Chapter 1
Introduction

Stationary linear reaction-convection-diffusion problems form the subject of this monograph:

$$-\varepsilon u'' - bu' + cu = f \text{ in } (0,1), \;\; u(0) = \gamma_0, \; u(1) = \gamma_1$$

and its two-dimensional analogue

$$-\varepsilon \Delta u - \boldsymbol{b} \cdot \nabla u + cu = f \text{ in } \Omega \subset I\!R^2, \;\; u|_{\partial\Omega} = g$$

with a small positive parameter ε.

Such problems arise in various models of fluid flow [52,53,73]; they appear in the (linearised) Navier-Stokes and in the Oseen equations, in the equations modelling oil extraction from underground reservoirs [32], flows in chemical reactors [3] and convective heat transport with large Péclet number [56]. Other applications include the simulation of semiconductor devices [130].

An Example

Consider the boundary-value problem of finding $u \in C^2(0,1) \cap C[0,1]$ such that

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

with $0 < \varepsilon \ll 1$. Formally setting $\varepsilon = 0$, yields

$$-u'(x) = 1 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0.$$

Unlike (1.1), this problem does not possess a solution in $C^2(0,1) \cap C[0,1]$. Consequently, when ε approaches zero, the solution of (1.1) is badly behaved in some way.

The solution of (1.1) is

$$u(x,\varepsilon) = \frac{\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-x/\varepsilon}}{1 - \mathrm{e}^{-1/\varepsilon}} + 1 - x.$$

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_1,

Due to the presence of the exponential $\mathrm{e}^{-x/\varepsilon}$, the solution u and its derivatives change rapidly near $x = 0$ for small values of ε. Regions where this happens are referred to as **layers**. Singularly perturbed problems are typically characterised by the presence of such layers. The term **boundary layer** was introduced by Ludwig Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904.

The solution of (1.1) may be regarded as a function of two variables:

$$u : [0,1] \times (0,1] : (x,\varepsilon) \mapsto u(x,\varepsilon).$$

Taking limits of u for $(x,\varepsilon) \to (0,0)$, we see that

$$\lim_{x\to 0}\lim_{\varepsilon\to 0} u(x,\varepsilon) = 1 \neq 0 = \lim_{\varepsilon\to 0}\lim_{x\to 0}. \tag{1.2}$$

Thus, u as a function of two variables possesses a *classical singularity* at the point $(0,0)$ in the (x,ε)-plane. For this reason we may call (1.1) a singularly perturbed boundary-value problem.

What Is a Singularly Perturbed Problem?

Miller et al. [121] give the following characterisation:

> *The justification for the name 'singular perturbation' is that the nature of the differential equations changes completely in the limit case, when the singular perturbation parameter is equal to zero. For example, … equations change from being nonlinear parabolic equations to nonlinear hyperbolic equations.*

This describes a phenomenon that *can* lead to the formation of boundary layers and *typically will*—if appropriate boundary conditions are imposed. Roos et al. [141] describe singularly perturbed problems as follows.

> *They are differential equations (ordinary or partial) that depend on a small positive parameter ε and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be* singularly perturbed*, where we regard ε as a perturbation parameter.*

Both sources avoid a formal definition:

In the present monograph we propose the following definition.

Definition 1.1. Let B be a function space with norm $\|\cdot\|_B$. Let $D \subset I\!R^d$ be a parameter domain. The continuous function $u : D \to B, \varepsilon \mapsto u(\varepsilon)$ is said to be *regular* for $\varepsilon \to \varepsilon^* \in \partial D$ if there exists a function $u^* \in B$ such that:

$$\lim_{\varepsilon\to\varepsilon^*} \|u_\varepsilon - u^*\|_B = 0,$$

otherwise u_ε is said to be **singular** for $\varepsilon \to \varepsilon^*$.

Let $(\mathcal{P}_\varepsilon)$ be a problem with solution $u(\varepsilon) \in B$ for all $\varepsilon \in D$. We say $(\mathcal{P}_\varepsilon)$ is **singularly perturbed** for $\varepsilon \to \varepsilon^* \in \partial D$ in the norm $\|\cdot\|_B$ if u is singular for $\varepsilon \to \varepsilon^*$. ♡

Remark 1.2. The definition is norm dependent. For example (1.1), is singularly perturbed in the C^0 norm and the L_∞ norm because of (1.2). However, it is not singularly perturbed in the L_2 norm. There exists a function $u^* : x \mapsto 1 - x$ with

$$\|u_\varepsilon - u^*\|_0 = \mathcal{O}\big(\varepsilon^{1/2}\big).$$

The L_2 norm fails to capture the boundary layer in u. ♣

Remark 1.3. Boundary conditions play an important role. Consider the boundary-value problem

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } x \in (0,1), \quad u'(0) = u(1) = 0.$$

This problem is singularly perturbed in the C^1 norm, but it is *not* perturbed in the C^0 norm. The Neumann boundary condition at $x = 0$ leads to the formation of a weak layer only. The first-order derivative remains bounded when $\varepsilon \to 0$. ♣

Uniform Convergence

Classical convergence results for numerical methods for boundary-value problems have the structure

$$\left\|u - u^h\right\| \le K h^k,$$

with the maximum mesh size h. The constant K depends on certain derivatives of u and typically tends to infinity as the perturbation parameter ε approaches zero. This means that the maximal step size h has to be chosen proportional to some positive power of ε which is impractical. Therefore, we are looking for so-called *uniform* or *robust* methods where the numerical costs are independent of the perturbation parameter ε. More precisely, we are looking for robust methods in the sense of the following definition:

Definition 1.4. Let u_ε be the solution of a singularly perturbed problem, and let u_ε^N be a numerical approximation of u_ε obtained by a numerical method with N degrees of freedom. The numerical method is said to be **uniformly convergent** or **robust** with respect to the perturbation parameter ε in the norm $\|\cdot\|$ if

$$\left\|u_\varepsilon - u_\varepsilon^N\right\| \le \vartheta(N) \text{ for } N \ge N_0$$

with a function ϑ satisfying

$$\lim_{N\to\infty} \vartheta(N) = 0 \text{ and } \partial_\varepsilon \vartheta \equiv 0,$$

and with some threshold value $N_0 > 0$ that is independent of ε. ♡

Scope of the Monograph

Well-developed techniques are available for the computation of solutions outside layers [123, 141], but the problem of resolving layers—which is of great practical importance—is still under investigation. This field has witnessed a stormy development. Layer-adapted meshes have first been proposed by Bakhvalov [18] in the context of reaction-diffusion problems. In the late 1970s and early 1980s, special meshes for convection-diffusion problems were investigated by Gartland [45], Liseikin [113, 114, 116], Vulanović [163–166] and others in order to achieve uniform convergence. The discussion has been livened up by the introduction of special piecewise-uniform meshes by Shishkin [150]. They will be described in more detail in Section 2.1.3. Because of their simple structure, they have attracted much attention and are now widely referred to as Shishkin meshes. A small survey of these meshes can be found in the monograph [141], while [109, 121] and [134] are devoted exclusively to them.

The performance of Shishkin meshes is however inferior to that of Bakhvalov meshes, which has prompted efforts to improve them while retaining some of their simplicity, in particular, the mesh uniformity outside the layers and the choice of mesh transition point where the mesh changes from fine to coarse. For instance, Vulanović [169] uses a piecewise-uniform mesh with more than one transition point. Linß [81, 82] combines the ideas of Bakhvalov and Shishkin, while Beckett and Mackenzie [20] combine an equidistribution idea [31] with a Shishkin-type transition point. With all these various mesh-construction ideas a natural question is:

> *Can a general theory be derived that allows one to immediately deduce the robust convergence of standard schemes on special meshes and a guaranteed rate of convergence?*

A first attempt towards this can be found in [137], where a first-order upwind scheme and a Galerkin FEM are studied on a class of so-called Shishkin-type meshes. A more general criterion was derived in [84, 85] for an upwind-difference scheme in one dimension.

The main purpose of this monograph is to give a survey of recent developments and present the state of the art in the analysis of layer-adapted meshes for a wide range of reaction-convection-diffusion problems.