$

On the other hand, $g_1^c[\chi]$ stays *unspecified* because the functional equation only relates $L'_\chi(1)$ to $L''_\chi(0)$ when χ is even, leaving us no wiser: thus for an even character, $g_1^c[\chi]$ seems to generalize Euler's constant in a nontrivial way (actually related to the *Euler–Kronecker invariant* of the field K , see (10.83) in Sect. 10.6.2).

Example: The lowest modulus for an even real primitive Dirichlet character is $d = 5$, with $\chi_{+5}(\pm 1 \bmod 5) = +1$, $\chi_{+5}(\pm 2 \bmod 5) = -1$, else $\chi_{+5}(n) = 0$; then (10.70) only yields

$$g_0^c[\chi_{+5}] = -\log\left(\frac{2}{\sqrt{5}} \log\left[2 \cos \frac{\pi}{5}\right]\right) = -\log\left(\frac{2}{\sqrt{5}} \log \frac{\sqrt{5}+1}{2}\right). \quad (10.71)$$

Remark. The $g_n^c[\chi]$ for general n also relate, through (10.50), to the Laurent coefficients $\gamma_m(w)$ of the Hurwitz zeta function $\zeta(x, w)$ around $x = 1$ [5, 65, 114]; however, these generalized Stieltjes constants $\gamma_m(w)$ are even more elusive than the original ones $\gamma_m = \gamma_m(1)$ from (3.9).

The overall resulting special values of $\mathcal{Z}(s|0)$ and $\mathcal{Z}(s|\frac{1}{2})$ in Dirichlet- L cases are presented in Tables 10.3 and 10.4 respectively, drawn from [107].

10.6.2 Dedekind Zeta Functions

For any algebraic number field K , its *Dedekind zeta function* is defined as

$$\zeta_K(x) \stackrel{\text{def}}{=} \sum_{\mathfrak{a}} N(\mathfrak{a})^{-x} \equiv \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-x})^{-1} \quad (\text{Re } x > 1), \quad (10.72)$$

where \mathfrak{a} (resp. \mathfrak{p}) runs over all *integral* (resp. *prime*) ideals of K , and $N(\mathfrak{a})$ is the *norm* of \mathfrak{a} [49, Sect. 42][23, Sect. 10.5].

Their Admissibility

Dedekind zeta functions at once satisfy conditions (a)–(c) of Sect. 10.1 with

$$q \equiv 1, \quad \mathbf{G}(x) \equiv \frac{(4^{r_2} \pi^{n_K} / |d_K|)^{x/2}}{x \Gamma(x/2)^{r_1} \Gamma(x)^{r_2}}, \quad (10.73)$$

where r_1 (resp. $2r_2$) is the number of real (resp. complex) *conjugate fields* of K , $n_K \equiv r_1 + 2r_2$ is the *degree* of K , and d_K (≥ 0) its *discriminant* [49, Sect. 42]. With Laurent series at $x = 1$ having the form

$$\zeta_K(x) = \frac{\mathfrak{A}_K}{x - 1} + \mathfrak{C}_K + \dots, \quad (10.74)$$

condition (d) asks for $\mathfrak{A}_K > 0$; this is now ensured by *Dedekind’s class number formula* for this residue, involving further *positive invariants* of the field K [49, Theorems 121, 124][23, Theorem 10.5.1]:

$$\mathfrak{A}_K = \frac{2^{r_1+r_2} \pi^{r_2} h R}{W \sqrt{|d_K|}}, \quad (10.75)$$

h = the *class number*, R = the *regulator*, W = the *number of roots of unity*.

So, Dedekind zeta functions $\zeta_K(x)$ are admissible primary functions for us: any such choice will be denoted “a Dedekind- ζ case” here. For $K = \mathbb{Q}$, which has $r_1 = 1$, $r_2 = 0$, and $d_K = 1$, one recovers the *Riemann case*: $\zeta_K(x) \equiv \zeta(x)$, with the trivial factor $\mathbf{G}(x) \equiv \pi^{x/2} [x \Gamma(x/2)]^{-1}$ as in (3.24).

Results for Superzeta Functions in Dedekind- ζ Cases

Based on (10.73), we can specify the other quantities needed in Table 10.1:

- The shadow zeta function (10.11), counting all zeros of $\mathbf{G}(x)$ with their multiplicities, becomes

$$\mathbf{Z}(s|t) = r_1 2^{-s} \zeta(s, \frac{1}{4} + \frac{1}{2}t) + r_2 \zeta(s, \frac{1}{2} + t) - (\frac{1}{2} + t)^{-s} \quad (10.76)$$

- The leading coefficients in the Stirling formula (10.12) are

$$\begin{aligned} \tilde{\mathbf{a}}_1 &= \frac{1}{2} n_K, & \tilde{\mathbf{a}}_0 &= 1 - \frac{1}{4} r_1, \\ \mathbf{b}_1 &= \frac{1}{2} \log[|d_K| / (2\pi)^{n_K}], & \mathbf{b}_0 &= \frac{1}{4} \log[(8\pi)^{r_1} |d_K|]. \end{aligned} \quad (10.77)$$

This completes the tool kit needed to handle Dedekind- ζ cases for general t .

We then list further results only for the particular cases $t = 0$ and $\frac{1}{2}$, in Tables 10.5 and 10.6 respectively and in the following comments.

• $\mathbf{t} = \mathbf{0}$: the identities (10.29), resulting from $\mathcal{Z}_0(n) \equiv 0$ for odd $n \geq 1$, yield more explicit identities for the Dedekind zeta function ζ_K itself:

$$\begin{aligned}
 (\log |\zeta_K|)^{(n)}\left(\frac{1}{2}\right) &\equiv -2^{-n} r_1 \psi^{(n-1)}\left(\frac{1}{4}\right) - r_2 \psi^{(n-1)}\left(\frac{1}{2}\right) + \delta_{n,1} \frac{1}{2} \log \frac{4^{r_2} \pi^{n\kappa}}{|d_K|} \\
 &\qquad\qquad\qquad \text{for } n \geq 1 \text{ odd} \\
 &\equiv \begin{cases} \frac{1}{2}(n-1)! [n_K(2^n-1)\zeta(n) + r_1 2^n \beta(n)], & n > 1, \\ \frac{1}{2} n_K \gamma + \frac{1}{4} r_1 \pi + \frac{1}{2} \log[(8\pi)^{n\kappa}/|d_K|], & n = 1, \end{cases} \quad (10.78)
 \end{aligned}$$

in which, n being odd, $\frac{1}{2}(n-1)! 2^n \beta(n)$ reduces to $\frac{1}{4} \pi^n |E_{n-1}|$ by (3.32) while $\zeta(n)$ remains elusive. (For related identities bypassing $\zeta(n)$, see Appendix C.)

• $\mathbf{t} = \frac{1}{2}$: by (10.5) with $q = 1$ and (10.74), the lowest generalized Stieltjes cumulants are

$$g_0^c\{K\} \equiv -\log \mathfrak{R}_K, \quad g_1^c\{K\} \equiv \mathfrak{C}_K / \mathfrak{R}_K \quad (10.79)$$

(we restate their K -dependence to clarify the next formulae). Then, somewhat similarly to the previous example: $g_0^c\{K\}$ is *always* expressible – now through Dedekind’s class number formula (10.75); and $g_1^c\{K\}$, or equivalently the term \mathfrak{C}_K , is *sometimes* expressible – through certain *Kronecker limit formulae*; $g_1^c\{K\}$ (also known as the *Euler–Kronecker invariant* γ_K [52]) links to $\mathcal{Z}_*(1) = \sum \rho^{-1}$ [47, Theorem B(2)][52, (1.4.1)], as Table 10.6 shows.

– We first consider $K = \mathbb{Q}$: then $\zeta_K(x) \equiv \zeta(x)$, and by (3.15),

$$g_0^c\{\mathbb{Q}\} = 0, \quad g_1^c\{\mathbb{Q}\} = \gamma. \quad (10.80)$$

– Next, if K is a *quadratic* number field: letting $\chi = \chi_{d_K}$, the real primitive character of modulus $|d_K|$ given by the Kronecker symbol for the discriminant d_K (cf. Sect. 10.6.1), then [49, Sect. 49]

$$\zeta_K(x) \equiv \zeta(x) L_\chi(x). \quad (10.81)$$

(*Note*: this factorization nicely extends to their trivial factors, resp. completed functions, given the factor $(2\sqrt{\pi})^a$ we inserted in (10.55)). Now as a rule, zeta functions over zeros (and their linear invariants) obviously *add up* when their primary functions are *multiplied*. Thus, for their Stieltjes cumulants,

$$g_n^c\{K\} = g_n^c\{\mathbb{Q}\} + g_n^c[\chi] \quad (\forall n), \quad (10.82)$$

and here in particular,

$$g_0^c\{K\} = g_0^c[\chi], \quad g_1^c\{K\} = \gamma + g_1^c[\chi]. \quad (10.83)$$

Following a well-known practice, we can thereby *pass results* from the real primitive Dirichlet characters χ to the quadratic number fields K , and vice-versa. Results on the χ side were supplied before (Sect. 10.6.1). Now on the K side, we can add the class number formula for $g_0^c\{K\}$ – which for quadratic K reduces to Dirichlet's (10.56), and Kronecker limit formulae for $g_1^c\{K\}$, which come in two types – one simpler form handles *imaginary* quadratic K ($d_K < 0$) using the *Dedekind η -function* at specific points [19, Sect. 6][113, Sect. 2], while much more involved forms hold for *real* quadratic fields ($d_K > 0$) [113, Sect. 3]. All in all, we find the more reduced formulae to be still *those found on the χ side* when they exist:

- For g_0^c , either (10.66) or (10.70) (according to parity) is a more explicit expression of $L_\chi(1)$ than the class number formula for K (here, (10.56))
- For g_1^c in the *odd- χ* case ($d_K < 0$), (10.66) is more elementary than the simpler Kronecker limit formula (which calls Dedekind η -function values)
- Inversely, for g_1^c in the *even- χ* case ($d_K > 0$), only the K side provides *something new*, namely Kronecker limit formulae for $g_1^c\{K\}$ that are now exceedingly involved, but then Sect. 10.6.1 had left us without any closed expression for the corresponding $g_1^c[\chi]$.

Any further description of those aspects would carry us too far here, so we simply refer the reader to the above literature.

Example: Two basic quadratic number fields, both imaginary (i.e., with $r_1 = 0$ and $r_2 = 1$), are: $K = \mathbb{Q}(i)$ (for which $d_K = -4$, $\chi = \chi_{-4} : L_\chi(x) \equiv \beta(x)$ as in (3.27)), and: $K = \mathbb{Q}(\sqrt{-3})$ (for which $d_K = -3$, $\chi = \chi_{-3}$) [14]; hence their specific cumulants $g_0^c\{K\}$ and $g_1^c\{K\}$ most readily follow from (10.67) and (10.83).

– More generally, $g_1^c\{K\}$ will also be expressible: if K is a quadratic extension of another field F , in terms of $g_1^c\{F\}$ [45]; if ζ_K factorizes in Dirichlet L -functions (e.g., for cyclotomic fields and their subfields [23, Sect. 10.5.4]).

The overall resulting special values of $\mathcal{Z}(s|0)$ and $\mathcal{Z}(s|\frac{1}{2})$ in Dedekind- ζ cases are presented in Tables 10.5 and 10.6 extending Tables 7.3 and 7.4 from the Riemann case, respectively, as drawn from [107].

10.7 Tables of Formulae for the Special Values

Note. The superzeta functions of all three kinds also have explicitly computable polar decompositions, displayed in the main text.

10.7.1 For General Primary Functions $L(x)$ at General t

Table 10.1 Special values of the function of first kind $\mathcal{Z}(s|t)$ over the zeros $\{\rho\}$ of a general primary zeta function $L(x)$, admissible in the sense of Sect. 10.1, and having a pole of order $q \leq 1$ at $x = 1$. Notation: see (10.3) for $\Xi(x)$, (10.11) and (10.19) for $\mathbf{Z}(-n|t)$, (10.12) for $\tilde{\mathbf{a}}_j, \mathbf{b}_j$; n is an integer. For $L(x) = \zeta(x)$ (the Riemann case), cf. Table 7.1

s	$\mathcal{Z}(s t) = \sum_{\rho} (\frac{1}{2} + t - \rho)^{-s}$
$-n \leq 0$	$-\mathbf{Z}(-n t) + q(t - \frac{1}{2})^n \dagger$
0	$\tilde{\mathbf{a}}_1 t + \tilde{\mathbf{a}}_0 + q$
$\overset{0}{(s\text{-derivative})}$	$\mathcal{Z}'(0 t) = \mathbf{b}_1 t + \mathbf{b}_0 - \log \Xi(\frac{1}{2} + t)$
$\overset{+1}{(finite\ part)}$	$\text{FP}_{s=1} \mathcal{Z}(s t) = -\mathbf{b}_1 + (\log \Xi)'(\frac{1}{2} + t)$
$+n \geq 1$	$\frac{(-1)^{n-1}}{(n-1)!} (\log \Xi)^{(n)}(\frac{1}{2} + t)$

\dagger With $(t - \frac{1}{2})^0 \stackrel{\text{def}}{=} 1$ for $t = \frac{1}{2}$ (continuity in t is imperative)

Table 10.2 As Table 10.1, but for the function of second kind $\mathcal{Z}(\sigma|t)$; m is an integer. For $L(x) = \zeta(x)$ (the Riemann case), cf. Table 8.1

σ	$\mathcal{Z}(\sigma t) = \sum_{k=1}^{\infty} (\tau_k^2 + t^2)^{-\sigma}$
$-m \leq 0$	$\frac{1}{2} \left[q(t^2 - \frac{1}{4})^m - \sum_{j=0}^m \binom{m}{j} (-1)^j \mathbf{Z}(-2j 0) t^{2(m-j)} \right] \dagger$
0	$\frac{1}{2}(q + \tilde{\mathbf{a}}_0)$
$\overset{0}{(\sigma\text{-derivative})}$	$\mathcal{Z}'(0 t) = \mathbf{b}_0 - \log \Xi(\frac{1}{2} \pm t)$
$+m \geq 1$	$\frac{(-1)^{m-1}}{(m-1)!} \frac{d^m}{d(t^2)^m} \log \Xi(\frac{1}{2} \pm t)$

\dagger With $(t - t_0)^0 \stackrel{\text{def}}{=} 1$ for $t = t_0$ (continuity in t is imperative)

10.7.2 Dirichlet-L Cases, Functions of First Kind at $t = 0$ and $\frac{1}{2}$

Table 10.3 Special values of the zeta function $\mathcal{Z}(s|t)$ at $t = 0$ over the zeros $\{\rho\}$ of an L-function L_χ , for a real primitive Dirichlet character χ with modulus $d > 1$ and parity bit $a = 0$ or 1 . Notation: see (1.4)–(1.7), (10.49)–(10.51); n is an integer

s	$\mathcal{Z}_0(s) \equiv \sum_{\rho} (\rho - \frac{1}{2})^{-s} \quad [t = 0]$
$-n \leq 0$	$\begin{cases} \text{even} & 2^{-n-1}(a - \frac{1}{2})E_n \\ \text{odd} & -\frac{1}{2}(1 - 2^{-n})\frac{B_{n+1}}{n+1} \end{cases}$
0	$\frac{1}{2}(a - \frac{1}{2})$
$\overset{0}{\text{(derivative)}}$	$\mathcal{Z}'_0(0) = (\frac{3}{4} - a) \log 2 + (a - \frac{1}{2}) \log [\Gamma(\frac{1}{4})^2/\pi] - \log L_\chi(\frac{1}{2})$
$\overset{+1}{\text{(finite part)}}$	$\text{FP}_{s=1} \mathcal{Z}_0(s) = \frac{1}{2} [\log 2\pi - \log d]$
$+n \geq 1$	$\begin{cases} \text{odd} & 0^* \\ \text{even} & -\frac{1}{2} [(2^n - 1)\zeta(n) + (1 - 2a)2^n\beta(n)] - \frac{(\log L_\chi)^{(n)}(\frac{1}{2})}{(n-1)!} \end{cases} \dagger$

* This amounts to the formulae (10.59) yielding $(\log L_\chi)^{(n)}(\frac{1}{2})$ for n odd
 † Here $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$, while $\beta(n)$ (Sect. 3.5) and $(\log L_\chi)^{(n)}(\frac{1}{2})$ remain elusive

Table 10.4 As Table 10.3, but at $t = \frac{1}{2}$. For the generalized Stieltjes cumulants $g_n^c[\chi]$, see (10.5) with $q = 0$, (10.66) and (10.70)

s	$\mathcal{Z}_*(s) \equiv \sum_{\rho} \rho^{-s} \quad [t = \frac{1}{2}]$
$-n < 0$	$[(a - 1)(2^n - 1) + a2^n] \frac{B_{n+1}}{n+1}$
0	$\frac{1}{2}a$
$\overset{0}{\text{(derivative)}}$	$\mathcal{Z}'_*(0) = \frac{1}{2} [(1 - a) \log 2 + a \log \pi] + g_0^c[\chi]$
$\overset{+1}{\text{(finite part)}}$	$\text{FP}_{s=1} \mathcal{Z}_*(s) = (a - \frac{1}{2}) \log 2 - \frac{1}{2}\gamma + g_1^c[\chi]$
$+1$	$(a - 1) \log 2 - \frac{1}{2} \log(\pi/d) - \frac{1}{2}\gamma + g_1^c[\chi]$
$+n > 1$	$[(a - 1)(1 - 2^{-n}) - a2^{-n}]\zeta(n) + \frac{g_n^c[\chi]}{(n-1)!} \dagger$

† For n even, $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$

10.7.3 Dedekind- ζ Cases, Functions of First Kind at $t = 0$ and $\frac{1}{2}$

Table 10.5 Special values of the zeta function $\mathcal{Z}(s|t)$ at $t = 0$ over the zeros $\{\rho\}$ of a Dedekind zeta function ζ_K , for an algebraic number field K . Notation: see (1.4)–(1.7), (10.72) and (10.73); n is an integer. For $K = \mathbb{Q}$ (the Riemann case), cf. Table 7.3

s	$\mathcal{Z}_0(s) \equiv \sum_{\rho} (\rho - \frac{1}{2})^{-s} \quad [t = 0]$
$-n \leq 0$	$\begin{cases} \text{even} & 2^{-n+1}(1 - \frac{1}{8}r_1 E_n) \\ \text{odd} & -\frac{1}{2}n_K(1-2^{-n})\frac{B_{n+1}}{n+1} \end{cases}$
0	$2 - \frac{1}{4}r_1$
$\overset{0}{\text{(derivative)}}$	$\mathcal{Z}'_0(0) = (2 + \frac{3}{4}r_1 + \frac{1}{2}r_2) \log 2 - \frac{1}{2}r_1 \log[\Gamma(\frac{1}{4})^2/\pi] - \log \zeta_K (\frac{1}{2})$
$\overset{+1}{\text{(finite part)}}$	$\text{FP}_{s=1} \mathcal{Z}_0(s) = \frac{1}{2}[n_K \log 2\pi - \log d_K]$
$+n \geq 1$	$\begin{cases} \text{odd} & 0^* \\ \text{even} & 2^{n+1} - \frac{1}{2}[n_K(2^n - 1)\zeta(n) + r_1 2^n \beta(n)] - \frac{(\log \zeta_K)^{(n)}(\frac{1}{2})}{(n-1)!} \dagger \end{cases}$

*This amounts to the formulae (10.78) yielding $(\log|\zeta_K|)^{(n)}(\frac{1}{2})$ for n odd
 †Here $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$, while $\beta(n)$ (Sect. 3.5) and $(\log|\zeta_K|)^{(n)}(\frac{1}{2})$ remain elusive

Table 10.6 As Table 10.5, but at $t = \frac{1}{2}$. For the generalized Stieltjes cumulants $g_n^c\{K\}$, see (10.5) with $q = 1$, (10.79)–(10.83). For $K = \mathbb{Q}$ (the Riemann case), cf. Table 7.4

s	$\mathcal{Z}_*(s) \equiv \sum_{\rho} \rho^{-s} \quad [t = \frac{1}{2}]$
$-n < 0$	$[-r_1(2^n - 1) + r_2] \frac{B_{n+1}}{n+1} + 1$
0	$\frac{1}{2}r_2 + 2$
$\overset{0}{\text{(derivative)}}$	$\mathcal{Z}'_*(0) = \frac{1}{2}[(r_1 + r_2) \log 2 + r_2 \log \pi] + g_0^c\{K\}$
$\overset{+1}{\text{(finite part)}}$	$\text{FP}_{s=1} \mathcal{Z}_*(s) = 1 - \frac{1}{2}r_1 \log 2 - \frac{1}{2}n_K \gamma + g_1^c\{K\}$
$+1$	$1 + \frac{1}{2} \log d_K - (r_1 + r_2) \log 2 - \frac{1}{2}n_K \log \pi - \frac{1}{2}n_K \gamma + g_1^c\{K\}$
$+n > 1$	$1 - [r_1(1-2^{-n}) + r_2] \zeta(n) + \frac{g_n^c\{K\}}{(n-1)!} \dagger$

† For n even, $\zeta(n) \equiv (2\pi)^n |B_n|/(2n!)$

Chapter 11

Application: an Asymptotic Criterion for the Riemann Hypothesis

This chapter is the only one whose contents expressly link to the truth or falsity of the *Riemann Hypothesis* [92]: it deals with a new and sharp criterion for this conjecture. Many statements equivalent to the Riemann Hypothesis already exist; they typically involve inequalities such as infinitely many positivity conditions to be all obeyed without exception; our criterion may have an unusual form in that it gives a neat *asymptotic alternative* instead (actually in two variants). It links to an earlier criterion by Li, which states that the Riemann Hypothesis is true if and only if a specific real sequence has *all its terms positive*. We actually use the same sequence, but through a different filter: from the general framework of the previous chapters, we deduce that the Riemann Hypothesis can be expressed purely in terms of the *large- n* behavior of that sequence.

11.1 Introduction to the Result

The Riemann Hypothesis (to be abbreviated RH) states that all the nontrivial zeros ρ of $\zeta(x)$ lie on the critical axis $\{\operatorname{Re} x = \frac{1}{2}\}$, and it is still open. Here we will translate the Riemann Hypothesis into a clear-cut and explicit *asymptotic alternative* for a specific real sequence $\{\lambda_n\}_{n=1,2,\dots}$ introduced by Keiper [64] and used by Li's criterion [9, 73], which says that RH is true if and only if $\lambda_n > 0$ ($\forall n$); we already briefly discussed this sequence in Sects. 7.6.2 and 8.6.2. Our present goal is to show that the λ_n can only follow one of two sharply distinct and mutually exclusive asymptotic behaviors: if RH is true, λ_n will grow *tamely* as $\lambda_n \sim n(a \log n + b)$ (for some explicit $a > 0$ and b , see (11.10) below); if RH is false, λ_n will oscillate with an *exponentially growing* amplitude, in both $+$ and $-$ directions (see (11.11) below). This dichotomy thus provides a sharp criterion of a new *asymptotic* type for the Riemann Hypothesis (and it extends to other zeta-type functions as well).

To obtain our result, we will start from our exact representation (8.35) of λ_n as a finite oscillatory sum in terms of the superzeta-values $\mathcal{Z}_*(j)$, then convert it to an integral formula, (11.5), which we will finally evaluate by the saddle-point method in the $n \rightarrow +\infty$ limit.

This chapter overall reviews our results from [109, 110], which mainly connect to those of Keiper [64], Li [73], Bombieri and Lagarias [9], Mašlanka [77–79], and Lagarias [67]; see also [7, Sect. 2.3][20–22, 38, 39]. Helpful information from J.C. Lagarias, K. Mašlanka and J. Oesterlé is gratefully acknowledged.

We recall the main facts we will need about the λ_n , from Sects. 7.6.2 and 8.6.2:

- Their definition (7.55), and the equivalent generating function (8.34):

$$\lambda_n \stackrel{\text{def}}{=} \sum_{\rho} [1 - (1 - 1/\rho)^n] \quad (n = 1, 2, \dots) \quad (11.1)$$

$$\sum_{j=1}^{\infty} \lambda_j z^{j-1} \equiv \frac{d}{dz} \log \Xi\left(\frac{1}{1-z}\right) \quad (11.2)$$

(we use the notation of Li [73], whose λ_n are n times Keiper's [64]);

- And their relation (8.35) to the superzeta values of *second kind* [109, 110]:

$$\lambda_n = -n \sum_{j=1}^n \frac{(-1)^j}{j} \binom{n+j-1}{2j-1} \mathcal{Z}_*(j), \quad (11.3)$$

$$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} u_k^{-\sigma} \quad \left(u_k \stackrel{\text{def}}{=} \tau_k^2 + \frac{1}{4} = \rho(1-\rho)\right). \quad (11.4)$$

Analogous earlier expressions of the λ_n exist, in terms of the Stieltjes cumulants η_j [9, Theorem 2] or of the superzeta values of the first kind $\mathcal{Z}_*(j)$ as in (7.56) [64]. Nevertheless, the expression (11.3) for λ_n has some distinctive advantages: being derived from a *symmetric* Hadamard product (cf. Sect. 8.6.2), it embodies the Functional Equation $\Xi(\frac{1}{2} + t) = \Xi(\frac{1}{2} - t)$; and unlike the $\mathcal{Z}_*(j)$, the $\mathcal{Z}_*(j)$ appear positive and monotonically decreasing. Still, (11.3) is globally an *oscillatory* sum, hence difficult to control directly.

Now, there exists an integral representation fully equivalent to (11.3), as verified simply by the residue calculus:

$$\lambda_n = \frac{(-1)^n n i}{\pi} \oint_C I(\sigma) d\sigma, \quad I(\sigma) = \frac{\Gamma(\sigma+n)\Gamma(\sigma-n)}{\Gamma(2\sigma+1)} \mathcal{Z}_*(\sigma), \quad (11.5)$$

where C is a positive contour encircling just the subset of poles $\sigma = +1, \dots, +n$ of the integrand $I(\sigma)$; moreover, C can stretch to $+\infty$ (Fig. 11.1). While (11.5) strictly amounts to (11.3), it will prove much more flexible, and our analysis entirely rests on it.

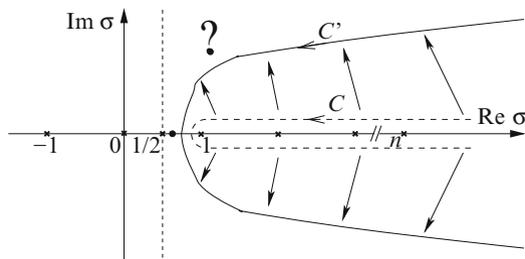


Fig. 11.1 Contour deformation for the saddle-point evaluation of the integral (11.5). Two saddle-points (\bullet) are sketched, the real one very near $\sigma = 1/2$, and a putative complex one (as the dot in “?”) in 1–1 correspondence with a Riemann zero off the critical axis

11.2 Asymptotic Alternative for $\lambda_n, n \rightarrow \infty$

The integral formula (11.5) will lend itself to an asymptotic ($n \rightarrow \infty$) evaluation by the classic *saddle-point method*. This method asymptotically evaluates contour integrals like $\int_C I(\sigma) d\sigma$ when $\log I$ is (roughly) proportional to a large parameter x [35, Sect. 2.5]: first, the integration path C is deformed in the σ -plane in the direction of decreasing $|I(\sigma)|$, as far down as possible; this forces it to pass through some saddle-points σ_* of $|I(\sigma)|$; then as $x \rightarrow \infty$, the integrand $I(\sigma)$ becomes more and more localized around each saddle-point σ_* where it gives a contribution to the integral of the order of magnitude $|I(\sigma_*)|$; as a result, the highest saddle-point(s) set(s) the dominant behavior of the integral. In parallel, the integrand $I(\sigma)$ may itself be asymptotically approximated; e.g., in (11.5) for $n \rightarrow +\infty$, the large parameter x will be $\log n$, and the Gamma factors may be replaced by their Stirling approximations.

This approach, for an integrand not controlled in fully closed form, partly retains an experimental character. We now use it for this problem as a heuristic, rather than rigorous, tool: it predicts the global structure of the results at once, and it correctly yields both behaviors that may take place; more established techniques confirm the results, but piecemeal and not as intuitively.

In the present problem, the landscape of the function $|I(\sigma)|$ for large n is dominantly shaped by the Gamma factors, which asymptotically read as $\pi [\sin \pi \sigma \Gamma(2\sigma + 1)]^{-1} n^{2\sigma-1}$ at fixed σ , except very near the poles of $I(\sigma)$ where the singular parts also contribute. For a saddle-point σ_* in the bulk, its contribution scales roughly like $n^{2\sigma_*-1}$, so that the saddle-points rank according to their real parts (the largest dominate). The ensuing contour deformation then starts as an expansion of C away from the poles $\{1, \dots, n\}$ in all directions, and the endpoints can go to infinity in the directions $|\arg \sigma| < \frac{\pi}{2} - \delta$ (Fig. 11.1). The saddle-points encountered here (once n is large enough) can be of two types, which will compete for domination.

1. For σ on the segment $(\frac{1}{2}, 1)$, $|I(\sigma)| \sim \pi [\sin \pi \sigma \Gamma(2\sigma + 1)]^{-1} n^{2\sigma-1} \mathcal{Z}_*(\sigma)$ always has one *real* minimum $\sigma_r(n)$ (tending to $\frac{1}{2}$ as $n \rightarrow \infty$), which will definitely retain the moving contour; any other real saddle-points lie below $\sigma = \frac{1}{2}$ hence cannot be reached and do not count.

2. *Complex* saddle-points may enter as well, for which we may focus just on the upper half-plane, and deduce the lower-half-plane contributions as the complex conjugates (“c.c.”). As long as the moving contour stays inside a half-plane $\{\operatorname{Re} \sigma > \frac{1}{2} + \varepsilon\}$, the integrand can be decomposed as $I = \sum_k I_k$ according to (11.4); then for each individual term and within the Stirling approximation for the Γ -ratio, the saddle-point equation is $0 = (d/d\sigma) \log |I_k(\sigma)| \sim \log(\sigma^2 - n^2) - 2 \log 2\sigma - \log u_k$, yielding the complex saddle-point location

$$\sigma_k(n) = n i / 2\tau_k. \quad (11.6)$$

Thus, for any zero which lies *on* the critical axis (τ_k real), the saddle-point $\sigma_k(n)$ comes out purely imaginary, but then it cannot be reached under the present restrictions placed upon the path: hence it should not be counted, and its real part is anyway dominated by $\sigma_r(n)$. So in the end, this paragraph is irrelevant to the summands of (11.4) for which τ_k is real.

The discussion then fundamentally depends on the presence or absence of zeros off the critical axis, as this determines which of the two types of saddle-points, (1) or (2), ultimately dominates.

11.2.1 The Case [RH False]

In case there is any zero $(\frac{1}{2} \pm i\tau_k)$ off the critical axis, we select $\arg \tau_k > 0$ for it, and assume it is a simple zero for argument’s sake (otherwise it will suffice to count it with its multiplicity). Then paragraph (2) above fully applies to each such zero: the complex saddle-point $\sigma_k(n)$ given by (11.6) lies inside the allowed domain $\{\operatorname{Re} \sigma > \frac{1}{2}\}$ as soon as $n > |\operatorname{Im} 1/\tau_k|^{-1}$, and for $n \rightarrow +\infty$ it gives an additive contribution $\sim [(\tau_k + i/2)/(\tau_k - i/2)]^n$ to λ_k (in the usual quadratic approximation of $\log I(\sigma)$ around $\sigma_k(n)$), which grows *exponentially* in modulus and fluctuates in phase; it will indeed exponentially dominate the contribution of the real saddle-point $\sigma_r(n)$ (computed later).

This result can also be confirmed rigorously via *Darboux’s theorem*: by the conformal mapping $x = \frac{1}{1-z} \mapsto z$ [9], the generating function (11.2) of the λ_n is seen to have precisely the points $z_k = (\tau_k - i/2)(\tau_k + i/2)^{-1}$ and z_k^* as simple poles of residue 1 in the unit disk (Fig. 11.2); then a general Darboux theorem [30, Chap. VII Sect. 2] applies here to the poles with $|z_k| < 1$, implying that the Taylor coefficients of that function (namely, the λ_n) indeed have the asymptotic form

$$\lambda_n \sim \sum_{\{|z_k| < 1\}} z_k^{-n} + \text{c.c.} \quad (\text{mod } o(e^{\varepsilon n}) \forall \varepsilon > 0), \quad n \rightarrow \infty. \quad (11.7)$$

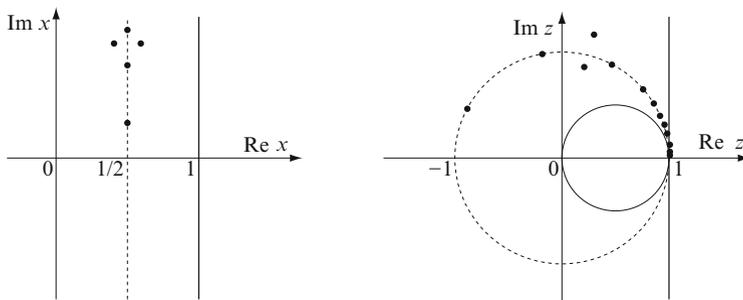


Fig. 11.2 The conformal mapping $x \mapsto z$; fictitious Riemann zeros (\bullet) are sketched, on and off the critical axis; the lower-half-plane zeros (*not drawn*) are their mirror images

Concretely, λ_n oscillates between *exponentially growing values of both signs*.

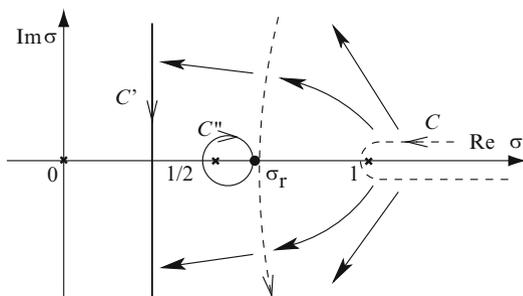
Infinitely many zeros ($\frac{1}{2} \pm i\tau_k$) *off the critical axis* are perfectly admissible in this argument: their z_k satisfy $|z_k| < 1$, $z_k \rightarrow 1$, hence the corresponding infinite sums $\sum_k z_k^{-n}$ still define valid $n \rightarrow +\infty$ asymptotic expansions.

On the other hand, this Darboux formula (11.7) becomes ill-defined if the infinitely many z_k^{-n} have identical and dominant modulus, which is precisely realized in the case [RH true], with all the z_k on the unit circle; we now turn to this case.

11.2.2 The Case [RH True]

Here, Darboux's theorem says that $\lambda_n = o(e^{\varepsilon n}) \forall \varepsilon > 0$, but it fails to give any clue as to an explicit asymptotic equivalent for λ_n . In contrast, the saddle-point treatment of the integral (11.5) itself remains thoroughly applicable. Simply now, all the τ_k are real, $\mathcal{Z}_*(\sigma) = O(\mathcal{Z}_*(\text{Re } \sigma) |\text{Im } \sigma|^{-3/2})$ in $\{\text{Re } \sigma > \frac{1}{2}\}$, and the contour C can be freely moved toward the boundary $\{\text{Re } \sigma = \frac{1}{2}\}$ without meeting any of the $\sigma_k(n)$, all purely imaginary (Fig. 11.3). The sole dominant saddle-point is then $\sigma_r(n) \in (\frac{1}{2}, 1)$; it is shaped by the double pole of $\mathcal{Z}_*(\sigma)$ at $\frac{1}{2}$ (itself generated by the totality of Riemann zeros), so that $\sigma_r(n) \sim \frac{1}{2} + \frac{1}{\log n}$. Since this saddle-point is non-isolated (it tends to a pole), the standard saddle-point evaluation using the quadratic approximation of $\log I(\sigma)$ around $\sigma_r(n)$ works very poorly. Here, it is at once simpler and more accurate to keep on deforming the contour C in the vicinity of $\sigma = \frac{1}{2}$ (up to only lower-order errors) until it fully encircles this pole clockwise, as C'' . Then a simple residue formula applies, that only uses the principal part at $\sigma = \frac{1}{2}$ of $\mathcal{Z}_*(\sigma)$, which is the same as for $\mathcal{Z}_0(\sigma)$, namely (10.32) (in general form). Then, as the end result under [RH true], λ_n obeys

Fig. 11.3 Contour deformation for the saddle-point method applied to λ_n (continued from Fig. 11.1): for [RH true], the contribution of the real saddle-point $\sigma_r(n)$ dominates



$$\lambda_n = (-1)^n 2n \operatorname{Res}_{\sigma=1/2} I(\sigma) + o(n),$$

$$\begin{aligned} \text{with } (-1)^n 2n \operatorname{Res}_{\sigma=1/2} I(\sigma) &\equiv n [\tilde{\mathbf{a}}_1 (\psi(\tfrac{1}{2} + n) - 1 + \gamma) + \mathbf{b}_1] & (11.8) \\ &= n [\tilde{\mathbf{a}}_1 (\log n - 1 + \gamma) + \mathbf{b}_1] + O(1/n) \end{aligned}$$

(by the Stirling formula (3.52) for ψ). In the Riemann case itself, by (8.4),

$$\tilde{\mathbf{a}}_1 = \tfrac{1}{2}, \quad \mathbf{b}_1 = -\tfrac{1}{2} \log 2\pi. \quad (11.9)$$

However, we kept the result (11.8) in general form, to be ready to cover all primary functions $L(x)$ compliant with Chap. 10 as well (see the closing section below).

Like (11.7) before, (11.8) can be derived quite rigorously but by still another method, written up for the Riemann zeros by Oesterlé [88] (private communication). We summarized his argument in [110]; it is purely real-analytic, and ignores the [RH false] case.

11.2.3 Recapitulation and Discussion

As we ended up with two mutually exclusive large- n behaviors for the λ_n , (11.7) and (11.8), together they provide a sharp equivalence result. For the Riemann zeros, using the explicit values (11.9):

Theorem (asymptotic criterion for the Riemann Hypothesis). *For $n \rightarrow +\infty$, the sequence $\{\lambda_n\}$ built from the Riemann zeros follows one of these asymptotic behaviors: [109, 110]*

- [RH true] gives a *tempered* growth to $+\infty$:

$$\lambda_n \sim \tfrac{1}{2}n (\log n - 1 + \gamma - \log 2\pi) \pmod{o(n)}; \quad (11.10)$$

- [RH false] gives *non-tempered* oscillations :

$$\lambda_n \sim \sum_{\{\arg \tau_k > 0\}} \left(\frac{\tau_k + i/2}{\tau_k - i/2} \right)^n + \text{c.c.} \pmod{o(e^{\varepsilon n}) \forall \varepsilon > 0}. \quad (11.11)$$

Such a comprehensive asymptotic statement was not known to us on the [RH false] side, and it appears to strengthen earlier results in the [RH true] case.

The main end formula (11.10), assuming RH, had actually been written by Keiper [64, (37)], but in a somewhat misleading context: in his words, (11.10) required not just RH but also “very evenly distributed” zeros, and was “much stronger than” RH (no details or proofs were ever supplied). All that hardly suggests the actual *equivalence* of (11.10) to RH.

Oesterlé had proved [RH true] \Rightarrow (11.10) (see above), but in a neither published nor even posted typescript [88].

Bombieri and Lagarias [9, Corollary 1(c)] showed rather weak exponential lower bounds $\lambda_n \geq -ce^{\varepsilon n}$ to imply RH; the converse statement [RH true] \Leftarrow (11.10) thus implicitly lies in [9] (but it cannot be inferred therefrom, as [9] never alludes to any asymptotic regarding the λ_n).

Furthermore, the above saddle-point analysis seems to be the only one yet to predict both behaviors (11.10) and (11.11) in parallel; while we do not see how Darboux’s approach could derive (11.10) in the [RH true] case, nor how Oesterlé’s approach could derive (11.11) in the [RH false] case. Here, in contrast, the alternative [RH true]/[RH false] boils down to finding which saddle-point wins dominance in one integral specified at the outset.

Numerical data [64, 78] agree well with (11.10) for $n < 7000$ (and even better in the mean if we add the contribution like (11.8) but from the next pole of $I(\sigma)$ at $\sigma = 0$ (cf. Table 8.4),

$$\delta\lambda_n = (-1)^n 2n \operatorname{Res}_{\sigma=0} I(\sigma) = 2 \mathcal{Z}_*(0) = +7/4, \quad (11.12)$$

even though this correction seems asymptotically negligible with respect to the oscillatory terms). However, the above numerical agreement is inconclusive regarding the Riemann Hypothesis: any currently possible violation of RH would be detectable only at *much higher* n (see end of the next section).

11.3 An Even More Sensitive Sequence

A slightly sharper difference of behavior follows for the special linear combinations (11.13) below of the cumulants η_n (for which we now switch back to the notation (3.12), as it is used in all related references).

Indeed, the definition $\Xi(x) = \Gamma(x/2)\pi^{-x/2}x(x-1)\zeta(x)$ substituted into (11.2) readily yields a decomposition $\lambda_n = S_n + \overline{S}_n$, where [9, Theorem 2]

$$S_n = - \sum_{j=1}^n \binom{n}{j} \eta_{j-1} \quad (11.13)$$

is the contribution of $(x - 1)\zeta(x)$, and

$$\bar{S}_n = 1 - \frac{1}{2}(\log 4\pi + \gamma)n + \hat{S}_n, \text{ with } \hat{S}_n = \sum_{j=2}^n \binom{n}{j} (-1)^j (1 - 2^{-j}) \zeta(j), \quad (11.14)$$

is the contribution of the remaining (more explicit) factor. Now the large- n behavior of the sum \hat{S}_n is also computable: we apply the same saddle-point technique we used from (11.3) to (11.8), but upon

$$\hat{S}_n = \frac{(-1)^n n!}{2\pi i} \oint_C J(\sigma) d\sigma, \quad J(\sigma) = \frac{\Gamma(\sigma - n)}{\Gamma(\sigma + 1)} (1 - 2^{-\sigma}) \zeta(\sigma), \quad (11.15)$$

integrated around the poles $\sigma = 2, \dots, n$ of $J(\sigma)$: the result for $n \rightarrow +\infty$ is

$$\hat{S}_n \sim (-1)^{n-1} n! \operatorname{Res}_{\sigma=1} J(\sigma) = \frac{1}{2}n [\psi(n) + \log 2 - 1 + 2\gamma], \quad (11.16)$$

now (mod $O(n^{-\infty})$) because $J(\sigma)$ has no singularities further to the left (Fig. 11.4); so that finally, using the Stirling expansion (3.52) for $\psi(n)$,

$$\bar{S}_n \sim \frac{1}{2}n (\log n - 1 + \gamma - \log 2\pi) + \frac{3}{4} - \sum_{k=1}^{\infty} \frac{B_{2k}}{4k} n^{1-2k} \quad (\text{for } n \rightarrow \infty) \quad (11.17)$$

unconditionally.

This at once confirms two empirical conjectures made by Maślanka [77, (2.5) and (2.8)]:

- mod $o(n)$, the sequence $\{\bar{S}_n\}$ expresses the “trend” (11.10) obeyed by the sequence $\{\lambda_n\}$ under [RH true]
- And to all orders in n , that same sequence $\{\bar{S}_n\}$ has the asymptotic expansion (11.17).

We add that the latter statement holds *unconditionally*, and that *whether RH holds or not*, it makes sense to subtract this fixed $\{\bar{S}_n\}$ -contribution from the previous formulae (11.10) and (11.11), with the results [109, 110]: either

$$S_n = o(n) \quad [\text{RH true}] \quad (11.18)$$

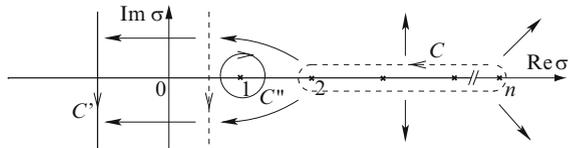


Fig. 11.4 Contour deformation for the saddle-point method applied to \hat{S}_n

(a case further discussed in [21, 78, 99]); or else $S_n \sim \lambda_n$, i.e.,

$$S_n \sim \sum_{\{\arg \tau_k > 0\}} \left(\frac{\tau_k + i/2}{\tau_k - i/2} \right)^n + \text{c.c.} \pmod{o(e^{\varepsilon n}) \forall \varepsilon > 0} \quad [\text{RH false}] \quad (11.19)$$

which gives oscillations that grow *exponentially* with n .

In absolute size, however, any contribution like (11.19) from a zero violating RH would remain considerably smaller than (11.18) (the background from all the other real τ_j) up to $n \approx \min_{\{\arg \tau_k > 0\}} \{[\text{Im } 1/\tau_k]^{-1}\}$: i.e., S_n can only

reliably signal zeros violating RH up to $|\text{Im } \rho| \lesssim \sqrt{n/2}$; as such zeros must currently have $|\text{Im } \rho| \gtrsim 10^9$, they could then only be detected at $n \gtrsim 10^{18}$ (see also [7, Sect. 2.3][88],[67, p. 5]).

Our asymptotic criteria thus may not surpass others in practical sensitivity, still their theoretical sharpness may be of interest. For instance, Li's criterion $\lambda_n > 0$ ($\forall n$) is now strengthened in that, beyond a finite n -range, it refers to ever larger amplitudes either way, i.e., delicate borderline situations such as $\lambda_n \rightarrow 0$ for $n \rightarrow \infty$ cannot occur.

11.4 More General Cases: a Summary

For the more general zeta-type functions compliant with Chap. 10, the asymptotic alternative for the associated λ_n persists in its general form: either (11.8) if the generalized Riemann Hypothesis (GRH) holds, or (11.7) otherwise. That includes, but is not limited to, Dedekind zeta functions (Sect. 10.6.2) and some Dirichlet L -functions (Sect. 10.6.1). The constants $\tilde{\mathbf{a}}_1$, \mathbf{b}_1 in (11.8) also enter the generalized Riemann–von Mangoldt formula (10.40) when it holds; this then ensures $\tilde{\mathbf{a}}_1 \geq 0$, in line with the corresponding generalized Li's criterion [9, 67, 74, 75].

The λ_n have also been generalized to L -functions defined by Hecke operators for the congruence subgroup $\Gamma_0(N)$ [74][75, specially Remark 5.4].

More recently and in a broader setting (the λ_n for automorphic L -functions), Lagarias presented an alternative approach to estimate the λ_n with greater accuracy, mod $O(\sqrt{n} \log n)$ under GRH [67]: in the notations of (11.13) and (11.14), he directly proves that \tilde{S}_n obeys (11.8) mod $O(1)$ unconditionally (Theorem 5.1), then that $S_n = O(\sqrt{n} \log n)$ under GRH (Theorem 6.1).

Appendix A

Numerical Explorations

We complement our analytical study by some *experimental* numerical evaluations of superzeta functions over the *Riemann* zeros, at real arguments. Apart from very few tables for special values (where special formulae apply), we had not seen such functions tabulated or plotted before, and we wanted to view them. (We have not tackled cases with other zeros.)

We found it easier to concentrate on $\mathcal{Z}_0(\sigma) = \mathcal{Z}(\sigma | 0)$ as the basic case, and to deduce other cases ($t \neq 0, \mathcal{L}, \dots$) by subsequent expansions.

The first step is to sum the defining series $\mathcal{Z}_0(\sigma) = \sum_k \tau_k^{-2\sigma}$ for $\sigma > \frac{1}{2}$. This needs large tables of Riemann zeros: we have used A.M. Odlyzko's list of the 100,000 first zeros he made freely available on the Web [86], for which we express our gratitude to him. Here, the Riemann Hypothesis is de facto implied throughout (there being no numerical counter example).

The other significant step is *numerical analytical continuation*, in order to reach values of $\mathcal{Z}_0(\sigma)$ for some $\sigma < \frac{1}{2}$. We tested two complementary approaches which agreed to several digits whenever they were both usable. However, we did not go below $\sigma = -2$, our numerical precision is rather modest and above all we cannot estimate it reliably. The rounded values we supply are just most likely to be correct; they intend to provide a first glimpse of superzeta functions. Better results are certainly reachable, should there arise a clear need.

A.1 Superzeta Functions of the Second Kind

We now sketch our computations for the function \mathcal{Z}_0 . We begin by using $\mathcal{Z}_0(\sigma) = \lim_{K \rightarrow \infty} \sum_{k \leq K} \tau_k^{-2\sigma}$ for $\sigma > \frac{1}{2}$, but the remainders $R_K(\sigma) \stackrel{\text{def}}{=} \sum_{k > K} \tau_k^{-2\sigma}$ ($\propto \tau_K^{1-2\sigma} \log \tau_K$) tend to 0 slowly, more and more so as $\sigma \rightarrow \frac{1}{2}^+$, where they diverge. We then approximate the sum defining $R_K(\sigma)$ by an integral $\bar{R}_K(\sigma)$,

through an Euler–Maclaurin formula (1.15) for the function $T(k)^{-2\sigma}$ (where $T(k) \stackrel{\text{def}}{=} \frac{1}{\bar{N}_0(T)}$ defined by (4.26)):

$$\begin{aligned} \mathcal{Z}_0(\sigma) &= \lim_{K \rightarrow +\infty} S_K(\sigma), \quad S_K(\sigma) \stackrel{\text{def}}{=} \sum_{k=1}^{K-1} \tau_k^{-2\sigma} + \frac{1}{2} \tau_K^{-2\sigma} + \bar{R}_K(\sigma), \quad (\text{A.1}) \\ \bar{R}_K(\sigma) &\stackrel{\text{def}}{=} \int_{\tau_K}^{+\infty} T^{-2\sigma} d\bar{N}_0(T) = \frac{1}{2\pi} \frac{\tau_K^{1-2\sigma}}{2\sigma-1} \left[\log \frac{\tau_K}{2\pi} + \frac{1}{2\sigma-1} \right] \end{aligned}$$

(and likewise for $\mathcal{Z}(\sigma | t)$, or derivatives, or finite parts at poles, etc.).

The residual error in (A.1), $\tilde{R}_K(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}_0(\sigma) - S_K(\sigma)$, is $O(\tau_K^{-2\sigma} S(\tau_K))$ from (4.28) (the error term from (1.15), $O(\tau_K^{-2\sigma})$, proves smaller). So, $\tilde{R}_K(\sigma) \ll R_K(\sigma)$: the use of $\bar{R}_K(\sigma)$ in (A.1) does accelerate convergence when $\sigma > \frac{1}{2}$, and extends it to $\{\sigma > 0\}$ [43, p. 116 last line]: therefore, (A.1) gives one method of numerical analytical continuation. The next obstacle to convergence, $\tilde{R}_K(\sigma)$, differs in type from $R_K(\sigma)$: it *fluctuates* in K , on a root-mean-square scale $\propto \tau_K^{-2\sigma} (\log \log \tau_K)^{1/2}$ according to [87, (2.5.7)], which diverges (as $K \rightarrow +\infty$) when $\sigma \leq 0$. Higher Euler–Maclaurin corrections are pointless for such fluctuating remainders, only some *damping* can help or restore convergence (as with “chaotic” spectra [3]). Here, *Cesaro averaging* (defined by $\langle S \rangle_K \stackrel{\text{def}}{=} K^{-1} \sum_1^K S_{K'}$) appears to work initially – it gives results verifiable at $\sigma = 0$ – but not far: already at $\sigma = -\frac{1}{4}$, even $\langle S \rangle_K(\sigma)$ retains fluctuations $> 10^{-3}$ (in standard deviation) up to $K \approx 10^5$.

So, instead of pursuing ever more severe (and unproven, after all) numerical dampings as σ decreases farther from $\frac{1}{2}$, we switched to another route: to compute the analytical continuation formulae (8.6) and (8.7) themselves. We thus first validated (8.7) against (A.1) for $0 < \sigma < \frac{1}{2}$, then used it for numerical analytical continuation below $\sigma = 0$. On paper this seems to be the obvious thing to do, but in practice it proves arduous: faced with a *Mellin integral*, over an *infinite path* which moreover goes *through a pole*, we achieved decent numerical stability only close to $\sigma = 0$ (and not near $\sigma = \frac{1}{2}$).

Turning to the other interesting case $\mathcal{Z}_*(\sigma) = \mathcal{Z}(s | t = \frac{1}{2})$, we evaluate it in terms of \mathcal{Z}_0 through the expansion (8.19) about $t = 0$. This way, first we capitalize on the preceding calculations, then we benefit from a very rapid numerical convergence: about three terms suffice to match our accuracy for \mathcal{Z}_0 itself. Indeed, numerically $\mathcal{Z}_*(\sigma)$ stays very close to $\mathcal{Z}_0(\sigma)$ for $\sigma \geq 0$, thanks to $\tau_k^2 \gg \frac{1}{4}$ ($\forall k$) (an empirical fact; $\tau_1^2 \approx 199.790455$). In the crudest approximation of (8.19), $\mathcal{Z}_*(\sigma) / \mathcal{Z}_0(\sigma) \approx 1 - \sigma / 4\tau_1^2 \approx 1 - \sigma / 800$. We numerically also found $0 < \mathcal{Z}_0(\sigma) - \mathcal{Z}_*(\sigma) < 3 \times 10^{-4}$ for all $\sigma > 0$, and

$$A \stackrel{\text{def}}{=} 4[\mathcal{Z}'_*(0) - \mathcal{Z}'_0(0)] \approx -0.0231003495 \quad \text{vs} \quad B \equiv -\mathcal{Z}_*(1) \approx -0.0230957090; \quad (\text{A.2})$$

not only A is small, but given that $A \equiv 4 \log \Xi(\frac{1}{2})$ by (8.25) and $B \equiv [\log \Xi]'(0)$ by (4.8), then $|B - A| < 5 \times 10^{-6}$ reflects how close the function $\log \Xi(x)$ stays to the parabolic shape $Ax(1 - x)$ over the interval $[0, 1]$.

A.2 Superzeta Functions of the First Kind

These functions are now most easily reached as follows: first, \mathcal{Z}_0 from \mathcal{Z}_0 through the confluence identity (5.8) (and the known exact values at $s \in -\mathbb{N}$); then, $\mathcal{Z}(s|t)$ for $t \neq 0$ by the analog of the expansion (8.19), namely

$$\mathcal{Z}(s|t) = \sum_{k=1}^{\infty} (\rho - \frac{1}{2})^{-s} \left[1 + \frac{t}{\rho - \frac{1}{2}} \right]^{-s} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-s)}{\Gamma(1-s-\ell) \ell!} \mathcal{Z}_0(s+\ell) t^\ell$$

(for $s \in -\mathbb{N}$, one has to cancel the poles of $\mathcal{Z}_0(s+\ell)$ and $\Gamma(1-s-\ell)$ at $\ell = 1-s$ by hand, or else directly use the special values from Table 7.1). Again, for $t = \frac{1}{2}$ about four terms sufficed to match our accuracy for \mathcal{Z} ; for $s > 1$, $\mathcal{Z}_*(s)$ stays very close to $\mathcal{Z}_0(s)$ but no longer for $s \approx 0$, contrary to $\mathcal{Z}_*(s)$ vs $\mathcal{Z}_0(s)$. Here, we could also have used Mellin integral representations like (7.6) for $s < 1$ at any t , but with the same difficulties as above (plus an extra one for $t = \frac{1}{2}$, when the pole merges with the integration endpoint).

We present a rough plot of the two particular functions of first kind $\mathcal{Z}_0(s)$ ($t = 0$) and $\mathcal{Z}_*(s)$ ($t = \frac{1}{2}$) (Fig. A.1). For $s \leq -2$ we show their leading asymptotic form, namely (7.28) cut at its ($n = 2$) term $(-\Gamma(s)^{-1}(\log 2)^s / 2^{\frac{1}{2}+t})$.

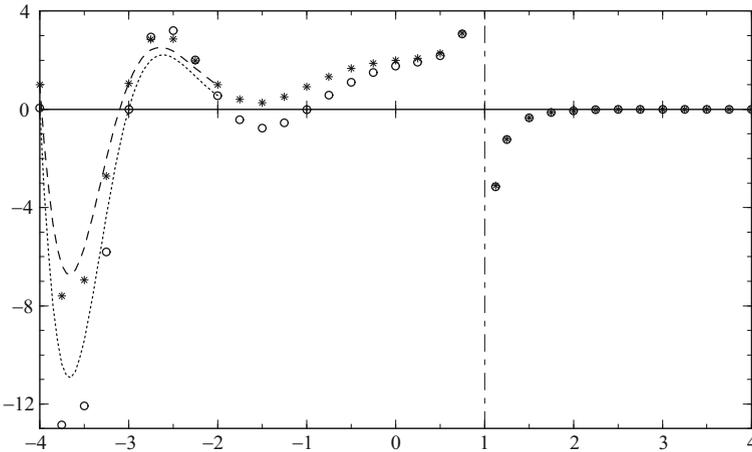


Fig. A.1 The superzeta functions of first kind \mathcal{Z}_0 (o) and \mathcal{Z}_* (*) over the Riemann zeros. The asymptotic forms (dashed curves) are shown for $s \leq -2$; they are exact at integer s

This form is exact at integer $s < 0$ due to the $1/\Gamma(s)$ factor in (7.28), otherwise the next asymptotic term ($n = 3$) is of order unity in the plotted region. (The corresponding graphs for functions of second kind would fall to the σ -axis very fast on the $\sigma > \frac{1}{2}$ side and be chopped up by poles on the whole $\sigma \leq \frac{1}{2}$ side, so we skip them.)

A.3 Numerical Tables

Here is a sample of our results, extending [106, Table 2]. We saw earlier such Tables for the *special values* $\sigma_n \stackrel{\text{def}}{=} \mathcal{Z}_*(n)$ ($n = 1, 2, \dots$) only, albeit with *much* greater accuracy and range ($n_{\text{max}} = 400$) [72, Table 5][64, Appendix].

Table A.1 Numerical values for the superzeta functions of first kind \mathcal{Z}_0 and \mathcal{Z}_* over the Riemann zeros. Implied precision is expected to hold, but not guaranteed (*: exact values)

s	$\mathcal{Z}_0(s) = \sum_{\rho} (\rho - \frac{1}{2})^{-s}$	$\mathcal{Z}_*(s) = \sum_{\rho} \rho^{-s}$
-2	$\frac{9}{16} = 0.5625^*$	1^*
-3/2	-0.76819	0.27288
-1	$-\frac{1}{48}^* \approx -0.0208333$	$\frac{11}{12}^* \approx 0.9166667$
-1/2	1.11061	1.66919
0	$\frac{7}{4} = 1.75^*$	2^*
<i>derivative at 0</i>	0.8118179	0.3465736
+1/2	2.19070	2.27780
<i>finite part at 1</i>	0.9189385	0.9420342
+1 (pole!)	0^*	0.0230957
+3/2	-0.350386	-0.346357
+2	-0.0462100	-0.0461543
+5/2	-0.0052348	-0.0055027
+3	0^*	-0.0001112
+7/2	0.0002259	0.0002022
+4	0.0000743	0.0000736

Table A.2 As above but for the functions of second kind \mathcal{Z}_0 and \mathcal{Z}_* (here the finite parts at $1/2$ have no analytic expressions to our knowledge, and are provided purely numerically)

σ	$\mathcal{Z}_0(\sigma) = \sum_{k=1}^{\infty} \tau_k^{-2\sigma}$	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} (\tau_k^2 + \frac{1}{4})^{-\sigma}$
-1	$-\frac{9}{32} = -0.28125^*$	$-\frac{1}{16} = -0.0625^*$
-3/4	0.54319	1.69388
-1/4	0.785321	0.800805
0	$\frac{7}{8} = 0.875^*$	$\frac{7}{8} = 0.875^*$
<i>Derivative at 0</i>	0.8118179	0.8060429
+1/4	1.549060	1.548829
<i>Finite part at 1/2</i>	0.251637	0.251546
+3/4	0.247760	0.247730
+1	0.0231050	0.0230957
+5/4	0.0037016	0.0036988
+3/2	0.0007295	0.0007287
+7/4	0.0001597	0.0001595
+2	0.0000372	0.0000371

Appendix B

The Selberg Case

We briefly describe how the our framework of Chap. 10 also successfully adapts to the somewhat different case of superzeta functions built over the zeros of *Selberg zeta functions* for compact hyperbolic surfaces.

The close formal analogy between this “Selberg case” and the general case (Chap. 10) allows us to keep the benefit of the whole procedure used there. However, several differences prevent us from transferring the results and Tables verbatim: mainly, the order of the Selberg zeta functions is $\mu_0 = 2$ vs 1 previously, and some of their nontrivial zeros are *real*, something we forbade before. (A few scattered details prove simpler, though.)

We will only recall selected results, referring to the literature for details, e.g., [12, 50, 91, 97], and to Sect. 6.3.1 for the basic background and notation; we just add that the Gauss–Bonnet formula is implicitly used henceforth: $\mathcal{A}_S/4\pi \equiv g-1$, where $g (= g_S)$ is the genus of the surface S (an integer ≥ 2).

For a compact hyperbolic surface S (henceforth fixed, with normalized curvature -1), its Selberg zeta function, specified by the Euler product

$$\zeta_S(x) \stackrel{\text{def}}{=} \prod_{\{\varpi\}} \prod_{k=0}^{\infty} (1 - e^{-\ell_{\varpi}(x+k)}) \quad (\text{Re } x > 1) \quad (\text{B.1})$$

continues to an entire function of order $\mu_0 = 2$, which has a functional equation,

$$\zeta_S\left(\frac{1}{2} + t\right) \equiv \exp\left[4(g-1) \int_0^t \pi t' \tan \pi t' dt'\right] \zeta_S\left(\frac{1}{2} - t\right), \quad (\text{B.2})$$

and two classes of zeros:

- Trivial zeros $\{-k\}$ with multiplicities $2(g-1)(2k+1)$, $k \in \mathbb{N}$
- Nontrivial zeros $\rho = \{\frac{1}{2} \pm i\kappa_k\}_{k=0,1,\dots}$ where $\frac{1}{4} + \kappa_k^2$ are the eigenvalues of the positive Laplacian on S (both zeros and eigenvalues being counted with multiplicities, as ever). The lowest eigenvalue 0 (the zero-mode), non-degenerate, yields two zeros $\rho = 1$ (simple) and 0 (then of total multiplicity $2(g-1) + 1$). To avoid complications, we exclude the exceptional case of an eigenvalue $= \frac{1}{4}$; now, however, finitely many other eigenvalues may exist in $(0, \frac{1}{4})$, adding zeros ρ on the real segment $(0, 1)$ save at $x = \frac{1}{2}$.

So, $\zeta_S(x)$ evokes a general primary function $L(x)$ as in Sect. 10.1, but with order $\mu_0 = 2$ and at least one pair of real nontrivial zeros (0 and 1). We can then still define generalized Stieltjes cumulants $g_n^c\langle S \rangle$ here by (10.5), but with $q = -1$:

$$\log \frac{\zeta_S(x)}{x-1} \equiv \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} g_n^c\langle S \rangle (x-1)^n \quad (\text{B.3})$$

$$(g_0^c\langle S \rangle = -\log \zeta_S'(1), \quad g_1^c\langle S \rangle = \frac{\zeta_S''(1)}{2\zeta_S'(1)}, \quad g_2^c\langle S \rangle = -\frac{\zeta_S'''(1)}{3\zeta_S'(1)} + \left[\frac{\zeta_S''(1)}{2\zeta_S'(1)} \right]^2, \dots).$$

Note: $\zeta_S'(1) > 0$, because $\zeta_S(x) > 0$ for $x > 1$ by (B.1) and its zero at $x = 1$ is simple. For $n \geq 1$, cf. also [47] (where $\gamma^{(n-1)} \equiv (-1)^{n-1} g_n^c\langle S \rangle / (n-1)!$).

B.1 Superzeta Functions of the First Kind

We can define $\mathcal{Z}(s|t)$ as in (10.6), and then the allowed t -domain Ω_1 also excludes the cut $(-\infty, +\frac{1}{2}]$ due to the real zeros of ζ_S starting with $x = 1$. A real $t < \frac{1}{2}$ now needs to be specified as $t \pm i0$ (only the special values $\mathcal{Z}(n|t)$ are continuous on this cut). The confluent case \mathcal{Z}_0 accordingly splits into two functions $\mathcal{Z}_{0\pm}$. The limit $t \rightarrow \frac{1}{2}$ will require us to remove the zero-mode, as

$$\mathcal{Z}_*(s|t) = \sum_{\rho \neq (0,1)} (\rho + t - \frac{1}{2})^{-s} = \mathcal{Z}(s|t) - (t - \frac{1}{2})^{-s} - (t + \frac{1}{2})^{-s}. \quad (\text{B.4})$$

The completion and Hadamard factorization of ζ_S usually involve the Barnes G -function and elaborate fudge factors [37, 95, 103]. Instead, we will decompose ζ_S directly in zeta-regularized factors. Specifically, the sequence

$$x_k = k + \frac{1}{2} \quad \text{with multiplicity } (2k + 1) \quad (\text{B.5})$$

is *theta-eligible* of order $\mu_0 = 2$, with the Theta function

$$\theta(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (2k + 1) e^{-(k+1/2)z} = \frac{d}{dz} \frac{1}{\sinh \frac{1}{2}z} \quad (\text{B.6})$$

$$= \sum_{m=-1}^{\infty} \mathbf{c}_{-2m} z^{2m}, \quad \mathbf{c}_{-2n+2} = \frac{2(2n-1)}{(2n)!} (1 - 2^{1-2n}) B_{2n}, \quad (\text{B.7})$$

the generalized zeta function

$$\mathbf{z}(s|t) = \sum_{k=0}^{\infty} (2k + 1) (k + \frac{1}{2} + t)^{-s} \quad (\text{Re } s > 2), \quad (\text{B.8})$$

and the zeta-regularized product $\mathbf{d}(\frac{1}{2} + t) = e^{-\mathbf{z}'(0|t)}$. The zeta function over the trivial zeros of ζ_S , patterned on (7.3), is then $\mathbf{Z} = 2(g - 1)\mathbf{z}$. Thereupon, in analogy with (10.15), we can write [104]

$$\zeta_S(x) \equiv \mathbf{d}(x)^{2(g-1)} \mathcal{D}(x) \equiv \mathbf{d}(x)^{2(g-1)} x(x - 1) \mathcal{D}_*(x) \quad (x \equiv \frac{1}{2} + t), \quad (\text{B.9})$$

defining complementary (S -dependent) factors $\mathcal{D}(x)$ (which keeps all the non-trivial zeros of ζ_S), and $\mathcal{D}_*(x)$ (without the zero-mode, as needed in the limit $x \rightarrow 1$). The same churning as in the Riemann case (Sect. 7.2.2) then converts (B.9) into integral representations similar to (7.4)–(7.6):

$$\mathcal{Z}(s|t) = -\mathbf{Z}(s|t) + \frac{\sin \pi s}{\pi} \int_0^\infty \frac{\zeta'_S(\frac{1}{2} + t + y)}{\zeta_S(\frac{1}{2} + t + y)} y^{-s} dy \quad (\text{Re } s < 1), \quad (\text{B.10})$$

$$\mathcal{Z}_*(s|t) = -\mathbf{Z}(s|t) - (t + \frac{1}{2})^{-s} + \frac{\sin \pi s}{\pi} \int_0^\infty \left[\frac{\zeta'_S(\frac{1}{2} + t + y)}{\zeta_S(\frac{1}{2} + t + y)} - \frac{1}{t - \frac{1}{2} + y} \right] y^{-s} dy,$$

which are, however, discontinuous across the t -plane cut $(-\infty, +\frac{1}{2}]$.

These entail that $\mathcal{Z}(s|t)$ and $\mathcal{Z}_*(s|t)$ have exactly the s -plane singularities of $-\mathbf{Z}(s|t)$, and that \mathcal{Z} has the special values

$$\mathcal{Z}(-n|t) = -\mathbf{Z}(-n|t) \quad (n \in \mathbb{N}), \quad (\text{B.11})$$

$$\mathcal{Z}'(0|t) = -\mathbf{Z}'(0|t) - \log \zeta_S(\frac{1}{2} + t), \quad (\text{B.12})$$

$$\mathcal{Z}(+n|t) = -\mathbf{Z}(+n|t) + \frac{(-1)^{n-1}}{(n-1)!} (\log \zeta_S)^{(n)}(\frac{1}{2} + t) \quad (n \in \mathbb{N}^*), \quad (\text{B.13})$$

where *finite parts apply to \mathcal{Z} , \mathbf{Z} for $1 \leq n \leq \mu_0 = 2$* ; and similarly for \mathcal{Z}_* using (B.4). Except for the derivative formula (B.12), the special values have (understandably) regained single-valuedness about the real- t axis.

The *glitch* in the Selberg case is that, unlike before, the functions $\mathbf{z}(s|t)$ and hence $\mathbf{Z}(s|t)$ are not known in closed form for general t , making the above formulae of little use. Rescue, however, comes from two directions.

- The *theta-eligibility* trivially transfers to the shifted sequence $\{x_k + t\}$,

$$\sum_{k=0}^\infty (2k + 1) e^{-(k+1/2+t)z} = e^{-tz} \theta(z) = \sum_{n=-2}^\infty \mathbf{c}_{-n}(t) z^n \quad (\text{B.14})$$

with computable polynomial $\mathbf{c}_{-n}(t)$, so all of the algebraic information can now flow from (2.58), yielding that $-\mathbf{Z}(s|t)$ and hence $\mathcal{Z}(s|t)$ have: *only two poles, $s = 2$ and $s = 1$, both simple and of respective residues $-4(g - 1)$ and $4(g - 1)t$, and the rational special values $-2(g - 1)(-1)^n n! \mathbf{c}_{-n}(t)$ at $s = -n$* ; the latter result completes Table B.1 for general t , as it is shown.

- The function $\mathbf{Z}(s|t)$ can be specified for particular values of t , including our favorite locations $t = 0$ and $\frac{1}{2}$, where by simple inspection,

$$\mathbf{z}(s|0) = 2(2^{s-1} - 1)\zeta(s - 1), \quad \mathbf{z}(s|\frac{1}{2}) = 2\zeta(s - 1) - \zeta(s), \quad (\text{B.15})$$

and this allows us to compute the corresponding special values as displayed in Tables B.2 and B.3. We recall that the confluent-case function now has two principal determinations $\mathcal{Z}_{0\pm}$ corresponding to $\text{Im } t \gtrless 0$ (Table B.2), and that the $t = \frac{1}{2}$ function (Table B.3) is now $\mathcal{Z}_*(s) \stackrel{\text{def}}{=} \mathcal{Z}_*(s|\frac{1}{2})$, excluding the zero mode which would cause divergence.

As a final remark, such functions of first kind for the Selberg case have never been considered before to our knowledge.

B.2 Superzeta Functions of the Second Kind

We can define $\mathcal{Z}(\sigma|t) = \sum_{k=0}^{\infty} (\kappa_k^2 + t^2)^{-\sigma}$ as in (10.7), but with the extra cut $(-\infty, +\frac{1}{2}]$ in the allowed t -domain due to the real zeros of ζ_S starting with $x = 1$. A real $t < \frac{1}{2}$ now needs to be specified as $t \pm i0$ (except for the special values $\mathcal{Z}(n|t)$). To accommodate the case $t = \frac{1}{2}$ it is useful to also define the sum without the zero-mode $k = 0$, denoted $\mathcal{Z}_*(\sigma|t) = \mathcal{Z}(\sigma|t) - (t^2 - \frac{1}{4})^{-\sigma}$.

In the Selberg case, only this kind of superzeta function has attracted some attention before, and mainly for $t = 0$ [15, 16][105, Sect. 4] and $t = \frac{1}{2}$ [47, 91, 100].

Our successive techniques of Chap. 8 all adapt to the present case.

For $t = 0$ (the confluent case), one must now specify on which side of the cut $(-\infty, \frac{1}{2}]$ the limit $t \rightarrow 0^+$ is taken; thus, two principal function determinations $\mathcal{Z}_{0\pm}$ emerge, and each satisfies the same confluence identity (8.2) as before with $\mathcal{Z}_{0\pm}$ (Table B.5; explicit values were given in [16, Sect. 6.4]). For general t , the expansion (8.19) about $t = 0$ works as previously; now it yields that $\mathcal{Z}(\sigma|t)$ has a *single pole* at $\sigma = 1$, *simple and of residue* $(g - 1)$, and rational special values $\mathcal{Z}(-m|t)$ as given in Table B.4 (upper part).

For the transcendental values, we again use the idea of zeta-regularizing the nontrivial factor \mathcal{D} of $\zeta_S(\frac{1}{2} + t)$ but in the variable $v = t^2$. The canonical large- t expansion (derivable from (B.7), (2.62), and $\log \zeta_S(\frac{1}{2} + t) = O(t^{-\infty})$),

$$\begin{aligned} \log \mathcal{D}(\frac{1}{2} + t) &\sim \tilde{a}_2 t^2 (\log t - \frac{3}{2}) + \tilde{a}_0 \log t + O(t^{-2}), \\ \tilde{a}_2 = -2(g - 1), \quad \tilde{a}_0 = -\frac{1}{6}(g - 1) &\quad (\text{from } \mathbf{c}_2 = 2, \mathbf{c}_0 = \frac{1}{12}), \end{aligned} \quad (\text{B.16})$$

becomes, in the variable $v = t^2$,

$$\log \mathcal{D}(\frac{1}{2} + v^{1/2}) \sim -(g - 1)[v(\log v - 1) - \underline{2}v + \frac{1}{12} \log v] (+O(v^{-1})), \quad (\text{B.17})$$

where we have readily underlined a newly born *banned* term in the v variable. This term must be killed to get the zeta-regularized form in v , which therefore has to be $\mathcal{D}(v) = e^{-2(g-1)v} \mathcal{D}(\frac{1}{2} + t)$; this enters a variant of the factorization (B.9), analogous to (8.18) [104],

$$\zeta_S(\frac{1}{2} + t) = [e^{t^2} \mathbf{d}(\frac{1}{2} + t)]^{2(g-1)} \mathcal{D}(v), \quad v \equiv t^2. \quad (\text{B.18})$$

As in the Riemann case (Table 8.2), the transcendental values of second kind are best expressed in terms of those of first kind, already found. We then use

$$\log \mathcal{D}(v) = \log \mathcal{D}(\frac{1}{2} + t) - 2(g-1)t^2. \quad (\text{B.19})$$

Immediately, $\log \mathcal{D}(v) \equiv -\mathcal{Z}'_{(\sigma)}(0|t)$ and $\log \mathcal{D}(\frac{1}{2} + t) \equiv -\mathcal{Z}'_{(s)}(0|t)$ entail

$$\mathcal{Z}'_{(\sigma)}(0|t) = \mathcal{Z}'_{(s)}(0|t) + 2(g-1)t^2. \quad (\text{B.20})$$

The v -derivative of (B.19), using (2.53) and (2.54) with $x = v$ then $x = t$, yields

$$\text{FP}_{\sigma=1} \mathcal{Z}(\sigma|t) = (1/2t) \text{FP}_{s=1} \mathcal{Z}(s|t) - 2(g-1) \quad (t \neq 0). \quad (\text{B.21})$$

As for higher v -derivatives, using (2.56) with \mathcal{D} for which $\mu_0 = 1$, they give

$$\mathcal{Z}(m|t) = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^m}{d(t^2)^m} \log \mathcal{D}(\frac{1}{2} + t) \quad (m = 2, 3, \dots), \quad (\text{B.22})$$

which reduces to the same linear combination of the $\log \mathcal{D}^{(n)}(\frac{1}{2} + t)$ as in (8.32). Now it matters that the general formula (2.53) for $\log \mathcal{D}^{(n)}$ outputs $\mathcal{Z}_\infty(n|t)$ and not $\text{FP}_{s=n} \mathcal{Z}(s|t)$: the two differ when $1 < n \leq \mu_0$, and in the t variable with $\mu_0 = 2$ there is one such n , namely $n = 2$. Then by (B.16),

$$\mathcal{Z}_\infty(n|y) - \text{FP}_{s=n} \mathcal{Z}(n|y) = (-1)^n n H_{n-1} \tilde{a}_n(y) = -\delta_{n,2} 4(g-1) \quad (\text{B.23})$$

(with $\tilde{a}_2(y) \equiv \tilde{a}_2$ constant here). All that accounts for Table B.4 (lower part).

As an illustration, we fully develop the first few transcendental values of the case $t = \frac{1}{2}$ (Table B.6), referring to our cumulants (B.3) and to Table B.3:

$$\begin{aligned} \mathcal{Z}'_{*(\sigma)}(0) &= \mathcal{Z}'_{*(s)}(0) + \frac{1}{2}(g-1) \\ &= -4(g-1) [\zeta'(-1) + \frac{1}{4} \log 2\pi] + g_0^c \langle S \rangle + \frac{1}{2}(g-1) \\ &= (g-1) (-4\zeta'(-1) - \log 2\pi + \frac{1}{2}) + g_0^c \langle S \rangle, \end{aligned} \quad (\text{B.24})$$

$$\begin{aligned} \text{FP}_{\sigma=1} \mathcal{Z}_*(\sigma) &= \text{FP}_{s=1} \mathcal{Z}_*(s) - 2(g-1) \\ &= 2(g-1)(1 + \gamma) + g_1^c \langle S \rangle - 1 - 2(g-1) \\ &= 2(g-1) \gamma + g_1^c \langle S \rangle - 1, \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned}
 \mathcal{Z}_*(2) &= 2 \mathcal{L}_{*,\infty}(1) + \mathcal{L}_{*,\infty}(2) = 2 \text{FP}_{\sigma=1} \mathcal{L}_*(s) + (\text{FP}_{\sigma=2} \mathcal{L}_*(s) + 2\tilde{a}_2) \\
 &= 2[2(g-1)(1+\gamma) + g_1^c(S) - 1] \\
 &\quad + [-4(g-1)(\gamma - \frac{1}{12}\pi^2) + g_2^c(S) - 1] - 4(g-1) \\
 &= \frac{1}{3}(g-1)\pi^2 + 2g_1^c(S) + g_2^c(S) - 3,
 \end{aligned}
 \tag{B.26}$$

using (B.23), and those agree with [100, (20), (21), (25)][47, Theorem A(2)].

Formulae for the general $\mathcal{Z}_*(m)$ are also given in [47, 100] but in recursive form, whereas Table B.6 exhibits a more closed structure.

B.3 Tables of Special-Value Formulae (Selberg Cases)

Table B.1 Special values of the function of first kind $\mathcal{Z}(s|t)$ over the zeros $\{\rho\}$ of the Selberg zeta function $\zeta_S(x)$ for a compact hyperbolic surface S of genus g . Notation: see (1.4), (1.7), (1.9), (B.1), (B.8); n is an integer

s	$\mathcal{Z}(s t) = \sum_{\rho} (\rho + t - \frac{1}{2})^{-s}$
$-n \leq 0$	$-\frac{4(g-1)}{(n+1)(n+2)} \sum_{0 \leq m \leq \frac{n}{2}+1} \binom{n+2}{2m} (2m-1)(1-2^{1-2m}) B_{2m} t^{n-2m+2}$
0	$-2(g-1)(t^2 + \frac{1}{12})$
0 (s -derivative)	$\mathcal{Z}'(0 t) = -2(g-1)\mathbf{z}'(0 t) - \log \zeta_S(\frac{1}{2} + t)$
$+n \geq 1$	$-2(g-1)\mathbf{z}(n t) + \frac{(-1)^{n-1}}{(n-1)!} (\log \zeta_S)^{(n)}(\frac{1}{2} + t)^\dagger$

[†] At $n = 1, 2$: finite parts $\text{FP}_{s=n}$ required for \mathcal{Z} and \mathbf{z}

Table B.2 Specialization of Table B.1 to $t = \pm i0$; all the values except the derivative are actually continuous at $t = 0$. Compare with Table 10.5

s	$\mathcal{Z}_{0\pm}(s) \equiv \sum_{\rho} (\rho \pm i0 - \frac{1}{2})^{-s}$
$-n \leq 0$	$-4(g-1)(1-2^{-n-1}) \frac{B_{n+2}}{n+2}^\dagger$
0	$-(g-1)/6$
0 (s -derivative)	$[\mathcal{Z}_{0\pm}]'(0) = 2(g-1) [\frac{1}{12} \log 2 + \zeta'(-1)] - \log \zeta_S(\frac{1}{2} \pm i0)$
1 $+2$ (finite part)	0^\dagger^* $\text{FP}_{s=2} \mathcal{Z}_{0\pm}(s) = -4(g-1)(\gamma + 2 \log 2) - (\log \zeta_S)''(\frac{1}{2})$
$+n \neq 2$	$-4(g-1)(2^{n-1} - 1) \zeta(n-1) + \frac{(-1)^{n-1}}{(n-1)!} (\log \zeta_S)^{(n)}(\frac{1}{2})^\dagger$

[†] For n odd, $\mathcal{Z}_0(n) \equiv 0$ which makes $(\log \zeta_S)^{(n)}(\frac{1}{2})$ fully explicit, as in (C.4)

* Exceptionally at $t = 0, s = 1$ is a regular point

Table B.3 As Table B.1, but at $t = \frac{1}{2}$ and with the zero-mode removed. For the generalized Stieltjes cumulants $g_n^c \langle S \rangle$, see (B.3). Compare with Table 10.6

s	$\mathcal{Z}_*(s) \equiv \sum_{\rho \neq 0,1} \rho^{-s}$
$-n < 0$	$-2(g-1)(-1)^n \left[2 \frac{B_{n+2}}{n+2} + \frac{B_{n+1}}{n+1} \right] - 1$
0	$\frac{2}{3}(g-1) - 2$
0 (derivative)	$\mathcal{Z}'_*(0) = -4(g-1) \left[\zeta'(-1) + \frac{1}{4} \log 2\pi \right] + g_0^c \langle S \rangle$
+1 (finite part)	$\text{FP}_{s=1} \mathcal{Z}_*(s) = 2(g-1)(1+\gamma) + g_1^c \langle S \rangle - 1$
+2 (finite part)	$\text{FP}_{s=2} \mathcal{Z}_*(s) = -4(g-1) \left(\gamma - \frac{1}{12} \pi^2 \right) + g_2^c \langle S \rangle - 1$
$+n > 2$	$-2(g-1) [2\zeta(n-1) - \zeta(n)] + \frac{g_n^c \langle S \rangle}{(n-1)!} - 1^\dagger$

† For k even, $\zeta(k) \equiv (2\pi)^k |B_k| / (2k!)$

Table B.4 As Table B.1, but for the function of second kind $\mathcal{Z}(\sigma | t)$; transcendental special values (lower part) are expressed in terms of those for $\mathcal{Z}(s | t)$, given by Table B.1; m, n are integers. Compare with Tables 8.1 (upper part) and 8.2 (lower part)

σ	$\mathcal{Z}(\sigma t) = \sum_{k=0}^{\infty} (\kappa_k^2 + t^2)^{-\sigma}$
$-m \leq 0$	$\frac{g-1}{m+1} \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} (1-2^{1-2n}) B_{2n} t^{2(m+1-n)}$
0	$-(g-1)(t^2 + \frac{1}{12})$
0 (σ -/ s -derivatives)	$\mathcal{Z}'_{(\sigma)}(0 t) = \mathcal{Z}'_{(s)}(0 t) + 2(g-1)t^2$
+1 (finite part)	$\text{FP}_{\sigma=1} \mathcal{Z}(\sigma t) = \frac{1}{2t} \text{FP}_{s=1} \mathcal{Z}(s t) - 2(g-1) \quad (t \neq 0)^*$
$+m \geq 2$	$\sum_{n=1}^m \binom{2m-n-1}{m-1} (2t)^{-2m+n} \mathcal{Z}_\infty(n t)^\dagger \quad (t \neq 0)^*$

* For $t = 0$, see Table B.5 below

† With $\mathcal{Z}_\infty(n | t) = \text{FP}_{s=n} \mathcal{Z}(s | t) - \delta_{n,2} 4(g-1)$, see (B.23) (FP is unneeded for $n > 2$)

Table B.5 As Table B.4, but at $t = (1 \pm i) 0$; all the values except the derivative are actually continuous at $t = 0$. Cf. Table B.2

σ	$\mathcal{Z}_{0\pm}(\sigma) = \sum_{k=0}^{\infty} (\kappa_k^2 \pm i0)^{-\sigma}$
$-m \leq 0$	$\frac{1}{2} (-1)^m \mathcal{Z}_{0\pm}(-2m)$
0 (σ -/ s -derivatives)	$[\mathcal{Z}_{0\pm}]'_{(\sigma)}(0) = [\mathcal{Z}_{0\pm}]'_{(s)}(0)$
+1 (finite part)	$\text{FP}_{\sigma=1} \mathcal{Z}_{0\pm}(\sigma) = -\frac{1}{2} \text{FP}_{s=2} \mathcal{Z}_{0\pm}(s)$
$+m \geq 2$	$\frac{1}{2} (-1)^m \mathcal{Z}_{0\pm}(+2m)$

Table B.6 As Table B.4, but at $t = \frac{1}{2}$ and with the zero-mode removed ($\mathcal{Z}_*(\sigma)$ is the Minakshisundaram–Pleijel zeta function); cf. Table B.3

σ	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} (\kappa_k^2 + \frac{1}{4})^{-\sigma} \equiv \sum_{k=1}^{\infty} v_k^{-\sigma}$
$-m < 0$	$\frac{g-1}{m+1} 2^{-2m-1} \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} (2^{2n-1} - 1) B_{2n}$
0	$-\frac{1}{3}(g-1) - 1$
0 (σ -/ s -derivatives)	$\mathcal{Z}'_{*(\sigma)}(0) = \mathcal{Z}'_{*(s)}(0) + \frac{1}{2}(g-1)$
+1 (finite part)	$\text{FP}_{\sigma=1} \mathcal{Z}_*(\sigma) = \text{FP}_{s=1} \mathcal{Z}_*(s) - 2(g-1)$
+ $m \geq 2$	$\sum_{n=1}^m \binom{2m-n-1}{m-1} \mathcal{Z}_{*,\infty}(n)^\dagger$

[†] With $\mathcal{Z}_{*,\infty}(n) = \text{FP}_{s=n} \mathcal{Z}_*(s) - \delta_{n,2} 4(g-1)$, see (B.23) (FP is unneeded for $n > 2$)

Appendix C

On the Logarithmic Derivatives at $\frac{1}{2}$

We briefly revisit the formulae that involve $(\log |L|)^{(n)}(\frac{1}{2})$ from the viewpoint of the primary function L itself when, as in Sect. 10.6, the latter is either the L -function L_χ of a real primitive Dirichlet character χ (= Dirichlet- L cases) or the Dedekind zeta function ζ_K of an algebraic number field K (= Dedekind- ζ cases, which include $\zeta(x)$ for $K = \mathbb{Q}$).

For even $n > 0$, those derivatives relate to the corresponding superzeta values $\mathcal{Z}_0(n)$ according to Tables 10.3 and 10.5 (last lines), which is far from an explicit evaluation. At $n = 0$, $\log |L|(\frac{1}{2})$ gets related to $\mathcal{Z}'_0(0)$. And even in the basic case: $L = \zeta$, $n = 0$, very little seems to be known about $\zeta(\frac{1}{2})$ [81][11, Knuth's sequence pp. 16–17].

In contrast, for odd $n > 0$, by virtue of (10.59) and (10.78) (which follow from the functional equation (10.3)), those derivatives *get specified*, as

$$(\log |L|)^{(n)}(\frac{1}{2}) \equiv \begin{cases} \frac{1}{2}(n-1)! [c(2^n-1)\zeta(n) + c'2^n\beta(n)], & n = 3, 5, \dots \\ \frac{1}{2}c\gamma + \frac{1}{4}c'\pi + \frac{1}{2}\log[(8\pi)^c/d], & n = 1, \end{cases} \quad (\text{C.1})$$

for suitable (integers) c, c', d : thus in a Dirichlet- L case, by (10.59), $c = 1$, $c' = 1 - 2a$ and d = the modulus (or period) of the character χ ; while in a Dedekind- ζ case, by (10.78), $c = n_K$, $c' = r_1$, and $d = |d_K|$. These formulae are more explicit than for even n but not yet fully: with n being odd, $\frac{1}{2}(n-1)!2^n\beta(n)$ reduces to $\frac{1}{4}\pi^n|E_{n-1}|$ by (3.32), but $\zeta(n)$ (and γ for $n = 1$) remain elusive (irreducible); e.g., $(\log |\zeta|)'(\frac{1}{2}) = \frac{1}{2}\gamma + \frac{1}{4}\pi + \frac{1}{2}\log 8\pi$.

However, related odd- n identities can manage to eliminate $\zeta(n)$ and γ : if (L_1, L_2) is any pair of such functions, each indifferently Dirichlet- L or Dedekind- ζ , then a linear combination (with obvious notation) yields the *fully explicit* relations

$$c_2(\log |L_1|)^{(n)}(\frac{1}{2}) - c_1(\log |L_2|)^{(n)}(\frac{1}{2}) \equiv (c_2c'_1 - c_1c'_2) \frac{1}{4}\pi^n|E_{n-1}| + \delta_{n,1} \frac{1}{2}\log(d_2^{c_1}/d_1^{c_2}) \quad \text{for odd } n \geq 1. \quad (\text{C.2})$$

An immediate example is $L_1 = \zeta$, $L_2 = \beta$, giving [106, (90)]

$$(\log |\zeta|)^{(n)}\left(\frac{1}{2}\right) - (\log \beta)^{(n)}\left(\frac{1}{2}\right) \equiv \frac{1}{2}\pi^n |E_{n-1}| + \delta_{n,1} \log 2 \quad \text{for odd } n \geq 1; \quad (\text{C.3})$$

see also (10.60) for another simple case.

Incidentally, the situation is simpler with Selberg zeta functions ζ_S as in Appendix B: for any odd $n \geq 1$, either the identity $\mathcal{Z}_0(n) = 0$, or Selberg's functional equation (B.2), entails the *fully* explicit values

$$\left. \begin{aligned} (\log |\zeta_S|)^{(n)}\left(\frac{1}{2}\right) &\equiv 4(g-1)(n-1)! (2^{n-1} - 1) \zeta(n-1) \\ &= 2(g-1)(2^{n-1} - 1)(2\pi)^{n-1} |B_{n-1}| \end{aligned} \right\} \text{ for odd } n \geq 1, \quad (\text{C.4})$$

where in particular $n = 1$ gives $(\log |\zeta_S|)'(\frac{1}{2}) \equiv 0$.

Appendix D

On the Zeros of the Zeta Function

by Hj. Mellin (1917)

Annotated translation from German, by A. Voros, of the article:

Über die Nullstellen der Zetafunktion
Ann. Acad. Sci. Fennicae **A10** Nr 11 (1917).

We include this English translation of Mellin's seminal paper featuring the first zeta functions over the Riemann zeros to our knowledge. The original is written in German and available in few libraries, hence we believe this Appendix can make that article accessible to a wider audience.

We indicate the errors and misprints we noticed, but we did not carry out a systematic verification. We have strictly preserved the author's notation, which is therefore not consistent with the rest of this book (cf. our discussion of this article in Sect. 5.5). We have not indexed the contents either.

Numbered footnotes are the author's, the others are translator's notes (A.V.).

On the zeros of the zeta function

§ 1.

The series, extended over all the complex zeros ρ of the zeta function,

$$\sum_{\rho} \frac{1}{|\rho|^{\kappa}},$$

is known to converge ¹⁾ for all $\kappa > 1$ and to diverge for $\kappa < 1$.

It follows that the series

$$\sum_{\rho} \frac{x^{\rho}}{\rho^s}, \quad x > 0,$$

where x is a positive parameter, converges absolutely and uniformly in every finite part of the half-plane

$$\Re(s) \geq 1 + \varepsilon \quad \varepsilon > 0.$$

In an important work ²⁾ dedicated to the above series, Mr LANDAU among others proved that the series converges for $0 < s < 1$ if and only if x is neither 1 nor p^m nor p^{-m} , where p is any prime number and m a positive integer.

I will now prove some of the *analytical continuation* properties of the same series in question, that I noticed several years ago.

Let these functions be defined by the following series:

$$(1) \quad Z(s, x) = \sum_{\rho} \frac{x^{\rho}}{\rho^s},$$

$$(2) \quad Z\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-\rho}}{\rho^s},$$

$$(3) \quad \bar{Z}(s, x) = \sum_{\rho} \frac{x^s}{(-\rho)^s}, \dagger$$

$$(4) \quad \bar{Z}\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-s}}{(-\rho)^s}, \dagger$$

$$(5) \quad Z(s) = \sum_{\rho} \frac{1}{\rho^s},$$

$$(6) \quad \bar{Z}(s) = \sum_{\rho} \frac{1}{(-\rho)^s}.$$

¹⁾ See LANDAU, Handbuch der Lehre von der Verteilung der Primzahlen [Lessons on the distribution of the prime numbers], I, p. 314.

²⁾ See LANDAU, Über die Nullstellen der Zetafunktion [On the zeros of the Zeta function], Math. Annalen 71, p. 548, 1911. [Actually: 1912. (A.V.)]

† *Misprints*: x^s should read x^{ρ} in (3); x^{-s} should read $x^{-\rho}$ in (4). (A.V.)

I always assume the parameter $x > 1$ in the first four series. The expressions

$$\rho^s, (-\rho)^s, x^\rho, x^{-\rho}$$

are defined through

$$\begin{aligned} \rho^s &= e^{s[\log |\rho| + i \operatorname{arc}(\rho)]}, \\ (-\rho)^s &= e^{s[\log |\rho| + i \operatorname{arc}(-\rho)]}, \\ x^\rho &= \frac{1}{x^{-\rho}} = e^{\rho \log x} \end{aligned}$$

with the arc taken between $-\pi$ and $+\pi$.

It now follows that:

The functions Z and \bar{Z} defined by the first four series (1) to (4) are, for all $x > 1$, entire functions of s , which become rational functions of x for $s = 0$ and for negative integer values $s = -n$, so that

$$(7) \quad \begin{cases} Z(0, x) = x - \sum_{\nu=1}^{\infty} x^{-2\nu} = x - \frac{1}{x^2 - 1}, \\ Z(-n, x) = x - \sum_{\nu=1}^{\infty} (-2\nu)^n x^{-2\nu} = x - \left(x \frac{d}{dx}\right)^n \frac{1}{x^2 - 1}, \end{cases}$$

$$(8) \quad \begin{cases} Z\left(0, \frac{1}{x}\right) = -1 - \sum_{\nu=1}^{\infty} x^{-2\nu-1} = -1 - \frac{1}{x(x^2 - 1)}, \\ Z\left(-n, \frac{1}{x}\right) = -\sum_{\nu=1}^{\infty} (2\nu + 1)^n x^{-2\nu-1} = (-1)^{n+1} \left(x \frac{d}{dx}\right)^n \frac{1}{x(x^2 - 1)}, \end{cases}$$

$$(9) \quad \begin{cases} \bar{Z}(0, x) = x - \sum_{\nu=1}^{\infty} x^{-2\nu} = x - \frac{1}{x^2 - 1}, \\ \bar{Z}(-n, x) = (-1)^n x - \sum_{\nu=1}^{\infty} (2\nu)^n x^{-2\nu} = (-1)^n \left[x - \left(x \frac{d}{dx}\right)^n \frac{1}{x^2 - 1} \right], \end{cases}$$

$$(10) \quad \begin{cases} \bar{Z}\left(0, \frac{1}{x}\right) = -1 - \sum_{\nu=1}^{\infty} x^{-2\nu-1} = -1 - \frac{1}{x(x^2 - 1)}, \\ \bar{Z}\left(-n, \frac{1}{x}\right) = -(-1)^n \sum_{\nu=1}^{\infty} (2\nu + 1)^n x^{-2\nu-1} = -\left(x \frac{d}{dx}\right)^n \frac{1}{x(x^2 - 1)}. \end{cases}$$

In these formulas $\left(x \frac{d}{dx}\right)^n$ means the n -th iteration of the operator $x \frac{d}{dx}$.

The functions Z and \bar{Z} defined by the series (5) and (6) are regular in the whole s -plane, except for the simple pole $s = 1$, with the common residue

$$(11) \quad \lim_{s \rightarrow 1} (s - 1)Z(s) = \lim_{s \rightarrow 1} (s - 1)\bar{Z}(s) = -\frac{1}{2}.$$

For $s = 0$ and for negative integers $s = -n$,

$$(12) \quad \begin{cases} Z(0) = 1, \dagger \\ Z(-n) = (2^n - 1)\zeta(-n), \dagger \end{cases}$$

$$(13) \quad \begin{cases} \bar{Z}(0) = -\frac{1}{2}, \ddagger \\ \bar{Z}(-n) = (-1)^{n+1} - 2^n \zeta(-n). \ddagger \end{cases}$$

§ 2.

The function

$$Z\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-\rho}}{\rho^s}.$$

I use the well-known formula ¹⁾

$$(14) \quad F(x, t) = \frac{x^{1-t}}{1-t} + \sum_{\nu=1}^{\infty} \frac{x^{-2\nu-t}}{2\nu+t} - \sum_{\rho} \frac{x^{\rho-t}}{\rho-t} - \frac{\zeta'(t)}{\zeta(t)},$$

which is valid for $x > 1$, and where

$$(15) \quad F(x, t) = \sum_{p^m \leq x} \frac{\log p}{p^{mt}}.$$

The dash means that when x is a prime power, the last term will carry the factor $\frac{1}{2}$.

First, I write (14) as follows:

$$(16) \quad F(x, t) x^t + \frac{\zeta'(t)}{\zeta(t)} x^t - \frac{x}{1-t} = \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{2\nu+t} + \sum_{\rho} \frac{x^{\rho}}{-\rho+t}.$$

Since the series considered here converge uniformly ²⁾ in any finite root-free regions of t , it is permitted to differentiate term by term, which yields:

$$(17) \quad G(x, t) - \frac{x}{(1-t)^2} = \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho+t)^2}.$$

[†] Error: (12) should read $Z(0) = 2$, $Z(-n) = (2^n - 1)\zeta(-n) + 1$, upon correcting (26) below. (A.V.)

[‡] Error: (13) should read $\bar{Z}(0) = \frac{3}{2}$, $\bar{Z}(-n) = (-1)^n - 2^n \zeta(-n)$ upon the corrections in §. 5 below. (A.V.)

¹⁾ See LANDAU, Handbuch, I. p. 353.

²⁾ See LANDAU, Handbuch, I. p. 354.

Here the abbreviation

$$(18) \quad G(x, t) = -\frac{\partial}{\partial t} \left\{ F(x, t) x^t + \frac{\zeta'(t)}{\zeta(t)} x^t \right\}$$

is used. Because of (15) and

$$\frac{\zeta'(t)}{\zeta(t)} = -\sum_{p,m} \frac{\log p}{p^{mt}} \quad \Re(t) > 1,$$

G can be represented in the half-plane

$$\Re(t) \geq 1 + \varepsilon \quad \varepsilon > 0$$

by the absolutely and uniformly convergent series

$$(19) \quad G(x, t) = \sum_{p^m \geq x} \left(\frac{x}{p^m} \right)^t \log p \log \left(\frac{x}{p^m} \right).$$

In case x = a prime power, the first term in (19) vanishes, so that in the angular sector

$$-\frac{\pi}{2} + \varepsilon \leq \arg(t) \leq +\frac{\pi}{2} - \varepsilon$$

for any constant k and any constant $x > 1$,

$$(20) \quad \lim_{t \rightarrow \infty} t^k G(x, t) = 0$$

uniformly. The corresponding expression in (16),

$$F(x, t) x^t + \frac{\zeta'(t)}{\zeta(t)} x^t = -\sum'_{p^m \geq x} \left(\frac{x}{p^m} \right)^t \log p,$$

does not have this property in the case mentioned, because it contains a t -independent, non-vanishing term.

In (17), I now replace t by $1 + t$, multiply by $t^{s-1} dt$ and integrate from $t = 0$ to $t = \infty$. By using the formula

$$\int_0^\infty \frac{t^{s-1} dt}{(a+t)^z} = \frac{\Gamma(s) \Gamma(z-s)}{\Gamma(z)} \cdot \frac{1}{a^{z-s}} \quad \begin{matrix} 0 < \Re(s) < z \\ -\pi < \arg(a) < +\pi \end{matrix}$$

i.e., in this case

$$\int_0^\infty \frac{t^{s-1} dt}{(a+t)^2} = \frac{\pi(1-s)}{\sin \pi s} \cdot \frac{1}{a^{2-s}},$$

then the result is

$$(21) \quad \int_0^\infty \{G(x, 1+t) - xt^{-2}\} t^{s-1} dt = \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^\infty \frac{x^{-2\nu}}{(2\nu+1)^{2-s}} + \sum_\rho \frac{x^\rho}{(1-\rho)^{2-s}} \right].$$

The first series on the right-hand side converges when $x > 1$ for all s , the second one is absolutely convergent for $\Re(s) < 1$. The left-hand side integral, which converges for

$$0 < \Re(s) < 2,$$

can be transformed as follows. In the vicinity of $t = 0$,

$$G(x, 1 + t) - xt^{-2}$$

can be developed in an ordinary power series:

$$G(x, 1 + t) - xt^{-2} = \sum_{\nu=0}^{\infty} C_{\nu} t^{\nu}.$$

Let r be positive and smaller than the radius of convergence of this series. Then

$$\int_0^r \{G(x, 1 + t) - xt^{-2}\} t^{s-1} dt = \sum_{\nu=0}^{\infty} C_{\nu} \frac{r^{s+\nu}}{s + \nu}.$$

The right-hand side is a regular function in the whole s -plane except at the poles $s = 0, -1, -2, \dots$. Next,

$$\int_r^{\infty} \{G(x, 1 + t) - xt^{-2}\} t^{s-1} dt = \int_r^{\infty} G(x, 1 + t) t^{s-1} dt + x \frac{r^{s-2}}{s-2}.$$

The right-hand side integral, because of (20), is an entire function $H(s)$ of s . Setting these expressions for

$$\int_0^{\infty} \{ \} t^{s-1} dt = \int_0^r + \int_r^{\infty}$$

in (21), one gets the formula

$$\sum_{\nu=0}^{\infty} C_{\nu} \frac{r^{s+\nu}}{s + \nu} + H(s) + x \frac{r^{s-2}}{s-2} = \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+1)^{2-s}} + \sum_{\rho} \frac{x^{\rho}}{(1-\rho)^{2-s}} \right],$$

or, if one substitutes s by $2 - s$ and observes that $1 - \rho$ simultaneously with ρ run over all nonreal roots,

$$(22) \quad \sum_{\nu=1}^{\infty} \frac{x^{-2\nu-1}}{(2\nu+1)^s} + \sum_{\rho} \frac{x^{-\rho}}{\rho^s} = x^{-1} \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} C_{\nu} \frac{r^{2-s+\nu}}{2-s+\nu} + H(2-s) - x \frac{r^{-s}}{s} \right].$$

The right-hand side is now manifestly an entire function of s , which has the value -1 for $s = 0$ and vanishes for all negative integer values of s . Since the first series on the left ($x > 1$) is also an entire function of s , which for

$s = 0, -1, -2, -3, \dots$ yields certain rational functions of x , so the function $Z\left(s, \frac{1}{x}\right)$ has the properties stated in §. 1.

§ 3.

The function

$$\bar{Z}(s, x) = \sum_{\rho} \frac{x^{\rho}}{(-\rho)^s}.$$

In this case I multiply (17) by $t^{s-1} dt$ and integrate along the straight lines

$$0 \text{ ----- } \infty e^{i\theta} \qquad 0 < \theta < \frac{\pi}{2}.$$

Let θ be small enough so that no root ρ lies in the angle

$$0 < \text{arc}(t) \leq \theta.$$

Then for $0 < \Re(s) < 1$,

$$\begin{aligned} (23) \quad \int_0^{\infty e^{i\theta}} \left\{ G(x, t) - \frac{x}{(1-t)^2} \right\} t^{s-1} dt &= \\ &= \int_0^{\infty e^{i\theta}} \left\{ \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho+t)^2} \right\} t^{s-1} dt \\ &= \int_0^{\infty} \left\{ \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho+t)^2} \right\} t^{s-1} dt \\ &= \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^{2-s}} \right]. \end{aligned}$$

The integral (23) can be transformed as follows:

$$\begin{aligned} \int_0^{\infty e^{i\theta}} \left\{ \right\} t^{s-1} dt &= \int_0^{\infty e^{i\theta}} G(x, t) t^{s-1} dt + x \int_0^{\infty e^{i\theta}} \frac{t^{s-1} dt}{(-1+t)^2} \\ &= \int_0^{re^{i\theta}} G t^{s-1} dt + \int_{re^{i\theta}}^{\infty e^{i\theta}} G t^{s-1} dt + x \int_0^{-\infty} \frac{t^{s-1} dt}{(-1+t)^2} \\ &= \int_0^{re^{i\theta}} G t^{s-1} dt + \int_{re^{i\theta}}^{\infty e^{i\theta}} G t^{s-1} dt + x e^{\pi i s} \int_0^{\infty} \frac{t^{s-1} dt}{(1+t)^2}. \end{aligned}$$

r is smaller than the radius of convergence of

$$G(x, t) = \sum_{\nu=0}^{\infty} b_{\nu} t^{\nu},$$

so that

$$\int_0^{re^{i\theta}} G t^{s-1} dt = \sum_{\nu=0}^{\infty} b_{\nu} \frac{(re^{i\theta})^{s+\nu}}{s+\nu}.$$

The second integral on the right is, because of (20), an entire function: $\bar{H}(s)$ of s :

$$\int_{re^{i\theta}}^{\infty e^{i\theta}} G t^{s-1} dt = \bar{H}(s),$$

while

$$\int_0^{\infty} \frac{t^{s-1} dt}{(1+t)^2} = \frac{\pi(1-s)}{\sin \pi s}.$$

This puts (23) in the form

$$\sum_{\nu=0}^{\infty} b_{\nu} \frac{(re^{i\theta})^{s+\nu}}{s+\nu} + \bar{H}(s) + x e^{\pi i s} \frac{\pi(1-s)}{\sin \pi s} = \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^{2-s}} \right].$$

Replacing s by $2-s$, one finally gets

$$(24) \quad \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^s} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^s} = \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} b_{\nu} \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + \bar{H}(2-s) \right] + x e^{\pi i s}.$$

The right-hand side is obviously an entire function of s , which is equal to $(-1)^n x$ for $s=0$ and for negative integer values $s=-n$. The first series on the left of (24) is manifestly ($x > 1$) also an entire function of s , which for $s=0, -1, -2, \dots$ gives some easily determined rational functions of x . It follows directly that $\bar{Z}(s, x)$ has the properties listed in §. 1.

§ 4.

The function

$$Z(s) = \sum_{\rho} \frac{1}{\rho^s}.$$

From the known formula ¹⁾

$$\begin{aligned} \frac{\zeta'(t)}{\zeta(t)} &= C - \frac{1}{t-1} - \frac{1}{2} \frac{\Gamma'(\frac{t}{2}+1)}{\Gamma(\frac{t}{2}+1)} + \sum_{\rho} \left(\frac{1}{t-\rho} + \frac{1}{\rho} \right) \\ &= C_1 - \frac{1}{t-1} + \frac{1}{t+2} + \sum_{\nu=1}^{\infty} \left(\frac{1}{t+2+2\nu} - \frac{1}{2\nu} \right) + \sum_{\rho} \left(\frac{1}{t-\rho} + \frac{1}{\rho} \right) \end{aligned}$$

¹⁾ See LANDAU, Handbuch, I. p. 316.

it follows by differentiation

$$(25) \quad -\left(\frac{\zeta'(t)}{\zeta(t)}\right)' + \frac{1}{(t-1)^2} = \sum_{\nu=1}^{\infty} \frac{1}{(t+2\nu)^2} + \sum_{\rho} \frac{1}{(t-\rho)^2}.$$

I replace t by $t + 1$ and, through integration for $0 < \Re(s) < 1$, obtain

$$-\int_0^{\infty} \left\{ \left(\frac{\zeta'(1+t)}{\zeta(1+t)}\right)' - \frac{1}{t^2} \right\} t^{s-1} dt = \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^{2-s}} + \sum_{\rho} \frac{1}{(1-\rho)^{2-s}} \right].$$

The integral allows exactly the same transformations as the corresponding integral in §. 2. One thus arrives, after replacing $2 - s$ by s , at the end result

$$(26) \quad \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^s} + \sum_{\rho} \frac{1}{\rho^s} = \left(1 - \frac{1}{2^s}\right) \zeta(s) + Z(s)^\dagger$$

$$= \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} C_{\nu} \frac{r^{2-s+\nu}}{2-s+\nu} + \frac{r^{-s}}{s} + H(2-s) \right],$$

where H is an entire function. This results in the properties listed in §. 2[‡] concerning $Z(s)$.

§ 5.

The function

$$\bar{Z}(s) = \sum_{\rho} \frac{1}{(-\rho)^s}.$$

From (25) one obtains, by integration along the lines

$$0 \text{-----} \infty e^{i\theta} \qquad 0 < \theta < \frac{\pi}{2},$$

$$\begin{aligned} & -\int_0^{\infty e^{i\theta}} \left(\frac{\zeta'(t)}{\zeta(t)}\right)' t^{s-1} dt = \\ & = -\int_0^{\infty e^{i\theta}} \frac{t^{s-1} dt}{(-1+t)^2} + \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+t)^2} dt + \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\rho} \frac{1}{(-\rho+t)^2} dt \\ & = -\int_0^{-\infty} \frac{t^{s-1} dt}{(-1+t)^2} + \int_0^{+\infty} t^{s-1} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+t)^2} dt + \int_0^{+\infty} t^{s-1} \sum_{\rho} \frac{1}{(-\rho+t)^2} dt \\ & = \frac{\pi(1-s)}{\sin \pi s} \left[e^{\pi i s} + \sum_{\nu=1}^{\infty} \frac{1}{(2\nu)^{2-s}} + \sum_{\rho} \frac{1}{(-\rho)^{2-s}} \right]^*, \end{aligned}$$

[†] Error: the middle member of (26) should read $(1 - \frac{1}{2^s})\zeta(s) - 1 + Z(s)$, and this affects the results (12) above. (A.V.)

[‡] Actually: §. 1. (A.V.)

* Error: the term $e^{\pi i s}$ inside the brackets should read $-e^{\pi i s}$, and this error propagates to (27) then to (13). (A.V.)

where θ is so small that it can be presumed there is no ρ in the angle

$$0 < \text{arc}(t) \leq \theta.$$

Just as in §. 3, the first integral can be brought to the form

$$\sum_{\nu=0}^{\infty} C_{\nu} \frac{(re^{i\theta})^{s+\nu}}{s+\nu} + \bar{H}(s),$$

where \bar{H} is an entire function. The final result is, after replacing $2-s$ by s ,

$$\begin{aligned} (27) \quad e^{-\pi is} + \sum_{\nu=1}^{\infty} \frac{1}{(2\nu)^s} + \sum_{\rho} \frac{1}{(-\rho)^s} \\ = e^{-\pi is} + \frac{1}{2^s} \zeta(s) + \bar{Z}(s) \dagger \\ = \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} C_{\nu} \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + \bar{H}(2-s) \right]. \end{aligned}$$

There follows the validity of the claims issued in §. 1 concerning $\bar{Z}(s)$.

§ 6.

The function

$$Z(s, x) = \sum_{\rho} \frac{x^{\rho}}{\rho^s}.$$

In formula (14) I replace t by $-t$ and use the formula

$$\frac{\zeta'(-t)}{\zeta(-t)} = -\log(2\pi) + \frac{\pi}{2} \cot \frac{\pi t}{2} + \frac{\Gamma'(1+t)}{\Gamma(1+t)} + \frac{\zeta'(1+t)}{\zeta(1+t)}.$$

Multiplying the resulting formula by x^{-t} and differentiating, one gets

$$\begin{aligned} (28) \quad \frac{\partial}{\partial t} \left\{ F(x, -t) x^{-t} + x^{-t} \log(2\pi) - x^{-t} \frac{\Gamma'(1+t)}{\Gamma(1+t)} - x^{-t} \left(\frac{\pi}{2} \cot \frac{\pi t}{2} + \frac{\zeta'(1+t)}{\zeta(1+t)} \right) \right\} \\ = -\frac{x}{(1+t)^2} + \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu-t)^2} + \sum_{\rho} \frac{x^{\rho}}{(\rho+t)^2}. \end{aligned}$$

In the vicinity of $t = 0$, the left-hand side can be expanded in a standard power series

$$\frac{\partial}{\partial t} \left\{ \right\} = \sum_{\nu=0}^{\infty} C_{\nu} t^{\nu}.$$

† *Error*: both terms $e^{-\pi is}$ in (27) should read $-e^{-\pi is}$, and this affects the results (13) above. (A.V.)

Next, in the angle

$$-\frac{\pi}{2} + \varepsilon \leq \arccos(t) \leq +\frac{\pi}{2} - \varepsilon$$

(ε positive, and arbitrarily small), this function behaves everywhere regularly, and upon multiplication with an arbitrary high power of t , it tends to the limit zero uniformly for increasing $|t|$ ($x > 1$). Also, r is positive and smaller than the radius of convergence of the above series, hence along the lines

$$0 \text{ ----- } \infty e^{i\theta} \qquad 0 < \theta < \frac{\pi}{2},$$

the integral of the left-hand side of (28) multiplied by $t^{s-1} dt$,

$$\int_0^{\infty e^{i\theta}} t^{s-1} \frac{\partial}{\partial t} \left\{ \right\} dt$$

can be brought to the form

$$\int_0^{\infty e^{i\theta}} t^{s-1} \frac{\partial}{\partial t} \left\{ \right\} dt = \sum_{\nu=0}^{\infty} C_{\nu} \frac{(re^{i\theta})^{s-\nu}}{s+\nu} + H(s), \dagger$$

where H is an entire function. The integrals performed on the right-hand side of (28) give the following results,

$$\begin{aligned} & -x \int_0^{\infty e^{i\theta}} \frac{t^{s-1}}{(1+t)^2} dt = -x \int_0^{\infty} \frac{t^{s-1}}{(1+t)^2} dt = -x \frac{\pi(1-s)}{\sin \pi s}, \\ & \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu-t)^2} dt = \int_0^{-\infty} t^{s-1} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu-t)^2} dt \\ & = e^{\pi i s} \int_0^{+\infty} t^{s-1} \sum_{\nu=0}^{\infty} \frac{x^{-2\nu}}{(2\nu+t)^2} dt \ddagger = e^{\pi i s} \frac{\pi(1-s)}{\sin \pi s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}}, \\ & \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\rho} \frac{x^{\rho}}{(\rho+t)^2} dt = \int_0^{+\infty} t^{s-1} \sum_{\rho} \frac{x^{\rho}}{(\rho+t)^2} dt = \frac{\pi(1-s)}{\sin \pi s} \sum_{\rho} \frac{x^{\rho}}{\rho^{2-s}}, \end{aligned}$$

where θ is so small that it can be presumed there is no ρ in the angle $0 < \arccos(t) \leq \theta$. One also has, upon replacing $2-s$ by s :

$$(29) \quad e^{-\pi i s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^s} + \sum_{\rho} \frac{x^{\rho}}{\rho^s} = x + \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} C_{\nu} \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + H(2-s) \right].$$

† *Misprint*: the exponent in the right-hand side should read $s + \nu$. (A.V.)

‡ *Misprint*: the summation $\sum_{\nu=0}^{\infty}$ should read $\sum_{\nu=1}^{\infty}$. (A.V.)

The right-hand side is obviously an entire function of s , which takes the value x for $s = 0$ and for negative integer values $s = -n$. The first series on the left-hand side is also ($x > 1$) an entire function of s , which for $s = 0$ and for negative integer values $s = -n$ turns into some rational functions of x , so that

$$\sum_{\nu=1}^{\infty} (-2\nu)^n x^{-2\nu} = \left(x \frac{d}{dx}\right)^n \frac{1}{x^2 - 1}.$$

Hence without further ado, one obtains the validity of the claims in §. 1 concerning $Z(s, x)$.

§ 7.

The function

$$\bar{Z}\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-\rho}}{(-\rho)^s}.$$

Replacing t by $t - 1$ in the formula (28), and bringing the term

$$-\frac{x}{t^2}$$

to the left-hand side, one is led by the above method (bearing in mind that $1 - \rho$ and ρ both run through the complex roots of $\zeta(t)$) to the formula

$$\begin{aligned} (30) \quad & e^{\pi i s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu-1}}{(2\nu+1)^s} + \sum_{\rho} \frac{x^{-\rho}}{(-\rho)^s} \\ & = x^{-1} \frac{\sin \pi s}{\pi(1-s)} \left[-x \frac{(re^{i\theta})^{-s}}{s} + \sum_{\nu=0}^{\infty} C_{\nu} \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + H(2-s) \right], \end{aligned}$$

which yields all that was said in §. 1 regarding $\bar{Z}\left(s, \frac{1}{x}\right)$.

§ 8.

The functions

$$\begin{aligned} Z_1(s, x) &= \sum_{\gamma>0} \frac{x^{\rho}}{\rho^s}, & Z_2(s, x) &= \sum_{\gamma<0} \frac{x^{\rho}}{\rho^s}, \\ Z_1\left(s, \frac{1}{x}\right) &= \sum_{\gamma>0} \frac{x^{-s}}{\rho^s}, \quad \dagger & Z_2\left(s, \frac{1}{x}\right) &= \sum_{\gamma<0} \frac{x^s}{\rho^s}, \quad \dagger \end{aligned}$$

† *Misprints*: both x^{-s} and x^s should read $x^{-\rho}$. (A.V.)

$$Z_1(s) = \sum_{\gamma > 0} \frac{1}{\rho^s}, \quad Z_2(s) = \sum_{\gamma < 0} \frac{1}{\rho^s},$$

$$\rho = \beta + i\gamma, \quad x > 1.$$

The nature of the functions defined by those series is now also easily obtained, if one notes that

$$(31) \quad \begin{cases} Z(s, x) = Z_1(s, x) + Z_2(s, x), \\ \bar{Z}(s, x) = e^{\pi is} Z_1(s, x) + e^{-\pi is} Z_2(s, x); \end{cases}$$

$$(32) \quad \begin{cases} Z\left(s, \frac{1}{x}\right) = Z_1\left(s, \frac{1}{x}\right) + Z_2\left(s, \frac{1}{x}\right), \\ \bar{Z}\left(s, \frac{1}{x}\right) = e^{\pi is} Z_1\left(s, \frac{1}{x}\right) + e^{-\pi is} Z_2\left(s, \frac{1}{x}\right); \end{cases}$$

$$(33) \quad \begin{cases} Z(s) = Z_1(s) + Z_2(s), \\ \bar{Z}(s) = e^{\pi is} Z_1(s) + e^{-\pi is} Z_2(s). \end{cases}$$

From (31), for instance, it follows that

$$2i Z_1(s, x) = \frac{\bar{Z}(s, x) - e^{-\pi is} Z(s, x)}{\sin \pi s}.$$

The numerator is an entire function of s , which vanishes not only at the points $s = 1, 2, 3, \dots, \infty$ but also at the points $s = 0, -1, -2, \dots, -\infty$. The former is due to the series ¹⁾ for $Z(s, x)$ and $\bar{Z}(s, x)$, the latter to the formulas (7) and (9). The same obviously apply also to the functions $Z_2(s, x)$, $Z_1\left(s, \frac{1}{x}\right)$, $Z_2\left(s, \frac{1}{x}\right)$.

¹⁾ By a result of VON MANGOLDT, the series

$$\sum_{\rho} \frac{x^{\rho}}{\rho}$$

converges for all $x > 0$. Concerning the point $s = 1$, one needs to prove that

$$\lim_{s \rightarrow 1} Z(s, x) = \sum_{\rho} \frac{x^{\rho}}{\rho} \quad \text{and} \quad \lim_{s \rightarrow 1} \bar{Z}(s, x) = \sum_{\rho} \frac{x^{\rho}}{-\rho},$$

but I will not elaborate on this further.

The functions $Z_1(s, x)$, $Z_2(s, x)$, $Z_1\left(s, \frac{1}{x}\right)$, $Z_2\left(s, \frac{1}{x}\right)$ are also entire functions of s for every $x > 1$.

From the formulae (33), in conjunction with (12) and (13), it follows that:

The functions $Z_1(s)$ and $Z_2(s)$ are meromorphic in the whole s -plane, with a double pole at the point $s = 1$ and with simple poles at the points $s = 0, -1, -2, \dots, -\infty$.

The results developed in this work can be generalized without difficulty. I will only allude to that. Setting

$$Z(s, x, a) = \sum_{\rho} \frac{x^{\rho+a}}{(\rho+a)^s},$$

where x is positive, and initially different from 1, then Z is an entire function of s . If $a = \frac{m}{n}$ is rational, then for $s = 0$ and for negative integer s , Z turns into certain rational functions of $\sqrt[n]{x}$.

Helsingfors, January 1917.

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