Chapter 10
Psycho-financial Model

This chapter presents a model that was proposed by Olga Choustova and me [53–62, 175]. She applied methods of quantum mechanics to mathematical modeling of price dynamics in the financial market. She pointed out that behavioral financial factors (e.g., expectations of traders) could be described by using the pilot wave (Bohmian) model of quantum mechanics; see Section 12.6 for a brief introduction to the mathematical formalism of Bohmian mechanics.

Trajectories of prices are determined by two financial potentials: classical-like $V(t, q)$ ("hard" market conditions, e.g., natural resources) and QL $U(t, q)$ (behavioral market conditions).

On the one hand, our Bohmian model is a QL model for the financial market, cf. publications of Segal and Segal [274], Baaquie [22, 23], Haven [132–135], Piotrowski et al. [252–258], and Danilov and Lambert-Mogiliansky [73–76].

On the other hand (since Bohmian mechanics provides the possibility of describing individual price trajectories), it belongs to the domain of extensive research on deterministic dynamics for financial assets (Granger [122], Barnett and Serletis [28], Benhabib [34], Brock and Sayers [43], Campbell et al. [52], Hsieh [152], and many others).

10.1 Deterministic and Stochastic Models of Financial Markets

10.1.1 Efficient Market Hypothesis

In economics and financial theory, analysts use random walks and more general martingale techniques to model the behavior of asset prices, in particular share prices on stock markets, currency exchange rates and commodity prices. This practice has its basis in the presumption that investors act *rationally and without bias,* and that at any moment they estimate the value of an asset based on future expectations. Under these conditions, all existing information affects the price, which changes only when new information comes out. By definition, *new information appears randomly and influences the asset price randomly.* Corresponding continuous time models are based on stochastic processes (this approach was initiated in the thesis

DOI 10.1007/978-3-642-05101-2_10,

of Bachelier [24] in 1890), see, e.g., the books of Mantegna and Stanley [237] and Shiryaev [281] for historical and mathematical details.

This practice was formalized through the *efficient market hypothesis*, which was formulated in the 1960s, see Samuelson [269, 270] and Fama [103] for details:

A market is said to be efficient in the determination of the most rational price if all the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded.

The efficient market hypothesis was supported by statistical investigations of Samuelson [269]. Mathematically the efficient markets hypothesis means that financial markets can be described by classical stochastic processes and they are of a very special type, namely, so-called martingales.

10.1.2 Deterministic Models for Dynamics of Prices

First we remark that empirical studies have demonstrated that prices do not completely follow a random walk. Low serial correlations (around 0.05) exist in the short term and slightly stronger correlations over the longer term. Their sign and the strength depend on a variety of factors, but transaction costs and bid–ask spreads generally make it impossible to earn excess returns. Interestingly, researchers have found that some of the biggest price deviations from a random walk result from seasonal and temporal patterns, see [237].

There are also a variety of arguments, both theoretical and obtained on the basis of statistical analysis of data, that question the general martingale model (and hence the efficient market hypothesis), see, e.g., [122, 28, 34, 52, 152]. It is important to note that efficient markets imply there are no exploitable profit opportunities. If this is true then trading on the stock market is a game of chance and not of any skill, but traders buy assets they think are undervalued in the hope of selling them at their true price for a profit. If market prices already reflect all available information, then where does the trader draw this privileged information from? Since there are thousands of very well informed, well-educated asset traders, backed by many data researchers, buying and selling securities quickly, logically assets markets should be very efficient and profit opportunities should be minimal. On the other hand, we see that there are many traders who successfully use their opportunities and continuously carry out very successful financial operations; see the book by Soros [282] for discussion.[1] Intensive investigations testing whether real financial data can be really described by the martingale model, have also been performed, see [122, 28, 34, 52, 152]. Roughly speaking, people try to understand the following on the basis of available financial data:

Do financial asset returns behave randomly (hence unpredictably) or deterministically (in which case one may hope to predict them and even to construct a deterministic dynamical system which would at least mimic the dynamics of the financial market)?

[1] It seems that Soros is sure he does not work at efficient markets.

Predictability of financial asset returns is a broad and very active research topic and a complete survey of the vast literature is beyond the scope of this work. We note, however, that there is a rather general opinion that financial asset returns are predictable, see [122, 28, 34, 52, 152].

10.1.3 Behavioral Finance and Economics

We point out that there is no general consensus on the validity of the efficient market hypothesis. As is pointed out in [52]: "...econometric advances and empirical evidence seem to suggest that financial asset returns are predictable to some degree. Thirty years ago this would have been tantamount to an outright rejection of market efficiency. However, modern financial economics teaches us that other, perfectly rational factors may account for such predictability. The fine structure of securities markets and frictions in trading process can generate predictability. Time-varying expected returns due to changing business conditions can generate predictability. A certain degree of predictability may be necessary to reward investors for bearing certain dynamic risks."

Therefore it would be natural to develop approaches which are not based on the assumption that investors act *rationally and without bias* and that, consequently, new information appears randomly and influences the asset price randomly. In particular, there are two well established (and closely related) fields of research, *behavioral finance and behavioral economics*, which apply scientific research on human and social cognitive and emotional biases[2] to better understand economic decisions and how they affect market prices, returns and the allocation of resources. The fields are primarily concerned with the rationality, or lack thereof, of economic agents. Behavioral models typically integrate insights from psychology with neo-classical economic theory. Behavioral analysis is mostly concerned with the effects of market decisions, but also those of public choice, another source of economic decisions with some similar biases.

Since the 1970s, the intensive exchange of information in the world of finances has become one of the main factors determining the dynamics of prices. Electronic trading (which has become the most important part of the environment of the major stock exchanges) induces huge information flows between traders (including the foreign exchange market). Financial contracts are made on a new time scale that differs essentially from the old "hard" time scale that was determined by the development of the economic basis of the financial market. Prices at which traders are willing

[2] Cognitive bias is any of a wide range of observer effects identified in cognitive science, including very basic statistical and memory errors that are common to all human beings and drastically skew the reliability of anecdotal and legal evidence. They also significantly affect the scientific method, which is deliberately designed to minimize such bias from any one observer. They were first identified by Amos Tversky and Daniel Kahneman as a foundation of behavioral economics, see, e.g., [293]. Bias arises from various life, loyalty and local risk and attention concerns that are difficult to separate or codify. Tversky and Kahneman claim that they are at least partially the result of problem-solving using heuristics, including the availability heuristic and the representativeness.

to buy (bid quotes) or sell (ask quotes) a financial asset are not only determined by the continuous development of industry, trade, services, the situation on the market of natural resources and so on. Information (mental, market-psychological) factors play a very important (and in some situations crucial) role in price dynamics. Traders performing financial operations work as a huge collective cognitive system. Roughly speaking, classical-like dynamics of prices (determined) by "hard" economic factors are permanently perturbed by additional financial forces, mental (or market-psychological) forces, see the book [282].

10.1.4 Quantum-like Model for Behavioral Finance

Olga Choustova has developed a new approach that is not based on the assumption that investors act rationally and without bias and that, consequently, new information appears randomly and influences the asset price randomly. Her approach can be considered as a special econophysical [237] model in the domain of behavioral finance. In her approach information about the financial market (including expectations of agents of the financial market) is described by an *information field* $\psi(t, q)$ – *a financial wave*. This field evolves deterministically, perturbing the dynamics of prices of stocks and options. The dynamics is given by Schrödinger's equation on the space of prices of shares. Since the psychology of agents of the financial market makes an important contribution to the financial wave $\psi(t, q)$, our model can be considered as a special *psycho-financial model*.

Choustova's model can be also considered as a contribution to applications of quantum mechanics outside the microworld, see [8, 161, 173, 176]. Her model is fundamentally based on investigations by Bohm, Hiley, and Pylkkänen [40, 144] of the *active information* interpretation of Bohmian mechanics and its applications to cognitive sciences, see also [176].

In her model Choustova used methods of Bohmian mechanics to simulate dynamics of prices on the financial market. She started with the development of the classical Hamiltonian formalism on the price/price-change phase space to describe the classical-like evolution of prices. This classical dynamics of prices is determined by "hard" financial conditions (natural resources, industrial production, services and so on). These conditions, as well as "hard" relations between traders at the financial market, are mathematically described by the classical financial potential. At the real financial market "hard" conditions are not the only source of price changes. Information and market psychology play an important (and sometimes determining) role in price dynamics.

Choustova proposed that these "soft" financial factors be described by using the pilot wave (Bohmian) model of quantum mechanics. The theory of financial mental (or psychological) waves was used to take into account market psychology. In Choustova's model the real trajectories of prices are determined (by the financial analogue of the second Newton law) by two financial potentials: classical-like ("hard" market conditions) and QL ("soft" market conditions).

This QL model of financial processes was strongly motivated by consideration by Soros [282] of the financial market as a complex cognitive system. Such an approach he called the theory of *reflexivity.* In this theory there is a large difference between market that is "ruled" by only "hard" economical factors and a market where mental factors play a crucial role (even changing the evolution of the "hard" basis, see [282]).

Soros rightly remarked that the "nonmental" market evolves due to classical random fluctuations. However, such fluctuations do not provide an adequate description of the mental market. He proposed that an analogy with quantum theory be used. However, it was noticed that quantum formalism could not be applied directly to the financial market [282]. Traders differ essentially from elementary particles. Elementary particles behave stochastically due to perturbation effects provided by measurement devices.

According to Soros, traders at the financial market behave stochastically due to free will of individuals. Combinations of a huge number of free wills of traders produce additional stochasticity at the financial market that could not be reduced to classical random fluctuations (determined by nonmental factors). Here Soros followed the conventional (Heisenberg, Bohr, Dirac) view of the origin of quantum stochasticity. However, in the Bohmian approach (that is the nonconventional one) quantum statistics is induced by the action of an additional potential, the quantum potential, that changes classical trajectories of elementary particles. Such an approach provides the possibility of applying quantum formalism to the financial market.

We remark that applications of the pilot-wave theory to financial option pricing were considered by Haven in [136]. There were also numerous investigations on applying quantum methods to the financial market, see, e.g., [132, 135], that were not directly coupled to behavioral modeling, but based on the general concept that randomness of the financial market can be better described by quantum mechanics, see, e.g., Segal and Segal [274]: "A natural explanation for extreme irregularities in the evolution of prices in financial markets is provided by quantum effects." Non-Bohmian quantum models for the financial market (in particular, based on quantum games) were developed by Piotrowski, Sladkowski, and coworkers, see [252, 254]. Some of those models can also be considered as behavioral QL models.

An interesting contribution to behavioral QL modeling is the theory of nonclassical measurements in behavioral sciences (with applications to economics) that was developed by Danilov and Lambert-Mogiliansky [73, 74].

10.2 Classical Econophysical Model of the Financial Market

10.2.1 Financial Phase Space

Let us consider a mathematical model in which a huge number of agents of the financial market interact with one another and take into account external economic (as well as political, social and even meteorological) conditions in order to determine the price to buy or sell financial assets. We consider the trade with shares of some corporations (e.g., Volvo, Saab, Ikea,...).[3]

We consider a *price system of coordinates.* We enumerate corporations that emitted shares in the financial market under consideration: $j = 1, 2, \ldots, n$ (e.g., Volvo:$j = 1$, Saab:$j = 2$, Ikea:$j = 3, \ldots$). Introduce the n-dimensional configuration space $Q = \mathbf{R}^n$ of prices, $q = (q_1, \ldots, q_n)$, where q_j is the price of a share of the jth corporation. Here $\mathbf{R}$ is the real line. The dynamics of prices is described by the trajectory $q(t) = (q_1(t), \ldots, q_n(t))$ in the configuration price space Q.

Another variable under consideration is the *price change variable*:

$$v_j(t) = \dot{q}_j(t) = \lim_{\Delta t \to 0} \frac{q_j(t + \Delta t) - q_j(t)}{\Delta t},$$

see, for example, the book [237] on the role of the price change description. In real models we consider the discrete time scale $\Delta t, 2\Delta t, \ldots$. Here we should use a discrete price change variable $\delta q_j(t) = q_j(t + \Delta t) - q_j(t)$.

We denote the space of price changes (price velocities) by the symbol $V(\equiv \mathbf{R}^n)$ with coordinates $v = (v_1, \ldots, v_n)$. As in classical physics, it is useful to introduce the phase space $Q \times V = \mathbf{R}^{2n}$, namely the *price phase space.* A pair (q, v) = (price, price change) is called the *state of the financial market.*

Later we shall consider QL states of the financial market. The state (q, v) that we consider at the moment is a classical state.

We now introduce an analogue m of mass as the number of items (in our case shares) that a trader emitted to the market.[4] We call m the *financial mass.* Thus each trader j (e.g., Volvo) has its own financial mass m_j (the size of the emission of its shares). The total price of the emission of the jth trader is equal to $T_j = m_j q_j$ (this is nothing other than *market capitalization*). Of course, it depends on time: $T_j(t) = m_j q_j(t)$. To simplify considerations, we consider a market in which any emission of shares is of fixed size, so m_j does not depend on time. In principle, our model can be generalized to describe a market with time-dependent financial masses, $m_j = m_j(t)$.

We also introduce *financial energy* of the market as a function $H : Q \times V \to \mathbf{R}$. Let us use the analogy with classical mechanics. (Why not? In principle, there is not so much difference between motions in "physical space" and "price space".) In this case we could consider (at least for mathematical modeling) the financial energy of the form

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[3] Similar models can be developed for trade with options, see Haven [136] for the Bohmian financial wave model for portfolios.

[4] 'Number' is a natural number $m = 0, 1, \ldots$, the price of share, e.g., in US dollars. However, in a mathematical model it can be convenient to consider real m. This can be useful for conversions from one currency to another.

$$H(q, v) = \frac{1}{2}\sum_{j=1}^{n} m_j v_j^2 + V(q_1, \ldots, q_n). \tag{10.1}$$

Here $K(q, v) = 1/2\sum_{j=1}^{n} m_j v_j^2$ is the *kinetic financial energy* and $V(q_1, \ldots, q_n)$ is the potential financial energy; m_j is the financial mass of the jth trader.

The kinetic financial energy represents efforts of agents of the financial market to change prices: higher price changes induce higher kinetic financial energies. If the corporation j_1 has higher financial mass than the corporation j_2, so $m_{j_1} > m_{j_2}$, then the same change of price, i.e., the same financial velocity $v_{j_1} = v_{j_2}$, is characterized by higher kinetic financial energy: $K_{j_1} > K_{j_2}$. We also remark that high kinetic financial energy characterizes rapid changes of the financial situation at the market. However, the kinetic financial energy does not give the sign of these changes. It could be rapid economic growth as well as recession.

The *potential financial energy* V describes the interactions between traders $j = 1, \ldots., n$ (e.g., competition between Nokia and Ericsson) as well as external economic conditions (e.g., the price of oil and gas) and even meteorological conditions (e.g., the weather conditions in Louisiana and Florida). For example, we can consider the simplest interaction potential:

$$V(q_1, \ldots, q_n) = \sum_{j=1}^{n}(q_i - q_j)^2.$$

The difference $|q_1 - q_j|$ between prices is the most important condition for *arbitrage.*

We could never take into account all economic and other conditions that influence the market. Therefore by using some concrete potential $V(t, q)$ we consider a very idealized model of financial processes. However, such an approach is standard for physical modeling, where we also consider idealized mathematical models of real physical processes.

10.2.2 Classical Dynamics

We apply Hamiltonian dynamics on the price phase space. As in classical mechanics for material objects, we introduce a new variable $p = mv$, the *price momentum* variable. Instead of the price change vector $v = (v_1, \ldots, v_n)$, we consider the price momentum vector $p = (p_1, \ldots, p_n)$, $p_j = m_j v_j$. The space of price momenta is denoted by the symbol P. The space $\Omega = Q \times P$ will also be called the price phase space. *Hamiltonian equations* of motion on the price phase space have the form $\dot{q} = \partial H/\partial p_j$, $\dot{p}_j = -\partial H/\partial q_j$, $j = 1, \ldots, n$.

If the financial energy has the form (10.1) then the Hamiltonian equations have the form

$$\dot{q}_j = \frac{p_j}{m_j} = v_j, \quad \dot{p}_j = -\frac{\partial V}{\partial q_j}.$$

The latter equation can be written as

$$m_j \dot{v}_j = -\frac{\partial V}{\partial q_j}.$$

It is natural to call the quantity

$$\dot{v}_j = \lim_{\Delta t \to 0} \frac{v_j(t + \Delta t) - v_j(t)}{\Delta t}$$

the *price acceleration* (rate of change of price rate of change). The quantity

$$f_j(q) = -\frac{\partial V}{\partial q_j}$$

is called the (potential) financial force. We get the financial variant of Newton's second law:

$$m\dot{v} = f \tag{10.2}$$

Law 10.1. The product of the financial mass and the price acceleration is equal to the financial force.

In fact, the Hamiltonian evolution is determined by the following fundamental property of the financial energy: *The financial energy is not changed in the process of Hamiltonian evolution:*

$$H(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t) = H(q_1(0), \ldots q_n(0), p_1(0), \ldots, p_n(0)).$$

We need not restrict our considerations to financial energies of form (10.1). First of all external (e.g. economic) conditions as well as the character of interactions between traders at the market depend strongly on time. This must be taken into account by considering time-dependent potentials:

$$V = V(t, q).$$

Moreover, the assumption that the financial potential depends only on prices, $V = V(t, q)$, is not so natural for the modern financial market. Financial agents have complete information on price changes. This information is taken into account by traders for acts of arbitrage, see [237] for details. Therefore, it can be useful to consider potentials that depend not only on prices, but also on price changes: $V = V(t, q, v)$, or in the Hamiltonian framework: $V = V(t, q, p)$. In such a case the financial force is not potential. Therefore, it is also useful to consider the financial Newton's second law for general financial forces: $m\dot{v} = f(t, q, p)$.

Remark 10.1 (On the form of the kinetic financial energy) We copied the form of kinetic energy from classical mechanics for material objects. It may be that such a form of kinetic financial energy is not justified by a real financial market. It might be better to consider our choice of the kinetic financial energy as just the basis for mathematical modeling (and look for other possibilities).

Remark 10.2 (Domain of price dynamics) It is natural to consider a model in which all prices are nonnegative, $q_j(t) \geq 0$. Therefore financial Hamiltonian dynamics should be considered in the phase space $\Omega_+ = \mathbf{R}^n_+ \times \mathbf{R}^n$, where $\mathbf{R}_+$ is the set of nonnegative real numbers. We shall not study this problem in detail, because our aim is the study of the corresponding quantum dynamics, but in the quantum case this problem is solved easily. One should just consider the corresponding Hamiltonian in the space of square integrable functions $L_2(\Omega_+)$. Another possibility in the classical case is to consider centered dynamics of prices: $z_j(t) = q_j(t) - q(0)$. The centered price $z_j(t)$ evolves in the configuration space $\mathbf{R}^n$.

10.2.3 Critique of Classical Econophysics

The model of Hamiltonian price dynamics on the price phase space can be useful to describe a market that depends only on "hard" economic conditions: natural resources, volumes of production, human resources and so on. However, the classical price dynamics cannot be applied (at least directly) to modern financial markets. It is clear that the stock market is not based only on these "hard" factors. There are other factors, soft ones (behavioral), that play an important (and sometimes even determining) role in forming of prices at the financial market. Market psychology should be taken into account. Negligibly small amounts of information (due to the rapid exchange of information) imply large changes of prices at the financial market. We can consider a model in which financial (psychological) waves are permanently present at the market. Sometimes these waves produce uncontrollable changes of prices disturbing the whole market (financial crashes). Of course, financial waves also depend on "hard economic factors." However, these factors do not play a crucial role in the formation of financial waves. Financial waves are merely waves of information.

We could compare the behavior of a financial market with the behavior of a gigantic ship that is ruled by a radio signal. A radio signal with negligibly small physical energy can essentially change (due to information contained in this signal) the motion of the gigantic ship. If we do not pay attention to (do not know about the presence of) the radio signal, then we will be continuously surprised by the ship's behavior. It can change its direction of motion without any "hard" reason (weather, destination, technical state of ship's equipment). However, if we know about the existence of radio monitoring, then we could find information that is sent by radio. This would give us a powerful tool to predict the ship's trajectory. This example on ship's monitoring was taken from the book by Bohm and Hiley [40] on so-called pilot wave quantum theory (or Bohmian quantum mechanics).

10.3 Quantum-like Econophysical Model of the Financial Market

10.3.1 Financial Pilot Waves

If we interpret the pilot wave as a field, then it is a rather strange field. It differs crucially from "ordinary physical fields," i.e., the electromagnetic field. We mention some of the pathological features of the pilot wave field. In particular, the force induced by this pilot wave field does not depend on the amplitude of the wave. Thus small waves and large waves equally disturb the trajectory of an elementary particle. Such features of the pilot wave make it possible to speculate [40] that this is just a wave of information (active information). Hence, the pilot wave field describes the propagation of information. The pilot wave is more similar to a radio signal that guides a ship. Of course, this is just an analogy (because a radio signal is related to an ordinary physical field, namely, the electromagnetic field). The more precise analogy is to compare the pilot wave with information contained in the radio signal.

We remark that the pilot wave (Bohmian) interpretation of quantum mechanics is not the conventional one. A few critical arguments against Bohmian quantum formalism can be mentioned:

1. Bohmian theory makes it possible to provide a mathematical description of the trajectory $q(t)$ of an elementary particle. However, such a trajectory does not exist according to conventional quantum formalism.
2. Bohmian theory is not local, namely, by means of the pilot wave field one particle "feels" another at large distances.

We say that these disadvantages of the theory will become advantages in our applications of Bohmian theory to the financial market. We also recall that Bohm and Hiley [40] and Hiley and Pilkkänen[144] have already discussed the possibility of interpreting the pilot wave field as a kind of information field. This information interpretation was essentially developed in my work, see, e.g., [176] devoted to pilot wave cognitive models.

Our fundamental assumption is that agents of the modern financial market are not just "classical-like agents." Their actions are ruled not only by classical-like financial potentials $V(t, q_1, \ldots, q_n)$, but also (in the same way as in the pilot wave theory for quantum systems) by an additional information (or psychological) potential induced by a financial pilot wave.

Therefore we cannot use classical financial dynamics (Hamiltonian formalism) on the financial phase space to describe the real price trajectories. Information (psychological) perturbation of Hamiltonian equations for price and price change must be taken into account. To describe such a model mathematically, it is convenient to use an object such as a *financial pilot wave* that rules the financial market.

In some sense $\psi(t, q)$ describes the psychological influence of the price configuration q on the behavior of agents of the financial market. In particular, $\psi(t, q)$ contains the expectations of agents.

We point out two important features of the financial pilot wave model:

1. All shares are coupled on the information level. The general formalism of the pilot wave theory says that if the function $\psi(t, q_1, \ldots, q_n)$ is not factorized, i.e.,

$$\psi(t, q_1, \ldots, q_n) \neq \psi_1(t, q_1) \ldots \psi_n(t, q_n),$$

then any change in the price q_i will automatically change the behavior of all agents of the financial market (even those who have no direct coupling with i-shares). This will imply a change in prices of j-shares for $i \neq j$. At the same time the "hard" economic potential $V(q_1, \ldots, q_n)$ need not contain any interaction term.

For example, let us consider for the moment the potential $V(q_1, \ldots, q_n) = q_1^2 + \ldots + q_n^2$. The Hamiltonian equations for this potential – in the absence of the financial pilot wave – have the form: $\dot{q}_j = p_j$, $\dot{p}_j = -2q_j$, $j = 1, 2, \ldots, n$. Thus the classical price trajectory $q_j(t)$, does not depend on the dynamics of prices of shares for other traders $i \neq j$ (for example, the price of Ericsson shares does not depend on the price of Nokia shares and vice versa).[5]

However, if, for example, the wave function has the form

$$\psi(q_1, \ldots, q_n) = c e^{i(q_1 q_2 + \ldots + q_{n-1} q_n)} e^{-(q_1^2 + \ldots + q_n^2)},$$

where $c \in C$ is some normalization constant, then financial behavior of agents of the financial market is nonlocal (see further considerations).

2. Reactions of the market do not depend on the amplitude of the financial pilot wave: waves $\psi, 2\psi, 100000\psi$ will produce the same reaction. Such a behavior of the market is quite natural (if the financial pilot wave is interpreted as an information wave, the wave of financial information). The amplitude of an information signal does not play so large a role in information exchange. Most important is the context of such a signal. The context is given by the shape of the signal, the form of the financial pilot wave function.

10.3.2 Dynamics of Prices Guided by Financial Pilot Wave

In fact, we do not need to develop a new mathematical formalism. We will just apply the standard pilot wave formalism to the financial market. The fundamental postulate of the pilot wave theory is that the pilot wave (field)

$$\psi(t, q_1, \ldots, q_n)$$

[5] Such a dynamics would be natural if these corporations operated on independent markets, e.g., Ericsson in Sweden and Nokia in Finland. Prices of their shares would depend only on local market conditions, e.g., on capacities of markets or consumer activity.

induces a new (quantum) potential

$$U(t, q_1, \ldots, q_n)$$

which perturbs the classical equations of motion. A modified Newton equation has the form

$$\dot{p} = f + g, \tag{10.3}$$

where $f = -\partial V/\partial q$ and $g = -\partial U/\partial q$. We call the additional financial force g a *financial mental force.* This force $g(t, q_1, \ldots, q_n)$ determines a kind of collective consciousness of the financial market. Of course, g depends on economic and other "hard" conditions given by the financial potential $V(t, q_1, \ldots, q_n)$. However, this is not a direct dependence. In principle, a nonzero financial mental force can be induced by the financial pilot wave ψ in the case of zero financial potential, $V \equiv 0$. So $V \equiv 0$ does not imply that $U \equiv 0$. *Market psychology is not totally determined by economic factors.* Financial (psychological) waves of information need not be generated by changes in the real economic situation. They are mixtures of mental and economic waves. Even in the absence of economic waves, mental financial waves can have a large influence on the market.

By using the standard pilot wave formalism we obtain the following rule for computing the financial mental force. We represent the financial pilot wave $\psi(t, q)$ in the form

$$\psi(t, q) = R(t, q)e^{iS(t,q)},$$

where $R(t, q) = |\psi(t, q)|$ is the amplitude of $\psi(t, q)$ (the absolute value of the complex number $c = \psi(t, q)$) and $S(t, q)$ is the phase of $\psi(t, q)$ (the argument of the complex number $c = \psi(t, q)$). Then the financial mental potential is computed as

$$U(t, q_1, \ldots, q_n) = -\frac{1}{R}\sum_{i=1}^{n} \frac{\partial^2 R}{\partial q_i^2}(t, q_1, \ldots, q_n)$$

and the financial mental force as

$$g_j(t, q_1, \ldots, q_n) = \frac{-\partial U}{\partial q_j}(t, q_1, \ldots, q_n).$$

These formulas imply that strong financial effects are produced by financial waves having significant variations of amplitude.

Example 10.1 (Financial waves with small variation have no effect) Let us start with the simplest example: $R \equiv const$. Then the financial (behavioral) force $g \equiv 0$. As $R \equiv const$, it is impossible to change expectations of the whole financial market by varying the price q_j of one fixed type of shares, j. The constant information field does not induce psychological financial effects at all. As we have already remarked, the absolute value of this constant does not play any role. Waves of constant amplitude $R = 1$, as well as $R = 10^{100}$, produce no effect.

Let now consider the case $R(q) = cq,\ c > 0$. This is a linear function; variation is not so large. As a result, $g \equiv 0$ here also. There are no financial behavioral effects.

Example 10.2 (Speculation) Let $R(q) = c(q^2 + d),\quad c, d > 0$. Here

$$U(q) = -\frac{2}{q^2 + d}$$

(it does not depend on the amplitude c !) and

$$g(q) = \frac{-4q}{(q^2 + d)^2}.$$

The quadratic function varies essentially more strongly than the linear function, and, as a result, such a financial pilot wave induces a nontrivial financial force.

We analyze financial drives induced by such a force. We consider the following situation: (the starting price) $q > 0$ and $g < 0$. The financial force g stimulates the market (which works as a huge cognitive system) to decrease the price. For small prices, $g(q) \approx -4q/d^2$. If the financial market increases the price q for shares of this type, then the negative reaction of the financial force becomes stronger and stronger. The market is pressed (by the financial force) to stop increasing the price q. However, for large prices, $g(q) \approx -4/q^3$. If the market can approach this range of prices (despite the negative pressure of the financial force for relatively small q) then the market will feel a decrease of the negative pressure (we recall that we consider the financial market as a huge cognitive system). This model explains well the speculative behavior of the financial market.

Example 10.3 Let now $R(q) = c(q^4 + b),\quad c, b > 0$. Thus

$$g(q) = \frac{bq - q^5}{(q^4 + b)^2}.$$

Here the behavior of the market is more complicated. Set $d =^4 \sqrt{b}$. If the price q is changing from $q = 0$ to $q = d$ then the market is motivated (by the financial force $g(q)$) to increase the price. The price $q = d$ is critical for its financial activity. For psychological reasons (of course, indirectly based on the whole information available at the market) the market "understands" that it would be dangerous to continue to increase the price. After approaching the price $q = d$, the market has a psychological stimulus to decrease the price.

Financial pilot waves $\psi(q)$ with $R(q)$ that are polynomials of higher order can induce very complex behavior. The interval $[0, \infty)$ is split into a collection of subintervals $0 < d_1 < d_2 < \ldots < d_n < \infty$ such that at each price level $q = d_j$ the trader changes his attitude to increase or to decrease the price.

In fact, we have considered just a one-dimensional model. In the real case we have to consider multidimensional models of huge dimension. A financial pilot wave $\psi(q_1, \ldots, q_n)$ on such a price space Q induces splitting of Q into a large number of domains $Q = O_1 \bigcup \ldots \bigcup O_N$.

The only problem that we have still to solve is the description of the time-dynamics of the financial pilot wave, $\psi(t, q)$. We follow the standard pilot wave theory. Here $\psi(t, q)$ is found as the solution of Schrödinger's equation. The Schrödinger equation for the energy

$$H(q, p) = \frac{1}{2} \sum_{j=1}^{n} \frac{p_j^2}{m_j} + V(q_1, \ldots, q_n)$$

has the form

$$i\hbar \frac{\partial \psi}{\partial t}(t, q_1, \ldots, q_n) =$$
$$-\sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \frac{\partial^2 \psi(t, q_1, \ldots, q_n)}{\partial q_j^2} + V(q_1, \ldots, q_n)\psi(t, q_1, \ldots, q_n), \tag{10.4}$$

with the initial condition

$$\psi(0, q_1, \ldots, q_n) = \psi(q_1, \ldots, q_n).$$

Thus if we know $\psi(0, q)$ then by using Schrödinger's equation we can find the pilot wave at any instant of time t, $\psi(t, q)$. Then we compute the corresponding mental potential $U(t, q)$ and mental force $g(t, q)$ and solve Newton's equation.

We shall use the same equation to find the evolution of the financial pilot wave. We have only to make one remark, namely, on the role of the constant $\hbar$ in Schrödinger's equation, see [132–136]. In quantum mechanics (which deals with microscopic objects) $\hbar$ is the Dirac constant, which is based on the Planck constant h. The latter constant plays the fundamental role in all quantum considerations. However, originally h appeared as just a scaling numerical parameter for processes of energy exchange. Therefore in our financial model we can consider $\hbar$ as a price scaling parameter, namely, the unit in which we would like to measure price change. We do not present any special value for $\hbar$. There are numerous investigations into price scaling. It may be that there can be recommended some special value for $\hbar$ related to the modern financial market, a *fundamental financial constant.* However, it seems that

$$\hbar = \hbar(t)$$

evolves depending on economic development.

We suppose that the financial pilot wave evolves according to the financial Schrödinger equation (an analogue of Schrödinger's equation) on the price space.

In the general case this equation has the form

$$i\hbar\frac{\partial\psi}{\partial t}(t,q) = \widehat{H}\psi(t,q), \quad \psi(0,q) = \psi(q),$$

where $\widehat{H}$ is a self-adjoint operator corresponding to the financial energy given by a function $H(q, p)$ on the financial phase space. Here we proceed in the same way as in ordinary quantum theory for elementary particles.

10.4 Application of Quantum Formalism to the Financial Market

We now turn back to the general scheme, concentrating on the configuration representation, $\psi : Q \to \mathbf{C}$; $\psi \in L_2(Q) \equiv L_2(Q, dx)$. This is the general QL statistical formalism on the price space.

As in ordinary quantum mechanics, we consider a representation of financial quantities, observables, by symmetric operators in $L_2(Q)$. By using Schrödinger's representation we define price and price change operators by setting

$$\hat{q}_j\psi(q) = q_j\psi(q),$$

the operator of multiplication by the q_j-price;

$$\hat{p}_j = \frac{\hbar}{i}\frac{\partial}{\partial q_j},$$

the operator of differentiation with respect to the q_j-price, normalized by the scaling constant $\hbar$. Operators of price and price change satisfy the canonical commutation relations

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar.$$

By using this operator representation of price and price changes we can represent every function $H(q, p)$ on the financial phase space as an operator $H(\hat{q}, \hat{p})$ in $L_2(Q)$. In particular, the financial energy operator is represented by the operator

$$\hat{H} = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2m_j} + V(\hat{q}_1, \dots, \hat{q}_n) = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j}\frac{\partial^2}{\partial q_j^2} + V(q_1, \dots, q_n).$$

Here $V(\hat{q}_1, \dots, \hat{q}_n)$ is the operator of multiplication by the function $V(q_1, \dots, q_n)$.

In this general QL formalism for the financial market we do not consider individual evolution of prices (in contrast to the Bohmian approach). The theory is purely statistical. We can only determine the average of a financial observable A for some fixed state ϕ of the financial market:

$$\langle A \rangle_\phi = \langle A\phi, \phi \rangle.$$

The use of the Bohmian model gives the additional possibility of determining individual trajectories.

10.5 Standard Deviation of Price

We are interested in the standard deviation of the price q_t. Let ψ be the mental state of the financial market. The quantum formalism gives us the following formula for the price dispersion:

$$\sigma^2_\psi(q_t) = E_\psi q_t^2 - (E_\psi q_t)^2, \tag{10.5}$$

where for an observable a the quantum average (with respect to the state ψ) is given by $E_\psi a = \langle a\psi, \psi \rangle$.

Since, for any observable a_t,

$$E_\psi a_t = E_{\psi(t)} a_0, \tag{10.6}$$

we have

$$\sigma^2_\psi(q_t) = E_{\psi(t)} q^2 - (E_{\psi(t)} q)^2. \tag{10.7}$$

So

$$\sigma^2_\psi(q_t) = \langle q^2\psi(t), \psi(t) \rangle - \langle q\psi(t), \psi(t) \rangle^2. \tag{10.8}$$

Suppose that at the initial instant of time the wave function has the form of a Gaussian packet:

$$\psi_0(q) \approx \int_{-\infty}^{+\infty} \exp\{-k^2(\Delta q)^2 + ikq\} dk,$$

where Δq is the width of packet in the price space. Here the mean value of price is equal to zero. It is well known that

$$\psi(t, q) \approx \int_{-\infty}^{+\infty} \exp\{-k^2(\Delta q)^2 + ikq - (ihk^2 t)/2m\} dk.$$

Here the mean value of price is equal to zero for any instance of time. By calculating this integral we see that

$$\sigma_\psi(q_t) = \sqrt{\langle q^2\psi(t), \psi(t) \rangle - \langle q\psi(t), \psi(t) \rangle^2} = \sqrt{\langle q^2\psi(t), \psi(t) \rangle} \approx ht/m\Delta q$$

for large t.

Thus for a Gaussian packet of prices its standard deviation evolves as a linear function with respect to t. Large financial mass (i.e., a higher level of emission of shares) induces smaller standard deviation – so the price does not fluctuate far from the mean value. If the level of emission is very small, then large deviations from the mean value can be expected.

10.6 Comparison with Conventional Models of the Financial Market

Our model of the stocks market differs crucially from the main conventional models. Therefore we should perform an extended comparative analysis of our model and known models. This is not a simple task and it takes a lot of effort.

10.6.1 Stochastic Model

Since the pioneer paper of Bachelier [24], various models of the financial market based on stochastic processes have been actively developed. We recall that Bachelier determined the probability of price changes $P(v(t) \leq v)$ by writing down what is now called the Chapman–Kolmogorov equation. If we introduce the density of this probability distribution $p(t, x)$, so $P(x_t \leq x) = \int_{-\infty}^{x} p(t, x) dx$, then it satisfies the Cauchy problem of the partial differential equation of the second order. This equation is known in physics as Chapman's equation and in probability theory as the direct Kolmogorov equation. In the simplest case, when the underlying diffusion process is the Wiener process (Brownian motion), this equation has the form (the heat conduction equation)

$$\frac{\partial p(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(t, x)}{\partial x^2}. \tag{10.9}$$

We recall again that in Bachelier's paper [24], $x = v$ was the price change variable.

For a general diffusion process we have the direct Kolmogorov equation

$$\frac{\partial p(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(t, x) p(t, x)) - \frac{\partial}{\partial x} (\mu(t, x) p(t, x)). \tag{10.10}$$

This equation is based on the diffusion process

$$dx_t = \mu(t, x_t)\, dt + \sigma(t, x_t)\, dw_t, \tag{10.11}$$

where $w(t)$ is the Wiener process. This equation should be interpreted as a slightly colloquial way of expressing the corresponding integral equation

$$x_t = x_{t_0} + \int_{t_0}^{t} \mu(s, x_s)ds + \int_{t_0}^{t} \sigma(s, x_s)dw_s. \tag{10.12}$$

We remark that Bachelier's original proposal of Gaussian-distributed price changes was soon replaced by a model in which prices of stocks are *log–normal distributed*, i.e., stock prices $q(t)$ are performing a *geometric Brownian motion.* In a geometric Brownian motion, the difference of the logarithms of prices are Gaussian distributed.

We recall that a stochastic process S_t is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation:

$$dS_t = u\, S_t\, dt + v\, S\, dw_t, \tag{10.13}$$

where w_t is a Wiener process (Brownian motion) and u ("the percentage drift") and v ("the percentage volatility") are constants. The equation has an analytic solution:

$$S_t = S_0 \exp\left((u - v^2/2)t + v w_t\right). \tag{10.14}$$

The $S_t = S_t(\omega)$ depends on a random parameter ω; this parameter is typically omitted. The crucial property of the stochastic process S_t is that the random variable

$$\log(S_t/S_0) = \log(S_t) - \log(S_0)$$

is normally distributed.

In contrast to such stochastic models, our Bohmian model of the stock market is not based on the theory of stochastic differential equations. In our model the randomness of the stock market cannot be represented in the form of some transformation of the Wiener process.

We recall that the stochastic process model has been intensely criticized for many reasons, see, e.g., [237]. First of all there are a number of difficult problems that could be interpreted as technical problems. The most important among them is the problem of the choice of an adequate stochastic process $\xi(t)$ describing price or price change. Nowadays it is widely accepted that the geometric Bohmian motion model provides only a first approximation of what is observed in real data. One should try to find new classes of stochastic processes. In particular, they would provide an explanation of the empirical evidence that the tails of measured distributions are longer than expected for a geometric Brownian motion. To solve this problem, Mandelbrot proposed that the price changes should be considered to follow a *Levy distribution* [237]. However, the Levy distribution has a rather pathological property: its variance is infinite. Therefore, as was emphasized in the book by Mantegna and Stanley [237], the problem of finding a stochastic process providing an adequate description of the stock market is still unsolved.

However, our critique of the conventional stochastic processes approach to the stock market has no direct relation to this discussion on the choice of an underlying stochastic process. We are closer to scientific groups that criticize this conventional model by questioning the possibility of describing price dynamics by stochastic processes at all.

10.6.2 Deterministic Dynamical Model

In particular, a lot of work has been done on applying deterministic nonlinear dynamical systems to simulate financial time series, see [237] for details. This approach is typically criticized through the following general argument: "the time evolution of an asset price depends on all information affecting the investigated asset and it seems unlikely to us that all this information can be essentially described by a small number of nonlinear equations," [237]. We support such a viewpoint.

We shall use only critical arguments against the hypothesis of the stochastic stock market that were provided by adherents of the hypothesis of a deterministic (but essentially nonlinear) stock market.

Only at first sight is the Bohmian financial model a kind of deterministic model. Of course, dynamics of prices (as well as price changes) are deterministic. It is described by Newton's second law, see the ordinary differential equation (10.3). It seems that randomness can be incorporated into such a model only through the initial conditions:

$$\dot{p}(t, \omega) = f(t, q(t, \omega)) + g(t, q(t, \omega)), \quad q(0) = q_0(\omega), \quad p(0) = p_0(\omega), \quad (10.15)$$

where $q(0) = q_0(\omega)$, $p(0) = p_0(\omega)$ are random variables (initial distribution of prices and momenta) and ω is a chance parameter.

However, the situation is not so simple. Bohmian randomness does not reduce to randomness of initial conditions or chaotic behavior of (10.3) for some nonlinear classical and quantum forces. These are classical impacts on randomness. But a really new impact is given by the essentially quantum randomness that is encoded in the ψ-function (i.e., pilot wave or wave function). As we know, the evolution of the ψ-function is described by an additional equation – Schrödinger's equation – and hence the ψ-randomness can be extracted neither from the initial conditions for (10.15) nor from possible chaotic behavior.

In our model the ψ-function gives the dynamics of expectations at the financial market. These expectations are a huge source of randomness at the market – mental (psychological) randomness. However, this randomness is not classical (so it is a non-Kolmogorov probability model).

Finally, we remark that in quantum mechanics the wave function is not a measurable quantity. It seems that we have a similar situation for the financial market. We are not able to measure the financial ψ-field (which is an infinite-dimensional object, since the Hilbert space has infinite dimension). This field contains thoughts and expectations of millions of agents and of course it could not be "recorded" (unlike prices or price changes).

10.6.3 Stochastic Model and Expectations of Agents of the Financial Market

Let us consider again the model of the stock market based on geometric Brownian motion:

$$dS_t = uS_t\,dt + vS\,dw_t.$$

We notice that in this equation there is no term describing the behavior of agents of the market. Coefficients u and v do not have any direct relation to expectations and the market psychology. Moreover, even if we introduce some additional stochastic processes

$$\eta(t, \omega) = (\eta_1(t, \omega), \dots, \eta_N(t, \omega))$$

describing the behavior of agents and additional coefficients (in stochastic differential equations for such processes) we would not be able to simulate the real market. A finite-dimensional vector $\eta(t, \omega)$ cannot describe the "mental state of the market", which is of infinite complexity. One can consider the Bohmian model as the introduction of the infinite-dimensional chance parameter ψ. And this chance parameter cannot be described by classical probability theory.