Chapter 1
Quantum-like Paradigm

The *Quantum-like paradigm* (QL) is based on understanding that the mathematical apparatus of quantum mechanics (and especially quantum probability) is not rigidly coupled with *quantum physics*, but can have a wider class of applications.

1.1 Applications of Mathematical Apparatus of QM Outside of Physics

Recall that differential and integral calculi were developed to serve classical Newtonian mechanics. However, nowadays nobody is surprised that these tools are widely used everywhere – in engineering, biology, economics, ... In the same way, although the mathematical apparatus of quantum mechanics was developed to describe phenomena in the microworld, it could be applied to the solution of various problems outside physics.

One of the interesting problems is to apply *quantum probability*, e.g., to cognitive science or to financial markets. The main distinguishing features of quantum probability is its representation by the *complex probability amplitude*. In the abstract approach, such an amplitude is represented by a normalized vector in a complex Hilbert space, while the so-called mixed state is represented by a *density matrix.* Probability (which is compared with experimental relative frequencies) is given by *Born's rule.* For example, for measurement of the coordinate x of a quantum particle, the probability of finding it at the point $x = x_0$ is equal to the square of the absolute value of the wave function at this point.

The main (merely psychological) barrier in applications of QL models outside quantum physics is a rather common opinion that the "unusualness" of the quantum formalism (compared with, e.g., classical statistical mechanics) is an exhibition of "unusualness" of quantum systems. Such rather mystical things as *quantum nonlocality* and *death of realism* are firmly coupled to the modern interpretation of quantum mechanics (in particular, through Bell's theorem and other "no-go" theorems). It seems that quantum formalism cannot be applied to "usual systems" (e.g., macroscopic biological systems or huge social systems), because in this case it is not

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easy to accept death of realism[1] or any sort of nonlocality. For instance, attempts to describe the brain's functioning by means of QM typically induce heavy discussions on time, space, and temperature scales. The aim of such scale studies is to couple the brain's scales to quantum scales.

My aim is to show that such a reduction to quantum physical scales is totally unnecessary. The hot macroscopic brain might be able to process information in a quantum-like way, exhibiting, e.g., the interference of probabilities of alternatives. Moreover, any sufficiently complex biological, social, or financial system might exhibit quantum-like probabilistic features. I shall explain the source of such features a little bit later after the discussion on quantum randomness and probability. At the moment I just point out *contexuality* as the main source of quantum-like probabilistic behavior.

1.2 Irreducible Quantum Randomness, Copenhagen Interpretation

During the past 70 years the development of quantum mechanics has been characterized by intense debates on the origin of quantum randomness and in particular on possibilities to reduce it to the classical ensemble randomness. For example, von Neumann was convinced that *quantum randomness is irreducible,* but Einstein had the opposite view on this problem: for him the discovery of quantum mechanics was merely a discovery of a special mathematical formalism (quantum formalism) for description of a special *incomplete representation of information* about microsystems.

According to the Copenhagen interpretation of QM a pure quantum state (wave function) describes an *individual quantum system*, not an ensemble of systems in the sense of classical probability. As a consequence of such an "individual interpretation", a concrete physical system, e.g., an electron, can be prepared in a physical superposition of pure states.

In the majority of textbooks on QM we can read about, e.g., an atom in a superposition of different energy states[2], or an electron in a superposition of spin-up and spin-down states; in the famous two-slit experiment a photon is in a superposition of passing through both slits.

An attempt to apply the mathematical formalism of QM outside of the microworld in combination with the Copenhagen interpretation would create obvious difficulties: it is not easy to imagine a macroscopic system, e.g., in economics, that is in a real, physical, superposition of two states. Of course, I am well aware of the existence of macroscopic quantum systems as well as of the attempts to use the Copenhagen interpretation even in this case – e.g., by Legget in superconductivity,

[1] For example, neurons in the brain definitely have objective properties.

[2] Unlike in QM, in classical mechanics the energy of a particle can (at least in principle) be exactly determined.

by Zeilinger in the two-slit experiments for macroscopic systems, by De Martini in experiments with “macroscopic Schrödinger cats” etc.

It is well known that such attempts to proceed with the Copenhagen interpretation for macroscopic quantum systems do not provide a clear physical picture of the phenomena. One of the possibilities is to use *de Broglie’s wavelength* for characterization of the wave features of a macroscopic system. Since it is very small for a large system, it is always possible to say that, although a macroscopic system has wave features, they are hardly observable.

This kind of compromise is hardly satisfactory as a solution to a conceptual problem, especially for justification of applications outside quantum physics, e.g., in psychology or economics. Moreover, as already pointed out by Pauli in the early times of QM, any attempt to interpret the wave function as a physical wave clashes against the fact that, for most interesting physically systems, these wave functions are defined in a multidimensional mathematical space.[3]

Thus the supporters of the *wave–particle duality* face the paradox of believing in *a physical wave in a nonphysical space* (already in the case of a two-particle system).

1.3 Quantum Reductionism in Biology and Cognitive Science

As a consequence of the above-mentioned difficulties with the interpretation of macroscopic quantum systems, a popular attitude today in attempts to apply quantum mechanics (e.g., in biology) is to proceed beyond conventional (e.g., biological) models that operate with states of macroscopic systems.

For example, in cognitive science a group of researchers (e.g., Penrose [249, 250] and Hameroff [128, 129]) developed the reductionist approach to the brain’s functioning. They moved beyond the *conventional neuronal paradigm* of cognitive science and tried to reduce processing of information in the brain to quantum microprocesses – on the level of quantum particles composing the brain. Penrose repeated many times that a neuron (as a macroscopic system) could not be in a physical superposition of two states: firing and nonfiring.

As was already mentioned, the majority of attempts to apply the mathematical formalism of quantum mechanics outside physics were based on the reduction of the processes under consideration to some underlying quantum processes in the microworld. This reductionist approach was heavily based on the following argument: since everything in this world is composed of quantum particles, any kind of process might be (at least in principle) reduced to a quantum process.

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[3] The wave function of, e.g., a pair of electrons is defined not on physical space described mathematically by the cartesian product of 3 real lines, but on configuration space, which is the cartesian product of 6 real lines. Thus, Schrödinger already understood well that two electrons cannot be embedded in physical space. Therefore he gave up with his interpretation of the wave function as charge density. In principle, already at this stage one might start to speak about “quantum nonlocality”, i.e., without any reference to Bell’s inequality.

The unification dream is in principle correct, and it has played an important role in the development of natural sciences; in this spirit any attempt to apply quantum mechanics to, e.g., cognitive science should be welcome. However, it is very difficult (if possible at all) to establish a natural correspondence between conventional macroscopic models and underlying quantum models.[4] There is a huge difference in scales of parameters in those models. Moreover, even in quantum physics the *correspondence principle* is vaguely formulated and not totally justified and, on the other hand, even in classical physics, the unification dream is far from being accomplished in spite of the important successes of statistical mechanics in the reduction of thermodynamics to mechanics. For example, structures such as crystals, which are relatively simple in comparison with biological structures, at the moment have not been deduced from first principles in either classical or quantum physics.

1.4 Statistical (or Ensemble) Interpretation of QM

I point out that it is possible to escape the above-mentioned difficulties by rejection of the Copenhagen interpretation and association of a pure quantum state (wave function) not with an individual quantum system, but with an ensemble of systems.

Such an interpretation is called the *statistical (or ensemble) interpretation* of QM. It was originally proposed by many authors, including Einstein, Popper, Margenau, de Broglie, Bohm, and Ballentine, but only with the development of quantum probability could it overcome the traditional criticism which prevented, for over 50 years, the majority of physicists from accepting this apparently natural interpretation. The main objection to it, to which the above-mentioned authors never gave a satisfactory answer, was that the statistical interpretation is contradicted by the experimental data. We recall that at the beginning Schrödinger was quite sympathetic to Einstein's attempts to proceed in the quantum framework on the basis of the statistical interpretation.[5]

Schrödinger wrote that if Einstein were able to derive *interference of probabilities* for the two-slit experiment on the basis of the statistical model, he (Schrödinger)

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[4] This whole "quantum approach" is very speculative because it is currently controversial whether quantum theoretical mechanisms can be experimentally identified in the neural correlates (or constituents) of cognitive processes such as, e.g., decision-making.

[5] It is practically forgotten that the famous *Schrödinger cat* was created to show the absurdness of the Copenhagen interpretation of QM. If one accepts superposition of states for an individual microscopic system, then superposition of states for an individual macroscopic system should also be accepted. We recall that Schrödinger just modified Einstein's example with a gun and a man by making it more peaceful (at least at that time) – by using poison and a cat. We remark that nowadays people (heavily infected by the Copenhagen interpretation) take Schrödinger cats seriously. A number of famous experimental groups produce Schrödinger cats (as they believe) in their labs. Of course, I have no doubt that such experiments as, e.g., done by the group of De Martini from the University of Rome ("La Sapienza") are great contributions in the domain of quantum foundations, but the belief that really Schrödinger cat-type states are produced is rather naive.

would definitely choose this model. However, neither Einstein nor anybody else was able to perform such a derivation.

Concerning this objection, my main point is that the experimental data *contradict the use of the Kolmogorov model* of probability and not the statistical interpretation by itself. If one keeps to the statistical interpretation, then one can assume that the quantum probabilistic description need not be based on irreducible quantum randomness. However, the classical probability model should be generalized to take into account *contextuality* of probability, i.e., its dependence on context (physical, biological, mental, social, financial) of observations (which could even be self-observations of, e.g., the brain).

The contextual probabilistic calculus can be used for an incomplete description of statistical data. One could not even exclude that in some cases a Kolmogorov model can be found beyond the QL contextual probabilistic description. The crucial point is that the role of the presence of a "hidden Kolmogorovian model" is negligible if one has no access to data described by the latter (typically unobservable joint probabilities). In such cases the only reasonable possibility is to use the quantum-like probabilistic description or different non-Kolmogorovian models.

Thus we propose testing in various applications the approach based on accepting Einstein's viewpoint: the mathematical formalism that was developed to serve quantum physics is a special form of *incomplete probabilistic description.* Of course, for QM (as a physical theory) Einstein's viewpoint implies its incompleteness.

1.5 No-Go Theorems

The natural question which is typically asked as the first reaction to my proposal is the following:

What about the known no-go theorems?

I will not enter here into a debate on the complicated problem of the validity of no-go theorems.[6] In fact, the main problem of the "no-go ideology" is that it is directed against *all possible* prequantum models (the so-called hidden variable models). Supporters of no-go activity formulate new theorems excluding various classes of models with hidden variables, but one can never be sure that a natural model that does not contradict any known no-go theorem will finally be found.

Remark 1.1 I do not agree with Bell's attempt to couple the so-called "quantum nonlocality" with the problem of completeness of quantum mechanics. Einstein, Podolsky and Rosen [99] considered "quantum nonlocality" as an absurd alternative to incompleteness. Unfortunately, nowadays quantum nonlocality has become extremely popular in quantum information theory. Moreover, this idea is diffusing

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[6] See, for example, my books [159, 161, 214] and papers [162, 164] as well as my papers with Igor Volovich [166] and Luigi Accardi [4]; see also stormy debates in the proceedings of Växjö conferences [165, 167, 5, 6].

outside quantum physics: it has become fashionable to refer to quantum nonlocality in cognitive and social sciences and even in parapsychology. In the latter case quantum nonlocality provides really great new possibilities. Conferences devoted to Quantum Mind have become a tribune for parapsychological speculations based on quantum nonlocality. Of course, people working in quantum information theory and trying to design quantum computers, cryptography and teleportation are not so happy to hold joint meetings with, e.g., "quantum buddhists" creating a powerful new religion, but they have no choice! By accepting "quantum nonlocality" they are in one camp with people providing the QM-interpretation for a nonlocal deity.

As a sign of inconsistency of the no-go activity, we mention the sharp criticism of the assumptions of known no-go theorems by newcomers – authors proposing new no-go statements. For instance, Bell criticized [31] quite aggressively assumptions of von Neumann's no-go theorem [301] (and other no-go theorems which existed at that time). Therefore it is surprising that nowadays the majority of the quantum community (especially people working in quantum information) reacts so painfully to critique of the assumptions of Bell's theorem – for such a critique see, e.g., Accardi [3], Accardi and Khrennikov [4], Adenier and Khrennikov [7], Andreev and Man'ko [17], De Baere [78, 79], De Muynck et al. [84, 85], Hess and Philipp [142, 143], Khrennikov [159, 161, 162, 164, 179], Khrennikov and Volovich [166, 193], Klyshko [216, 217]. We point out especially the practically forgotten papers by Klyshko. He was one of the best experimenters in the world in the domain of quantum optics. It is amazing that he came to the same conclusion as pure mathematicians, e.g., Accardi, Aerts, Khrennkiov, and recently Hess and Philipp.

Violation of Bell's inequality is merely an exhibition of non-Kolmogorovness of quantum probability, i.e., the impossibility of representing all quantum correlations as correlations with respect to a single Kolmogorov probability space [219]; there is no direct relation between this violation and such mysterious things as nonlocality and death of realism.

1.6 Einstein's and Bohr's Views on Realism

I emphasize that Einstein's realism is a quite naive form of realism. It does not take into account the dynamics of the process of interaction of a system with the measurement device. The father of the Copenhagen interpretation N. Bohr permanently pointed out that the *whole experimental arrangement* (context) should be taken into account. It is the crucial point not only for physics, but for other domains of science. For example, in the process of decision making the brain interacts with questions (problems). The resulting decision is the result of such an interaction.

I share Einstein's views only partially. I keep to the statistical (ensemble) interpretation of the quantum state and consequent incompleteness of QM, but I agree with Bohr in considering the values of some quantum observables as responses to interactions with apparatus rather than objective properties of quantum systems. The latter point has been emphasized by Luigi Accardi since the early 1980s in a series of works on the so-called *chameleon effect*, see [1–4]. This effect is nothing other than the dependence of a chameleon's color (its "property") on a surface's color. The latter is an analog of the state of apparatus. Of course, such a type of system–apparatus interaction is purely deterministic. To be closer to reality, one should consider random chameleons. Another similar approach is the *adaptive dynamics* developed by Masanori Ohya, see, e.g., [245, 246].

On the basis of such an ideology one can proceed successfully.

This combination of the views of Einstein and Bohr is known as the *Växjö interpretation* of QM, see [177].

Concerning no-go theorems my proposal could be summarized as follows:

"Do not be afraid to consider the quantum description as an incomplete one. Look for applications of quantum formalism outside quantum physics!".

1.7 Quantum and Quantum-like Models

As a comment on the use of the notion quantum-like (QL) behavior, I think that it would be useful to preserve the term "quantum" for quantum physics while, in other models which are still based on quantum or, more generally, non-Kolmogorovian, probabilistic description, we should use the term "quantum-like". In particular, in this way my approach can be distinguished from a purely reductionist one. For example, the quantum brain model is a reductionist model of the brain functioning, but the *QL brain* model is a model in which the wave function provides a (incomplete) probabilistic representation of information produced by the neurons and not a model of the actual physical state of them. In the same way a *quantum game* is based on randomness produced by quantum physical systems (e.g., photons), but a *QL game* can be performed by purely classical physical systems (e.g., people) exhibiting QL probabilistic behavior.

1.8 Quantum-like Representation Algorithm – QLRA

Quantum-like modeling immediately meets one complex problem: the creation of QL-representations (in complex and more general Hilbert spaces) of classical (contextual) probabilistic data. For example, looking for a QL model of the brain's functioning we should be able to answer the following question:

How does the brain represent statistical information by the wave function (complex probability amplitude)?

If one considers the brain as a kind of probabilistic machine, then this problem can be formulated as the *inverse Born problem:*

To construct a complex probability amplitude (or in the abstract framework a normalized vector of Hilbert space) on the basis of probabilistic data. This amplitude should produce probabilities by Born's rule.

An attempt to solve this problem was made in a series of my works, starting with [163]. There was created the so-called *QL representation algorithm* – QLRA.[7] It transforms probabilistic data of any origin (which should satisfy some natural restrictions) into a complex probability amplitude.

1.9 Non-Kolmogorov Probability

We now couple the QL paradigm with another important probabilistic paradigm. Nonclassical statistical data are not covered completely by the conventional quantum model. The main distinguishing feature of quantum probability is its *non-Kolmogorovinity* expressed in the form of contextuality.[8]

It was emphasized (by Luigi Accardi, Diederik Aerts, Stan Gudder and me[9]) that in the same way as in geometry (where, starting with Lobachevsky, Gauss, Riemann, . . . , various non-Euclidean geometries were developed and widely applied, e.g., in relativity theory), in probability theory various non-Kolmogorov models may be developed to serve applications. The QM probabilistic model was one of the first non-Kolmogorovian models that had important applications. Thus one may expect development of other types of probabilistic models which would be neither Kolmogorovian nor quantum.

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[7] Improvement of this algorithm, its generalization, and creation of new QL representation algorithms is an important problem in the realization of the QL paradigm.

[8] We remind the reader that the main distinguishing feature of the Kolmogorov [219], 1933, model of probability is the possibility of embedding everything (probabilities and obsevables – called random variables) in a single space Ω. In particular, all probabilities are reduced from a single probability measure on this space, say $\mathbf{P}$, and all random variables are realized by functions on this space. We also point out that, in spite of rather common opinion, the Kolmogorov model is not so simple mathematically. The measure-theoretic considerations are complicated – much more complicated than the linear algebra of quantum probability. In fact, the main difficulties in quantum probability are related to the infinite dimension of the state space, Hilbert space. However, in many applications, e.g., quantum information, one can proceed with linear algebra in finite-dimensional spaces.

[9] There are some debates on priority. However, I think that such debates are totally meaningless. These are debates about the first two bright stars in the complete darkness of the traditional quantum kingdom. I was strongly influenced by all of them. Conversations with Stan Gudder and Luigi Accardi during their visits to Växjö played an important role in the formation of my views on nonclassical probability. Although I did not succeed in inviting Diederik Aerts to one of the conferences of the Växjö series on foundations of quantum theory and quantum information, e-mail exchange with him also was very important for me.

1.10 Contextual Probabilistic Model – Växjö Model

In this book a general scheme of probabilistic description of experimental probabilistic data is presented. We call this model the contextual probabilistic model (the *Växjö model*) [185]. The origin of the data does not play any role. It could be statistical data from quantum mechanics or equally from classical statistical mechanics, biology, sociology, economics, or meteorology. Then it will be shown that the interference of probabilities (even more generally than in quantum mechanics) can be found for any kind of data [163, 170–174, 176–178, 186–188]. As was pointed out, one can easily obtain the linear space representation of probabilities from the interference of probabilities (by applying QLRA) and then recover Born's rule (which will not be a postulate anymore). Thus, in our approach the quantum probabilistic calculus is just a special linear space representation of given probabilistic data [163]. One of the advantages of the QL representation of probabilistic data is an essential simplification of operating with this data – *linearization* of a model always induces simplification.

Basic to our approach is the notion of *context* – a complex of conditions under which the measurement is performed. Contexts of different kinds can be considered: physical, biological or even political. Our approach to the subject of probability is contextual. It is meaningless to consider a probability not specifying the context of consideration.

Kolmogorov was well aware of the *contextuality* of his probability space [219]. He emphasized that any experiment should be described by its own (Kolmogorov) probability space. Unfortunately, this ideological recognition of contextuality of probabilities did not imply any form of the mathematical formalization in his work (or in the work of later users of the Kolmogorov probability model). One of the sources of quite misleading (for applications) development of the mathematical theory of probability was the presence of conditioning in the standard probability model. Kolmogorov defined conditional probabilities by using Bayes formula. It became a custom to identify context-dependence with such *Bayesian conditioning*. I think that it restricted essentially the range of context-dependent phenomena. Already in QM, one should go beyond Bayesian conditioning. It is amazing that QM was created at the same time as the Kolmogorovian model. But mathematicians were able to proceed for at least 50 years without understanding that this model suffers from a very special definition of conditioning.

I should also mention the model of Hungarian mathematician Renyi [267], which is nowadays practically forgotten. He tried to extend Bayes-Kolmogorov conditioning. But the closest to my approach were the probabilistic studies by Mackey [235], who tried to reconstruct quantum probability on the basis of contextual (conditional) probabilities. Unfortunately, his attempt was not completely successful. Finally, he postulated the Hilbert space representation of probabilities.

As was pointed out, generally the QL paradigm is not reduced to application of conventional quantum probability (based on the complex Hilbert space representation) outside quantum physics. Other (non-Kolmogorovian) models might be applied as well. One such model was presented – it is the model with so-called *hyperbolic interference* of probabilities [163]. It is based on representation of probabilities by amplitudes taking values in the algebra of so-called hyperbolic numbers (a two-dimensional Clifford algebra).[10]

1.11 Experimental Verification

Finally, one should analyze several different families of empirical data in order to realize concretely the outlined programme. It is clear that without experimental justification the QL paradigm is simply a philosophical-mathematical principle. It can give rise to a huge variety of interesting mathematical models (as was done, e.g., in string theory [123]), but finally these models should be verified by comparison with experimental statistical data.

The first steps of the experimental verification have already been done, see Conte et al. [66, 67]. It was experimentally confirmed that the brain behaves as a QL system in the process of decision making (in tests with ambiguous figures): the interference of probabilities related to incompatible questions was found. Moreover, it was natural to suppose that some well-known experiments that have already been done in cognitive science or psychology might produce nonclassical statistical data.[11] Recently Busemeyer et al. [48, 49] pointed out that statistical data collected in famous experiments in cognitive psychology performed by Shafir and Tversky [275, 295] do not agree with the standard Markov chain model based on classical probability theory. He proposed using the quantum model. Stimulated by discussions with Jerome Busemeyer (professor of cognitive psychology), I applied my QL approach to Shafir-Tversky statistical data. It was represented (via QLRA) by complex (and in some case more general hyperbolic!) probability amplitudes. The interference of probabilities was found [208, 213, 206], see also the joint work with Emmanuel Haven [207]. I remark that, opposite to the original conjecture of Busemeyer, Shafir-Tversky statistical data do not match the conventional quantum model. The corresponding matrices of *transition probabilities* are not *doubly stochastic*, but they should be by QM (in the case of observers with nondegenerate spectra).

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[10] This example motivates extension of the QL paradigm by attempting to develop and apply models in which probability amplitudes take values in various commutative and even noncommutative algebras. The corresponding generalizations of Born's rule should be presented, analogues of the QL representation algorithms should be created. These are interesting and complex problems!

[11] Luigi Accardi pointed out such a possibility many years ago (during my visit to Centro V. Volterra, University of Rome - 2, in 1994). We suspected that nonclassical data might be found in economics, finances, and psychology. The same point was expressed in long discussions with Stan Gudder and Karl Gustafson during my visit to the Universities of Denver and Boulder, in 2001. However, it was really impossible to find such data without competence in the corresponding domains of science. Therefore I was happy when Elio Conte proposed performing experiments to test my QL model [180] of cognition.

1.12 Violation of Savage's Sure Thing Principle

We recall that experimental studies by Shafir and Tversky [275, 295] were about behavior of people in games such as the well-known Prisoner's Dilemma, which illustrate Savage's *Sure Thing Principle* (STP) [271]. Violation of this principle is known as the *disjunction effect,* see Shafir and Tversky [275, 295] and also Rapoport [266], Hofstadter [145, 146] and Croson [71]. We recall that STP is one of the basic principles of modern theoretical economics. In fact, it is a form of the postulate on *rational behavior* of agents acting at the market. Thus experimental violation of STP (disjunction effect) is a sign of *irrational behavior* of agents.

The QL model, see Chapter 7, explains this irrationality. The QL extension of probabilistic description of natural and social phenomena shows that the notion of rationality is itself dependent on a probabilistic model. The conventional rationality of economic agents is in fact the classical probabilistic rationality. If the brain processes information by using some QL representation of probabilistic data, then there are no reasons to expect exhibitions of conventional rationality. Of course, such cognitive systems behave irrationally from the conventional (say Kolmogorovian) viewpoint. However, they are completely rational in the corresponding QL sense.

1.13 Quantum-like Description of the Financial Market

In the conventional financial models, rationality of agents of the financial market is formalized through the *efficient market hypothesis*, which was formulated in the 1960s, see Samuelson [269, 270] and Fama [103] for details:

A market is said to be efficient in the determination of the most rational price if all the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded.

From the viewpoint of the QL paradigm the most rational price in the sense of conventional theory of the financial market is the most rational in the framework of the classical probabilistic model (the Kolmogorovian measure-theoretic model).[12] If agents of the financial market behave nonclassically (and that is my conjecture), then they use another type of rationality, the QL rationality. QL-rational behavior may look irrational in the conventional framework. (We remark that the experiments of Shafir and Tversky [275, 295] were done to show the irrationality of agents.) We recall once again that QL models describe the following situation. For each context C, only a special part of (statistical) information on this context can be measured and hence be available, for example, to traders. And we emphasize once again that such a viewpoint on the conventional quantum model contradicts the Copenhagen interpretation. However, the latter does not disturb us much. Debates on hidden

[12] "All the available information" has Boolean structure. The corresponding probability is described by the Kolmogorovian model.

variables in QM may continue for the next few hundred years. Even if physical quantum randomness is irreducible, one cannot exclude the possibility that quantum and more general QL formalisms can describe reducible (but contextual) randomness in other domains of science, for example, psychology or economics.

In Chapter 10 of this book we shall present a QL model of the functioning of agents of the financial market which is based on the mathematical formalism of Bohmian mechanics.[13] The "financial pilot wave" describes expectations of agents. Nonlocality of this model is purely classical: the common field of expectations is created through classical communication channels (TV, internet, newspapers, private communications). The presence of such a field provides a possibility to "rule" the financial market – in the same way as one can rule quantum particles by manipulating the pilot wave.

We remark that the original Bohmian model of QM is totally deterministic. Each quantum particle has well-defined position and momentum. Motion is described by a kind of Newton's equation (say *Bohm-Newton equation*), which contains not only the classical force generated by the physical potential V (for example, the Coulomb potential describing interaction of two charged particles, e.g., electron and proton), but also an additional force – the so-called *quantum force*. The latter is produced by the pilot wave, which is described by Schrödinger's equation (so mathematically it is just the wave function). It can be changed by modification of the physical potential V (which can depend on time) in Schrödinger's equation. The main point is that a slight modification of V can produce a modification of the pilot wave such that the corresponding *quantum force* will be essentially different (from that expected for non-modified V). Applying this formalism to the financial market, we see that by slight modification of the financial potential V one can totally change financial forces. Of course, it should be the right modification.[14]

The original Bohmian model can be essentially improved by adding a classical stochastic term to the Bohm-Newton equation describing motion of the quantum particle. This is the Vigier-Bohm model [40]. Such an improvement is especially useful for description of financial processes. The financial version of the original Bohmian model produces price trajectories that are solutions of the ordinary differential equation (*Bohm-Newton equation*) that is "controlled" by the field of expectations, the pilot wave. The latter makes the model essentially more realistic than classical financial models. Nevertheless, the presence of deterministic price trajectories might be considered a weakness of the Bohmian financial model.[15]

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[13] This model was invented by the author and Olga Choustova [53, 62, 175] and it was generalized by Emmanuel Haven [136] who made it closer to applications.

[14] Development of the present financial crisis might be considered as indirect confirmation of our model. The effects of special manipulations in the mass media can be considered as modifications of V. Such modifications need not involve many financial resources, so they are really small from the financial point of view.

[15] The problem of whether the financial market is deterministic (a huge deterministic dynamical system, maybe chaotic) or random is still the subject of debate, see the introduction to Chapter 10

The Vigier-Bohm model for the financial market was developed by Olga Choustova [60, 62] and Emmanuel Haven [136].

In fact, the financial Bohmian model is an attempt to go beyond the QL description. One can also proceed with the standard mathematical formalism of QM: Segal and Segal [274], Haven [132–135], Piotrowski et al. [252–258], Danilov and Lambert-Mogiliansky [73, 74], Khrennikov [176]. The model of the financial market based on the use of the Hilbert space representation of financial quantities developed by Baaquie [22] can also be mentioned. The quantum ideology does not play a large role in *Baaquie's model.* It is closer to classical models of *signal analysis* based on the Hilbert space representation.

Finally, we point to the book by Soros [282], which was definitely one of the first works on applications of methods of QM to the financial market. It is amazing that Soros, who was so far from quantum science, claimed that the financial market is a huge quantum system and not at all a collection of classical random systems (as claimed by conventional financial science). It is well known that Soros senses the financial market very well. His heuristic justification of its quantumness is a strong argument in favor of the QL approach. In particular, his book was the starting point for Olga Choustova's and my research.

1.14 Quantum and Quantum-like Games

Game theory plays an important role in various models of economics and evolution theory (including genetic evolution). *Quantum games* naturally arose in the process of development of quantum information, see, e.g., Ekert [100]. In principle, quantum games might be used for modeling of evolution and even in economics. However, one meets the same problems as in attempts to proceed with quantum reductionism in cognitive modeling. Maybe in genetics one could still appeal to conventional quantum games (induced by quantum systems), but in higher level cognitive processes or economics attempts to use quantum games (based on the irreducible randomness of quantum physical systems) are not sufficiently justified. In complete accordance with the QL paradigm, the QL probabilistic behavior can be exhibited in games that are not coupled directly to quantum physical systems – in particular, by (macroscopic) cognitive systems. Thus biological organisms and populations are able to play QL games in the process of evolution. Moreover, the evolution of social and economic systems might be based on QL games.[16] In particular, agents of the

for details. Although nowadays various models based on (classical) randomness dominate in theoretical finances, a large amount in real finances is done by technical analysis.

[16] By the QL paradigm the main reason for such games is operating with incomplete information. It can be profitable for a population not to spend too many resources on obtaining complete information about context, but to proceed by operating with incomplete information. Of course, such "nonclassical" operating should be consistent. Some rules should be established. By the QL paradigm various QL rules can be elaborated in this way.

market can play QL games, compare the experimental studies of Shafir and Tversky [275, 295]; see also publications by Piotrowski et al. [252–258].

Even in genetics the quantum-likeness (and not directly quantumness) may be crucial for evolution. Genes and genomes are macroscopic structures. In principle, their functioning can be based on incomplete information processing. In such a case the right model of genetic evolution would be based on a QL game between genes as well as between genomes. However, the latter conjecture is rather speculative. The corresponding experimental studies of quantum-likeness of genetic processes should be performed. On the other hand, simple QL games between people exhibiting interference effects (and even violating Bell's inequality) were proposed by Grib et al. [124, 125] and me [203]; see also the recent paper by Aerts et al. [9].

1.15 Terminology: Context, Contextual Probability, Contextuality

A few remarks regarding the terminology in this book are called for. The notion of the context can be related to the notion of the *preparation procedure*, which is widely used in quantum measurement theory [234, 46, 148]. Of course, preparation procedures – devices preparing systems for subsequent measurements – give a wide class of contexts. However, the context is a more general concept. For example, we can develop models operating with social, political or historical contexts, e.g., socialism context, victorian context. To give another example, one can consider the context "Leo Tolstoy" in literature. The latter context can be represented by various kinds of physical and mental systems – by books, readers, movies.

Contextual probability can be coupled to a *conditional probability.* However, once again the direct identification can be rather misleading, since the conventional meaning of the conditional probability $\mathbf{P}(B|A)$ is the probability that *event* B occurs under the condition that *event* A has occurred [219]. Thus, conventional conditioning is *event-conditioning.* Our conditioning is a *context-conditioning*: $\mathbf{P}(b = \beta|C)$ is the probability that observable b takes the value β in the process of measurement under context C. In principle, we are not against the term "conditional probability" if it is used in the contextual sense.

The main terminological problem is related to the notion of the *contextuality.* The use of the term "contextual" is characterized by a huge diversity of meanings, see Bell [31], Svozil [289] or Beltrametti and Cassinelli [32] for the notion of contextuality in quantum physics as well as Light and Butterworth [228] and Bernasconi and Gustafson [35] for the notion of contextuality in cognitive science and artifical intelligence (AI). In quantum physics the contextuality is typically reduced to a rather specific contextuality – "Bell contextuality." Bell invented this notion in the framework of the EPR-Bohm experiment [31, 32]. We recall that such *quantum contextuality* ("Bell's contextuality") is defined as follows:

The result of the measurement of an observable a depends on another measurement on observable b, although these two observables commute with each other.

It should be emphasized that *nonlocality* in the framework of the EPR-Bohm experiment is a special case of quantum contextuality. Our contextuality is essentially more general than Bell's. In a very special case one can determine the context for the measurement of a by fixing an observable b that is compatible with a. However, in the general case there is nothing about a mutual dependence or compatibility of observables. The context is simply a complex of conditions (e.g. physical or biological). Our description of the EPR-Bohm experiment is contextual, but there is no direct coupling with nonlocality. Our approach to the contextuality is closer to the one used in cognitive science and AI, see [228, 35].

1.16 Formula of Total Probability

The basis of linear representations of probabilities (performed by QLRA) is a generalization of the well-known formula of *total probability* (FTP) [280]. We recall that in the case of two dichotomous variables $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$ this basic formula has the form

$$\mathbf{P}(b = \beta) = \mathbf{P}(a = \alpha_1)\, \mathbf{P}(b = \beta|a = \alpha_1) \;\; + \;\; \mathbf{P}(a = \alpha_2)\, \mathbf{P}(b = \beta|a = \alpha_2), \tag{1.1}$$

where $b = \beta_1$ or $b = \beta_2$.

This formula is widely used in statistics and especially in statistical decision making. One wants to predict probabilities for values of the b-variable on the basis of probabilities for values of the a-variable and conditional probabilities to get the value $b = \beta$ under the assumption that $a = \alpha$. In decision making one makes decisions depending on magnitudes of probabilities $\mathbf{P}(b = \beta)$ given by FTP (1.1). Probabilities on the right hand side of FTP could have various interpretations. They could be *objective probabilities* calculated on the basis of the previous statistical experience. They also could be *subjective probabilities* assigned to, e.g., values of the a-variable. Further considerations do not depend on interpretation of probabilities.

1.17 Formula of Total Probability with Interference Term

Starting with the contextual statistical model (Växjö model) we will obtain a generalization of the conventional FTP (1.1) that is characterized by the appearance of an additional term, an *interference term*

$$\mathbf{P}(b = \beta) = \mathbf{P}(a = \alpha_1)\, \mathbf{P}(b = \beta|a = \alpha_1) \;\; + \;\; \mathbf{P}(a = \alpha_2)\, \mathbf{P}(b = \beta|a = \alpha_2) + \delta(b = \beta|a). \tag{1.2}$$

Depending on the magnitude of this term (relative to the magnitudes of probabilities on the right-hand side of (1.1)), we obtain either the conventional *trigonometric interference* ("cos-interference"), which is well known in classical wave mechanics as well as in quantum mechanics, or a *hyperbolic interference* ("cosh-interference"), which was not predicted by conventional physical theories, neither by classical wave theory nor by quantum mechanics. Such a new type of interference arises naturally in the Växjö model.

The possibility of violating the FTP is one of the important consequences of non-Kolmogorovness of probabilistic data. One of the main consequences of QL modeling is that cognitive and social systems can process probabilistic data violating FTP. Such a violation can be negligible for some contexts (which is why conventional FTP has been applied so successfully in many domains of science), but in general it cannot be completely neglected. By neglecting the interference term one comes to paradoxical conclusions, as in the case of data from the experiments of Shafir and Tversky [275, 295]. It is natural to suppose that cognitive and social systems should take into account (to survive in the process of evolution) the mentioned possibility of violation of FTP. Thus they should develop the ability to use a more general probabilistic model than the classical model. We speculate that they use special representations of the contextual statistical model. They may be able to apply QLRA and to represent contextual probabilities in complex or more general linear spaces.

1.18 Quantum-like Representation of Contexts

We recall once again that all probabilities in (1.2) are contextual. They depend on a complex of conditions, context C, for measurements of observables a and b. Starting with FTP with the interference term (1.1) and applying QLRA we obtain two basic types of representations of contexts, $C \to \psi_C$, in linear spaces:

a) representation of some special collection of contexts $\mathcal{C}^{\text{tr}}$ ("trigonometric contexts")[17] in complex Hilbert space, see Chap. 2;
b) representation of some special collection of contexts $\mathcal{C}^{\text{hyp}}$ ("hyperbolic contexts")[18] in the so-called hyperbolic Hilbert space.

The complex and hyperbolic representations can be combined in a single representation over a little bit more complicated algebraic structure – the algebra of complex hyperbolic numbers.

We emphasize that in general the collections of trigonometric and hyperbolic contexts, $\mathcal{C}^{\text{tr}}$ and $\mathcal{C}^{\text{hyp}}$, are just proper subsets of the complete collection of contexts $\mathcal{C}$ of a Växjö model M (contextual statistical model). Depending on model M there

[17] They produce the ordinary cos-interference.

[18] They produce hyperbolic cosh-interference.

can exist contexts that cannot be represented algebraically: neither in complex nor in hyperbolic Hilbert spaces.

We can speculate that some cognitive and social systems might restrict information processing by taking into account only trigonometric contexts. Such systems would process probabilistic information through its representation in the complex Hilbert space.