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Modeling Decisions for Artificial Intelligence

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Preface

This volume contains papers presented at the 6th International Conference on Modeling Decisions for Artificial Intelligence (MDAI 2009), held in Awaji Island, Japan, November 30 – December 2, 2009. This conference followed MDAI 2004 (Barcelona, Catalonia), MDAI 2005 (Tsukuba, Japan), MDAI 2006 (Tarragona, Catalonia), MDAI 2007 (Kitakyushu, Japan), and MDAI 2008 (Sabadell, Catalonia) with proceedings also published in the LNAI series (Vols. 3131, 3558, 3885, 4617, and 5285).

The aim of this conference was to provide a forum for researchers to discuss the theory and tools for modeling decisions, as well as applications that encompass decision-making processes and information-fusion techniques.

The organizers received 61 papers from 15 different countries, from Asia, Europe, and America, 28 of which are published in this volume. Each submission received at least two reviews from the Program Committee and a few external reviewers. We would like to express our gratitude to them for their work. The plenary talks presented at the conference are also included in this volume.

The conference was supported by the Commemorative Organization for The Japan World Exposition '70, the Tsutomu Nakauchi Foundation, Hyogo International Association, the Institute of Systems, Control and Information Engineers (ISCIE), the Operations Research Society of Japan (ORSJ), the UNESCO Chair in Data Privacy, the Japan Society for Fuzzy Theory and Intelligent Informatics (SOFT), the Catalan Association for Artificial Intelligence (ACIA), the European Society for Fuzzy Logic and Technology (EUSFLAT), and the Spanish MEC (ARES - CONSOLIDER INGENIO 2010 CSD2007-00004).

August 2009

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Interactive Robust Multiobjective Optimization Driven by Decision Rule Preference Model

Roman Słowiński

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Interactive procedures for MultiObjective Optimization (MOO) consist of a sequence of steps alternating calculation of a sample of non-dominated solutions and elicitation of preference information from the Decision Maker (DM). We consider **three types of procedures**, where in preference elicitation stage, the DM is just asked to indicate which solutions are relatively good in the proposed sample. In all three cases, the preference model is a set of "*if* ..., *then* ..." decision rules inferred from the preference information using the Dominance-based Rough Set Approach (DRSA) ([3]; [4]; [11]).

As proved in ([5]; [10]), the set of "*if* ..., *then* ..." decision rules is the most general and the most comprehensible preference (aggregation) model. The rules obtained using DRSA have a syntax adequate to multiobjective decision problems: the condition part of a rule compares a solution in the objective space to a dominance cone built on a subset of objectives; if the solution is within this cone, then the rule assigns the solution to either a class of "good" solutions (the case of a positive dominance cone) or to a class of "other" solutions (the case of a negative dominance cone). The main advantage of decision rules is their simplicity and human-interpretable form. Moreover, they are able to model interactions between objectives.

The **first** case considered is a deterministic MOO problem. Selected decision rules permit to focus progressively on the most interesting region of the Pareto-optimal set ([6]).

The **second** case considered is an optimization problem under uncertainty, exemplified by portfolio selection. Feasible portfolios are evaluated in terms of meaningful quantiles of the distribution of return. Using stochastic dominance on these quantiles, DRSA is producing decision rules guiding convergence to the most interesting region of the Pareto-optimal set ([7]).

The **third** optimization problem involves both multiple objectives and uncertainty. Some coefficients in the objective functions and/or constraints of this problem are not precisely known and given as interval values. The proposed interactive procedure is called DARWIN ([8]). In the calculation stage of DARWIN, a sample of feasible solutions is generated together with a sample of vectors of possible values of the imprecise coefficients, called scenarios. Each feasible solution from the current sample is characterized by a distribution over generated scenarios. Some representative quantiles of these distributions are presented to the DM in the preference elicitation stage. The DM is indicating relatively good

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solutions, and then DRSA is producing decision rules exploited by an evolutionary search of a better sample of solutions. Experiments with a hypothetical DM's value function used to select relatively good solutions, prove that DARWIN converges to the same region of the Pareto-optimal set as an evolutionary procedure optimizing just this value function.

In all three cases, the DM gives preference information by answering easy questions, and obtains transparent feedback in a learning oriented perspective (see (1)).

As to the third procedure, remark that most of past research on Evolutionary Multiobjective Optimization (EMO) attempts to approximate the complete Pareto-optimal front by a set of well-distributed representatives of Pareto-optimal solutions. The underlying reasoning is that in the absence of any preference information, all Pareto-optimal solutions have to be considered equivalent. On the other hand, in most practical applications, the DM is eventually interested in only a small subset of good solutions, or even a single most preferred solution. In order to come up with such a result, it is necessary to involve the DM. This is the underlying idea of Interactive Multiobjective Optimization (IMO). IMO deals with the identification of the most preferred solution by means of a systematic dialogue with the DM. Only recently, the scientific community has discovered the great potential of combining the two paradigms (see (2)). From the point of view of EMO, involving the DM in an interactive procedure allows to focus the search on the area of the Pareto front which is most relevant to the DM. This, in turn, may allow to find preferred solutions faster. In particular, in the case of many objectives, EMO has difficulties, because the number of Pareto-optimal solutions becomes huge, and Pareto-optimality is not sufficiently discriminative to guide the search into better regions. Integrating user's preferences promises to alleviate these problems, allowing to converge faster to the preferred region of the Pareto-optimal front.

It is also worth stressing that, in practice, not all data needed to formulate the MOO problem are known as precise numbers. Rather the opposite, they are often not precisely known, and thus the coefficients of the multiobjective optimization problem are given as intervals of possible values. In this situation, instead of seeking for the best solution with respect to the considered objectives, one is rather interested in the best robust solution with respect to the considered objectives and uncertainties.

In DARWIN, due to imprecision of some coefficients of the MOO problem, the evaluation of a population of solutions takes place in a transformed evaluation space, where for each solution $\mathbf{x}$, the DM gets information about $\mathbf{x}$ in terms of some meaningful quantiles of the distribution of objectives over scenarios, e.g., for maximized objectives: 1% probability of obtaining at most $f_1^{1\%}(\mathbf{x}), \ldots, f_k^{1\%}(\mathbf{x})$ on particular objectives, 25% probability of obtaining at most $f_1^{25\%}(\mathbf{x}), \ldots, f_k^{25\%}(\mathbf{x})$, 50% probability of obtaining at most $f_1^{50\%}(\mathbf{x}), \ldots, f_k^{50\%}(\mathbf{x})$, 75% probability of obtaining at most $f_1^{75\%}(\mathbf{x}), \ldots, f_k^{75\%}(\mathbf{x})$, and 99% probability of obtaining at least $f_1^{99\%}(\mathbf{x}), \ldots, f_k^{99\%}(\mathbf{x})$; of course $f_j^{1\%}(\mathbf{x}) \leq f_j^{25\%}(\mathbf{x}) \leq f_j^{50\%}(\mathbf{x}) \leq f_j^{75\%}(\mathbf{x}) \leq f_j^{99\%}(\mathbf{x})$, $j = 1, \ldots, k$.

In this way, DM's preferences are expressed in terms of the transformed evaluation space which combines the preferences on performances of the solutions with the preferences on the risk of attaining these performances. This is why the most preferred solution finally found is considered as robust. In DARWIN, the robustness of solutions is ensured twofold, since: (*a*) DRSA decision rules are immune to inconsistencies in preference information, and (*b*) DARWIN takes into account many possible scenarios, and involves preferences on distribution of values of objective functions over possible scenarios.

An important comment on the representation of risk by consideration of quantiles follows. Remark that when the best values of objectives in the meaningful quantiles are considered, then the comparison of two vectors of these best values using a dominance relation is equivalent to first-order stochastic dominance. On the other hand, when the mean values of objectives in the meaningful quantiles are considered, then the comparison of two vectors of these mean values using a dominance relation is equivalent to second-order stochastic dominance, also called Lorenz dominance (9). To underline the difference between the first- and second-order stochastic dominance in this context, remark that considering the mean instead of the best values in quantiles, we take into account bad evaluations in mean values of all quantiles, while good evaluations are taken into account in mean values of high quantiles only. In other words, in the second-order stochastic (Lorenz) dominance, worse scenarios gain importance. For this reason, the second-order stochastic dominance is more risk averse than the first-order stochastic dominance.

Depending on the consideration of the best or the mean values of objectives in the meaningful quantiles, the dominance relation used in DRSA for inferring the decision rules from the preference information is either the first- or the second-order stochastic dominance. In consequence, the decision rules involving the second-order stochastic dominance are more risk averse than the decision rules involving the first-order stochastic dominance. A further consequence of using these rules in the evolutionary procedure is that in the first case the procedure converges to a less risky region of solutions than in the second case.

Remark, finally, that DRSA decision rules do not convert ordinal information into numeric one, which implies that: (*i*) from the point of view of multiobjective optimization, no scalarization is involved, and (*ii*) from the point of view of decision under uncertainty, no specific model, such as expected utility, Choquet integral, Max-min expected utility, cumulative prospect theory, etc., has been imposed, and only a very general principle of the first- or second-order stochastic dominance in the space of meaningful quantiles is considered.

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g-BDI: A Graded Intensional Agent Model for Practical Reasoning

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Abstract. In intentional agents, actions are derived from the mental attitudes and their relationships. In particular, preferences (positive desires) and restrictions (negative desires) are important proactive attitudes which guide agents to intentions and eventually to actions. In this paper we overview recent developments about a multi-context based agent architecture g-BDI to represent and reasoning about gradual notions of desires and intentions, including sound and complete logical formalizations. We also show that the framework is expressive enough to describe how desires, together with other information, can lead agents to intentions and finally to actions. As a case-study, we will also describe the design and implementation of recommender system on tourism as well as the results of some experiments concerning the flexibility and performance of the g-BDI model.

1 Introduction

In the recent past, an increasing number of theories and architectures have been proposed to provide multiagent systems a formal support for their reasoning and decision making models, among them the so-called BDI architectures [16,9,15]. We consider that making the BDI architecture more flexible will allow for designing and developing intensional agents potentially capable to have a better performance in uncertain and dynamic environments. Along this research line we have developed a general model for *graded BDI agents* (g-BDI agents for short), specifying an architecture able to deal with the environment uncertainty (via graded beliefs) and with graded mental proactive attitudes (via desires and intentions). In the g-BDI model, belief degrees represent the extent to which the agent believes formulas hold true. Degrees of positive or negative desires allow the agent to set different levels of preference or rejection respectively. Intention degrees also give a preference measure but, in this case, modelling the

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cost/benefit trade off of achieving an agent's goal. Then, agents having different kinds of behaviour can be modelled on the basis of the representation and interaction of their graded beliefs, desires and intentions.

The formalization of the g-BDI agent model is based on multi-context systems (MCS) [10], and in order to represent and reason about the beliefs, desires and intentions, we followed a many-valued modal approach, following the approach in [11,12,13], where uncertainty reasoning is dealt with by defining suitable modal-like extensions of suitable many-valued logics. The logical framework of this model has been presented in [4,6] and it will be summarized in Section 2, while in Section 3 we present a small example of how the model works. Finally, in Section 4 we describe a prototype of a tourism recommender system which has been developed as a proof concept, where the g-BDI model has been used to design a Travel Assistant agent, which recommends tourist packages and destinations according to the user's preferences and constraints. The implementation details system have been described in [5,8] and experimentation and validation of the system is reported in [7]. We end up with some conclusions n Section 5.

2 Graded BDI Agent Model

The specification of the g-BDI agent model is based on multi-context systems (MCS) and is an extension of the work of Parsons et al. [15] about multi-context BDI agents. Multi-context systems were introduced by Giunchiglia and Serafini [10] to allow different formal (logical) components to be defined and interrelated. The MCS specification contains two basic components: units (or contexts) and bridge rules, which channel the propagation of consequences among unit theories. Thus, a MCS is defined as a group of interconnected units $\langle \{C_i\}_{i\in I}, \Delta_{br} \rangle$. Each context C_i is specified by a 3-tuple $C_i = \langle L_i, A_i, \Delta_i \rangle$ where L_i, A_i and Δ_i are its language, axioms, and inference rules respectively. Δ_{br} can be understood as rules of inference with premises and conclusions in different contexts, for instance a bridge rule like

$$\frac{C_1 : \psi, C_2 : \varphi}{C_3 : \theta}$$

specifies that if formula ψ is deduced in context C_1 and formula φ is deduced in context C_2 then formula θ is added to context C_3. When a theory $T_i \subseteq L_i$ is associated with each unit, the specification of a particular MCS is complete.

The deduction mechanism of a multi-context system $\langle \{C_i\}_{i\in I}, \Delta_{br} \rangle$ is therefore based on two kinds of inference rules, internal rules Δ_i, and bridge rules Δ_{br}, which allow to embed formulae into a context whenever the conditions of the bridge rule are satisfied.

In the basic specification of the g-BDI agent model as a MCS we have two kinds of contexts: three *mental* contexts, to represent beliefs (BC), desires (DC) and intentions (IC), as well as two *functional* contexts, for planning (PC) and communication (CC). The overall behavior of the system will depend of the logical representation of each intentional notion in their corresponding contexts and the particular set of bridge rules Δ_{br} used. Thus, a g-BDI agent model will

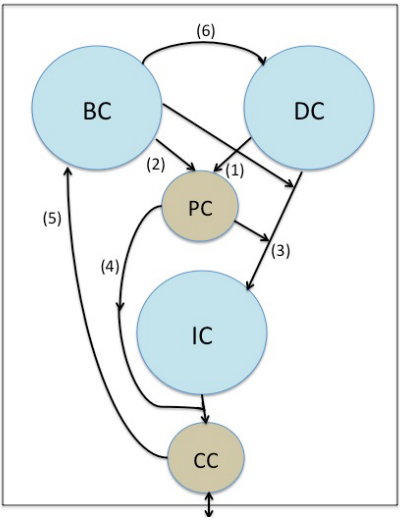

Fig. 1. Multi-context model of a graded BDI agent

be defined as a MCS of the form $A_g = (\{BC, DC, IC, PC, CC\}, \Delta_{br})$. Figure 1 illustrates such a g-BDI agent model with the different five contexts and six bridge rules relating them.

Next, we synthesize the purpose and formalization of each component (i.e. contexts and bridge rules) in the agent model. For full details the reader is referred to [3].

2.1 Belief Context (BC)

The aim of this context is to model the agent's uncertain beliefs about the environment. Since the agent needs to reason about her possible actions and the environment transformations they cause and their associated cost, this knowledge must be part of any situated agent's belief set. To represent knowledge related to action execution, we use Dynamic Propositional logic (PDL) as the base propositional logic (PDL has been proposed to model agent's actions e.g. in [14].) To account for the uncertainty or belief on the result of actions, either a probability-based approach or possibilistic-based approach (based on necessity degrees) can be adopted in the Belief Context BC. To do so, a many-valued modal-like logic (BC_{prob} or BC_{nec} respectively) is defined over a propositional dynamic language $\mathcal{L}_{PDL}$ to reason about the probability or necessity on dynamic logic formulas.

For instance, let us consider a Belief context BC_{prob} where belief degrees are to be modeled as probabilities. Then, for each classical formula φ, we consider a modal formula $B\varphi$ which is interpreted as "φ is probable". This modal formula $B\varphi$ is then a *fuzzy* formula which may be more or less true, depending on the

probability of φ. In particular, we can take as truth-value of $B\varphi$ the probability degree of φ. This is the guiding idea exploited in the probabilistic logic BC_{prob}, which is formally defined as a modal-like extension of Rational Pavelka logic (RPL) [12], an expansion of $[0,1]$-valued Łukasiewicz logic with a truth-constant $\overline{r}$ for every rational $r \in [0,1]$, following the approach proposed in [13,12]. We use this logic to reason about the $B\varphi$'s formulas since the probability axioms are easily expressible using Łukasiewicz logic connectives.

The modal language (B-formulas) of the logic BC_{prob} is built from propositional variables of the form $B\varphi$ for each $\varphi \in \mathcal{L}_{PDL}$. Compound formulae are defined in the usual way in the Rational Pavelka logic (RPL) using the Łukasiewicz connectives $\rightarrow_L$ and $\neg_L$, and truth-constants $\overline{r}$, for each rational $r \in [0,1]$ (note that nesting of the operator B is not allowed). For instance, if an agent has formula $\overline{0.6} \rightarrow_L B[\alpha]\varphi$ in its BC context, it means that he believes that the probability of having a goal φ true after perfoming action α is at least 0.6.

The semantics for this language is given by probabilistic Kripke structures of the following form: $M_{BC} = \langle W, \{R_\alpha : \alpha \in \Pi\}, e, \mu\rangle$ where $\langle W, \{R_\alpha : \alpha \in \Pi\}, e\rangle$ is regular Kripke model of PDL and $\mu : F \rightarrow [0,1]$ is a probabilistic measure on a Boolean algebra $F \subseteq 2^W$ such that for each crisp φ, the set $\{w \mid e(\varphi, w) = 1\}$ is μ-measurable. The e evaluation is extended as usual to PDL-formulae and it is extended to B-modal formulas by means of the folowing probabilistic interpretation of atomic belief formulas,

- $e(B\varphi, w) = \mu(\{w' \in W \mid e(\varphi, w') = 1\})$

and by means of Łukasiewicz logic truth-functions for compound modal formulas.

The axioms and rules for BC_{prob} are built in layers according to the nature of the language $\mathcal{L}_{BC}$ and the particular uncertainty model chosen, here probability. Namely, the set of axioms consists of: (i) axioms of propositional Dynamic logic for PDL-formulas; (ii) axiom of RPL for B-formulas, and (iii) the following probabilistic axioms for B-formulas:

(BC1) $B(\varphi \rightarrow \psi) \rightarrow_L (B\varphi \rightarrow_L B\psi)$
(BC2) $B(\varphi \vee \psi) \leftrightarrow_L B\varphi \oplus (B\psi \ominus B(\varphi \wedge \psi))$
(BC3) $\neg_L B(\bot)$
(BC4) $B\varphi$, for each theorem φ of PDL

where $\Phi \oplus \Psi$ is a shorthand for $\neg_L \Phi \rightarrow_L \Psi$ and $\Psi \ominus \Phi$ is a shorthand for $\neg_L(\Phi \rightarrow_L \Psi)$.[1] Deduction rules for BC are Modus Ponens (both for $\rightarrow$ of PDL and for $\rightarrow_L$ of RPL) and Necessitation for the modality B.

In [6] it is proved that the logic BC_{prob} is sound and Pavelka-style complete with respect to the above probabilistic semantics.

2.2 Desire Context (DC)

Desires represent the *ideal* agent's preferences regardless of the agent's current perception of the environment and regardless of the cost involved in actually

[1] Note that in Łukasiewicz logic $(x \Rightarrow_L 0) \Rightarrow_L y = \min(1, x + y)$ and $(x \Rightarrow_L y) \Rightarrow_L 0 = max(0, x - y)$.

achieving them. Positive desires represent what the agent would like to be the case. Negative desires correspond to what the agent rejects or does not want to occur. In this setting, one can also express indifference in a natural way just by expressing that has neither a positive nor a negative preference over an object. Furthermore, positive and negative desires can be graded to represent different levels of preference or rejection, respectively.

In the g-BDI agent model, following the approach on bipolarity representation of preferences in [1,2], we model in the DC context positive and negative information in the framework of possibilistic logic. In a similarly way as we do in the BC context, to represent and reason about the agent bipolar preferences in the DC context a modal many-valued approach is used to deal with the (positive and negative) desire degrees and a corresponding layered structure of axioms is set. As for combining one kind of desires (positive or negative) usually the conjunction of independent positive (resp. negative) preferences should produce a higher positive (resp. negative) preference. The degree of a disjunction of positive (resp. negative) preferences is computed as the minimum of the preference degrees, following the intuition that if the disjunction is satisfied at least the minimum of the satisfaction (rejection) levels is guaranteed. This corresponds to the use of the so-called *guaranteed possibility measures* to model the strength of the preferences [1]. In this way, a basic logic framework for the Desire context (DC schema) to capture these combination properties for positive and negative desires is independently defined.

The language $\mathcal{L}_{DC}$ in the DC context is defined over a classical propositional language $\mathcal{L}$ (built from a countable set of propositional variables Var with connectives $\wedge$, $\rightarrow$ and $\neg$) expanded with two (fuzzy) modal-like operators D^+ and D^-. $D^+\varphi$ reads as "φ is positively desired" and its truth degree represents the agent's level of satisfaction would φ become true. $D^-\varphi$ reads as "φ is negatively desired" (or "φ is rejected") and its truth degree represents the agent's level of disgust on φ becoming true. Notice that, as in BC, we do not allow nesting of the D^+ and D^- operators. As in the BC_{prob} logic, we use Rational Pavelka logic as the fuzzy logic to reason about the $D^+\varphi$ and $D^-\varphi$'s formulas.

The intended DC models are Kripke structures $M = \langle W, e, \pi^+, \pi^- \rangle$ where W and e are defined as in the BC semantics and π^+ and π^- are preference distributions over worlds, which are used to give semantics to positive and negative desires:

- $\pi^+ : W \rightarrow [0,1]$ is a distribution of positive preferences over the possible worlds. In this context $\pi^+(w) < \pi^+(w')$ means that w' is more preferred than w.
- $\pi^- : W \rightarrow [0,1]$ is a distribution of negative preferences over the possible worlds: $\pi^-(w) < \pi^-(w')$ means that w' is more rejected than w.

The truth evaluation for non-modal formulae $e : \mathcal{L} \times W \rightarrow \{0,1\}$ is defined in the usual (classical) way, and it is extended to atomic modal formulae $D^-\varphi$ and $D^+\varphi$ by:

- $e(D^+\varphi, w) = \inf\{\pi^+(w') \mid e(\varphi, w') = 1\}$
- $e(D^-\varphi, w) = \inf\{\pi^-(w') \mid e(\varphi, w') = 1\}$

The basic set of axioms axioms and inference rules aim capturing these combination properties, considering positive or negative desires independently, are: axioms of classical logic for non-modal formulae, axioms of Rational Pavelka logic for modal formulas, the following preference handling axioms

(DC0$^+$) $D^+(\varphi \vee \psi) \equiv_L D^+\varphi \wedge_L D^+\psi$
(DC0$^-$) $D^-(\varphi \vee \psi) \equiv_L D^-\varphi \wedge_L D^-\psi$

and modus ponens for $\rightarrow$ and for $\rightarrow_L$, together with rules of introduction of D^+ and D^- for implications:

(ID$^+$) from $\varphi \rightarrow \psi$ derive $D^+\psi \rightarrow_L D^+\varphi$
(ID$^-$) from $\varphi \rightarrow \psi$ derive $D^-\psi \rightarrow_L D^-\varphi$.

Soundness and completeness results have been also proved for this basic logic for graded, independent positive and negative desires. It is also possible to extend this framework to deal with different forms of interaction between positive and negative desires by adding some suitable axiom schemes. In [6], we have considered three additional schemes:

(DC1$^+$) $D^+\varphi \wedge_L D^+(\neg\varphi) \rightarrow_L \bar{0}$
(DC1$^-$) $D^-\varphi \wedge_L D^-(\neg\varphi) \rightarrow_L \bar{0}$
(DC2) $(D^+\varphi \otimes D^-\varphi) \rightarrow_L \bar{0}$
(DC3) $(D^+\varphi \wedge_L D^-\varphi) \rightarrow_L \bar{0}$

Intuitively, axioms (DC1$^+$) and (DC1$^-$) capture the constraint that an agent cannot have simultaneously a positive (negative) degree for a goal φ and for its contrary $\neg\varphi$. On the other hand, axiom (DC2) captures a constraint stipulating that the positive and negative degree for a same goal cannot sum more than 1, while axiom (DC3) is stronger and forbids having a positive and a negative desire for a same goal.

2.3 Intention Context (IC)

This unit is used to represent the agent's intentions. Together with the desires, they represent the agent's preferences. However, we consider that intentions cannot depend just on the benefit of reaching a goal φ, but also on the world's state and the cost of transforming it into one where the formula φ is true. By allowing degrees in intentions we represent a measure of the cost/benefit relation involved in the agent's actions towards the goal.

We represent in this context two kinds of graded intentions, intention of a formula φ considering the execution of a particularly plan α, noted $I_\alpha\varphi$, and the final intention to φ, noted $I\varphi$, which takes into account the best path to reach φ. As in the other contexts, if the degree of $I\varphi$ is δ, it may be considered that the truth degree of the expression "φ is intended" is δ. The intention to make φ true must be the consequence of finding a feasible plan α, that permits to achieve a state of the world where φ holds. Indeed, a suitable bridge rule (described in Subsection 2.5 as bridge rule (3)) infers these degrees of intention towards a goal φ for each plan α that allows to achieve the goal.

The agent intentions will be represented in the IC context by a theory $\mathcal{T}_{\mathcal{I}}$ over Rational Pavelka logic RPL. The language used is built in a similar way as done in the BC and DC contexts. We start from a classical propositional language $\mathcal{L}$ with a finite set of actions or plans Π^0 at the agent disposal to achieve her desires. Then, for each $\alpha \in \Pi^0$ we introduce a modal operator I_α such that the truth-degree of a formula $I_\alpha\varphi$ will represent the strength the agent intends φ by means of the execution of the particular action α.[2] We also introduce another modal operator I with the idea that $I\varphi$ will represent that the agent intends φ by means of the best plan in Π^0. These atomic modal formulas are then combined using Łukasiewicz connectives and rational truth-constants. Then, for instance, if the agent IC theory $\mathcal{T}_{\mathcal{I}}$ contains the formula $I_\alpha\varphi \to_L I_\beta\varphi$ then the agent will try φ by executing the plan β before than executing plan α.

Models for IC are Kripke structures $M = \langle W, e, \{\pi_\alpha\}_{\alpha\in\Pi^0}\rangle$ where W is a set of worlds and $\pi_\alpha : W \times W \to [0,1]$ is the utility distribution corresponding to action α: $\pi_\alpha(w,w')$ is the utility of applying α to transform world w into world w'.[3] Then e is extended to Boolean formulae as usual and to atomic modal formulae by

- $e(w, I_\alpha\varphi) = \inf\{\pi_\alpha(w,w') \mid w' \in W, e(w',\varphi) = 1\}$
- $e(w, I\varphi) = \max\{e(w, I_\alpha\varphi) \mid \alpha \in \Pi^0\}$

and to compound modal formulae using the truth functions of Rational Łukasiewicz logic.

The set of axioms for the IC logic consists of: axioms of classical logic for the non-modal formulas, axioms of Rational Pavelka logic for the modal formulas and the following specific axioms for the I_α and I modalities:

(IC0) $I_\alpha(\varphi \vee \psi) \equiv_L I_\alpha\varphi \wedge_L I_\alpha\psi$
(IC1) $I\varphi \equiv_L \bigvee_{\alpha\in\Pi^0} I_\alpha\varphi$

The rules are modus ponens for $\to$ and for $\to_L$ and introduction of I_α for implications: from $\varphi \to \psi$ derive $I_\alpha\psi \to_L I_\alpha\varphi$ for each $\alpha \in \Pi$.

Again, suitable soundness and completeness results can be proven for such a logic.

2.4 Planner and Communication Contexts (CC and PC)

The Planner Context (PC) has to look for feasible plans, these plans are generated from actions that are believed to fulfill a given positive desire and avoiding negative desires as post-conditions. These feasible plans are computed within this unit using an appropriate planner that takes into account beliefs and desires injected by bridge rules from the BC and DC units respectively.

[2] The IC context is not concerned about the question of whether a given desire can be reached by the execution of a particular action, this is left for the Planner Context, see next subsection.

[3] Indeed, it can be seen as a kind of refinement of the R_α relations of the action dynamic logic semantics considered in the BC context.

The Communication unit (CC) makes it possible to encapsulate the agent's internal structure by having a unique and well-defined interface with the environment. The theory inside this context will take care of the sending and receiving of messages to and from other agents in the multiagent society where our graded BDI agent lives.

Due to their functional aspect, we will not go into further details of these two units.

2.5 Bridge Rules (BRs)

A collection of basic bridge rules is considered to establish the necessary interrelations between context theories. We describe them next:

1. There are bridge rules from DC to PC that, from the positive and negative desires (pro-active attitudes), generate predicate instances in the PC unit that are used by the planner program to build the feasible plans:

$$\frac{DC : (D^+\varphi, d)}{PC : \lceil (D^+\varphi, d) \rceil} \quad \text{and} \quad \frac{DC : (D^-\psi, n)}{PC : \lceil (D^-\psi, n) \rceil} \tag{1}$$

2. The agent knowledge about the world state and about actions that change the world, is introduced from the belief context into the Planner as first order formulas:

$$\frac{BC : B\varphi}{PC : \lceil B\varphi \rceil} \tag{2}$$

3. Regarding intentions, there is a bridge rule that infers the degree of $I_\alpha\varphi$ for each feasible plan α that allows to achieve φ. The intention degree is thought as a trade-off among the benefit of reaching a desire, the cost of the plan and the belief degree in the full achievement of φ after performing α. The following bridge rule computes this value from the degree of $D^+\varphi$ (d), the degree of belief $B[\alpha]\varphi$ (r), the cost of the plan α (c):

$$\frac{DC : (D^+\varphi, d), BC : (B[\alpha]\varphi, r), PC : fplan(\varphi, \alpha, P, A, c)}{IC : (I_\alpha\varphi, f(d, r, c))} \tag{3}$$

   Different functions f allow to model different agent behaviors. For instance, if we consider an *equilibrated agent*, where all the factors involved are equally taken into account, the function might be defined as the average among these factors. In other cases, a weighted average may be used where the different weights w_i are set according to the agent expected behavior:

$$f(d, r, c) = (w_d d + w_r r + w_c\,(1 - c))\,/\,(w_d + w_r + w_c)$$

   For example, for a *greedy agent*, w_c may be set greater than the other weights: w_d and w_r.
4. The information supplied by the above bridge rule to the IC unit allows this unit to derive, for each desire φ, a formula $(I\varphi, i)$ where i is the maximum degree of all the $(I_\alpha\varphi, i_\alpha)$ formulae, where α is a feasible plan for φ. The

plan α_b that allows to get the maximum intention degree i_{max} considering all the agent desires, will be set by the PC unit as the *best plan* (see the definitional axiom (IC1) for I in Subsection 2.3). Finally, we also need rules to establish the agent interaction with the environment, meaning that if the agent intends φ at degree i_{max}, the maximum degree of all the intentions, then the agent will choose to execute the plan α_b —*bestplan*— that will allow him to reach the most intended goal φ:

$$\frac{IC : (I_{\alpha_b}\varphi, i_{max}), PC : bestplan(\varphi, \alpha_b, P, A, c)}{CC : C(does(\alpha_b))} \tag{4}$$

5. Through the communication unit the agent perceives all the changes in the environment that are introduced by the following bridge rule in the belief context:

$$\frac{CC : \beta}{BC : B\beta} \tag{5}$$

6. *Bridge rules to generate desires in a dynamic way.* In the desire context DC different schemas to represent and reason about desires were presented but how desires are derived was not discussed. In a dynamic environment the agent desires will change, depending on her beliefs and also on the set of current desires. Notably, Rahwan and Amgoud in their argumentation-based approach to practical reasoning [17] provide an argumentation-based framework for generating consistent desires, among other tasks, see also [18]. The basic elements of this argumentation framework are the desire-generation rules. We introduce in our g-BDI model a multi-context and many-valued version of these rules. As the desire and belief formulae in the premise are coming from different contexts, we define the following bridge rules for desire generation:

$$\frac{BC : (B\varphi_1 \wedge ... \wedge B\varphi_n, b), DC : (D^+\psi_1 \wedge ... \wedge D^+\psi_m, c)}{DC : (D^+\psi, d)} \tag{6}$$

Namely, if the agent has the beliefs $B\varphi_1, ..., B\varphi_n$ in degree greater or equal then a threshold b and positively desires $D^+\psi_1, ..., D^+\psi_m$ in degree at least c, she also desires ψ in degree at least d.

With the description of this set of bridge rules (BR) we have finished a short description of all components of the g-BDI agent model.

3 A Small Example

Here we present a simple example as to show how the proposed agent model works.

Peter, who lives in Rosario, wants to planify his activities for the next week. He activates a personal assistant agent based on our *g-BDI model* to find an adequate travel plan (transportation + accommodation). He would be very happy

attending to a conference on his speciality scheduled to take place in Buenos Aires (φ_1) and he would be rather happy visiting a friend living near this city (φ_2). But he would indeed prefer much more to be able to do both things. Besides, he doesn't like to travel during the night (ψ). This assistant has Peter's positive and negative desires represented by the following formulae in the theory T_{DC} of the *agent*'s Desire context:

$$T_{DC} = \{(D^+\varphi_1, 0.8), (D^+\varphi_2, 0.6), (D^+(\varphi_1 \wedge \varphi_2), 0.9), (D^-\psi, 0.7)\}$$

This means that the agent has rather high positive desires on achieving φ_1 and φ_2 but he even has a higher desire to achieve both (0.9). At the same time, the agent he rather rejects ψ, represents by a rather negative desire on ψ (0.7).

The agent also has knowledge about the conference schedule, his friend's agenda and transportation information, that is represented in the theory T_{BC} of the Belief context BC. Moreover, from this information and the set of positive and negative desires in T_{DC}, the planner context (PC) looks for *feasible* travel plans that are believed to satisfy φ_1 and/or φ_2 by their execution, but avoiding ψ as post-condition. Assume both α and β are found as feasible plans, whose normalized costs are $c_\alpha = 0.6$ and $c_\beta = 0.5$ respectively.

On the other hand, assume the Belief context (BC) is able to estimate the following beliefs (modelled as probabilities) about the achievement of the different goals by the feasible plans α and β, represented by the following formulae in the theory T_{BC}:

$$\begin{aligned} T_{BC} \supseteq \{&(B[\alpha]\varphi_1, 0.7),\ (B[\alpha]\varphi_2, 0.6),\ (B[\alpha](\varphi_1 \wedge \varphi_2), 0.42),\\ &B[\beta]\varphi_1, 0.5),\ (B[\beta]\varphi_2, 0.6),\ (B[\beta](\varphi_1 \wedge \varphi_2), 0.3)\} \end{aligned}$$

Then, using Bridge rule (3) and choosing the function f as

$$f(d, r, c) = r \cdot (1 - c + d)/2,$$

which computes an expected utility (taking the value $(1-c+d)/2$ as the global utility of reaching a goal with desire degree d and cost c, and 0 otherwise), the agent computes the different intention degrees towards the goals by considering the different feasible plans (i.e. α or β). In this example, the intention degrees for the goal with the highest desire degree, i.e. $\varphi_1 \wedge \varphi_2$, are:

$$(I_\alpha(\varphi_1 \wedge \varphi_2), 0.273) \text{ and } (I_\beta(\varphi_1 \wedge \varphi_2), 0.210)$$

From these results, the *assistant agent* choses to recommend Peter the plan α that would allow him to attend the conference and to visit his friend ($\varphi_1 \wedge \varphi_2$).

4 A Case Study: An Application of a Tourism Recommender System

In this section,as a matter of application of the previously introduced main components of the g-BDI agent model, we succinctly describe the general architecture

of a Tourism Recommender system that has been developed (see [8,5] for more details). The goal of the system is to recommend the best tourist packages on Argentinian destinations according to the user's preferences and restrictions. The packages are provided by different tourist operators. This system has been designed using a multiagent architecture and we particularly use the g-BDI model to specify one of its agents, the Travel Assistant Agent (T-Agent). The purpose of this prototype implementation was to show that the g-BDI agent model is useful to develop concrete agents on a real domain.

Inspired in the different components of a tourism chain, in the analysis phase we have identified the following roles: the Provider role (tourist package providers), the Travel Assistant role and Service roles (hotel chains, airlines, etc.). However, in this case study we don't deal with the service roles, we only mention them as necessary collaborators of the Provider role. Other functional roles have been identified as well, like for instance the Interface role, to manage the user interface, and the repository Maintenance role (R-Maintenance), to update and code into the system format the packages sent by the provider roles. In this simplified version of Recommender System, we define two agent's types: the Provider agent and the Travel Assistant Agent. We assign the interface role, the repository maintenance role and the travel assistant role to the Travel Assistant Agent (*T-Agent*).

The internal architecture of the Provider agents (*P-Agent*) is not considered in our implementation and for our purposes they are considered only as tourist packages suppliers. The multi-agent architecture of the prototype version of the tourism recommender system, composed by a *T-Agent* and two *P-Agents*, together with the main source of information they interact with (the destination ontology and the package repository) is illustrated in Figure 2. This multiagent system is easily scalable to include other providers.

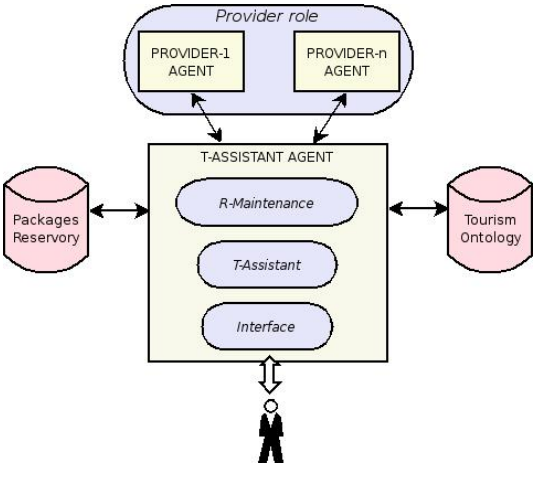

Fig. 2. Multiagent architecture of the Tourism Recommender System

The implementation of the Recommender system was developed using SWI-Prolog, a multi-threaded version of prolog which is a suitable language both to deal with logical deduction and allowing an independent execution of the different contexts (i.e. in different threads). Moreover, each *P-Agent* runs in a different thread, so in this way being independent from each other and from the *T-Agent*. When the *T-Agent* requests for information, the *P-Agents* send to *T-Agent* all the current packages they can offer. The communication between agents is by message exchange.

Next, we briefly describe how the contexts have been implemented in order to obtain the desired behaviour of the *T-agent* (for a detailed description see [8]).

Communication Context (CC): The CC is the agent's interface and is in charge of interacting with the tourism operators (*P-Agents*) and with the tourist that is looking for recommendation. The *T-Agent*, before beginning its recommendation task, updates its information about current packages (carrying out its reservory maintenance role). It behaves as a wrapper translating the incoming packages into the *T-Agent* format and sends them to the Planner context. The user's interface has been developed as a Web service application and it is responsible for:

- *Acquiring user's preferences:* they are explicitly obtained from the user by filling in a form. The tourist can set her preferences (positive desires) and restrictions (negative desires) and assign them a natural number from 1 to 10 to represent the level of preference (resp. restriction) for the selected item. Preferences are given about the following issues: geographic zone, natural resources, infrastructure, accommodation, transport or activities. The constraints are related to the maximum cost she is able to afford, the days available for traveling and the maximum total distance she is willing to travel. Once the user finishes his selection, the CC sends all the acquired information to the Desire context DC.

- *Showing the resulting recommendation:* as a result of the *T-Agent* deliberation process, the CC receives from the Intention context a ranking of feasible packages that satisfies some or all of the tourist preferences. Then, he can visualize the information about them (i.e. the description of the transport, destination, accommodation, activities) opening suitable files.

- *Receiving Tourist's feedback:* After analyzing the ranking of the recommended packages, the user can express through the CC interface her opinion about the recommendation. Namely, the user can select one of the following three possible evaluations:

1. *Correct*: When the user is satisfied with the ranking obtained.
2. *Different order*: When the recommended packages are fine for the user, but they are ranked in a different order than the user's own order. In such a case, the user is able to introduce the three best packages in the right order.
3. *Incorrect*: The user is not satisfied with the given recommendation. Then, the interface enables him to introduce a (textual) comment with his opinion.

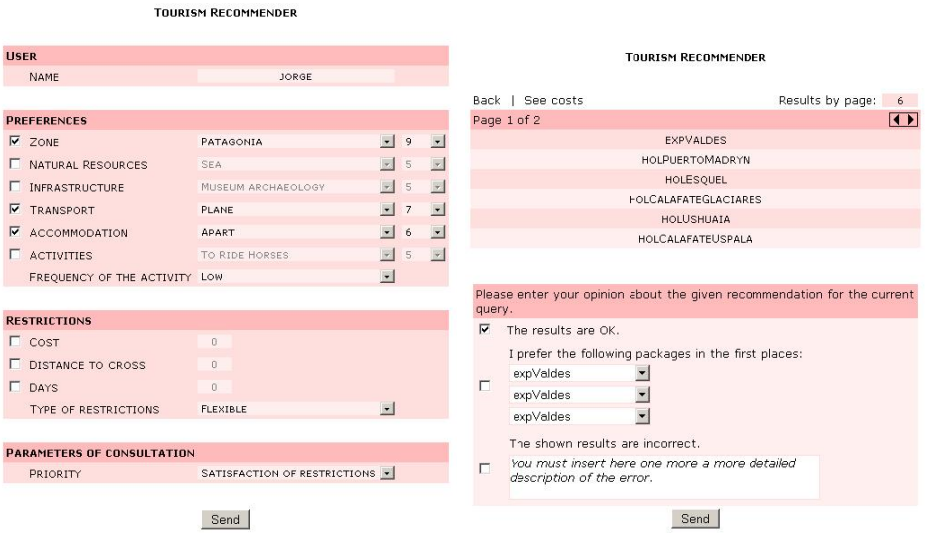

Fig. 3. Two screenshots of the user interface. Left: acquisition of user's preferences. Right: package recommendation and user feedback.

All the information resulting from the user data entry is stored to evaluate the system behaviour.

An example of a tourist's preferences specification and the system recommendation using this interface is shown in Figure 3.

Desire Context (DC): As the *T-Agent* is a *personal agent*, its overall desire is to maximize the satisfaction of the tourist's preferences. Thus, in this context the different tourist's graded preferences and restrictions are respectively represented as positive and negative desires. For instance, the preferences of a tourist that would like to go to a mountain place and to travel by plane but not more than 2000 kms could be represented by the following theory:

$$\mathcal{T}_{DC} = \{(D^{+}resources\_mountain, 0.9), (D^{+}transport\_air, 0.7),\\ (D^{+}(resources\_mountain \wedge transport\_air), 0.92), (D^{-}(distance \geq 2000), 0.5)\}$$

The *T-Agent* uses the desires as pro-active elements, and are passed by a bridge rule to the Planner context that looks for feasible packages.

Belief Context (BC): In this context the *T-Agent* represents all the necessary knowledge about tourism and the Argentinian domain: tourist packages (each package is represented as a list containing an identifier, a tour provider, the package cost and a travel-stay sequence), information about destinations (represented by a destination ontology) and rules to infer how much preferences can be satisfied (to some degree) by the feasible tourist packages. This context also contains knowledge about *similarity relations* between concepts to extend the possibility

of satisfying a tourist with similar preferences than the actually selected ones. Besides, the BC is in charge of estimating the extent (the belief) $B([\alpha_P]\varphi)$ to which a desire (preference) φ will be achieved when selecting a given package α_P.

Planner Context (PC): The Planner Conext (PC) is fundamental for the *T-Agent* implementation. The PC unit is assumed to contain a set of available plans, coded as instances of the predicate *planner* with *paq* formulae (see below). The Planner context is responsible for looking among them for *feasible packages*. By *feasible package* we mean a package that fulfills, to some degree, one of the positive desires (elementary or combined) and avoids, as post-condition, the satisfaction of the agent's negative desires above to a given threshold. The set of feasible plans is determined within this context using an appropriate searching method that takes into account information injected by bridge rules from the BC and DC units, including positive and negative desires, information about packages (including their cost), the agent's beliefs about package destinations and the estimation of the agent's desires fulfillment by the different plan executions.

After the PC has identified the set of feasible packages, they are passed to the Intention context, which is in charge of ranking of these packages according to the user's preferences.

Intention Context (IC): In order to rank the feasible packages to be offered to the user, the Intention context IC of the *T-Agent* is in charge of estimating the intention degree for each feasible package as a trade off between the benefit (expected satisfaction) and the cost of reaching the user's desires through that package. Thus, first, this context estimates the expected satisfaction $E(D, \alpha)$ of a tourist's desire D assuming she selects a package α. Second, using a suitable bridge rule, it computes the intention degree (the truth degree of the formula $I_\alpha D$) towards the desire D by executing a tourist package α using a function that combines the expected satisfaction $E(D, \alpha)$ and the normalized package cost CN. In the following Subsections we give some insights of how this estimations are implemented in the *T-Agent*.

A first experimentation of this prototype has been carried out with promising results (see [7] for a preliminary report). Considering 52 queries, 73% of the user's opinions were satisfactory (namely 40.4% with *correct order* and 32.7% with *different order* as user feedbacks). Furthermore, we have performed some experimentations using this recommender agent with the aim of proving different properties of the g-BDI model of agents. On the one hand, we have performed a sensitivity analysis to show how the g-BDI agent model can be tuned to have different behaviors by modifying some of its component elements. On the other hand, we have also done some experiments in order to compare the performance of recommender agents using the g-BDI model with agents without graded attitudes.

5 Conclusions

In this paper we have overviewed the main characteristics of a general graded BDI agent model. In this model, the agent graded attitudes have an explicit and

suitable representation. Belief degrees represent the extent to which the agent believes a formula to be true. Degrees of positive or negative desires allow the agent to set different levels of preference or rejection respectively. Intention degrees also give a preference measure but, in this case, modelling the cost/benefit trade off of achieving an agent's goal. Then, agents having different kinds of behaviour can be modelled on the basis of the representation and interaction of their graded beliefs, desires and intentions. In this respect, the role of preference representation is fundamental in this agent model as they are the agent proactive attitudes which lead agent to intentions and then, to actions.

As proof of concept, a prototype of multiagent Tourism Recommender system has been developed, where the g-BDI architecture has been used for modelling the T-Agent, showing in this way that the model is useful to develop concrete agents in real domains. We remark that the graded model of information representation and reasoning in the g-BDI agent has several advantages for this implementation. For instance, this model enables an expressive representation of the domain knowledge (agent beliefs), the user's preferences (desires) and the resulting intentions. Also, it makes it possible to compute in a graded way the expected satisfaction of the different tourist's preferences by the execution of several packages, so providing rankings of recommendations. Indeed, some validation and experimentation results reported in [7] show that (i) g-BDI agents are useful to build recommender systems in a real domains such as tourism, (ii) they provide satisfactory results, and (iii) the distinctive feature of recommender systems modelled using g-BDI agents, which is using graded mental attitudes, allows them to provide better results than those obtained by non-graded BDI models.

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Modeling Ambiguity Averse Behavior of Individual Decision Making: Prospect Theory under Uncertainty

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Abstract. Firstly, a behavioral model based on the "Prospect Theory" developed by Kahneman and Tversky is described. In this model weighting function of non-additive probabilities are introduced where probability of each event occurring is known. The effective application of this approach to the public sector is shown in modeling risks of extreme events with low probability and high outcome. Next, a behavioral model based on our "Prospect Theory under Uncertainty" is described where basic probability of a set of events is known but occurrence probability of each event is not known. It is shown that this model could properly explain the Ellsberg paradox of ambiguity aversion. Potential applicability of this approach to assessing global environmental-economic policies is described.

Keywords: Individual decision making; Behavioral (descriptive) model; Utility theory; Expected utility paradox; Prospect theory under uncertainty.

1 Introduction

Since when Prof. Daniel Kahneman received the Nobel Memorial Prize in Economics for his work in “Prospect Theory” in 2002, the area of behavioral (descriptive) decision theory, behavioral economics, or economic psychology [1] has been highly thought of in many respects.

A normative (prescriptive) model of decision making prescribes optimal behavior of how decisions should be made. It is concerned with identifying the best decision to be made, assuming an ideal decision maker who is fully rational. On the other hand, since people do not typically behave in optimal ways, a descriptive model is concerned with understanding how people actually behave when making decisions.

The expected utility model has been widely used as a normative model of decision analysis under risk for modeling individual decision making. However, various paradoxes [2,3] have been reported for the expected utility model, and it is argued that the expected utility model is not an adequate behavioral (descriptive) model. As a model to explain the violations of the expected utility hypothesis for individual decision making, outcome-dependent, non-additive probabilities are introduced in a measurable value function under risk where the probability of each event occurring is postulated to be known. The effective application of this approach to the public sector is mentioned in

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modeling risks of extreme events with low probability and high outcome. Prospect theory (PT) [4] under risk is embedded in the measurable value function under risk, where "under risk" means that the probability of each event occurring is known.

Prospect theory under risk is extended to the prospect theory under uncertainty (PTU) [5] where the basic probability of a set of event is known but the probability of each event occurring is not known. It is shown that the Ellsberg paradox [3] is consistently resolved by using this model. This is a paradox in decision theory and experimental economics in which people's choices violate the expected utility hypothesis because of their tendency for ambiguity aversion. Potential applicability to evaluating global environmental-economic policies is mentioned.

2 Expected Utility Model

Utility functions could provide a means of modeling value judgment of a decision maker quantitatively. The scientific approach for value judgment has been discussed rigorously in the area of economics. People get a feeling of psychological satisfaction by consuming economic goods, by receiving service and so forth. This degree of satisfaction obtained is called "utility." This concept plays a fundamental role in the theory of consumers' behavior.

Let x_1 and x_2 be the amount of goods A and B consumed, respectively, $u(x_1, x_2)$ be the corresponding value of the consumer's utility function, p_1 and p_2 be the price of a unit amount of goods A and B, respectively, and b be the budget. Then, the consumer may want to

$$\text{maximize } u(x_1, x_2) \tag{1}$$

$$\text{subject to } p_1 x_1 + p_2 x_2 \le b. \tag{2}$$

That is, the consumer behavior has been explained in such a way that they would act to maximize their own utility, Eq. (1), subject to the budget constraint, Eq. (2).

In order to derive the equilibrium condition of a consumer's behavior, an ordinal utility function would be enough to evaluate it, but, for obtaining a preferred solution for a multiple criteria decision-making problem, we need a cardinal utility function. Furthermore, since, in a decision making problem under risk, the outcome would be obtained under some probability distribution, we need to evaluate the so-called expected utility [6]. For this we need to provide a cardinal utility function. Von Neumann and Morgenstern [7] first developed axioms such that the expected utility hypotheses for the decision making problem under risk are meaningful.

Let X be a set of outcomes, $u : X \to \mathrm{Re}$ be a cardinal utility function, then the expected utility with respect to the probability on X is

$$E(u, p) = \sum_{x \in X} p(x) u(x) \tag{3}$$

which is called the expected utility. Let $P = \{p_1, p_2, \ldots\}$ be the set of probabilities on X. Then the following theorem provides the existence and uniqueness of a cardinal utility function [7].

Theorem: Existence and uniqueness of a cardinal utility function
Let P be the set of all probabilities on X, $(P,\succeq)$ be preference structure on P, then for any $p,q\in P$, the necessary and sufficient condition of the existence of a cardinal utility function $u: X \to \mathrm{Re}$ such that

$$p \succeq q \;\Leftrightarrow\; E(u,p) \geq E(u,q), \quad \forall p,q \in P \tag{4}$$

is given as follows:

NM1: $(P,\succeq)$ is weak order.
NM2: $p \succ q \;\Rightarrow\; \alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r, \quad \forall r \in P, \quad \alpha \in (0,1)$
NM3: $p \succ q \succ r \;\Rightarrow\; \alpha p + (1-\alpha) r \succ q \succ \beta p + (1-\beta) r$, for some $\alpha, \beta \in (0,1)$

Furthermore, such u is unique within the positive linear transformation (there exist h and $k>0$ such that $u' = h + ku$) and is called a von Neumann-Morgenstern utility function.

The expected utility model based on the expected utility hypothesis is useful as a normative model, that is, to find a decision to be made. However, since various paradoxes [2,3] have been reported for the expected utility model, it is argued that the expected utility model is not an adequate behavioral (descriptive) model. Actually, there exist many phenomena that violate the expected utility hypothesis such as the Allais paradox [2] and the Ellsberg paradox [3]. In the following sections we describe a generalized model of a measurable value function under risk [8], prospect theory under risk [4] and prospect theory under uncertainty [5] to overcome the difficulty of the expected utility paradoxes.

3 Behavioral Models to Resolve Expected Utility Paradoxes: Measurable Value Function under Risk

The expected utility model has been widely used as a normative model of decision analysis under risk. However, various paradoxes have been reported for the expected utility model, and it is argued that the expected utility model is not an adequate descriptive model.

In this section a descriptive extension of the expected utility model to account for various paradoxes is shown using the concept of strength of preference [8]. Let *X* be a set of all outcomes, $x \in X$, and *A* be a set of all risky alternatives; a prospect (risky alternative) $\ell \in A$ is written as

$$\ell = (x_1, x_2, \ldots, x_n; p_1, p_2, \ldots, p_n) \tag{5}$$

that yields outcome $x_i \in X$ with probability $p_i, i = 1,2,\ldots,n$, where $\sum p_i = 1$.

Let *A** be a nonempty subset of $A \times A$, and $\succeq$ and $\succeq^*$ be binary relations on *A* and *A**, respectively. Relation $\succeq$ could also be a binary relation on *X*. We interpret

$\ell_1 \succeq \ell_2 \,(\ell_1, \ell_2 \in A)$ to mean that ℓ_1 is preferred than or indifferent to ℓ_2, and $\ell_1 \ell_2 \succeq^* \ell_3 \ell_4 \,(\ell_1, \ell_2, \ell_3, \ell_4 \in A)$ to mean that the strength of preference for ℓ_1 over ℓ_2 is greater than or equal to the strength of preference for ℓ_3 over ℓ_4.

We postulate that $(A, A^*, \succeq^*)$ takes a positive difference structure that is based on the axioms described by Kranz et al. [9]. The axioms imply that there exists a real-valued function F on A such that for all $\ell_1, \ell_2, \ell_3, \ell_4 \in A$, if $\ell_1 \succeq \ell_2$ and $\ell_3 \succeq \ell_4$, then

$$\ell_1 \ell_2 \succeq^* \ell_3 \ell_4 \;\Leftrightarrow\; F(\ell_1) - F(\ell_2) \geq F(\ell_3) - F(\ell_4) \tag{6}$$

Since F is unique up to a positive linear transformation, it is a cardinal function. It is natural to hold for $\ell_1, \ell_2, \ell_3 \in A$ that

$$\ell_1 \ell_3 \succeq^* \ell_2 \ell_3 \;\Leftrightarrow\; \ell_1 \succeq \ell_2 . \tag{7}$$

Then from Eq. (6) we obtain

$$\ell_1 \succeq \ell_2 \;\Leftrightarrow\; F(\ell_1) \geq F(\ell_2). \tag{8}$$

Thus, F is a value function on A and, in view of Eq. (6), it is a measurable value function.

We assume that the decision maker will try to maximize the value (or utility) of a prospect (risky alternative) $\ell \in A$, which is given by the general form as follows:

$$\max_{\ell \in A} F(\ell) = \max_{\ell \in A} \sum_i f(x_i, p_i) \tag{9}$$

where $f(x, p)$ denotes the value (strength of preference) for an outcome x which comes out with probability p. This function is called the *measurable value function under risk*. The main objectives here are to give an appropriate decomposition and interpretation of $f(x, p)$ and to explore its descriptive implications to account for the various paradoxes.

The model of Eq. (9) is reduced to the expected utility form [7] by setting

$$f(x, p) = pu(x) \tag{10}$$

when $u(x)$ is regarded as a von Neumann-Morgenstern utility function. The prospect theory of Kahneman and Tversky [4] is obtained by setting

$$f(x, p) = \pi(p) v(x) \tag{11}$$

where $\pi(p)$ denotes a weighting function for probability and $v(x)$ a value function for outcome. In this model the value of each outcome is multiplied by a decision weight for probability (not by probability itself).

Extending this Kahneman-Tversky model we obtain a decomposition form

$$f(x, p) = w(p \mid x)\, v(x) \tag{12}$$

where

$$w(p \mid x) \equiv \frac{f(x, p)}{f(x,1)} \tag{13a}$$

$$v(x) \equiv v(x \mid 1) \tag{13b}$$

$$v(x \mid p) \equiv \frac{f(x, p)}{f(x^*, p)} \tag{13c}$$

and x^* denotes the best outcome. The expected utility model, Eq. (10), and Kahneman-Tversky model, Eq. (11), are included in our model, Eq. (12), as a special case. Eq. (13b) implies that $v(x)$ denotes a measurable value function under certainty. Therefore, our model, Eq. (12), also includes Dyer and Sarin's model [10] as a special case.

The model of Eq. (12) could also be written as

$$f(x, p) = w(p)\, v(x \mid p) \tag{14}$$

where

$$w(p) \equiv w(p \mid x^*). \tag{15}$$

We assume that

$$f(x,0) = 0, \quad \forall x \in X \tag{16a}$$

$$f(x^R, p) = 0, \quad \forall p \in [0,1] \tag{16b}$$

where $x^R \in X$ denotes the reference point (e.g. status quo). The better region on X compared with x^R is called the gain domain and the worse region the loss domain. We also assume that $f(x, p) \geq 0$ in the gain domain and $f(x, p) < 0$ in the loss domain.

It will be shown that the conditional weighting function $w(p \mid x)$, that is an outcome-dependent, non-additive probability, describes the strength of preference for probability under the given conditional level of outcome, and $v(x \mid p)$ describes the strength of preference for outcome under the given conditional level of probability.

In interpreting the descriptive model $f(x, p)$ we need to interpret F such that Eq. (6) holds. For all $x_1, x_2, x_3, x_4 \in X$, $\alpha \in [0,1]$ and $y \in X$ such that $x_1 \succeq x_2 \succeq x_3 \succeq x_4$, we consider four alternatives:

$$\ell_1 = (x_1, y; \alpha, 1-\alpha), \quad \ell_2 = (x_2, y; \alpha, 1-\alpha), \tag{17a}$$

$$\ell_3 = (x_3, y; \alpha, 1-\alpha), \quad \ell_4 = (x_4, y; \alpha, 1-\alpha). \tag{17b}$$

In this case we obtain

$$\ell_1\ell_2 \succeq^* \ell_3\ell_4 \quad \Leftrightarrow \quad f(x_1,\alpha)-f(x_2,\alpha)\geq f(x_3,\alpha)-f(x_4,\alpha) \tag{18a}$$

$$\Leftrightarrow \quad v(x_1\,|\,\alpha)-v(x_2\,|\,\alpha)\geq v(x_3\,|\,\alpha)-v(x_4\,|\,\alpha). \tag{18b}$$

Therefore, the value function $v(x\,|\,p)$ defined by Eq. (13c) represents the strength of preference for the four risky alternatives in Eq. (17).

On the other hand, for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in[0,1]$, $x\in X$ and $x^R\in X$, we consider four alternatives:

$$\ell_1'=(x,x^R;\alpha_1,1-\alpha_1),\quad \ell_2'=(x,x^R;\alpha_2,1-\alpha_2), \tag{19a}$$

$$\ell_3'=(x,x^R;\alpha_3,1-\alpha_3),\quad \ell_4'=(x,x^R;\alpha_4,1-\alpha_4), \tag{19b}$$

then we obtain

$$\ell_1'\ell_2' \succeq^* \ell_3'\ell_4' \quad \Leftrightarrow \quad f(x,\alpha_1)-f(x,\alpha_2)\geq f(x,\alpha_3)-f(x,\alpha_4) \tag{20a}$$

$$\Leftrightarrow \quad w(\alpha_1\,|\,x)-w(\alpha_2\,|\,x)\geq w(\alpha_3,x)-w(\alpha_4,x). \tag{20b}$$

Therefore, the weighting function defined by Eq. (13a) represents the strength of preference for the four risky alternatives in Eq. (19).

The above discussion asserts that the descriptive model $f(x,p)$ represents the measurable value function under risk to evaluate the outcome $x\in X$ that comes out with probability p. The Kahneman-Tversky model of Eq. (11) could explain a so-called certainty effect to resolve the Allais paradox. Our descriptive model $f(x,p)$ could also resolve the Allais paradox.

It is well known that the expected utility model is not an appropriate model for modeling extreme events with low probability and high outcome. In Tamura, *et al.* [11] it is shown that our descriptive model could resolve such paradox in application to the public sector.

4 Prospect Theory under Risk

Kahneman and Tversky proposed prospect theory [4] in order to explain people's decision making such that

(a) People's value judgment is highly dependent on the *reference point*, that is, people are more focused on changes in their value (utility) states than the states themselves.
(b) People's marginal value (utility) is diminishing both in gain domain and in loss domain.

(c) Value function in loss domain is steeper than in gain domain, that is, losses looms larger than gains and people have a tendency of *loss aversion*.
(d) People feel that weight for very small probability is disproportionate and they have a tendency to overestimate for low probability and underestimate for higher probability, that is, subjective probability is severely biased by anchoring.

We denote the prospect that yields an outcome x_j with probability $p_j, j = 1,2,\ldots,n$ by Eq. (5). In prospect theory (PT), the value V for the prospect (5) is evaluated using the evaluation function

$$V = \sum_{j=1}^{n} \pi(p_j) v(x_j) \tag{21}$$

where the value function v is convex with a gentle curve in the gain domain, while it is concave and its curve is steeper in the loss domain, as shown in Figure 1. This shows that people, in general, are loss averse.

The weighting function π is a convex function as shown in Figure 2, so a small probability is weighted higher and middle or large probabilities are weighted lower. However, this weighting function is not defined near the end points 0 and 1. The dotted line in Figure 2 shows the case for the expected utility (EU) model.

In Tamura, *et al.* [12] the value for the sense of security provided by nursing care robots is evaluated by using various utility theoretic approaches. As the result of comparison it is found that prospect theory under risk is more appropriate for this problem.

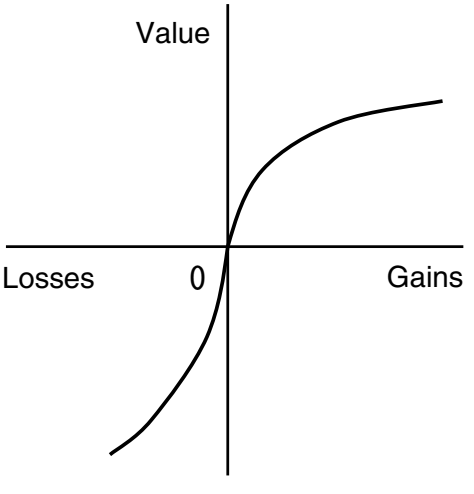

Fig. 1. Value function

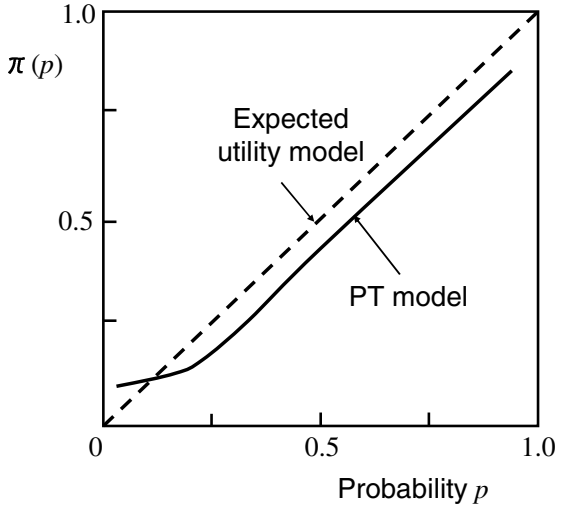

Fig. 2. Weighting function used in PT

5 Prospect Theory under Uncertainty

5.1 Basic Principle

In this section we deal with the case where the probability of occurrence for each event is unknown. When we describe the degree of ignorance and uncertainty by the basic probability of Dempster and Shafer theory [12], the problem is how to represent the value of a set element in constructing a measurable value function under uncertainty based on this concept.

In the Dempster-Shafer theory of probability let $\mu(A_i)$ be basic probability which could be assigned by any subset A_i of Θ, where Θ denotes a set containing every possible element. The basic probability $\mu(A_i)$ can be regarded as a semimobile probability mass. Let $\Lambda = 2^{\Theta}$ be a set containing every subset of Θ. Then the basic probability $\mu(A_i)$ is defined on Λ and takes a value contained in [0,1]. When $\mu(A_i)>0$, A_i is called the focal element or the set element and the following conditions hold:

$$\mu(\phi)=0\,, \qquad \sum_{A_i \in A} \mu(A_i)=1 \tag{22}$$

where ϕ denotes an empty set.

Let the value function under uncertainty based on this basic probability be

$$f^*(B,\mu)=\pi'(\mu)\,v^*(B) \tag{23}$$

where B denotes a set element, μ denotes the basic probability, π' denotes the weighting function for the basic probability, and v^* denotes the value function with respect to a set element. The set element B is a subset of $\Lambda = 2^{\Theta}$. Eq. (23) is an extended version of the value function, Eq. (14), where an element is extended to a set element and the Bayes' probability is extended to the Dempster-Shafer basic probability.

For identifying v^*, we need to find the preference relations among set elements, which is not an easy task. If the number of elements contained in the set Θ is getting larger, and the set element B contains a considerable number of elements, it is not practical to find v^* as a function of B. To cope with this difficulty we could use some appropriate axiom of dominance as follows:.

Axiom of Dominance 1:

In the set element B let the worst outcome be m_B and the best outcome be M_B. For any $B1, B2 \subset \Lambda = 2^{\Theta}$

$$m_{B1} \preceq m_{B2},\ M_{B1} \preceq M_{B2} \ \Rightarrow\ B1 \preceq B2 \tag{24}$$

and

$$m_{B1} \sim m_{B2},\ M_{B1} \sim M_{B2} \ \Rightarrow\ B1 \sim B2. \tag{25}$$

Our descriptive model $f^*(B, \mu)$ could resolve the Ellsberg paradox by restricting a set element B to

$$\Omega = \{(m, M) \in \Theta \times \Theta : m \preceq M\} \tag{26}$$

where m and M denote the worst and the best outcome in the set element B, respectively. In this case Eq. (22) is reduced to

$$f^*(\Omega, \mu) = \pi'(\mu) v^*(\Omega). \tag{27}$$

Suppose we look at an index of pessimism $\alpha(m, M)$ such that the following two alternatives are indifferent [14]. (The index of optimism $\beta(m, M) = 1 - \alpha(m, M)$ may be defined instead of the index of pessimism depending upon the situation.)

Alternative 1: One can receive m for the worst case and M for the best case. There exists no other information.

Alternative 2: One receives m with probability $\alpha(m, M)$ and receives M with probability $1 - \alpha(m, M)$, where $0 < \alpha(m, M) < 1$.

If one is quite pessimistic, $\alpha(m, M)$ becomes nearly equal to 1, or if one is quite optimistic, $\alpha(m, M)$ becomes nearly equal to zero. If we incorporate this pessimism index $\alpha(m, M)$ in Eq. (26), the value function is described as

$$v^*(\Omega) = v^*((m, M)) = \alpha(m, M)\, v'(m) + (1 - \alpha(m, M))\, v'(M) \tag{28}$$

where v' denotes a value function for a single element.

Incorporating Dempster-Shafer probability theory in the descriptive model $f^*(\Omega, \mu)$ of a value function under uncertainty, we could model the lack of belief which cannot be modeled by Bayes' probability theory. As the result our descriptive model $f^*(\Omega, \mu)$ could resolve the Ellsberg paradox as follows.

5.2 Resolving Ellsberg Paradox

Suppose an urn contains 30 balls coloured red, black or white. We know that 10 of 30 balls are red, but for the other 20 balls we know only that each of these balls is either black or white. Suppose we pick a ball from this urn, and consider four events as follows:

a. We will get 100 dollars if we pick a red ball.
b. We will get 100 dollars if we pick a black ball.
c. We will get 100 dollars if we pick a red or white ball.
d. We will get 100 dollars if we pick a black or white ball.

Many people show the preference (Ellsberg, 1961),

$$a \succ b, \quad d \succ c. \tag{29}$$

The probability of picking up a red ball is 1/3. Let p_b and p_w be the probability of picking up a black ball and a white ball, respectively. Then

$$p_b + p_w = \frac{2}{3}. \tag{30}$$

The expected utility theory says that

$$a \succ b \Rightarrow \frac{1}{3} u(1M) > p_b u(1M) \Rightarrow p_b < \frac{1}{3} \tag{31}$$

$$d \succ c \Rightarrow \frac{2}{3} u(1M) > \frac{1}{3} u(1M) + p_w u(1M) \tag{32a}$$

$$\Rightarrow p_w < \frac{1}{3} \Rightarrow p_b > \frac{1}{3} \tag{32b}$$

where u denotes a von Neumann-Morgenstern utility function and $1M = 100$ dollars. Eqs. (31) and (32b) are obviously contradictory. This phenomenon is called the Ellsberg paradox. Therefore, the expected utility theory cannot represent the preference

when the probability of each event is not known but only the basic probability for a set of events is known. This phenomenon shows that one prefers the events with known probability and is called the sure-thing principle [9].

How can we explain the preference of this Ellsberg paradox by using the descriptive model $f^*(\Omega,\mu)$ of a prospect theory under uncertainty? Let $\{R\}$ be the event of picking a red ball and $\{B,W\}$ be the set element of picking a black or white ball. Then the basic probability is written as

$$\mu(\{R\})=\frac{1}{3},\quad \mu(\{B,W\})=\frac{2}{3} \tag{33}$$

In this case a set Θ containing every possible event is written as

$$\Theta=\{0,1M\}. \tag{34}$$

Table 1 shows the basic probability of getting each event for each alternative. The value for each alternative is given by

$$V(a)=\pi'\left(\frac{2}{3}\right)v'(\{0\})+\pi'\left(\frac{1}{3}\right)v'(\{1M\}) \tag{35a}$$

$$V(b)=\pi'\left(\frac{1}{3}\right)v'(\{0\})+\pi'\left(\frac{2}{3}\right)v'(\{0,1M\}) \tag{35b}$$

$$V(c)=\pi'\left(\frac{1}{3}\right)v'(\{1M\})+\pi'\left(\frac{2}{3}\right)v'(\{0,1M\}) \tag{35c}$$

$$V(d)=\pi'\left(\frac{1}{3}\right)v'(\{0\})+\pi'\left(\frac{2}{3}\right)v'(\{1M\}). \tag{35d}$$

Table 1. Basic probability for each event

	Event		
Alternative	**{0}**	**{1*M*}**	**{0,1*M*}**
a	**2/3**	**1/3**	**0**
b	**1/3**	**0**	**2/3**
c	**0**	**1/3**	**2/3**
d	**1/3**	**2/3**	**0**

In the set Θ let x^0 and x^* be the worst outcome and the best outcome, respectively, then

$$x^0 = 0, \quad x^* = 1M \ . \tag{36}$$

Therefore, we obtain

$$v'(\{0\}) = 0, \quad v'(\{1M\}) = 1, \quad v'(\{0,1M\}) = \alpha(0,1M) \tag{37}$$

where $\alpha = \alpha(0,1M)$ denotes an index of pessimism. Then,

$$a \succ b \ \Rightarrow \ V(a) > V(b) \tag{38a}$$

$$\Rightarrow \pi'\left(\frac{1}{3}\right) > \pi'\left(\frac{2}{3}\right) v'(\{0,1M\}) \tag{38b}$$

$$\Rightarrow \ \pi'\left(\frac{1}{3}\right) > (1-\alpha)\pi'\left(\frac{2}{3}\right) \tag{38c}$$

$$d \succ c \ \Rightarrow \ V(d) > V(c) \tag{39a}$$

$$\Rightarrow \pi'\left(\frac{2}{3}\right) > \pi'\left(\frac{1}{3}\right) + w'\left(\frac{2}{3}\right) v'(\{0,1M\}) \tag{39b}$$

$$\Rightarrow \ \pi'\left(\frac{1}{3}\right) < \alpha\, \pi'\left(\frac{2}{3}\right). \tag{39c}$$

To hold these preference relation we need to have $\alpha = \alpha(0,1M)$ such that

$$\frac{\pi'\left(\frac{1}{3}\right)}{\pi'\left(\frac{2}{3}\right)} < \alpha, \quad \frac{\pi'\left(\frac{2}{3}\right) - \pi'\left(\frac{1}{3}\right)}{\pi'\left(\frac{2}{3}\right)} < \alpha \, . \tag{40}$$

If $\alpha = \alpha(0,1M) > 0.5$, Eq. (40) holds. This situation shows that, in general, one is pessimistic about events with unknown probability. The Ellsberg paradox is resolved by the descriptive model $f^*(\Omega, \mu)$ based on the prospect theory under uncertainty.

5.3 Further Axiom of Dominance

There exist some cases for which Axiom of Dominance 1 is unsuitable. Then we introduce Axiom of Dominance 2 [15] which is more strict than Axiom of Dominance 1 as follows:

Axiom of Dominance 2:

Let the worst outcome be m_1 and the best outcome be M_1 in the set element B_1, and let the worst outcome be m_2 and the best outcome be M_2 in the set element B_2. Moreover, let the hypothetical elements whose values are equal to the average values of B_1, B_2 be g_1, g_2, respectively. Then

$$m_1 \prec m_2,\ M_1 \prec M_2,\ g_1 \prec g_2 \Rightarrow B_1 \prec B_2 \tag{41}$$

$$m_1 \sim m_2,\ M_1 \sim M_2,\ g_1 \sim g_2 \Rightarrow B_1 \sim B_2 \tag{42}$$

where

$$v(g_1) = \frac{\sum_{i=1}^{n_1} v(a_i)}{n_1}, \quad v(g_2) = \frac{\sum_{i=1}^{n_2} v(b_i)}{n_2}, \tag{43}$$

n_1 denotes the number of elements in the set element B_1 and n_2 denotes the number of elements in the set element B_2.

Axiom of Dominance 2 is too strict to use practically, so we try to relax it. Someone attaches importance to the best outcome and chooses an alternative, someone attaches importance to the worst outcome, and someone pays attention to the whole. We introduce the model that properly describes this situation.

Definition:

Let the elements in the set element B be $a_1, a_2, \ldots, a_n$ such that $a_i \prec a_{i+1}$, $i = 1,2,\ldots,n-1$, the value of element a_i, $i = 1,2,\ldots,n$ be $v(a_i)$ and the average value of elements be

$$v(g) = \sum_{i=1}^{n} v(a_i) \tag{44}$$

Further, let the pessimism index decided by the question for the worst element $a_1 = m$ and the best element $a_n = M$ be $\alpha(m,M)$. We assume the value h of the set element B to be

$$h(B \mid \alpha) = a + be^{-c\alpha(m,M)} \quad \text{if } v(g) \neq \frac{v(M)+v(m)}{2} \tag{45}$$

and

$$h(B \mid \alpha) = a + b\alpha(m,M) \quad \text{if } v(g) = \frac{v(M)+v(m)}{2} \tag{46}$$

where unknown parameters a, b, c are decided by

$$h(B \mid 0) = v(M), \quad h(B \mid 0.5) = v(g), \quad h(B \mid 1) = v(m).$$

We introduce Axiom of Dominance 3 in order to evaluate values based on the above definition as follows:

Axiom of Dominance 3:

$$h(B_1 \mid \alpha) < h(B_2 \mid \alpha) \Rightarrow B_1 \prec B_2 \tag{47}$$

and

$$h(B_1 \mid \alpha) = h(B_2 \mid \alpha) \Rightarrow B_1 \sim B_2 \tag{48}$$

By using Axiom of Dominance 3, we are able to write the value function in the prospect theory under uncertainty as

$$f^*(h(B \mid \alpha), \mu) = \pi'(\mu) h(B \mid \alpha). \tag{49}$$

We could properly describe the value judgment of pessimistic people and optimistic people, respectively, by using Eq. (49).

We are able to evaluate the value V of the prospect that includes the case where the probability of occurrence for each element is unknown but the basic probability of occurrence for each set element is known through the evaluation function

$$V = \sum_{j=1}^{n} \pi(\mu_j) v^*(B_j) \tag{50}$$

where π denotes the weighting function of prospect theory and v^* denotes the value function with respect to a set element B_j.

6 Assessing Global Environmental-Economic Policies

We could show quantitatively how the hybrid policy of carbon tax and emissions trading would be effective to achieve the targeted reduction of the Kyoto Protocol [16] where the Kyoto Protocol is an agreement made under the United Nations Framework Convention on Climate Change [17]. Countries that ratify this protocol would commit to reducing their emissions of CO_2 and five other greenhouse gases (GHG), or would engage in emissions trading if they maintain or increase emissions of these gases.

We evaluate the cost of reducing CO_2 emissions for three scenarios as follows using a model of prospect theory under uncertainty:

Scenario 1: Each sector reduces its emissions by taking into account the carbon tax only without the emissions trading.

Scenario 2: Each sector reduces its emissions by taking into account the emissions trading only without the carbon tax.

Scenario 3: Each sector reduces its emissions by taking into account both the carbon tax and the emission trading.

The cost of reducing CO_2 emissions in Scenario 1 is set to the reference cost. We evaluate the saving cost in Scenarios 2 and 3 compared to the reference cost in Scenario 1 where we postulate uncertainty in the cost of reducing unit CO_2 emissions. For each sector it is found that the undesirable influence on the profit would be reduced by joining the emissions trading, and it would become more profitable for all the sectors under Scenario 3.

7 Concluding Remarks

Behavioral models of individual decision making have been described. In the model, outcome-dependent, non-additive probabilities are introduced as a measurable value function under risk where probability of each event occurring is postulated to be known. The effective application of this approach to the public sector is shown in modeling risks of extreme events with low probability and high outcome. It is shown that the measurable value function under risk is an extended version of prospect theory under risk.

We extended the prospect theory (PT) of Kahneman-Tversky to the prospect theory under uncertainty (PTU) where the basic probability for a set of events is known but the probability of each event occurring is not. It is shown that the Ellsberg paradox is resolved by using PTU, that is, PTU could model properly the property of ambiguity averse behavior of individual decision making. In other words, people feel anxious if outcomes are not clear. It shows that publishing information is important to give people a peace of mind. The potential applicability to assessing the global environmental-economic policies is described.

In this article we have described value judgment of an individual, but the value judgment of society is yet to be developed, under further research.

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Generalized Bags, Bag Relations, and Applications to Data Analysis and Decision Making

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Abstract. Bags alias multisets have long been studied in computer science, but recently more attention is paid on bags. In this paper we consider generalized bags which include real-valued bags, fuzzy bags, and a region-valued bags. Basic definitions as well as their properties are established; advanced operations such as s-norms, t-norms, and their duality are also studied. Moreover bag relations are discussed which has max-plus and max-min algebras as special cases. The reason why generalized bags are useful in applications is described. As two applications, bag-based data analysis and decision making based on convex function optimization related to bags are discussed.

Keywords: Generalized bag, s-norm, bag relation, data analysis, decision making, convex function.

1 Introduction

Bags which are also called multisets have long been studied by computer scientists as a basic data structure [6,10]. More recently, Calude and others [2] showed various aspects of multiset processing including a new paradigm of computation.

Since Yager [32] have proposed fuzzy bag, its theory and applications have been studied by several researchers [7,8,26,27,28,29,33] in the field of soft computing.

The author has redefined and re-established basic operations for fuzzy bags [14,15,16,17,20], and considered further generalizations [19]. Moreover we have applied fuzzy bags to data clustering [18] and text data analysis [21].

In this paper we overview bags and their generalizations with basic operations and their fundamental properties. Advanced operations such as s-norms and bag relations with new compositions are considered. We also discuss applications of generalized bags to data analysis, where classical methods as well as more recent techniques using kernel functions [30] are considered. Moreover a decision making aspect based on optimization of convex functions derived from set operations is considered, which are inspired from toll sets [4,13].

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Although we show many propositions, we omit the proofs for the most part, as they are straightforward.

2 Bags and Generalized Bags

We begin with a review of classical bags and their generalizations.

2.1 Crisp Bags

Assume that the universal set $X = \{x_1, \ldots, x_n\}$ is finite for simplicity. A (crisp) bag M of $X = \{x_1, \ldots, x_n\}$ is characterized by a function $C_M(\cdot)$ (called count of M) whereby a natural number including zero corresponds to each $x \in X$ ($C_M \colon X \to \boldsymbol{N}$), where $\boldsymbol{N} = \{0, 1, 2, \ldots\}$. $C_M(\cdot)$ is called a *count function*.

We may express a crisp bag as

$$M = \{k_1/x_1, \ldots, k_n/x_n\}$$

or

$$M = \{\overbrace{x_1, \ldots, x_1}^{k_1}, \ldots, \overbrace{x_n, \ldots, x_n}^{k_n}\}.$$

In this way, an element of X may appear more than once in a bag.

Example 1. Consider an example in which $X = \{a, b, c, d\}$ and

$$C_M(a) = 2, \quad C_M(b) = 3, \quad C_M(c) = 1, \quad C_M(d) = 0.$$

In other words, $M = \{a, a, b, b, b, c\}$. This means that a, b, c, and d are included 2, 1, 3, and 0 times, respectively, in M. We can write $M = \{2/a, 3/b, 1/c\}$, ignoring an element of zero occurrence. Other expressions such as $M = \{3/b, 2/a, 1/c\}$ and $M = \{c, a, b, b, a, b\}$ are also used.

Basic relations and operations for crisp bags:

1. (inclusion): $M \subseteq N \Leftrightarrow C_M(x) \leq C_N(x), \quad \forall x \in X.$
2. (equality): $M = N \Leftrightarrow C_M(x) = C_N(x), \quad \forall x \in X.$
3. (union): $C_{M \cup N}(x) = \max\{C_M(x), C_N(x)\}.$
4. (intersection): $C_{M \cap N}(x) = \min\{C_M(x), C_N(x)\}.$
5. (addition or sum): $C_{M+N}(x) = C_M(x) + C_N(x).$
6. (scalar multiplication): $C_{\alpha M} = \alpha C_M(x)$, where α is a nonnegative integer.
7. (Cartesian product): Let P is a bag of Y. $C_{M \times P}(x, y) = C_M(x) C_P(y).$

We use $\vee$ and $\wedge$ for max and min, respectively. Note that the relations and operations are similar to those for fuzzy sets. However, bags have the addition operation that fuzzy sets do not have, and the Cartesian product for bags is different from that for fuzzy sets.

2.2 *R*-Bags, *F*-Bags, and *G*-Bags

We discuss three generalizations. The first generalization to real-valued bags is simple, but useful in applications, and second is fuzzy bags, while the third is a minimum extension including the former two. We call them R-bags, F-bags, and G-bags for simplicity.

***R*-Bags.** The first generalization is straightforward. We assume a count function can take an arbitrary positive real value. Moreover the value of infinity should be included into the range of a count function, as we show its usefulness later. Thus, $C_M : X \to [0, +\infty]$ (note $[0, +\infty] = [0, \infty) \cup \{+\infty\}$). Since count function takes real values, we say real-valued bags, or shortly R-bags. The above definitions of basic relations and operations 1–7 are unchanged.

***F*-Bags.** Fuzzy bags are abbreviated as F-bags here. They were first studied by Yager [32], and basic relations and operations have been reconsidered by the authors [14,15].

In a fuzzy bag an element of X may occur more than once with possibly the same or different membership values.

Example 2. Consider a fuzzy bag

$$A = \{(a, 0.2), (a, 0.3), (b, 1), (b, 0.5), (b, 0.5)\}$$

of $X = \{a, b, c, d\}$, which means that a with the membership 0.2, a with 0.3, b with the membership 0.5, and two b's with 0.5 are contained in A.

We may write

$$A = \{\{0.2, 0.3\}/a, \{1, 0.5, 0.5\}/b\}$$

in which the bag of membership $\{0.2, 0.3\}$ corresponds to a and $\{1, 0.5, 0.5\}$ corresponds to b. Thus, $C_A(x)$ is a bag of the unit interval [32].

For an $x \in X$, the membership sequence is defined to be the decreasingly ordered sequence of elements in $C_A(x)$. It is denoted by

$$\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x),$$

$(\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^p(x))$.

When we handle a finite number of fuzzy bags in a finite universal set, the length p of the membership sequences is set to be a constant for all members and for all the concerned fuzzy bags, by appending appropriate numbers of 0 at the end of the membership sequences.

Example 3. For the above example, we can set $p = 3$, $\mu_A^1(a) = 0.3$, $\mu_A^2(a) = 0.2$, $\mu_A^3(a) = 0$, $\mu_A^1(b) = 1$, $\mu_A^2(b) = \mu_A^3(b) = 0.5$, $\mu_A^1(c) = \mu_A^2(c) = \mu_A^3(c) = \mu_A^1(d) = \mu_A^2(d) = \mu_A^3(d) = 0$. By the representation of the membership sequence,

$$A = \{(0.3, 0.2)/a, (1, 0.5, 0.5)/b\},$$

or appending 0,

$$A = \{(0.3, 0.2, 0)/a, (1, 0.5, 0.5)/b, (0, 0, 0)/c, (0, 0, 0)/d\}.$$

The followings are basic relations and operations for fuzzy bags [14].

1. **inclusion:**
   $A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x),\ j = 1, \ldots, p,\ \ \forall x \in X.$
2. **equality:**
   $A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x),\ j = 1, \ldots, p,\ \ \forall x \in X.$
3. **sum:**
   $A + B$ is defined by the sum operation in $X \times [0, 1]$ for crisp bags [32].
4. **union:**
   $\mu_{A \cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1, \ldots, p,\ \forall x \in X.$
5. **intersection:**
   $\mu_{A \cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1, \ldots, p,\ \forall x \in X.$

***G*-Bags.** A further generalization of fuzzy bags is useful from theoretical viewpoint. It has been studied by the author [19] and is called G-bags here (this name is an abbreviation of generalized bags).

We introduce a G-bag using a closed region on a first quadrant $[0, +\infty]^2$ of a plane. The horizontal and vertical axes are called y-axis and z-axis, respectively. We define

$$C_A(x) = \nu_A(x) \tag{1}$$

where $\nu_A(x)$ is a closed region of $[0, +\infty]^2$ that satisfies the following conditions.

(I) For each $y \in [0, +\infty]$, the intersection between $\nu_A(x)$ and $\{y\} \times [0, +\infty]$ (the vertical line starting from y) is either empty or a segment starting from 0 and ending up to a point. We call this point $Z\nu_A(y; x)$. Thus,

$$\nu_A(x) \cap (\{y\} \times [0, +\infty]) = \{y\} \times [0, Z\nu_A(y; x)].$$

$Z\nu_A(y; x)$ as a function of y is monotonically non-increasing and

$$\lim_{y \to \infty} Z\nu_A(y; x) = 0.$$

(II) For each $z \in [0, +\infty]$, the intersection between $\nu_A(x)$ and $[0, +\infty] \times \{z\}$ (the horizontal line starting from z) is either empty or a segment starting from 0 and ending up to a point. We call this point $Y\nu_A(z; x)$. Thus,

$$\nu_A(x) \cap ([0, +\infty] \times \{z\}) = [0, Y\nu_A(z; x)] \times \{z\}.$$

$Y\nu_A(z; x)$ as a function of z is monotonically non-increasing and

$$\lim_{z \to \infty} Y\nu_A(z; x) = 0.$$

We illustrate an example of $\nu_A(x)$ in Figure 1. Note that when we are given either one of $Z\nu_A(y; x)$ or $Y\nu_A(z; x)$, $\nu_A(x)$ can uniquely be determined.

The basic relations and operations for two G-bags are defined as follows.

(I) (inclusion)

$$A \subseteq B \Leftrightarrow \nu_A(x) \subseteq \nu_B(x),\ \forall x \in X.$$

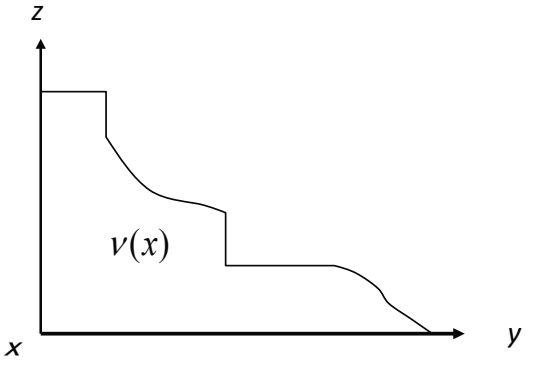

Fig. 1. Region $\nu(x)$ as a count function for G-bag

(II) (equality)

$$A = B \Leftrightarrow \nu_A(x) = \nu_B(x),\ \forall x \in X.$$

(III) (sum) Define

$$Y\nu_{A+B}(z;x) = Y\nu_A(z;x) + Y\nu_B(z;x)$$

and derive $\nu_{A+B}(x)$ from $Y\nu_{A+B}(z;x)$.

(IV) (union) Define

$$\nu_{A\cup B}(x) = \nu_A(x) \cup \nu_B(x).$$

(V) (intersection) Define

$$\nu_{A\cap B}(x) = \nu_A(x) \cap \nu_B(x).$$

(VI) (α-cut and ℓ-cut) Let a G-bag be A and $\alpha \in [0,1]$ and $\ell \in [0,+\infty)$ are given. An α-cut $[A]_\alpha$ is an R-bag with count function

$$C_{[A]_\alpha}(x) = Y\nu_A(\alpha;x).$$

On the other hand, a ℓ-cut $\langle A\rangle^\ell$ is a fuzzy set with membership

$$\mu_{\langle A\rangle^\ell}(x) = Z\nu_A(\ell;x)$$

A crisp bag, an R-bag, and a fuzzy bag can be regarded as a special case of G-bags by taking regions under the bars defined from count functions. If we have an R-bag with $C_M(x) = a$, then we define

$$\nu_A(x) = \{(y,z) : 0 \le y \le a,\ 0 \le z \le 1\} \tag{2}$$

and this R-bag is transformed into an equivalent G-bag. If we have a fuzzy bag, then we define

$$\nu_A(x) = \bigcup_{i=1}^{\infty} \{\ (y,z)\ :\ i-1 \le y \le i,\ 0 \le z \le \mu_A^i(x)\ \} \tag{3}$$

We have the next proposition.

Proposition 1. *Let A and B be arbitrary G-bags of X.*

$$[A+B]_\alpha = [A]_\alpha + [B]_\alpha, \tag{4}$$
$$[A \cup B]_\alpha = [A]_\alpha \cup [B]_\alpha, \tag{5}$$
$$[A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha, \tag{6}$$
$$\langle A \cup B \rangle^\ell = \langle A \rangle^\ell \cup \langle B \rangle^\ell, \tag{7}$$
$$\langle A \cap B \rangle^\ell = \langle A \rangle^\ell \cap \langle B \rangle^\ell. \tag{8}$$

The proof is straightforward and omitted.

Note 1. G-bags have a close relation to fuzzy interval-valued bags of which future studies are expected, but we omit the detail (see [19]).

2.3 Complement, *s*-Norm and *t*-Norm

This section is mainly concerned with R-bags. We state propositions without proofs. They are found in [22].

Complementation of *R*-Bags. A function $\mathcal{N}\colon [0,+\infty] \to [0,+\infty]$ with the next properties is used to define a complementation operation:

(i) $\mathcal{N}(0) = +\infty, \quad \mathcal{N}(+\infty) = 0.$
(ii) $\mathcal{N}(x)$ is strictly monotonically decreasing on $(0,+\infty)$.

A typical example is $\mathcal{N}(x) = const/x$ with $const > 0$.

An operation for the complement is then defined:

9.(complement):
$C_{\bar{M}}(x) = \mathcal{N}(C_M(x))$.

This operation justifies the generalization into R-bags, i.e., even when we start from crisp bags, the result of complementation is generally real-valued.

We immediately have the next two propositions; the proof is easy and omitted.

Proposition 2. *For arbitrary R-bags M, N, the next properties are valid:*

$$\overline{(\overline{M})} = M \tag{9}$$
$$\overline{M \cup N} = \bar{M} \cap \bar{N}, \qquad \overline{M \cap N} = \bar{M} \cup \bar{N}. \tag{10}$$

Proposition 3. *Let an empty bag $\emptyset$ and the maximum bag* **Infinity** *in R-bags be*

$$C_\emptyset(x) = 0, \quad \forall x \in X, \tag{11}$$
$$C_{\mathbf{Infinity}}(x) = +\infty, \quad \forall x \in X. \tag{12}$$

Then we have

$$\bar{\emptyset} = \mathbf{Infinity}, \quad \overline{\mathbf{Infinity}} = \emptyset. \tag{13}$$

***s*-Norms and *t*-Norms.** There have been studies on t-norms for crisp bags [11,3], but generalization into R-bags admits a broader class of s-norms and t-norms. For this purpose we introduce two functions $t(a,b)$ and $s(a,b)$ like those in fuzzy sets, but the definitions are different.

Definition 1. *Two functions* $t\colon [0,+\infty]\times[0,+\infty]\to[0,+\infty]$ *and* $s\colon [0,+\infty]\times[0,+\infty]\to[0,+\infty]$ *having the next properties (I)–(IV) are called a t-norm and an s-norm for R-bags, respectively. An s-norm is also called a t-conorm for bags.*

(I)[monotonicity] For $a\le c$, $b\le d$,

$$t(a,b)\le t(c,d),$$
$$s(a,b)\le s(c,d).$$

(II)[symmetry]

$$t(a,b)=t(c,d),\quad s(a,b)=s(b,a).$$

(III)[associativity]

$$t(t(a,b),c)=t(a,t(b,c)),$$
$$s(s(a,b),c)=s(a,s(b,c)).$$

(IV)[boundary condition]

$$t(0,0)=0,\quad t(a,+\infty)=t(+\infty,a)=a,$$
$$s(+\infty,+\infty)=+\infty,\quad s(a,0)=s(0,a)=a.$$

A purpose to introduce such norms for bags is to generalize the intersection and union operations. First we note that $s(a,b)=a+b$, $s(a,b)=\max\{a,b\}$, and $t(a,b)=\min\{a,b\}$ satisfy the above conditions (I)–(IV). Thus the s-norms and t-norm represent the operations of addition, union, and intersection.

We moreover introduce a generating function $g(x)$ for s-norm.

Definition 2. *A function* $g\colon [0,+\infty]\to[0,+\infty]$ *is called a generating function for s-norm if it satisfies the next (i)–(iii):*

(i) it is strictly monotonically increasing,
(ii) $g(0)=0,\quad g(+\infty)=+\infty$,
(iii) $g(x+y)\ge g(x)+g(y),\quad \forall x,y\in[0,+\infty]$.

We have the next proposition.

Proposition 4. *Let*

$$s(a,b)=g^{-1}(g(a)+g(b)). \tag{14}$$

Then $s(a,b)$ *is an s-norm.*

An example of the generation function is

$$g(x)=x^p\quad (p\ge 1). \tag{15}$$

Moreover, note the following.

Proposition 5. *Let $s(a,b)$ is an s-norm and $\mathcal{N}$ is a complementation operator. Then*

$$t(a,b) = \mathcal{N}(s(\mathcal{N}(a), \mathcal{N}(b))) \tag{16}$$

is a t-norm. Suppose $t(a,b)$ is a t-norm, then

$$s(a,b) = \mathcal{N}(t(\mathcal{N}(a), \mathcal{N}(b))) \tag{17}$$

is an s-norm.

If a pair of t-norm and s-norm has the above property stated in Proposition 5, we say (s,t) has the *duality* of norm and conorm. The duality has the next property.

Proposition 6. *Suppose $s_0(a,b)$ is an s-norm and $t_0(a,b)$ is derived from $s_0(a,b)$ by the operation (16). Let*

$$s(a,b) = \mathcal{N}(t_0(\mathcal{N}(a), \mathcal{N}(b)))$$

Then $s(a,b) = s_0(a,b)$. Suppose also that $t_0(a,b)$ is a t-norm and $s_0(a,b)$ is derived from $t_0(a,b)$ by the operation (16). Let

$$t(a,b) = \mathcal{N}(s_0(\mathcal{N}(a), \mathcal{N}(b)))$$

Then $t(a,b) = t_0(a,b)$.

We apply s-norm and t-norm to define bag operations $M\mathcal{S}N$ and $M\mathcal{T}N$, respectively.

$$C_{M\mathcal{S}N}(x) = s(C_M(x), C_N(x)). \tag{18}$$
$$C_{M\mathcal{T}N}(x) = t(C_M(x), C_N(x)). \tag{19}$$

Let us consider examples of s-norms and t-norms.

Example 4. The standard operators

$$s(a,b) = \max\{a,b\} \tag{20}$$
$$t(a,b) = \min\{a,b\} \tag{21}$$

are an s-norm and a t-norm, respectively. This pair has the duality stated in Propositions 5 and 6. Note, however, that s-norm (20) does not have a generating function that satisfies (14), while the next example uses the generating function.

Example 5. Let $g(x)$ be given by (15). Then we have

$$s(a,b) = (a^p + b^p)^{\frac{1}{p}}, \tag{22}$$
$$t(a,b) = (a^{-p} + b^{-p})^{-\frac{1}{p}}. \tag{23}$$

are an s-norm and a t-norm, respectively. This pair has the duality stated in Proposition 5 when $\mathcal{N} = const/x$ is used.

The second example has interesting properties. First, $s(a,b) = a + b$ is a particular case of (22) for $p = 1$. Moreover $s(a,b) = \max\{a,b\}$ and $t(a,b) = \min\{a,b\}$ are obtained from (22) and (23) when $p \to +\infty$.

Generalization to *G*-Bags. Apparently the complementation $\mathcal{N}$ cannot be generalized to G-bags. However, it is possible to define s-norms and t-norms as follows.

Definition 3. *Given two G-bags A, B of X, and an s-norm and t-norm, we define $Z\nu_{A\mathcal{S}B}(z;x)$ and $Z\nu_{A\mathcal{T}B}(z;x)$ are defined by*

$$Y\nu_{A\mathcal{S}B}(z;x) = s(Y\nu_A(z;x), Y\nu_B(z;x)), \tag{24}$$
$$Y\nu_{A\mathcal{T}B}(z;x) = t(Y\nu_A(z;x), Y\nu_B(z;x)), \tag{25}$$

Using $Y\nu_{A\mathcal{S}B}(z;x)$ and $Y\nu_{A\mathcal{T}B}(z;x)$, we generate $\nu_{A\mathcal{S}B}(x)$ and $\nu_{A\mathcal{T}B}(x)$.

The next proposition justifies the above definition.

Proposition 7.

$$[A\,\mathcal{S}\,B]_\alpha = [A]_\alpha\,\mathcal{S}\,[B]_\alpha, \tag{26}$$
$$[A\,\mathcal{T}\,B]_\alpha = [A]_\alpha\,\mathcal{T}\,[B]_\alpha \tag{27}$$

3 Bag Relations for Generalized Bags

A bag relation is a concept that corresponds to fuzzy relation. We define algebras for bag relations for R-bags, and then generalize them to G-bags. Proofs of the propositions in this section are shown in [22].

3.1 Max-*s* and Max-*t* Algebras

Let us introduce a new notation of $\boxplus$ and $\boxdot$ for

$$a \boxplus b = \max\{a, b\}, \quad a \boxdot b = s(a, b) \tag{28}$$

where $s(a, b)$ is an s-norm for R-bags. We call this *max-s algebra*.

It is easy to see that the following properties hold.

$$a \boxplus b = b \boxplus a, \tag{29}$$
$$a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c, \tag{30}$$
$$a \boxplus 0 = a, \tag{31}$$
$$a \boxdot b = b \boxdot a, \tag{32}$$
$$a \boxdot (b \boxdot c) = (a \boxdot b) \boxdot c, \tag{33}$$
$$a \boxdot 0 = a. \tag{34}$$

We moreover define $\boxplus$ and $\boxdot$ for

$$a \boxplus b = \max\{a, b\}, \quad a \boxdot b = t(a, b) \tag{35}$$

where $t(a, b)$ is a t-norm for R-bags. The latrter is called *max-t algebra*. We see that (29)–(33) hold, while (34) should be replaced by

$$a \boxdot +\infty = a. \tag{36}$$

We have the following.

Proposition 8. *Let a, b, c be real numbers. Then*

$$a \boxdot (b \boxplus c) = (a \boxdot b) \boxplus (a \boxdot c). \tag{37}$$

where $\boxdot$ is either an s-norm or a t-norm.

3.2 Bag Relations

Assume that all bags in this section are R-bags, unless otherwise stated.

Definition 4. *A bag relation R on $X \times Y$ is a bag R of $X \times Y$. The count function is denoted by $R(x, y)$ instead of $C_R(x, y)$ for simplicity.*

We define composition operation using max-s or max-t algebra.

Definition 5. *Let X, Y, Z be three universal sets. Assume R is a bag relation of $X \times Y$ and S is a bag relation of $Y \times Z$. Then a max-s composition $R \circ S$ is defined as follows.*

$$(R \circ S)(x, z) = \boxplus_{y \in Y} \{R(x, y) \boxdot S(y, z)\} \tag{38}$$

where $\boxdot$ is defined by an s-norm. Note that

$$\boxplus_{y \in \{a_1, \ldots, a_L\}} = a_1 \boxplus a_2 \boxplus \cdots \boxplus a_L.$$

A max-t composition is defined by the same equation (49) except that $\boxdot$ uses a t-norm.

Note also that the addition is straightforward

$$(R_1 \boxplus R_2)(x, y) = R_1(x, y) \boxplus R_2(x, y), \tag{39}$$

for bag relations on $X \times Y$.

We have the following.

Proposition 9. *The composition satisfies the associative property*

$$(R \circ S) \circ T = R \circ (S \circ T). \tag{40}$$

and the distributive property

$$(R_1 \boxplus R_2) \circ S = (R_1 \circ S) \boxplus (R_2 \circ S), \tag{41}$$

$$R \circ (S_1 \boxplus S_2) = (R \circ S_1) \boxplus (R \circ S_2). \tag{42}$$

In short, the composition is calculated like ordinary matrix calculations when the universes are finite.

We introduce the unit relations for the max-s and max-t compositions. For this purpose we define O_{XY} and Ω_{XY} on $X \times Y$.

$$O_{XY}(x, y) = 0, \quad \forall (x, y) \in X \times Y, \tag{43}$$

$$\Omega_{XY}(x, y) = +\infty, \quad \forall (x, y) \in X \times Y. \tag{44}$$

Frequently we omit the subscripts like O and Ω when we have no ambiguity. We then have

Proposition 10. *Assume that the max-s algebra is used. For arbitrary bag relation R on $X \times Y$,*

$$R \uplus O = O \uplus R = R, \tag{45}$$
$$R \circ O = O \circ R = R. \tag{46}$$

In contrast, assume that the max-t algebra is used. For arbitrary bag relation R on $X \times Y$,

$$R \uplus O = O \uplus R = R, \tag{47}$$
$$R \circ \Omega = \Omega \circ R = R. \tag{48}$$

Note 2. Max-s algebra is a generalization of max-plus algebra [5] and max-t algebra generalizes max-min algebra [12].

3.3 Relations of G-Bags

It is possible to generalize bag relations to G-bags. The idea is the same as that for s-norms of G-bags.

Definition 6. *Let X, Y, Z be three universal sets. Assume R is a G-bag relation of $X \times Y$ and S is a G-bag relation of $Y \times Z$. Then a max-s composition $R \circ S$ is defined as follows.*

$$Y\nu_{(R \circ S)}(w; x, z) = \uplus_{y \in Y} \{Y\nu_R(w; x, y) \boxdot Y\nu_S(w; y, z)\} \tag{49}$$

where $\boxdot$ is defined by an s-norm.

A max-t composition is defined by the same equation (49) except that $\boxdot$ uses a t-norm. Using $Y\nu_{(R \circ S)}(w; x, z)$, we generate $\nu_{(R \circ S)}(x, z)$.

This definition is justified by the next proposition.

Proposition 11. *Let R is a G-bag relation of $X \times Y$ and S is a G-bag relation of $Y \times Z$. Assume $R \circ S$ is either max-s or max-t composition. We then have*

$$[R \circ S]_\alpha = [R]_\alpha \circ [S]_\alpha. \tag{50}$$

4 Data Analysis Based on Bag Models

We briefly overview bag-based models for data analysis. A typical bag model is used in document analysis, where frequency of a term in a document is regarded as a bag.

A less-known but useful model is fuzzy bags (F-bags), that is, weighted terms with many occurrences. Since discussion of F-bags include that of classical bags, we focus on F-bags.

4.1 Distance between *F*-Bags

An important point in data analysis is the measurement of a distance between two F-bags. We consider two distances. For this purpose we introduce additional symbols.

(i) Cardinal number of F-bag:

$$|A| = \sum_{x \in X} \sum_{j} \mu_A^j(x). \tag{51}$$

(ii) Product:

$$\mu_{A \cdot B}^j(x) = \mu_A^j(x) \mu_B^j(x), \quad j = 1, 2, \ldots \tag{52}$$

We now define two distances:

$$d_1(A, B) = |A \cup B| - |A \cap B|, \tag{53}$$
$$d_2(A, B) = |A \cdot A| + |B \cdot B| - 2|A \cdot B|. \tag{54}$$

The next proposition is useful.

Proposition 12.

$$d_1(A, B) = \sum_{x \in X} \sum_{j} |\mu_A^j(x) - \mu_B^j(x)|, \tag{55}$$
$$d_2(A, B) = \sum_{x \in X} \sum_{j} |\mu_A^j(x) - \mu_B^j(x)|^2. \tag{56}$$

Proof. We have

$$\begin{aligned} d_1(A, B) &= |A \cup B| - |A \cap B| \\ &= \sum_{x \in X} \sum_{j} \max\{\mu_A^j(x), \mu_B^j(x)\} - \sum_{x \in X} \sum_{j} \min\{\mu_A^j(x), \mu_B^j(x)\} \\ &= \sum_{x \in X} \sum_{j} |\mu_A^j(x) - \mu_B^j(x)|. \end{aligned}$$

$$\begin{aligned} d_2(A, B) &= |A \cdot A| + |B \cdot B| - 2|A \cdot B| \\ &= \sum_{x \in X} \sum_{j} \{(\mu_A^j(x))^2 + (\mu_B^j(x))^2\} - 2 \sum_{x \in X} \sum_{j} \mu_A^j(x) \mu_B^j(x) \\ &= \sum_{x \in X} \sum_{j} |\mu_A^j(x) - \mu_B^j(x)|^2. \end{aligned}$$

□

Miyamoto [18] applied these measures to fuzzy c-means clustering of documents and terms when terms have weights. The weighted terms with many occurrences were interpreted as fuzzy bags. It should be noted that cluster centers for both $d_1(A, B)$ and $d_2(A, B)$ are well-defined fuzzy bags and their calculations are not difficult [18].

Kernel Functions. Recently, kernel functions have been remarked by many researchers (e.g. [31,30]). It is possible to apply kernel functions to the set of F-bags. An effect of kernel functions is that nonlinear classification boundaries are easily obtained. The best-known kernel is the Gaussian kernel:

$$K(x, y) = \exp(-\lambda \|x - y\|^2)$$

where $\|x - y\|$ is the Euclidean distance between two points of the Euclidean space. When we use a kernel function, we assume an implicit mapping $\phi(\cdot)$ from a data space into an implicit high-dimensional space. Note that the high-dimensional space and mapping ϕ need not be known, but their inner product $\langle \phi(x), \phi(y) \rangle$ is given by an explicit kernel function:

$$K(x, y) = \langle \phi(x), \phi(y) \rangle.$$

For F-bags, it is not difficult to see that

$$K(A, B) = \exp(-\lambda d_2(A, B)) \tag{57}$$

is a positive-definite kernel, and hence we can use this kernel to data analysis of F-bags. On the other hand, $K(A, B) = \exp(-\lambda d_1(A, B))$ does not necessarily define a positive-definite kernel. Mizutani *et al.* [24] applied the Gaussian kernel to a set of documents and performed kernel fuzzy c-means clustering [23]. The results showed the kernel function better separates clusters than the ordinary fuzzy c-means clustering.

There is another point that kernel functions are useful. The original set of fuzzy bags is not a vector space, but after the mapping, F-bags are represented as points in a high-dimensional space. It is true that the high-dimensional space itself is invisible, but the method of kernel principal components [30] projects the points onto a low-dimensional subspace. Using such a method, we can visualize F-bags as points on a plane when two principal axes are used.

5 Application to Decision Making Using *R*-Bags

A classical work by Bellman and Zadeh [1] showed how fuzzy set framework is used in decision making, where an objective and a constraint are represented by fuzzy sets and a point that maximizes the membership of their intersection should be an optimal solution.

When we contrast bags and fuzzy sets, we should study decision making using bags instead of fuzzy sets, and consider if we have an essential difference between the two approaches.

Example 6. Let us review a simple example in the framework of fuzzy decision making. For simplicity we handle an objective and a constraint, but generalization to many objectives and constraints are straightforward.

An objective is represented by a fuzzy set G of X, while a *soft* constraint C is also a fuzzy set of the same universe. A larger $G(x)$ (we write $G(x)$ instead

of $\mu_G(x)$) means that the objective is better satisfied. In the same way, a larger $C(x)$ means that the constraint is more satisfied. Hence we should consider maximization:

$$\max_{x \in X} (G \cap C)(x) \tag{58}$$

because both the objective and the constraint should be satisfied. That is, decision should be fuzzy set $D = G \cap C$.

If we use bags in the above formulation in just the same way, the result is the same as that by fuzzy sets. Only difference is that we do not have the ceiling of unity when handling bags.

We have, however, another formulation which is complementary to fuzzy decision making. It is more classical and yet employs a feature of fuzzy decision making. We describe an example using R-bags.

Example 7. Let G and C be R-bags of X, but they have different meanings:

- $G(x) = n$ means that n people are unsatisfied concerning the objective.
- $C(x) = m$ means that m people say the constraint is unsatisfied.
- We should minimize the number of unsatisfied people.

Since we have two bags G and C, minimization of

$$D(x) = G(x) + C(x) \tag{59}$$

is reasonable. Note that we write $D(x)$ instead of $C_D(x)$ for simplicity.

The above equation (59) means that total number of unsatisfied people is estimated to be $D(x) = G(x) + C(x)$ when decision variable is x. This means that there is no overlap between people unsatisfied to G and those unsatisfied to C.

In contrast, if we consider maximum overlap between those people unsatisfied to G or C, the decision is represented by

$$D(x) = \max\{G(x), C(x)\} = (G \cup C)(x) \tag{60}$$

These equations show a complementary formulation to that of fuzzy decision (58). If we should handle multiple goals $G_1, \ldots, G_m$ and constraints $C_1, \ldots, C_n$, we consider either

$$D(x) = \sum_{i=1}^{m} G_i(x) + \sum_{j=1}^{n} C_j(x), \tag{61}$$

or

$$\begin{aligned} D(x) &= \max_{1 \le i \le m, 1 \le j \le n} \{G_i(x), C_j(x)\} \\ &= (G_1 \cup \cdots \cup G_m \cup C_1 \cup \cdots \cup C_n)(x). \end{aligned} \tag{62}$$

Moreover we can use an s-norm of Minkowski type as a generalization:

$$D(x) = (G_1(x)^p \cup \cdots \cup G_m(x)^p \cup C_1(x)^p \cup \cdots \cup C_n(x)^p)^{\frac{1}{p}}, \quad (p \ge 1). \tag{63}$$

Note that (63) includes (61) when $p = 1$ and also approaches (62) as $p \to \infty$.

Note that in any case of (61), (62), and (63), we consider the minimization:

$$\min_{x \in X} D(x). \tag{64}$$

5.1 Convexity of Bags

In order to handle convex functions, we assume $X = \boldsymbol{R}^h$, the h-dimensional Euclidean space. Note that a convex function F defined on $X = \boldsymbol{R}^h$ means that

$$F(\lambda x + (1-\lambda)y) \leq (\lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in X$ and all $\lambda \in [0,1]$. A necessary and sufficient condition for the convexity of $F(x)$ is that its epigraph

$$\mathrm{epi}(F) = \{(x,\beta) \in \boldsymbol{R}^{h+1} : F(x) \leq \beta\}$$

is a convex set.

A drawback in fuzzy decision making is that we cannot use the theory of convex functions, i.e., even when we handle convex fuzzy sets, they are quasi-convex but never convex, since the membership of a fuzzy set is limited to $[0,1]$.

In contrast, we can assume convex R-bags G and C, since the membership value is in $[0,+\infty]$. It is easy to see the next properties are valid.

Proposition 13. *Assume R-bags G and C of $X = \boldsymbol{R}^h$ are convex. Then,*

$$D(x) = G(x) + C(x) \tag{65}$$

and

$$D'(x) = (G \cup C)(x) \tag{66}$$

are convex functions.

Proof. The convexity of $D(x)$ from (65) follows from the well-known fact that addition of two convex functions are also convex. The convexity of $D'(x)$ from (66) is based on the property that the intersection of two convex epigraphs is convex. □

We moreover have the next proposition.

Proposition 14. *Assume R-bags G and C of $X = \boldsymbol{R}^h$ are convex. Then,*

$$D''(x) = (G(x)^p + C(x)^p)^{\frac{1}{p}}, \quad (p \geq 1) \tag{67}$$

is convex.

Proof. We first note the Minkowski inequality [9]:

$$((a_1+b_1)^p + (a_2+b_2)^p)^{\frac{1}{p}} \leq (a_1^p + a_2^p)^{\frac{1}{p}} + (b_1^p + b_2^p)^{\frac{1}{p}}$$

for $a_1, a_2, b_1, b_2 \geq 0$. noting that $G(x)$ and $C(x)$ are convex, we have

$$\begin{aligned} D''(\lambda x + (1-\lambda)y) &= [G(\lambda x + (1-\lambda)y)^p + C(\lambda x + (1-\lambda)y)^p]^{\frac{1}{p}} \\ &\leq [(\lambda G(x) + (1-\lambda)G(y))^p + (\lambda C(x) + (1-\lambda)C(y))^p]^{\frac{1}{p}}. \end{aligned}$$

Using the Minkowski inequality, we have

$$\begin{aligned} D''(\lambda x + (1-\lambda)y) &\leq [(\lambda G(x))^p + (\lambda C(x))^p]^{\frac{1}{p}} + [((1-\lambda)G(y))^p + ((1-\lambda)C(y))^p]^{\frac{1}{p}} \\ &= \lambda(G(x)^p + C(x)^p)^{\frac{1}{p}} + (1-\lambda)(G(y)^p + C(y)^p)^{\frac{1}{p}} \\ &= \lambda D''(x) + (1-\lambda)D''(y). \end{aligned}$$

Thus the convexity of $D''(x)$ is proved. □

It is straightforward to generalize the above propositions to decisions with multiple objectives and constraints using (61), (62), and (63). We omit the detail.

Thus if we use R-bags, we can handle convex decision functions.

6 Conclusion

We have overviewed generalizations of classical bags. Three types of generalizations have been studied. For R-bags, complementation, s-norm and t-norm, and bag relations have directly been defined, while they are more complicated for G-bags. Fuzzy bags (F-bags) can be handled as a special case of G-bags. Using s-norms and t-norms, we have defined max-s and max-t compositions for bag relations.

We have shown applications of F-bags to data analysis with discussion of kernel functions. It has been known that kernel-based methods of data analysis work well in many applications, and hence more studies are necessary concerning this topic.

Moreover it was shown that decision functions can be convex using s-norms in contrast to fuzzy decision making, where convexity property does not hold.

We have omitted many other applications, for example, application to rough sets [25] is also possible and we can develop rough bags and their generalizations (see, e.g., [19]).

Overall, bags have great potential to produce new useful tools in soft computing. There are many unsolved problems both in theory and applications. Many future researches are needed.

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The Relationship between Interval, Fuzzy and Possibilistic Optimization

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Abstract. The relationship between fuzzy set theory (in particular fuzzy arithmetic) and interval analysis is well-know. This study explores the interconnections between interval analysis, fuzzy interval analysis, and interval and fuzzy/possibilistic optimization. Two key ideas are considered herein: (1) constraint set computation and (2) the clear distinctions and relationships between interval, fuzzy, and possibilistic entities as they are used in optimization within an historical and taxonomic context. Constraint fuzzy interval arithmetic is used to compute constraint sets.

Keywords: Fuzzy optimization, possibilistic optimization, interval analysis, constraint fuzzy interval arithmetic.

1 Introduction

This study explores interval analysis, fuzzy set theory, and possibility theory as they related to optimization of mathematical programming which is both a synthesis of the author's previous work bringing together, transforming, and updating many ideas found in [5],[6],[7],[8],[9], [10],[11], and [12] as well as some new material dealing with the computation of constraint sets. The real-valued mathematical programming problem for this presentation is

$$\begin{aligned} z &= \min f(x, c) \\ \text{subject to } g_i(x, a) &\leq b \quad i = 1, \ldots, M_1 \\ h_j(x, d) &= e \quad j = 1, \ldots, M_2. \end{aligned} \tag{1}$$

The constraint set is denoted $\Omega = \{x \mid g_i(x, a) \leq b \; i = 1, \ldots, M_1, \; h_j(x, d) = e, \; j = 1, \ldots, M_2\}$. It is assumed that $\Omega \neq \emptyset$. The values of a, b, c, d and e are inputs (data), parameters of the programming problem. Given our interest, we restrict these parameters to be intervals, fuzzy intervals or possibilistic distributions which we defined below. Moreover, the operator min and relationships $=$ and $\leq$ can take on flexible or fuzzy meaning becoming soft relationships or

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constraints. For example, the equality and inequality relationships may be aspirations, that is, they may take on the meaning, "Come as close as possible to satisfying the relationships with some degree of violation being permissible." Of course, if the value of a, b, d, or e are interval, fuzzy interval, or possibilistic distribution, the meaning of the inequality must be specified as will be discussed. It is noted that when the objective function and/or constraints parameters are not real numbers, the optimization problem may not be (undoubted is not) convex in the classical sense even when the real-valued equivalent problem is so that the usual solution methods are local. For example, in very simple cases where the constraint is linear, of the form $Ax \leq b$, and the coeffcients of the matrix are intervals, the solution set can be a star-shaped region even for two variables and two constraints(see [3]). When the parameters a, b, d, and/or e are interval, fuzzy, or possibilistic, the underlying model is not known exactly or it may be that the model is precise but knowledge of what the values of the data are, is incomplete or deficient in some sense.

There is an older point of view in which the optimization problem statement begins as a fuzzy optimization statement rather than a real-valued optimization model some (all) of whose parameters are fuzzy and/or intervals and/or possibilistic (see [13], [14]). Possibility theory was not well-defined at the time. For the purposes of our research, we begin with (1). This study restricts itself to real-valued interval, fuzzy interval, possibilistic coefficients, and to soft constraints.

1.1 Intervals, Fuzzy Intervals and Possibility

We outline the relationship between intervals, fuzzy intervals, and possibility distributions before looking at interval analysis as it relates to fuzzy interval analysis and possibility theory applied to optimization since there is often confusion between interval optimization, fuzzy optimization, and possibilistic optimization especially as found in the literature. An interval is a connected set of real numbers $X = [\underline{x}, \overline{x}] = \{x \mid \underline{x} \leq x \leq \overline{x}\}$. There is no "fuzziness" in an interval though an interval may be considered as a type of fuzzy number (just as a real number may be considered as a type of complex number). Either an element belongs to an interval or it does not. An interval possesses a dual nature, that of a *"new" type of number* $X = \{\underline{x}, \overline{x}\}$ consisting of two elements (the lower bound and the upper bound) or as a *set* $X = \{x \mid \underline{x} \leq x \leq \overline{x}\}$. This dual nature is exploited in interval analysis as we shall see in the sequel. Intervals capture non-specificity and are thus possibilistic as we discuss in the next paragraph. An interval belongs to uncertainty theory in so far as uncertainty models non-specificity or lack of (complete) information.

It is well-know by this time that fuzzy captures the transitional nature both in the abstract (a gradual membership function rather than a Boolean one) and in reality (the coast "line" of Japan). Possibility captures lack of information, lack of specificity, lack of precise information about an entity. Fuzzy sets are *not* inherently (by nature) uncertain in meaning or semantics. They are transitional or gradual. A fuzzy interval, as we shall see, can be automatically translated into a possibility distribution and thus reflect lack of specificity as well as transition.

Thus, fuzzy sets *may* have or take on a dual nature – that of capturing gradualness of belonging and capturing non-specificity. That is, a fuzzy set that is used in modelling uncertainty must be translatable into a possibility given the inherent nature of optimization which requires a complete order relationship. This dual nature (gradualness and uncertainty/lack of specificity) only occurs when the entity (variable) is decomposable into mutually exclusive elements. If coefficients are fuzzy numbers (fuzzy membership functions indicate, semantically, that the coefficients are transitional), the fuzzy numbers are by their very nature decomposable into mutually exclusive elements and thus possess a dual nature (gradual and uncertain). This means that they will be (except semantically) possibility distributions.

> (From [2]) Limited (minimal) specificity can be modelled in a natural way by possibility theory. The mathematical structure of possibility theory equips fuzzy sets with set functions, conditioning tools, notions of independence/dependence, decision-making capabilities [lattices]. Lack [deficiency] of information or lack of specificity means we do not have "the negation of a proposition is improbable if and only if the proposition is probable." In the setting of lack of specificity, "the negation of a proposition is impossible if and only if the proposition is necessarily true." Hence, in possibility theory pairs of possibility and necessity are used to capture the notions of plausibility [possibility] and certainty [necessity]. When pairs of functions are used we may be able to capture or model lack of information. *A membership function is a possibility only when the domain of a fuzzy set is decomposable into mutually exclusive elements.* A second difference [between probability and possibility] lies in the underlying assumption regarding a probability distribution. Namely all values of positive probability are mutually exclusive. A fuzzy set is a conjunction of elements. For instance, in image processing, imprecise regions are often modelled by fuzzy sets. However, the pixels in the region are not mutually exclusive although they do not overlap. Namely the region contains several pixels, not a single unknown one. *When the assumption of mutual exclusion of elements of a fuzzy set is explicitly made, then, and only then, the membership function is interpreted as a possibility distribution; this is the case of fuzzy numbers [intervals] describing the ill-located unique value of a parameter.* (italics, my emphasis with square brackets being my addition)

Thus, in the context of an image composed of distinct pixels, "conjunctive" means that a pixel may be more than one thing at once and the sum of the membership values need not be one. So, for example, a pixel could be classified as being a part of the stomach lining with membership value of 0.30 and stomach muscle with membership value of 0.75 with these being the only two classification for this pixel. Mutual exclusivity in this context would means that we can decompose, for example, cancerous and non-cancerous cells into pixels having possibility $x > 0$ (say 0.25) of being cancerous and a possibility $1-x$ (say

0.75) of being non-cancerous with no more classification. Moreover, possibility is always normalized since the semantics of possibility is tied to an existential entity. *Thus, not all fuzzy set membership functions are possibility distributions.*

1.2 A Taxonomy of Fuzzy and Possibilistic Optimization for Our Generic Problem

The structure of (1) when it is a linear programming problem may be considered to be formed by (i) the *rim* $f(x, c) = c^T x$, and b, e, (ii) the *body* $f(x, a), h(x, d) = Ax$, and (iii) the *relationship*, $\leq$, $=$ ([4]). Note that in the context of interval, fuzzy, possibility, $r \leq s$ and $s \leq r$ does **not** imply $r = s$. For the generic form of the mathematical programming problem (1), we consider a parameterized decomposition of (i) rim objective function parameters, c, (ii) rim right-hand side parameters, b and e, (iii) body parameters a and d, and (iv) relationship $\leq$, $=$. A real number and an interval may be considered as a fuzzy interval. A right-hand side value that is fuzzy may be interpreted in two ways depending on the context of the problem. First, a fuzzy right-hand side may indicate flexibility. Second, it may indicate (true) decomposable transition modelled by a fuzzy interval. For the former, the constraint becomes a flexible constraint. For the latter, it becomes a possibility.

The types of optimization in the presence of interval, fuzzy interval, possibilistic coefficients and soft constraints are as follows:

1. **Flexible Programming** – soft constraints where the relationships $\leq$ and/or $=$ take on a flexible meaning
2. **Utility Programming** – interval, fuzzy interval, possibilistic cost coefficients of the objective function rim parameter c and real-valued coefficient constraint coefficient a, b, d, $e \in \mathbb{R}$.
3. Real-valued (deterministic) objective function (cost) coefficients $c \in \mathbb{R}$, and (some/all) interval, fuzzy interval, possibilistic coefficients a, b, d, e.
   (a) **Flexible Programming** – body parameters real, a, $d \in \mathbb{R}$ and rim right-hand side values fuzzy interval, transformed into flexible right-hand side values
   (b) **Possibilistic Programming** – body parameters interval, fuzzy interval, possibilistic a, d and/or rim right-hand values b, e are possibilistic
   (c) **Mixed Fuzzy/Possibilistic Programming** – mixture of soft constraints and interval, fuzzy interval, possibilistic coefficients a, b, d, e.
4. **Random Set Programming** – all coefficients a, b, c, d, e may be interval, fuzzy, possibilistic where there may be a mixture of types within one constraint statement.

One might also classify fuzzy and possibilistic programming according to whether nor not the solution is a real-valued fuzzy interval vector or a real-valued vector. For this study, solutions will be real-valued. Possibilistic programming methods of Buckley and his colleagues and Delgado and his colleagues have considered fuzzy interval-valued solutions. The methods to obtain fuzzy interval solutions are different than those that obtain a real-valued solution. Nevertheless, they

fall under possibilistic programming or random set programming of the above taxonomy. In this study we focus only on 1, 2, and 3 of the above taxonomy which will suffice to illustrate the relationships between interval, fuzzy, and possibility as it impacts optimization.

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Decision Making in Voting Games: An Insight into Theory and Practice

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Abstract. The paper introduces basics of a notion of voting game theory and illustrates its application in practice. Examples outlined in the paper show how the theory works in practice of parliamentary institutions. A few simple examples explain the difference between voting weight and voting power and show how important analysis of voting games is in practice. A study of selected decisive systems explains meaning of such notions as winning and blocking coalitions, initiative and preventive power of a player etc.

Keywords: decision making, voting games, winning and blocking coalitions, power indexes, square root law.

1 Introduction

The theory of voting games is a case of decision making. The voting games are mathematical models utilized in analysis of the decision power of members of organizations like decision committees of institutions, executive boards of corporations, national parliaments, agendas of United Nations, Council and Parliament of European Union.

All modern democracies rely on the idea of representation. A certain body of representatives, a parliament for example, makes decisions on behalf of the voters. In most parliaments each of its members represents roughly the same number of people, namely the voters in his or her constituency, [6]. The simplest and most common system relies on assigning a weight to every member. The decision is taken when the sum of weights of members supporting decision reaches or exceeds an assumed threshold.

A voting game is a conflict in which the only objective is winning and the only rule is an algorithm to decide which coalitions are winning. These games have been used to study the distribution of power in voting situations. Two power indices have received the most theoretical attention as well as application to political structures, e.g. [1,2,6].

The analysis of power is central in political science. In general, it is difficult to define the idea of power, but for the special case of voting situations several quantitative measures for evaluating the power of a voter or coalition have

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been proposed. In this paper we will discuss methods of power's measuring and application of them to the Council of European Union.

2 Voting Games

Let us consider a system

$$G = (P, W)$$

where:

- P is a finite set of elements called *players*. Finiteness of P is always assumed in this paper.
- $W \subset 2^P$ is a family of subsets of the set P of players. W is called the family of *winning coalitions*.

A system G satisfying the following conditions is called *voting game*:

1. W is nonempty, i.e. at least one coalitions is winning,
2. if $C \in W$ and $C \subset C'$ then $C' \in W$, i.e. an extension of a winning coalition is the winning coalition,

A voting game is simple if it satisfies the condition:

3) if $C \in W$ then $P \setminus C \notin W$, i.e. there is no pair of non-intersecting winning coalitions

Note that first two conditions imply that P, the *grand coalition*, is winning.

Complement of a winning coalition in simple game is called *losing coalition*. Coalitions that neither are winning, nor losing are called *blocking coalitions*.

A voting game is said to be *strong* if it does not have blocking coalitions. The ratio $e = |W|/|2^P|$ is said to be game *effectiveness* parameter. Effectiveness cannot exceed 0.5 and only effectiveness of strong games reaches 0.5. Of course, effectiveness measures ability of decision making: the higher the value of effectiveness parameter, the easier to find a winning coalition.

A player p is called a *swing player* in a given winning (blocking) coalition C if and only if the coalition $C \setminus \{p\}$ is not winning (not blocking).

A player is called *dummy player* if it is not swing in any winning (blocking) coalition.

A coalition is said to be *minimal coalition* if every player of this coalition is swing.

A game is called *weighted voting game* if every player $p_i \in P$ is assigned a natural number w_i and a number q, $0 < q \leq \sum_{p_i \in P} w_i$, is given. Numbers w_i are called *weights* and q is called *qualified majority*. A subset C of the set P of players is a winning coalition if and only if its weight is not less than the qualified majority, i.e. if $\sum_{w_i \in C} w_i \geq q$.

We will denote such games as a system:

$$(\sum_{p_i \in P} w_i;\ q;\ w_1, w_2, \ldots, w_n)$$

A weighted voting game is simple if and only if qualified majority is greater than 50% of total sum of weights: $q > 0.5 \cdot \sum_{p_i \in P} w_i$.

3 Voting Power

Strength of a players depends on weights. However, the analysis of strength of a player cannot be based on weights only. Let us consider an example of a game with three players:

$$(100;\ 51;\ 49, 49, 2)$$

A winning coalition is either grand coalition or it includes any two players. This means that strength of every player is the same, despite of disproportion of weights between first two player and the third player.

Let us consider another example:

$$(100;\ 51;\ 26, 26, 25, 23)$$

In this case weights of all players are similar. Despite of this the fourth player is a dummy player.

Observing the above examples we see that it is important to clearly differentiate between weight of a player and its strength, i.e. ability to affect decisions taken in the game.

Many measures of players' strength have been invented. We will focus our attention on Shapley-Shubik, Banzhaf, Coleman initiative and Coleman prevention power indexes. Power indexes are based on number of coalitions, in which a given player is swing. So then strength of a player can be expressed by numbers of winning and blocking coalitions, in which the player is swing. In this case we usually consider the smallest coalitions in term of number of players in the coalition.

3.1 Power Indexes

The *Shapley-Shubik power index* was formulated by Lloyd Shapley and Martin Shubik in 1954, [8]. The Shapley-Shubik power index for a given player p_i is defined as a proportion:

$$ShSh(p_i) = \frac{sws_{ShSh}(p_i)}{\sum_{w_j \in P} sws_{ShSh}(p_j)}$$

where $sws_{ShSh}(p_i)$ is the number of winning coalitions in which the player p_i is swing. Of course, the term $sws_{ShSh}(p_j)$ stands for the number of winning coalitions in which the player p_j is swing.

The definition of swing player in terms of Shapley-Shubik power index is specific. We consider all sequences of n players, i.e. $n!$ sequences. A player p_{i_k} is swing if and only if the coalition including first k players of the sequence is winning while the coalition including first $k-1$ players of the sequence is not winning, see Figure 1. In other words, a winning coalition in terms of Shapley-Shubik power index is a sequence rather than a set. This definition of a swing player is not compatible with the definition of voting game (let us recall that - in terms of voting game definition - coalition is a set of players).

$$\frac{\overbrace{w_{i_1}, w_{i_2}, w_{i_3}, ..., w_{i_{k-1}}}^{not\ winning}, \overbrace{w_{i_k}, w_{i_{k-1}}, ..., w_{i_n}}^{winning}}{}$$

Fig. 1. Swing player in terms of the definition of the Shapley-Shubik power index

Note, that every sequence of n players has a swing player in terms of Shapley-Shubik power index, what implies that the denominator in the formula of Shapley-Shubik power index is equal $n!$.

The definition of the *Banzhaf power index* is similar to that of Shapley-Shubik index, c.f. [3]:

$$Bz(p_k) = \frac{sws_{Bz}(p_k)}{\sum_{p_j \in P} sws_{Bz}(p_j)}$$

The difference is in definition of winning coalitions. In case of Banzhaf index a winning coalition is a set rather than a sequence of players. This definition of a swing player is compatible with the definition of voting games, in which a coalition is a set rather than a sequence. We can observe that $sws_{Bz}(k) = sws_{ShSh}(k)/((k-1)!)$.

The Coleman power indexes define ability of a player to initiate or to prevent a decision, c.f. [4]. The *Coleman initiative power index* is a proportion:

$$Ci(p_k) = \frac{sws_{Bz}(p_k)}{number\ of\ not\ winning\ coalitions}$$

while the *Coleman preventive power index* is a proportion:

$$Cp(p_k) = \frac{sws_{Bz}(p_k)}{number\ of\ winning\ coalitions}$$

where, of course, a coalition (winning or not winning) is a set of players rather than a sequence of players.

The Coleman initiative power index tells how often a player can turn a not winning coalition to a winning coalition. On the other hand, the Coleman preventive power index expresses how often a player can turn a winning coalition to a not winning coalition.

The above power indexes are the most important and the most popular. Many other indexes could be found in literature.

3.2 Numbers of Coalitions

Besides power indexes we can analyze number of winning or blocking coalitions in which a given player is a swing player. Most important are small coalitions, i.e. coalitions which include minimal number of players, minimal number plus one player up to minimal number plus a few players. Since creating a small coalition is easier than creating a bigger one, these numbers define an ability of a player to create a winning or a blocking coalition.

3.3 Banzhaf Indexes in Work, an Example

A typical parliament includes several hundred envoys. Usually decisions are taken if majority of envoys vote for it. In other words, we have a game with given number of n player with weight equal to 1 assign to each of them. So, analysis of the system seems to be not very complex. However, the number of possible coalitions, which is equal to 2^n, makes analysis hard in practice. Such a number is rather abstract when we consider a typical parliament having a few hundred of envoys. Also, some of envoys may be absent, other may abstain, what increases number of all possible coalitions and makes the situation even worse.

To be able to do any analysis of a parliamentary voting system we make assumptions simplifying the system. First of all we can notice that most of envoys belong to political parties and usually vote in the same way. Based on this observation we assume that envoys of every party cast in the same way. Second, we assume that votes of abstain and absent envoys are against (since they are not pro). These assumptions allow considering every party as a player with weight equal to number of envoys in this party.

It may seem that number of envoys in a party, i.e. weight of the party, articulates a strength of the party. This is true that a strength depends on its weight. However, dependence is not straightforward. Considering the example $(100;\ 51;\ 49, 49, 2)$ we can see that weights of players can be widely changed without affecting strength of players.

Let us consider a real life example outlined in Table 1. It explains how closely the theory of voting games explains the practice. It is worth to note that the country, parties etc. are not important and can be dropped in our discussion. But a following paragraph of explanations is aimed on readers who may be interested in real situation.

In 2001 fall election to Sejm, the Polish Parliament, Democratic Left Alliance / Labor Union (SLD/UP) won, but did not reach the majority. Other parties winning seats in the parliament are: Civic Platform (PO), Law and Justice (PiS), the Polish Peasant(s) Party (PSL), Self-Defence of the Republic of Poland (SRP), the League of Polish Families (LPR) and German Minority (MN). The SLD/UP formed a government along with PSL. And then during next two and half a year both aliens straggled between each other rather then making law. In that time tens of envoys left their parliamentary clubs creating a new club and several groups of envoys. Also, 15 single envoys declared independence on any parliamentary group. Finally, on February 19th 2004 the Prime Minister L. Miller decided to fire PSL and to form a minority SLD/UP government. Surprisingly, minority government had no problem winning in votes. Why? We may speculate that it was easier to convince single envoys or small groups of envoys than to come to agreement with the alien. This situation is well modelled by Banzhaf power index. Notice that SLD/UP lost 10 seats (216 seats won in election and 206 seats two and half a year later) and significantly earned 10 percent

Table 1. Polish Parliament *Sejm* - results of 2001 fall election and later after changes in parties. Legend: SLD/UP - Democratic Left Alliance / Labor Union, PO - Civic Platform, PiS - Law and Justice, PSL - the Polish Peasant(s) Party, SRP - Self-Defence of the Republic of Poland, LPR - the League of Polish Families, MN - German Minority and Niez. - independent envoys.

Player's power in Polish Parliament						
Parties	Fall election 2001			February 19th, 2004		
	Number of envoys		Banzhaf Index	Number of envoys		Banzhaf Index
	Numbers	Percentage		Numbers	Percentage	
SLD/UP	**216**	**47.0**	**75.0**	**206**	**44.8**	**85.0**
PO	65	14.1	5.0	56	12.2	1.9
PiS	44	9.6	5.0	43	9.3	1.9
PSL	42	9.1	5.0	37	8.0	1.9
SRP	53	11.5	5.0	31	6.7	1.9
LPR	38	8.3	5.0	30	6.5	1.9
FKP	0	0	0	15	3.3	1.7
SKL	0	0	0	8	1.7	0.7
PBL	0	0	0	6	1.3	0.5
RKN	0	0	0	5	1.1	0.4
PP	0	0	0	3	0.7	0.3
ROP	0	0	0	3	0.7	0.3
MN	2	0.4	0	2	0.4	0.2
Niez.	0	0	0	15*1	15*0.22	15*0.1
Total	460; 231	100.0	100.0	460; 231	100.0	100.0

points of Banzhaf index (75% after election and 85% in 2004). On the other hand big opposition parties significantly lost Banzhaf index (5% to 1.9%). Notice also that Banzhaf index of big opposition parties was equal despite that percentage of seats of PO is twice as much as of LPR. Is not it a surprising consistency between theory and practice?

4 Making Decisions in the Council of the European Union

As mentioned before, it is not important whether examples presented are based on decision systems practiced in Europe, America, United Nations or anywhere else. What is important is that examples presented illustrate real life applications of voting games. Moreover, European Union institutions might be interesting for some readers. This is why we allocate some parts of the paper explaining basic notions of EU.

Table 2. Voting system in EEC and EU9

European Economic Community and early European Union								
Countries	Number of votes		Power index					
			Shapley-Shubick		Banzhaf		Coleman init.	
	1958	1973	1958	1973	1958	1973	1958	1973
France	4	10	0.233	0.179	0.238	0.167	0.204	0.122
Germany	4	10	0.233	0.179	0.238	0.167	0.204	0.122
Italy	4	10	0.233	0.179	0.238	0.167	0.204	0.122
Belgium	2	5	0.150	0.081	0.143	0.091	0.122	0.067
Netherland	2	5	0.150	0.081	0.143	0.091	0.122	0.067
Luxembourg	1	2	0	0.010	0	0.016	0	0.011
Denmark	-	3	-	0.057	-	0.066	-	0.048
Ireland	-	3	-	0.057	-	0.066	-	0.048
United Kingdom	-	10	-	0.179	-	0.167	-	0.122
	17;12	58;41	1.000	1.000	1.000	1.000		

4.1 European Economic Community and Early European Union

European Economic Community formed in 1958 included six countries. The distribution of votes is outlined in Table 2. Note that Luxembourg was given 1 voice while Germany - 4 voices. Comparing number of citizens (several hundred thousand against tens of millions) it may seem that Luxembourg was overrepresented. Nevertheless, there was no coalition in which Luxembourg is critical. In other words, Luxembourg did not affect any decision. This fact is consistent

with the value of Banzhaf index. Was it a mistake of designer of the voting system? It is difficult to say. Anyway, after the first enlargement done in 1973 the problem was removed, see Table 2.

The first enlargement was done in 1973. It is worth to notice that distribution of votes between nine countries, as shown in Table 2 corrected the Luxembourg's dummy player problem. No member was dummy in the European Union enlarged to nine members.

4.2 Blocking in Mid European Union

Enlargements done in 1981, 1986 and 1995 brought EU to have 15 members. The decision system in the Council of the European Union was not changed. It was still weighted voting game. Distribution of weights is outlined in Table 3: four big countries got 10 votes, Spain got 8 votes, other countries got 5 to 2 votes. What is interesting is a blocking ability of members. There was two levels' blocking structure. There were 10 smallest, three members' blocking coalitions. Every of five biggest countries belonged to six such coalitions. No other country could form three members' blocking coalitions. All countries could form four members' blocking coalitions. Notice, that the smaller weight of a player, the smaller number of blocking coalition is. Such (in our case - two levels') blocking system is regular.

Let us have a look at the qualified majority of that game. It seems that designers of the system carefully considered it. The obvious constrain put at qualified majority is to ensure that game was simple. Another constrain required was to guarantee that decision could be made by majority of members. We can notice that the value 59 assured both constrains giving the best efficiency. However, it excluded Spain, the moderately big country, from any three members' blocking coalition. On the other hand, qualified majority equal to 60 or 61 gave irregular blocking in four members' coalitions. The qualified majority equal to 62 satisfied the above conditions.

4.3 Enlarged European Union

The last two enlargements increased the number of states to twenty seven. Since 2005 decision-making system is based on the Treaty of Nice. It was intended to replace the Treaty of Nice by a new Treaty. However, so called Constitutional Treaty was rejected by **people** in France and Netherland. A mutation of the Constitutional Treaty, so called Lisbon Treaty, recently has also been rejected by **people** in Ireland. So, future of EU decision-making system is still opened. Now it is still based on the Nice Treaty.

The Treaty of Nice. *The decision-making system by qualified majority will be changed as from 1 January 2005. In future, a qualified majority will be obtained if:*

- *the decision receives at least a specified number of votes (the qualified majority threshold),*
- *the decision is approved by a majority of Member States,*

Table 3. Blocking system in EU 15

Blocking structure in EU 15			
Countries	Weights	b3	b4
Germany	10	6	153
France	10	6	153
United Kingdom	10	6	153
Italy	10	6	153
Spain	8	6	108
Netherland	5	0	86
Greece	5	0	86
Portugal	5	0	86
Belgium	5	0	86
Sweden	4	0	74
Austria	4	0	74
Denmark	3	0	64
Finland	3	0	64
Ireland	3	0	64
Luxembourg	2	0	36
	87;62	10	369

The Treaty also provides for the possibility for a member of the Council to request verification that the qualified majority represents at least 62% of the total population of the European Union. If this condition is not met, the decision will not be adopted. However, this condition applies only if verification is requested.

As we can see, the Nice Treaty is a conjunction of tree simple games: weighted voting games, majority of countries and population majority. In practice, the first game overheads the last ones. The last two games may affect decision only in few cases. Weights were distributed as follow: 29 (Germany, France, Great Britain, Italy), 27 (Spain, Poland), 14 (Romania), 13 (Netherland), 12 (Greece, Portugal, Belgium, Czechia, Hungary), 10 (Sweden, Austria, Bulgaria), 7 (Denmark, Slovakia, Finland, Ireland, Lithuania), 4 (Latvia, Slovenia, Estonia, Cyprus, Luxembourg), 3 (Malta). The total sum of weighs is equal to 345 while qualified majority is equal to 255.

Banzhaf and Coleman power indexes of countries are shown in Figure 2 (left bars). We can see that values of indexes of four biggest countries (Germany, France, Great Britain and Italy) are equal. Values of power indexes of next two countries (Spain and Poland) are only slightly smaller.

The Treaty of Constitution. *Definition of qualified majority within the European Council and the Council*

1. *A qualified majority shall be defined as at least 55% of the members of the Council, comprising at least fifteen of them and representing Member States comprising at least 65% of the population of the Union.*
   A blocking minority must include at least four Council members, failing which the qualified majority shall be deemed attained.
2. *By way of derogation from paragraph 1, when the Council does not act on a proposal from the Commission or from the Union Minister for Foreign Affairs, the qualified majority shall be defined as at least 72% of the members of the Council, representing Member States comprising at least 65% of the population of the Union.*

Design of the Constitutional Treaty was based on so called double majority, i.e. majority of countries and majority of population with qualified majorities set as above and with excluded smallest blocking coalitions. Banzhaf and Coleman power indexes of countries are shown in Figure 2 (middle bars). Note that this decision-making system privileged Germany while Spain and especially Poland significantly lost their voting power comparing to the Nice Treaty.

The Square Root Law and Jagiellonian Compromise. Disproportions in gains and loses raised discussions and protests in several countries and societies. In 2004 Zyczkowski and Slomczynski proposed a decision-making system based on the Penrose's square-root law. The system, called *Jagiellonian compromise*, was supported by a wide gremium of European researchers, c.f. [7]. It was a weighted voting game in which distribution of weights was proportional to square root of population of countries. As it was shown in [10], the optimal qualified majority equal to 62% of total sum of weights.

The square-root law states that the distribution of power in a heterogenous voting systems is fair if the power (index) of each council member i is proportional to $\sqrt{N_i}$, where N_i is the population of the state which i represents, c.f. [6]. The optimal qualified majority q is given by the formula, cf. [9]:

$$q = \frac{1}{2}(1 + \frac{\sqrt{N_1 + \ldots + N_M}}{\sqrt{N_1} + \ldots + \sqrt{N_M}})$$

It is easily seen in Figure 2 that distribution of the Banzhaf index of power in the Jagiellonian Compromise is a compromise between Nice Treaty and Constitution Treaty. Beside this, values of the Coleman initiative index in Jagiellonian Compromise overhead values in Nice and Constitutional Treaties and values of the Coleman preventive index in Jagiellonian Compromise are smaller then values in Nice and Constitutional Treaties, what makes Jagiellonian Compromise more effective. As a result, Jagiellonian Compromise has not been accepted.

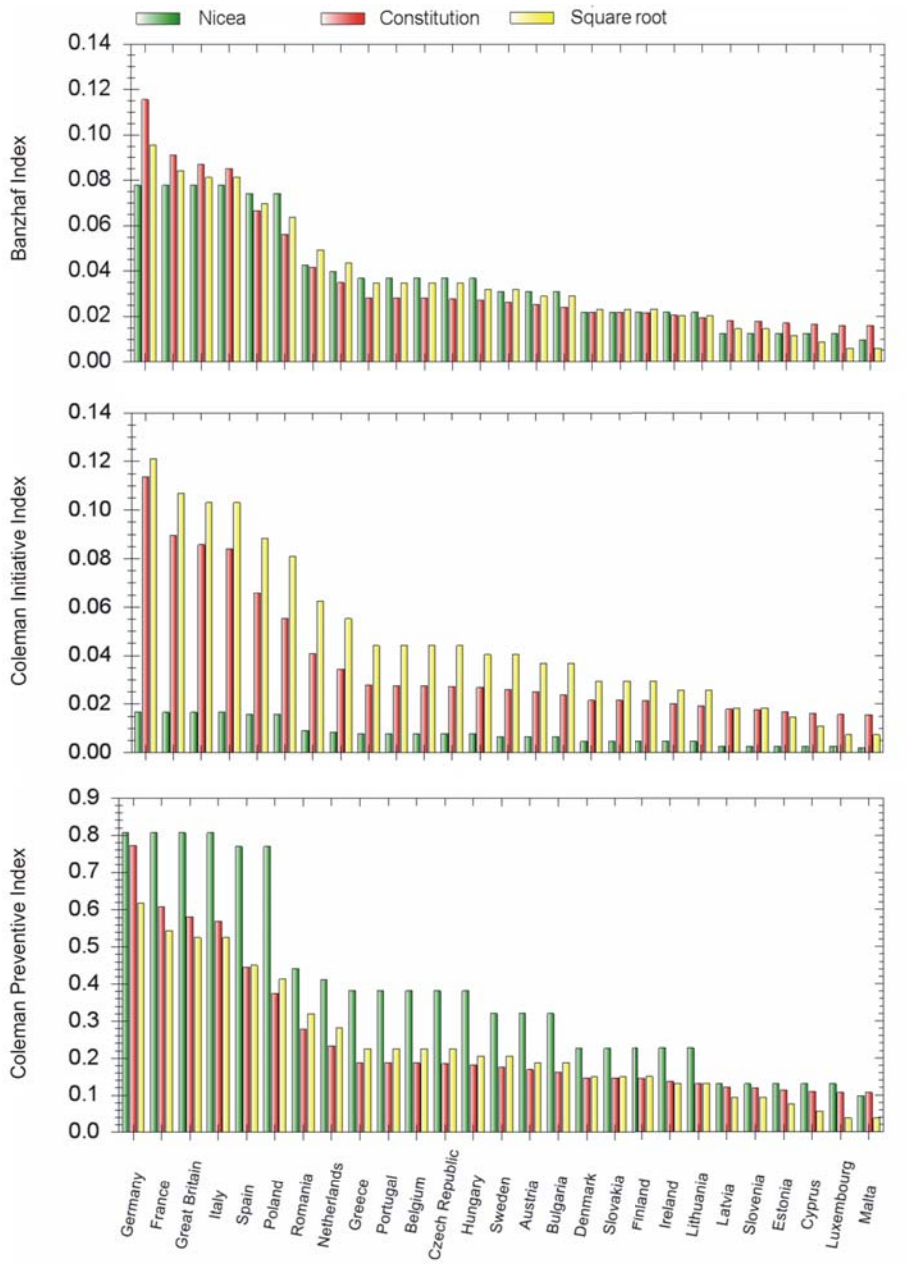

Fig. 2. UE 27 Indexes

5 Conclusions

The analysis of power is central in political science. In general, it is difficult to define the idea of power, but for the special case of voting situations several quantitative measures for evaluating the power of a voter or coalition have been proposed. In the paper we shortly presented Banzhaf and Coleman power indexes as well as blocking abilities of several decision-making institutions. The extended discussion on selected topics could be found in the literature.

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A Lyapunov-Type Theorem for Nonadditive Vector Measures*

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Abstract. We prove the convexity and compactness of the closure of the lower partition range of an $\mathbb{R}^n$-valued, nonatomic, supermodular capacity, employing a useful relationship between convex games and their Choquet integrals. The main result is applied to fair division problems, and the existence of Pareto optimal α-fair partitions is demonstrated for the case of nonadditive measures.

Keywords: Nonatomic vector measure, Lyapunov's convexity theorem, Capacity, Supermodularity, Choquet integral, Convex game, Core, Fair division.

MSC 2000: Primary: 28B05, 28E10, secondary: 91A12.

1 Introduction

Let $\mu_1, \ldots, \mu_n$ be finite measures of a measurable space $(\Omega, \mathscr{F})$. The range of an $\mathbb{R}^n$-valued vector measure $(\mu_1, \ldots, \mu_n)$ is given by

$$\mathscr{R}(\mu_1, \ldots, \mu_n) = \{(\mu_1(A), \ldots, \mu_n(A)) \in \mathbb{R}^n \mid A \in \mathscr{F}\}.$$

The integral of a measurable function f on Ω with respect to the measure μ_i is denoted by $\hat{\mu}_i(f)$. For the finite measure given by $\mu = \sum_{i=1}^n \mu_i$, we denote by $L^\infty(\Omega, \mathscr{F}, \mu)$ the space of μ-essentially bounded functions on Ω with the μ-essential sup norm.

The following result is attributed to Lyapunov [10].

Lyapunov's Convexity Theorem. *If $\mu_1, \ldots, \mu_n$ are nonatomic finite measures of a measurable space $(\Omega, \mathscr{F})$, then $\mathscr{R}(\mu_1, \ldots, \mu_n)$ is convex and compact in $\mathbb{R}^n$ and it coincides with the set given by*

$$\{(\hat{\mu}_1(f), \ldots, \hat{\mu}_n(f)) \in \mathbb{R}^n \mid 0 \leq f \leq 1,\ f \in L^\infty(\Omega, \mathscr{F}, \mu)\}.$$

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There is a large number of elaborated proofs of Lyapunov's convexity theorem. For example, Halmos [8] presented a purely measure-theoretic proof and Lindenstrauss [11] provided a proof based on fundamental results of functional analysis. Useful extensions of this theorem for fair division problems were given by Akin [1], Dubins and Spanier [4], Dvoretsky et al. [5] and Gouweleeuw [7], incorporating the partition range of a nonatomic vector measure.

The purpose of this paper is to establish a Lyapunov-type convexity theorem for an important class of set functions on σ-algebras, which has been investigated independently for many years in game theory, discrete convex analysis, fuzzy measure theory and statistical decision theory, namely, the class of supermodular set functions (convex games). We prove the convexity and compactness of the closure of the lower partition range of an $\mathbb{R}^n$-valued, nonatomic, continuous, supermodular set function, employing a useful relationship between cores and Choquet integrals for convex games studied by Choquet [2], Delbaen [3], Kelley [9], Marinacci and Montrucchio [12] and Schmeidler [17, 18]. The main result is applied to partitioning a measurable space among a finite number of players, and the existence of Pareto optimal α-fair partitions is demonstrated for the case of nonadditive measures.

2 Main Result

Throughout this paper, a *set function* is a real-valued function on a σ-algebra $\mathscr{F}$ of subsets of a nonempty set Ω that vanishes at the empty set.

A set function ν is *monotone* if $\nu(A) \leq \nu(B)$ for every $A, B \in \mathscr{F}$ with $A \subset B$; ν is *supermodular* (or *convex*) if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ for every $A, B \in \mathscr{F}$. A supermodular set function is monotone if and only if it is nonnegative.

A set function ν is *bounded* if $\sup_{A \in \mathscr{F}} |\nu(A)| < \infty$. A monotone set function is bounded. A set function ν is of *bounded variation* if $\sup \sum_{i=1}^{k} |\nu(A_i) - \nu(A_{i-1})|$ is finite, where the supremum is taken over all finite chains $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = \Omega$ in $\mathscr{F}$. A bounded supermodular set function is of bounded variation (see Marinacci and Montrucchio [12]).

Given a set function ν, an element $N \in \mathscr{F}$ is *ν-null* if $\nu(A \cup N) = \nu(A)$ for every $A \in \mathscr{F}$. If $N \in \mathscr{F}$ is ν-null, then $\nu(N) = 0$. A set function ν is *null-additive* if $A \cap N = \emptyset$, and $\nu(N) = 0$ implies $\nu(A \cup N) = \nu(A)$. For a null-additive monotone set function ν, an element $N \in \mathscr{F}$ is ν-null if and only if $\nu(N) = 0$ (see Pap [13], Theorem 2.1).

A set function ν is *nonatomic* if, for every ν-nonnull element, $A \in \mathscr{F}$ there exists a measurable subset B of A such that both $A \setminus B$ and B are ν-nonnull. A set function ν is *absolutely continuous* with respect to a set function μ if every μ-null set is ν-null; ν is *equivalent* to μ if an element in $\mathscr{F}$ is ν-null if and only if it is μ-null.

A set function ν is *continuous from above* at A if $\nu(A_k) \to \nu(A)$ for every sequence $\{A_k\}$ in $\mathscr{F}$ with $A_k \downarrow A$; ν is *continuous from below* at A if $\nu(A_k) \to \nu(A)$ for every sequence $\{A_k\}$ in $\mathscr{F}$ with $A_k \uparrow A$; ν is *continuous* if it is both

continuous above and continuous below at every element in $\mathscr{F}$. A monotone continuous set function is called a *capacity* (or a *fuzzy measure*).

A set function ν is *normalized* if $0 \leq \nu \leq 1$ and $\nu(\Omega) = 1$.

Let $B(\Omega, \mathscr{F})$ be the space of bounded measurable functions on Ω with the supremum norm. The *Choquet integral* $\hat{\nu} : B(\Omega, \mathscr{F}) \to \mathbb{R}$ of a set function ν is defined by an improper Riemann integral of the form

$$\hat{\nu}(f) = \int_0^{+\infty} \nu(f \geq t)dt + \int_{-\infty}^{0} [\nu(f \geq t) - \nu(\Omega)]dt.$$

Here, $(f \geq t)$ denotes the measurable set $\{\omega \in \Omega \mid f(\omega) \geq t\}$. Note that this integral exists whenever ν is of bounded variation (see Pap [13], Theorem 7.21); $\nu(A) = \hat{\nu}(\chi_A)$ for every $A \in \mathscr{F}$, where χ_A is the characteristic function of $A \in \mathscr{F}$; $\hat{\nu}$ is *positively homogeneous*; that is, $\hat{\nu}(\alpha f) = \alpha\hat{\nu}(f)$ for every $\alpha \geq 0$ and $f \in B(\Omega, \mathscr{F})$.

For the Choquet integral $\hat{\nu}$ of a set function ν of bounded variation, there are equivalent conditions: (i) ν is supermodular; (ii) $\hat{\nu}$ is *supermodular*; that is, $\hat{\nu}(f) + \hat{\nu}(g) \leq \hat{\nu}(f \vee g) + \hat{\nu}(f \wedge g)$ for every $f, g \in B(\Omega, \mathscr{F})$; (iii) $\hat{\nu}$ is concave; (iv) $\hat{\nu}$ is *superadditive*; that is, $\hat{\nu}(f) + \hat{\nu}(g) \leq \hat{\nu}(f + g)$ for every $f, g \in B(\Omega, \mathscr{F})$. (For a proof, see Marinacci and Montrucchio [12].)

Let $B(\Omega, \mathscr{F}; \mathbb{R}^m)$ be the space of $\mathbb{R}^m$-valued bounded measurable functions on Ω with the sup norm, where its generic element is denoted componentially by $(f_1, \ldots, f_m)$. We introduce subsets S^m and S_0^m of $B(\Omega, \mathscr{F}; \mathbb{R}^m)$ as follows.

$$S^m = \left\{ (f_1, \ldots, f_m) \in B(\Omega, \mathscr{F}; \mathbb{R}^m) \mid \sum_{j=1}^{m} f_j = 1,\ f_1, \ldots, f_m \geq 0 \right\};$$

$$S_0^m = \left\{ (\chi_{A_1}, \ldots, \chi_{A_m}) \in B(\Omega, \mathscr{F}; \mathbb{R}^m) \mid \sum_{j=1}^{m} \chi_{A_j} = 1,\ A_1, \ldots, A_m \in \mathscr{F} \right\}.$$

Note that an m-tuple of measurable sets $(A_1, \ldots, A_m)$ is a partition of Ω if and only if $\sum_{j=1}^{m} \chi_{A_j} = 1$. Thus, S_0^m denotes the set of m-tuples of measurable partitions of Ω.

Let $\nu_1, \ldots, \nu_n$ be nonnegative set functions of bounded variation. Define the *lower range* of an $\mathbb{R}^n$-valued set function $(\nu_1, \ldots, \nu_n)$ under S^m by

$$\underline{\mathscr{R}}^m(\nu_1, \ldots, \nu_n) = \left\{ (x_{ij}) \in \mathbb{R}^{nm} \,\middle|\, \begin{array}{l} \exists (f_1, \ldots, f_m) \in S^m : 0 \leq x_{ij} \leq \hat{\nu}_i(f_j) \\ i = 1, \ldots, n;\ j = 1, \ldots, m \end{array} \right\}$$

and the *lower partition range* of $(\nu_1, \ldots, \nu_n)$ under S_0^m by

$$\underline{\mathscr{R}}_0^m(\nu_1, \ldots, \nu_n) = \left\{ (x_{ij}) \in \mathbb{R}^{nm} \,\middle|\, \begin{array}{l} \exists (\chi_{A_1}, \ldots, \chi_{A_m}) \in S_0^m : 0 \leq x_{ij} \leq \hat{\nu}_i(\chi_{A_j}) \\ i = 1, \ldots, n;\ j = 1, \ldots, m \end{array} \right\}.$$

The main result of this paper is as follows.

Theorem 1. *If $\nu_1, \ldots, \nu_n$ are nonatomic supermodular capacities, then the closure of $\underline{\mathscr{R}}_0^m(\nu_1, \ldots, \nu_n)$ is convex and compact in $\mathbb{R}^{nm}$ and it coincides with $\underline{\mathscr{R}}^m(\nu_1, \ldots, \nu_n)$.*

3 Preliminary Result

Let $ba(\Omega, \mathscr{F})$ be the space of finitely additive set functions on $\mathscr{F}$ of bounded variation with the total variation norm, which is the dual space of $B(\Omega, \mathscr{F})$ (see Dunford and Schwartz [6], Theorem IV.5.1), with the corresponding duality denoted by $\langle f, \lambda \rangle$ for $f \in B(\Omega, \mathscr{F})$ and $\lambda \in ba(\Omega, \mathscr{F})$.

Let ν be a set function. Define the subset $\mathscr{C}(\nu)$ of $ba(\Omega, \mathscr{F})$ by

$$\mathscr{C}(\nu) = \{\lambda \in ba(\Omega, \mathscr{F}) \mid \nu \leq \lambda \text{ and } \lambda(\Omega) = \nu(\Omega)\}.$$

The set $\mathscr{C}(\nu)$ is called the *core* of ν in game theory. Note that $\mathscr{C}(\nu)$ is (possibly empty) weak*-compact in $ba(\Omega, \mathscr{F})$ because it is bounded in the total variation norm and weak*-closed (see Dunford and Schwartz [6], Corollary V.4.3).

The following characterization of continuous supermodular set functions indicates a profound relationship between their cores and Choquet integrals.

Theorem 2. *For every bounded set function $\nu : \mathscr{F} \to \mathbb{R}$, the following conditions are equivalent.*

(i) *ν is continuous and supermodular;*
(ii) *ν is of bounded variation and for every finite measure μ satisfying*

$$\lim_{\mu(A)\to 0} \sup_{\lambda \in \mathscr{C}(\nu)} \lambda(A) = 0, \tag{1}$$

the Choquet integral $\hat{\nu} : B(\Omega, \mathscr{F}) \to \mathbb{R}$ of ν has a unique extension to $L^\infty(\Omega, \mathscr{F}, \mu)$ such that $\hat{\nu}$ is concave on $L^\infty(\Omega, \mathscr{F}, \mu)$ and weak-continuous on bounded subsets of $L^\infty(\Omega, \mathscr{F}, \mu)$.*

(A measure μ satisfying condition (1) is called a *control measure* for $\mathscr{C}(\nu)$.)

Proof. (i) $\Rightarrow$ (ii): Under the hypotheses of the theorem, $\mathscr{C}(\nu)$ is nonempty (see Kelley [9] and Schmeidler [17]); every element in $\mathscr{C}(\nu)$ is countably additive (see Schmeidler [17]); there exists a finite control measure for $\mathscr{C}(\nu)$ (see Schmeidler [17] and Delbaen [3]); ν is *exact*; i.e., $\nu(A) = \min_{\lambda \in \mathscr{C}(\nu)} \lambda(A)$ for every $A \in \mathscr{F}$ (see Kelley [9] and Schmeidler [17]); more generally, $-\hat{\nu}$ is the support function of $\mathscr{C}(\nu)$ in the sense that $\hat{\nu}(f) = \min_{\lambda \in \mathscr{C}(\nu)} \langle f, \lambda \rangle$ for every $f \in B(\Omega, \mathscr{F})$ (see Schmeidler [18]). (All of these statements have been proven in full generality by Marinacci and Montrucchio [12].)

Choose any finite measure μ that is a control measure for $\mathscr{C}(\nu)$. Since every element in $\mathscr{C}(\nu)$ is absolutely continuous with respect to μ, we have that if $\mu(N) = 0$, then $\lambda(N) = 0$ for every $\lambda \in \mathscr{C}(\nu)$. We then have $\nu(A \cup N) = \min_{\lambda \in \mathscr{C}(\nu)} \lambda(A \cup N) = \min_{\lambda \in \mathscr{C}(\nu)} \lambda(A) = \nu(A)$ for every $A \in \mathscr{F}$, which demonstrates that every μ-null set is ν-null. Take any $f \in L^\infty(\Omega, \mathscr{F}, \mu)$. Then $f = g$ a.e. for some $g \in B(\Omega, \mathscr{F})$. Since

$$(f \geq t) = [(f \geq t) \cap (f = g)] \cup [(f \geq t) \cap (f \neq g)]$$

for every $t \in \mathbb{R}$ and the sets $(f \geq t) \cap (f \neq g)$ and $(g \geq t) \cap (f \neq g)$ are μ-null, we have $\nu(f \geq t) = \nu(g \geq t)$. Hence, $\hat{\nu}(f) = \hat{\nu}(g)$ and the value $\hat{\nu}(f)$ is well

defined because ν is of bounded variation. It follows from this argument that the Choquet integral $\hat{\nu}$ defined on $B(\Omega, \mathscr{F})$ has a unique extension to $L^\infty(\Omega, \mathscr{F}, \mu)$ (which we do not relabel).

The concavity of $\hat{\nu}$ on $L^\infty(\Omega, \mathscr{F}, \mu)$ follows easily from the observation that $\hat{\nu}$ is the pointwise minimum of the family of linear functionals $f \mapsto \langle f, \lambda \rangle$ on $B(\Omega, \mathscr{F})$ over $\lambda \in \mathscr{C}(\nu)$.

Let $\{f_\alpha\}$ be a net in a bounded subset of $L^\infty(\Omega, \mathscr{F}, \mu)$ such that $f_\alpha \to f$ in $\sigma(L^\infty, L^1)$. Since for each α there exists an element $\lambda_\alpha \in \mathscr{C}(\nu)$ such that $\hat{\nu}(f_\alpha) = \langle f_\alpha, \lambda_\alpha \rangle$ and $\mathscr{C}(\nu)$ is weak*-compact in $ba(\Omega, \mathscr{F})$, we can extract a weak*-convergent subnet $\{\lambda_\alpha\}$ (which we do not relabel) with $\lambda_\alpha \to \lambda_* \in \mathscr{C}(\nu)$. We thus have

$$\begin{aligned} |\langle f_\alpha, \lambda_\alpha \rangle - \langle f, \lambda_* \rangle| &= |\langle f_\alpha - f, \lambda_* \rangle + \langle f_\alpha, \lambda_\alpha - \lambda_* \rangle| \\ &\le \left| \int (f_\alpha - f) d\lambda_* \right| + \|f_\alpha\|_\infty \left| \int \chi_\Omega d\lambda_\alpha - \int \chi_\Omega d\lambda_* \right| \\ &\le \left| \int (f_\alpha - f) g_* d\mu \right| + C \left| \int \chi_\Omega d\lambda_\alpha - \int \chi_\Omega d\lambda_* \right| \to 0, \end{aligned}$$

where $g_* = \frac{d\lambda_*}{d\mu} \in L^1(\Omega, \mathscr{F}, \mu)$ and the constant C is such that $\|f_\alpha\|_\infty \le C$ for each α. Therefore, $\hat{\nu}(f_\alpha) \to \langle f, \lambda_* \rangle$. Choose any $\lambda \in \mathscr{C}(\nu)$. We then have

$$\langle f_\alpha, \lambda_\alpha \rangle = \hat{\nu}(f_\alpha) \le \langle f_\alpha, \lambda \rangle = \int f_\alpha g d\mu \quad \text{for each } \alpha,$$

where $g = \frac{d\lambda}{d\mu} \in L^1(\Omega, \mathscr{F}, \mu)$. Taking the limit for this inequality yields $\langle f, \lambda_* \rangle \le \langle f, \lambda \rangle$ for every $\lambda \in \mathscr{C}(\nu)$. Therefore, $\hat{\nu}(f) = \min_{\lambda \in \mathscr{C}(\nu)} \langle f, \lambda \rangle = \langle f, \lambda_* \rangle$, and hence $\hat{\nu}(f_\alpha) \to \hat{\nu}(f)$.

(ii) $\Rightarrow$ (i): Let μ be a finite measure stated in condition (ii). Since the concavity of $\hat{\nu}$ is equivalent to the supermodularity of ν, it suffices to show that ν is continuous.

To this end, let $A_k \uparrow A$ in $\mathscr{F}$. Then $\{\chi_{A_k}\}$ is a bounded sequence in $L^\infty(\Omega, \mathscr{F}, \mu)$ with $\chi_{A_k} \uparrow \chi_A$ a.e. For every $f \in L^1(\Omega, \mathscr{F}, \mu)$, we have $\chi_{A_k} f \to \chi_A f$ a.e. with $|\chi_{A_k} f| \le |f|$ for each k. Then from Lebesgue's dominated convergence theorem, $\int \chi_{A_k} f d\mu \to \int \chi_A f d\mu$ for every $f \in L^1(\Omega, \mathscr{F}, \mu)$, which implies $\chi_{A_k} \to \chi_A$ in $\sigma(L^\infty, L^1)$. Therefore, $\nu(A_k) = \hat{\nu}(\chi_{A_k}) \to \hat{\nu}(\chi_A) = \nu(A)$, and hence ν is continuous from below at every $A \in \mathscr{F}$.

The verification of the continuity from above is similar.

4 Proof of Theorem 1

Under the hypothesis of the theorem, every element in $\mathscr{C}(\nu_i)$ is a nonatomic finite measure and there exists a nonatomic finite measure μ_i in $\mathscr{C}(\nu_i)$ such that μ_i is a control measure for $\mathscr{C}(\nu_i)$ (see Marinacci and Montrucchio [12]). Let μ be the finite measure given by $\mu = \sum_{i=1}^n \mu_i$. From Theorem 2, each $\hat{\nu}_i$ defined on $B(\Omega, \mathscr{F})$ has a unique extension to $L^\infty(\Omega, \mathscr{F}, \mu)$ such that $\hat{\nu}_i$ is concave on $L^\infty(\Omega, \mathscr{F}, \mu)$

and weak*-continuous on bounded subsets of $L^\infty(\Omega,\mathscr{F},\mu)$. Thus, in the definition of S^m and S_0^m, we may replace $B(\Omega,\mathscr{F};\mathbb{R}^m)$ with $L^\infty(\Omega,\mathscr{F},\mu;\mathbb{R}^m)$, which does not change the lower range and lower partition range of $(\nu_1,\ldots,\nu_n)$. Note that S^m is weak*-compact in $L^\infty(\Omega,\mathscr{F},\mu;\mathbb{R}^m)$ because it is bounded in the essential sup norm and weak*-closed (see Dunford and Schwartz [6], Corollary V.4.3).

We denote by $\operatorname{int}\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)$ the interior of the lower range of the vector capacity $(\nu_1,\ldots,\nu_n)$. By virtue of the trivial inclusion $\underline{\mathscr{R}}_0^m(\nu_1,\ldots,\nu_n)\subset\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)$, to prove the theorem, it suffices to show that $\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)$ is convex and compact in $\mathbb{R}^{nm}$ and the inclusion

$$\operatorname{int}\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)\subset\underline{\mathscr{R}}_0^m(\nu_1,\ldots,\nu_n). \tag{2}$$

Since S^m is a convex and weak*-compact subset of $L^\infty(\Omega,\mathscr{F},\mu;\mathbb{R}^m)$, the convexity and compactness of $\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)$ are derived readily from Theorem 2.

To prove the remaining inclusion (2), the technique derived by Lindenstrauss [11] for proving Lyapunov's convexity theorem, and its variant derived by Akin [1], are helpful.

Choose $(x_{ij})\in\operatorname{int}\underline{\mathscr{R}}^m(\nu_1,\ldots,\nu_n)$ arbitrarily. Then there is a $(g_1,\ldots,g_m)\in S^m$ and $\delta_{ij}>0$ such that $x_{ij}+\delta_{ij}=\hat{\nu}_i(g_j)$ for each i,j. Define the set W_{ij} by

$$W_{ij}=\{(f_1,\ldots,f_m)\in S^m \mid x_{ij}\le\hat{\nu}_i(f_j)\}.$$

Since W_{ij} is nonempty, convex and weak*-compact in $L^\infty(\Omega,\mathscr{F},\mu;\mathbb{R}^m)$ for each i,j, so is the set $W=\bigcap_{i=1}^n\bigcap_{j=1}^m W_{ij}$. Thus, W has an extreme point $(f_1,\ldots,f_m)$ according to the Krein–Milman theorem (see Dunford and Schwartz [6], Lemma V.8.2). It suffices to show that each f_j is a characteristic function because if this is shown, then $(x_{ij})\in\underline{\mathscr{R}}_0^m(\nu_1,\ldots,\nu_n)$, and hence inclusion (2) is true.

Suppose to the contrary that some f_j is not a characteristic function. By virtue that $(f_1,\ldots,f_m)\in S^m$, we may assume without loss of generality that there exist some $\varepsilon>0$ and $A\in\mathscr{F}$ with $\mu(A)>0$ such that $\varepsilon<f_1,f_2<1-\varepsilon$ on A. Since $\mu_1,\ldots,\mu_n$ are nonatomic finite measures, from Lyapunov's convexity theorem, there exists some $B\subset A$ such that $\frac{1}{2}(\mu_1(A),\ldots,\mu_n(A))=(\mu_1(B),\ldots,\mu_n(B))$. Define the measurable function $h=s(\chi_A-2\chi_B)$ with $0<s\le\varepsilon$. Then $h\ne0$, $0\le f_1\pm h,f_2\pm h\le1$ and $\langle h,\mu_i\rangle=\int h\,d\mu_i=0$ for each i. In view of $\mu_i\in\mathscr{C}(\nu_i)$, we have $\hat{\nu}_i(\pm h)=\min_{\lambda\in\mathscr{C}(\nu_i)}\langle\pm h,\lambda\rangle\le\langle\pm h,\mu_i\rangle=0$. Define $f_j^\alpha=\alpha g_j+(1-\alpha)f_j$ for $\alpha\in(0,1)$. Then $\hat{\nu}_i(f_j^\alpha)\to\hat{\nu}_i(f_j)$ as $\alpha\to0$ because $f_j^\alpha\to f_j$ in $\sigma(L^\infty,L^1)$ as $\alpha\to0$. Hence, there exists a subnet $\{f_j^\alpha\}$ (which we do not relabel) such that $\hat{\nu}_i(f_j^\alpha)-\frac{\alpha\delta_{ij}}{2}<\hat{\nu}_i(f_j)$ for every $\alpha\in(0,1)$ and i,j. Since $\hat{\nu}_i$ is positively homogeneous, we have $\hat{\nu}_i(\pm h)=s\hat{\nu}_i(\chi_A-2\chi_B)$. Therefore, for arbitrarily fixed $\alpha\in(0,1)$, by choosing s sufficiently small, $-\frac{\alpha\delta_{ij}}{2}<\hat{\nu}_i(\pm h)\le0$ for each i,j. Since $\hat{\nu}_i$ is concave and superadditive, we have

$$\begin{aligned}\hat{\nu}_i(f_j\pm h)\ge\hat{\nu}_i(f_j)+\hat{\nu}_i(\pm h)&>\hat{\nu}_i(f_j^\alpha)-\frac{\alpha\delta_{ij}}{2}-\frac{\alpha\delta_{ij}}{2}\\&\ge\alpha\hat{\nu}_i(g_j)+(1-\alpha)\hat{\nu}_i(f_j)-\alpha\delta_{ij}\ge x_{ij}\end{aligned}$$

for each i, j. This implies that $(f_1 \pm h, f_2 \mp h, f_3, \ldots, f_m)$ is in W, and hence

$$(f_1, \ldots, f_m) = \frac{1}{2}[(f_1 + h, f_2 - h, f_3, \ldots, f_m) + (f_1 - h, f_2 + h, f_3, \ldots, f_m)] \in W,$$

which contradicts that $(f_1, \ldots, f_m)$ is an extreme point in W. □

5 Application to Fair Division Problems

The fair division problem is the partitioning of a measurable space $(\Omega, \mathscr{F})$ among n players so as to fulfill equity and efficiency in which each player i evaluates measurable sets in terms of a set function ν_i.

We denote by $\alpha = (\alpha_1, \ldots, \alpha_n)$ a generic element in the unit simplex Δ^{n-1} of $\mathbb{R}^n$.

Definition 1. *Let $\nu_1, \ldots, \nu_n$ be normalized set functions. A measurable partition $(A_1, \ldots, A_n)$ of Ω is:*

(i) *α-fair if $\nu_i(A_i) \geq \alpha_i$ for each $i = 1, \ldots, n$;*
(ii) *Pareto optimal if there exists no partition $(B_1, \ldots, B_n)$ of Ω such that $\nu_i(A_i) \leq \nu_i(B_i)$ for each $i = 1, \ldots, n$ and $\nu_j(A_j) < \nu_j(B_j)$ for some j.*

The following result is a generalization of that of Dubins and Spanier [4] for the case where $\nu_1, \ldots, \nu_n$ are nonatomic probability measures to the case for nonatomic, normalized, supermodular capacities.

Theorem 3. *Let $\nu_1, \ldots, \nu_n$ be normalized capacities that are mutually absolutely continuous, nonatomic and supermodular. If some ν_i is null-additive and $\underline{\mathscr{R}}_0^n(\nu_1, \ldots, \nu_n)$ is closed in $\mathbb{R}^{n^2}$, then for every $\alpha \in \Delta^{n-1}$, there exists a Pareto optimal α-fair partition.*

Proof. If $P_k = (E_1, \ldots, E_n)$ is a partition of Ω in which $E_k = \Omega$ and $E_j = \emptyset$ for $j \neq k$, then a partition matrix $M(P_k) = (\nu_i(E_j))$ has values of 1 in the kth column and values of zero elsewhere. Since $M(P_1), \ldots, M(P_n)$ belong to $\underline{\mathscr{R}}_0^n(\nu_1, \ldots, \nu_n)$, Theorem 1 implies that $\sum_{i=1}^n \alpha_i M(P_i)$ is in $\underline{\mathscr{R}}_0^n(\nu_1, \ldots, \nu_n)$ for every $\alpha \in \Delta^{n-1}$. Therefore, there exists a partition $P = (A_1, \ldots, A_n)$ of Ω such that $M(P) \geq \sum_{i=1}^n \alpha_i M(P_i)$; that is, $\nu_i(A_j) \geq \alpha_j$ for each i, j. As a result, solutions to the maximization problem

$$\begin{aligned} &\max x_{11} \\ &\text{s.t. } x_{ii} \geq \alpha_i, \quad i = 1, \ldots, n \\ &\qquad (x_{ij}) \in \underline{\mathscr{R}}_0^n(\nu_1, \ldots, \nu_n) \end{aligned} \tag{P_α}$$

exist for every $\alpha \in \Delta^{n-1}$ owing to the compactness of $\underline{\mathscr{R}}_0^n(\nu_1, \ldots, \nu_n)$. Here, we assume without loss of generality that ν_1 is null-additive.

Take any solution (x_{ij}) to the problem (P_α). Then there exists a partition $(A_1, \ldots, A_n)$ of Ω such that $\nu_i(A_i) \geq \alpha_i$ for each i. It suffices to show that

$(A_1, \ldots, A_n)$ is Pareto optimal. Suppose to the contrary that there exists a partition $(B_1, \ldots, B_n)$ of Ω such that $\nu_i(A_i) \leq \nu_i(B_i)$ for each i and $\nu_j(A_j) < \nu_j(B_j)$ for some j. If $j = 1$, then $x_{11} \leq \nu_1(A_1) < \nu_1(B_1)$, which obviously contradicts (x_{ij}) being a solution to (P_α). Thus, we investigate the case for $j \neq 1$.

¿From the hypothesis of the theorem, there exists a control measure μ_i for $\mathscr{C}(\nu_i)$ that is a nonatomic probability measure equivalent to ν_i (see Marinacci and Montrucchio [12]). Thus, from the continuity of ν_j and the nonatomicity of μ_j, there exists a measurable subset B'_j of B_j such that $\nu_j(A_j) < \nu_j(B'_j)$ and $\nu_j(B_j \setminus B'_j) > 0$. Hence, $B_j \setminus B'_j$ is ν_1-nonnull in view of the mutual absolute continuity of ν_1 with respect to ν_j and the null-additivity of ν_1. We thus have $\nu_1(B_j \setminus B'_j) > 0$. Define a partition $(C_1, \ldots, C_n)$ of Ω by $C_1 = B_1 \cup (B_j \setminus B'_j)$, $C_j = B'_j$ and $C_i = B_i$ for $i \neq 1, j$. We then have $\nu_i(C_i) \geq \alpha_i$ for each i and the superadditivity of ν_1 yields $\nu_1(C_1) \geq \nu_1(B_1) + \nu_1(B_j \setminus B'_j) > \nu_1(B_1) \geq x_{11}$, which contradicts (x_{ij}) being a solution to (P_α).

6 Concluding Remarks

We have established the convexity and compactness of the closure of the lower partition range $\underline{\mathscr{R}}_0^m(\nu_1, \ldots, \nu_n)$ for nonatomic supermodular capacities $\nu_1, \ldots, \nu_n$. There is another significant class of nonadditive set functions on σ-algebras that guarantees the convexity of the lower partition range, that is, the class of *concave measures* proposed by Sagara and Vlach [15, 16]. A simple application of Lyapunov's convexity theorem plays a crucial role to prove the convexity of the lower partition range for concave measures. This is another variant of the Lyapunov-type theorem for nonadditive vector measures. Under alternative hypotheses, Sagara [14] demonstrates the existence of Pareto optimal α-fair partitions when $\nu_1, \ldots, \nu_n$ are concave measures and the lower partition range is closed.

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A Formal Theory of Cooperative TU-Games

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Abstract. Results of game theory are often the keys to decisions of economical and political executives. They are also used to create internal tools of many decision making software. For example, coordination games may be cooperative games, when the players choose the strategies by a consensus decision making process, and game trees are used to represent some key cooperative games. Our theory of cooperative games with transferable utilities makes it possible to deliver a formal certificate that contains statements and proofs with each result of any procedure in theory of cooperative TU-games. Such formal certificates can be archived and audited by independent experts to guarantee that the process that lead to the decision is sound and pertaining. As we use an automated proof checker, the review only has to guarantee that the statements of the certificate are correct. The proofs contained in the certificate are guaranteed automatically by the proof checker and our formal theory.

Keywords: formalization, cooperative games, automated proof checker.

1 Introduction and Motivations

Game theory was first introduced in a famous book by von Neumann and Morgenstern(1944) and it has fostered a wide variety of theoretical as well as applied research in many domains of social sciences. A game consists in a set of interactive and fully educated decisions in which the utilities (cost or benefit) of each participant (later referred as player) depends on decisions taken by the other participants.

Two game theories are developed separately. On one hand, players solely focus on their interest in non-cooperative games. On the other hand, players may form binding commitments and build coalitions in cooperative games. In theory, players participate to the welfare of their coalition (maximize profit or minimize cost). In practice, the existence of cooperative games depend on referees that are able to enforce all the commitments between players once signed.

A game where the utilities can be perfectly transferred between players and their values are identical for all players, is called a game with transferable utilities or a TU-game. For example, a game where players exchange goods is not a TU-game since the value of each good for each player may be a part of the decisions.

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In a TU-game, a coalition may receive all the utilities of its members in one transfer and later redistribute or collect its utilities among its members.

A legitimate distribution or collect is the key to a fair coalition. For this reason, results of game theory (Fougères, Truffert, and Ventou, 1999, 2000a,b) are often used in practice to propose such distributions or collects and support them on theoretical unbiased grounds.

To mechanize our proofs, we have been using PVS proof assistant (Owre, Rushby, and Shankar, 1992). Systems like PVS allow the user to define new objects and to derive consequences of these definitions formally. The language of PVS is based on a higher-order logic. With such an expressive logic, it is possible to state properties in their most general form. For example, universal quantification has been used to state properties that are true for an arbitrary game and/or an arbitrary set of players. Proofs are built interactively using high-level strategies. Theorem provers have already been successfully used in many domains of sciences including floating-point arithmetic and probability (Daumas et al., 2009a,b,d).

2 Cooperative Games with Transferable Utilities

TU-games are modeled by a set of players N and a function v that gives the utility of each subset S (earlier referred as a coalition) of N such that $v(\emptyset) = 0$. As the utility can be perfectly transferred between players $v(S)$ has values in $\mathbb{R}$ or $\mathbb{Z}$ depending whether the utility can be indefinitely cut into parts. Restricting the values of $v(S)$ to $\mathbb{Q}$ would make some result of game theory unreachable though it would have led to some easier developments in formal methods (first order logic).

In a profit game, v is the payoff for each coalition that may be formed and the players want to maximize it by committing themselves to the best coalition, the one that will redistribute the highest payoff to them. Inversely, in a cost game, v is a tax or a fee that must be collected by each formed coalition and the players want to minimize it.

2.1 Applications

One common example of profit games is the assignment game (Shapley and Shubik, 1971; Kern and Paulusna, 2003). The set of players N is build from two distinct sets: the set of traders or producers and the set of consumers. In this case, coalitions are sets of producers and consumers such that no transaction involve players from two different coalitions.

Another example is raised by the creation and/or the extension of a water supply network. We should detail it a little more. The price of water paid by customers is defined independently for each city. Yet a reliable and cost-efficient supply network usually involve more than one city. As cities are not evenly spread on a flat land, costs and benefits of connecting each individual city to an existing or a projected network vary. The practical purpose of game theory in this case is to propose the graph of a water supply network and a distribution

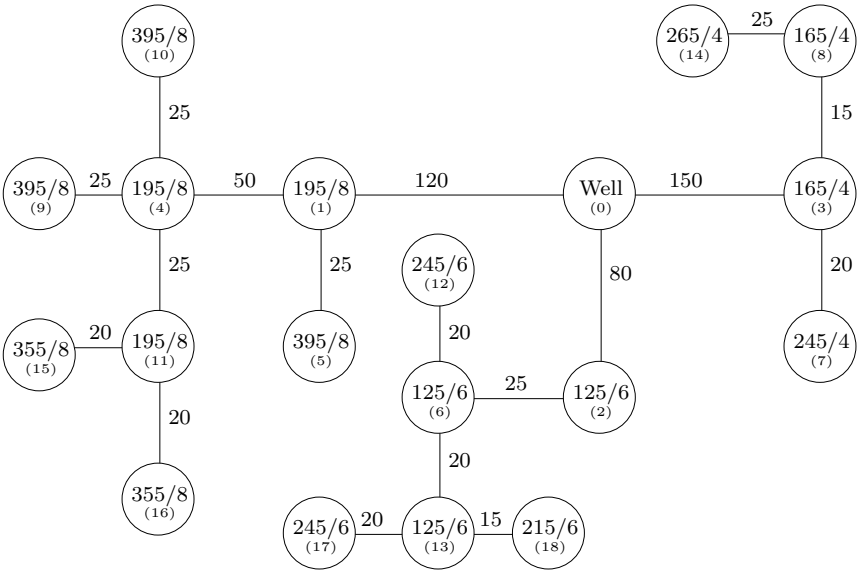

Fig. 1. A water supply network showing segments (pipelines) and nodes (cities)

of its cost that can be accepted by all the participating cities. Assuming that all the cities have to be connected to some water supply network, the game can be represented by a weighted graph. The cost of each coalition $v(S)$ is the cost of its minimum spanning tree (Bird, 1976; Granot and Huberman, 1981; Aarts and Driessen, 1993).

In Figure 1 we present a water supply network assuming that the spanning tree has already been fixed. The node numbers appear between parentheses. A cost is attached to each segment but for example the segment between the well and node 1 is also used by nodes 4, 9, 10, 11, 15 and 16. These nodes should pay some amount of the cost of this specific segment. The numbers appearing in each node is the fair fee that each city should pay to enter the network. For example, node 13 will have to pay 125/6 units to get connected to the well. That is 6 times less than what it would have to pay should it connect directly to the well $20 + 25 + 80 = 125$ instead of sharing a network.

Many similar examples appear in the different tasks handled by cities and regional administrations. For example Northern (French) Catalunya was drowned by heavy rains in 1999. The drowning was partly due to a poorly maintained drainage network. After the disaster, the cost of maintenance was significantly raised. A similar change also occurred among the cities of the Ouvèze valley in (French) Provence after the drowning in Vaison-la-Romaine. Though the regional administration pays for most of the maintenance, some part of the cost is billed to the cities.

In all these examples, all the parties involved easily agree that some work has to be done and paid. Yet, the main issue is to make sure that taxpayers and

citizens agree that the distribution of the cost among the different cities is fair and beyond any reasonable doubt.

A payoff of a game v defined on $\mathcal{P}(N)$ (the powerset of N, equivalently denoted by 2^N), is modeled as a function $x : N \to \mathbb{R}$. It is first defined on elements on N and later extended to subsets of N with $x(S) = \sum_{i \in S} x(i)$. In a profit game coalitions cannot distribute more than the total amount generated and $x(N) \leq v(N)$ for feasible payoffs. This condition is necessary but not sufficient and we define preimputations and imputations that may be accepted by all the players.

Game theory uses a set of properties (usually referred as axioms) to characterize acceptable payoffs. A solution concept σ is a function (not necessarily known or implementable) that suggests a (possibly empty) set of feasible payoffs for each game v. Solution concepts were introduced because most properties cannot be expressed on a given payoff but need to refer to a larger set. Some of the most studied solution concepts are

- the Shapley value (Shapley, 1953; Driessen, 1985),
- the core (Gillies, 1953; Bondareva, 1963; Shapley, 1967; Maschler et al., 1979),
- the kernel (Davis and Maschler, 1965; Maschler et al., 1971),
- and the prenucleolus (Schmeidler, 1969; Sobolev, 1975; Potters, 1991; Snijders, 1995).

2.2 Mathematical Properties

We recall now more formally and sometimes complete some definitions that have been introduced earlier in this text.

Definition 1. *A* **cooperative game with transferable utilities** *or a TU-game is a pair (N, v) such that N is a nonempty finite set (the set of players) and $v : 2^N \to \mathbb{R}$ is a mapping (the coalition function) satisfying $v(\emptyset) = 0$.*

Definition 2. *The set of* **feasible payoffs** *for a TU-game (N, v) is $X(N, v) = \left\{x \in \mathbb{R}^N \mid x(N) \leq v(N)\right\}$ where $x(S)$ is a shorthand for $\sum_{i \in S} x_i$.*

Definition 3. *The* **preimputations** *$\mathcal{I}^*(N, v)$ of a TU-game (N, v) is the subset of the feasible payoffs $x \in X(N, v)$ such that $x(N) = v(N)$.*

Definition 4. *The* **imputations** *$\mathcal{I}(N, v)$ of a TU-game (N, v) is the subset of the preimputations $x \in \mathcal{I}^*(N, v)$ satisfying $x_i \geq v(\{i\})$ for all $i \in N$.*

Definition 5. *A* **solution concept** *σ is a mapping that associates to every game (N, v) a subset $\sigma(N, v) \subseteq X(N, v)$.*

The elements of a solution concept of a game are interpreted as proposals on how to distribute the payoffs to each members of the coalitions. For example, the core of a TU-game (N, v) introduced earlier is the set

$$\mathcal{C}(N, v) = \{x \in \mathcal{I}^*(N, v) \mid x(S) \geq v(S),\ \forall S \subseteq N\}$$

The notion of solution concept is voluntarily vast. We detail in Table 1 properties on solution concepts σ to characterize the fair and acceptable ones. Sudhölter and Peleg (1998), as well as some other authors, have presented slight variations on these definitions and axioms.

Table 1. Properties (axioms) usually needed from solutions concepts

Short name	Definition
PO	σ is **pareto optimal** if for all (N,v), $\sigma(N,v) \subseteq \mathcal{I}^*(N,v)$.
AN	σ is **anonymous** if for all (N,v), and all bijective mapping τ of N onto N' then $$\sigma(N', \tau v) = \tau(\sigma(N,v))$$ where $\tau v(S') = v(\tau^{-1}(S'))$ and $\tau(x)_i = x_{\tau^{-1}i}$ $\quad (x \in \mathbb{R}^N,\ i \in N'$ and $S' \subseteq N')$.
ETP	σ satisfies **equal treatment property** if for all (N,v), all x in $\sigma(N,v)$ and all player i and j interchangeable, $x_i = x_j$. i and j are *interchangeable* if : $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.
DES	σ respect **desirability** if for all (N,v), all x in $\sigma(N,v)$ and all player i more desirable than j, $x_i \geq x_j$. i is more *desirable* than j if : $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.
NPP	σ satisfies the **nullplayer property** if for all (N,v), all x in $\sigma(N,v)$ and all nullplayer i, $x_i = v(\{i\})$. i is a *nullplayer* if : $v(S \cup \{i\}) = v(S) + v(i)$ for all $S \subseteq N \setminus \{i\}$.
COV	σ is **covariant under strategic equivalence** if for all (N,v) and (N,w) with $w = \alpha v + \beta$, $\alpha > 0$ and $\beta \in \mathbb{R}^N$: $$\sigma(N,w) = \alpha\sigma(N,v) + \beta.$$
SIVA	σ is **single valued** if $\lvert\sigma(N,v)\rvert = 1$ for all (N,v).
NE	σ satisfies **nonemptiness** if $\sigma(N,v) \neq \emptyset$ for all (N,v).
REAS	σ satisfies **reasonableness (on both side)** if for all (N,v), all x in $\sigma(N,v)$ and all i in N: • $x_i \geq \min_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$, • $x_i \leq \max_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$.

3 Building a Formal Theory of Cooperative TU-Games

3.1 A Brief Overview of PVS (Prototype Verification System)

We give a quick overview of PVS. For a more complete introduction, we refer the reader to Owre et al. (1992).

PVS is a generic prover. In this system, users can define new objects *and* prove properties that derive logically from these definitions. Objects in PVS are typed and functions are first-class objects. The system is distributed with standard libraries. Types like `int` and `real` that correspond to the integer and the real numbers are built in PVS. We define the type `tu_game` below to give an example from our theory. It represents the pair (2-tuple) composed of the player set N and the coalition function v:

```
tu_game: TYPE = [N: players_set, v: coalition_fun[U,N]]
```

PVS allows that the type of the second member of the pair depends on the first member of the pair. U and N are parameters of the theory in which the `coalition_fun` type is defined. Furthermore, for an n-ary function PVS makes no difference between providing n variables independently or providing an n-tuple (Owre et al., 2001, p. 41 and 50). Elements of a tuple are numbered starting with 1.

```
core(g): set_vect(g‘1) = core[U,g‘1,g‘2]
```

defines the core of a game as a subset of $\mathbb{R}^N$ specified in the `imputation` theory, parametrized by U (population), N (players set) and v (coalition function). Arguments between square brackets `[U,g‘1,g‘2]` provide the parameters of the theory, arguments between parentheses `(g‘1)` are arguments of functions or predicates.

Objects are first named, then they are typed after a colon. Then they are defined but syntax varies depending on the kind of object.
Functions, including predicates (when the codomain is `bool`), are stated just after the `=` sign,

```
PO(s): bool = FORALL (g: tu_game): subset?(s(g), setPI(g))
```

whereas the statement of lemmas, theorems, and so on, start just after the `LEMMA` or `THEOREM` keyword, e.g.:

```
DES_impl_ETP: LEMMA FORALL (s:solution_concept): DES(s)=>ETP(s)
```

In practice, formula statements are allowed to contain free variables, in which case PVS uses “the universal closure” of these statements, as explained in (Owre et al., 2001, p. 26). Thus, the previous formula can be shortened to

```
DES_implies_ETP: LEMMA DES(s) => ETP(s)
```

provided there is an appropriate variable declaration before, such as:

```
s: VAR solution_concept
```

3.2 Our Formal Theory and Properties of Cooperative TU-Games

Defining the types of the various objects involved in a mathematical theory is usually benign. Often, two of more definitions can be applied to some objects

relating to different points of view. People start with one of such definitions and prove that the others ones are equivalent.

We have to work differently with formal methods. Each object is strongly typed. This means that we have to apply the equivalence theorem in order to use a property or a theorem on a type equivalent to the one used in its statement. This has to be done independently for each variable appearing in the statement.

Another major difficulty in working with equivalent types is that most known equivalences become false on some rares cases considered as degenerate. For example, the definition that *a parallelogram is a quadrilateral with two sets of parallel sides*[1] is no longer sound if the four points defining the quadrilateral are on the same line. Working with formal methods, we have to formally exclude the degenerate cases before each use of an equivalence.

The first object defined in our theory is the set of players. One may use a *finite non-empty type* to populate the set of players but this is not permitted by PVS typechecker. We used the definition of Listing 1, where the set of players is a *non-empty finite subset* of a possibly infinite "universe" of players (Sudhölter and Potters, 2001).

Listing 1. First type used in our formal theory

```
1 players_set[U: TYPE+]: THEORY
2  BEGIN
3   elt: U
4   players_set: TYPE+ = non_empty_finite_set[U]
5     CONTAINING singleton(elt)
6  END players_set
```

It is possible to enforce the fact that the population of possible players is finite with an `ASSUMING` statement in PVS. Yet any task that cannot be handled automatically by the typechecker is left to be done by the user in the form of a Type Correctness Condition (TCC). Focusing on a finite universe of players would generate many additional TCCs.

As soon as we have defined the set of players, we define in two theories omitted here the coalition function v, then the payoffs, the imputations, and the core. Some of our decisions in the development of these theories are similar to the ones of software engineering. For example, we have to coherently decide the theory where each notion and formula belongs. We also have to decide on global parameters (parameters theories) vs. local ones (parameters of predicates and functions). Finally, some notions such as subset and subtype introduce slight differences in formal proofs.

We define in Listing 2 what is a solution concept and which properties such an object might satisfy. The identifiers chosen for these properties correspond to the

[1] Citation from Wikipedia at `http://en.wikipedia.org/wiki/Parallelogram`

Listing 2. Extracts from our formal theory of cooperative TU-games

```
tu_game[U: TYPE+]: THEORY
 BEGIN
  IMPORTING players_set[U], coalition_fun, imputations

  tu_game: TYPE = [N:players_set, v:coalition_fun[U,N]]

  g: VAR tu_game
  N,NN: VAR players_set
  set_vect(N): TYPE = set_vect[U,N]
  setFP(g): set_vect(g`1) = setFP[U,g`1,g`2]
  setPI(g): set_vect(g`1) = setPI[U,g`1,g`2]
  setI(g):  set_vect(g`1) =  setI[U,g`1,g`2]
  core(g):  set_vect(g`1) =  core[U,g`1,g`2]

  ss: VAR [g:tu_game -> set_vect(g`1)]
  solution_concept?(ss): bool =
    FORALL (g:tu_game): subset?(ss(g), setFP(g))
  solution_concept: TYPE+ =
    (solution_concept?) CONTAINING core

  s: VAR solution_concept
  PO(s): bool = FORALL g: subset?(s(g),setPI(g))
  AN(s): bool = FORALL g,NN:
    FORALL (tau:tau_type(g`1,NN)):
    s(NN,tau_v(g,NN,tau)) = image(tau_X(g,NN,tau),s(g))
  ETP(s): bool = FORALL g: FORALL (x:(s(g))):
    FORALL (i,j:(g`1)): interchangeable?(g)(i,j) =>
      x(i) = x(j)
  DES(s): bool = FORALL g: FORALL (x:(s(g))):
    FORALL (i,j:(g`1)): more_desirable?(g)(i,j) =>
      x(i) >= x(j)
  NPP(s): bool = FORALL g: FORALL (x:(s(g))):
    FORALL (i:(g`1)): nullplayer?(g)(i) =>
      x(i) = g`2(singleton(i))
  COV(s): bool = FORALL (N:players_set):
    FORALL (a: posreal, b: [(N) -> real]):
    FORALL (v: coalition_fun[U,N]):
      s(N,affinestar(N,a,b)(v)) =
      image(affine(N,a,b), s(N,v))
  SIVA(s): bool = FORALL g: is_finite(s(g)) AND
                            card(s(g)) = 1
  NE(s):   bool = FORALL g: nonempty?(s(g))
  REAS(s): bool =
    FORALL (N:players_set, v:coalition_fun[U,N]):
    FORALL (x:(s(N,v))): FORALL (i:(N)):
        (EXISTS (S1:setof[(N)]): (NOT member(i,S1))
           AND x(i) >= v(add(i,S1))-v(S1))
    AND (EXISTS (S2:setof[(N)]): (NOT member(i,S2))
           AND x(i) <= v(add(i,S2))-v(S2))
 END tu_game
```

names presented in Table 1, in the same order. Some intermediate definitions are omitted in this text but they are visible in the corresponding PVS file. `bstar` (i.e. β_*) is one such definition introduced by Sudhölter and Peleg (2002). The whole specification is supported by about thirty theorems, including the following ones.

```
DES_implies_ETP: LEMMA DES(s) => ETP(s)
SIVA_implies_NE: LEMMA SIVA(s) => NE(s)
PO_core: LEMMA PO(core)
COV_core: THEOREM COV(core)
```

4 Perspectives and Concluding Remarks

We have presented a formal theory of cooperative games with transferable utilities implemented in PVS automated proof checker. Our theory will be submitted as one of the NASA Langley PVS libraries[2] as soon as it is stable. Our work can be used to support the wide acceptation of some suggestions computed on instances of games for a fair distribution of the cost of the development of some infrastructures. It can be applied to the development or the extension of some water supply networks and the maintenance of some flood-resilient water drainage and dike systems, for example.

Providing a formal proof of correctness of the method used to obtain the distribution of the costs is useful even when there is no legitimate reason to suspect a flow in the method or its software implementation. The development of infrastructures for example is a very expensive task with some political and legal aspects. For a mild cost, a formal proof of correctness helps convince citizens and taxpayers that the distribution is fair.

Moreover, a formal proof of correctness is a more efficient deterrent of lawsuits than a normal process involving peer-review, audit and interviews. Some aspects of game theory are still not well settled and any method that contains some subtleties, can only be reviewed by an expert in the field. If a pencil-and-paper proof of correctness is written for example in French, Catalan or Japanese to accommodate legal and political requirements, chances are that the number of experts able to review it will be very small and it would be impossible to prevent any suspicions of conflicting interests.

By using a formal proof checker, we guarantee that all the proofs are correct provided that the statements are appropriate. Formal methods lead to much shorter reports containing a lot of abstract statements that do not need to be translated in another language. It would be much easier for a foreign expert to work only on these statements.

Our team has some experience with two different formal proof checkers, PVS and Coq (Huet et al., 2004). Some emerging developments in Coq (Gonthier and Mahboubi, 2008) seems to make the latter system very efficient for game theory. We expect to port our library to this system with the help of

[2] `http://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/pvslib.html`

their authors. We will then be able to draw some comparisons between these systems. We also expect that some other aspects of modeling decision in artificial intelligence can benefit from formal methods to strengthen their perception to non-specialist including political and economical executives.

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The Functionality-Security-Privacy Game

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Abstract. Privacy preservation in the information society is in many respects parallel to environment preservation in the physical world. In this way, "green ICT services" are those achieving functionality and security with minimum invasion of the privacy of individuals, where such an invasion can be regarded as a kind of pollution as harmful in the long run to their moral welfare as physical pollution is to their physical welfare. Depending on the type of service, individuals can be users, data owners or respondents having supplied data. We show that the conflict between functionality, security and privacy can be viewed as a game between several players whose interests differ. If the game is properly formulated, its equilibria can lead to protocols conciliating the functionality, security and privacy interests of all players.

Keywords: Privacy, Security, Functionality, Game theory, Mechanism design.

1 Introduction

The starting point of this paper is that *privacy preservation in the information society is analogous to environment preservation* in the physical world. With this idea in mind, "green" or "clean" information and communications technologies (ICT) are those offering functionality and security with minimum invasion of the privacy of individuals. Such an invasion can be regarded as a virtual pollution as harmful in the long run to the moral welfare of individuals as physical pollution is to their physical welfare. The moral value of privacy was of course previous to the information society: privacy is a fundamental right of the individual, acclaimed by the United Nations in article 12 of the Universal Declaration of Human Rights (1948). In fact, the lack of privacy undermines most of the remaining fundamental rights (freedom of speech, democracy, etc.).

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1.1 The Pollutants of Privacy

One source of privacy pollution has to do with *privacy-unfriendly security*. Security in general and information security in particular are fields of increasing relevance. The advent of R+D funding priorities related to security is an indication and a confirmation of the previous statement: the European Union's 7th Framework Programme (FP7) includes, among its ten thematic priorities, one named "Security". Additionally, security and privacy are nearly ubiquitous in a second FP7 thematic priority among those ten, the one named "Information and Communication Technologies (ICT)". Especially focused to security is the first of the seven challenges addressed by this priority, named "Pervasive and Trusted Network and Service Infrastructures".

This interest of governments and corporations in security is partly justified by the social alarm in front of the global threat of international terrorism. With the argument of such a threat, the European Union and several overseas states have adopted shock measures on information security. Beyond the obvious technological challenge of securing and analyzing communications on a mass scale, a new, subtler and unaddressed challenge arises: security must be increased while minimizing the loss of privacy for the citizens. This second challenge becomes especially pressing after the measures adopted in Europe about keeping track of phone calls and e-mail messages. The tendency of governments is to sacrifice privacy for security (*e.g.* the former UK security and intelligence coordinator recently asserted that anti-terror fight will need privacy sacrifice [15]). Similar conflicts between privacy and security appear in connection with the identity theft in bank transactions by the organized crime [24]. The attraction of focusing on security technologies while putting aside privacy and other rights of the individuals is very strong as it can be inferred even from the FP7 stance about privacy, which is mentioned as an ethical-legal issue rather than as technological objective *per se*. In general, increasing security without significantly decreasing privacy is one of the main challenges faced by the information society.

A second source of privacy pollution is *privacy-unaware* (let alone privacy-unfriendly) *functionality*. Many ICT services, like search engines (*e.g.* Google, Yahoo, etc.), social networks and most Web 2.0 services, the constellation of Google additional services (Calendar, Streetview, Latitude), etc., concentrate on offering enticing functionality for users while completely disregarding their privacy. They are like powerful cars which pollute a lot. Whenever one of such services boasts a privacy pledge, it refers to privacy in front of third parties (*e.g.* the service provider commits to abstaining from unauthorized transfer of user data to third parties) but not to privacy of the user in front of the service provider itself (*e.g.* Google learns and records all queries submitted to its search engine, all events entered in the Calendar, all e-mails sent/received with Gmail, etc.). Hence, each service provider (Google, Yahoo, Facebook, etc.) becomes a big brother in the purest Orwellian sense.

1.2 Contribution and Plan of This Paper

We show that the conflict between functionality, security and privacy can be expressed as a game between some players, whose number, nature and utility functions depend on the specific application scenario. If the game is designed to deter rational deviations by the players, its equilibria can lead to protocols conciliating the functionality, security and privacy interests of the players.

Section 2 states the general game-theoretic approach. Section 3 illustrates the approach in several application scenarios, namely, statistical disclosure control, car-to-car communication, private information retrieval and social networks. Conclusions and lines for future research are summarized in Section 4.

2 A Game-Theoretic Framework

The trade-off between privacy, security and functionality can be viewed as a game [23,19,21] between an individual and a system (which may be an organization, a computer system or an ICT service):

- The individual wishes to obtain *functionality* with minimum *privacy* loss. Think of an e-commerce transaction by way of illustration: regarding functionality, the buyer expects the electronic shop to have a good catalog and be convenient to use; regarding privacy, the buyer wants to pay without her credit card number being stolen, and she wants the system to keep her purchase record confidential or, even better, not to keep any record about her at all, unless she is offered some rewards, like improved customer relationship or discounts.
- The system's primary goals are *functionality* and *security* of its own strategic information (accounting, inventory levels, digital content if the system trades with information as a commodity, etc.); this kind of security could also be termed system privacy as opposed to privacy of individuals, but we stick to the term security to avoid confusion. Individual privacy (related to customers or subjects the system holds information about) is, at best, a secondary goal. In the e-commerce example, the system is the e-shop, which wants to offer functionality to customers, while keeping its backoffice information secure and confidential (stocks, sales, etc.); regarding customers, the usual aim of the e-shop is to profile them as much as possible in view of improving the customer relationship management.

Hence, functionality is a goal shared by the individual and the system, but privacy and security are not. We have so far described the functionality-security-privacy game as one between two players: the individual user and the system. However, when the system holds data about third parties, *e.g.* if the system is a database holding records about individual respondents, then those respondents are a third player (or a community of "third players"), whose main goal is to preserve their privacy; the information released by the system to individual users should not be linkable to specific individual respondents. Key questions whose answer could be obtained with such a game-theoretic framework include the following:

1. How much privacy is the individual user willing to surrender in exchange for increased functionality? How much functionality can be offered for a certain level of user privacy?
2. Can the system offer the required functionality and user privacy while guaranteeing its own security (and the privacy of respondents, if there are respondents)?

In [8], this author presented a three-dimensional framework for privacy in databases. The three dimensions correspond to the three types of privacy sought: user privacy, system privacy (called above system security) and respondent privacy. That paper showed the independent nature of those types of privacy and the need to trade them off; it also assessed how well various technologies for database privacy managed to provide the three types of privacy; the comparison included privacy information retrieval (PIR, [5]), noise-based privacy-preserving data mining (noise-based PPDM, [1]), cryptographic privacy-preserving data mining (crypto PPDM, [17]) and statistical disclosure control (SDC, [14]). The main limitations of [8] were the following: i) only database privacy was discussed, but not privacy in other ICT applications; ii) functionality was not considered; and iii) the assessment of technologies was qualitative and lacked quantitative criteria.

Progressing to a general game-theoretic framework requires going through the steps below:

1. If there are n players, $\{1, 2, \cdots, n\}$ in the functionality-security-privacy game, identify the set of possible strategies S_i of each player i. If player i selects a strategy $s_i \in S_i$, denote by $s = (s_1, \cdots, s_n)$ the vector of strategies selected by the players.
2. For each player i, find functionality metrics $f_i(s)$, security metrics $sec_i(s)$ and privacy metrics $p_i(s)$. Such metrics will be application-dependent, as argued below.
3. For each player i, find utility functions $u_i(s)$ mixing the above functionality, security and privacy metrics. The utility for the individual user is likely to be a mixture of functionality and privacy; for the system, it is likely to be a mixture of functionality and security; for the respondent, it is likely to be basically privacy. Finding the optimal mixture function is in itself a decision-theoretic problem faced by each player. If we denote by s_i the strategy played by player i and by s_{-i} the $(n-1)$-dimensional vector of the strategies played by the other players, we can write $s = (s_i, s_{-i})$. Whatever the utility mixture, if s_i' is an alternate strategy to s_i, it should hold that:

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \text{ if } f_i(s_i) = f_i(s_i') \text{ and } sec_i(s_i) = sec_i(s_i') \text{ and } p(s_i) = p(s_i')$$

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ if } \begin{cases} f_i(s_i) > f_i(s_i') \text{ and } sec_i(s_i) \geq sec_i(s_i') \text{ and } p(s_i) \geq p(s_i') \\ f_i(s_i) \geq f_i(s_i') \text{ and } sec_i(s_i) > sec_i(s_i') \text{ and } p(s_i) \geq p(s_i') \\ f_i(s_i) \geq f_i(s_i') \text{ and } sec_i(s_i) \geq sec_i(s_i') \text{ and } p(s_i) > p(s_i') \end{cases}$$

4. Use mechanism design [19] to find game rules which, combined with the players' utility functions, ensure that:
   (a) Players will not deviate from the prescribed rules;
   (b) The game results in equilibria acceptable to all players.

3 Specific Application Scenarios

For the sake of concreteness, we discuss the above game-theoretic framework in several specific application scenarios, namely: statistical disclosure control, user-private information retrieval, car-to-car communication and social networks.

3.1 Statistical Disclosure Control

Statistical disclosure control (shortened as SDC and known as Privacy-Preserving Data Mining in the database community, see *e.g.* [10]) seeks to protect statistical data in such a way that they can be publicly released and mined by users without giving away private information which can be linked to specific individuals or entities (the respondents who supplied the data). This is a case of functionality-security-privacy game, which was partially stated for the specific case of tabular data in [18].

One challenge is to model SDC as a functionality-security-privacy game for any kind of data, which requires quantifying risks and pay-offs in order to construct suitable utility functions. Players are the database owner, users and respondents:

- The owner (typically a national statistical office) wants security, that is, to make sure that no data misuse will occur; hence, the owner's utility function u_o is proportional to the probability of users *not* misbehaving.
- The respondent wants privacy, that is, guarantees that no user or intruder will be able to link her responses with her identity; hence, the respondent's utility function u_r is proportional to the disclosure risk in the published data set.
- The user wants functionality, that is, maximum analytic flexibility and accuracy; hence, the user's utility function u_u is inversely proportional to the information loss caused by the anonymization process (from the user's point of view, the best anonymization is just releasing the original data unaltered).

Thus, SDC turns out to be a case in which the utilities of the players are "pure", that is, each player type is interested in one and only one property. However, at a closer look, the owner's and the respondent's utility do have some correlation, because they are both maximized when the user is "under control": if disclosure risk is low, the user has less chances to misbehave.

The strategies of the SDC game can be summarized as: i) for the data owner, the anonymization procedure and the computer security defenses chosen; ii) for the respondent, whether to respond accurately, to respond inaccurately or not to respond at all; iii) for the user, whether to make use of the anonymized data (and maybe pay for them) or to reject the anonymized data.

The utility functions should be such that it is in each player's own interest to behave rationally in order to maximize their utility. In this case, the equilibria of the game would yield parameterizations of the anonymization procedures which optimize the trade-off between functionality, security and privacy, that is, between information loss, security and disclosure risk.

3.2 User-Private Information Retrieval

Private information retrieval (PIR) is normally modeled as a game between two players: a user and a database. The user retrieves some item from the database without the latter learning which item was retrieved. Most current PIR protocols are ill-suited to provide PIR from a search engine or large database: i) their computational complexity is linear in the size of the database; ii) they assume active cooperation by the database in the PIR protocol. If the database cannot be assumed to cooperate, two pragmatic approaches can be adopted by the user to keep her query interests private:

1. *Standalone*. A program in the user's computer keeps submitting fake queries to the search engine at random times, so that when the user submits a real query, the search engine cannot distinguish it from the fake queries; this is the TrackMeNot approach [13]. An alternative standalone approach which is less resource-consuming is for the user to mask her real query keywords by adding some fake keywords with similar frequency; in this way, the number of queries submitted equals the number of real user queries; this is the way our GooPIR prototype operates [11].
2. *Peer-to-peer (P2P)*. A user gets her queries submitted on her behalf by other users in the P2P community. In this way, the database still learns which item is being retrieved, but it cannot obtain the real query histories of users, which become diffused among the peer users. We named this relaxation user-private information retrieval and published it in [12].

PIR and its standalone and P2P relaxations can be modeled as functionality-privacy games. The user wants to query the search engine or database with as much flexibility and speed as possible (functionality), while keeping her query history private (privacy). In fact, the main problem of strict PIR is that, while it achieves very good privacy, it offers only very restricted functionality. On the other hand, the P2P relaxation is a game with several players: the peers and the database, and user privacy means privacy in front of the database and in front of the rest of peer users. Determining the optimal values of parameters such as the number of peers and the connectivity among peers can be an outcome of this modeling process.

A further challenge is to design a new relaxation of PIR offering also system security. If the database information items are not free and have different prices, system security means that users should not be able to retrieve items without paying the established fees. Thus, the game should not be just functionality-privacy, but functionality-security-privacy.

3.3 Car-to-Car Communication

Vehicular *ad hoc* networks (VANETs) allowing car-to-car communication are expected to be commercially available in cars manufactured from 2011 onwards [3]. There are several standards for VANET communication under way both in the United States (DSRC, Dedicated Short Range Communications, IEEE 802.11p)

and Europe (Car2Car Consortium). VANETs will allow vehicles to disseminate announcement messages about road conditions to other vehicles (icy road, traffic jam, etc.), in order to improve traffic safety and efficiency.

Security is a clear requirement of such announcement messages: these must be trustworthy, because false messages could seriously disrupt traffic, cause accidents or cause certain areas to become deserted and hence easy targets for criminals. There are two approaches in the literature: *a posteriori* and *a priori*. Under the *a posteriori* approach, punitive action is taken against vehicles which have been proven to have originated false messages (*e.g.* [16]); hence, means are required to identify malicious vehicles. Under the *a priori* approach, the goal is to prevent the generation of false messages (*e.g.* [20]): a message is given credit only if it is endorsed by a number of nearby vehicles greater than a certain threshold.

Privacy is also a requirement in VANET communication, although perhaps less compelling for carmakers and policy makers. Indeed, it would not be very fair if the collaboration of a driver to improve traffic safety and efficiency by generating or endorsing announcements forced her to disclose her identity and location. Note that knowing the mobility pattern of someone reveals a lot of private information: the way of driving tells a lot about an individual's character (nervous, calm, etc.), her whereabouts give information on her work and social habits, etc. *Thus VANET communication is a case of the functionality-security-privacy game* mentioned above: functionality for individual cars and the overall traffic system, security for the traffic system and privacy for the individual cars.

Privacy can be added to the *a posteriori* security approach by using pseudonyms or more advanced cryptographic primitives like group signatures. A trusted third party is needed who can open the identities of honest vehicles.

Adding vehicle privacy to the *a priori* security approach can imply vulnerability against the Sybil attack, in which a vehicle generates a false message and takes advantage of anonymity to compute itself as many endorsements as required. A private *a priori* scheme for VANET announcements based on threshold signatures and resistant against the Sybil attack was recently proposed in [6]. In that paper, irrevocable anonymity for cars generating or endorsing messages is provided.

A posteriori countermeasures alone are not sufficient. They can indeed deter some rational attackers, but they cannot prevent damage by irrational attackers (*e.g.* terrorists willing to risk anything to disseminate false messages aimed at causing an accident or a massive jam). On the other hand, *a priori* countermeasures alone are not sufficient either: if a number of nearby attackers greater than the preset threshold collude, they can generate a valid false message. If they enjoy irrevocable anonymity, those colluders cannot even be traced.

The above shortcomings have been recently addressed in [25] by:

- Designing new group signatures such that: i) they offer anonymity revocability; ii) signatures on different messages by different signers are indistinguishable; iii) signatures on the same message by different signers are distinguishable (so that the Sybil attack can be detected); iv) they are computationally efficient.

– Based on the above signatures, giving a solution to the VANET functionality-privacy-security game offering both *a priori* and *a posteriori* security and privacy for honest vehicles. A traffic load-dependent threshold for *a priori* security and a trusted third party to handle revocation in the case of a posteriori security are used. The solution allows finding the optimal threshold for given traffic conditions.

3.4 Social Networks

Social networks have become an important web service with a broad range of applications: collaborative work, collaborative service rating, resource sharing, friend search, etc. Facebook, MySpace, Xing, etc. are well-known examples. In a social network, a user publishes and shares information and services. In some social networks, the user can specify how much it trusts other users, by assigning them a trust level. It is also possible to establish several types of relationships among users (like "colleague of", "friend of", etc.). The trust level and the type of a relationship are used to decide whether access is granted to resources and services being offered (*access rule*).

The availability of information on relationships (trust level, relationship type) has increased with the advent of the Semantic Web and raises privacy concerns: knowing who is trusted by a user and to what extent discloses a lot about the user's thoughts and feelings. For a list of related abuses see [2]. Also, it is known that some human resources departments use to look up job applicants in a well-known and privacy-weak social network like Facebook to find out more about their personality. Hence, *social networks are another instance of the functionality-security-privacy game*; functionality means resource availability and flexibility of access, which should be possible even if there are only indirect relationships between the resource owner and the resource requestor (as considered in [4]); security refers to the resource owner, who wants to make sure that her resource will be accessed only by users whom she trusts enough; privacy refers to users, who should be able to access resources (or help other users in accessing them) with minimum disclosure of the identities of their relationships.

In [9], a new protocol was described which offers private relationships in a social network while allowing resource access through indirect relationships without requiring a mediating trusted third party. Thanks to homomorphic encryption, this scheme prevents the resource owner from learning the relationships and trust levels between the users who collaborate in the resource access (these users are intermediate relationships between the resource owner and the resource requestor). In this way, the number of users who might refuse collaboration due to privacy concerns is minimized. This results in increased functionality, *i.e.* availability.

Research avenues in social networks are:

– State the problem of resource access via indirect private relationships as a functionality-security-privacy game.
– Use that game-theoretic formulation to design access rules which optimize the pay-offs of users in terms of functionality, security and privacy.

4 Conclusions and Future Research

We have presented the conflict between functionality, security and privacy as a game, and we have illustrated the possible ramifications of such a game-theoretic framework in a number of specific application scenarios.

Future research will involve: i) turning those ramifications into concrete results for each scenario by addressing the challenges identified above; and ii) tackling scenarios not considered here. This requires specifying utility functions and strategies for players.

A final caution is in order, though. As pointed out in [7], there are some fundamental differences between game theory, where players are not supposed to deviate from the game rules, and cryptographic or security protocols, where deviation must be accounted for. Such a shortcoming can be sometimes mitigated with a careful design of the game mechanism (the game rules), *i.e.* so that it is in the players' own interest not to deviate (take for example the Vickrey auction mechanism, [22]). However, for some practical application scenarios, no game mechanism may exist which discourages all possible player deviations: such is the case in the scenario of secure multiparty computation considered in [19], where the class of non-cooperatively computable functions currently seems to be the only one whose computation can be modeled as a game.

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Toward the Theory of Cooperative Games under Incomplete Information

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Abstract. In the conventional cooperative games, it is assumed that the payoff of each coalition is known. However, in the real world problems, there may exist situations in which some coalitional values are unknown. In this paper, we take the first step toward the theory of cooperative games under incomplete information of coalitional values. We define concepts related to such incomplete games. We investigate the solution concepts in a special case when only values of the grand coalition and singleton coalitions are known. We show that there exists a focal point solution which is commonly suggested in many points of view.

Keywords: Cooperative game, lower game, upper game, reference point game, Shapley value, nucleolus.

1 Introduction

The cooperative game theory provides useful tools to analyze cost allocation, voting power, distribution of profit, and so on. The problems to be analyzed by the cooperative game theory include n entities called players and are usually expressed by characteristic functions called games which map each subset of players to a real number. The solutions to the problems are given by the set of n-dimensional real numbers or value functions which assign a real number to each player. Such a real number can show the cost borne by the player, power of influence, an allocation of the shared profits, and so on. Several solution concepts for cooperative games have been proposed. As representative examples of solution concepts, the core, the Shapley value [6] and the nucleolus [7] are well-known. The core can be represented by a set of solutions while the Shapley value and the nucleolus are one point solutions.

A classical approach of von Neumann and Morgenstern [3] to cooperative games assumes that the values of all coalitions are given. However, in the real world problems, there may exist situations in which the values of some coalitions are unknown. Such cooperative games under incomplete information have not yet investigated considerably. Polkowski and Araszkiewicz [5] considered the estimation problem of coalitional values as well as the Shapley value of such a game from the partial data about coalition structures. They applied the rough

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set ideas of approximations [4] and defined the lower and upper values of a game as well as the Shapley values.

In this paper, we treat cooperative games under incomplete information in a simpler way. While Polkowski and Araszkiewicz estimate coalitional values from partial data about the coalition structures, we assume that some coalitional values are known but the others are unknown. For the sake of simplicity, in this paper, a cooperative game under incomplete information is called "an incomplete game" while a cooperative game with complete information is called "a complete game". We defined the lower and upper games associated with the given incomplete game. We assume the superadditivity, and show that the lower game is superadditive but the upper game is not. We show the existence of superadditive games which attain values of the upper game. In order to initiate the investigation of the solution concepts to incomplete games, we consider the simplest case when only values of grand and singleton coalitions are known. We apply the Shapley value and the nucleolus to complete games related with the given incomplete games. We show that there exists a solution agreed with many view points.

This paper is organized as follows. In Section 2, we introduce the classical cooperative game and well-known solution concepts. In Section 3, we present cooperative games under incomplete information and investigate the lower and upper games as well as some games associated with them. To consider the solution concepts, we concentrate a special case when values of grand and individual coalitions are only known. We investigate the Shapley values of complete games related with the given incomplete game in Section 4, while we investigate the nucleoluses in Section 5. In Section 6, concluding remarks and future directions are given.

2 Classical Cooperative Game Theory

Let $N = \{1, 2, \ldots, n\}$ be the set of players and $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. A classical cooperative game, i.e., a coalitional game with transferable utility (a TU game) is characterized by a pair (N, v). A set $S \subseteq N$ is regarded as a coalition of players and function value $v(S) \in \mathbb{R}$ shows a collective payoff that players in S can gain by forming coalition S.

A cooperative game (N, v) is said to be monotone if and only if

$$v(T) \geq v(S),\ \forall S \subseteq T \subseteq N. \tag{1}$$

A cooperative game (N, v) is said to be superadditive if and only if

$$v(S \cup T) \geq v(S) + v(T),\ \forall S,\ T \subseteq N \text{ such that } S \cap T = \emptyset. \tag{2}$$

Moreover, a cooperative game (N, v) is said to be convex if and only if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T),\ \forall S,\ T \subseteq N. \tag{3}$$

The superadditivity is a natural property for giving players incentives to form a bigger coalition. The monotonicity is a weaker property than the superadditivity.

On the other hand, the convexity is a stronger property than the superadditivity so that a convex cooperative game is superadditive. The convexity can be also characterized by

$$v(T) - v(T \setminus i) \geq v(S) - v(S \setminus i),\ \forall S \subseteq T \subseteq N \setminus i,\ \forall i \in N, \tag{4}$$

where $S \setminus i$ denotes $S \setminus \{i\}$ for the sake of simplicity. This implies that the marginal contribution of a player to a coalition is nondecreasing as the coalition enlarges in the sense of set-theoretic inclusion if and only if the cooperative game is convex.

Now let us introduce basic solution concepts in cooperative games. In cooperative games, it is assumed that the grand coalition N forms. The problem is how to allocate the collective payoff $v(N)$ to all players. The solution is a payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that its component $x_i \in \mathbb{R}$ represents the allocation to player i. Many solution concepts have been proposed. We describe the core, the Shapley value and the nucleolus.

The solution is often selected from the following set of imputations of (N, v):

$$I(N,v) = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i \in N} x_i = v(N),\ x_i \geq v(\{i\}),\ \forall i \in N \right\}. \tag{5}$$

The first requirement $\sum_{i \in N} x_i = v(N)$ is called efficiency which implies that the payoff vector splits the total value $v(N)$. On the other hand, a set of requirements $x_i \geq v(\{i\}), \forall i \in N$ is called individual rationality which implies that the payoff vector guarantees a payoff x_i not worse than solo worth $v(\{i\})$.

The core $C(N, v)$ of (N, v) is the set of all payoff vectors that cannot be improved by any coalition $S \subseteq N$, i.e.,

$$C(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S),\ \forall S \subseteq N \right\} \tag{6}$$

Obviously, we have $C(N, v) \subseteq I(N, v)$. The requirements $\sum_{i \in S} x_i \geq v(S), \forall S \subseteq N$ implies that no coalition can obtain a better payoff than the sum of members' payoffs. Therefore no player is given an incentive to leave the grand coalition under a solution in the core.

An element of the core is often considered as a highly stable payoff vector. However the core can be empty and a large set of payoff vectors. While set of solutions could be useful to estimate the ranges of payoffs, one point solution rather than set of solutions would be useful as reference of the expected payoffs allocated to players.

The Shapley value and the nucleolus are known as one point solution concepts in cooperative games and these solutions are always included in the set of imputations. If the core exists, the nucleolus is always included in it while the Shapley value is not always. The Shapley value is included in the core when the cooperative game is convex.

Let $G(N)$ be the set of cooperative games with the player set N. For convenience, because the set of players is fixed as N, cooperative game (N, v) is

denoted simply by v. Let π be a vector function from $G(N)$ to $\mathbb{R}^n$ specifying a payoff vector to a cooperative game. The i-th component of π is denoted by π_i.

The Shapley value is characterized by four axioms; axioms of null player, symmetry, efficiency and additivity. Player i is said to be a *null player* if and only if $v(S) - v(S \setminus i) = 0$ for all $S \subseteq N$ such that $i \in S$. Then the axiom of null player means $\pi_i(v) = 0$ if i is a null player. The axiom of symmetry means if players i and j are equivalent in the sense that $v(S \setminus i) = v(S \setminus j)$ for all $S \subseteq N$ such that $\{i, j\} \subseteq S$ then $\pi_i(v) = \pi_j(v)$. The axiom of efficiency means the satisfaction of the efficiency, i.e., $\sum_{i \in N} \pi_i(v) = v(N)$. The axiom of additivity means $\pi(v + w) = \pi(v) + \pi(w)$ for all v, $w \in G(N)$, where the sum of games $v + w \in G(N)$ is defined by $(v + w)(S) = v(S) + w(S)$, $\forall S \subseteq N$.

The Shapley value is known as a unique function $\phi : G(N) \to \mathbb{R}^n$ satisfying these four axioms (see Shapley [6]) and its explicit form is

$$\phi_i(v) = \sum_{\substack{S \ni i \\ S \subseteq N}} \frac{(|S| - 1)!(n - |S|)!}{n!}(v(S) - v(S \setminus i)), \ \forall i \in N, \tag{7}$$

where ϕ_i is the i-th component of ϕ and $|S|$ is the cardinality of the set S.

While the Shapley value can be seen as an average of the marginal contributions $v(S) - v(S \setminus i)$, $S \subseteq N$ such that $i \in S$, the nucleolus [7] is defined as a lexicographical minimal imputation of the excesses. The excess of a payoff vector $x = (x_1, x_2, \ldots, x_n)$ for a coalition $S \subseteq N$ is defined by the difference $e(S, x) = v(S) - \sum_{i \in S} x_i$. For a payoff vector x, consider a $(2^n - 2)$ vector $\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{2^n-2}(x))$ of excesses $e(S, x)$, $S \subset N$ such that $S \neq \emptyset$, arranged in non-increasing order. Then x is said to be lexicographically smaller than y if and only if there exists $h \leq 2^n - 2$ such that $\theta_i(x) = \theta_i(y)$, for all $i < h$ and $\theta_h(x) < \theta_h(y)$. The nucleolus of v is a lexicographically minimal imputation.

It is known that the nucleolus of v denoted by $\eta(v) = (\eta_1(v), \eta_2(v), \ldots, \eta_n(v))$, can be obtained by solving a series of linear programming problems (see Kopelowitz [1]).

The following type of linear programming problem is solved:

$$\begin{aligned} &\text{minimize } \varepsilon, \\ &\text{subject to } v(S) - \sum_{i \in S} x_i \leq \varepsilon, \ \forall S \in \mathcal{V}, \\ &\qquad\qquad v(S) - \sum_{i \in S} x_i \leq e_S, \ \forall S \in \mathcal{F}, \\ &\qquad\qquad x_1 + x_2 + \cdots + x_n = v(N), \\ &\qquad\qquad x_i \geq v(\{i\}), \ i = 1, \ldots, n, \end{aligned} \tag{8}$$

where $\mathcal{V}$ and $\mathcal{F}$ are families of coalitions such that $\mathcal{V} \cap \mathcal{F} = \emptyset$ and neither $\mathcal{V}$ nor $\mathcal{F}$ include $\{N, \{i\}, i = 1, 2, \ldots, n\}$. $\mathcal{V}$ is a set of coalitions whose excesses are further minimized while $\mathcal{F}$ is a set of coalitions whose excesses have been already minimized in the previous stages. If the problem has a unique optimal solution (x, ε), we obtain the nucleolus as $\eta(v) = x$ and then the procedure is

terminated. Otherwise, we proceed to the next stage by updating $\mathcal{V}$ and $\mathcal{F}$. In order to proceed the next stage, $\mathcal{F}$ is composed of coalitions corresponding to active constraints at the current optimal solution and $\mathcal{V}$ is composed of the other coalitions.

Finally we remark the relations between solutions of a cooperative game (N, v) and its zero-normalized game $(N, \bar{v})$. The zero-normalized game of (N, v) is defined by $\bar{v}(S) = v(S) - \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. For a zero-normalized game $(N, \bar{v})$, we obviously have $\bar{v}(\{i\}) = 0$, $i = 1, 2, \ldots, n$. For the solution concepts described above we have

$$S(N, v) = S(N, \bar{v}) + (v(\{1\}), v(\{2\}), \ldots, v(\{n\})), \tag{9}$$

where $S(N, v)$ represents a set of solutions or a one point solution described above.

3 Cooperative Games under Incomplete Information

In classical cooperative games, we assume that all coalitional values are known. However, in the real world problems, there may exist situations in which some coalitional values are unknown. Therefore, we would treat such cooperative games under incomplete information. To avoid the confusion, we call cooperative games under incomplete information "incomplete games" and the conventional cooperative games "complete games".

The incomplete games can be characterized by a set of players $N = \{1, 2, \ldots, n\}$, a set of coalitions whose values are known, say $\mathcal{K} \subseteq 2^N$, and a function $\nu : \mathcal{K} \to \mathbb{R}$, where we basically assume that $\emptyset \in \mathcal{K}$ and $\nu(\emptyset) = 0$. We assume that values of singleton and the grand coalitions are at least known, i.e., $\{i\} \in \mathcal{K}$, $i = 1, 2, \ldots, n$ and $N \in \mathcal{K}$. Moreover, as is often assumed in classical cooperative game theory, we assume the superadditivity of ν, i.e.,

$$\nu(S) \geq \sum_{i=1}^{s} \nu(T_i), \ \forall S, T_i \ (i = 1, 2, \ldots, s) \in \mathcal{K} \text{ such that } \bigcup_{i=1,2,\ldots,s} T_i \subseteq S \\ \text{and } T_i,\ i = 1, 2, \ldots, s \text{ are disjoint.} \tag{10}$$

As defined above, a triple $(N, \mathcal{K}, \nu)$ can identify an incomplete game. When we consider only games under fixed N and $\mathcal{K}$, incomplete game $(N, \mathcal{K}, \nu)$ is simply written as ν.

Associated with a given incomplete game $(N, \mathcal{K}, \nu)$, we may define two complete games $(N, \underline{\nu})$ and $(N, \overline{\nu})$:

$$\underline{\nu}(S) = \max_{\substack{T_i \in \mathcal{K},\ i=1,2,\ldots,s \\ \cup_i T_i \subseteq S,\ T_i \text{ are disjoint}}} \sum_{i=1}^{s} \nu(T_i), \tag{11}$$

$$\overline{\nu}(S) = \min_{\hat{S} \in \mathcal{K},\ \hat{S} \supseteq S} \left(\nu(\hat{S}) - \underline{\nu}(\hat{S} \setminus S) \right) \tag{12}$$

From the superadditivity of ν, we have $\underline{\nu}(S) = \nu(S)$, $\forall S \in \mathcal{K}$. Moreover, when $\nu(\{i\}) \geq 0$, $i = 1, 2, \ldots, n$, from $\{i\} \in \mathcal{K}$, $i = 1, 2, \ldots, n$, we have

$$\underline{\nu}(S) = \max_{\substack{T_i \in \mathcal{K},\ i=1,2,\ldots,s \\ \cup_i T_i = S,\ T_i \text{ are disjoint}}} \sum_{i=1}^{s} \nu(T_i). \tag{13}$$

As is seen easily, $\underline{\nu}(S)$ shows the lower bound of the payoff of coalition S in the superadditive coalitional game under the incomplete information expressed by ν. On the other hand, $\overline{\nu}(S)$ shows the upper bound of the payoff of coalition S in the coalitional game. Indeed, if $\overline{\nu}(S) > \nu(\hat{S}) - \underline{\nu}(\hat{S} \setminus S)$ for some $\hat{S} \in \mathcal{K}$ such that $\hat{S} \supseteq S$, there is no superadditive complete game v satisfying $v(S) = \overline{\nu}(S)$, $v(\hat{S}) = \nu(\hat{S})$ and $v(\hat{S} \setminus S) \geq \underline{\nu}(\hat{S} \setminus S)$.

We obtain the following theorem.

Theorem 1. *Given incomplete game $(N, \mathcal{K}, \nu)$, the lower game $(N, \underline{\nu})$ is superadditive while the upper game $(N, \overline{\nu})$ is not always superadditive but monotone.*

From the incomplete information expressed by ν, we may consider a set $V(\nu)$ of possible complete games. Because we assume the superadditivity, the set eligible for the incomplete information is given by

$$V(\nu) = \{v : 2^N \to \mathbb{R} \mid v \text{ is superadditive, } v(S) = \nu(S),\ \forall S \in \mathcal{K}\}. \tag{14}$$

From the discussion above, we do not always have $\overline{\nu} \notin V(\nu)$ but $\underline{\nu} \in V(\nu)$. However, for any $T \subseteq N$, there exists $v \in V(\nu)$ such that $v(T) = \overline{\nu}(T)$. Indeed, such a complete game $v^T(S)$ for $T \neq \emptyset$ can be defined by

$$v^T(S) = \begin{cases} \overline{\nu}(S) & \text{if } S \supseteq T, \\ \underline{\nu}(S) & \text{otherwise.} \end{cases} \tag{15}$$

Note that for $T = N$, we have $v^T = \underline{\nu}$. The following theorem guarantees the superadditivity of v^T.

Theorem 2. *Game (N, v^T) defined by (15) for $T \neq \emptyset$ is a superadditive complete game.*

Because the condition for the superadditivity is represented by linear inequalities with respect to variables $v(S)$, $S \notin \mathcal{K}$, $V(\nu)$ is polyhedral. Moreover, from $\underline{\nu}(S) \leq v(S) \leq \overline{\nu}(S)$, $\forall S \subseteq N$, $V(\nu)$ is bounded. Therefore, $V(\nu)$ is a polytope. It is not easy to enumerate all vertices of $V(\nu)$ but v^T's are a part of them.

As described earlier, upper game $\overline{\nu}$ is not included in $V(\nu)$. However, it can be interpreted as a complete game that all players insist their strong powers so that all coalitions take the maximal possible values.

For the incomplete games, we would like to investigate their solution concepts. Because we assume $\mathcal{K} \supseteq \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$, the set of imputations of the incomplete game $(N, \mathcal{K}, \nu)$, denoted by $I(N, \mathcal{K}, \nu)$, can be defined by the set of imputations of any complete game in $V(\nu)$, e.g., by that of the lower

game, $I(N, \underline{\nu})$. For other solution concepts, because of the complexity, it will not be appropriate to study the general incomplete games at the beginning of the research. Therefore, in this paper, we mainly focus on the simplest case when coalitional values of only singleton and the grand coalitions are known, i.e., $\mathcal{K} = \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$. Such a case may occur in real world problems. For example, we know the value in cooperation among all players (e.g., total cost of joint implementation among n entities) and values of individual activities (e.g., cost of individual implementation) but values in cooperation among some of players are unknown without further costly investigation. Paying the further investigation expenses will not be advantageous because the allocated payoffs to many of players will be decreased by the cost.

When $\mathcal{K} = \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$, the lower and upper games become simpler as

$$\underline{\nu}(S) = \begin{cases} \sum_{i \in S} \nu(\{i\}) & \text{if } S \neq N, \\ \nu(N) & \text{if } S = N, \end{cases} \tag{16}$$

$$\overline{\nu}(S) = \begin{cases} \nu(S) & \text{if } S \text{ is a singleton,} \\ \nu(N) - \sum_{i \in N \setminus S} \nu(\{i\}) & \text{otherwise.} \end{cases} \tag{17}$$

We can observe the following relationships: $\underline{\nu}(S) \leq \overline{\nu}(S),\ \forall S \subseteq N$, $\overline{\nu}(S) = \nu(N) - \underline{\nu}(N \setminus S), \forall S \subset N$ such that $|S| > 1$, or equivalently, $\underline{\nu}(S) = \nu(N) - \overline{\nu}(N \setminus S),\ \forall S \subset N$ such that $|S| < n - 1$, and $\underline{\nu}(S) = \overline{\nu}(S) = \nu(S),\ \forall S \in \mathcal{K}$. These relationships are known as the duarity between $\overline{\nu}$ and $\underline{\nu}$. (See, for example, Honda and Okazaki [2]).

Because of the simplicity of $\mathcal{K}$, vertices of set $V(\nu)$ of possible complete games, v^T can be also represented explicitly but we omit them. Moreover, we obtain the convexity of complete games associated to ν as is shown in the following theorem.

Theorem 3. *The lower game $(N, \underline{\nu})$ as well as complete games (N, v^T), $T \subseteq N$ such that $|T| > 1$ are convex.*

From Theorem 3, the cores of $\underline{\nu}$ and v^T, $T \subseteq N$ such that $|T| > 1$ are non-empty and their Shapley values are their centers of gravity.

Moreover, as shown in the following theorem, we obtain all vertices of the set $\hat{V}(\nu)$ of all convex complete games in $V(\nu)$ in the special case.

Theorem 4. *Let $\hat{V}(\nu)$ be the set of all convex complete games in $V(\nu)$ with $\mathcal{K} = \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$. Then we have*

$$\hat{V}(\nu) = \left\{ v : 2^N \to \mathbb{R} \;\middle|\; v = \sum_{T \subseteq N,\ |T|>1} k_T v^T, \sum_{T \subseteq N,\ |T|>1} k_T = 1, \right. \\ \left. k_T \geq 0,\ \forall T \subseteq N \text{ such that } |T| > 1 \right\}. \tag{18}$$

Theorem 4 shows that the vertices of $\hat{V}(\nu)$ are v^T, $\forall T \subseteq N$ such that $|T| > 1$. Because of the significance of the convexity as well as the understandability of v^T, we might restrict ourselves in considerations of $v \in \hat{V}(\nu)$ rather than $V(\nu)$.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{K} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, N\}$. Let us consider incomplete game $(N, \mathcal{K}, \nu)$ with $\nu : \mathcal{K} \to \mathbb{R}$ defined by

$$v(\{1\}) = 8,\ v(\{2\}) = 7,\ v(\{3\}) = 3,\ v(\{4\}) = 1,\ v(\{1,2,3,4\}) = 30.$$

The lower game $\underline{v}$ and the upper game $\overline{v}$ are obtained as follows:

$$\begin{aligned}
&\underline{\nu}(\{1\}) = 8,\ \underline{\nu}(\{2\}) = 7,\ \underline{\nu}(\{3\}) = 3,\ \underline{\nu}(\{4\}) = 1,\ \underline{\nu}(\{1,2,3,4\}) = 30,\\
&\underline{\nu}(\{1,2\}) = 15,\ \underline{\nu}(\{1,3\}) = 11,\ \underline{\nu}(\{1,4\}) = 9,\ \underline{\nu}(\{2,3\}) = 10,\ \underline{\nu}(\{2,4\}) = 8,\\
&\underline{\nu}(\{3,4\}) = 4,\ \underline{\nu}(\{1,2,3\}) = 18,\ \underline{\nu}(\{1,2,4\}) = 16,\ \underline{\nu}(\{1,3,4\}) = 12,\\
&\underline{\nu}(\{2,3,4\}) = 11,\\
&\overline{\nu}(\{1\}) = 8,\ \overline{\nu}(\{2\}) = 7,\ \overline{\nu}(\{3\}) = 3,\ \overline{\nu}(\{4\}) = 1,\ \overline{\nu}(\{1,2,3,4\}) = 30,\\
&\overline{\nu}(\{1,2\}) = 26,\ \overline{\nu}(\{1,3\}) = 22,\ \overline{\nu}(\{1,4\}) = 20,\ \overline{\nu}(\{2,3\}) = 21,\ \overline{\nu}(\{2,4\}) = 19,\\
&\overline{\nu}(\{3,4\}) = 15,\ \overline{\nu}(\{1,2,3\}) = 29,\ \overline{\nu}(\{1,2,4\}) = 27,\ \overline{\nu}(\{1,3,4\}) = 23,\\
&\overline{\nu}(\{2,3,4\}) = 22,
\end{aligned}$$

We can observe that the lower game $\underline{\nu}$ satisfies the convexity while the upper game $\overline{\nu}$ does not satisfy even the superadditivity but the monotonicity.

Moreover, as an example of v^T, let us consider $T = \{1, 2\}$. v^T is obtained as

$$\begin{aligned}
&v^T(\{1\}) = 8,\ v^T(\{2\}) = 7,\ v^T(\{3\}) = 3,\ v^T(\{4\}) = 1,\ v^T(\{1,2,3,4\}) = 30,\\
&v^T(\{1,2\}) = 26,\ v^T(\{1,3\}) = 11,\ v^T(\{1,4\}) = 9,\ v^T(\{2,3\}) = 10,\\
&v^T(\{2,4\}) = 8,\ v^T(\{3,4\}) = 4,\ v^T(\{1,2,3\}) = 29,\ v^T(\{1,2,4\}) = 27,\\
&v^T(\{1,3,4\}) = 12,\ v^T(\{2,3,4\}) = 11.
\end{aligned}$$

Then we can observe the convexity of v^T.

4 The Shapley Value

In this section, we consider solutions of incomplete games $(N, \mathcal{K}, \nu)$ with $\mathcal{K} = \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$ through investigation of solutions of complete games related to it.

Before the investigation, let us define the total excess by forming the grand coalition by

$$\Delta = v(N) - \sum_{i \in N} v(\{i\}). \tag{19}$$

First let us investigate the Shapley values of lower and upper games $(N, \underline{\nu})$ and $(N, \overline{\nu})$. We have the following results.

Theorem 5. *Let $(N,\underline{\nu})$ and $(N,\overline{\nu})$ be lower and upper games associated with incomplete game $(N,\mathcal{K},\nu)$ with $\mathcal{K}=\{\emptyset,N,\{i\},\ i=1,2,\ldots,n\}$. Then, the following holds:*

$$\phi_i(\underline{\nu})=\phi_i(\overline{\nu})=\nu(\{i\})+\frac{\Delta}{n},\ \forall i\in N. \tag{20}$$

This theorem shows that the Shapley value of upper game is the same as that of lower game.

The Shapley value of (N,v^T) for $T\subseteq N$ such that $T\neq\emptyset$ is obtained as

$$\phi_i(v^T)=\begin{cases}\nu(\{i\})+\dfrac{1}{|T|}\Delta & \text{if } i\in T \text{ and } T\neq\{i\},\\ \nu(\{i\}) & \text{if } i\notin T \text{ and } |T|>1,\\ \nu(\{i\})+\dfrac{(n-1)}{n}\Delta & \text{if } T=\{i\},\\ \nu(\{i\})+\dfrac{1}{n(n-1)}\Delta & \text{if } T=\{j\}\subset N \text{ and } i\neq j.\end{cases} \tag{21}$$

From this result, Theorem 4 and the linearity of Shapley value, the set of Shapley values of player i under complete games in $\hat{V}(\nu)$ is obtained as a polytope,

$$\hat{\Phi}_i(\nu)=\left\{\theta_i=\sum_{T\subseteq N,\ |T|>1}k_T\phi_i(v^T)\ \middle|\ \sum_{T\subseteq N,\ |T|>1}k_T=1,\ k_T\geq 0,\forall T\subseteq N \text{ such that } |T|>1\right\}. \tag{22}$$

Restricting ourselves in considerations of complete games in $\hat{V}(\nu)$, we have the following result. In order to interpret k_T specifying a complete game in $\hat{V}(\nu)$, let us consider T in v^T. From the definition of v^T, T is a carrier of the zero-normalization of (N,v^T), i.e., $\bar{v}^T(T\cup S)=\bar{v}^T(T)$, $\forall S\subseteq N\setminus T$, where $\bar{v}^T$ is the zero-normalized function of v^T. To put this differently, each $i\in N\setminus T$ is a null player in complete game $(N,\bar{v}^T)$. Then k_T can be interpreted as the ratio of power showing how much excess is governed by coalition T.

If there is no further information on the incomplete game $(N,\mathcal{K},\nu)$, there is no sufficient reason why some coalition T can govern more excess than the other coalitions with the same size as T. Therefore, at reference points in $\hat{V}(\nu)$, we may assume $k_T=k_S$ if $|T|=|S|$. The Shapley values of reference point games under this assumption are shown in the following theorem.

Theorem 6. *Let $k_T\geq 0$, $T\subseteq N$ such that $|T|>1$ satisfy $\sum_{T\subseteq N,\ |T|>1}k_T=1$ and $k_T=k_S$ if $|T|=|S|$. The Shapley value of player i under a complete game $(N,\hat{v})$ with*

$$\hat{v}(S)=\sum_{T\subseteq N,\ |T|>1}k_Tv^T(S) \tag{23}$$

is obtained as

$$\phi_i(\hat{v})=\nu(\{i\})+\frac{1}{n}\Delta,\ i\in N. \tag{24}$$

Note that there are infinitely many $k_T \geq 0$, $T \subseteq N$ such that $|T| > 1$ satisfying $\sum_{T\subseteq N,\ |T|>1} k_T = 1$ and $k_T = k_S$ if $|T| = |S|$. Theorem 6 shows that, for all such combinations of k_T's, the Shapley value $\phi_i(\hat{v})$ is $\nu(\{i\}) + \frac{1}{n}\Delta$ which equals to the Shapley value $\phi_i(\underline{\nu})$ (resp. $\phi_i(\overline{\nu})$) under lower game $\underline{\nu}$ (resp. upper game $\overline{\nu}$).

The solution $x_i = \nu(\{i\}) + \frac{1}{n}\Delta$, $i \in N$ can be also interpreted as a solution where all players equally divided total excess Δ and added it to their values of individual coalitions. This can be understood more easily by considering zero-normalized version of ν. By the zero-normalization, the incomplete game is reduced to an incomplete game taking the value of total exceed at grand coalition and zero at all individual coalitions. The solution corresponding to the zero-normalization becomes $x_i = \frac{1}{n}\Delta$, $i \in N$ which shows the equal sharing of total excess Δ.

5 The Nucleolus

Now let us apply the nucleolus to incomplete game $(N, \mathcal{K}, \nu)$. The nucleoluses of lower and upper games $(N, \underline{\nu})$ and $(N, \overline{\nu})$ are obtained in the following theorem.

Theorem 7. *Let $(N,\underline{\nu})$ and $(N,\overline{\nu})$ be lower and upper games associated with incomplete game $(N, \mathcal{K}, \nu)$ with $\mathcal{K} = \{\emptyset, N, \{i\},\ i = 1, 2, \ldots, n\}$. Then, the nucleoluses $\eta(\underline{\nu})$ and $\eta(\overline{\nu})$ are obtained as*

$$\eta_i(\underline{\nu}) = \eta_i(\overline{\nu}) = \nu(\{i\}) + \frac{\Delta}{n},\ \forall i \in N. \tag{25}$$

Similar to the Shapley values of lower and upper games, the nucleoluses of upper game is the same as that of lower game. From Theorem 5, we know that the Shapley values and nucleoluses of lower and upper games are the same.

Corresponding to (21), the nucleolus of (N, v^T) for $T \subseteq N$ such that $T \neq \emptyset$ is obtained in the following theorem.

Theorem 8. *The nucleolus $\eta(v^T) = (\eta_1(v^T), \eta_2(v^T), \ldots, \eta_n(v^T))$ of (N, v^T) for $T \subseteq N$ such that $T \neq \emptyset$ is obtained as*

$$\eta_i(v^T) = \begin{cases} \nu(\{i\}) + \dfrac{1}{|T|}\Delta & \text{if } i \in T, \\ \nu(\{i\}) & \text{if } i \notin T. \end{cases} \tag{26}$$

Instead of nucleoluses of all complete games in $\hat{V}(\nu)$, we investigate the nucleoluses of the reference point games. As discussed in the previous section, the set of reference point games is defined by

$$\hat{V}^{\mathrm{ref}}(\nu) = \Bigg\{ v : 2^N \to \mathbb{R} \;\Bigg|\; v = \sum_{T\subseteq N,\ |T|>1} k_T v^T,\ \sum_{T\subseteq N,\ |T|>1} k_T = 1,$$
$$k_T \geq 0,\ \forall T \subseteq N \text{ such that } |T| > 1,$$
$$k_T = k_S,\ \forall T, S \subseteq N \text{ such that } |T| = |S| \Bigg\}. \tag{27}$$

The nucleoluses of reference point games are obtained in the following theorem.

Theorem 9. *For any $v \in \hat{V}^{\mathrm{ref}}(\nu)$, the nucleolus of pleayer i, $\eta_i(v)$ is obtained as*

$$\eta_i(v) = \nu(\{i\}) + \frac{1}{n}\Delta. \tag{28}$$

As shown in Theorem 9, the nucleolus of player i under any reference point game is $\nu(\{i\}) + \frac{1}{n}\Delta$ which is often appeared in the discussion above.

As demonstrated in the previous and this sections, the solution $x = (x_1, x_2, \ldots, x_n)$ defined by $x_i = \nu(\{i\}) + \frac{1}{n}\Delta$ is reasonable in many viewpoints for the incomplete game $(N, \mathcal{K}, \nu)$ with $\mathcal{K} = \{\emptyset, N, \{i\}, i = 1, 2, \ldots, n\}$.

Example 2. Consider the incomplete game ν discussed in Example 1. The Shapley value and the nucleolus of each player under lower game coincide and are obtained as

$$\phi(\underline{\nu}) = \eta(\underline{\nu}) = \left(\frac{43}{4}, \frac{39}{4}, \frac{23}{4}, \frac{15}{4}\right).$$

This solution is equal to the Shapley values and nucleoluses of upper game, reference point games in $\hat{V}(\nu)$, i.e., $\phi(\underline{\nu}) = \eta(\underline{\nu}) = \phi(\overline{\nu}) = \eta(\overline{\nu}) = \phi(v) = \eta(v)$, $\forall v \in \hat{V}^{\mathrm{ref}}(\nu)$. On the other hand, in the incomplete game, we obtain the total excess $\Delta = \nu(N) - \sum_{i=1}^{4} \nu(\{i\}) = 11$. We can confirm

$$\begin{aligned}\phi(\underline{\nu}) &= (v(\{1\}), v(\{2\}), v(\{3\}), v(\{4\})) + \frac{1}{4}(\Delta, \Delta, \Delta, \Delta) \\ &= (8, 7, 3, 1) + \frac{1}{4}(11, 11, 11, 11) = \left(\frac{43}{4}, \frac{39}{4}, \frac{23}{4}, \frac{15}{4}\right).\end{aligned}$$

6 Concluding Remarks and Future Research

We investigated cooperative games under incomplete information about coalitional values. Assuming the superadditivity, we introduced lower and upper games. While the lower game is superadditive, the upper game is only monotone. Each value of the upper game is attained by a superadditive complete game.

Because we were at the first stage of the study of solution concepts to incomplete games, we focused on a special case when the values of grand and individual coalitions are known. We investigated the Shapley values and the nucleoluses under incomplete games in the special case. We showed that four solutions, the Shapley values of lower and upper games and the nucleolus of them, are equal one another. The Shapley values and nucleoluses of $(2^n - 1)$ extreme games are shown explicitly. Because the set of all convex complete games is expressed as a polytope of at most $(2^n - n - 1)$ extreme games, the Shapley values of all convex games becomes a polytope spanned by those of the extreme games. We considered reference point games which players may use to evaluate the values of the given incomplete game assuming the convexity. We proved that the Shapley values and the nucleoluses of reference point games are all equal to the solution agreed by the Shapley values of lower and upper games and the nucleolus of

them. The solution would be considered one of the most reasonable solutions to the incomplete games, which could be a common focal point solution.

We believe that this study open up a new direction of cooperative games toward the theory under incomplete information about coalitional values. We have just taken the first step into the frontier. There still remain many topics for future investigation. We should research solution concepts in general cases. In general cases, it would be difficult to find a focal point solution which is commonly suggested in many viewpoints. One approach is constraining solutions to those to reference point games. In the approach, the definition of reference point games is a key issue. The second approach is an axiomatic approach to qualify solutions to incomplete games. There would exist other approaches.

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Comparison of Data Structures for Computing Formal Concepts*

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Abstract. Presented is preliminary study of the role of data structures in algorithms for formal concept analysis. Studied is performance of selected algorithms in dependence on chosen data structures and size and density of input object-attribute data. The observations made in the paper can be seen as guidelines on how to select data structures for implementing algorithms for formal concept analysis.

Keywords: formal concept analysis, data structures, algorithms, performance, comparison.

1 Introduction

Formal concept analysis (FCA) [17] is a method of qualitative data analysis with a broad outreach to other analytical disciplines. Formal concepts, i.e. maximal rectangular submatrices of Boolean object-attribute matrices, which are the basic patterns studied by formal concept analysis, are important for various data-mining and decision-making tasks. For instance, formal concepts can be used to obtain nonredundant association rules [18] and minimal factorizations of Boolean matrices [2]. Recently, it has been shown in [1] that formal concepts can be used to construct decision trees. From the computational point of view, these applications of FCA depend on algorithms for computing all formal concepts (possibly satisfying additional constraints) given an input data set. It is therefore important to pay attention to algorithms for FCA especially in case of large input data where the performance of algorithms becomes a crucial issue.

In this paper we focus on the data structures used in algorithms for computing formal concepts. Selection of the appropriate data structure has an important impact virtually on any algorithm and the decision of which structure is the optimal one for given algorithm is often uneasy. Usually, the decision depends on many factors, especially on the data being processed. Such decision has to be done wisely since selection of an inappropriate structure may lead to a poor performance or to an excessive use of resources in real programs. Algorithms for computing formal concepts are not an exception. Moreover, the situation is complicated since algorithms are usually described in pseudocode that is a

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language combining (vague) human and (formal) programming languages. This gives certain freedom to a programmer but with this freedom is tightly coupled a big piece of responsibility. For example, if a description of an algorithm contains the term "store $B \cap C$ into A", from the point of view of the algorithm description, the statement is clear and sufficiently descriptive. The term says: "store intersection of sets B and C into set A". On the other hand, from the implementation point of view, such description is ambiguous because it does not provide any information on how such intersection should be computed and how the sets should be represented.

Interestingly, the data representation issues are almost neglected in literature on FCA. The well-known comparison study [11] of FCA algorithms mentions the need to study the influence of data structures on practical performance of FCA algorithms but it does not pay attention to that particular issue. This paper should be considered a first step towards this direction. Recall that the limiting factor of computing all formal concepts is that the problem is $\#P$-complete [8]. The theoretical complexity of algorithms for FCA is usually expressed in terms of time delay [7] and all commonly used FCA algorithms have polynomial time delay [8]. Still, the asymptotic complexity does not say which algorithm is faster as many different algorithms belong to the same class. With various data structures, the problem becomes even more complicated. Therefore, there is a need for experimental evaluation which may help users decide which FCA algorithm should be used for particular type of data, cf. [11]. In this paper, we try to answer a related question: "Which data representation should be chosen for particular type of data?"

The paper is organized as follows. Section 2 presents a survey of notions of FCA and used algorithms. Section 3 describes used data structures and set-theoretical operations on these data structures. Finally, Section 4 presents experimental evaluation showing the impact of data structures on the performance and concluding remarks.

2 Formal Concept Analysis

In this section we recall basic notions of the formal concept analysis (FCA). More details can be found in monographs [6] and [3].

2.1 Survey of Basic Notions

FCA deals with binary data tables describing relationship between objects and attributes, respectively. The input for FCA is a data table with rows corresponding to objects, columns corresponding to attributes (or features), and table entries being 1's and 0's, indicating whether an object given by row has or does not have an attribute given by column. The input is formalized by a binary relation $I \subseteq X \times Y$, $\langle x, y \rangle \in I$ meaning that object x has attribute y, and I being called a *formal context* [6]. Each formal context $I \subseteq X \times Y$ induces a couple of concept-forming operators $^{\uparrow}$ and $^{\downarrow}$ defined, for each $A \subseteq X$ and $B \subseteq Y$, by

$$A^{\uparrow} = \{y \in Y \mid \text{for each } x \in A\colon \langle x, y\rangle \in I\}, \tag{1}$$
$$B^{\downarrow} = \{x \in X \mid \text{for each } y \in B\colon \langle x, y\rangle \in I\}. \tag{2}$$

Operators $^{\uparrow}\colon 2^X \to 2^Y$ and $^{\downarrow}\colon 2^Y \to 2^X$ defined by (1) and (2) form a so-called Galois connection [6]. By definition (1), $A^{\uparrow}$ is a set of all attributes shared by all objects from A and, by (2), $B^{\downarrow}$ is a set of all objects sharing all attributes from B. A pair $\langle A, B\rangle$ where $A \subseteq X$, $B \subseteq Y$, $A^{\uparrow} = B$, and $B^{\downarrow} = A$, is called a *formal concept* (*in* $I \subseteq X \times Y$). Formal concepts can be seen as particular clusters hidden in the data. Namely, if $\langle A, B\rangle$ is a formal concept, A (called an *extent of* $\langle A, B\rangle$) is the set all objects sharing all attributes from B and, conversely, B (called an *intent of* $\langle A, B\rangle$) is the set of all attributes shared by all objects from A. From the technical point of view, formal concepts are fixed points of the Galois connection $\langle^{\uparrow}, ^{\downarrow}\rangle$ induced by I. Formal concepts in $I \subseteq X \times Y$ correspond to so-called maximal rectangles in I. In a more detail, any $\langle A, B\rangle \in 2^X \times 2^Y$ such that $A \times B \subseteq I$ shall be called a rectangle in I. Rectangle $\langle A, B\rangle$ in I is a maximal one if, for each rectangle $\langle A', B'\rangle$ in I such that $A \times B \subseteq A' \times B'$, we have $A = A'$ and $B = B'$. We have that $\langle A, B\rangle \in 2^X \times 2^Y$ is a maximal rectangle in I iff $A^{\uparrow} = B$ and $B^{\downarrow} = A$, i.e. maximal rectangles = formal concepts.

The set of all formal concepts in I is denoted by $\mathcal{B}(X, Y, I)$. In this paper, we will be interested in performance of algorithms computing (listing all concepts in) $\mathcal{B}(X, Y, I)$. Note that $\mathcal{B}(X, Y, I)$ can optionally be equipped with a partial order $\leq$ modeling the subconcept-superconcept hierarchy: We put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, equivalently, iff $B_2 \subseteq B_1$). If $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ then $\langle A_1, B_1\rangle$ is called a subconcept of $\langle A_2, B_2\rangle$. The set $\mathcal{B}(X, Y, I)$ together with $\leq$ form a complete lattice whose structure is described by the Main Theorem of Formal Concept Analysis [6].

2.2 Algorithms for Computing Formal Concepts

Several algorithms for computing formal concepts have been proposed. In our experiments, we have considered three well-known algorithms—Ganter's NextClosure [5], Lindig's UpperNeighbor [13], and Kuznetsov's CloseByOne [9,10] which is conceptually close to the algorithm of Norris [14]. These algorithms are commonly used for computing formal concepts and, therefore, their efficient implementation is crucial.

A detailed description of the algorithms is outside the scope of this paper. Interested readers can find details in the papers cited above and in a survey paper [11] presenting a comparison of various algorithms for FCA. Just to recall, NextClosure and CbO are algorithms which are conceptually close because they perform the same canonicity test to prevent listing the same formal concept multiple times. The fundamental difference of the algorithm is in the strategy in which they traverse through the search space containing all formal concepts. Although mutually reducible, the algorithms are different from the practical efficiency point of view as we will see later and as it is also shown in [11]. Lindig's algorithm belongs to a different family of algorithms that compute

formal concepts and the subconcept-superconcept ordering $\leq$ at the same time. The algorithm keeps track of all formal concepts that have been computed, i.e. it stores them in a data structure. Usually, a balanced tree or a hash table is used to store concepts. The concepts are stored in a data structure for the sake of checking whether a formal concept has been found in previous steps of computation.

2.3 Representation of Formal Contexts and Computing Closures

Representation of the input data (a formal context) is crucial and has an important impact on performance of real applications. To increase the speed of our implementations, we store each context in two set-theoretical forms. This allows us to (i) increase speed of computing closures for certain algorithms and (ii) we are able to use a uniform data representation for contexts, extents, and intents. The first form is an array of sets containing, for each object $x \in X$, a set $\{x\}^\uparrow$ of all attributes of object x. Dually, the second form is an array of sets containing, for each attribute $y \in Y$, a set $\{y\}^\downarrow$ of all objects having attribute y. This redundant representation of contexts can significantly improve the speed of UpperNeighbor and CbO. Namely, given a formal concept $\langle A, B\rangle$ and $y \notin B$, we can compute a new formal concept $\langle A \cap \{j\}^\downarrow, (A \cap \{j\}^\downarrow)^\uparrow\rangle$ by intersecting sets of objects and attributes from both context representations [15].

3 Used Data Structures and Algorithms: An Overview

The most critical operations used in the algorithms for computing formal concepts are set operations and predicates that are needed to manipulate extents and intents of computed formal concepts. This means, operations of *intersection*, *union*, *difference* and predicate of *membership* ($\cap$, $\cup$, $\setminus$, and $\in$, respectively). Therefore, we focus on data structures that allow to efficiently implement these operations. In the sequel, we provide a brief overview of five data structures we deem suitable to represent sets and which will be used in our performance measurements.

3.1 Bit Array

The first data structure we use to represent a set is an array of bits. A bit array is a sequence of 0's and 1's and is very suitable for representing characteristic function of a set. If the element is present in a set, the bit at the appropriate position has value 1, otherwise it has value 0. Let us consider universe $U = \{a, b, c, d, e\}$ and bit array `01001` such bit array may represent set $\{b, e\}$. Obviously, one has to fix a total order on U in order to make such representation unambiguous. In the sequel, we are going to use a set $X = \{0, 1, \ldots, m\}$ of objects and a set $Y = \{0, 1, \ldots, n\}$ of attributes, respectively, with the natural ordering of numbers. In other words, each element in X or Y can be used as an index (in a bit array). Note that there is no danger of confusing objects with attributes because we do not mix elements from the sets X and Y in any way.

An important feature of this data structure is that all operations may be reduced to few bitwise operations performed directly by CPU. For instance, if we consider two sets from universe of size 64, their intersection may be computed on contemporary computers in one operation (bitwise logical AND). On the other hand, the size of the data structure representing a set is determined by the size of the universe and not by the size of the set itself. This may be a serious disadvantage while dealing with small sets defined in large universes—a situation that may frequently occur when dealing with sparse data sets with low densities of 1's (i.e., low percentages of 1's in the context, meaning that $|I|$ is small compared to $|X| \cdot |Y|$). In such a case, sets occupy large segments of memory and operations may not have to be so efficient as expected.

3.2 Sorted Linked List

Linked lists represent another type of a data structure suitable and frequently used for representing sets. The usage is obvious, an element belongs into a set, if and only if it is present in the list representing the set (we allow no element duplicities in lists). In the sequel, we consider a variant of linked list, where all elements are sorted w.r.t. the fixed total order (see Section 3.1). This allows us to implement set operations more efficiently. For instance, while computing an intersection of two sets, we may use so called *merging*. This means, we take both lists and repeatedly apply the following procedure:

> If the first elements of both the lists are the same, we put this element into the resulting set and we remove both the elements from the considered lists, otherwise we remove the least element.

We repeat this procedure until one of the lists is empty and then the resulting set contains only the elements that are present in both sets. Other set operations can be implemented analogously taking into account the total ordering of elements in the lists.

From the point of view of memory requirements, linked lists have certain overhead since with each element of a list we have to allocate an additional space for pointer to the next element of the list.

3.3 Array

In much the same way as in case of lists, we may store elements of a set into an array. This makes the representation of a set more space efficient since we do not have to store pointers between elements. Furthermore, if the set elements are ordered, we may optimize particular operations with the *divide et impera* approach by employing binary search [12]. For instance, to compute intersection of two sets we can go through all elements in the smaller set and using binary search we can check whether given element is also in the second set.

On the other hand, the advantages of arrays are counterweighted by the fact that arrays need some additional effort to shrink or expand their size while adding or removing elements from the set.

3.4 Binary Search Tree

Binary search tree is a data structure that has similar time complexity of the essential operations as an array. For example, when computing intersection of two sets, we proceed in a similar way as in case of arrays. We go through all elements in the smaller set and check if the elements is also in the second one. If the tree is balanced, we can do such check in a logarithmic time. This means, computation of the intersection has time complexity $O(n \cdot \log m)$.

Besides the performance, other advantages of binary trees include more efficient insertion and deletion of elements than in case of arrays. On the other hand, trees are less space efficient and need additional effort to keep them balanced to provide adequate performance. Several variants of binary search trees were proposed. For our experiments we have selected self-balancing lean-left red-black tree [4,16] for its efficiency and briefness of its implementation.

3.5 Hash Table

The last data structure we consider is a hash table. The hash tables are usually not so space efficient as the previously mentioned data structures but the time complexity of operations with hash tables is comparable. For example, computation of intersection is similar as in the previous cases: We go through all elements in the first set and check whether the elements are also in the second one. We have included hash tables since they are frequently used to implement sets in standard libraries of programming languages. There is therefore a temptation of using such library structures for representing sets. In our experiments, we have considered a variant of hash table with separate chaining [4].

3.6 Time Complexity of Operations

Fig. 1 depicts asymptotic time complexities of the elementary set operations with respect to data structures representing sets. The the time complexities are expressed in terms of the O-notation.

	$\cap$	$\cup$	$\in$
bit array	$O(\|U\|)$	$O(\|U\|)$	$O(1)$
sorted linked list	$O(m+n)$	$O(m+n)$	$O(n)$
sorted array	$O(n \cdot \log m)$	$O(n \cdot \log m)$	$O(\log n)$
binary search tree	$O(n \cdot \log m)$	$O(n \cdot \log m)$	$O(\log n)$
hash table	$O(m \cdot n)$	$O(m \cdot n)$	$O(n)$

Fig. 1. Worst-case time complexity of set operations

Remarks. The O-notation captures only a particular aspect of the time complexity and real performance of data structures may significantly differ. Sometimes, it may be useful to join several elementary operations into a single compound operation. For instance, Ganter's algorithm and CbO perform a canonicity test that prevents computing concepts multiple times. This test consists of several operations

and it seems to be practical to implement this test as one operation that takes advantage of the underlying data structure. For example, in the CloseByOne (CbO) algorithm, the test is defined as $B \cap Y_j = D \cap Y_j$, where $D = (B \cup \{y\})^{\downarrow\uparrow}$ is an intent of a newly generated concept, B is the intent of previously generated concept, and Y_j represents first j attributes. In some cases, it may be inefficient to compute the intersections of sets B and D with the set Y_j and then compare the results. In fact, it suffices to compare just one set inclusion $B \cap Y_j \supseteq D \cap Y_j$ as the converse inclusion follows from the monotony of the closure operator $^{\downarrow\uparrow}$. In other cases, however, it may be more efficient to perform the test without computing the intersections first: we check whether each $y \in D$ such that $y \leq j$ is present in B. From the point of view of the asymptotic complexity, such optimization is not essential because its complexity is the same. On the other hand, the impact on the practical performance of such optimization may be significant as we will see in the next section.

4 Experimental Performance Measurements

In this section we discuss the behavior of algorithms under various conditions. This means, we have tested algorithms for various input datasets (with various sizes and densities) and data structures.

4.1 Implementation

In order to compare properties of algorithms and data structures, we have implemented all tests in the C language. All programs share the same code base, i.e., the implementation of each algorithm is shared and only the implementation of (operations with) data structures differs. We have almost directly translated the usual pseudocode of algorithms to an equivalent code in C. In the codes of the algorithms, we have not employed any specific optimizations but to emulate environment of real applications and to reflect strengths of each data structure, we have optimized particular operations. For instance, while computing the outcomes of the concept-forming operators $^{\downarrow}$ and $^{\uparrow}$, it is necessary to compute an intersection of multiple sets, see Section 2.3. One option to compute such intersection is to repeatedly apply an operation of intersection of two sets. This approach is for example suitable for bit arrays. On the other hand, in some cases it is possible to compute intersection of multiple sets more efficiently if we consider all sets in the intersection. For instance, this applies for ordered lists, i.e., we perform a merge of all the lists simultaneously.

Remarks. The C language has been selected for testing purposes since it allows equally efficient implementations of all considered data structures. If anyone is going to use other programming language, he or she should be aware of its specific properties. For instance, in case of Java or C#, particular data structures may be less efficient due to issues connected to auto-boxing, etc.

All experiments were done on an otherwise idle computer equipped with two quad-core Intel Xeons E5345, 12 GB RAM, GNU/Linux and we have compiled all programs with GCC 4.1.2 with only `-O3` option turned on.

4.2 Performance

In our experiments, we compare running times needed to compute all formal concepts using considered algorithms with particular data structures. Since the time of computation is dependent on many factors, especially the size of the data

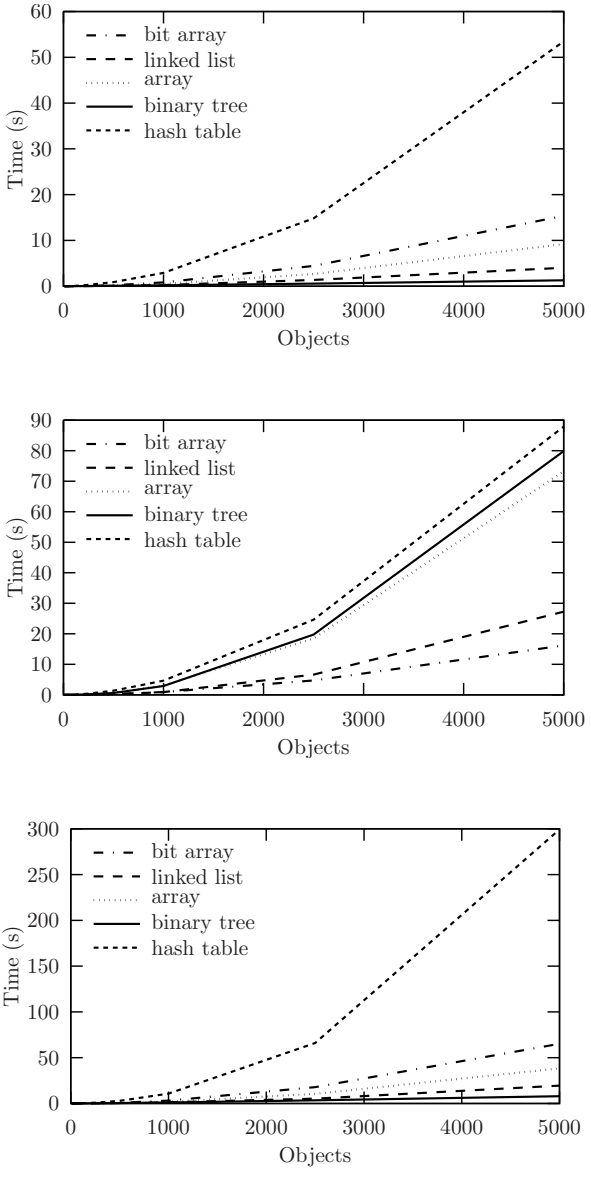

Fig. 2. Efficiency of data structures for particular algorithms—CloseByOne (top); NextClosure (middle); UpperNeighbor (bottom)

and density of 1's present in the data matrix, we have used randomly generated data tables of various properties for our experiments.

In the first set of experiments, we tried to answer the question if some structure is better for particular algorithm then other structures. To find the answer, for each algorithm we compared time it takes to compute all formal concepts in data tables with various numbers of objects, 50 attributes, where the density of 1's in the data table is 15 %. We have selected 15 % because data used in formal concept analysis are usually sparse (but there can be exceptions). The results are presented in Fig. 2. One can see that for CloseByOne and Lindig's UpperNeighbor, the tree representation of sets is the optimal one and for Ganter's NextClosure it is the bit array representation. Notice that the linked list representation provides reasonable results for all algorithms. On the contrary, hash table representation seems to provide a poor performance under all circumstances.

The previous experiment involved a fixed number of attributes and a fixed density of 1's in the data matrix. Since the dimension and density of contexts have considerable impact on performance, we performed additional experiments where the dimensions and/or density are variable.

In the next experiment, we selected tables of size 100×100 and 500×100 with various densities of 1's and compared time it takes to compute all formal concepts. Fig. 3 and Fig. 4 show times for the CloseByOne algorithm (the results for other algorithms turned out to be analogous and are therefore omitted). Notice, that we have used logarithmic scale for the time axes. This allows us to identify a point where some data structures become less efficient. Fig. 3 and Fig. 4 also indicate that linked list, binary tree and array are suitable for sparse data. Contrary to that, bit arrays are more suitable for dense data.

The point, i.e., the density, for which bit array outperforms other representations is dependent on other factors. As we can see in Fig. 3 and Fig. 4, for larger data tables this point shifts to higher densities.

The last property of data tables we are going to consider is the number of attributes. The results are shown Fig. 5, presenting computation time of NextClosure algorithm with data tables consisting of 100 objects, various numbers of attributes and 15 % density of 1's. With the exception of hash tables and binary

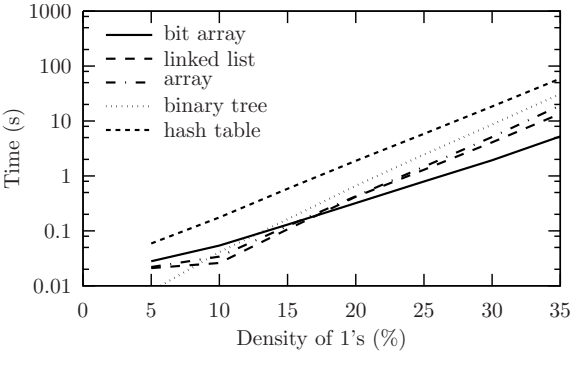

Fig. 3. Efficiency of data structures for various densities; table 100×100

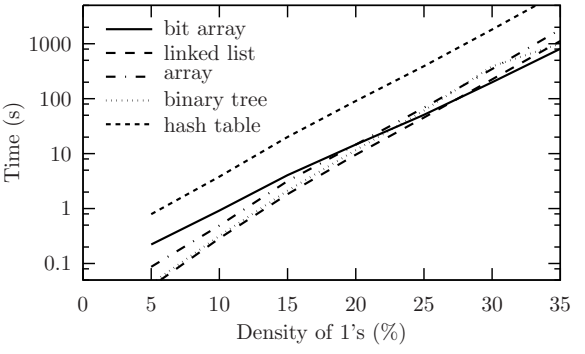

Fig. 4. Efficiency of data structures for various densities; table 500×100

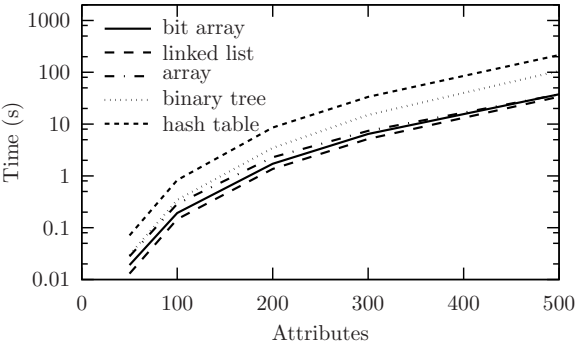

Fig. 5. Efficiency of data structures for various numbers attributes in table with 100 objects

trees, the number of attributes does not have a significant impact on the time of the computation. Notice that in case of bit arrays, linked lists, and arrays, the impact is so insignificant that we had to use logarithmic scale to make the corresponding lines distinct.

4.3 Memory Efficiency

In previous sections, we have focused on time efficiency of algorithms and data structures. Another important aspect of data structures is their space efficiency. Common feature of the Ganter's NextClosure and Kuznetsov's CloseByOne algorithms is that they are not extensively memory demanding. NextClosure has a constant memory complexity and CloseByOne has a (worst-case) linear memory complexity depending on the number of attributes (in practice, the memory consumption is typically strongly sublinear).

This means, the size of particular data structure chosen for representing sets affects practically just the size of the (the representation of) the context. The size of (the representation of) the context does not have significant influence on the

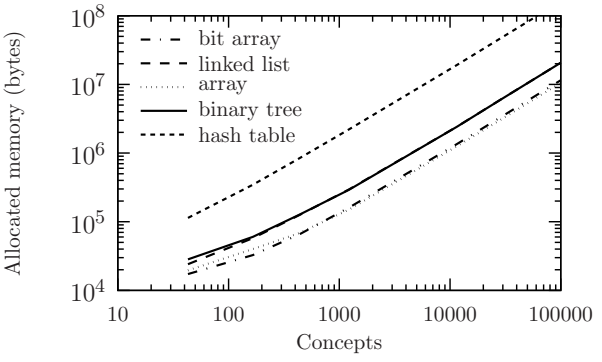

Fig. 6. Memory consumption

overall memory consumption. On the other hand, Lindig's UpperNeighbor needs to store generated concepts (or at least their intents or extents) to check whether a newly computed concept have already been generated or not. This feature may seriously affect the size of data that may be processed by this algorithm, i.e., all concepts present in the data have to fit into available memory.

In the following experiment, we have focused on the memory consumption. We have selected random data matrix with 100 attributes, various counts of objects and density of 1's 10 %. Fig. 6 shows the growth of the allocated memory dependent on the number of concepts present in the data. One can see that the bit array and array representations require approximately the same amount of memory. Furthermore, this applies also for linked list and binary tree representations. The disproportion between the assumed memory consumption and the real one may be caused by the memory management. Memory allocators in modern operating systems usually do not allocate the exact amount of memory that is requested for an object but allocate rather a larger amount.

Conclusions

This paper addresses an important but overlooked issue: which data structures should be chosen to compute formal concepts. As expected, there is no "the best" structure suitable for all types of data tables and data structures have to be wisely selected. The paper provides a survey with guidelines on how to select such data structure in dependence on the data size, used algorithm, and density. It contains our initial observations on the role of data structures in FCA in terms of the efficiency. If your data is sparse or if you have to deal with large dataset, binary search trees or linked lists are good choices. If you have dense data or smaller data table, the bit array seems to be an appropriate structure. Definitely, usage of hash tables should be avoided as it has shown to be inappropriate for computing formal concepts.

Future research will focus on considering more data structures, mixed data representations, and statistical description of factors and conditions that may have a (hidden) influence on the choice of data structure.

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Using Conditional Random Fields for Decision-Theoretic Planning

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Abstract. We propose a means of extending Conditional Random Field modeling to decision-theoretic planning where valuation is dependent upon fully-observable factors. Representation is discussed, and a comparison with existing decision problem methodologies is presented. Included are exact and inexact message passing schemes for policy making, examples of decision making in practice, extensions to solving general decision problems, and suggestions for future use.

Keywords: Utility and Decision Theory, Graphical Modeling.

1 Introduction

Although there exist various approaches to utility calculation of a decision sequence, we desire a means of describing a series of potentially-inhomogenous decisions that are fully-, partially-, or un-ordered, along with an efficient means of computing and comparing decision valuations. Traditional formulations are inadequate, as the introduction of decision reordering results in a blowup in graph complexity and size. We have devised a means of applying traditional discriminative graphical model inference methods to the task of compact exact and approximate planning under full observation, and extend it to the generation of optimal strategies for general decision problems.

For the purposes of comparing our model to existing generative decision formulations, we will make use of Shenoy's Valuation Network **(VN)** [11] approach as it is the most intuitively similar means of modeling potentially-asymmetric decision problems ("scenarios" in equivalent terminology), using complimentary graphical and numerical representations most similar to our Conditional Random Field **(CRF)** [6] approach. Additionally, we will compare our approach with the more general Markov Logic Network **(MLN)** formalism [10] in its use of absolute valuations for hard constraints, although we prefer the CRF description for compactness (see Figure 1 and later discussion in Section 3). Other decision problem formulations such as decision trees, asymmetric Influence Diagrams **(IDs)** [2], and Sequential Valuation Networks **(SVNs)** [1] will briefly appear as they relate to relative strengths or weaknesses of our approach. Additional general motivation may be found in planning for large Markov Decision Processes **(MDPs)** [7,3], although this formulation does not suit our purposes without modifying a number of limiting factors (*e.g.*, temporally-universal decision space, additive-only valuation) that compromise expressibility or compactness.

V. Torra, Y. Narukawa, and M. Inuiguchi (Eds.): MDAI 2009, LNAI 5861, pp. 126–136, 2009.

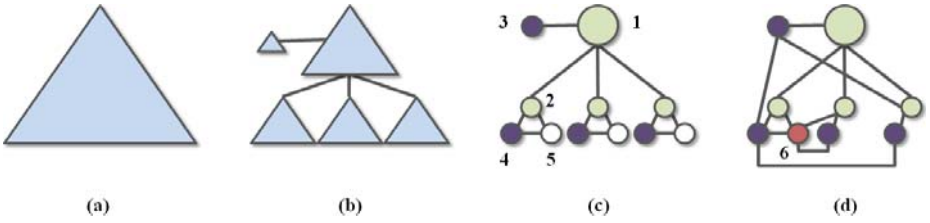

Fig. 1. General Graph Construction for a Decision Problem: **(a)** Judging potential solutions is initially viewed as a single complex operation. **(b)** However, it is possible to describe a solution as a series of solutions to subproblems plus a temporal ordering. To judge an overall solution, then, is to judge these results together. **(c)** More specifically, a solution can be judged by a numerical score, where this score depends upon the benefit from each disparate decision. The final score (1) is computed from the decision scores (2) and the ordering (3), while the decision scores depend upon the decision selected (4) and other relevant factors (5), including decision order and outside influences. The decision may be constrained by the same factors, preventing a setting of impossible decisions. **(d)** Concretely, we can name the variables (6), decisions, and orderings that influence each score, expanding out the factors and removing duplicates. All data needed to compute an overall scoring are known, so it is possible to compute and compare these scores. A model should therefore be complete (represent all data points using graph nodes), correct (represent all data point connections using edges), clear (represent unique data points and connections using one or more disparate nodes or edges), and compact (contain a minimal number of nodes and edges). We suggest that the DARN model is all four, while earlier models and other graphical formulations are not clear or compact for problems where ordering is not fixed.

In short, our approach is to generate a CRF from the full problem description, producing a discriminative graphical model for policy making that we have dubbed a Decision-Action-Reward Network **(DARN)**. Decision problems are easily expressed using this framework, and instance-specific inference is based upon an efficient utilization of Loopy Belief Propagation **(BP)** [9] using message passing [14]. Finally, optimal strategies for the general problem can be found by including additional instance likelihood factors and selecting the set of variable assignments with highest expected utility over all instances.

The remainder of this document is structured as follows: we define key terms in Section 2, explain the details of the DARN model and its inference in Sections 3 through 5, present modifications for general decision problem solving in Section 6, and conclude in Section 7.

2 Definitions

To begin, we define a generic terminology for discussion. A *decision problem* is defined as an ordered tuple $(\mathbf{D}, \mathbf{C}, \mathbf{A}, \mathbf{O}, u(\cdot))$, where:

- $\mathbf{D}$ is an ordered list of finite domains for m discrete variables
- $\mathbf{C}$ is a probability distribution for the simultaneous assignment of values of n discrete finite-domain variables

- $\mathbf{A}$ is an alphabet of m unique symbols
- $\mathbf{O}$ is a set of $m!$ or fewer unique ordered tuples of m unique elements of $\mathbf{A}$
- $u(\cdot)$ is a function whose domain is the Cartesian product of $\mathbf{O}$ and the m and n variable domains, and whose range is a set where at least one computable binary relation defines a strict total order

Specifically, $\mathbf{D}$ corresponds to all possible options ("alternatives" in equivalent terminology) for m finite decisions, $\mathbf{C}$ corresponds to the influence of outside factors in the form of n chance variables (independent of the m decisions), $\mathbf{A}$ corresponds to a unique labeling of the m decisions, $\mathbf{O}$ corresponds to allowed orderings of the m decisions (listing their labels in their temporal order), and $u(\cdot)$ corresponds to the scoring of the m decisions given the n chance variables and the order of decisions made (minimally evaluating for unallowed decisions). That is, when faced with a series of decisions to make (in one or more allowed orders), it is possible to compute the relative value of the outcome and therefore determine what decisions guarantee a highly-valued result.

Using the terminology of VNs, $\mathbf{D}$ is the set of all assignments to decision nodes (with nodes named by $\mathbf{A}$), $\mathbf{C}$ is the joint probability distribution of all of the chance nodes, $\mathbf{O}$ is a summary of the directed paths between decision nodes, and $u(\cdot)$ is the joint utility function; the values of all relevant indicator nodes are encoded in $u(\cdot)$ by evaluating to a minimal term when given a "disallowed" assignment of variables.

A *strategy* $\mathbf{S}$ for a decision problem is a set of variable assignments from $\mathbf{D}$ along an ordering appearing in $\mathbf{O}$. Incorporating the influence of $\mathbf{C}$, it is possible to determine an *expected utility* corresponding to each strategy, where this value is a member of the range of $u(\cdot)$ and is equal to an occurrence-weighted average of the values of $u(\cdot)$ produced using the strategy over an infinite timespan. When using real-valued numbers for the range of $u(\cdot)$, this expected utility is defined as $\int p(u(\cdot) = x|S)\, x\, dx$, although other formulations are used when the range of $u(\cdot)$ is otherwise defined. An *optimal strategy* is a strategy whose expected utility is maximally ordered in relation to the expected utility of all other strategies. In other words, an optimal strategy is the set of decisions expected to perform the best over time.

When solving a decision problem, the goal is to produce one or more optimal strategies. However, this may not be possible unless $\mathbf{D}$, $\mathbf{C}$, $\mathbf{O}$ and $u(\cdot)$ are all known during *policy making* (*i.e.*, reasoning performed to produce a strategy). In many applications, in fact, $\mathbf{C}$ is not known and may not even be well-approximated by a small number of samples. Rather than attempting to determine a setting and ordering of decisions that is expected to do well over all $\mathbf{C}$, it is often a useful task to instead perform policy making on specific *decision instances*, where $\mathbf{C}$ is replaced by a set of unary spike distributions whose impulses are defined by a single joint sample from the original $\mathbf{C}$. In practical terms, this corresponds to the selection of decisions that will perform well under a given setting of outside factors; we refer to this as *decision-theoretic planning*. We describe a method for performing fast exact and approximate reasoning in that case, then demonstrate how it can be extended to solve full decision problems.

3 The DARN Model

The DARN model of a decision problem is specified at two levels - graphical and numerical. The numerical and graphical specifications of the DARN model together form a CRF (expressed as a factor graph [5]) where $p(Y \mid X) = u(\cdot)$; we will not provide a full introduction to CRF modeling, but instead instruct the reader to review the definitions and examples presented by [12] if unfamiliar. The strength of this approach depends upon a factorization of $u(\cdot)$ such that $u(\cdot) = h(g_1(\cdot), g_2(\cdot), \ldots, g_m(\cdot), O)$, where $g_i(\cdot)$ is the value gained (or lost) by the ith decision, O is an ordering with $O \in \mathbf{O}$, and $h(\cdot)$ is a function that unifies these terms to reflect their impact upon the final utility valuation. This approach allows for a larger domain of representable problems than traditional approaches (as final valuation is not restricted to being purely additive or multiplicative, and as decisions may be fully-, partially-, or un-ordered by modifying $h(\cdot)$)

The details of the CRF structure are as follows:

- Y consists of
  - $y_1^{dec}, y_2^{dec}, \ldots, y_m^{dec}$ whose domains form $\mathbf{D}$
  - y_i^{val} to record each of the $g_i(\cdot)$ factors of $u(\cdot)$
  - y_i^{ord} to record the value of O
- X consists of
  - n observed variables $x_1, x_2, \ldots, x_n$ with joint probability distribution $\mathbf{C}$
- The set of features (F, as factor nodes) consists of
  - $f_i^{dec}(Y_i^{dec}, Y, X)$, whose parameters are Y_i^{dec} and the subset of $X \cup Y$ that determines if a decision is "allowed" (*i.e.*, feasible given previous actions, environmental constraints, and decision ordering) according to the problem. Output is 0 if the decision is allowed, otherwise $-\infty$.
  - $f_i^{val}(Y_i^{dec}, Y_i^{val}, Y, X)$, whose parameters are Y_i^{dec}, Y_i^{val}, and the subset of $X \cup Y$ that form the parameters of $g_i(\cdot)$ (*i.e.*, variables influencing the value of the ith decision). Output is 0 if $Y_i^{val} = g_i(\cdot)$, otherwise $-\infty$.
  - $f^{tot}(Y_1^{val}, Y_2^{val}, \ldots, Y_m^{val}, Y_1^{ord}, Y_2^{ord}, \ldots, Y_m^{ord})$, whose parameters are the Y variable assignments (*i.e.*, factors of the joint utility) and the decision ordering. Output is $\ln(h(\cdot))$ if the ordering is allowed, otherwise $-\infty$.

Since $p(Y \mid X) = \exp\left(f^{tot}(\cdot) + \sum_i f_i^{dec}(\cdot) + f_i^{val}(\cdot)\right)$, we use variable assignments of $-\infty$ to assign a probability of 0 in cases where decision problem constraints (*e.g.*, correct valuation, decision restrictions) are violated; otherwise, $p(Y \mid X) = u(\cdot)$. As a result, an optimal strategy may be found by maximum-likelihood inference for Y (explained in Section 5). Factor weights are not used, although replacing $f^{tot}(\cdot) = ln(h(\cdot))$ with $f^{tot}(\cdot) = h(\cdot)$ or normalizing the computed utility is allowed if valuations are only used in order-based comparison; other DARN uses (*e.g.*, remorse quantification with approximate inference) will void their use.

4 Example: Sandwich-Making

We present an example for illustration, where the graphical level is shown in Figure 2. In this example, we are tasked with creating the ultimate sandwich. The available ingredients are: chicken, ham, cheese (Swiss and American), lettuce, tomato, butter, olive

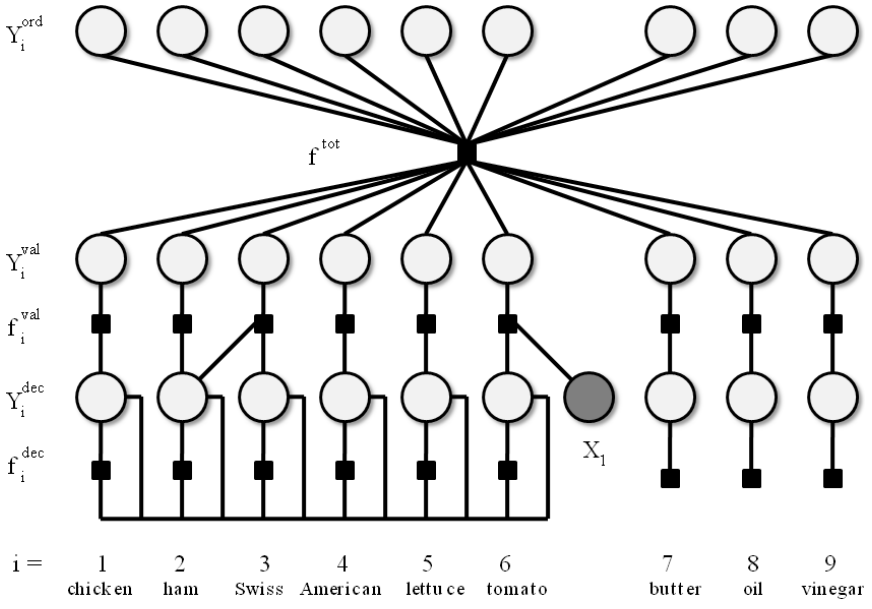

Fig. 2. Example sandwich-making DARN model (graphical level)

oil, and vinegar. Other than the last three, ingredients come in 100g increments up to 500g; the final three are used in negligible quantity and are required. Scoring depends upon the ingredients as well as the order of their placement. Total weight is limited to 1kg, including a 200g bun.

The rules that determine sandwich quality are as follows:

1. Butter is the key to a good sandwich: its appearance as the first ingredient solidifies its impact on the flavor of the sandwich (1 point).
2. Chicken is tasty (2 points per 100g), but ham is delicious (3 points per 100g).
3. Swiss cheese works well if there is an equal or greater amount of ham to pair it (2 points per 100g), but overpowers otherwise (-1 points per 100g unmatched).
4. American cheese is simple and unrefined, and provides a reasonable flavor no matter what else is present (1 point per 100g).
5. Oil and vinegar need to mix for proper impact, and will lessen the sandwich's impact if not placed immediately adjacent (-4 points).
6. Tomato is yummy (2 point per 100g), as is lettuce (2 point per 100g), but both should be used sparingly (resulting in no additional points beyond 100g).
7. Be careful about accidentally using bad tomatoes (there is a 50% chance of -1 point if tomato is included).

The details of the numerical level (using ingredient names in place of i) are below:

- $Y_{butter}^{dec}, Y_{oil}^{dec}, Y_{vinegar}^{dec} \in \{0\}$
- $Y_i^{dec} \in \{0, 1, 2, 3, 4, 5\}$ for all other i

- $g_{chicken}(Y^{dec}_{chicken}) = 2 * Y^{dec}_{chicken}$
- $g_{ham}(Y^{dec}_{ham}) = 3 * Y^{dec}_{ham}$
- $g_{swiss}(Y^{dec}_{swiss}, Y^{dec}_{ham}) = 2 * Y^{dec}_{swiss}$ if $Y^{dec}_{swiss} \leq Y^{dec}_{ham}$, otherwise $2 * Y^{dec}_{ham} - (Y^{dec}_{swiss} - Y^{dec}_{ham})$
- $g_{american}(Y^{dec}_{american}) = Y^{dec}_{american}$
- $g_{lettuce}(Y^{dec}_{lettuce}) = 2$ if $Y^{dec}_{lettuce} > 0$, otherwise 0
- $g_{tomato}(Y^{dec}_{tomato}) = 2$ if $Y^{dec}_{tomato} > 0$ and $X_{tomato} = 0$, 1 if $Y^{dec}_{tomato} > 0$ and $X_{tomato} = 1$, otherwise 0
- $g_{butter}(Y^{dec}_{butter}) = g_{oil}(Y^{dec}_{oil}) = g_{vinegar}(Y^{dec}_{vinegar}) = 0$
- $f^{val}_i(Y^{val}_i, Y^{dec}_i, \cdot) = 0$ if $Y^{val}_i = g_i(\cdot)$, otherwise $-\infty$
- $f^{dec}_i(Y) = 0$ if $\sum_i Y^{dec}_i \leq 8$, otherwise $-\infty$
- $h(\cdot) = \sum_i Y^{val}_i$ if Y^{ord}_i is pairwise unique, $-\infty$ otherwise
  (add 1 if $Y^{ord}_{butter} = 1$)
  (subtract 4 if $\mid Y^{ord}_{oil} - Y^{ord}_{vinegar} \mid > 1$)
- $f^{tot} = \ln(h(\cdot))$

One optimal choice of sandwich ingredients and order, then, is: butter, 500g ham, 300g Swiss cheese, 0g of the rest, olive oil, vinegar. Assuming that the tomatoes are not bad, another optimal sandwich replaces 100g of Swiss cheese with 100g of tomatoes; in Section 5, we will show how to calculate these arrangements.

As a sidenote, we will consider how this problem would be expressed using existing decision-theoretic constructs. A decision tree would need branches for all 362880 feasible orderings. Likewise, an ID, VN, or SVN would require linear node growth in the number of orderings, as non-loopy paths are required by their various fusion algorithms [11,1]. Similarly, an MDP would require a state space that encoded all previous actions (to restrict the next decision to remaining ingredients). In all of these cases, the result is a graph that is incomprehensibly large for human purposes. The undirected orderable approach taken by our model is therefore a serious advantage, as it allows for compact representation where decision structure has not changed beyond reordering, and where each decision has a separate decision space. The next step is to show how to use this representation for planning.

5 Inference

Using existing approaches for CRFs, inference in the DARN model is performed by Loopy BP via message passing. Although a number of optimizations are effective in reducing message density for general CRFs (*e.g.*, Sparse Belief Propagation, a generalization of linear-chain Sparse Forward-Backward [8]), sparsification is not feasible for the DARN model without compromising the encoding of hard decision constraints.[1] Instead, we propose two customized message ordering schemes designed to reduce the number of required factor function computations.

[1] Since the exponentiation of $F \setminus f^{tot}$ results in a constant value for allowed and zero for disallowed assignments, the removal of message entries is arbitrary and may disregard high-valuation assignments. This argument also rules out Sparse Mean Field approximation [13].

However, it is first necessary to describe the general message passing process used for DARN inference. Since $p(Y \mid X) = u(\cdot)$, an optimal strategy is where the CRF variable encoding, $Y_{optimal}$, is such that $Y_{optimal} = \text{argmax}_Y\, p(Y|X)$. This probability may be approximated by message accumulation, as follows:

$$
\begin{aligned}
m^{(1)}_{V\to F}(x) &= 1 \\
m^{(k)}_{V\to F}(x) &= \prod_{F'\in N(V)\setminus F} m^{(k-1)}_{F'\to V}(x) \\
m^{(k)}_{F\to V}(x) &= \sum_{X':x\notin X'} \exp(F(x,X')) \prod_{V'\in N(F)\setminus V} m^{(k-1)}_{V'\to F}(x) \\
p^{(k)}(x) &= \sum_{X':x\notin X'} \exp(f^{tot}(x,X')) \prod_{V'\in N(f^{tot})} m^{(k-1)}_{V'\to f^{tot}}(x)
\end{aligned}
$$

where:

- k is the number of an arbitrary iteration
- x is one or more variable assignments; messages with no trailing x cover all assignments
- $m^{(k)}_{A\to B}(\cdot)$ is the message from graph node A to graph node B during iteration k (regarding the parameters)
- F and F' are arbitrary factor nodes
- V and V' are arbitrary variable nodes
- $N(\cdot)$ is a function that returns the set of all adjacent graph nodes
- $\sum_{X':x\notin X'}$ refers to the summation over all parameter assignments excluding x
- $F(\cdot)$ is the computation of the function whose factor node is F (ignoring x if it is not in the parameter list of $F(\cdot)$)

The desire is to compute $p^{(k)}(Y,X)$ (for some k) where $p^{(k)}(Y,X) = p(Y \mid X)$ (or $\mid p^{(k)}(Y,X) - p(Y \mid X) \mid < \epsilon$ for some small ϵ, when approximating). Messages propagating from the $f^{dec}_i(\cdot)$ and $f^{val}_i(\cdot)$ nodes indicate variable assignment constraints that must be taken into account during final valuation. Exact inference will include the propagation of all of these messages, while inexact (approximate) inference will propagate a growing proportion.

The impact of message ordering on general loopy CRFs is still poorly understood, although recent publications have suggested approaches aimed at reducing the number of messages without compromising precision. For instance, [12] support the adaptation of Tree Reparameterization (**TRP**) for general CRFs, where messages are generated along cross-cutting spanning trees and tree selection is performed randomly. Although TRP is expected to perform well at propagating decision ordering information and generally passing information between local trees in a DARN model, the introduction of cross-decision restrictions or valuations (from the parametrization of Y^{dec}_i and Y^{val}_i) suggests a theoretically large expected number of iterations required to converge if using random tree selection. Using TRP as our motivation, we propose message ordering schemes designed to use small spanning trees for message propagation while using custom tree orderings that outperform random selection.

We begin with an observation: although it is not possible to reduce the number of computations of $f_i^{dec}(\cdot)$, it is possible to reduce the number of computations of $f_i^{val}(\cdot)$ and $f^{tot}(\cdot)$ by ensuring that messages sent outwards from those nodes follow certain incoming messages. Specifically, if $m^{(k)}_{f_i^{val} \to V}(x)$ is not computed before $m^{(j)}_{f_i^{dec} \to Y_i^{dec}}(x)$ and $m^{(j')}_{Y_i^{dec} \to f_i^{val}}(x)$ (for $j < j' < k$), then it is possible to register 0-valued entries for assignments in $m^{(k)}_{f_i^{val} \to V}(x)$ where $f_i^{dec}(\cdot) = -\infty$ without computing $f_i^{val}(\cdot)$. So, it is ideal to pass messages among decision-local nodes before local valuation nodes, and local valuation nodes before the global valuation node, to reduce the number of functions computed for disallowed decisions.

However, it is still possible that valuations will be calculated for sets of decisions that are not allowed, as there may be situations where setting one decision adds constraints upon another decision; these can be eliminated if $m^{(k)}_{f_i^{dec} \to V}$ propagate prior to further calculation. These two notions give us the exact message passing scheme shown as Algorithm 1 that terminates after the transmission of all $m^{(7)}_{V \to f^{tot}}$ messages.

Algorithm 1. Exact message passing

for each i **do**
  Compute and send $m^{(1)}_{X \to f_i^{dec}}$, $m^{(2)}_{f_i^{dec} \to Y_i^{dec}}$
  for each $V \in N(Y_i^{dec}) \setminus \{X, f_i^{val}, f_i^{dec}, \forall i' f_{i'}^{val}\}$ **do**
    Compute and send $m^{(3)}_{Y_i^{dec} \to V}$
  end
  Compute and send $m^{(3)}_{X \to f_i^{dec}}$, $m^{(4)}_{f_i^{dec} \to Y_i^{dec}}$, $m^{(5)}_{Y_i^{dec} \to f_i^{val}}$, $m^{(5)}_{X \to f_i^{val}}$
  for each $V \in N(Y_i^{dec}) \setminus \{X, f_i^{val}, f_i^{dec}, \forall i' f_{i'}^{dec}\}$ **do**
    Compute and send $m^{(5)}_{Y_i^{dec} \to V}$
  end
  Compute and send $m^{(6)}_{f_i^{val} \to Y_i^{val}}$
  Compute and send $m^{(7)}_{Y_i^{val} \to f^{tot}}$, $m^{(7)}_{Y_i^{ord} \to f^{tot}}$
end

The result of Algorithm 1 is to produce a set of messages such that $p^{(8)}(Y, X) = p(Y \mid X)$ using a near-optimal number of function computations; improvement is only possible when the algorithm is tailored to the specific constraints present, and can require overwhelming computational cost to determine an optimal ordering as the number of decisions and connections increases. It is important to note that messages sent more than once (*e.g.*, from X to f_i^{dec}) may be stored without recomputing, and that modifications to these messages (due to other incoming messages) is performed by zeroing existing entries without recomputing any factor functions. This "storing" may be applied to future inference performed on the same DARN model but with partially-varying X, thereby allowing for a reduction in the number of computations when only a portion of

the decision space has been changed, even to the point where no additional functions need be calculated. This property may be viewed as "training" by storing valuations on given data, then using these valuations to perform fast inference on test data (by substituting training valuations for approximately-equivalent test situations while attempting to minimize computational error). We will explore this property in greater depth in aditional publications, especially as it pertains to inference in large DARN models for Computer Vision tasks.

An approximate version of Algorithm 1 is provided in Algorithm 2, with the additional requirements that:

- $m^{(0)}_{A\to B}(x) = 0$ for all nodes A and B where $A \notin X$
- $m^{(j+1)}_{A\to B}(x) = m^{(j)}_{A\to B}(x)$ $(\forall j \geq 0)$ unless specified
- $\hat{i}$ is the set of all values not chosen from i so far
- If $\hat{i} = \emptyset$ at the end of any iteration, $\hat{i} = i$ at the start of the next iteration

These properties ensure that the set of valuated assignments is initialized to be empty and grows as k increases past m. This may incorrectly assume high-valued assignments to be disallowed, but will converge exactly as k approaches $2m$ (where worst-case selection will require $2m$ iterations for all inter-decision messages). The rate of theoretical convergence is dependent upon the structure of the modeled problem and the sampling schema used for each iteration; we suggest an initial scheme where the selection from $\hat{i}$ maximizes the number of $m^{(k)}_{V\to F}$ messages sent during each iteration.

Once $p(Y \mid X)$ is computed or approximated for all feasible decisions, and given X, inference terminates by selecting a setting of Y with highest conditional probability. If there are multiple such settings, selection is arbitrary. Algorithm 2 must be

Algorithm 2. Inexact message passing (k iterations)

for each i **do**
  Compute and send $m^{(1)}_{Y_i^{ord}\to f^{tot}}$
end
for $r = 0$ **to** $(k-1)$ **do**
  for a constant-size sample of $\hat{i}$ **do**
    Compute and send $m^{(5r+1)}_{X\to f_i^{dec}}$, $m^{(5r+2)}_{f_i^{dec}\to Y_i^{dec}}$
    for each $V \in N(Y_i^{dec}) \setminus \{X, f_i^{val}, f_i^{dec}, \forall i' f_{i'}^{val}\}$ **do**
      Compute and send $m^{(5r+3)}_{Y_i^{dec}\to V}$
    end
    Compute and send $m^{(5r+3)}_{Y_i^{dec}\to f_i^{val}}$, $m^{(5r+3)}_{X\to f_i^{val}}$
    for each $V \in N(Y_i^{dec}) \setminus \{X, f_i^{val}, f_i^{dec}, \forall i' f_{i'}^{dec}\}$ **do**
      Compute and send $m^{(5r+3)}_{Y_i^{dec}\to V}$
    end
    Compute and send $m^{(5r+4)}_{f_i^{val}\to Y_i^{val}}$, $m^{(5r+5)}_{Y_i^{val}\to f^{tot}}$
  end
end

performed for a larger value of k if no values for Y are approximated as having non-zero conditional probability. Cases where there are no non-zero valuations (*i.e.*, where approximation will never terminate) are treated as degenerate.

Using this inference mechanism, it is now possible to compute an optimal sandwich construction, albeit only if tomato quality is already known. We will next show how to take this "tomato uncertainty" into account.

6 Solving General Problems

To convert the instance-specific framework into one that will solve the general decision problem, we need only introduce information about $\mathbf{C}$ into the existing graphical structure. Specifically, we convert the CRF to a Markov Random Field (**MRF**) [4], adding a new factor function $f^{prob}(X) = p(X)$. Graphically, this factor node is connected to all (now unobserved) X_i nodes; if X contains independent subsets, each subset may have a disparate new factor node that is connected only to its corresponding X_i nodes, returning the joint probability of the elements of the subset. The message-passing computation with these additional factors (introducing $m^0_{f^{prob} \to X_i}$ messages and replacing constant $m^1_{X_i \to V}$ with computed $m^k_{X_i \to V}$ for $k = 1$) results in the joint probability of Y and X, so an optimal strategy can be determined by selecting assignments to Y that maximize this probability; this is equivalent to finding $argmax_Y \mathrm{E}\left[u(\cdot)\right]_X$.

Returning to the sandwich-making example, we set $f^{prob}(X_1) = 0.5$, so the expected utility of the ham-and-Swiss sandwich (22) is strictly greater than the ham-Swiss-tomato sandwich (21) as anticipated. Thus, it is best to avoid tomatoes if quality is unknown. We have also applied this approach to Towers of Hanoi and chess "opening book" planning, with similar findings: decisions are correctly selected to maximize utility while including the influence of priors on stone arrangements and opponent movements respectively.

Although it is possible to produce alternate graphical models that compute similar results without relying on existing (incompact) tools, we prefer the use of CRFs (and MRFs). Of particular interest, given recent trends, is the possibility of using an MLN to encode the hard constraints of $g_i(\cdot)$ computations as well as the comparable utility score. While it is provably feasible to produce an MLN that embodies the same probability distribution as our CRF-based approach (as MLNs subsume all discrete graphical models), it is not possible to produce one with comparable compactness. The reason for this lies in the encoding of the distribution such that the ordering of probabilities of variable assignments (groundings) preserves the ordering of the full utility valuation. For this, the probability of a particular ground MLN (where all variables have received assignments) must appear in the same order relative to all other ground MLNs as its utility valuation does to all other feasible valuations. As the probability of a particular ground MLN is equal to the exponentiation of summed weighted formula truths, this means that an MLN that preserves valuation ordering must have formula weights that produce an equivalent ordering. However, according to the definition of an MLN, "all groundings of the same formula will have the same weight". To produce correctly-ordered ground probabilities, therefore, the MLN must include disparate formulas for each possible utility valuation (with according weight and limited satisfiability), resulting in an exponential growth in the number of graph nodes.

7 Conclusions

Overall, the proposed DARN model is an effective tool for modeling decisions where ordering may or may not be constrained and where valuation is not necessarily additive or multiplicative. In comparison, existing approaches do not guarantee clarity and compactness on the same domain of application.

We suggest efficient algorithms for exact and inexact inference that are demonstrably correct; future work will study more sophisticated guidance during inexact inference and applications of the "training" mechanism suggested in Section 5.

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Interactive Decision Making for Hierarchical Multiobjective Linear Programming Problems

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Abstract. In this paper, we focus on hierarchical multiobjective linear programming problems where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints, and propose an interactive decision making method to obtain the satisfactory solution which reflects not only the hierarchical relationships between multiple decision makers but also their own preferences for their objective functions. In the proposed method, instead of Pareto optimal concept, the generalized Λ-extreme point concept is introduced. In order to obtain the satisfactory solution from among the generalized Λ-extreme point set, an interactive decision making method based on the linear programming is proposed, and an interactive processes are demonstrated by means of an illustrative numerical example.

1 Introduction

In the real-world decision making situations, it is often required that the goal of the overall system is achieved in the hierarchical structure, where many decision makers who belong to its sections or divisions are in action to seek their own goals independently and are affected each other. The Stackelberg games [1,9] can be regarded as multilevel programming problems with multiple decision makers. In such a Stackelberg problem, each decision maker agrees on the order in which decisions are made and seeks his/her own benefits independently. But each of their benefits is affected by the decision of the other decision makers and they do not negotiate each other. As a result, in general, the solution is not Pareto optimal. Although many kinds of techniques to obtain the Stackelberg solution have been proposed, almost all of such techniques are unfortunately not efficient in computational aspects.

In order to circumvent the computation inefficiency to obtain such a Stackelberg solution and the paradox that the lower level decision power often dominates the upper level decision power, Lai [3] and Shih et al.[8] introduced concepts of memberships of optimalities and degrees of decision powers and proposed the fuzzy approach to multilevel linear programming problems. In their approaches, each decision maker elicits his/her own membership functions for not only the objective functions but also the decision variables. Following the fuzzy decision [4] together with the membership functions, the mathematical programming problem of finding the maximum decision is formulated and solved to obtain

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the candidate of the satisfactory solution. As a generalized version of the above fuzzy approaches, Shih [6] proposed an interactive approach for integrated multilevel programming problems in a fuzzy environment, where not only imprecise information for the preference of the decision makers but also the imprecision of coefficients in the mathematical model are considered. The proposed approach was extended to apply to multilevel knapsack problems [7].

However, in such proposed fuzzy approaches for multilevel linear programming problems, the decision makers are required to elicit each of the membership functions for not only the objective functions but also the decision variables, and to update them in each of the iterations. It seems to be very difficult to elicit the membership functions for the decision variables, because, in general, the decision maker does not have preference for each of the decision variables and the number of the decision variable is rather larger than the number of the objective functions.

In this paper, we especially focus on hierarchical multiobjective linear programming problems where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints. In section 2, hierarchical multiobjective linear programming problems are formulated and the corresponding solution concept called the generalized Λ-extreme point is introduced. In section 3, an interactive algorithm is proposed to obtain the satisfactory solution from among the generalized Λ-extreme point set, where the corresponding hyperplane problem [5,10] is solved. In section 4, an interactive process of the proposed method is demonstrated by means of an illustrative numerical example.

2 Hierarchical Multiobjective Linear Programming Problems

We consider the following hierarchical multiobjective linear programming problems (HMOLP), where each decision maker (DM_r) has his/her own multiple objective linear functions together with common linear constraints.

[HMOLP]

first level decision maker : $\mathbf{DM}_1$

$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_1\boldsymbol{x} = (\boldsymbol{c}_{11}\boldsymbol{x}, \boldsymbol{c}_{12}\boldsymbol{x}, \cdots, \boldsymbol{c}_{1k_1}\boldsymbol{x})^T \tag{1}$$

second level decision maker : $\mathbf{DM}_2$

$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_2\boldsymbol{x} = (\boldsymbol{c}_{21}\boldsymbol{x}, \boldsymbol{c}_{22}\boldsymbol{x}, \cdots, \boldsymbol{c}_{2k_2}\boldsymbol{x})^T \tag{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

p-th level decision maker : $\mathbf{DM}_p$

$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_p\boldsymbol{x} = (\boldsymbol{c}_{p1}\boldsymbol{x}, \boldsymbol{c}_{p2}\boldsymbol{x}, \cdots, \boldsymbol{c}_{pk_p}\boldsymbol{x})^T \tag{3}$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is n-dimensional decision vector, $X \in \mathbf{E}^n$ is a linear constraint set of $\boldsymbol{x}$, and $\boldsymbol{c}_{ri} = (c_{ri1}, c_{ri2}, \cdots, c_{rin})$, $i = 1, \cdots, k_r, r = 1, \cdots, p$ are

n-dimensional row vectors, $\boldsymbol{C}_r = (\boldsymbol{c}_{r1}, \boldsymbol{c}_{r2}, \cdots, \boldsymbol{c}_{rk_r})^T$, $r = 1, \cdots, p$ are $(k_r \times n)$-dimensional matrices.

In this paper, we assume that each decision maker (DM_r) in HMOLPs finds his/her satisfactory solution from among Λ_r-extreme point set which is a generalized version of Pareto optimal solution set. Λ_r-extreme point [11] is defined by a cone Λ_r in objective space of DM_r as follows.

Definition 1. *$\boldsymbol{y}_r^* \in \boldsymbol{C}_r X$ is said to be a Λ_r-extreme point of $\boldsymbol{C}_r X$ to $MOLP_r$, if there is no $\boldsymbol{y}_r \in \boldsymbol{C}_r X$ such that $\boldsymbol{y}_r^* \in \boldsymbol{y}_r + \Lambda_r, \boldsymbol{y}_r^* \neq \boldsymbol{y}_r$, where $\boldsymbol{C}_r X = \{\boldsymbol{C}_r \boldsymbol{x} \in E^{k_r} \mid \boldsymbol{x} \in X\}$, $\Lambda_r \subset E^{k_r}$ is a cone, and $MOLP_r$ is DM_r's multiobjective linear programming problem formulated as follows:*

[$\mathbf{MOLP}_r$]

$$\min_{\boldsymbol{x} \in X} \boldsymbol{C}_r \boldsymbol{x} = (\boldsymbol{C}_{r1}\boldsymbol{x}, \boldsymbol{C}_{r2}\boldsymbol{x}, \cdots, \boldsymbol{C}_{rk_r}\boldsymbol{x})^T \tag{4}$$

According to the notation of Yu [11], let us denote a set of Λ_r-extreme points in DM_r's objective space as $\mathrm{Ext}[\boldsymbol{C}_r X \mid \Lambda_r]$. Unfortunately, although $\mathrm{Ext}[\boldsymbol{C}_r X \mid \Lambda_r]$ can be applied to MOLP_r, $\mathrm{Ext}[\boldsymbol{C}_r X \mid \Lambda_r]$ can not to be directly applied to HMOLP, because multiple decision makers $\mathrm{DM}_r, r = 1, \cdots, p$ in the hierarchical structure have to seek their common satisfactory solution in HMOLP. Therefore, in order to deal with HMOLP, we introduce the following extended concept called a generalized Λ-extreme point where cones Λ_r, $r = 1, \cdots, p$ are integrated in objective space of $\mathrm{DM}_r, r = 1, \cdots, p$.

Definition 2. *$\boldsymbol{y}* \in \boldsymbol{C}X$ is said to be a generalized Λ-extreme point of $\boldsymbol{C}X$ to HMOLP, if there is no $\boldsymbol{y} \in \boldsymbol{C}X$ such that $\boldsymbol{y}* \in \boldsymbol{y} + \Lambda, \boldsymbol{y}* \neq \boldsymbol{y}$, where $\boldsymbol{C}X = \{\boldsymbol{C}\boldsymbol{x} \in \mathbf{E}^{\sum_{r=1}^{p} k_r} \mid \boldsymbol{x} \in X\}$, and a cone Λ is defined as follows.*

$$\Lambda = \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_p \tag{5}$$

where $\otimes$ means Cartesian product.

Similar to $\mathrm{Ext}[\boldsymbol{C}_r X \mid \Lambda_r]$, let us denote a set of generalized Λ-extreme points in objective space of all decision makers as $\mathrm{Ext}[\boldsymbol{C}X \mid \Lambda]$, and the corresponding set of Λ-extreme points in decision space as $\mathrm{Ext}[X \mid \Lambda]$, respectively.

Since it is very difficult to deal with a cone Λ directly, in the following, let us assume that $\Lambda_r, r = 1, \cdots, p$ are polyhedral cones defined as follows:

$$\Lambda_r = \{\sum_{i=1}^{k_r} \alpha_{ri} \boldsymbol{v}_{ri},\ \alpha_{ri} \geq 0, \alpha_{ri} \in \mathbf{R}^1\} \tag{6}$$

where $\boldsymbol{v}_{ri}, i = 1, \cdots, k_r$ are generators of a cone Λ_r, *i.e.*,

$$\boldsymbol{v}_{ri} = (v_{ri1}, v_{ri2}, \cdots, v_{rik_r})^T, \tag{7}$$

and $\boldsymbol{v}_{ri}$ is assumed to satisfy the following condition.

$$\| \boldsymbol{v}_{ri} \| = \sqrt{\sum_{j=1}^{k_r} v_{rij}^2} = 1. \tag{8}$$

Using generators $\boldsymbol{v}_{ri}, i = 1, \cdots, k_r$, $(k_r \times k_r)$-dimensional generator matrix $\boldsymbol{V}_r$ of a cone Λ_r can be formulated.

$$\boldsymbol{V}_r = (\boldsymbol{v}_{r1}, \boldsymbol{v}_{r2}, \cdots, \boldsymbol{v}_{rk_r}) = \begin{pmatrix} v_{r11} & v_{r21} & \cdots & v_{rk_r1} \\ v_{r12} & v_{r22} & \cdots & v_{rk_r2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r1k_r} & v_{r2k_r} & \cdots & v_{rk_rk_r} \end{pmatrix} \tag{9}$$

Moreover, on the basis of matrices $\boldsymbol{V}_r, r = 1, \cdots, p$, $(\sum_{r=1}^{p} k_r \times \sum_{r=1}^{p} k_r)$-dimensional generator matrix $\boldsymbol{V}$ of a cone Λ can be formulated as follows.

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{V}_p \end{pmatrix} \tag{10}$$

Then, a integrated cone Λ defined by (5) can be expressed as follows.

$$\Lambda = \boldsymbol{V} \cdot \boldsymbol{\alpha}^T \tag{11}$$

where $\boldsymbol{\alpha_r} = (\alpha_{r1}, \alpha_{r2}, \cdots, \alpha_{rk_r}) \geq \boldsymbol{0}, r = 1, \cdots, p$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_p) \in \mathbf{E}^{\sum_{r=1}^{p} k_r}$.

If inverse matrices $\boldsymbol{V}_r^{-1}$ for $\boldsymbol{V}_r, r = 1, \cdots, p$ exist, an inverse matrix $\boldsymbol{V}^{-1}$ for $\boldsymbol{V}$ becomes as follows.

$$\boldsymbol{V}^{-1} = \begin{pmatrix} \boldsymbol{V}_1^{-1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_2^{-1} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{V}_p^{-1} \end{pmatrix} \tag{12}$$

In the following, (i, j)-element of $\boldsymbol{V}_r^{-1}$ is denoted as q_{rij}, *i.e.*,

$$\boldsymbol{V}_r^{-1} = \begin{pmatrix} q_{r11} & q_{r21} & \cdots & q_{rk_r1} \\ q_{r12} & q_{r22} & \cdots & q_{rk_r2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1k_r} & q_{r2k_r} & \cdots & q_{rk_rk_r} \end{pmatrix} \tag{13}$$

For generating a candidate of the satisfactory solution from among a generalized Λ-extreme point set $\mathrm{Ext}[\boldsymbol{C}X \mid \Lambda]$, each decision maker ($\mathrm{DM}_r$) is asked to specify his/her reference values $\bar{\boldsymbol{z}}_r = (\bar{z}_{r1}, \bar{z}_{r2}, \cdots, \bar{z}_{rk_r})$ [4] which are reference levels of achievement of the objective functions. Once the reference values $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{z}}_1, \bar{\boldsymbol{z}}_2, \cdots, \bar{\boldsymbol{z}}_p)$ are specified, the corresponding generalized Λ-extreme point, which is, in a sense, close to their requirement, is obtained by solving the following hyperplane problem [5].

[**HP1($\bar{\boldsymbol{z}}$)**]

$$\min_{\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1} \; x_{n+1} \tag{14}$$

subject to

$$\boldsymbol{V}^{-1} \cdot \{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \tag{15}$$

where $\boldsymbol{x}_{n+1} = (x_{n+1}, x_{n+1}, \cdots, x_{n+1})^T \in \mathrm{E}^{\sum_{r=1}^{p} k_r}$. It should be noted in HP1($\bar{\boldsymbol{z}}$) that (15) is the inequality set whose elements are $\sum_{j=1}^{k_r} q_{rij}\{-\bar{z}_{rj} + \sum_{\ell=1}^{n} c_{rj\ell} x_\ell - x_{n+1}\} \leq 0, i = 1, \cdots, k_r, r = 1, \cdots, p$.

The relationships between the optimal solution to HP1($\bar{\boldsymbol{z}}$) and the the corresponding generalized Λ-extreme point set Ext$[X \mid \Lambda]$ can be characterized by the following theorems.

Theorem 1. *If $(\boldsymbol{x}^*, x^*_{n+1})$ is a unique optimal solution to HP1($\bar{\boldsymbol{z}}$), then $\boldsymbol{x}^* \in$ Ext$[X \mid \Lambda]$.*

Proof. Assume $\boldsymbol{x}^* \notin \mathrm{Ext}[X \mid \Lambda]$, then there exist $\boldsymbol{x} \in X$ and $\boldsymbol{\lambda} \in \Lambda$ (or equivalently $\boldsymbol{\alpha} \geq \boldsymbol{0}$) such that

$$\begin{aligned} \boldsymbol{C}\boldsymbol{x}^* &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\lambda} \\ &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T. \end{aligned}$$

If $(\boldsymbol{x}^*, x^*_{n+1})$ is an optimal solution to HP1($\bar{\boldsymbol{z}}$),

$$\begin{aligned} &\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{x}^*_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ &\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T - \boldsymbol{x}^*_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ &\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x} - \boldsymbol{x}^*_{n+1}\} \leq -\boldsymbol{\alpha}^T \leq \boldsymbol{0}. \end{aligned}$$

This implies that $\boldsymbol{x}^*$ is not a unique optimal solution to HP1($\bar{\boldsymbol{z}}$). □

Theorem 2. *If $\boldsymbol{x}^* \in$ Ext$[X \mid \Lambda]$, then $(\boldsymbol{x}^*, x^*_{n+1})$ is an optimal solution to HP1($\bar{\boldsymbol{z}}$) for some reference values $\bar{\boldsymbol{z}}$, where $(-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{x}^*_{n+1}) = \boldsymbol{0}$.*

Proof. Assume that $(\boldsymbol{x}^*, x^*_{n+1})$ is not an optimal solution to HP1($\bar{\boldsymbol{z}}$). Then, there exist $\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1$ such that

$$\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0},\ x_{n+1} < x^*_{n+1},$$

where $\boldsymbol{x}_{n+1} = (x_{n+1}, \cdots, x_{n+1})^T \in \mathbf{E}^{\sum_{r=1}^{p} k_r}$. Moreover, because of $(-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{x}^*_{n+1}) = \boldsymbol{0}$, the following inequality relations must be satisfied.

$$\begin{aligned} &\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T + \boldsymbol{C}\boldsymbol{x} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ &\boldsymbol{V}^{-1}\{-\bar{\boldsymbol{z}}^T - \boldsymbol{x}^*_{n+1} + \boldsymbol{C}\boldsymbol{x} + \boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ &\boldsymbol{V}^{-1}\{-\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{C}\boldsymbol{x} + \boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{0} < \boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1} \in \Lambda$, there exists $\boldsymbol{\alpha} \geq \boldsymbol{0}$ such that $\boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1} = \boldsymbol{V} \cdot \boldsymbol{\alpha}^T$. Therefore, it holds that

$$\begin{aligned} &\boldsymbol{V}^{-1}\{-\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{C}\boldsymbol{x} + \boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0} \\ \Leftrightarrow\ &\boldsymbol{V}^{-1}\{-\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{C}\boldsymbol{x}\} \leq -\boldsymbol{V}^{-1} \cdot (\boldsymbol{x}^*_{n+1} - \boldsymbol{x}_{n+1}) \\ \Leftrightarrow\ &\boldsymbol{V}^{-1} \cdot \{-\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{C}\boldsymbol{x}\} \leq -\boldsymbol{\alpha}^T \leq \boldsymbol{0}. \end{aligned}$$

There exists $\boldsymbol{\beta} \in \Lambda$ such that $\boldsymbol{V}^{-1} \cdot \{-\boldsymbol{C}\boldsymbol{x}^* + \boldsymbol{C}\boldsymbol{x}\} = -\boldsymbol{\beta}^T \leq \boldsymbol{0}$. This implies that $\boldsymbol{C}\boldsymbol{x} + \boldsymbol{V}\boldsymbol{\beta}^T = \boldsymbol{C}\boldsymbol{x}^*$, *i.e.*, $\boldsymbol{x}^* \notin \mathrm{Ext}[X \mid \Lambda]$. □

It should be noted here that, in general, the generalized extreme point obtained by solving HP1($\bar{\boldsymbol{z}}$) does not reflect the hierarchical structure between p decision makers where the upper level decision maker can take priority for his/her objective functions over the the lower level decision makers. In order to cope with such a hierarchical preference structure between p decision makers, we introduce the decision powers $\boldsymbol{w} = (w_1, w_2, \cdots, w_p)^T \in \mathbf{E}^p$ [3] in HP1($\bar{\boldsymbol{z}}$), where the r-th level decision maker (DM_r) can specify the decision power w_{r+1} in his/her subjective manner and the last decision maker (DM_p) has no decision power. In order to reflect the hierarchical preference structure between multiple decision makers, the decision powers $\boldsymbol{w} = (w_1, w_2, \cdots, w_p)^T$ have to satisfy the following inequality condition.

$$w_1 = 1 \geq w_2 \geq \cdots\cdots \geq w_{p-1} \geq w_p > 0 \tag{16}$$

Then, the corresponding modified HP1($\bar{\boldsymbol{z}}$) is reformulated as follows:

[**HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$)**]

$$\min_{\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1} \quad x_{n+1} \tag{17}$$

subject to

$$\boldsymbol{V}^{-1} \cdot \begin{pmatrix} -\bar{\boldsymbol{z}}_{\boldsymbol{1}} + \boldsymbol{C}_1\boldsymbol{x} - \boldsymbol{x}_{n+1}/w_1 \\ -\bar{\boldsymbol{z}}_{\boldsymbol{2}} + \boldsymbol{C}_2\boldsymbol{x} - \boldsymbol{x}_{n+1}/w_2 \\ \vdots \\ -\bar{\boldsymbol{z}}_{\boldsymbol{p}} + \boldsymbol{C}_p\boldsymbol{x} - \boldsymbol{x}_{n+1}/w_p \end{pmatrix} \leq \boldsymbol{0} \tag{18}$$

The constraints (18) in HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$) are equivalently expressed as follows.

$$\sum_{j=1}^{k_r} q_{rij}(-\bar{z}_{rj} + \sum_{\ell=1}^{n} c_{rj\ell}x_\ell - x_{n+1}/w_r) \leq 0, \ i = 1, \cdots, k_r, r = 1, \cdots, p. \tag{19}$$

The relationships between the optimal solution of HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$) and generalized Λ-extreme points to HMOLP can be characterized by the following theorem.

Theorem 3. *If $(\boldsymbol{x}^*, x^*_{n+1})$ is a unique optimal solution to HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$), then $\boldsymbol{x}^* \in$ Ext[$X \mid \Lambda$].*

Proof. Assume $\boldsymbol{x}^* \notin \mathrm{Ext}[X \mid \Lambda]$, then there exist $\boldsymbol{x} \in X$ and $\boldsymbol{\lambda} \in \Lambda$ (or equivalently $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_p) \geq \boldsymbol{0}$) such that

$$\begin{aligned} \boldsymbol{C}\boldsymbol{x}^* &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\lambda} \\ &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T. \end{aligned}$$

If $(\boldsymbol{x}^*, x^*_{n+1})$ is an optimal solution to HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$),

$$\begin{aligned} & \boldsymbol{V}_r^{-1}\{-\bar{\boldsymbol{z}}_r^T + \boldsymbol{C}_r\boldsymbol{x}^* - \boldsymbol{x}^*_{n+1}/w_r\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}_r^{-1}\{-\bar{\boldsymbol{z}}_r^T + \boldsymbol{C}_r\boldsymbol{x} + \boldsymbol{V}_r \cdot \boldsymbol{\alpha}_r^T - \boldsymbol{x}^*_{n+1}/w_r\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}_r^{-1}\{-\bar{\boldsymbol{z}}_r^T + \boldsymbol{C}_r\boldsymbol{x} - \boldsymbol{x}^*_{n+1}/w_r\} \leq -\boldsymbol{\alpha}_r^T \leq \boldsymbol{0}, r = 1, \cdots, p. \end{aligned}$$

This implies that $\boldsymbol{x}^*$ is not a unique optimal solution to HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$). □

It must be observed here that for generating a generalized Λ-extreme point to HMOLP using the above theorem, uniqueness of solution must be verified. In order to test whether a current optimal solution $\boldsymbol{x}^*$ of HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$ is a generalized Λ-extreme point or not, we formulate and solve the following linear programming problem.

[**Test problem for** $\boldsymbol{x}^*$]

$$\max_{\boldsymbol{x} \in X} \sum_{r=1}^{p} \sum_{i=1}^{k_r} \epsilon_{ri} \tag{20}$$

subject to

$$\boldsymbol{V}^{-1} \cdot (\boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{C}\boldsymbol{x}) = \boldsymbol{\epsilon}^T \tag{21}$$

$$\boldsymbol{\epsilon} = (\epsilon_{11}, \cdots, \epsilon_{1,k_1}, \cdots, \epsilon_{p1}, \cdots, \epsilon_{p,k_p}) \geq \boldsymbol{0} \tag{22}$$

The following theorem guarantees that the optimal solution $\bar{\boldsymbol{x}}$ of the above test problem is a generalized Λ-extreme point to HMOLP.

Theorem 4. *Let $\boldsymbol{x}^* \in X$ be an optimal solution to HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$, and $\bar{\boldsymbol{x}} \in X$ and $\bar{\boldsymbol{\epsilon}} \geq \boldsymbol{0}$ be an optimal solution to test problem (21) for $\boldsymbol{x}^* \in X$. Then, if all $\bar{\epsilon}_{ri} = 0, r = 1, \cdots, p, i = 1, \cdots, k_r$, then $\boldsymbol{x}^* \in Ext[X \mid \Lambda]$. If at least one $\bar{\epsilon}_{ri} > 0$, then $\bar{\boldsymbol{x}} \in Ext[X \mid \Lambda]$.*

Proof. Let all $\bar{\epsilon}_{ri} = 0, r = 1, \cdots, p, i = 1, \cdots, k_r$. Then, there is no $\boldsymbol{x} \in X$ and $\bar{\boldsymbol{\epsilon}} \geq \boldsymbol{0} (\bar{\boldsymbol{\epsilon}} \neq \boldsymbol{0})$ such that $\boldsymbol{C}\boldsymbol{x}^* = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T$. This means that $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$. Let some $\bar{\epsilon}_{ri} > 0$. Then, it holds that $\boldsymbol{C}\boldsymbol{x}^* = \boldsymbol{C}\bar{\boldsymbol{x}} + \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T$. Assume $\bar{\boldsymbol{x}} \notin \mathrm{Ext}[X \mid \Lambda]$. Then, there are some $\boldsymbol{x} \in X$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$ such that $\boldsymbol{C}\bar{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T$. This means that

$$\begin{aligned} & \boldsymbol{C}\bar{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T = \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T, \\ \Leftrightarrow\ & \boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{C}\boldsymbol{x} = \boldsymbol{V} \cdot (\bar{\boldsymbol{\epsilon}}^T + \boldsymbol{\alpha}^T), \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}(\boldsymbol{C}\boldsymbol{x}^* - \boldsymbol{C}\boldsymbol{x}) = (\bar{\boldsymbol{\epsilon}}^T + \boldsymbol{\alpha}^T). \end{aligned}$$

This contradicts that $\bar{\boldsymbol{\epsilon}}$ is an optimal solution of test problem (20)-(22). □

3 An Interactive Algorithm

After obtaining a generalized Λ-extreme point $\boldsymbol{x}^*$ by solving HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$, each decision maker must either be satisfied with the current values of the objective functions, or update his/her decision power w_r and/or his/her reference values $\bar{\boldsymbol{z}}_{\boldsymbol{r}} = (\bar{z}_{r1}, \cdots, \bar{z}_{rk_r})$.

In order to help each decision maker update his/her reference values $\bar{\boldsymbol{z}}_{\boldsymbol{r}} = (\bar{z}_{r1}, \cdots, \bar{z}_{rk_r})$, trade-off information [2] between a standing objective function and each of the other objective functions is very useful. Such trade-off information between the objective functions is obtainable since it is related to the simplex multipliers of HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$.

Theorem 5. *Let $(\boldsymbol{x}^*, x^*_{n+1})$ be a unique and nondegenerate optimal solution of HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$), and let two constraints with the reference values $\bar{z}_{rj_1}$ and $\bar{z}_{rj_2}$ be active. Then, the following relations hold.*

$$-\frac{\partial(\boldsymbol{c}_{rj_1}\boldsymbol{x})}{\partial(\boldsymbol{c}_{rj_2}\boldsymbol{x})}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^*} = \frac{\sum_{i=1}^{k_r} \pi^*_{ri} q_{rij_2}}{\sum_{i=1}^{k_r} \pi^*_{ri} q_{rij_1}}, \tag{23}$$

where $\pi^*_{ri} > 0$ is the corresponding simplex multipliers for the constraint (19) of HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$).

Proof. In the generalized hyperplane problem GHP(Λ, t) of the reference (see p.372, [5]), let us set

$$\boldsymbol{D}_r(x_{n+1}, \bar{\boldsymbol{z}}_r) = \boldsymbol{x}_{n+1}/w_r + \bar{\boldsymbol{z}}_r^T,$$

$$\boldsymbol{D}(x_{n+1}, \bar{\boldsymbol{z}}) = (\boldsymbol{D}_1(x_{n+1}, \bar{\boldsymbol{z}}_1), \boldsymbol{D}_2(x_{n+1}, \bar{\boldsymbol{z}}_2), \cdots, \boldsymbol{D}_p(x_{n+1}, \bar{\boldsymbol{z}}_p))^T,$$

for some fixed degree $\boldsymbol{w} \geq \boldsymbol{0}$. Define $(\sum_{r=1}^{p} k_r \times (\sum_{r=1}^{p} k_r + 1))$-dimensional Jacobian matrix $A(x_{n+1}, \bar{\boldsymbol{z}})$ (see p.370, [5]) of $\boldsymbol{D}(x_{n+1}, \bar{\boldsymbol{z}})$ on $(x_{n+1}, \bar{\boldsymbol{z}})$ as follows:

$$A(x_{n+1}, \bar{\boldsymbol{z}}) = \begin{pmatrix} 1/w_1 & & & & \\ \cdots & \mathbf{I}_1 & & & \\ 1/w_1 & & & & \\ 1/w_2 & & & & \\ \cdots & & \mathbf{I}_2 & & \\ 1/w_2 & & & & \\ & & & \ddots & \\ \cdots & & & & \\ 1/w_p & & & & \\ \cdots & & & & \mathbf{I}_p \\ 1/w_p & & & & \end{pmatrix}, \tag{24}$$

where $\mathbf{I}_r, r = 1, \cdots, p$ are $(k_r \times k_r)$-dimensional identity matrices. By leaving out the last column of $A(x_{n+1}, \bar{\boldsymbol{z}})$, the following $(\sum_{r=1}^{p} k_r \times (\sum_{r=1}^{p} k_r))$-dimensional matrix $\bar{A}(x_{n+1}, \bar{\boldsymbol{z}})$ is defined.

$$\bar{A}(x_{n+1}, \bar{\boldsymbol{z}}) = \begin{pmatrix} 1/w_1 & & & & \\ \cdots & \mathbf{I}_1 & & & \\ 1/w_1 & & & & \\ 1/w_2 & & & & \\ \cdots & & \mathbf{I}_2 & & \\ 1/w_2 & & & & \\ & & & \ddots & \\ \cdots & & & & \\ 1/w_p & & & & \\ \cdots & & & & \mathbf{I}_{p*} \\ 1/w_p & & & & \\ 1/w_p & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \tag{25}$$

where $\mathbf{I}_{p*}$ is $((k_r - 1) \times (k_r - 1))$-dimensional identity matrix. Then, it holds that $\det \bar{A}(x_{n+1}, \bar{\boldsymbol{z}}) \neq 0$ for any values $w_r > 0, r = 1, \cdots, p$. This means that det

$\bar{A}(x_{n+1}, \bar{\boldsymbol{z}})$ satisfies Property 1 (see p.370, [5]), and $\boldsymbol{D}(x_{n+1}, \bar{\boldsymbol{z}})$ satisfies Properties 2 and 3 of (see p.370, [5]). As a result, Theorem 8 (see p378, [5]) can be directly applied to HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$). □

Theorem 6. *Let $(\boldsymbol{x}^*, x^*_{n+1})$ be a unique and nondegenerate optimal solution of HP2($\boldsymbol{w}^*, \bar{\boldsymbol{z}}$), and let the constraint with the reference values $\bar{z}_{rj}$ be active. Then, the following relation holds.*

$$\left.\frac{\partial(\boldsymbol{c}_{rj}\boldsymbol{x})}{\partial w_r}\right|_{\boldsymbol{x}=\boldsymbol{x}^*} = -\frac{x^*_{n+1}}{w^{*2}_r} + \frac{x^*_{n+1}}{w^{*3}_r}\left\{\sum_{i=1}^{k_r}\pi^*_{ri}\sum_{j=1}^{k_r} q_{rij}\right\} \tag{26}$$

where $\lambda^*_{ri} > 0$ is a simplex multiplier for the constraints (18) in HP2($\boldsymbol{w}^*, \bar{\boldsymbol{z}}$).

Proof. Let $(\boldsymbol{x}^*, x^*_{n+1})$ be a unique and nondegenerate optimal solution of HP2($\boldsymbol{w}*, \bar{\boldsymbol{z}}$). Then, the corresponding Lagrangian function can be defined as follows.

$$L(\boldsymbol{x}, x_{n+1}, \boldsymbol{w}) = x_{n+1} + \sum_{r=1}^{p}\sum_{i=1}^{k_r}\pi_{ri}\left\{\sum_{j=1}^{k_r} q_{rij}(-\bar{z}_{rj} + \sum_{\ell=1}^{n} c_{rj\ell}x_\ell - x_{n+1}/w_r)\right\} \tag{27}$$

From the basic sensitivity theorem [2], on some neighborhood $N(\boldsymbol{w}*)$ of $\boldsymbol{w}*$, it holds that

$$\frac{\partial x_{n+1}(\boldsymbol{w})}{\partial w_r} = \frac{\partial L(\boldsymbol{x}, x_{n+1}, \boldsymbol{w})}{\partial w_r} = \sum_{i=1}^{k_r}\pi_{ri}(\boldsymbol{w})\left\{\sum_{j=1}^{k_r} q_{rij}\frac{x_{n+1}(\boldsymbol{w})}{w^2_r}\right\},\ \boldsymbol{w} \in N(\boldsymbol{w}*)$$

where $(\boldsymbol{x}(\boldsymbol{w}), x_{n+1}(\boldsymbol{w}))$ and $\pi_{ri}(\boldsymbol{w}), i = 1, \cdots, k_r, r = 1, \cdots, p$ are a corresponding optimal solution and the simplex multipliers of HP2($\boldsymbol{w}*, \bar{\boldsymbol{z}}$), and $(\boldsymbol{x}(\boldsymbol{w}), x_{n+1}(\boldsymbol{w}))$ and $\pi_{ri}(\boldsymbol{w}), i = 1, \cdots, k_r, r = 1, \cdots, p$ are continuously differentiable vector valued functions defined on the neighborhood $N(\boldsymbol{w}*)$. On the other hand, on the neighborhood $N(\boldsymbol{w}*)$, it holds that $-\bar{z}_{rj} + \boldsymbol{c}_{rj}\boldsymbol{x}(\boldsymbol{w}) - x_{n+1}(\boldsymbol{w})/w_r = 0,\ \boldsymbol{w} \in N(\boldsymbol{w}*)$. By differentiating $\boldsymbol{c}_{rj}\boldsymbol{x}(\boldsymbol{w})$ by w_r, it follows that

$$\begin{aligned}\frac{\partial \boldsymbol{c}_{rj}\boldsymbol{x}(\boldsymbol{w})}{\partial w_r} &= -\frac{x_{n+1}(\boldsymbol{w})}{w^2_r} + \frac{1}{w_r}\cdot\frac{\partial x_{n+1}(\boldsymbol{w})}{\partial w_r} \\ &= -\frac{x_{n+1}(\boldsymbol{w})}{w^2_r} + \frac{x_{n+1}(\boldsymbol{w})}{w^3_r}\left\{\sum_{i=1}^{k_r}\pi_{ri}(\boldsymbol{w})\sum_{j=1}^{k_r} q_{rij}\right\},\ \boldsymbol{w} \in N(\boldsymbol{w}*).\end{aligned}$$

□

Now, we can construct the interactive algorithm to derive the satisfactory solution of multiple decision makers in a hierarchical organization from among the generalized Λ-extreme point set.

Step 1: Set the initial decision powers $w_r = 1, r = 1, \cdots, p$ and each decision maker sets his/her initial reference values $\bar{z}_{ri}, i = 1, \cdots, k_r, r = 1, \cdots, p$ in his/her subjective manner.

Step 2: For the specified decision powers $w_r, r = 1, \cdots, p$ and the specified reference values $\bar{z}_{ri}, i = 1, \cdots, k_r, r = 1, \cdots, p$, solve HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$), and obtain the corresponding generalized Λ-extreme point $(\boldsymbol{x}^*, x^*_{n+1})$ and trade-off information. If $x^*_{n+1} \geq 0$, then go to Step 3. If $x^*_{n+1} < 0$, then update reference values as follows, $\hat{z}_{ri} \leftarrow \bar{z}_{ri} + x^*_{n+1}/w_r$, $i = 1, \cdots, k_r, r = 1, \cdots, p$, and solve HP2($\boldsymbol{w}, \hat{\boldsymbol{z}}$) again, where $\hat{\boldsymbol{z}} = (\hat{z}_{11}, \cdots, \hat{z}_{1k_1}, \cdots, \hat{z}_{p1}, \cdots, \hat{z}_{pk_p})$.

Step 3: If each decision maker (DM_r) is satisfied with the current values of his/her objective functions, then stop. Otherwise, let the s-th level decision maker (DM_s) be the uppermost of the decision makers who are not satisfied with the current values. Considering the current values of his/her objective functions and two kinds of trade-off rates, DM_s updates his/her decision power w_{s+1} and/or his/her reference values $\bar{z}_{si}, i = 1, \cdots, k_s$ according to the following rules, and return to Step 2.

(1) the rule of updating w_{s+1}: In order to satisfy the condition (16), w_{s+1} must be set as the value which is equal to or less than w_s ($w_{s+1} \leq w_s$). If $w_{s+1} < w_t, s+1 < t \leq p$, w_t is replaced by w_{s+1} ($w_t \leftarrow w_{s+1}$). Here, it should be noted for DM_s that the less value of the decision power w_{s+1} gives better values of the objective functions of $\mathrm{DM}_r (1 \leq r \leq s)$ at the expense of the ones of $\mathrm{DM}_r (s+1 \leq r \leq p)$ for some fixed reference values.
(2) the rule of updating $\bar{z}_{si}, i = 1, \cdots, k_s$: After setting $\bar{z}_{ri} \leftarrow \boldsymbol{c}_{ri}\boldsymbol{x}*, i = 1, \cdots, k_r, r = 1, \cdots, p, r \neq s$, DM_s updates his/her reference values $\bar{z}_{si}, i = 1, \cdots, k_s$. Here, it should be stressed for DM_s that any improvement of one objective function can be achieved only at the expense of at least one of the other objective functions for some fixed decision powers.

4 A Numerical Example

In order to demonstrate the proposed method and the interactive process, we consider the following hierarchical two-objective linear programming problem.

[HMOLP]
first level decision maker : DM_1

$$\min \boldsymbol{C}_1\boldsymbol{x} = \begin{pmatrix} \boldsymbol{c}_{11}\boldsymbol{x} \\ \boldsymbol{c}_{12}\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} -10x_1 - 2x_2 - x_3 - x_4 \\ -x_1 - 13x_2 - 2x_3 - x_4 \end{pmatrix}$$

second level decision maker : DM_2

$$\min \boldsymbol{C}_2\boldsymbol{x} = \begin{pmatrix} \boldsymbol{c}_{21}\boldsymbol{x} \\ \boldsymbol{c}_{22}\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} -x_1 - 2x_2 - 11x_3 - x_4 \\ -2x_1 - x_2 - x_3 - 14x_4 \end{pmatrix}$$

subject to

$$\boldsymbol{X} = \{\boldsymbol{x} = (x_1, x_2, x_2, x_4)^T \geq \boldsymbol{0} \mid x_1 + x_2 + x_3 + x_4 \leq 30\}$$

In the above HMOLP, let us assume that the hypothetical decision makers (DM_1 and DM_2) find their satisfactory solution from Λ_r-extreme point sets $\mathrm{Ext}[\boldsymbol{C}_1 X \mid \Lambda_1]$ and $\mathrm{Ext}[\boldsymbol{C}_2 X \mid \Lambda_2]$, where the generators of the polyhedral cones Λ_1 and Λ_2 are defined as follows:

$$\boldsymbol{V}_1 = (\boldsymbol{v}_{11}, \boldsymbol{v}_{12}) = \begin{pmatrix} 5/\sqrt{26} & -1/\sqrt{65} \\ -1/\sqrt{26} & 8/\sqrt{65} \end{pmatrix}$$

$$\boldsymbol{V}_2 = (\boldsymbol{v}_{21}, \boldsymbol{v}_{22}) = \begin{pmatrix} 7/\sqrt{50} & -1/\sqrt{17} \\ -1/\sqrt{50} & 4/\sqrt{17} \end{pmatrix}$$

According to Step 1, the initial decision powers are set as $\boldsymbol{w} = (w_1, w_2)^T = (1, 1)^T$, and let us assume that the hypothetical decision makers DM_1 and DM_2 specify their initial reference values as $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{z}}_1, \bar{\boldsymbol{z}}_2)^T = (-150, -150, -150, -150)^T$. Then, at Step 2, HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$ is formulated to obtain the corresponding generalized Λ-extreme point.

$$\min_{\boldsymbol{x} \in X, x_5 \in E^1} \quad x_5$$

subject to

$$\begin{pmatrix} 1.045953 & 0.1307441 & 0 & 0 \\ 0.2067246 & 1.033623 & 0 & 0 \\ 0 & 0 & 1.047566 & 0.2618914 \\ 0 & 0 & 0.1527076 & 1.068953 \end{pmatrix} \cdot \begin{pmatrix} -z_{11} + \boldsymbol{c}_{11}\boldsymbol{x} - x_5/w_1 \\ -z_{12} + \boldsymbol{c}_{12}\boldsymbol{x} - x_5/w_1 \\ -z_{21} + \boldsymbol{c}_{21}\boldsymbol{x} - x_5/w_2 \\ -z_{22} + \boldsymbol{c}_{22}\boldsymbol{x} - x_5/w_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The optimal solution of HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$, which is the generalized Λ-extreme point, is obtained as follows:

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (9.067850, 6.677235, 8.161065, 6.093849, 31.71211) \\ (\boldsymbol{c}_{11}\boldsymbol{x}, \boldsymbol{c}_{12}\boldsymbol{x}, \boldsymbol{c}_{21}\boldsymbol{x}, \boldsymbol{c}_{22}\boldsymbol{x}) &= (-118.2879, -118.2879, -118.2879, -118.2879) \\ (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) &= (0.2573014, 0.1582485, 0.2397975, 0.1530266) \end{aligned}$$

According to Theorems 5 and 6, the trade-off rates between the objective functions and the decision power w_2 become as follows:

$$-\frac{\partial(\boldsymbol{c}_{11}\boldsymbol{x})}{\partial(\boldsymbol{c}_{12}\boldsymbol{x})} = \frac{\pi_{11}q_{112} + \pi_{12}q_{122}}{\pi_{11}q_{111} + \pi_{12}q_{121}} = 0.6533613$$

$$-\frac{\partial(\boldsymbol{c}_{21}\boldsymbol{x})}{\partial(\boldsymbol{c}_{22}\boldsymbol{x})} = \frac{\pi_{21}q_{212} + \pi_{22}q_{222}}{\pi_{21}q_{211} + \pi_{22}q_{221}} = 0.8244804$$

$$\frac{\partial(\boldsymbol{c}_{2j}\boldsymbol{x})}{\partial w_2} = -\frac{x_5}{w_2^2} + \frac{x_5}{w_2^3}\left\{\pi_{21}(q_{211} + q_{212}) + \pi_{22}(q_{221} + q_{222})\right\} = -15.82589$$

At Step 3, let us assume that, DM_1 updates his/her decision power in order to improve his/her objective functions as $w_2 = 0.7$, and go to Step 2. Then, the corresponding problem HP2$(\boldsymbol{w}, \bar{\boldsymbol{z}})$ is solved and the corresponding generalized Λ-extreme point is obtained as $(x_1, x_2, x_3, x_4, x_5) = (9.633082, 7.195167, 7.550899, 5.620852, 26.10709)$, $(\boldsymbol{c}_{11}\boldsymbol{x}, \boldsymbol{c}_{12}\boldsymbol{x}, \boldsymbol{c}_{21}\boldsymbol{x}, \boldsymbol{c}_{22}\boldsymbol{x}) = (-123.8929, -123.8929, -112.7042, -112.7042)$, $(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) = (0.211824, 0.130279, 0.197414, 0.125980)$, where DM_1's objective functions $\boldsymbol{c}_{11}\boldsymbol{x}$ and $\boldsymbol{c}_{12}\boldsymbol{x}$ are improved at the expense of DM_2's objective functions $\boldsymbol{c}_{21}\boldsymbol{x}$ and $\boldsymbol{c}_{22}\boldsymbol{x}$. At Step 3, let us assume that DM_1 is satisfied with the current values, and DM_2 updates his/her reference values as $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{z}}_1, \bar{\boldsymbol{z}}_2)^T = (-123.8929, -123.8929, -120, -108)^T$ according to the rule (2) of Step 3 in order to improve $\boldsymbol{c}_{21}\boldsymbol{x}$ at the expense of $\boldsymbol{c}_{22}\boldsymbol{x}$. At Step

2, the corresponding problem HP2($\boldsymbol{w}, \bar{\boldsymbol{z}}$) is solved and the corresponding generalized Λ-extreme point is obtained as $(x_1, x_2, x_3, x_4, x_5) = (9.560246, 7.078217, 8.181826, 5.179710, 0.772465)$, $(\boldsymbol{c}_{11}\boldsymbol{x}, \boldsymbol{c}_{12}\boldsymbol{x}, \boldsymbol{c}_{21}\boldsymbol{x}, \boldsymbol{c}_{22}\boldsymbol{x}) = (-123.1204, -123.1204, -118.8965, -106.8965)$. At this point, both DM_1 and DM_2 are satisfied, and the interactive processes are terminated.

5 Conclusions

In this paper, hierarchical multiobjective linear programming problems (HMOLP) have been formulated, where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints. In order to deal with HMOLP, concepts of a generalized Λ-extreme point and decision powers have been introduced and a linear programming based interactive algorithm has been proposed to obtain the satisfactory solution. In the proposed method, not only the hierarchical relationships between multiple decision makers but also their own preferences for their objective functions can be reflected for the satisfactory solution. Applications of the proposed method will require further investigation.

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A Perception-Based Portfolio Under Uncertainty: Minimization of Average Rates of Falling

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Abstract. A perception-based portfolio model under uncertainty is discussed. In the proposed model, randomness and fuzziness are evaluated respectively by the probabilistic expectation and the mean values with evaluation weights and λ-mean functions. The means, the variances and the covariances fuzzy numbers/fuzzy random variables are evaluated in the possibility case and the necessity case, and the rate of return with portfolios is estimated by the both random factors and imprecise factors. In the portfolio model, the average rate of falling is minimized using average value-at-risks as a coherent risk measure. By analytical approach, we derive a solution of the portfolio problem to minimize the average rate of falling. A numerical example is given to illustrate our idea.

1 Introduction

This paper deals with a risk management model with perception-based representation under uncertainty. Soft computing like fuzzy logic works effectively for financial models in uncertain environment. To represent uncertainty in this paper, we use *fuzzy random variables* which have two kinds of uncertainties, i.e. randomness and fuzziness. In this model, randomness is used to represent the uncertainty regarding the belief degree of frequency, and fuzziness is applied to linguistic imprecision of data because of a lack of knowledge regarding the current stock market. At the financial crisis in October 2008, we have observed the serious distrust of the market that the risky information regarding banks and security companies, for example the amounts of trouble loans, risky accounts, debts and so on, may not disclose to the investors and the public, and it is surely a kind of risks occurring from the imprecision of information. The fuzziness comes from the imprecision of data because of a lack of knowledge, and such serious distrust in the stock market will be represented by the fuzziness of information in finance models.

In financial market, the portfolio is one of the most useful risk allocation technique for stable asset management. The minimization of the financial risk as well as the maximization of the return are important themes in the asset management. In a classical portfolio theory, *Markowitz's mean-variance model* is studied by many researchers and fruitful results have been achieved, and then the variance is investigated as the risk for portfolios ([9,10,13]). In this paper we

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focus on the drastic decline of asset prices. Recently, *value-at-risk (VaR)* is used widely to estimate the risk that asset prices decline based on worst scenarios. VaR is a risk-sensitive criterion based on percentiles, and it is one of the standard criteria in asset management ([19]). VaR is a kind of risk-level values of the asset at a specified probability of decline and it is used to select portfolios after due consideration of worst scenarios in investment. Many researchers and financial traders usually use VaR by mathematical programming since it is not easy to analyze the VaR portfolio model mathematically. Because Markowitz's mean-variance criterion and the variance-minimizing criterion are represented by quadratic programming, but VaR criterion in portfolio is neither linear nor quadratic ([19]). *Average value-at-risk (AVaR)* is also studied as one of coherent risk measures derived from VaR. In this paper, by use of AVaR regarding the average rates of return, we discuss a *portfolio selection problem not only to minimize the rates of falling and but also to maximize the expected rates of return.* This paper derives an analytical solution for the portfolio problem to minimize the average rate of falling, extending the results for VaR in Yoshida [18].

We extend the AVaR for real random variables to one regarding fuzzy random variables from the viewpoint of perception-based approach in Yoshida [16]. We formulate a portfolio problem with fuzzy random variables, and we discuss the fundamental properties of the extended AVaR. Estimation of uncertain quantities is important in decision making. Recently, Yoshida [15] introduced the mean, the variance and the covariances of fuzzy random variables, using *evaluation weights* and λ-mean functions. This paper estimates fuzzy numbers and fuzzy random variables by the probabilistic expectation and these criteria, which are characterized by *possibility and necessity criteria* for subjective estimation and a *pessimistic-optimistic index* for subjective decision. These parameters are decided by the investor and are based on the degree of his certainty regarding the current information in the market. In this portfolio model, we use fuzzy random variables with triangle-type fuzzy numbers for computation in actual models.

2 A Portfolio Model and the Rate of Falling

In this section, we explain a portfolio model with n stocks, where n is a positive integer. Let $\mathbb{T} := \{0, 1, 2, \cdots, T\}$ be the time space with an expiration date T, and $\mathbb{R}$ denotes the set of all real numbers. Let (Ω, P) be a probability space, where P is a non-atomic probability on a sample space Ω. For an asset $i = 1, 2, \cdots, n$, a *stock price process* $\{S_t^i\}_{t=0}^T$ is given by *rates of return* R_t^i as follows. Let

$$S_t^i := S_{t-1}^i(1 + R_t^i) \tag{1}$$

for $t = 1, 2, \cdots, T$, where $\{R_t^i\}_{t=1}^T$ is assumed to be an integrable sequence of independent real random variables. Hence $w_t = (w_t^1, w_t^2, \cdots, w_t^n)$ is called a *portfolio weight vector* if it satisfies $w_t^1 + w_t^2 + \cdots + w_t^n = 1$, and further a portfolio $(w_t^1, w_t^2, \cdots, w_t^n)$ is said to *allow for short selling* if $w_t^i \geq 0$ for all $i = 1, 2, \cdots, n$. Then the rate of return with a portfolio $(w_t^1, w_t^2, \cdots, w_t^n)$ is given by

$$R_t := w_t^1 R_t^1 + w_t^2 R_t^2 + \cdots + w_t^n R_t^n. \tag{2}$$

Therefore, the reward at time $t = 1, 2, \cdots, T$ follows

$$S_t := S_{t-1} \sum_{i=1}^{n} w_t^i (1 + R_t^i) = S_{t-1}(1 + R_t). \tag{3}$$

In this paper, we present a portfolio model where stock price processes S_t^i take fuzzy values using fuzzy random variables. The falling of asset prices is one of the most important risks in stock markets. In this section, we discuss a portfolio model where the risk is estimated by the rate of falling. Regarding the asset (3) with the portfolio w_t, the theoretical *bankruptcy* at time t occurs on scenarios ω satisfying $S_t(\omega) \leq 0$, i.e. it follows $1 + R_t(\omega) \leq 0$ from (3). Similarly, for a constant $\bar{\delta}$ satisfying $0 \leq \bar{\delta} \leq 1$, a set of sample paths

$$\{\omega \in \Omega \mid 1 + R_t(\omega) \leq 1 - \bar{\delta}\} = \{\omega \in \Omega \mid R_t(\omega) \leq -\bar{\delta}\} \tag{4}$$

is the event of scenarios where the asset price S_t will fall from the current price S_{t-1} to a lower level than $100(1 - \bar{\delta})\,\%$ of the current price S_{t-1}, i.e. the rate of falling is $100\,\bar{\delta}\,\%$. The parameter $\bar{\delta}$ is called *the rate of falling*. Then the probability of falling is also given by

$$p_{\bar{\delta}} := P(R_t \leq -\bar{\delta}). \tag{5}$$

For example, $p_{\bar{\delta}}$ denotes the probability of the falling below par value if '$\bar{\delta} = 0$' and it indicates the probability of the bankruptcy if '$\bar{\delta} = 1$'. In this paper, we discuss portfolios regarding the rate of falling $\bar{\delta}$.

For a positive probability p, a value-at-risk (VaR) regarding the rate of return R_t at the probability p is given by a real number $\bar{v}$ satisfying

$$P(R_t \leq \bar{v}) = p \tag{6}$$

since P is non-atomic. The value-at-risk $\bar{v}$ is the upper bound of the rate of return R_t at the worst scenarios under a given risk probability p, and then the value-at-risk $\bar{v}$ in (6) is denoted by $\mathrm{VaR}_p(R_t)$. From (5) and (6), for a risk probability $p = p_{\bar{\delta}}$, the rate of falling is

$$\bar{\delta} = -\mathrm{VaR}_p(R_t). \tag{7}$$

To minimize the average rate of falling derived from (7) under a fuzzy and random environment, in next section we discuss the fundamental properties of value-at-risks.

3 A Portfolio Model with Fuzzy Random Variables

In this section, we introduce fuzzy numbers and fuzzy random variables to give a portfolio model under uncertainty. In this paper, a fuzzy number is represented by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous and quasi-concave and has a compact support ([20]). $\mathcal{R}$ denotes the set of all

fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\}$ $(\alpha \in (0,1])$ and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. An α-cut $\tilde{a}_\alpha$ is a closed interval since the membership function is is normal, upper-semicontinuous and quasi-concave, and it is written as $\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ for $\alpha \in [0,1]$. Hence we also introduce a partial order $\succeq$, so called the *fuzzy max order*, on fuzzy numbers $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. Then, $\tilde{a} \succeq \tilde{b}$ means that $\tilde{a}_\alpha^- \geq \tilde{b}_\alpha^-$ and $\tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+$ for all $\alpha \in [0,1]$. An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined by Zadeh's extension principle as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\xi \in \mathbb{R}$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\xi\tilde{a}$ of ξ and $\tilde{a}$ are fuzzy numbers given by their α-cuts $(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+]$, $(\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]$ and $(\xi\tilde{a})_\alpha := [\xi\tilde{a}_\alpha^-, \xi\tilde{a}_\alpha^+]$ if $\xi \geq 0$.

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* if the maps $\omega \mapsto \tilde{X}_\alpha^\pm(\omega)$ are measurable for all $\alpha \in (0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ ([8,11]). We need to introduce expectations of fuzzy random variables in order to describe a portfolio model. A fuzzy random variable $\tilde{X}$ is said to be integrably bounded if both $\omega \mapsto \tilde{X}_\alpha^\pm(\omega)$ are integrable for all $\alpha \in (0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number $E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}$ for $x \in \mathbb{R}$, where $E(\tilde{X})_\alpha := [\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)]$ for $\alpha \in (0,1]$ ([7,11]).

Now we deal with a case where the rate of return $\{R_t^i\}_{t=1}^T$ has some imprecision. In this paper we use triangle-type fuzzy random variables for computation though we can apply similar approaches with general fuzzy random variables. We define a *rate of return process with imprecision* $\{\tilde{R}_t^i\}_{t=0}^T$ by a sequence of triangle-type fuzzy random variables

$$\tilde{R}_t^i(\cdot)(x) = \begin{cases} 1 - |x - R_t^i|/c_t^i & \text{if } |x - R_t^i| \leq c_t^i \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where c_t^i is a positive number. We call c_t^i a *imprecise factor* for asset i at time t. Hence we can represent $\tilde{R}_t^i$ by the sum of the real random variable R_t^i and a fuzzy number $\tilde{a}_t^i$:

$$\tilde{R}_t^i(\omega)(\cdot) := 1_{\{R_t^i(\omega)\}}(\cdot) + \tilde{a}_t^i(\cdot) \tag{9}$$

for $\omega \in \Omega$, where $1_{\{\cdot\}}$ denotes the characteristic function of a singleton and $\tilde{a}_t^i$ is a triangle-type fuzzy number defined by

$$\tilde{a}_t^i(x) = \begin{cases} 1 - |x|/c_t^i & \text{if } |x| \leq c_t^i \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

For assets $i = 1, 2, \cdots, n$, we define *stock price processes* $\{\tilde{S}_t^i\}_{t=0}^T$ by the *rates of return with imprecision* $\tilde{R}_t^i$ as follows: $\tilde{S}_0^i := S_0^i$ is a positive number and

$$\tilde{S}_t^i = \tilde{S}_0^i \prod_{s=1}^t (1 + \tilde{R}_s^i) \tag{11}$$

for $t = 1, 2, \cdots, T$. For a portfolio $w = (w^1, w^2, \cdots, w^n)$, the rate of return with imprecision is given by a linear combination of fuzzy random variables

$$\tilde{R}_t := w^1 \tilde{R}_t^1 + w^2 \tilde{R}_t^2 + \cdots + w^n \tilde{R}_t^n. \tag{12}$$

In Section 4 we investigate the average value-at-risk to apply the fuzzy random variable (12), and in Section 5 we discuss the portfolio problem to minimize the average rate of falling regarding (12).

4 A Perception-Based Extension of Average Value-at-Risks

First we introduce mathematical notations of the value-at-risk for real random variables to apply it to the rates of return (12). Let $\mathcal{X}$ be the set of all integrable real random variables X on Ω with a continuous distribution function $x \mapsto F_X(x) := P(X < x)$ for which there exists a non-empty open interval I such that $F_X(\cdot) : I \mapsto (0,1)$ is a strictly increasing and onto. Then there exists a strictly increasing and continuous inverse function $F_X^{-1} : (0,1) \mapsto I$. We note that $F_X(\cdot) : I \mapsto (0,1)$ and $F_X^{-1} : (0,1) \mapsto I$ are one-to-one and onto, and we put $F_X(\inf I) := \lim_{x \downarrow \inf I} F_X(x) = 0$ and $F_X(\sup I) := \lim_{x \uparrow \sup I} F_X(x) = 1$. Then, the *value-at-risk*, shortly for *VaR*, at a risk probability p is given by the $100\,p$-percentile of the distribution function F_X: For a probability p $(0 < p < 1)$,

$$\mathrm{VaR}_p(X) := \sup\{x \in I \mid F_X(x) \le p\}, \tag{13}$$

and then we have $F_X(\mathrm{VaR}_p(X)) = p$ and $\mathrm{VaR}_p(X) = F_X^{-1}(p)$ for $0 < p < 1$. In this paper, we assume that VaR $\bar{v}$ in (6) has the following representation (14).

$$(\text{VaR } \bar{v}) = (\text{the mean}) - (\text{a positive constant } \kappa(p)) \times (\text{the standard deviation}), \tag{14}$$

where the positive constant $\kappa(p)$ is given corresponding to the probability p. The details are as follows: For any probability p satisfying $0 < p < 1$, there exists a positive constant $\kappa(p)$ such that a real number $\bar{v} := \mu_t - \kappa(p)\,\sigma_t$ satisfies (14) for all portfolios, where μ_t and σ_t are the expectation and the standard deviation of R_t respectively. One of the most popular sufficient condition for (14) is what the distribution of the rate of return R_t is Gaussian ([3,19]).

The *average value-at-risk (AVaR)* at a probability level p (*Expected Shortfall* with at a confidence probability level $1 - p$) is given by

$$\mathrm{AVaR}_p(X) := \frac{1}{p} \int_0^p \mathrm{VaR}_q(X)\, dq \tag{15}$$

if $0 < p \le 1$ and $\mathrm{AVaR}_p(X) := \inf I$ if $p = 0$ ([12]) It is known that AVaR has the following properties ([18]), which imply AVaR is a *coherent risk measure* ([1]).

Lemma 1. *Let $X, Y \in \mathcal{X}$ and let p be a positive probability. Then the average value-at-risk* AVaR_p *defined by (15) has the following properties:*

(i) *If* $X \leq Y$, *then* $\mathrm{AVaR}_p(X) \leq \mathrm{AVaR}_p(Y)$. *(monotonicity)*
(ii) $\mathrm{AVaR}_p(\zeta X) = \zeta\,\mathrm{AVaR}_p(X)$ *for* $\zeta > 0$. *(positive homogeneity)*
(iii) $\mathrm{AVaR}_p(X+\theta) = \mathrm{AVaR}_p(X) + \theta$ *for* $\theta \in \mathbb{R}$. *(translation invariance)*
(iv) $\mathrm{AVaR}_p(X+Y) \geq \mathrm{AVaR}_p(X) + \mathrm{AVaR}_p(Y)$. *(super-additivity)*

On the other hand, *the super-additivity for the value-at-risk* $\mathrm{VaR}_p(\cdot)$ *does not hold in general* ([1]). Next we denote $\tilde{\mathcal{X}}$ the set of all fuzzy random variables $\tilde{X}$ on Ω such that their α-cuts $\tilde{X}_\alpha^{\pm}$ are integrable and $\lambda\tilde{X}_\alpha^{-} + (1-\lambda)\tilde{X}_\alpha^{+} \in \mathcal{X}$ for all $\lambda \in [0,1]$ and $\alpha \in [0,1]$. Hence, from (15) we introduce an average value-at-risk for a fuzzy random variable $\tilde{X}(\in \tilde{\mathcal{X}})$ at a positive risk probability p as follows.

$$\mathrm{AVaR}_p(\tilde{X})(x) := \sup_{X \in \mathcal{X}\,:\,\mathrm{AVaR}_p(X)=x} \inf_{\omega\in\Omega} \tilde{X}(\omega)(X(\omega)) \tag{16}$$

for $x \in \mathbb{R}$. Yoshida [16] has studied *perception-based estimations* extending the concept of the expectations in Kruce and Meyer [7]. This definition (16) is an extension from the average value-at-risk on real random variables to one on fuzzy random variables based on the perception. The average value-at-risk (16) on fuzzy random variables is characterized by the following representation and properties, which are from the continuity and the monotonicity of $\mathrm{AVaR}_p(\cdot)$.

Lemma 2. ([16]). *Let* $\tilde{X} \in \tilde{\mathcal{X}}$ *be a fuzzy random variable and let* p *be a positive probability. Then the average value-at-risk* $\mathrm{AVaR}_p(\tilde{X})$ *defined by (16) is a fuzzy number whose* α*-cuts are*

$$\mathrm{AVaR}_p(\tilde{X})_\alpha = [\mathrm{AVaR}_p(\tilde{X}_\alpha^{-}), \mathrm{AVaR}_p(\tilde{X}_\alpha^{+})] \quad \textit{for } \alpha \in (0,1]. \tag{17}$$

Lemma 3. ([16]). *Let* $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$ *be fuzzy random variables and let* p *be a positive probability. Then the average value-at-risk* AVaR_p *defined by (16) has the following properties:*

(i) *If* $\tilde{X} \preceq \tilde{Y}$, *then* $\mathrm{AVaR}_p(\tilde{X}) \preceq \mathrm{VaR}_p(\tilde{Y})$. *(monotonicity)*
(ii) $\mathrm{AVaR}_p(\zeta\tilde{X}) = \zeta\,\mathrm{AVaR}_p(\tilde{X})$ *for* $\zeta > 0$. *(positive homogeneity)*
(iii) $\mathrm{AVaR}_p(\tilde{X}+\tilde{a}) = \mathrm{AVaR}_p(\tilde{X}) + \tilde{a}$ *for* $\tilde{a} \in \mathcal{R}$. *(translation invariance)*
(iv) $\mathrm{AVaR}_p(\tilde{X}+\tilde{Y}) \succeq \mathrm{AVaR}_p(\tilde{X}) + \mathrm{AVaR}_p(\tilde{Y})$. *(super-additivity)*

Next since the average value-at-risk $\mathrm{AVaR}_p(\tilde{R}_t)$ for the rate of return (12) with a portfolio is a fuzzy number, we need to evaluate the fuzziness of fuzzy numbers and fuzzy random variables. There are many studies regarding the evaluation of fuzzy numbers. Two major approaches of them are as follows. One is to use weighting functions([2,4,15]) and the other is to use possibility and necessity criteria([5]). Here we adopt the former evaluation method of fuzzy numbers and fuzzy random variables. In the rest of this section we introduce the definitions from [15,17], and in the next section we estimate the AVaR regarding the rate of return (12) by the evaluation method. Yoshida [15] has studied an evaluation of fuzzy numbers by *evaluation weights* which are induced from fuzzy measures to evaluate a confidence degree that a fuzzy number takes values in an interval.

With respect to fuzzy random variables, the randomness is evaluated by the probabilistic expectation and the fuzziness is estimated by the evaluation weights and the following function. Let $g^\lambda : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g^\lambda([x,y]) := \lambda x + (1-\lambda)y, \quad [x,y] \in \mathcal{I}, \tag{18}$$

where λ is a constant satisfying $0 \leq \lambda \leq 1$ and $\mathcal{I}$ denotes the set of all bounded closed intervals. This scalarization is used for the estimation of fuzzy numbers to give a mean value of the interval $[x,y]$ with a weight λ, and g^λ is called a λ-*mean function* and λ is called a *pessimistic-optimistic index* which indicates the pessimistic degree of attitude in decision making ([4]). Let a fuzzy number $\tilde{a} \in \mathcal{R}$. A mean value of the fuzzy number $\tilde{a}$ with respect to λ-mean functions g^λ and an evaluation weight $w(\alpha)$, which depends only on $\tilde{a}$ and α, is given as follows ([17]):

$$\tilde{E}(\tilde{a}) := \frac{\int_0^1 g^\lambda(\tilde{a}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}, \tag{19}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ is the α-cut of the fuzzy number $\tilde{a}$. In (19), $w(\alpha)$ indicates a *confidence degree that the fuzzy number* $\tilde{a}$ *takes values in the interval* $\tilde{a}_\alpha$ *at each level* α. Hence, an evaluation weight $w(\alpha)$ is called the *possibility evaluation weight* $w^P(\alpha)$ if $w^P(\alpha) := 1$ for $\alpha \in [0,1]$, and $w(\alpha)$ is called the *necessity evaluation weight* $w^N(\alpha)$ if $w^N(\alpha) := 1-\alpha$ for $\alpha \in [0,1]$. Especially, for a fuzzy number $\tilde{a} \in \mathcal{R}$, the means in the possibility and necessity cases are represented respectively by $\tilde{E}^P(\tilde{a})$ and $\tilde{E}^N(\tilde{a})$, and we consider their combination $\nu\tilde{E}^P(\tilde{a}) + (1-\nu)\tilde{E}^P(\tilde{a})$ with a parameter $\nu \in [0,1]$ ([15,16,18]). The mean $\tilde{E}$ has the following natural properties of the linearity and the monotonicity regarding the fuzzy max order.

Lemma 4. ([15,17]). *Let* $\lambda \in [0,1]$. *For fuzzy numbers* $\tilde{a}, \tilde{b} \in \mathcal{R}$ *and real numbers* θ, ζ, *the following (i) – (iv) hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$.
(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$ *if* $\zeta \geq 0$.
(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.
(iv) *If* $\tilde{a} \succeq \tilde{b}$, *then* $\tilde{E}(\tilde{a}) \geq \tilde{E}(\tilde{b})$ *holds.*

For a fuzzy random variable $\tilde{X}$, *the mean of the expectation* $E(\tilde{X})$ is a real number

$$E(\tilde{E}(\tilde{X})) = E\left(\frac{\int_0^1 g^\lambda(\tilde{X}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}\right). \tag{20}$$

Then, from Lemma 4, we obtain the following results.

Lemma 5. ([15,17]). *Let* $\lambda \in [0,1]$. *For a fuzzy number* $\tilde{a} \in \mathcal{R}$, *integrable fuzzy random variables* $\tilde{X}, \tilde{Y}$, *an integrable real random variable* Z *and a nonnegative real number* ζ, *the following (i) – (v) hold.*

(i) $E(\tilde{E}(\tilde{X})) = \tilde{E}(E(\tilde{X}))$.
(ii) $E(\tilde{E}(\tilde{a})) = \tilde{E}(\tilde{a})$ *and* $E(\tilde{E}(Z)) = E(Z)$.
(iii) $E(\tilde{E}(\zeta\tilde{X})) = \zeta E(\tilde{E}(\tilde{X}))$.
(iv) $E(\tilde{E}(\tilde{X}+\tilde{Y})) = E(\tilde{E}(\tilde{X})) + E(\tilde{E}(\tilde{Y}))$.
(v) *If* $\tilde{X} \succeq \tilde{Y}$, *then* $E(\tilde{E}(\tilde{X})) \geq E(\tilde{E}(\tilde{Y}))$ *holds.*

Finally we introduce variances and covariances of fuzzy random variables from the viewpoint of λ-mean functions and evaluation weights. From the results in [15], for fuzzy random variables $\tilde{X}$ and $\tilde{Y}$, we define variances and covariances as follows. For $\lambda \in [0,1]$,

$$V(\tilde{X}) := E\left(\frac{\int_0^1 (g^\lambda(\tilde{X}_\alpha) - E(g^\lambda(\tilde{X}_\alpha)))^2\, w(\alpha)\,\mathrm{d}\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha}\right), \tag{21}$$

$$Cov(\tilde{X},\tilde{Y}) := E\left(\frac{\int_0^1 (g^\lambda(\tilde{X}_\alpha) - E(g^\lambda(\tilde{X}_\alpha)))(g^\lambda(\tilde{Y}_\alpha) - E(g^\lambda(\tilde{Y}_\alpha)))\, w(\alpha)\,\mathrm{d}\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha}\right). \tag{22}$$

Lemma 6. ([15]). *Let* $\lambda \in [0,1]$. *For fuzzy numbers* $\tilde{a}, \tilde{b} \in \mathcal{R}$, *integrable fuzzy random variables* $\tilde{X}, \tilde{Y}$ *and a nonnegative real number* ζ, *the following (i) – (v) hold.*

(i) $V(\tilde{a}) = 0$.
(ii) $V(\tilde{X}+\tilde{a}) = V(\tilde{X})$.
(iii) $V(\zeta\tilde{X}) = \zeta^2 V(\tilde{X})$.
(iv) $Cov(\tilde{X},\tilde{a}) = Cov(\tilde{a},\tilde{X}) = 0$.
(v) $Cov(\tilde{X}+\tilde{a},\tilde{Y}+\tilde{b}) = Cov(\tilde{X},\tilde{Y})$.

5 The Minimization of the Average Risk of Falling

In this section, we discuss portfolio problems under uncertainty. First we estimate the rate of return with imprecision for a portfolio. Let the mean, the variance and the covariance of the rate of return R_t^i, which are the real random variables in (1), by $\mu_t^i := E(R_t^i)$, $(\sigma_t^i)^2 := E((R_t^i - \mu_t^i)^2)$, and $\sigma_t^{ij} := E((R_t^i - \mu_t^i)(R_t^j - \mu_t^j))$ for $i,j = 1,2,\cdots,n$. Hence we assume that the determinant of the variance-covariance matrix $\Sigma := [\sigma_t^{ij}]$ is not zero and there exists its inverse matrix Σ^{-1}. This assumption is natural and it can be realized easily by taking care of the combinations of assets. For a portfolio $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ $(i = 1,2,\cdots,n)$, we calculate the expectation and the variance regarding $\tilde{R}_t = w^1\tilde{R}_t^1 + w^2\tilde{R}_t^2 + \cdots + w^n\tilde{R}_t^n$. From Lemma 5, the expectation $\tilde{\mu}_t := E(\tilde{E}(\tilde{R}_t))$ follows

$$\tilde{\mu}_t = E(\tilde{E}(\tilde{R}_t)) = \sum_{i=1}^n w^i E(\tilde{E}(\tilde{R}_t^i)) = \sum_{i=1}^n w^i \tilde{\mu}_t^i, \tag{23}$$

where $\tilde{\mu}_t^i := E(\tilde{E}(\tilde{R}_t^i))$ for $i = 1, 2, \cdots, n$. On the other hand, regarding this model, from [15] we can find that the variance $(\tilde{\sigma}_t)^2 := E((\tilde{E}(\tilde{R}_t) - \tilde{\mu}_t)^2)$ of $\tilde{R}_t$ equals to the variance $(\sigma_t)^2 := E((R_t - \mu_t)^2)$ of R_t:

$$(\tilde{\sigma}_t)^2 = (\sigma_t)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w^i w^j \sigma_t^{ij}. \tag{24}$$

By (14), (23) and (24), the mean of $\mathrm{AVaR}_p(\tilde{R}_t)$ is

$$\tilde{E}(\mathrm{AVaR}_p(\tilde{R}_t)) = \sum_{i=1}^{n} w^i \tilde{\mu}_t^i - \kappa \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w^i w^j \sigma_t^{ij}}, \tag{25}$$

where $\kappa := \frac{1}{p} \int_0^p \kappa(q)\, dq$ with $\kappa(q)$ defined by (14). Now step by step we discuss a portfolio problem to minimize the average rate of falling $\delta = -\tilde{E}(\mathrm{AVaR}_p(\tilde{R}_t))$. First, we deal with a variance-minimizing model. For a given constant γ, which is the minimum expected rate of return to be guaranteed for the portfolio, we consider the following quadratic programming with respect to portfolios with allowance for short selling.

Variance-minimizing problem (P1): Minimize the variance

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w^i w^j \sigma_t^{ij} \tag{26}$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ under the following condition regarding the expected rate of return: $\sum_{i=1}^{n} w^i \tilde{\mu}_t^i = \gamma$.

We start from the following classical results regarding the variance-minimizing problem (P1), and we analyze the portfolio problem to minimize the average rate of falling.

Lemma 7. ([18]). *The solution of the variance-minimizing problem (P1) is given by*

$$w = \xi \Sigma^{-1} \mathbf{1} + \eta \Sigma^{-1} \tilde{\mu} \tag{27}$$

and then the corresponding variance is

$$\tilde{\rho} := \frac{A\gamma^2 - 2B\gamma + C}{\Delta}, \tag{28}$$

where $\tilde{\mu}^i := \mu_t^i + \tilde{E}^{\lambda}(\tilde{a}_t^i)$, $\tilde{\sigma}^{ij} := \sigma_t^{ij}$ $(i, j = 1, 2, \cdots, n)$, $\tilde{\Sigma} := [\tilde{\sigma}^{ij}]$, $\tilde{\mu} := [\tilde{\mu}^1\ \tilde{\mu}^2\ \cdots\ \tilde{\mu}^n]^{\mathrm{T}}$, $\mathbf{1} := [1\ 1\ \cdots\ 1]^{\mathrm{T}}$, $\xi := \frac{C - B\gamma}{\Delta}$, $\eta := \frac{A\gamma - B}{\Delta}$, $A := \mathbf{1}^{\mathrm{T}} \tilde{\Sigma}^{-1} \mathbf{1}$, $B := \mathbf{1}^{\mathrm{T}} \tilde{\Sigma}^{-1} \tilde{\mu}$, $C := \tilde{\mu}^{\mathrm{T}} \tilde{\Sigma}^{-1} \tilde{\mu}$, $\Delta := AC - B^2$ *and* $^{\mathrm{T}}$ *denotes the transpose of a vector.*

Hence, we consider a risk-sensitive model, which is not of mean-variance types but of mean-standard deviation types, in order to deal with a portfolio problem to minimize the average rate of falling in the third step. For a constant γ and

a positive constant κ, we discuss the following risk-sensitive portfolio problem with allowance for short selling.

Risk-sensitive problem (P2): Maximize a risk-sensitive expected rate of return

$$\sum_{i=1}^{n} w^i \tilde{\mu}_t^i - \kappa \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w^i w^j \sigma_t^{ij}} \tag{29}$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ $(w^1 + w^2 + \cdots + w^n = 1)$ under the condition $\sum_{i=1}^{n} w^i \tilde{\mu}_t^i = \gamma$.

Now we discuss the following AVaR portfolio problem without allowance for short selling. The following form (30) comes from the average value-at-risk $\tilde{E}(\mathrm{AVaR}_p(\tilde{R}_t))$ given in (25).

Portfolio problem to minimize the average rate of falling (P3): Minimize the risk of falling

$$\delta := -\sum_{i=1}^{n} w^i \tilde{\mu}_t^i + \kappa \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w^i w^j \sigma_t^{ij}} \tag{30}$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ for $i = 1, 2, \cdots, n$.

Since we have $\inf_w (30) = \inf_\gamma \left(\inf_{w:\sum_{i=1}^{n} w^i \tilde{\mu}_t^i = \gamma} (30) \right) = -\sup_\gamma (29)$, in the same way as [18] we arrive at the following analytical solutions of the portfolio problem to minimize the average rate of falling (P3).

Lemma 8. *Let A and Δ be positive. Let κ satisfy $\kappa^2 > C$. The solution of the portfolio problem to minimize the average rate of falling (P3) is given by $w^* = \xi \Sigma^{-1}\mathbf{1} + \eta \Sigma^{-1}\tilde{\mu}$, and then the corresponding average rate of falling is $\delta(\gamma^*) = -\frac{B - \sqrt{A\kappa^2 - \Delta}}{A}$ at the expected rate of return $\gamma^* := \frac{B}{A} + \frac{\Delta}{A\sqrt{A\kappa^2 - \Delta}}$, where $\xi := \frac{C - B\gamma^*}{\Delta}$ and $\eta := \frac{A\gamma^* - B}{\Delta}$. Further, if $\Sigma^{-1}\mathbf{1} \geq \mathbf{0}$ and $\Sigma^{-1}\tilde{\mu} \geq \mathbf{0}$, then the portfolio w^* satisfies $w^* \geq \mathbf{0}$, i.e. the portfolio w^* is a trading strategy without allowance for short selling. Here, $\mathbf{0}$ denotes the zero vector.*

In Lemma 8, we note that the optimal portfolio w^* not only to minimize the average rate of falling $\delta(\gamma^*)$ but also to maximize the expected rate of return γ^*.

Theorem 1. *Let A and Δ be positive. Let δ satisfy $\delta > -2B/A$. Then, the assumptions in Lemma 8 are satisfied, and the following (i) and (ii) hold for the optimal portfolio in Lemma 8.*

(i) *For an average rate of falling δ, the corresponding constant κ_δ and the expected rate of return γ_δ are given by*

$$\kappa_\delta := \sqrt{A\delta^2 + 2B\delta + C} \quad \text{and} \quad \gamma_\delta := \frac{B\delta + C}{A\delta + B}. \tag{31}$$

Then the risk probability is $p_\delta = P(\tilde{E}(\tilde{R}_t) \leq -\delta)$.

(ii) *If R_t^i $(i = 1, 2, \cdots, n)$ have Gaussian distributions, the risk probability p_δ in (i) is given by*

$$p_\delta := \Phi(-\kappa_\delta), \tag{32}$$

where κ_δ is defined by (31) and Φ is the cumulative Gaussian distribution function $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt$ for $z \in \mathbb{R}$.

6 A Numerical Example

In this session, we give a simple example to illustrate our idea. For the numerical computation, we need to evaluate fuzzy numbers representing the rates of return (8). From [15,16], we have evaluations of the fuzzy numbers $\tilde{a}_t^i$ by a combination $\nu \tilde{E}^P(\tilde{a}) + (1-\nu)\tilde{E}^P(\tilde{a})$ with a parameter $\nu \in [0,1]$ with a parameter $\nu (\in [0,1])$, which is called a *possibility-necessity weight* ([15]). Then from (23) we obtain the expected rate of return

$$\tilde{\mu}_t = \sum_{i=1}^{n} w^i \left(\mu_t^i + \frac{(1-2\lambda)(4-\nu)}{6} c_t^i \right) \tag{33}$$

for the possibility-necessity weight $\nu (\in [0,1])$ and the pessimistic-optimistic index $\lambda (\in [0,1])$. In (33), the decision maker may choose the parameters $\lambda (\in [0,1])$ and $\nu (\in [0,1])$. The pessimistic-optimistic index is taken as $\lambda = 1$ if he has pessimistic personal forecast in the market and he takes careful decision, and $\lambda = 0$ if he has optimistic personal forecast and he is not nervous. The possibility-necessity weight is taken as $\nu = 1$ when he has enough confidential information about the market, and $\nu = 0$ when he does not have confidential information. In this model, $\nu = 0$ is reasonable since our objective function is VaR, which is a kind of risk, and we need to take into account of the fuzziness of information in the market. While λ depends on the decision maker's attitude in his investment. In this example, we compute the pessimistic case $\lambda = 1$.

Let $n = 4$ be the number of assets. Take the expected rate of return, a variance-covariance matrix and imprecise factors as Table 1. We assume that the rate of return R_t^i has the Gaussian distributions. We discuss a risk probability 1% in the Gaussian distribution, and then the corresponding constant is $\kappa = 2.66521$, which is given in (14) for VaR. Then, the conditions in Theorem 1 are

Table 1. Variances-covariances σ_t^{ij} of R_t^i, expected rates of return μ_t^i and imprecise factors c_t^i

σ_t^{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0.35	0.03	0.02	−0.08
$i=2$	0.03	0.25	−0.06	0.08
$i=3$	0.02	−0.06	0.33	−0.02
$i=4$	−0.08	0.08	−0.02	0.24

	μ_t^i
$i=1$	0.05
$i=2$	0.07
$i=3$	0.06
$i=4$	0.04

	c_t^i
$i=1$	0.006
$i=2$	0.008
$i=3$	0.007
$i=4$	0.005

satisfied and by formulae of Lemma 8 we easily obtain the optimal portfolio $w^* = (w^1, w^2, w^3, w^4) = (0.23026, 0.21391, 0.251467, 0.304363)$ for the portfolio problem to minimize the rate of falling (P3), and then the corresponding rate of falling is $\delta(\gamma^*) = 0.645537$ and the expected rate of return is $\gamma^* = 0.0497492$.

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A Differential Evolution Based Time-Frequency Atom Decomposition for Analyzing Emitter signals*

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Abstract. This paper discusses the use of time-frequency atom decomposition based on a differential evolution to analyze radar emitter signals. Decomposing a signal into an appropriate time-frequency atoms is a well-known NP-hard problem. This paper applies a differential evolution to replace the traditional approach, a greedy strategy, to approximately solve this problem within a tolerable time. A large number of experiments conducted on various radar emitter signals verify the feasibilities that the time-frequency characteristics are shown by using a small number of decomposed time-frequency atoms, instead of traditional time-frequency distributions.

1 Introduction

The time-frequency atom decomposition, introduced by Mallat and Zhang in 1993 [1], is a kind of analysis technique for non-stationary signals. The main idea of this technique is to select a subset of elementary components with good time and frequency resolution, called time-frequency atom, from a redundant collection of waveforms of a signal, called time-frequency atom dictionary, to match the local structures of the signal [1] [2]. A large number of studies [3]-[5] show that the time-frequency atom decomposition is a good and flexible representation for various signals.

The common algorithm applies a greedy strategy to decompose a signal into atom components. This strategy selects atoms one by one from a very large dictionary in an iterative way. In [6], Davis et al. showed that selecting a subset of atoms from a redundant time-frequency atom dictionary to optimally approximate a signal is NP-hard problem. The extremely high computational load becomes a bottleneck for the application of the time-frequency atom decomposition, and moreover, if a time-frequency dictionary is relatively large, it turns out to be almost impossible to conduct the full search and represent the signals within a finite time [7].

In [2] [8]-[10], conventional genetic algorithms were used to solve the time-frequency atom decomposition of a signal and preliminary experiments show that genetic algorithms are attractive approaches to decrease the computational load of this

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decomposition. This paper uses a differential evolution, which has good global search capability, high precision of solutions and good convergence, to implement the time-frequency atom decomposition for analyzing radar emitter signals. Extensive experiments show that this approach has huge potentials of sketching the time-frequency characteristics of radar emitter signals with a small number of atoms and hence with a small quantity of computing time.

2 Problem Statement

In the time-frequency atom decomposition, an appropriate countable subset of time-frequency atoms $g_\gamma(t)_{\gamma\in\Gamma}$ is selected from an over-complete time-frequency atom dictionary $\mathcal{D} = (g_\gamma(t))$ to expand a signal $f(t)$ into a linear sum of $g_\gamma(t)$ [1], i.e.,

$$f(t) = \sum_{h=1}^{+\infty} a_h g_{\gamma_h}(t), \tag{1}$$

where a_h is the expansion coefficient of the atom $g_{\gamma_h}(t)$. It is reported in the study [6] that finding the optimal atoms and their corresponding optimal expansion coefficients in (1) to optimally approximate a signal in a redundant time-frequency dictionary is an NP-hard problem. In [1], an iterative approach was presented to perform time-frequency atom decomposition. This approach starts by projecting $f(t)$ on an atom $g_{\gamma_0} \in \mathcal{D}$ and computing the residual Rf:

$$f = \langle f, g_{\gamma_0} \rangle g_{\gamma_0} + Rf, \tag{2}$$

where Rf is the residual vector after approximating f in the direction of g_{γ_0}. By projecting the residual Rf on an atom of $\mathcal{D}$, this iterative approach can subdecompose the residual Rf sequentially. At the h th iteration, a best time-frequency atom g_{γ_h} is searched from a time-frequency atom dictionary $\mathcal{D}$ to maximize the module $\left|\langle R^h f, g_{\gamma_h} \rangle\right|$, where $R^h f$ is the h th order residual of the signal $f(t)$ and $R^0 f = f$. Finally, the signal f can be represented as

$$f = \sum_{h=0}^{H-1} \langle R^h f, g_{\gamma_h} \rangle g_{\gamma_h} + R^H f, \tag{3}$$

where H is the maximal number of iterations. To reduce the intolerable computational complexity, this paper uses the differential evolution (DE) to replace the greedy strategy [1] to search the suboptimal time-frequency atom from redundant time-frequency atom dictionaries. The pseudocode algorithm for solving this problem is shown in Fig.1 [11].

Begin

$R^0 f = f$; $h = 0$;

While (not termination condition) **do**

Set parameters of time-frequency atom;

(*) Search the suboptimal time-frequency atom g_{γ_h} in $\mathcal{D}$ using DE;

Compute $\left| \langle R^h f, g_{\gamma_h} \rangle g_{\gamma_h} \right|$;

$R^h f \leftarrow (R^h f - \langle R^h f, g_{\gamma_h} \rangle g_{\gamma_h})$;

$h \leftarrow h + 1$;

End

End

Fig. 1. Pseudocode algorithm for time-frequency atom decomposition using differential evolution (DE)

3 Differential Evolution

A differential evolution (DE), proposed by Storn and Price in 1995 [12], is a version of evolutionary computation for numeric optimization problems. In a DE, the evolutionary operators consist of mutation, crossover and selection ones. In these three operators, the mutation operation, which generates a vector by adding the base vector with a scaled difference of a randomly sampled pair of vectors, plays a very important role to guide the individuals toward the optimal solution [13]. A DE has several characteristics, such as good global search capability, high precision of solutions resulted from real-valued coding, good convergence, small number of parameters to adjust in the process of evolution, etc. [14]-[16]. As shown in Fig.2, this paper uses a modified DE to implement the time-frequency atom decomposition [12]-[17]. In what follows a brief description about the DE is listed.

(1) Initialization: this step produces a set of initial candidate solutions, which forms an initial population with N individuals described as $P^0 = [X_1^0, X_2^0, ..., X_N^0]$, where $X_i^0 = [x_{i,1}^0, x_{i,2}^0, ..., x_{i,D}^0]$ and $i = 1, 2, ..., N$, where $x_{i,j}^0$ denotes the j th variable of i th individual and D represents the number of variables. If there is not prior knowledge available about the problem, the initial population is randomly generated in the search space by using uniform distribution, i.e.,

$$x_{i,j}^0 = a_j + rand(0,1)(b_j - a_j) \tag{4}$$

where $i = 1, 2, ..., N$, $j = 1, 2, ..., D$, a_j and b_j are the lower and upper bounds of the j th variable and $rand(0,1)$ is a real-valued random number in the range $[0,1]$.

```
Begin
(1)   Initialization;    % generation G=0
      While (not termination condition) do
         G=G+1;
(2)      Determine Differential factor F;
(3)      Mutation;
(4)      Crossover;
(5)      Selection;
      End while
End Begin
```

Fig. 2. The pseudocode algorithm for DE

(2) This step uses a dynamic adjustment method for differential factor F, which is defined as

$$F_G = F_{\max} - \frac{G}{IterNum}(F_{\max} - F_{\min}) \tag{5}$$

where $IterNum$ is the number of the iterations; G denotes the G th iteration and $G \le InterNum$; F_G is the differential factor for G th iteration; $F_{\max}$ and $F_{\min}$ stands for the upper limit and lower limit of differential factor F, respectively.

(3) Except for the best vector, the information of other individuals in the population is considered to modify the mutation operator. We use the following mutation strategy

$$V_i^{G+1} = X_i^G + F \cdot (X_{best}^G - X_{r1}^G) + F \cdot (X_{aver}^G - X_{r2}^G) \tag{6}$$

$$X_{aver}^G = \frac{1}{n}\sum_{i=1}^{n} \overline{X}_i^G \tag{7}$$

where $r1 \neq r2 \neq i$, $r1, r2 \in \{1, 2, ..., N\}$, $\overline{X}_i^G$ is the i th individual of the sorted population in an descending order according to the fitness values, and X_{aver}^G is the center of the first n best vectors.

(4) In this step, the recombination between target vector X_i^G and mutant vector V_i^{G+1} is performed to produce an offspring, U_i^{G+1}, $U_i^{G+1} = [u_{i,1}^{G+1}, u_{i,2}^{G+1}, ..., u_{i,D}^{G+1}]$ where $u_{i,j}^{G+1}$ is

$$u_{i,j}^{G+1} = \begin{cases} v_{i,j}^{G+1}, & if \quad rand_i^{G+1}[0,1] \le C_r \quad or \quad j = k \\ x_{i,j}^G, & otherwise \end{cases}, \quad j = 1, 2, ..., D \tag{8}$$

where C_r is a predefined crossover rate in the range of $(0,1)$, and $rand_i^{G+1}[0,1]$ is a real-valued random number in [0,1], and k is a random parameter index $k \in \{1,2,...,D\}$. Each value of k corresponds to each i. The condition $j = k$ guarantees that at least one component of the vector U_i^{G+1} comes from the mutant vector V_i^{G+1} so that U_i^{G+1} will not be identical with the vector X_i^G.

(5) If an offspring is produced, we will determine which one of the two vectors X_i^G and U_i^{G+1} goes to the next generation. If a minimal optimization problem is considered, we will perform the following selection operation

$$X_i^{G+1} = \begin{cases} U_i^{G+1} & if \ \ f(U_i^{G+1}) < f(X_i^G) \\ X_i^G & otherwise \end{cases} \tag{9}$$

4 Experiments and Results

This section uses the above approach, the time-frequency atom decomposition based on a differential evolution, to analyze various radar emitter signals (RESs) including conventional (CON) RESs, binary phase shift-key (BPSK) RESs, linear frequency modulation (LFM) RESs and nonlinear frequency modulation (NLFM) RESs. The sampling frequency is set to 100 MHz. In the experiments, the population size, the crossover rate, the maximum and minimum values of differential factors in the DE are set to 20, 0.5, 0.9 and 0.3, respectively. The maximal number of generations is set to 300 as the termination condition of the DE. The termination condition of the time-frequency atom decomposition is the number of atoms required, 10 for CON and 15 for the others. The time-frequency atom dictionary is constructed by using the real Gabor atom with Gaussian modulation

$$g_\gamma(t) = \frac{k}{\sqrt{s}} g(\frac{t-u}{s}) \cos(vt + w) \tag{10}$$

$$g(t) = e^{-\pi t^2} \tag{11}$$

where k is the normalized constant and $\gamma = (s,u,v,w)$ is parameter set, in which s,u,v,w represent scale, translation, modulation and initial phase of an atom [1].

First, we apply 10 CON RESs to conduct the experiments. The pulse widths (PW) of these RESs are 10 *u*s and the frequencies are chosen as 12 MHz, 14 MHz, 16 MHz, 18 MHz, 20 MHz, 22 MHz, 24 MHz, 26 MHz, 28 MHz and 30 MHz, respectively. To each signal, we choose 10 time-frequency atoms decomposed to outline its time-frequency characteristics, shown in Fig.3, by using their translation and modulation parameters obtained.

Second, 10 LFM, 4 NLFM with parabola frequency modulations (NLFM/p) and 4 NLFM with sinusoidal frequency modulations (NLFM/s) are used to carry out the experiments. The parameters for the three types of RESs are listed in Table 1, Table 2 and Table 3, respectively, and their corresponding time-frequency characteristics are shown in Fig.4, Fig.5 and Fig.6, respectively.

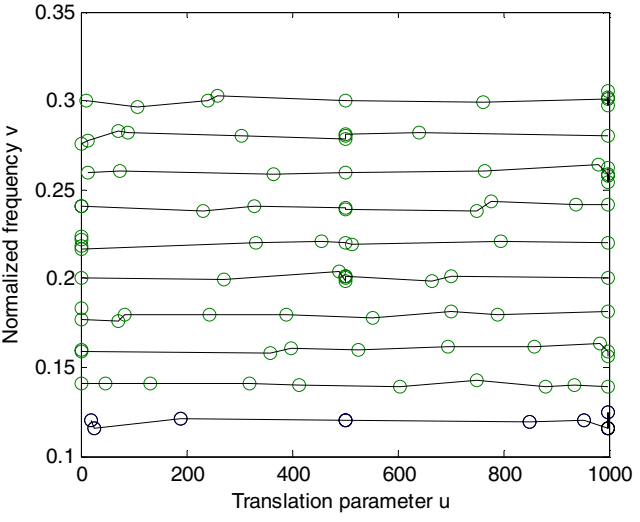


Fig. 3. Time-frequency characteristics of CON

Table 1. Parameters for LFM

No.	Starting frequency f_1 (MHz)	Ending frequency f_2 (MHz)	Frequency slope (f_2-f_1) (MHz)	Sampling frequency (MHz)	PW (us)
1	10	20	10	100	10
2	10	25	15	100	10
3	10	30	20	100	10
4	10	35	25	100	10
5	10	40	30	100	10
6	20	30	10	100	10
7	20	35	15	100	10
8	20	40	20	100	10
9	20	45	25	100	10
10	30	45	15	100	10

Table 2. Parameters for NLFM/p

No.	Frequency of the 1st point (MHz)	Frequency of the 400th point (MHz)	Frequency of the 1000th point (MHz)	Sampling frequency (MHz)	PW (us)
1	2	8	35	100	10
2	4	10	35	100	10
3	6	10	40	100	10
4	10	15	40	100	10

Finally, we go further to use the introduced approach to detect the code sequences employed in the BPSK RESs. Unlike the ways that the information of a small number of atoms is applied to show the time-frequency characteristics of CON, LFM and NLFM, the residual information obtained by subtracting a certain number of atoms

Table 3. Parameters for NLFM/s

No.	Minimal frequency (MHz)	Maximal frequency (MHz)	Sampling frequency (MHz)	PW (μs)	Number of periods
1	15	35	100	10	1
2	10	40	100	10	1
3	5	45	100	10	1
4	12	42	100	10	1

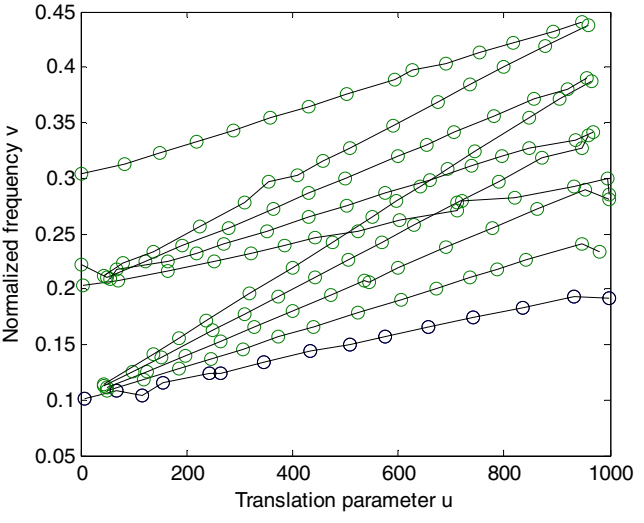


Fig. 4. Time-frequency characteristic of LFM

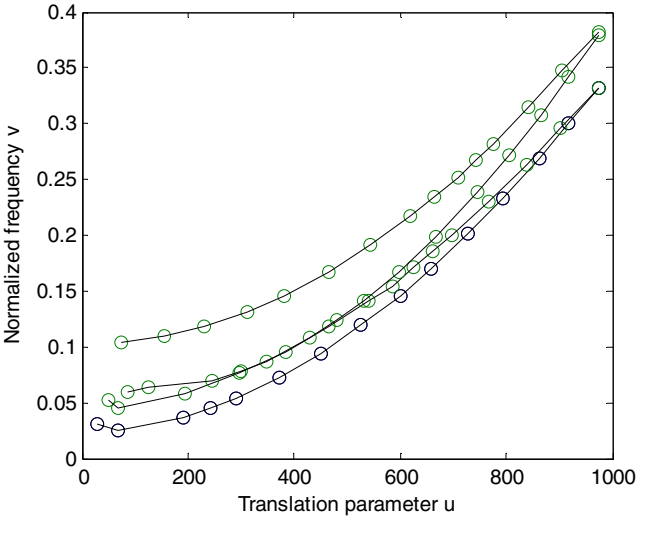


Fig. 5. Time-frequency characteristics of NLFM/p

from the original signal is utilized to illustrate the sharp changes of phases due to code sequences. The parameters used in BPSK RESs are provided in Table 4 and their residual energies are shown in Fig.7.

As shown in Fig.3, 10 time-frequency atoms can outline a brief time-frequency feature of each CON RES. The 10 CON RESs with different carrier frequencies have distinct positions in the time-frequency plane (translation-modulation plane). Furthermore, we can easily read the specific frequency value of each signal, which approximately corresponds to the given frequency parameter. To be specific, 10 types of carrier frequencies ranged from 12 to 30 MHz can be identified.

The three figures, Fig.4, Fig.5 and Fig.6, show that the time-frequency characteristics of RESs with more complex modulations such as linear, parabola and sinusoidal frequency modulations can obtained by using 15 atoms decomposed. Figure 4 clearly shows the changing trends of the 10 different LFM RESs. Figure 5 and Fig.6 demonstrate that frequency-modulated laws of 4 NLFM/p and NLFM/s RESs, respectively.

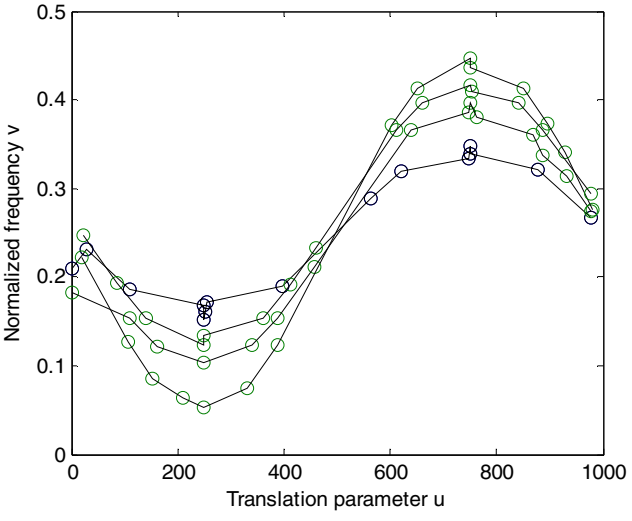
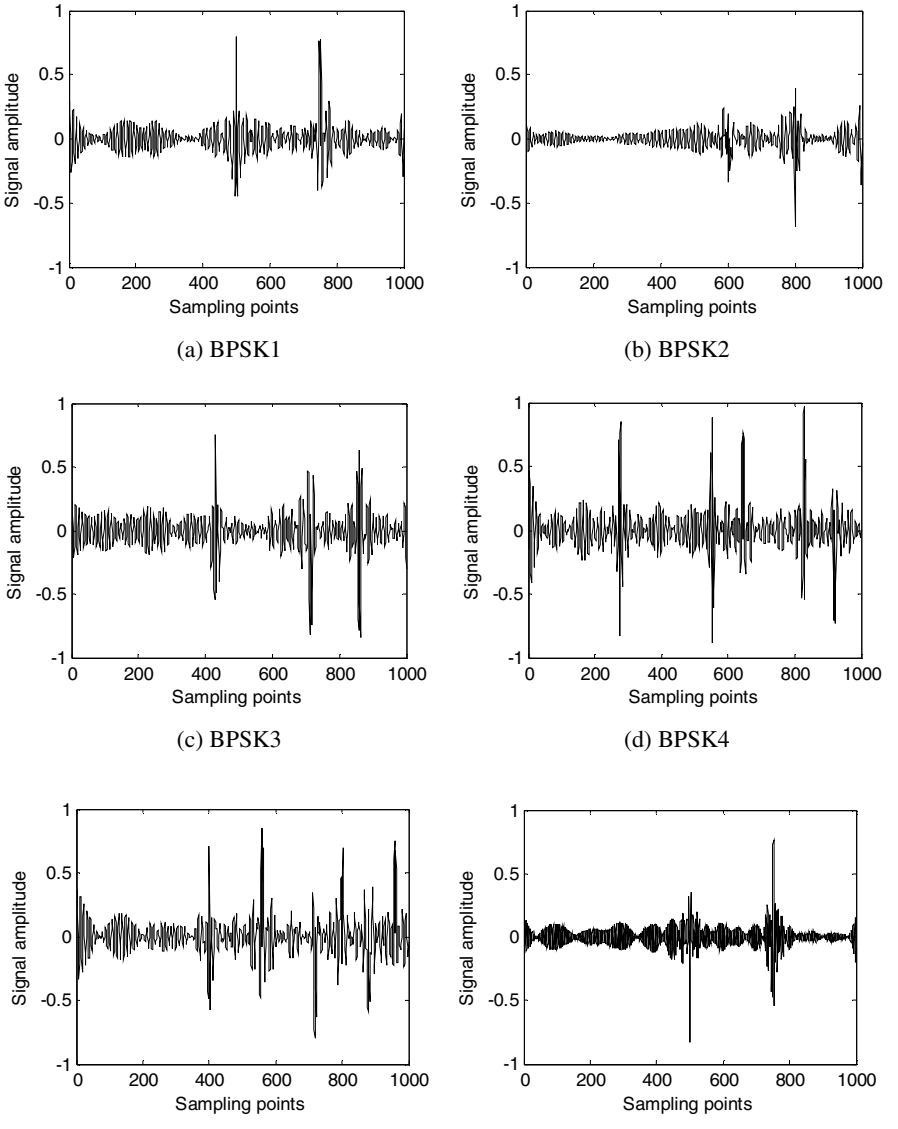


Fig. 6. Time-frequency characteristics of NLFM/s

Table 4. Parameters for BPSK

Signal	Code width (*u*s)	Frequency (MHz)	Sampling frequency (MHz)	Barker codes
BPSK1	2.5	10	100	+ + - +
BPSK 2	2	10	100	+ + + - +
BPSK 3	1.43	10	100	+ + + - - + -
BPSK 4	0.92	10	100	+ + + - - - + - - + -
BPSK 5	0.8	10	100	+++++- - ++-+-+
BPSK 6	2.5	20	100	+ + - +
BPSK 7	2	20	100	+ + + - +
BPSK 8	1.43	20	100	+ + + - - + -
BPSK 9	0.92	20	100	+ + + - - - + - - + -
BPSK 10	0.8	20	100	+++++- - ++-+-+

It is ongoing issue to identify the specific code sequences used in a BPSK RES. It can be seen from Fig.7 that the phase changes of Barker code sequences hide in the residual signal obtained by subtracting the energy of time-frequency atoms from the original signal, instead of in the time-frequency atoms decomposed. In Fig. 7 (a) and (f), there are two sharp changes at sampling points 500 and 750 corresponding to the phase from positive to negative and then to positive. Figure 7 (b) and (g) show that

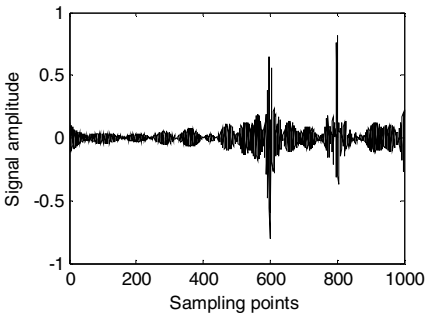


(a) BPSK1 (b) BPSK2

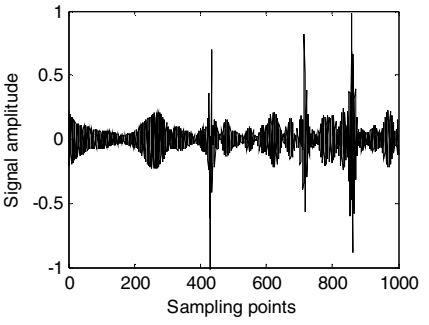
(c) BPSK3 (d) BPSK4

(e) BPSK5 (f) BPSK6

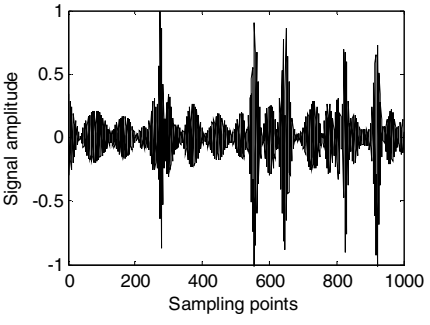
Fig. 7. Residual energies of different BPSK RESs



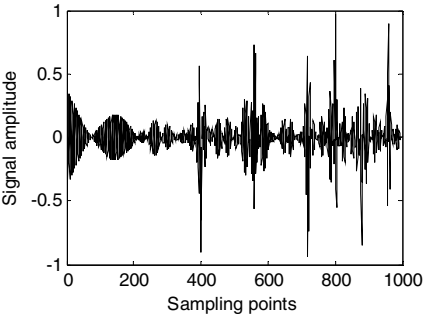
(g) BPSK7



(h) BPSK8



(i) BPSK9



(j) BPSK10

Fig. 7. (*continued*)

there are phase changes at sampling points 600 and 800. The changing cases in Barker code with length 7 can be identified in Fig. 7 (c) and (h). The Barker codes with length 11 and 13 have five and six turns in phase, respectively, Fig. 7 (d) and (i), (e) and (j), however, still clearly show the positions of these turns. These results indicate that the time-frequency atom decomposition based on a differential evolution can be applied to analyze the phase changes of signals and also to locate the positions of phase changes.

5 Conclusions

It is a time-consuming process to use time-frequency transforms to illustrate the time-frequency characteristic of a signal, which is a common technique to analyze a time-variant signal in the literature. Beyond this idea, this paper uses a novel opinion to arrive at the same aim. The time-frequency atom decomposition based on a differential evolution is introduced to analyze radar emitter signals and a small number of time-frequency atoms decomposed from the original signal or the residual signals after the decomposition process are employed to sketch the time-frequency

characteristics or phase changes of radar emitter signals. Experiments carried out on various radar emitter signals with different modulations verify the feasibility and effectiveness of this presented idea.

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Combining Various Methods of Automated User Decision and Preferences Modelling

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Abstract. In this paper we present a proposal of a system that combines various methods of user modelling. This system may find its application in e-commerce, recommender systems, etc. The main focus of this paper is on automatic methods that require only a small amount of data from user. The different ways of integration of user models are studied. A proof-of-concept implementation is compared to standard methods in an initial experiment with artificial user data...

Keywords: user preferences learning, recommender systems.

1 Introduction

The search for the best product is a very usual task for most people. Be it a digital camera, a notebook, a kitchen desk, a song, a photo or a flat, user has to decide what is best for her. Then, she has to search for the product that most satisfies her preferences. The task is how to transform the preferences sensed in human way into the language the computer would also understand.

The use case for us is a user looking for a notebook. She has some preferences about its attributes and wants that the system will recommend her the most appropriate notebook. There are various ways for recommending things on the web. They perform differently in different conditions.

Our contribution is to study their strengths and weaknesses and propose a way to combine them efficiently according to the current state of the system.

In Section 2 some user models and methods for their acquisition are described. Our system for combining preferences is studied in Section 3 and tested in Section 4. Section 5 contains conclusions and some future work.

2 User Model Acquisition and Related Work

In this section, some of the methods for acquisition of user model will be described. Our model of user preferences is partial ordering of objects. The ordered list (or only the first 10 objects) is then presented to the user, who can easily see the most interesting objects at the top of the list as it is usual in search engines. The user

V. Torra, Y. Narukawa, and M. Inuiguchi (Eds.): MDAI 2009, LNAI 5861, pp. 172–181, 2009.

can proceed to the following objects, but they are supposed to be of less interest for her. We assume that every method is able to give its recommendation in the form of a rating of a given object. Ratings of objects are then used to order them. The following four types of methods are taken into account:

Content-based (in further text as CB) methods are based on the attributes of objects. In our case of notebooks, these are size of RAM, harddisk, display, manufacturer and price of the notebook. User model based on fuzzy logic is used in our system proposal. It is two-step model; the first step is normalisation of the attribute values of the object, and the second is combination of these normalised values into the overall rating of the whole object.

User ratings are used for construction of this model. User associates a rating to some objects. Ratings may be represented in various ways: as stars, smileys, school marks etc, but they have to be totally ordered. Our method Statistical, described in [1,2], is used for acquisition of this user model.

CB techniques *need meaningful attributes of objects*. They are unsuitable for example for finding a good joke or a book. These methods also require *some objects to be rated by the user*. There is *no requirement on other users*; this method depends only on objects' attributes and the number of ratings user has made.

Collaborative filtering (CF) is based on similarity of users. The idea behind CF is expressed in the sentence "Users who are similar to you have rated this object with ". CF needs *a lot of users and a lot of ratings* so that a recommendation can be made. CF shows *good precision* under these conditions. Collaborative filtering was described for example in [3,4,5].

Clickstream analyse (CA) is the method that *requires the least effort* from the user. Her behaviour when working with the system is gathered; actions like clicking on a link, scrolling down a document, adding an item to the cart, closing the page etc., are collected. They are interpreted as if they were motivated by user's preferences. However, CA needs *a lot of information about the user* and the predictions are *not always very reliable*. This method was studied among others in [6,7,8].

Direct query (DQ) can be viewed as a user preferences definition. These queries may have a traditional form of keyword search, but in e-shop environment, they will have typically a form of conjunctive restrictions of objects' attributes. For example, in notebook area, user may specify that she wants notebooks with display from 10" to 12", harddisk at least 160GB and the only producer allowed is HP. Example of an interface for such queries is in Figure 1. When the user issues a query, objects that satisfy the query obtain the highest rating and the objects that do not fit into receive the worst rating. Querying *requires most effort from the user* - she has to specify as many restrictions as possible for obtaining the result set of acceptable size. Many queries *return no object or far too many objects* to be suitable for human processing.

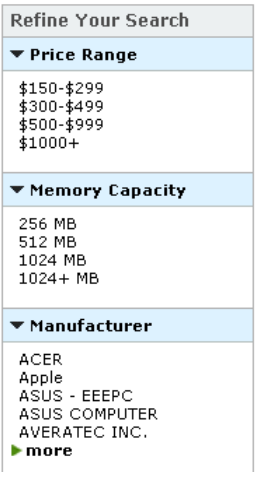

Fig. 1. Direct query interface

Every method will be addressed by its abbreviation in following text and we expect that they give recommendations as ratings of objects:

Content based	$CB(o) : Object \to [0,1]$,
Collaborative filtering based	$CF(o) : Object \to [0,1]$,
Click stream analysis based	$CA(o) : Object \to [0,1]$ and
Direct query result	$DQ(o) : Object \to [0,1]$

3 Combining Recommendation from Various Sources

The proposal of a system that combines preferences from various sources will be studied in this section. We are considering four different methods for preference acquisition and three different inputs from the user - ratings, behaviour and querying the system as they were described in Section 2. There is one module that combines the outputs of these methods. We will now classify methods that are used according to [9].

As we can see in Figure 2, we are trying to combine recommendations from rather automatic (no effort from the user) and persistent (user dependent, they require some information about user or her history) methods with a method that is manual (some effort from user is expected) and ephemeral (does not depend on the user, only on the query issued). As we will see, the added information from DQ will improve the recommendation from traditional recommendation methods. The system uses only the highest rating of the objects that satisfy the direct query, because the lowest rating is influencing badly the stability of combined rating. It is simply too low and it penalises objects that are mediocre.

We are concentrating more on automatic methods, because they require little effort from user. This is an alternative to [10], where every category is studied

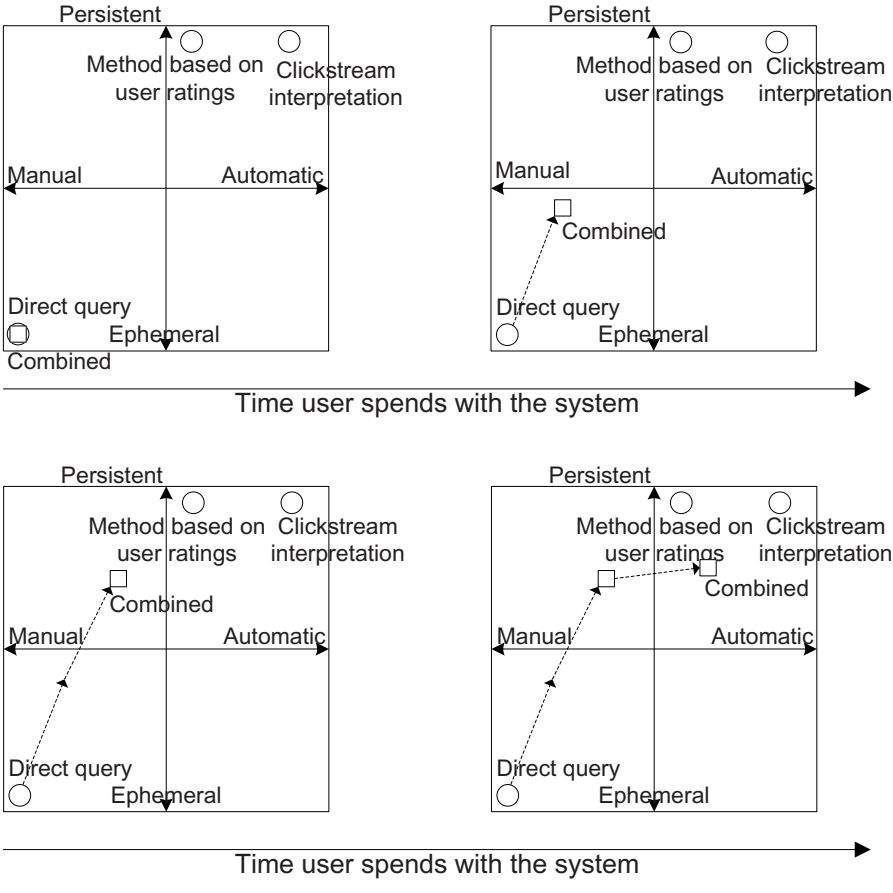

Fig. 2. Classification of recommendation methods

in depth. According to [11] very few users tend to fill complicated forms even if it enhances search results and recommendation accuracy.

The way to combine recommendations given by each method is influenced by the amount of information the system has about the user. Combined method starts as purely Manual and Ephemeral method (see Figure 2), as in the beginning it has no information about the user. The more time the user spends with the system the more information the system acquires and the more automatic and persistent Combined method can be. There is a weight associated with each method: wCA, wCB, wCF and wDQ. Overall rating is computed as weighted average:

$$Combined(o) = \frac{w_{CA} * CA(o) + w_{CB} * CB(o) + w_{CF} * CF(o) + w_{DQ} * DQ(o)}{w_{CA} + w_{CB} + w_{CF} + w_{DQ}}.$$

Table 1. The weights of different recommendation methods

Weight	1) New user	2) Some ratings	3) Users spends some time with the system	4) Different query
wCB	0	1	1	1/0
wCF[1)]	0	1/0	1/0	1/0
wCA	0	0	1	1/0
wDQ	1	0.8	0.2	1

[1)] Only if there are enough users, otherwise 0.

3.1 Weights of Methods

In the following paragraph, we will describe some typical scenarios and how this system will behave at them. The weights are summarised in Table 1.

1) When a new user comes to the system, it has no information about her. She issues a query to the system. In this case, only DQ is taken into account, because we cannot use any other method without any information about the user.
2) After she got some results to her query, the user rates some small number of objects. Now we have some information about her. CB will make some recommendation. If our system already has a large number of other users, CF can be used also. Results from DQ are taken with less importance. If there are few users that are similar enough to the current user, CF stays out of the way.
3) After some time the user spends with the system, enough user behaviour has been accumulated for CA so that it can be incorporated into the result.
4) If the user makes a new query, then the weight of direct query is refreshed. According to [6] it can be estimated whether the user is using the system systematically or just browsing with no interest. In the first case, the weights of other methods remain the same, in the latter case we take into account only DQ.

4 Experiments

4.1 Experiment Settings

A set of 198 notebooks extracted from the web was used for experiments. Notebooks have five attributes: harddisk, display, price, producer and ram. We have created 4 artificial users with different preferences. Result is 4 ratings on the scale {0,1,2,3,4,5} for every object. Artificial users were represented by a set of methods that compute preference over attributes and an aggregation function. We manually set the preference of producers for nominal attribute producer. An example of assigning the preference is :

Table 2. Preference of producers

Producer	Preference
ACER	0.8
FUJITSU	0.3
HP	0.4
IBM	0.1
TOSHIBA	0.7
ASUS	0.6
MSI	0.2
SONY	0.4
LENOVO	0.3

A function that normalises the domain was used for numerical attribute; this function has an ideal value, for which the preference is 1 and the preference linearly decreases with the increasing distance from this ideal value. An example of normalisation function is $|DisplaySize - 15|$ for the size of display.

Testing was performed with a traditional cross-validation method. Because we are dealing with user ratings, it can not be expected that the user will rate many objects. That is why we limited the size of the training set (TSS in following text) to 40 ratings at maximum. Methods were tested on the rest of the set.

There are six methods tested: **Mean, Statistical, Collaborative filtering, Multilayer perceptron, Direct query and CombinedMethod**. Mean is a baseline. This method always returns the average rating from the training set. Multilayer perceptron is a traditional data mining method. We also tested Support Vector Machine, but it gave almost identical results as Multilayer perceptron. Implementation from Weka [12] was used. Direct query returns either 5 or 0 according to whether the object fits into the query issued by the user or not. It does not depend on the TSS, because it simply has no training phase. CombinedMethod is the proposed method that combines Statistical, CF and DQ. In our case, collaborative filtering plays small role (because of a small number of users), so the major influence is from Statistical and Direct query. Every method gets the same weight, but CF fails to predict often, as we will see.

4.2 Results

RMSE. In Figure 3 there are results for RMSE (root of mean squared error), a traditional data-mining error measure. To explain the strange behaviour of CF, we studied the number of objects for which the method failed to give prediction. For all TSSes, CF failed to predict more than 50% of objects, and until TSS = 20 it was 90%. So we do not take CF into account from now on. For TSS = 2, Statistical is the best. For other TSSes, the combined method is the best and it is about the same as Multilayer perceptron for TSS = 40. DQ is not in the graph, because it has RMSE about 2.5 for every TSS.

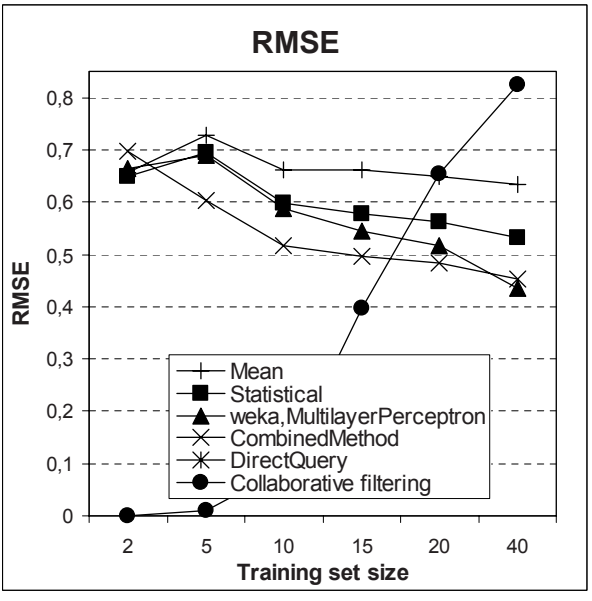

Fig. 3. Root of mean squared error of recommendation methods

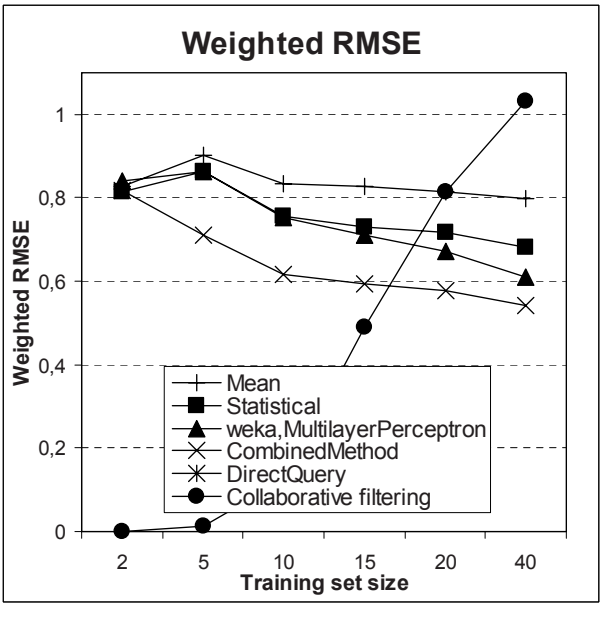

Fig. 4. Root of mean square of weighted error of recommendation methods

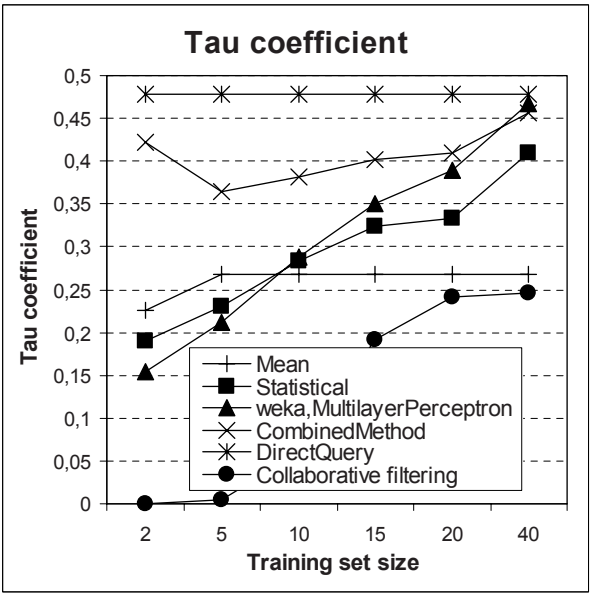

Fig. 5. Tau coefficient of recommendation methods

Weighted RMSE. We also studied weighted RMSE in Figure 4. Weighted RMSE is the same as RMSE except that the errors are weighted with the original ratings - the higher the rating is, the higher weight it gets

$$\frac{\sum_{o \in Trainingset} r(o) * |r(o) - \widehat{r(o)}|}{\sum_{o \in Trainingset} r(o)},$$

where $r(o)$ is user rating of object o and $\widehat{r(o)}$ is the rating determined by the tested method.

The motivation behind this is that we are more in interested in recommending good notebooks and the preferences of not-preferred ones are not so important. For Weighted RMSE, our Combined method was even better than for RMSE, but not significantly.

Tau coefficient. Tau correlation coefficient measures similarity of two ordered lists. The higher the coefficient is, the better the method corresponds to real user preferences. The first list contains all notebooks from test set ordered by real user preferences, the second one is ordered by the method. Results are in Figure 5. Our combined method is the second best for every TSS except 40. The best one is DQ. It is because it uses the real user preferences for generation of the query. This result can be viewed as a real success of combination of a manual recommendation method and an automatic one.

5 Conclusion

The proposal of a system that combines various sources of user preferences was described and tested. It is motivated by the fact that a system gets various inputs from the user and each method uses different one. However the combination of these methods performs better than the methods individually.

In experiments, combined method have proven itself worth trying, it outperformed almost every other method in every criteria.

As of future work, we have to design an experimental setting, where collaborative filtering will perform better. This will also affect Combined method. To let clickstream analyse influence the result, we would have to collect information about some user behaviour and interpret it, which seems like a long term task.

We would like also study the influence of our method Phases [13] on the behaviour of the system, especially on changing weights of methods.

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Target-Oriented Decision Analysis with Different Target Preferences

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Abstract. Decision maker's behavioral aspects play an important role in human decision making, and this was recognized by the award of the 2002 Nobel Prize in Economics to Daniel Kahneman. Target-oriented decision analysis lies in the philosophical root of bounded rationality as well as represents the S-shaped value function. In most studies on target-oriented decision making, monotonic assumptions are given in advance to simplify the problems, e.g., the attribute wealth. However, there are three types of target preferences: "the more the better" (corresponding to benefit target preference), "the less the better" (corresponding to cost target preference), and equal/range targets (too much or too little is not acceptable). Toward this end, two methods have been proposed to model the different types of target preferences: cumulative distribution function (cdf) based method and level set based method. These two methods can both induce four shaped value functions: S-shaped, inverse S-shaped, convex, and concave, which represents decision maker's psychological preference. The main difference between these two methods is that the level set based method induces a steeper value function than that by the cdf based method.

Keywords: Satisfactory-oriented decision, S-shaped function, Target-oriented decision analysis, Cumulative distribution function, Level set, Target preference type.

1 Introduction

Traditionally, when modeling a decision maker's (DM for short) rational choice between acts with uncertainty, it is assumed that the uncertainty is described by a probability distribution on the space of states, and the ranking of acts is based on the expected utilities of the consequences of these acts. This utility maximization principle was justified axiomatically in Savage [18] and von Neumann and Morgenstern [17]. However, substantial empirical evidence and recent research have shown that it is difficult to build mathematically rigorous utility functions based on attributes [3] and the conventional attribute utility function often does not provide a good description of individual preferences [12]. As a substitute for utility theory, Kahneman and Tversky [12] propose an S-shaped value function, and Heath et al. [9] suggest that the inflection point in this S-shaped value

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function can be interpreted as a target. To develop this concept further, target-oriented decision analysis involves interpreting an increasing, bounded function, properly scaled, as a cumulative distribution function (cdf) and relating it to the probability of meeting or exceeding a target value.

Berhold [2] notes that "there are advantages to having the utility function represented by a distribution" (p. 825), arguing that it permits the use of the known properties of distribution functions to find analytic results. Manski [16] calls this the "utility mass model". Castagnoli and LiCalzi [5] prove that expected utility can be expressed in terms of "expected probability", with the utility function for performance interpreted as a cdf in the case of a single attribute (see also Bordley and LiCalzi [4]). Interestingly, the Savage's utility function [17] can always be interpreted as the probability of achieving a target [4,5]. In maximizing expected utility, a DM behaves as if maximizing the probability that performance is greater than or equal to a target, whether the target is real or just a convenient interpretation. More details on target-oriented decision analysis could be referred to [11,14,21].

In general, target-oriented decision analysis lies in the philosophical root of bounded rationality [19] as well as represents the S-shaped value function [12]. However, in most studies on target-oriented decision analysis, monotonic assumption of attribute is given in advance to simplify the decision problems. In the context of decision analysis involving targets/goals, usually there are three types of goals: "the more the better" (corresponding to benefit target), "the less the better" (corresponding to cost target), and goal values are fairly fixed and not subject to much change, i.e., too much or too little is not acceptable (we shall call this type of targets as equal/range targets). Thus it is important to consider these three types of targets. Based on the above observations, we summarize our primary contributions as follows. Two methods have been proposed to model the different target preference types: cdf based method and level set based method. No matter which method is selected, both of these two methods can induce four shaped value functions: S-shaped, inverse S-shaped, convex, and concave, which represents DM's psychological (behavioral) preference. The main difference between these two methods is that the level set based model induces a stricter value function than the cdf based model.

The rest of this paper is as follows. Section 2 proposes a target-oriented decision model with different target preferences based the cumulative distribution function. In Section 3 we propose a target-oriented decision model based on the level sets of probability distributions. In Section 4, two examples are used to illustrate the proposed models. Finally, some concluding remarks are also given in Section 5.

2 Target-Oriented Decision Analysis Based on Cumulative Distribution Function

For notational convenience, let us designate an evaluation attribute by X, and an arbitrary specific level of that evaluation attribute by x. We also restrict

the variable x to a bounded domain $D = [X_{\min}, X_{\max}]$. Suppose that a DM has to rank several possible decisions. Assume for simplicity that the set $\mathcal{A}$ of consequences is finite and completely ordered by a preference relation $\succeq$. Denote by p_d his probability distribution for the random consequence X_d associated with a decision d. The expected utility model suggests that the ranking be obtained by using the following value function

$$V(d) = \mathrm{EU}(X_d) = \sum_x U(x) \cdot p_d(x) \tag{1}$$

where $U(x)$ is a Von Neumann and Morgenstern (NM-)utility function over consequences.

An expected utility DM is defined to be target oriented for a single attribute decision if the DM's utility for an outcome depends only on whether a target is achieved with respect to x [3], thus a target-oriented DM has only two different utility levels, and because a utility function is only specified to within a positive affine transformation, these two utility levels can be set to one (if the target is achieved) and zero (if the target is not achieved). Then a target-oriented DM's expected utility for decision d is

$$\begin{aligned} V(d) &= \Pr(X_d \succeq T) \\ &= \sum_x \left[\Pr(x \succeq T) * 1 + (1 - \Pr(x \succeq T)) * 0\right] p_d(x) \\ &= \sum_x \Pr(x \succeq T) p_d(x) \end{aligned} \tag{2}$$

where $\Pr(x \succeq T)$ is the probability of meeting the uncertain target T and T is stochastically independent of X_d. The idea that the NM-utility function U should be interpreted as a probability distribution may appear unusual but, in fact, NM-utilities are probabilities [1,4]. With the assumption that the attribute is monotonically increasing, x and t are mutually independent, Bordley and Kirkwood [3] suggest the following function

$$\Pr(x \succeq T) = \int_{X_{\min}}^{x} p(t)dt, \tag{3}$$

where $p(t)$ is the probability density function of uncertain target T.

In most studies on target-oriented decision making, monotonic assumptions of attributes (e.g., wealth) are given to simplify the problems. In many decision problems involving goals/targets, usually there are three types of goal preferences [13].

- Goal values are adjustable: "more is better" (we shall call benefit target);
- Goal values are adjustable: or "less is better" (with respect to cost target);
- Goal values are fairly fixed and not subject to much change, i.e. too much or too little is not acceptable (we shall call this type of target as equal/range target).

The target-oriented decision model assumes that the probability distribution with respect to the uncertain target is unimodal as well as views the mode value of probability distribution of the uncertain target as the reference point, denoted as T_m [4]. To model the three types of goal preference, similar with Bordely and Kirkwood [3], we define

$$\Pr(x \succeq T) = \int_{X_{\min}}^{X_{\max}} u(x,t)p(t)dt. \tag{4}$$

As target-oriented decision model has only two different utility levels, we can define $u(x,t)$ as follows.

In case of benefit target preference, the DM has a monotonically increasing preference. As target-oriented model assumes that there are only two levels of utility (1 or 0), thus we define $u(x,t) = \begin{cases} 1, x \geq t; \\ 0, \text{otherwise.} \end{cases}$ Then we can obtain the probability of meeting uncertain target T as the following function

$$\Pr(x \succeq T) = \Pr(x \geq T) = \int_{X_{\min}}^{x} p(t)dt. \tag{5}$$

This is consistent with the target-oriented model in the literature [4,5], i.e. the target-oriented model views the cdf as the probability of meeting uncertain target T.

Similar with the benefit target, for cost target we define $u(x,t) = \begin{cases} 1, x \leq t; \\ 0, \text{otherwise.} \end{cases}$ Then the probability of meeting cost target is as follows

$$\Pr(x \succeq T) = \Pr(x \leq T) = \int_{x}^{X_{\max}} p(t)dt = 1 - \int_{X_{\min}}^{x} p(t)dt \tag{6}$$

In case of equal/range target preference, the mode value T_m is the reference point. There will be added loss of value for missing the reference point on the low side, or added loss for exceeding the reference point. When $x = T_m$ the probability of meeting target should be equivalent to one. Based on this observation, we define the target achievement function as follows:

$$\Pr(x \succeq T) = \Pr(x \cong T) = \begin{cases} \frac{\int_{X_{\min}}^{x} p(t)dt}{\int_{X_{\min}}^{T_m} p(t)dt}, & \text{if } x < T_m; \\ 1, & \text{else if } x = T_m; \\ \frac{\int_{x}^{X_{\max}} p(t)dt}{\int_{T_m}^{X_{\max}} p(t)dt}, & \text{otherwise.} \end{cases} \tag{7}$$

Generally speaking, when the DM has an equal/range target preference, the value domain below the reference point T_m can be viewed as a pseudo benefit attribute; the value domain upper than the reference point T_m can be viewed as pseudo cost attribute. As a generation of this type of target preference, the reference point value T_m may have a interval range, such that $T_m \equiv [T_{ml}, T_{mu}]$.

In this case, Eq. (7) becomes

$$\Pr(x \succeq T) = \Pr(x \cong T) = \begin{cases} \frac{\int_{X_{\min}}^{x} p(t)dt}{\int_{X_{\min}}^{T_{ml}} p(t)dt}, & \text{if } x < T_{ml}; \\ 1, & \text{else if } x \in [T_{ml}, T_{mu}]; \\ \frac{\int_{x}^{X_{\max}} p(t)dt}{\int_{T_{mu}}^{X_{\max}} p(t)dt}, & \text{otherwise.} \end{cases} \tag{8}$$

3 Target-Oriented Decision Analysis Based on the Level Sets of Probability Density Function

The random target has an interval or range domain and the level set of probability density distribution (pdf) provides a convenient way to represent the probability distribution. Dubois et al. [6] call this level set "confidence interval", which is different from the confidence interval in measurement theory. Due to this observation, we shall use the level sets of probability density functions to model the three types of target preferences.

3.1 Random Variable and Its Level Set Representation

Probability theory is the branch of mathematics that studies the behavior of random events. Let Ω be a nonempty set, χ be a σ-algebra of subsets of Ω, and Pr be a probability measure. Then the triplet $(\Omega, \chi, \Pr)$ is called a probability space [15].

Let ξ be a random variable defined on a probability space $(\Omega, \chi, \Pr)$. The probability distribution $\Phi : \Re \to [0, 1]$ of a random variable is defined by

$$\Phi(x) = \Pr\{\omega \in \Omega | \xi(\omega) \leq x\} \tag{9}$$

$\Phi(x)$ is the probability that the random variable ξ takes a value less than or equal to x [15].

Let $\Phi : \Re \in [0, 1]$ be the cdf of the random variable ξ. The pdf $p : \Re \to [0, +\infty]$ of a random variable ξ is a function satisfying $\Phi(x) = \int_{-\infty}^{x} p(t)dt$, and this holds for all $x \in \Re$ [15].

Let ξ be a random variable defined on a probability space $(\Omega, \chi, \Pr)$, σ be any given probability level, where $0 \leq \sigma \leq \sup \xi$. Let $p(x)$ be the probability distribution of the random variable ξ, and ξ_σ consists of all the elements whose probabilities are greater than or equal to σ such that [15]

$$\xi_\sigma = \{x \in \Omega | p(x) \geq \sigma\}. \tag{10}$$

Then ξ_σ is called the σ-level sets of random variable ξ. If the random variable ξ has a unimodal distribution, then $\xi_\sigma = [\xi_\sigma^l, \xi_\sigma^r]$, where ξ_σ^l is the left bound and ξ_σ^r is the right bound. The distribution function of level sets of the pdf can be defined as follows [7]

$$\Phi(\xi_\sigma) = \int_0^\sigma \xi_t dt. \tag{11}$$

This mapping characterizes the relationship between the corresponding level sets of a pdf and their probabilities.

3.2 A Level Set Based Approach to Target-Oriented Decision Analysis

Let T be a random uncertain target having a bounded domain $D = [X_{\min} X_{\max}]$, $p(t)$ be the pdf of the random target T, T_σ consists of all the elements whose probabilities are greater than or equal to σ such that $T_\sigma = \{t \in D | p(t) \geq \sigma\}$, T_σ is called the σ-level sets of random target T. It should be noted that target-oriented decision analysis assumes that the uncertain target has a unimodal probability distribution, thus we can express as $T_\sigma = [T_\sigma^l, T_\sigma^r]$, where T_σ^l and T_σ^r are the left and right bound of level cut, respectively. Based on the distribution function of level sets of pdf provided before, we define the following function:

$$\Pr(x \succeq T) = \int_0^{\sup T} u(x, T_\sigma) T_\sigma d\sigma, \tag{12}$$

where $u(x, T_\sigma)$ indicates the degree that the target achievement in the level set T_σ, $\sup T$ denotes the support of the pdf of uncertain target, and $u(x, T_\sigma) \in [0, 1]$, $\sup u(x, T_\sigma) = 1$. Considering different target preferences, defined as

$$\Pr(x \succeq T) = \begin{cases} \Pr(x \geq T) = \int_0^{\sup T} u(x \geq T_\sigma) T_\sigma d\sigma, & \text{for benefit target;} \\ \Pr(x \leq T) = \int_0^{\sup T} u(x \leq T_\sigma) T_\sigma d\sigma, & \text{for cost target;} \\ \Pr(x \cong T) = \int_0^{\sup T} u(x \cong T_\sigma) T_\sigma d\sigma, & \text{for equal/range target.} \end{cases} \tag{13}$$

Now let us consider these three cases in great detail.

Benefit Target Preference. If the DM assumes a monotonically increasing target preference, for an interval $T_\sigma = [T_\sigma^l, T_\sigma^r]$, to ensure that $u(x, T_\sigma) \in [0, 1]$ and $\sup u(x, T_\sigma) = 1$, we define

$$u(x \geq T_\sigma) = \frac{\int_{T_\sigma^l}^{T_\sigma^r} u(x, t) p(t) dt}{\int_{T_\sigma^l}^{T_\sigma^r} p(t) dt} \tag{14}$$

As target-oriented model assumes that there are only two levels of utility (1 or 0), thus we define $u(x, t) = \begin{cases} 1, \text{ if } x \geq t; \\ 0, \text{ otherwise.} \end{cases}$ where $u(x, t)$ denotes whether the attribute level achieves target level or not. Then we can obtain $u(x \geq T_\sigma)$ as follows:

$$u(x \geq T_\sigma) = \begin{cases} 0, & \text{if } x < T_\sigma^l; \\ \frac{\int_{T_\sigma^l}^{x} p(t) dt}{\int_{T_\sigma^l}^{T_\sigma^r} p(t) dt}, & \text{if } T_\sigma^l \leq x \leq T_\sigma^r; \\ 1, & \text{if } x > T_\sigma^r. \end{cases} \tag{15}$$

Cost Target Preference. If the DM assumes a monotonically decreasing target preference, similarly we define $u(x, t) = \begin{cases} 1, \text{ if } x \leq t; \\ 0, \text{ otherwise.} \end{cases}$ and then we can obtain $u(x \leq T_\sigma)$ as follows:

$$u(x \le T_\sigma) = \begin{cases} 1, & \text{if } x < T_\sigma^l; \\ \frac{\int_x^{T_\sigma^r} p(t)dt}{\int_{T_\sigma^l}^{T_\sigma^r} p(t)dt}, & \text{if } T_\sigma^l \le x \le T_\sigma^r; \\ 0, & \text{if } x > T_\sigma^r. \end{cases} \tag{16}$$

It is clear that $u(x \le T_\sigma) = 1 - u(x \ge T_\sigma)$, thus we obtain

$$\Pr(x \ge T) = \int_0^{\sup T} u(x \ge T_\sigma) T_\sigma d\sigma = 1 - \Pr(x \le T) \tag{17}$$

Equal/Range Target Preference. In case of non-monotonic target preference, there exists an "ideal" level. Recall that target-oriented decision analysis views the mode value T_m of the pdf as reference point (reflection point), then there will be added loss of value for missing the reference point on the low side, or added loss for exceeding the reference point. In other words, when $x < T_m$ it can be viewed as pseudo benefit attribute; when $x = T_m$ the probability of meeting target should be equivalent to one; and when $x > T_m$ it can be viewed as pseudo cost attribute. Due to this observation, we can define the following function:

1. When $x < T_m$,

$$u(\cong\in T_\sigma) = u(x \ge T_\sigma) = \begin{cases} 0, & \text{if } x < T_\sigma^l; \\ \frac{\int_{T_\sigma^l}^{x} p(t)dt}{\int_{T_\sigma^l}^{T_\sigma^r} p(t)dt}, & \text{otherwise.} \end{cases} \tag{18}$$

2. When $x = T_m$,

$$u(x \cong T_\sigma) = 1 \tag{19}$$

   It should be noted that if the modal value T_m is an interval range, such that $T_m = [T_{m1}, T_{m2}]$, then we can define $u(x \cong T_\sigma) = 1$ if $T_{m1} \le x \le T_{m2}$. Typical examples of this case are the trapezoidal distributions.
3. When $x > T_m$,

$$u(x \cong T_\sigma) = u(x \le T_\sigma) = \begin{cases} 0, & \text{if } x > T_\sigma^r; \\ \frac{\int_x^{T_\sigma^r} p(t)dt}{\int_{T_\sigma^l}^{T_\sigma^r} p(t)dt}, & \text{otherwise.} \end{cases} \tag{20}$$

4 Illustrative Examples

In this section, we shall consider two special cases to illustrate the proposed models.

4.1 Normally Distributed Target

In real applications, choosing a suitable probability distribution for uncertain targets is due to specific problems. As the normal distribution is widely used as a model of quantitative phenomena in the natural and behavioral sciences, we shall assume that the uncertain target is normally distributed in the bounded domain D and with mode value T_m.

Benefit Target Preference. Firstly, let us consider the benefit case. According to the two target achievement method in previous sections, we can obtain the target achievements with respect to these three target preference types, as shown in Fig. 1. To distinguish these two methods, $\Pr_I(x \geq T)$ is used to denote the target-oriented model based on the cdf, whereas $\Pr_{\mathrm{II}}(x \geq T)$ is used to denote the target-oriented model based on the level sets of probability distributions.

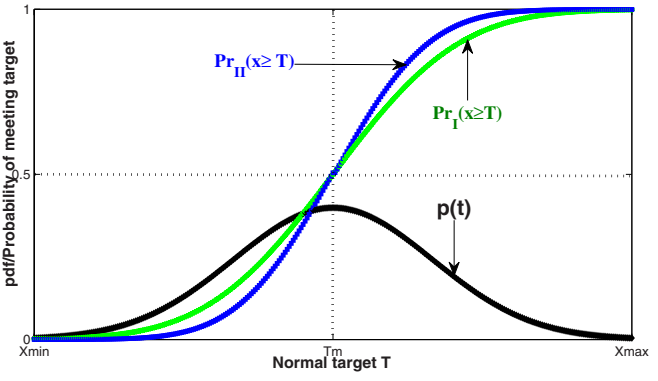

Fig. 1. Benefit target achievements under normal distribution by means of cdf and level set based approaches

Looking at the target achievement of benefit attribute, $\Pr_{\mathrm{I}}(x \geq T)$ and $\Pr_{\mathrm{II}}(x \geq T)$, as shown in Fig. 1. No matter which method is chosen, the induced probability of meeting target (utility function) corresponds to the S-shaped function, which is equivalent to the S-shaped utility function of prospect theory [12] as well as is consistent with "Goals as reference point" by Heath et al. [9]. The induced value functions have the following two properties:

1. **Gain and loss:** The target divides the space of outcomes into regions of gain and loss (or success and failure). Thus, the value function assumes that people evaluate outcomes as gains or losses relative to the reference point T_m.
2. **Diminishing sensitivity:** The value function draws an analogy to psychophysical process and predicts that outcomes have a smaller marginal impact when they are more distant from the reference point T_m.

Remark 1. It should be noted that Kahneman and Tversky [12] assume another principle: outcomes that are encoded as losses are more painful than the similar sized gains are pleasurable. In their words, "losses loom larger than gains". The induced value function by target-oriented model does not entirely satisfy the prospect theory.

In Fig. 1, it is clearly that both the level set based approach and the cdf based approach have the S-shaped value function, whereas the behavior of value function of the level set based approach $\Pr_{\mathrm{II}}(x \geq T)$ is stepper towards the mode

value of the corresponding target than that of the cdf function $\Pr_{\mathrm{I}}(x \geq T)$. This practically implies that the value function based on the level set reflects a stronger decision attitude towards the target than that defined with the cdf function [10].

Cost Target Preference. In case of monotonically decreasing preference, the induced utility function should have an inverse S-shaped function. According to Eq. (6), we can obtain the utility function based on the cdf of probability distribution, denoted as $\Pr_{\mathrm{I}}(x \leq T)$. By means of Eqs. (13) and (16) we can obtain the probability of meeting target based on the level sets of probability distributions, denoted as $\Pr_{\mathrm{II}}(x \leq T)$. The induced utility functions by those two approaches are graphically depicted in Fig. 2.

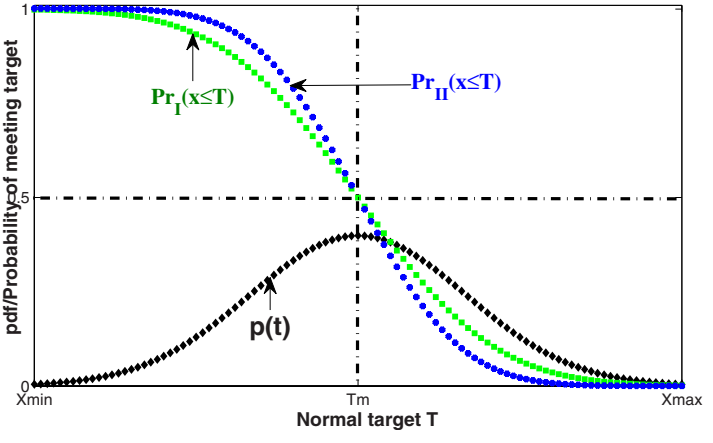

Fig. 2. Cost target achievements under normal distribution by means of cdf and level set based approaches

From Fig. 2, it is clear that whatever which approach is chose, both of these two approach induce an inverse S-shaped value function. The reference point T_m divides the value function into two parts: gains and losses (the value below T_m can be viewed as a kind of gains; the value upper than T_m can be viewed a losses). In addition, the value function draws an analogy to psychophysical process and predicts that outcomes have a smaller marginal impact when they are more distant from the reference point T_m. Furthermore, this type of induced value function has an inverse S-shaped value function. Finally, the behavior of value function $\Pr_{\mathrm{II}}(x \leq T)$ based on the level set is stepper towards the modal value of the corresponding target than that of the cdf function $\Pr_{\mathrm{I}}(x \leq T)$.

Equal/Range Target Preference. When the DM has an interval target preference, DM will evaluate outcomes as losses relative to the reference point T_m.

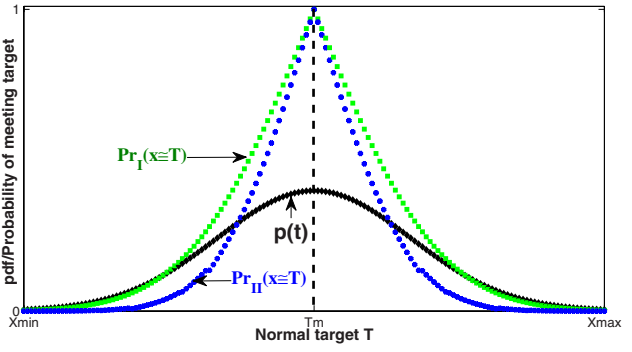

Fig. 3. Equal/range target achievements under normal distribution by means of cdf and level set based approaches

The attribute value below or exceeding the reference point is viewed as a loss, in which the value function is convex, as shown in $\Pr(x \cong T)$ of Fig. 3.

According to Eq. (7) we can obtain the induced value function based on the cdf of probability distributions. By means of Eqs. (13) and (18)-(20) we can induce the value function based on the level set of probability distributions. The induced value functions are graphically depicted in Fig. 3. Looking at Fig. 3, the reference point value T_m will be the reflection point. As the DM assumes interval/range target preference, there will be added loss of value for missing the reference point on the low side, or added loss for exceeding the reference point T_m. As illustrated in Fig. 3, the induced value function below or upper than the modal value T_m has a convex shape. It is clear that the behavior of value function $\Pr_{\mathrm{II}}(x \cong T)$ based on the level sets of probability distributions is stepper towards the modal value of the corresponding target than that of the cdf function $\Pr_{\mathrm{I}}(x \cong T)$.

4.2 Uniformly Distributed Target

Furthermore, let us consider a special case. Without additional information about the target distribution, we can assume that the random target T has a uniform distribution on D with the probability density function $p(t)$ defined by

$$p(t) = \begin{cases} \frac{1}{X_{\max} - X_{\min}}, & X_{\min} \le t \le X_{\max}; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Under the assumption that the random target T is stochastically independent of any alternative, by means of the cumulative distributive function based approach and the level set based approach we can obtain the same probability of meeting uncertain target for benefit and cost attributes as follows

$$\Pr(x \succeq T) = \begin{cases} \Pr(x \ge T) = \frac{x - X_{\min}}{X_{\max} - X_{\min}}, & \text{for benifit target;} \\ \Pr(x \le T) = \frac{X_{\max} - x}{X_{\max} - X_{\min}}, & \text{for cost target.} \end{cases} \tag{22}$$

It is easily seen that, for benefit and cost attribute there is no way to tell whether the DM selects an alternative by traditional normalization method or by target-oriented model. In other words, in this case the target-based decision model with the decision function is equivalent to the traditional normalization function.

5 Conclusions

Decision analysis with targets/goals has a long history in the literature. Distance-based approach is one widely used method in decision analysis problems. However, different distances to the target value should have different impacts on DMs' preferences, which is missed in the distance-based approach.

In this paper, two methods have been proposed for target-oriented decision analysis with different types of target preferences: cdf based method and level set based method. These two methods can induce four shaped value functions: S-shaped, inverse S-shaped, convex, and concave, which represents DM's psychological preferences. The main difference between these two methods is that the level set based model induces a stricter value function than the cdf based model. Our research research verifies the proposition that probability can be viewed as a psychological distance [20].

Differed from the original target-oriented decision model [3,4,5,14], our two proposed models add a target achievement level function $u(x,t)$ into the value function such that

$$\Pr(x \succeq T) = \int_{-\infty}^{\infty} u(x,t)p(t)dt.$$

It is interesting that while assuming that x and t are stochastically independent, we can rewrite the function as

$$\Pr(x \succeq T) = \int_{-\infty}^{\infty} u(x,t)d\Phi(t), \tag{23}$$

where $\Phi(t)$ is the cdf of the random target. This representation has a similar structure with the Choquet fuzzy integral [8]. Viewing $\Phi(t)$ as the fuzzy density, the general target-oriented decision model can be a special case of Choquet fuzzy integral, where $u(x,t)$ is the Heaviside utility function, i.e. only two target achievement levels exist (1 and 0). Due to the limitation of paper space, a further study on the relationship between target-oriented decision model and Choquet fuzzy integral will be left for the future work.

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A Novel Method for Multibiometric Fusion Based on FAR and FRR

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Abstract. Based on the fusion of multiple biometric sources, Multibiometric systems can be expected to be more accurate due to the presence of multiple pieces of evidence. Multibiometric system design is a challenging problem because it is very difficult to choose the optimal fusion strategy. Score level fusion is the most commonly used approach in Multibiometric systems. The distribution of genuine and imposter scores are very important for score fusion of Multibiometric systems. FRR (False Reject Rate) and FAR (False Accept Rate) are two key parameters to cultivate the distribution of genuine and imposter scores. In this paper, we first present a model for Multibiometric fusion and then proposed a novel approach for score level fusion which is based on FAR and FRR. By this method, the match scores first are transformed into LL1s and then the sum rule is used to combine the LL1s of the scores. The experimental results show that the new fusion scheme is efficient for different Multibiometric systems.

Keywords: biometrics, Multibiometric, score level fusion, multi-modal, FRR, FAR.

1 Introduction

Biometric recognition refers to the use of distinctive physiological or behavioral characteristics for automatically confirming the identity of a person. Typically, sources of physiological characteristics include face, fingerprint, iris, palmprint, hand geometry and ear shape; and sources of behavioral characteristics include gait, signature and keystroke dynamics and speak. Multibiometrics which combines more information is expected to improve the performance of biometric system efficiently [1]. Based on the nature of the sources, Multibiometric systems can be subdivided into five classes: multi-sensor systems (multiple acquisitions of the same biometry with the same sensor), multi-sample systems (multiple representations of the same biometric trait), multi-unit systems (e.g. the fingerprints of left and right index fingers), multi-algorithm systems, multi-modal systems and hybrid systems which refers to systems that integrate a subset of the five classes above [2]. Depending on the level of information that is fused, the fusion scheme can be classified as sensor

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level, feature level, score level and decision level fusion. Apart from the raw data and feature sets, the match scores contain the richest information about the input pattern. Also, it is relatively easy to get and combine the scores generated by biometric matchers. Consequently, score level fusion is the most commonly used approach in Multibiometric systems. Information fusion has a long history and the theory of multiple classifier systems (MCS) has been rigorously studied over the past several years. Scores contain less but more useful information than the raw data and feature vectors. Scores generated by different matchers are not homogeneous often. For example, scores of different matchers may not be on the same numerical range and may follow different probability distributions. For the above factors, score level fusion in Multibiometric systems is still a challenging problem.

In this paper, we first propose a model for multibiometric systems and give a clear definition for different kind of multibiometric systems based on this model. Then we propose a novel approach for score level fusion which is based on FAR and FRR. By this method, the match scores first are transformed into LL1s. Different from traditional normalization techniques, the scores transformed by LL1 transformation function are comparable. Based on the LL1s of match scores, different combination rules can be used to give the final decision. Since sum rule show good performance in multi-modal system [9], we use sum rule to give the final decision.

The remainder of this paper is organized as follows. In Section 2, the related work and motivation is introduced. In Section 3, a novel fusion model for Multibiometric systems are described. The new fusion strategy is proposed in Section 4. To study the effectiveness of the proposed technique, section 5 gives the experimental results. The last section summarizes the results of this work and provides future directions for research.

2 Related Work and Motivation

The match score is a measure of similarity between the template and the input biometric feature vectors. Combining different match scores with fusion strategy to give the final decision is called score level fusion. There are usually three kinds techniques of score fusion: transformation-based score fusion, classifier-based score fusion and density-based score fusion [2]. Transformation-based score fusion first transforms the match score into a common domain and then combines the normalized scores to gain the final score and the final decision. Several normalization techniques are proposed: min-max, z-score, tanh, sigmoid. However, the scores presented by the different matchers usually have different distributions; the performance of transformation-based fusion can not be conformed in multi-modal systems. In classifier-based score fusion, the scores of the different matchers can be non-homogeneous. A limitation of the classifier-based score fusion approach is that it is difficult to get different FRRs and FARs which are useful to draw the ROC curves and to choose different thresholds for different applications by operators. In density-based score fusion, the match scores are first transformed into probabilities and then the Bayesian decision rule can be used to make a decision.

Among the three approaches, density-based fusion is a more principled approach because it achieves optional recognition performance if the densities of genuine and

imposter scores are estimated accurately. Since the match scores are limited, the difficulty of density-based sore fusion approach is the estimation of the conditional densities of the match scores. Density estimation techniques can be classified as non-parametric and parametric [3]. For non-parametric approaches, Histogram [4], Parzen window [5] and ROC curve [6] were used to estimate the match score densities. In reference [7], a likelihood ratio (LR)-based framework is proposed for optimal combination of match scores that is based on the likelihood ratio test. The distributions of genuine and impostor match scores are modeled as finite Gaussian mixture model which belongs to parametric approaches. Although we have mentioned several density estimation techniques, score density estimation is still the most complex work in multibiometric fusion because match scores are limited especially for genuine match scores. In this paper, we try to cultivate the distribution or densities of genuine and imposter scores based on FAR and FRR indirectly but not to estimate the densities of match scores directly.

3 A Fusion Model for Multimodal Fusion

3.1 The Property of Biometric Match Scores

The match score is a measure of similarity between the template and the input biometric feature vectors. Verlinde et al., 1999[8] proposed that the match score s is related to $P(\omega_k \mid x_j)$ where x_j is the feature vector and ω_k refers to the class label as follows:

$$s = g(P(\omega_k \mid x_j)) + \beta(x_j), \tag{1}$$

Where g is a monotonic function and β is the error made by the biometric systems which depends on the input features. But the function *g* and the error function β are unknown. We give a simple relationship between the match score *s* and posteriori probabilities $P(g \mid s)$. A biometric system f gives one score s for each verification between biometric template T and the input patterns I. For example,

$$s_i = f(T_i, I_i) \tag{2}$$

is the similarity between T_i and I_i based on the recognition system *f*. Suppose two scores,

$$s_i < s_j \quad then \quad P(g \mid s_i) \le P(g \mid s_j). \tag{3}$$

$P(g \mid s_i)$ refers to the posteriori probabilities which the input patterns and the templates are from the sample user that means the input user is a genuine. And we call the formula (3) as the property of match scores.

3.2 A Formal Definition of Multibiometric Systems

Suppose there are *N* times verification, for each verification, we get *M* match scores and then score level fusion is conducted to get the final decision.

$$\begin{gathered} s_i^m = f^{(m)}(T_i^m, I_i^m) \quad i = 1,2,\ldots,N, m = 1,2,\ldots M \\ u_i = fusion(s_i^1, s_i^2, s_i^3, \ldots s_i^M) \end{gathered} \tag{1}$$

In formula (4), ***i*** refers to the ***i***th verification. $f^{(m)}$ refers to different biometric recognition system. T_i^m, I_i^m refer to template and input biometric features for the ***m***th biometric recognition system during the ***i***th verification. From formula (4), we can also give the definition of different Multibiometric systems which are classified by the source of biometric systems. To give a definition of Multibiometric system, we define functions ***type*** and ***sensor***. $type(T_i^m)$ and $type(I_i^m)$ refer to the biometric type which means fingerprint, face, iris, voice and so on. $sensor(T_i^m)$ and $sensor(I_i^m)$ refer to the sensor types which capture the traits. For biometric systems, $type(T_i^m) = type(I_i^m)$ and $sensor(T_i^m) = sensor(I_i^m)$ usually. So in equations (5)-(9), only $type(T_i^m)$ and $sensor(T_i^m)$ are used.

Multi-algorithm biometric systems:

$$f^{(m_1)} \neq f^{(m_2)} \quad (T_i^{m_1} = T_i^{m_2} \quad and \quad I_i^{m_1} = I_i^{m_2})(1 \leq m_1 < m_2 \leq M). \tag{5}$$

Multi-unit biometric system: (e.g. multi-finger fingerprint recognition system)

$$T_i^{m_1} \neq T_i^{m_2} \quad type(T_i^{m_1}) = type(T_i^{m_2}) \quad (1 \leq m_1 < m_2 \leq M). \tag{6}$$

Multi-sample biometric systems:

$$f^{(m_1)} \neq f^{(m_2)} \quad (T_i^{m_1} = T_i^{m_2} \quad or \quad I_i^{m_1} = I_i^{m_2})(1 \leq m_1 < m_2 \leq M). \tag{7}$$

Multi-sensor biometric systems:

$$T_i^{m_1} = T_i^{m_2} \quad sensor(T_i^{m_1}) \neq sensor(T_i^{m_2}) \quad (1 \leq m_1 < m_2 \leq M). \tag{8}$$

Multi-modal biometric systems:

$$f^{(m_1)} \neq f^{(m_2)} \quad type(T_i^{m_1}) \neq type(T_i^{m_2}) \quad (1 \leq m_1 < m_2 \leq M) \tag{9}$$

If we consider function ***fusion*** as a classifier and $(s_i^1, s_i^2, s_i^3, \ldots s_i^M)$ as a feature vector, the fusion modal belongs to a classifier-based fusion. If we first translate the match scores into post probabilities by the density of imposter and genuine scores, and then get the final decision by Bayes rule, the fusion modal is density-based fusion. The transformation-based rule first translate the match score by normalized functions and then use a fixed fusion rule such as mean, max, min, median and product.

3.3 Constraints for the Fusion Function in Multibiometric Systems

The fusion function is defined as follows:

$$fusion(f^{(m)}(T_i^m, I_i^m)) = fusion(s_i^m) = t_i^m \tag{10}$$

And the fusion function should satisfies formulas (11) and (12)

$$t_i^m \in [0,1], s_{i_1}^m < s_{i_2}^m \Rightarrow t_{i_1}^m < t_{i_2}^m \tag{11}$$

$$t_{i_1}^{m_1} < t_{i_2}^{m_2} \Rightarrow Decision(s_{i_1}^{m_1}) < Decision(s_{i_2}^{m_2}) \tag{12}$$

The function $Decision(s)$ refers to the very decision function which must satisfy formula (11). Usually, the function ***fusion*** is not the same as the function ***Decision***. For example,

$$fusion(s_i^m) = \frac{P(genuine \mid s_i^m)}{P(impostor \mid s_i^m) + P(genuine \mid s_i^m)} \tag{13}$$

$$Decision(s_i^m) = \frac{P(genuine \mid s_i^m)}{P(impostor \mid s_i^m)} \tag{14}$$

After transformation by the fusion function, the match scores can be combined with the fixed fusion rules such as sum, product, max, min and median.

4 Score Fusion Based on FRR and FAR

4.1 The Definition of LL1_Fusion Function

A likelihood ratio (LR)-based framework for score level fusion was proposed in [3] for the verification scenario. This LR framework is based on the Neyman-Pearson theorem and it maximizes the genuine accept rate (GAR) at any desired false accept rate (FAR), provided genuine and impostor match densities are known or can be estimated accurately. LR-based fusion framework needs to compute the density of genuine and imposter scores accurately. Since the match scores are limited especially for genuine scores, the accurate density is very difficult to compute. And the densities are affected by data noise seriously. Also, the fusion function based on likelihood could not satisfy the constraint (12). However, based on this theory, we give a new transformation function based on FRR and FAR. The equations (15) and (16) are the definitions of FAR, FRR. More details can be found in Ref. [2]

$$FAR(t) = p(\mathrm{s} \geq t \mid impostor) = \int_t^{\infty} f_{imp}(s)ds \tag{15}$$

$$FRR(t) = p(\mathrm{s} < t \mid genuine) = \int_{-\infty}^{t} f_{gen}(s)ds \tag{16}$$

For real systems, the match score is limited and the definition of FRR and FAR is as follows.

$$FAR(t) = \frac{Number\{s \geq \mathrm{t}\ AND\ s \in impostor\}}{Number(impostor)} \tag{17}$$

$$FRR(t) = \frac{Number\{s < \mathrm{t}\ AND\ s \in genuine\}}{Number(genuine)} \tag{18}$$

First, we define a function ***L_fusion*** for a single biometric(means for the very ***m***) systems as follows:

$$L\_fusion(s_i^m) = \begin{cases} FRR0 / FAR(s_i^m) & if\ \ FRR(s_i^m) = 0 \\ FRR(s_i^m) / FAR0 & if\ \ FAR(s_i^m) = 0 \\ FRR(s_i^m) / FAR(s_i^m)\ otherwise \end{cases} \tag{19}$$

Where ***FRR0=1/(10*Number(genuine))*** and ***FAR0=1/(10*Number(impostors)).*** ***FRR0*** and ***FAR0*** are used to keep the discriminative ability of ***LL1_fusion*** when ***FRR*** or ***FAR*** equal to zero.

The final fusion function LL1_fusion is defined:

$$\begin{aligned} LL1\_fusion(s_i^m) &= \frac{FRR(s_i^m)}{FAR(s_i^m) + FRR(s_i^m)} \\ &= \frac{L\_fusion(s_i^m)}{L\_fusion(s_i^m) + 1} \end{aligned} \tag{20}$$

And the Decision function is defined as follows:

$$Decision(s_i^m) = L\_fusion(s_i^m) = FRR(s_i^m) / FAR(s_i^m) \tag{21}$$

In the rest of this paper, we simply use ***LL1*** instead of ***LL1_fusion***. We can verify that the function ***LL1*** satisfies the constraint (11) and (12) on the training set.

Proof. We proof the function ***LL1*** satisfies formulas (11) and (12)

1. It is easy to proof that $LL1\_fusion(s_i^m) \in [0,1]$

If

$$s_{i_1}^m < s_{i_2}^m \in S^m, \tag{22}$$

we have

$$FAR(s_{i_2}^m) < FAR(s_{i_1}^m) \text{ and } FRR(s_{i_1}^m) < FRR(s_{i_2}^m), \tag{23}$$

then

$$t_{i_1}^m - t_{i_2}^m = \frac{FRR(s_{i_1}^m)}{FAR(s_{i_1}^m) + FRR(s_{i_1}^m)} - \frac{FRR(s_{i_2}^m)}{FAR(s_{i_2}^m) + FRR(s_{i_2}^m)} < 0. \tag{24}$$

So

$$t_{i_1}^m < t_{i_2}^m. \tag{25}$$

2. We proof LL1 satisfies formula (12)
If

$$t_{i_1}^{m_1} < t_{i_2}^{m_2}, \tag{26}$$

Then

$$\frac{L\_fusion(s_{i_1}^{m_1})}{L\_fusion(s_{i_1}^{m_1}) + 1} < \frac{L\_fusion(s_{i_2}^{m_2})}{L\_fusion(s_{i_2}^{m_2}) + 1}, \tag{27}$$

and we get

$$L\_fusion(s_{i_1}^{m_1}) < L\_fusion(s_{i_2}^{m_2}). \tag{28}$$

4.2 The Computation of LL1

For the training set, we can get the LL1(s) for each $S^{(m)} = [s_1^m, s_2^m, s_3^m, \ldots s_N^m]$. We assume that $s_i^m < s_{i+1}^m$ or else we sort the data by ascend order and if one score appear more than once, only kept one in the score set. For any score s_i^m in $S^{(m)}$, we can get $LL1(s_i^m)$. For any new score $s_{N^+}^m$, we compute the LL1 of $s_{N^+}^m$ by the following step.

1. If there has $s_{i^*}^m (1 \le i^* \le N)$ that $s_{N^+}^m = s_{i^*}^m$

$$LL1(s_{N^+}^m) = LL1(s_{i^*}^m). \tag{29}$$

2. Else if $s_{N^+}^m < s_1^m$,

$$LL1(s_{N^+}^m) = 0. \tag{30}$$

3. Else if $s_{N^+}^m > s_N^m$,

$$LL1(s_{N^+}^m)=1. \tag{31}$$

4. Else there must have $s_{i^*}^m, s_{i^*+1}^m (1 \le i^* \le N-1)$ and $s_{i^*}^m < s_{N^+}^m < s_{i^*+1}^m$,

$$LL1(s_{N^+}^m) = \frac{s_{i^*+1}^m - s_{N^+}^m}{s_{i^*+1}^m - s_{i^*}^m} LL1(s_{i^*}^m) + \frac{s_{N^+}^m - s_{i^*}^m}{s_{i^*+1}^m - s_{i^*}^m} LL1(s_{i^*+1}^m). \tag{32}$$

We search the $s_{i^*}^m, s_{i^*+1}^m (1 \le i^* \le N-1)$ by binary search. The complexity of this search is $O(\log_2 N)$. Hence, we get the value of LL1 for any score. After that, we use the fixed rules for fusion to get the final score. Since sum rule works better in most applications [9-11], we used the sum rule in our experiments.

Based on the proof in 4.1, it is not difficult to verify that the LL1 function satisfies formula (11) and (12). Then we get the final score u_i by formula (33).

$$u_i = mean(LL1(s_i^1), LL1(s_i^2), \ldots LL1(s_i^M)) \tag{33}$$

5 Experimental Results

5.1 Database

The XM2VTS database [12] contains synchronized video and speech data from 295 subjects, recorded during four sessions taken at one month intervals. On each session, two recordings were made, each consisting of a speech shot and a head shot. The speech shot consisted of frontal face and speech recordings of each subject during the recital of a sentence.

The XM2VTS-Benchmark database [13] consists of five face matchers and three speech matchers and was partitioned into training and evaluation sets according to the Lausanne Protocol-1(LPI). The benchmark of LPI includes two files, one is dev.label and the other is eva.label. We use dev.label as training data and eva.label as test data. Our experiments are conducted based on this match score benchmark. We sign the face matchers as face-1, face-2, face-3, face-4 and face-5 and the speech matchers as speech-1, speech-2 and speech-3 respectively.

5.2 Experimental Results

From Fig 1 and Fig 2, we can find the performance of each face matcher and speech matcher. The Equal Error Rates (EERs) of all the matchers are shown in Table 1. Fig 1, Fig 2 and Table 1 show the order of the performance of face and speech matcher respectively. Among face matchers, matcher face-3 and face-5 gain the best and worst performance respectively. And among speech matchers, the performance order is speech-1, speech-3 and speech-2.

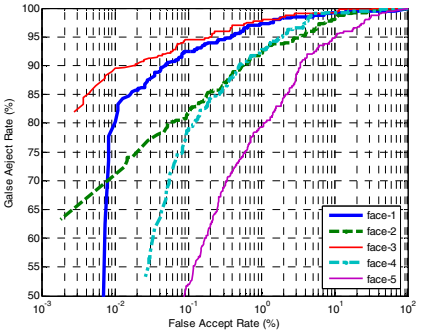

Fig. 1. ROC curves show comparison among 5 face matchers

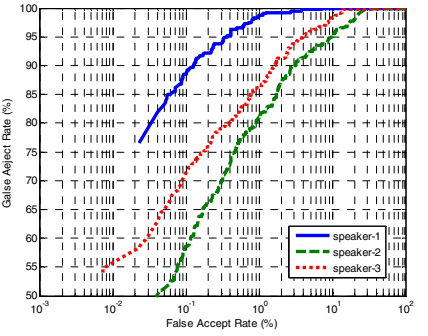

Fig. 2. ROC curves show comparison among 3 speech matchers

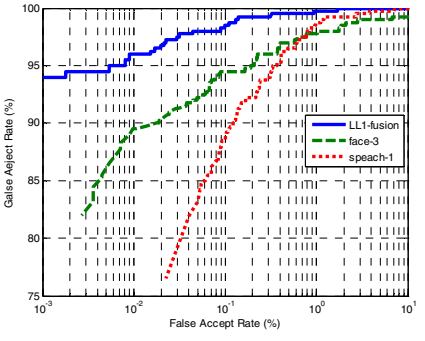

Fig. 3. ROC curves show the LL1 based fusion between face matcher 3 and speech matcher 1

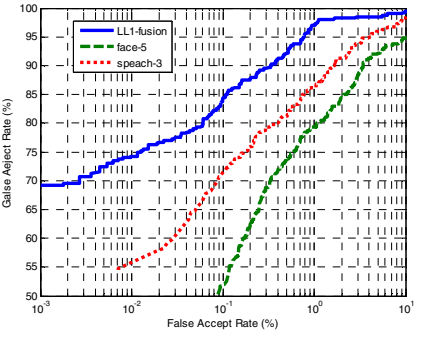

Fig. 4. ROC curves show LL1-based fusion between face matcher 5 and speech matcher 3

Fig 3 shows the ROC curves combining the best two matchers, face-3 and speech-1 and Fig 4 shows the ROC curves combining the worst two matchers, face-5 and speech-3. As shown in Fig 3 and Fig 4, the LL1 fusion based on FRR and FAR can improve the performance of Multibiometric system efficiently.

Table 1. EER of the multi-modal fusion

matcher	Face	Face-1	Face-2	Face-3	Face-4	Face-5
Speech	EER(%)	1.81	4.11	1.76	3.50	6.49
Speech-1	1.11	0.27	0.74	0.47	0.72	0.498
Speech-2	6.49	0.81	1.67	1.26	1.01	2.95
Speech-3	4.49	0.75	1.10	0.97	0.79	1.66

Also, other multi-modal fusion experiments are conducted with the LL1 fusion scheme. The EERs of the experimental results are shown in Table 1. All the EERs of multi-modal are smaller than the EERs of the best single matcher. Hence, our algorithms can improve the performance steadily.

6 Conclusions and Future Work

In this paper, we propose a novel modal for Multibiometric systems. The most important part of the modal is the relationship between match scores and the posteriori probabilities which is shown in formula (3) in Section 3.1. The fusion modal of multibiometric systems is given in Section 3.2. Based on this modal, we give an exact definition of different types of Multibiometric systems. And then the constraints for the fusion function are given in Section 3.3. Then we propose the fusion function ***LL1_fusion*** which is based on FRR and FAR. This function satisfies the constraints for fusion functions. We first compute the ***LL1*** of each match score in the training sets. And then we get the ***LL1*** score of one match in the test data by searching the ***LL1*** of training data with binary search. The experimental results show that our algorithms can improve the performance efficiently and steadily. The ***LL1*** fusion scheme, which is based on FRR and FAR, can be used in real systems easily.

Currently, the experiments are only conducted on the benchmark of database XM2VTS and our algorithms are not compared with other algorithms. In future, we will carry out the experiments on more databases and will try to compare our algorithms with other algorithms.

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Performance Evaluation of TEWA Systems for Improved Decision Support

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Abstract. In air defense situations, decision makers have to protect defended assets through assigning available firing units to threatening targets in real-time. To their help they have decision support systems known as threat evaluation and weapon allocation (TEWA) systems. The problem of performance evaluation of such systems is of great importance, due to their critical role. Despite this, research on this problem is close to non-existing. We are discussing the use of survivability and resource usage cost as comparative performance metrics, which can be used for comparing the effectiveness of different system configurations, by using simulations. These metrics have been implemented into a testbed, in which we have performed some comparative experiments. Our results show that changes of individual parts of the threat evaluation and weapon allocation system configuration can have a large effect on the effectiveness of the system as a whole, and illustrate how the metrics and the testbed can be used.

Keywords: Performance evaluation, testbed, TEWA, threat evaluation, weapon allocation.

1 Introduction

Threat evaluation and weapon allocation (TEWA) operators are in air defense situations assigning threat values to detected targets, values which are used for in real-time deciding on allocations of defensive resources to threatening targets [1] [2]. Human TEWA operators perform worse as the number of targets increase, calling for automatic control from TEWA systems in high-intensity battles [3]. Since TEWA systems are operating in a critical environment where wrong decisions can have fatal consequences, systematic performance evaluation of automated (and semi-automated) TEWA systems becomes very important. This can be illustrated by Operation Iraqi Freedom, during which a lot of Iraqi tactical ballistic missiles were successfully engaged by U.S. Patriot missiles, but where the automatic air defense systems also were involved in two fratricide incidents, in which coalition aircrafts were accidentally shot down since they were classified as attacking missiles [4]. Despite their critical role, very little research is to be found on the topic of performance evaluation of TEWA systems. Earlier

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research on evaluation of weapon allocation algorithms exist (see e.g. [5], [6]), but the algorithms are evaluated independently of the rest of the system, since the threat values of incoming targets are typically either ignored or assigned randomly. Unclassified research on performance evaluation of threat evaluation algorithms is much more uncommon [2]. We argue that there is a high degree of interdependence between the threat evaluation and weapon allocation parts of a TEWA system, and that the ingoing parts of such a system should not be evaluated in a reductionistic manner, but rather as a whole. Therefore we propose a holistic methodology for performance evaluation of TEWA systems.

In this paper, we describe a testbed in which we use simulations to make a comparative performance evaluation of TEWA systems, based on metrics measuring survivability of defended assets and the use of defensive resources. The use of simulations for performance evaluation of TEWA systems has earlier been described as a cost-effective approach by Chapman and Benke [7]. The usage of survivability as a measure of the effectiveness of weapon allocation solutions has earlier been suggested in [8] by Carling. However, their definition of survivability is based upon the ratio between the number of weapon-target assignments and the number of targets submitted to TEWA. Hence, their use of survivability as a performance metric is quite different from ours.

The rest of this paper is structured as follows. Firstly, Section 2 gives a precise description of what a TEWA system is, and the purpose of such a system. Secondly, Section 3 gives a summary of earlier work within performance evaluation of TEWA systems and their ingoing parts, and a description of our suggested holistic methodology for comparative performance evaluation of TEWA systems is given in Section 3.1. A presentation of the testbed into which the suggested metrics have been implemented is provided in Section 3.2. This is followed by Section 4, in which we describe two experiments for which the developed testbed has been used. Section 5 is devoted for a discussion of the suggested metrics, the testbed, and the outcome of the performed experiments. We conclude the paper in Section 6, where we also discuss future work.

2 TEWA Systems

The role of a TEWA system is to evaluate and assess the level of threat posed by detected targets with respect to defended assets (DAs), and based on this threat evaluation propose allocations of available weapon systems to threatening targets [2], [9]. In this paper, we will focus the attention to fully automated TEWA systems in which the proposed allocations are implemented without human intervention. Furthermore, we will in our examples focus on ground based air defense (GBAD) TEWA systems, but the general description will make sense even for other kinds of TEWA systems.

A TEWA system, obviously, is situated in a real world *environment*. This environment is continuously observed using *sensors*. For a GBAD TEWA system, a typical example of a sensor is a surveillance radar system such as the Swedish Giraffe AMB. The sensors are used to detect and track *targets*. There are two

main classes of targets we are interested in here, aircrafts (i.e., both fixed-wing and rotary-wing) and missiles (e.g., cruise missiles and anti-tank guided missiles). Once an aerial target has been detected by our sensors, we would like to determine the level of threat posed by the target to a set of *defended assets*, e.g., air bases, bridges, radars, etc. This threat evaluation is often based upon parameters measuring the proximity between the target and the defended asset, or other physical parameters such as altitude or speed [10]. The threat evaluation process typically results in a threat value in the unit interval $[0, 1]$ for each (target, DA) pair, which can be combined into an individual threat value for each target, in which also the protection values of the defended assets are taken into account [2]. Once the threat evaluation is completed, the resulting threat values can be used as a basis for the weapon allocation process, where blue force *weapon systems* are allocated to threatening targets. These weapon systems are typically firing units equipped with surface to air missiles (SAMs), but can also be more short range weapon systems such as close-in weapon systems (CIWSs). In addition to the calculated threat values, another important input to the weapon allocation process is estimates of kill probabilities, i.e., the probability that a specific weapon destroys a target, if assigned to it. Such kill probabilities are abstractions of a lot of different factors, such as the distance between the weapon system and the target, and the target's radar cross-section. Using information regarding estimated kill probabilities and threat values, the weapon allocation problem can be seen as an optimization problem where we search a feasible solution that minimizes the total expected threat value of surviving targets [11]. For scenarios consisting of a small number of weapon systems and targets, the optimal solution can be computed in real-time, while we for larger problems have to rely on heuristic algorithms. The chosen allocation is realized into a number of engagements. The effect of the engagements on the environment is observed using the sensors, and so on. Hence, threat evaluation and weapon allocation are continuous processes.

3 Performance Evaluation of TEWA Systems

It is easily understood that the topic of performance evaluation of TEWA systems is important, due to the critical role of such systems. Wrong decisions can have very fatal consequences. Despite this, very little research has been published on the topic of performance evaluation of TEWA systems. To the extent that such research exist, all focus tend to be on either the weapon allocation or the threat evaluation part, and not on the performance of the TEWA system as a whole. This reductionistic perspective on TEWA systems is in our view problematic, since there is a strong interdependence between the different parts of a TEWA system.

An initial study of a comparison between two threat evaluation algorithms is described in [12]. In that paper, certain characteristics of the threat evaluation algorithms are compared to each other theoretically, e.g., the algorithms' abilities to handle missing information and uncertain evidence, their transparency,

and their mathematical foundation. In addition to this, the algorithms are implemented into a testbed in which air defense scenarios can be run, and where the algorithms' calculated threat values are logged. Both threat evaluation algorithms are run on the same scenario, and their calculated threat values are compared to each other as well as to human knowledge. As far as we know, the procedure of running threat evaluation algorithms on a set of scenarios and comparing the output to human expert knowledge is also what is commonly used within industry. A problem with this qualitative approach is the daunting task of elicitation of expert knowledge. Due to this bottleneck, the amount of scenarios on which the threat evaluation algorithms can be tested will always be quite small, raising the issue of robustness of the threat evaluation algorithms. Another problem with this approach is that the threat evaluation algorithms are evaluated independently of the rest of the TEWA system, in a reductionistic manner.

3.1 A Holistic Performance Evaluation Measure

We have recently suggested a quantitative *survivability* metric, where we define survivability as [2]:

$$S = \frac{\sum_{j=1}^{|\mathbf{A}|} \omega_j u_j}{\sum_{j=1}^{|\mathbf{A}|} \omega_j}, \tag{1}$$

where $|\mathbf{A}|$ is the number of defended assets, ω_j is the protection value (weight) of the defended asset A_j, and $\boldsymbol{u} \in \{0,1\}^{|\mathbf{A}|}$ is a binary vector defined as

$$u_j = \begin{cases} 1 \text{ if defended asset } A_j \text{ survived;} \\ 0 \text{ otherwise.} \end{cases}$$

The original idea with this approach was to compare threat evaluation algorithms to each other, by fixing the remaining parts of the TEWA system and only change the threat evaluation algorithm. However, other parts of the TEWA system can be changed and compared as well, e.g., weapon allocation algorithms. Hence, we argue that the same approach can be used to compare the effectiveness of different TEWA implementations. By letting a complete TEWA system observe, orient, decide, and act within a simulated environment, we can use the survivability of defended assets in an air defense scenario as a relative metric for measuring the performance of the TEWA system, to which the performance of other TEWA system implementations can be compared. The strength of this way of performance evaluation is that we remove the need for elicitation of threat values from experts, and that the interdependences between different parts of the TEWA system are acknowledged and taken into consideration. Also, we open up the possibility for more systematic comparisons on a larger set of scenarios, making the performance evaluation more robust.

A potential problem with using the survivability metric is that it does not explicitly take misguided allocations against targets with non-hostile intent into consideration. We can often assume that identification, friend or foe (IFF)

transponders can be used to avoid blue force targets from becoming candidates for weapon allocation, but this does not always hold true, as illustrated in the fratricide example in Section 1. There is also a risk that we can not always differentiate between red force and civilian targets. This must therefore be taken into consideration somehow. If the scenario is complex enough, e.g., the ratio of the number of threatening targets to the number of SAMs is high, the allocation of weapons to harmless targets will be reflected in the resulting survivability, since valuable weapons will be "wasted" on non-hostile targets. However, if there is only a few targets and a lot of SAMs, the engagement of harmless targets will not decrease the survivability.

As a solution to this problem, we introduce a cost for each engagement. This does not only punish the engagement of non-hostile targets, but all kind of unnecessary allocations of weapon systems to targets. The saving of SAMs is important, both for protection against possible future attacks and for the high price of a single missile [5]. Hence, we introduce a cost C_k for each weapon system W_k. This cost can either be assumed to be the same for all weapon systems, or vary between weapon systems. A TEWA system's effectiveness E on a specific scenario can therefore be expressed as

$$E = \alpha S - \beta C_{tot}, \tag{2}$$

where C_{tot} is the sum of all weapon usage costs during the scenario, and α and β are adjustable weights. This effectiveness measure is our proposed holistic performance measure for comparison of different TEWA systems.

We have here used a simple linear model for calculating the effectiveness from the survivability and weapon usage cost, although there exist many other models that can be used for such multi-criteria decision making problems. We have presented our suggested measure for air defense experts who have confirmed its appropriateness and appreciated its simplicity and directness, but it is fully possible that more advanced aggregation operators (cf. [13]) will be used in the future.

3.2 Testbed

Our proposed effectiveness measure has been implemented into a testbed, consisting of two modules. One of the two modules is STAGE Scenario, in which we can create and run air defense scenarios. The observations regarding the entities in the scenarios are communicated in real-time to our TEWA module, in which we simulate a TEWA system where different threat evaluation and weapon allocation algorithms can be chosen. The proposed weapon allocations are communicated back to STAGE Scenario and are realized into engagements, where the fired weapons become part of the simulation.

At the moment two threat evaluation algorithms are implemented into the TEWA module. The first is a Bayesian network approach, while the other is based on fuzzy logic. These algorithms are thoroughly described in [10] and [12], but examples of the used fuzzy inference rules are shown below, and the structure of the Bayesian network is illustrated in Fig. 1.

```
TargetType == Mig21 --> Threat = medium (1)
TargetType == B747 --> Threat = low (1)
TBH == short --> Threat = high (1)
TBH == medium --> Threat = medium (1)
TBH == long --> Threat = low (1)
Speed == low AND Distance == far --> Threat = low (1)
Speed == medium AND Distance == medium --> Threat = medium (1)
Speed == high AND Distance == close --> Threat = high (1)
```

Both algorithms use information regarding a target's speed, type, and calculated distances between the target and the defended assets to calculate the threat value V_i for a target T_i. The set of targets $\mathbf{T}_{wa}$ which are taken into consideration for weapon allocation is consisting of all the individual targets with a threat value higher or equal to an adjustable threshold τ. The use of weapon resources has been constrained so that a weapon system cannot make a new allocation until its last engagement is finished (i.e., until the SAM destroys or miss its intended target). This make perfect sense for missiles guided by the firing unit's radar, while it is less realistic for fire-and-forget missiles, i.e., missiles that do not require further guidance after launch. This constraint has been implemented into our testbed by the creation of a set $\mathbf{W}_{eng}$ of current engagements. If a weapon system is allocated to a target, the weapon system is added to this set, and information regarding the unique object ID of the weapon implementing the allocation is stored. The weapon system is not removed from this set until the testbed gets information from the simulation environment that the weapon with a matching

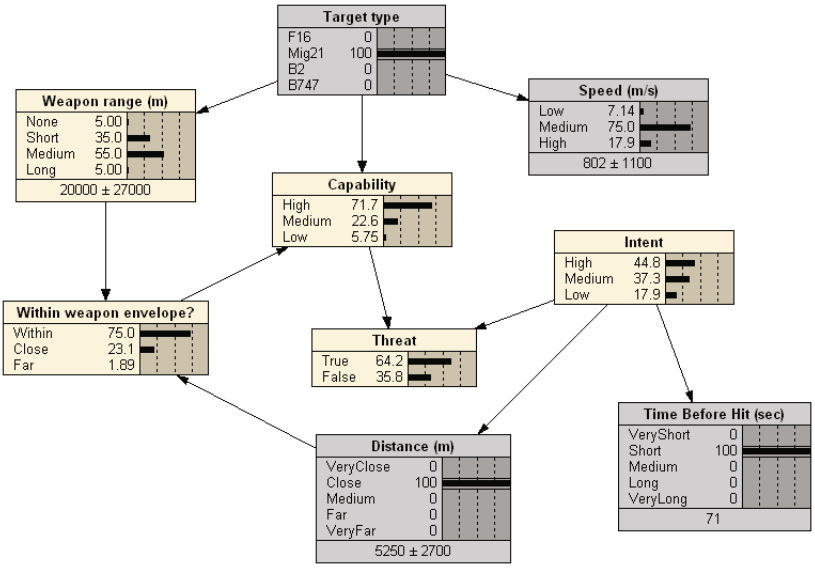

Fig. 1. The structure of the Bayesian network used for threat evaluation

object ID has been destroyed. Hence, we exclude all members of $\mathbf{W}_{eng}$ from the set $\mathbf{W}_{wa}$ of potential weapon system candidates for weapon allocation, i.e., $\mathbf{W}_{wa} = \mathbf{W} - \mathbf{W}_{eng}$.

For each pair (T_i, W_k), a corresponding kill probability $P_{i,k} \in [0,1]$ is associated, i.e., the probability that a weapon W_k destroys its target T_i if assigned to it. Given the calculated threat values and the kill probabilities, the NP-complete optimization problem the weapon allocation algorithm has to solve becomes [11]:

$$F^* \equiv \min_{x_{i,k} \in \{0,1\}} F = \sum_{i=1}^{|\mathbf{T}_{wa}|} V_i \prod_{k=1}^{|\mathbf{W}_{wa}|} (1 - P_{i,k})^{x_{i,k}}, \tag{3}$$

$$\text{subject to} \sum_{i=1}^{|\mathbf{T}_{wa}|} x_{i,k} = 1, \quad k = 1, \ldots, |\mathbf{W}_{wa}|, \tag{4}$$

where a decision variable $x_{i,k} \in \{0,1\}$ takes on the value 1 if weapon W_k is assigned to target T_i, and 0 otherwise. This far we have implemented only one weapon allocation algorithm into the testbed, and the pseudo code for this algorithm is given in Algorithm 1. The largest problem instances that can be handled in real-time by the algorithm consist of 8 targets and 6 weapon systems. Each solution checked by the algorithm is represented as a vector of length $|\mathbf{W}_{wa}|$, where each element W_k in the vector points out the target T_i, to which the weapon should be assigned. The best solution is communicated back to STAGE Scenario, in which weapons are fired against targets, in accordance to the suggested allocation. Once a scenario is over, the testbed calculates the survivability of the defended assets and the resource usage cost. The outcome of the engagements are not deterministic (as long as the kill probability does not equal one, which is quite unrealistic). Therefore we have implemented the possibility to design experiments in which a scenario automatically is repeated a large number of times, in order to get statistically significant results.

4 Experiments

In order to demonstrate how the testbed and the suggested metrics work, we have created an air defense scenario consisting of two firing units (which also are the defended assets within the scenario), three hostile fighters, and a neutral civilian aircraft. Both firing units are assigned a protection value of 1.0. The scenario is of quite high intensity and is approximately 8 minutes long. The blue force firing units are equipped with three SAMs each. The hostile fighters have one missile each, which they fire according to preassigned missions. For each simulation run, we log the survivability and the number of weapons used. If a weapon is fired against a target, there is an associated kill probability to such an engagement. Hence, the SAM will hit and destroy its target for some scenario runs, while it will not in some others. Therefore, we run each simulation hundred times for each TEWA system configuration, in order to get more reliable results.

Algorithm 1. Pseudo code for the weapon allocation algorithm

```
F_min ← ∞
count ← 0
while count < |T_wa|^|W_wa| do
   sol ← createSolution(count)
   F_sol ← 0
   for all i such that 1 ≤ i ≤ |T_wa| do
      P_survival ← 1.0
      for all k such that 1 ≤ k ≤ |W_wa| do
         P_survival ← P_survival × (1 − P_i,k)^x_i,k
      end for
      F_sol ← F_sol + V_i × P_survival
   end for
   if F_sol < F_min then
      F_min ← F_sol
      opt_solution ← sol
   end if
   count ← count + 1
end while
return opt_solution
```

Hence, a TEWA system's effectiveness is calculated as:

$$E = \frac{\sum_{t=1}^{100} \alpha S_t - \beta C_t}{100}$$

In the experiments, we have used a constant kill probability of 0.7 within the range 0-500 kilometers. If the range is larger, the kill probability becomes 0. This holds true for both red force and blue force weapons in the simulations.

In the first experimental setup, we have used two identical TEWA system configurations, except for the threat evaluation algorithms. The threat evaluation algorithm used in TEWA configuration 1 is our implemented Bayesian network, while configuration 2 uses the threat evaluation algorithm based on fuzzy logic. Both configurations use our static weapon allocation algorithm which make an exhaustive search for the optimal solution, minimizing the total expected threat value of the surviving targets. The weapon allocation algorithm is run with regular intervals, and takes targets with threat values higher than 0.5 into consideration. The results obtained for the two configurations are shown in the two first rows of Table 1 (we have used a uniform cost of 1 for each weapon usage).

Looking at the survivability, we can see that the average is 0.255 for configuration 1, and 0.17 for configuration 2 (while the standard deviations are 0.26 and 0.25 respectively). Hence, if we only consider survivability, the first configuration is the best of the two. If we instead look at the weapon usage cost, the average for configuration 1 is 4.83, while it for configuration 2 becomes 6 (i.e., it uses all defensive resources in each run of the scenario). From this we can see that the first configuration also is more selective with its resources than the second configuration. Assuming use of the user-defined parameters $\alpha = 0.99$ and

Table 1. Results for the different configurations

Configuration	μ_S	σ_S	μ_C	σ_C	$E(\alpha = 0.99, \beta = 0.01)$
BN ($\tau = 0.5$)	0.255	0.26	4.83	0.51	0.204
FL ($\tau = 0.5$)	0.17	0.25	6	0	0.108
BN ($\tau = 0.7$)	0.48	0.33	3.76	0.82	0.438
FL ($\tau = 0.7$)	0.465	0.34	3.58	0.75	0.425

$\beta = 0.01$, we can calculate the effectiveness of each TEWA configuration for the specific scenario. Doing this, we end up with an effectiveness of 0.204 for configuration 1. The corresponding number for configuration 2 becomes 0.108. The actual effectiveness number does not say anything on its own, but if we compare the numbers to each other, we can conclude that the first TEWA configuration performs better than the second, on the described scenario.

In the second experiment, we have used an identical setup to the one in experiment 1, except for the threshold τ for weapon allocation. This threshold has been changed from 0.5 to 0.7, in order to investigate what effect small changes of the TEWA configuration can have on the overall performance.

As can be seen in the two last rows of Table 1, the average survivability of the third configuration (i.e., the same configuration as configuration 1, except for the used threshold) has increased to 0.48, while it for the fourth configuration (the same as configuration 2 except for the used threshold) has increased to 0.465. The number of used weapons is noticeable lower in this second experiment, compared to the first one. This is expected, since we demand higher threat values before targets are considered as potential candidates for weapon allocation. Using the same user-defined parameters as in experiment 1, i.e., $\alpha = 0.99$, $\beta = 0.01$, the resulting effectiveness becomes 0.438 for configuration 3 and 0.425 for configuration 4.

From the above results, it is obvious that quite small changes of a TEWA system configuration can have major impact upon the resulting weapon allocation, which in its turn affects survivability and resource usage, and thereby also the effectiveness.

5 Discussion

In our experiments we have created a specific scenario in which we are interested, and compared a number of different configurations of TEWA systems. In this way, we can find out which TEWA system configuration (among a set of configurations) that is the best for a specific scenario, with respect to survivability and resource usage. Using this kind of experiments the expected performance and the decision support quality of the TEWA system can be improved. For the chosen scenario, it is clearly better to use a higher threshold ($\tau = 0.7$) for when to consider targets for weapon allocation, compared to the lower one ($\tau = 0.5$).

To run this kind of experiments on a single scenario can be very useful if we know what kind of scenario to expect, but this is often not the case. We

therefore would like to be able to test different TEWA system configurations on a larger number of different scenarios, and in that way test the robustness of the configurations. It is at the moment possible to create a number of different scenarios manually in the scenario generator, and to test the configurations on all these scenarios, but in order to get really robust results a larger number of different scenarios are needed. This gives raise to the question of how we quickly can construct a large number of scenarios. One of the major reasons for using survivability as a performance metric is to avoid the bottleneck of knowledge elicitation. This bottleneck has been removed from the judgment part of whether a TEWA system has performed well or not, but we still have the bottleneck of the manual construction of realistic scenarios on which to measure the survivability and resource usage. An important issue therefore becomes how to automatically generate a large number of realistic scenarios. We are currently discussing this issue with air defense experts, and have some initial ideas for how to automatically generate a large number of scenarios in which the incoming threats are cruise missiles.

Given a trustworthy estimate of kill probabilities, taking all important factors into consideration, another potential use of the testbed is to use the simulations for finding the best emplacement of available weapon systems, once a TEWA system configuration has been established. In this way, the defending forces can optimize their setup, based on likely attack scenarios.

6 Conclusions and Future Work

Automated TEWA systems operate in very complex and critical environments, in which wrong decisions can have fatal consequences. A prerequisite for improving TEWA systems' decision support is the ability to evaluate their performance. Earlier research on the topic of performance evaluation of TEWA systems is sparse, and the existing research tends to focus on either threat evaluation or weapon allocation, but not on the TEWA system as a whole.

Our suggested holistic approach to performance evaluation of TEWA systems is to use simulations of air defense scenarios in order to measure the survivability of the TEWA system, i.e., the ratio of the protection value of the surviving defended assets to the protection value of all the defended assets within the scenario. This survivability metric has also been complemented with the cost of resource usage. The reason for this is to penalize unnecessary or unwanted resource allocations that are not needed. An extreme example of this is the engagement of friendly or civilian aircrafts. Combining the survivability and resource usage metrics, we end up with a measure of a TEWA system's effectiveness, which we use for the comparative performance evaluation. The appropriateness of the suggested measure has been confirmed by military air defense experts. The metrics have been implemented into our testbed, in which we can create scenarios on which different TEWA system configurations can be run. At the moment, there are two different threat evaluation algorithms implemented, and one static weapon allocation algorithm.

The testbed has been used for comparing the effectiveness of different TEWA system configurations on a manually created air defense scenario. As evident from the experimental results, changes of individual parameters (in this case the threshold for how high a target's threat value must be before it becomes a potential candidate for weapon allocation) can have a large impact upon the outcome of the scenario, and the corresponding effectiveness of the TEWA system. This strengthens our view that we can not evaluate individual parts of a TEWA system in a reductionistic manner, but rather, have to evaluate the TEWA system as a whole.

We have in our experiments used *ground truth* data (referring to the reality of the tactical situation) as input to our TEWA systems. Hence, we have assumed the existence of a sensor making perfect observations of all aerial entities within the scenario. Obviously, such a perfect sensor does not exist, and we will therefore always have to deal with imperfect and uncertain data in a TEWA system situated within the real world. Hence, there is a need for exchangeable sensor models, in which we can model the behavior of different sensors under different circumstances. Our plan for future work is therefore to incorporate sensor models into our simulation environment, in which we can simulate the characteristics of real surveillance radar systems. We would also like to attack the problem of automatic scenario generation described in Section 5. To compare different TEWA system configurations on a specific scenario can be very valuable, but to say something about how good they are in general on a larger set of scenarios would be even more beneficial, since we do not know in advance which kind of scenarios a TEWA system will be used for.

It would also be interesting to perform a large number of simulations for a specific scenario, in which the threshold settings are tested more rigorously than in this paper. We have shown that small changes can have a large impact upon the resulting effectiveness, but we have not tested this systematically. For future work, it can be of interest to test and plot the changes in effectiveness for each possible threshold setting, for a more detailed study of the sensitivity to a chosen threshold.

The threat evaluation algorithms which are implemented into our testbed are quite advanced in comparison to the weapon allocation algorithm. Our implemented weapon allocation algorithm makes an exhaustive search for the optimal allocation, but such a search becomes infeasible when the number of weapon systems and/or targets with a threat value higher than the specified thresholds are to great. We would therefore want to implement better weapon allocation algorithms, which make guided heuristic searches for a good solution, as well as take the time for an engagement into consideration.

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The STAGE Scenario tool, which has been used for the air defense simulations in this project, is a tool from Presagis Inc. (see www.presagis.com). The core of the BN implementation in this project is based on Netica's C++-API (see www.norsys.com). For the implementation of the fuzzy inference rules, we have used the Fuzzy Logic Toolbox in MATLAB (see www.mathworks.com).

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Discounting and Combination Scheme in Evidence Theory for Dealing with Conflict in Information Fusion*

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Abstract. Recently combination rules as well as the issue of conflict management in Dempster-Shafer theory have received considerable attention in information fusion research. Mostly these studies considered the combined mass assigned to the empty set as the conflict and have tried to provide alternatives to Dempster's rule of combination, which mainly differ in the way of how to manage the conflict. In this paper, we introduce a hybrid measure to judge the difference between two bodies of evidence as a basis for conflict analysis, and argue that using the combined mass assigned to the empty set *as a whole* to quantify conflict seems inappropriate. We then propose to use the *discounting* operator in association with the *combination* operator to resolve conflict when combining evidence, in which the discount rate of a basic probability assignment is defined using the entropy of its corresponding pignistic probability function. Finally, an application of this *discounting and combination scheme* to fusion of decisions in classifier combination is demonstrated.

1 Introduction

The Dempster-Shafer theory of evidence (D-S theory, for short), originated from the work by Dempster [6] and then developed by Shafer [32], has appeared as one of the most popular theories for modeling and reasoning with uncertainty and imprecision in intelligent systems. In the D-S theory, Dempster's rule of combination plays a pivotal role serving as a powerful tool for combining evidence from distinct sources of information. According to Dempster's rule [32], the combined mass assigned to the empty set considered as the conflict is distributed proportionally to the other masses. Critically, Zadeh [41] presented an example showing that applying Dempster's rule to conflicting evidence yields counterintuitive results. After Zadeh's example, many alternatives have been proposed in the literature, most notably Smets' unnormalized combination rule [33], Yager's combination rule [39], Dubois and Prade's disjunctive combination rule [10].

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Lefevre *et al.* [21] have proposed a generic framework for evidence combination which provides a flexible way of distributing the conflict, i.e. the combined mass assigned to the empty set, among subsets of the frame of discernment and allows Dempster's rule as well as the three just mentioned rules of combination to be retrieved within the framework. Recently, motivated by the practical difficulty of verifying the distinctness assumption imposed on combined sources of evidence, Denoeux [9] has proposed two new rules of combination, namely the cautious conjunctive rule and its dual bold disjunctive rule, which are suggested to be suitable for combining belief functions from possibly overlapping bodies of evidence. Although there have been a numerous number of combination rules developed so far, Dempster's rule of combination [32] together with its unnormalized version [33] have been well justified theoretically and have greatly dominated the other rules in information fusion applications, e.g., [2,3,5,8,7,19,31,38].

In most previous studies on conflict management, it is mainly assumed that the conflict is identified by using the combined mass assigned to the empty set before normalization, denoted by $m_{\oplus}(\emptyset)$, and the thinking of how to manage this mass has basically raised interesting ideas for developing alternatives such as in [13,21,39]. Recently, Liu [26] has argued that the use of $m_{\oplus}(\emptyset)$ alone to quantify the conflict might lead to a wrong claim when considering what combination rule would be appropriate for combining conflicting evidence. Instead, Liu proposed to use a pair of quantitative measures, the combined mass allocated to the empty set before normalization, i.e. $m_{\oplus}(\emptyset)$, and the so-called distance between betting commitments, to justify when two pieces of evidence are in conflict. This formal definition of conflict can be served as a prerequisite for selecting appropriate combination rules [26]. Smets [36] has eventually provided a throughout examination of perhaps all existing combination rules and proposed an expert system approach for resolving conflict in evidence combination.

Note that the difference between two distinct bodies of evidence may be not only due to the conflict between two sources of evidence but also due to the complement of each other. For example, different sensors observe an object from different angles may provide different but complementary evidence about it. Although disjunctive consensus rules proposed in the literature such as Dubois and Prade's disjunctive combination rule [10] may be properly applied for combining complementary sources of evidence, the issue of detection of complement between combined bodies of evidence has been completely ignored so far. In the following of this paper, we first introduce a hybrid measure consisting of two components, the *quantitative distance between two mass assignments* and the *qualitative distance between two families of focal sets*, to judge the difference between two bodies of evidence. This hybrid measure can be used as a basis for conflict and complement analysis later on. We then argue that only a part of $m_{\oplus}(\emptyset)$ reflecting the conflict whilst the remainder representing the mass of uncommitted belief as a result of combination.

On the other hand, observing from the previous studies on the conflict analysis which mostly cited Zadeh's famous counterexample [41] to criticize Dempster's rule, we can see that ones assumed combined sources of evidence are still fully

reliable to be combined even a large conflict has been identified between them. Naturally, once realized that there is a conflict between sources of evidence, one should behave as if at least one of the sources would be not fully reliable. This issue has been critically discussed by Haenni in [23,24]. One of reasonable solutions to tackle such situations is to use discounting operator in association with combination [24,30]. A problem naturally arises here is how to determine which source of evidence is not fully reliable and to what discount rate it should be applied. Haenni [24] and Smets [36] suggested to use a meta-belief structure on combined sources of evidence for modeling this problem. However, it seems practically difficult to obtain such a meta-belief especially in information fusion for pattern recognition applications. In this paper, motivated from Smets' two-level model of belief [34], we propose to define the discount rate of a basic probability assignment based on how sure its commitment is if we use it alone for decision making. More particularly, the discount rate applied to a body of evidence is defined using the entropy of its corresponding pignistic probability function and intuitively, the more committed a basic probability assignment is, the lower discount rate it is applied.

The rest of this paper is organized as follows. In Section 2, we recall necessary concepts in the D-S theory. Section 3 devotes to the analysis of conflict and difference between two bodies of evidence. We particularly ague that the conventional view of $m_{\oplus}(\emptyset)$ as a whole to reflect conflict may be inappropriate. In Section 4, we propose to use the *discounting and combination scheme* for resolving conflict when combining evidence. Section 5 then illustrates an application of this scheme to ensemble learning for the problem of word sense disambiguation. Finally, some conclusions are presented in Section 6.

2 Basic of Dempster-Shafer Theory of Evidence

In the D-S theory [32], a problem domain is represented by a finite set Θ of mutually exclusive and exhaustive hypotheses, called *frame of discernment*. An important concept of the theory is the so-called *basic probability assignment* (BPA, for short), also called *mass function* or *basic belief assignment* (Smets [34]), $m : 2^{\Theta} \rightarrow [0, 1]$ satisfying

$$m(\emptyset) = 0, \text{ and } \sum_{A \in 2^{\Theta}} m(A) = 1$$

The quantity $m(A)$ can be interpreted as a measure of the belief that is committed exactly to A, given the available evidence. Note that the condition of $m(\emptyset) = 0$ corresponding to the "closed-world assumption" is not required in the Transferable Belief Model (TBM) introduced by Smets [33]. A subset $A \in 2^{\Theta}$ with $m(A) > 0$ is called a *focal element* of m. A BPA m is called to be *vacuous* if $m(\Theta) = 1$ and $m(A) = 0$ for all $A \neq \Theta$.

Let us denote $\mathcal{F}_m$ the set of focal elements of m, i.e.

$$\mathcal{F}_m = \{A \in 2^{\Theta} | m(A) > 0\}$$

Union of all elements in $\mathcal{F}_m$ defines the core of m and the pair $\mathcal{B} = (\mathcal{F}_m, m)$ is called a *body of evidence* (BOE).

Two useful operations that especially play an important role in the evidential reasoning are *discounting* and *Dempster's rule of combination* [32]. The discounting operation is used when a source of information provides a BPA m, but knowing that this source has probability α of reliability. Then one may adopt $(1-\alpha)$ as one's *discount rate*, resulting in a new BPA m^α defined by

$$m^\alpha(A) = \alpha \times m(A), \quad \text{for any } A \subset \Theta \tag{1}$$

$$m^\alpha(\Theta) = (1-\alpha) + \alpha \times m(\Theta) \tag{2}$$

Consider now two pieces of evidence on the same frame Θ represented by two BPAs m_1 and m_2. Dempster's rule of combination is then used to generate a new BPA, denoted by $m_\oplus = (m_1 \oplus m_2)$ (also called the orthogonal sum of m_1 and m_2), which is defined, for any $A \in 2^\Theta \setminus \emptyset$, as follows

$$m_\oplus(A) = \frac{\sum\limits_{B \cap C = A} m_1(B) m_2(C)}{1 - \sum\limits_{B \cap C = \emptyset} m_1(B) m_2(C)} \tag{3}$$

where

$$\sum_{B \cap C = \emptyset} m_1(B) m_2(C) \triangleq m_\oplus(\emptyset) \tag{4}$$

is the combined mass assigned to the empty set before normalization. Note that the orthogonal sum combination is only applicable to such two BPAs that verify the condition $m_\oplus(\emptyset) < 1$.

According to Smets' two-level view in TBM [34], when a decision needs to be made, a BPA m encoded the available evidence must be transformed into a so-called *pignistic probability function* $BetP_m : \Theta \to [0,1]$ defined by

$$BetP_m(\theta) = \sum_{A \subseteq \Theta, \theta \in A} \frac{m(A)}{|A|} \tag{5}$$

where $|A|$ is the cardinality of A. A justification for the necessity of the pignistic transformation in TBM framework is provided in [35]. Here we assume, however, to work under the closed-world assumption, i.e. $m(\emptyset) = 0$.

3 Conflict and Difference between Two BOEs

3.1 Conflict Revisited

In the research community of Dempster-Shafer theory, the mass associated with $m_\oplus(\emptyset)$ when combining two bodies of evidence with Dempster's rule has long been commonly taken as the only quantity indicating the conflict between two sources of information. The extreme case of fully conflict appears when $m_\oplus(\emptyset) = 1$. Recently, Liu [26] argued that value $m_\oplus(\emptyset)$ cannot be used as a quantitative measure of conflict between two bodies of evidence but only represents the mass of uncommitted belief as a result of combination.

Example 1. Let us consider Liu's example of two identical BPAs $m_1 = m_2$ on frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ and $m_1(\theta_i) = 0.2$ for $i = 1, ..., 5$. Then we get $m_\oplus(\emptyset) = 0.8$, which is quite high whilst it appears the total absence of conflict as two BPAs are identical.

More generally, we always get $m_\oplus(\emptyset) > 0$ with two identical BPAs whenever their focal elements define a partition of the frame. Simultaneously, Liu also proposes to use an addition criterion based on the difference between the pignistic probabilities together with value $m_\oplus(\emptyset)$ for judging whether two bodies of evidence are in conflict. Formally, two BPAs m_1 and m_2 are said to be *in conflict* if and only if

$$m_\oplus(\emptyset) > \epsilon \text{ and } \text{difBetP}(m_1, m_2) > \epsilon \tag{6}$$

where $\epsilon \in [0, 1]$ is a threshold of conflict tolerance and $\text{difBetP}(m_1, m_2)$ is defined by

$$\text{difBetP}(m_1, m_2) = \max_{A \subseteq \Theta}(|BetP_{m_1}(A) - BetP_{m_2}(A)|)$$

and called the *distance between betting commitments* of the two BPAs [26].

Basically, by the conclusion that "value $m_\oplus(\emptyset)$ cannot be used as a quantitative measure of conflict between two beliefs, contrary to what has long been taken as a *fact* in the Dempster-Shafer theory community." ([26], page 913) Liu tries to look into an addition criterion, namely $\text{difBetP}(m_1, m_2)$, in order to use in association with value $m_\oplus(\emptyset)$ for revealing the relationship between two BPAs.

Let us consider the following example.

Example 2. Suppose that we have the following pair of BPAs on the same frame $\Theta = \{\theta_i | i = 1, \ldots, 7\}$

$$m_1(\{\theta_1, \theta_2, \theta_3, \theta_4\}) = 1; \text{ and } m_2(\{\theta_4, \theta_5, \theta_6, \theta_7\}) = 1$$

Then, combining these two BPAs produces $m_\oplus(\emptyset) = 0$. That is, in the qualitative view of conflict defined by Liu [26], they do not contradict with each other, or in other words these two BPAs are not in conflict at all. However, using the second criterion we easily get $\text{difBetP}(m_1, m_2) = 0.75$.

In this example, note that m_1 and m_2 have assigned, by definition, the total mass exactly to $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ and $\{\theta_4, \theta_5, \theta_6, \theta_7\}$, respectively, and *to none of the proper subsets of them*. So intuitively these two BPAs are partly in conflict. Clearly, such a partial conflict does not be judged by means of $m_\oplus(\emptyset)$ but $\text{difBetP}(m_1, m_2)$ as shown above. However, they are not in conflict in the sense of (6).

On the other hand, in some information fusion situations, evidence come from different sources may offer complementary information each other but not only being in conflict.

Example 3. Consider the following two BPAs on the frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

$$m_1(\{\theta_1, \theta_2\}) = 0.4, m_1(\Theta) = 0.6$$
$$m_2(\{\theta_3, \theta_4\}) = 0.6, m_2(\Theta) = 0.4$$

That is, while the event $\{\theta_1, \theta_2\}$ is observable from the first source and becomes unseen from the second one, its complementary event $\{\theta_3, \theta_4\}$ is vice versa. The masses assigned to these events are based on available evidence of corresponding sources, and the unassigned masses are attributed to the whole frame due to ignorance. Intuitively, these two sources of evidence provide complementary information each other rather than they are in conflict. However, we obtain

$$m_{\oplus}(\emptyset) = 0.24, \text{ and } \mathrm{difBetP}(m_1, m_2) = 0.4$$

which allows us, in light of Liu's definition above, to conclude that two BPAs are in conflict to some extent.

The above observations suggest that taking $m_{\oplus}(\emptyset)$ as a whole for identifying the conflict seems inappropriate, except the extreme case of fully conflict, i.e. when $m_{\oplus}(\emptyset) = 1$. In the following subsection, we propose a more direct approach to judging the difference between two bodies of evidence, which then together with value $m_{\oplus}(\emptyset)$ can serve for conflict analysis. In the other words, we need to look at the difference between two bodies of evidence before using value $m_{\oplus}(\emptyset)$ for analyzing conflict.

3.2 Difference between Two BOEs

Let $\mathcal{B}_1 = (\mathcal{F}_{m_1}, m_1)$ and $\mathcal{B}_2 = (\mathcal{F}_{m_2}, m_2)$ be two bodies of evidence on the same frame Θ derived from two distinct sources of information. We first directly define the distance between two BPAs m_1 and m_2, denoted by $d(m_1, m_2)$, as follows

$$d(m_1, m_2) = \max_{A \subseteq \Theta}(|m_1(A) - m_2(A)|) \tag{7}$$

Obviously, $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$. This distance is considered as a *quantitative measure* for judging the difference between two bodies of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$. Now let us denote $\mathrm{dif}_{\mathcal{F}}(m_1, m_2)$ the symmetric difference between two families of focal elements $\mathcal{F}_{m_1}$ and $\mathcal{F}_{m_2}$, i.e.

$$\mathrm{dif}_{\mathcal{F}}(m_1, m_2) = (\mathcal{F}_{m_1} \setminus \mathcal{F}_{m_2}) \cup (\mathcal{F}_{m_2} \setminus \mathcal{F}_{m_1}) \tag{8}$$

It is easily seen that if $\mathrm{dif}_{\mathcal{F}}(m_1, m_2) = \mathcal{F}_{m_1} \cup \mathcal{F}_{m_2}$, and $A \cap B = \emptyset$ for any $A \in \mathcal{F}_{m_1}$ and $B \in \mathcal{F}_{m_2}$, then $m_{\oplus}(\emptyset) = 1$, which corresponds to the extreme case of fully conflict mentioned above. If $\mathrm{dif}_{\mathcal{F}}(m_1, m_2) = \emptyset$ and $d(m_1, m_2) > 0$, then qualitatively two sources are not in conflict but having different preferences in distributing their masses to focal elements. This qualitative measure $\mathrm{dif}_{\mathcal{F}}(m_1, m_2)$ allows us to see how different between two sources in realization of the question of where the true hypothesis lies.

Let us denote

$$\mathrm{dif}(\mathcal{B}_1, \mathcal{B}_2) = \langle d(m_1, m_2), \mathrm{dif}_{\mathcal{F}}(m_1, m_2)\rangle \tag{9}$$

and call it the *difference measure of two bodies of evidence*. It is clearly that the conflict between two bodies of evidence originates from either or both of

$d(m_1, m_2)$ (quantitative) and $\text{dif}_{\mathcal{F}}(m_1, m_2)$ (qualitative). Actually, Liu's criterion of using $\text{difBetP}(m_1, m_2)$ is somewhat weaker than using the direct distance of $d(m_1, m_2)$. For example, consider the pair of BPAs given in Example 2 we have $d(m_1, m_2) = 1$ whilst $\text{difBetP}(m_1, m_2) = 0.75$. Note further that if $m_1 = m_2$ we have $\text{difBetP}(m_1, m_2) = 0$ but the reverse does not hold in general.

We now argue that only a part of value $m_{\oplus}(\emptyset)$ should be used to quantify a conflict qualitatively stemming from $\text{dif}_{\mathcal{F}}(m_1, m_2)$. Let

$$m_{\oplus}^{\text{comb}}(\emptyset) = \sum_{A,B \in \mathcal{F}_1 \cap \mathcal{F}_2, A \cap B = \emptyset} m_1(A) m_2(B) \tag{10}$$

Clearly, $m_{\oplus}^{\text{comb}}(\emptyset)$ is a part of $m_{\oplus}(\emptyset)$ and intuitively representing the mass of uncommitted belief as a result of combination rather than a conflict, which, however, may be properly represented by the remainder of $m_{\oplus}(\emptyset)$, i.e.

$$m_{\oplus}(\emptyset) - m_{\oplus}^{\text{comb}}(\emptyset) \triangleq m_{\oplus}^{\text{conf}}(\emptyset) \tag{11}$$

Interestingly enough, with this formulation of conflict, the fact used to question the validity of Dempster's rule that two identical probability measures are always conflicting becomes inappropriate.

Example 4. Consider again two BPAs considered in Example 1, which are identical. Then we get $m_{\oplus}^{\text{comb}}(\emptyset) = 0.8$ and $m_{\oplus}^{\text{conf}}(\emptyset) = 0$, and hence no conflict appears between the two sources at all. Generally, we always get $m_{\oplus}^{\text{conf}}(\emptyset) = 0$ whenever two BPAs being combined are identical. Now, looking at Zadeh's famous counterexample with two BPAs m_1 and m_2 defined on $\Theta = \{a, b, c\}$ as: $m_1(a) = 0.99$, $m_1(b) = 0.01$ and $m_2(c) = 0.99$, $m_2(b) = 0.01$, we have $m_{\oplus}^{\text{conf}}(\emptyset) = 0.98$, which accurately reflects a very high conflict between two BPAs. With such a high conflict but still assuming both sources are fully reliable to proceed with directly applying Demspter's rule on them (to get unsatisfactory results) seems irrational.

Intuitively, the information from $\text{dif}(\mathcal{B}_1, \mathcal{B}_2)$ and $m_{\oplus}^{\text{conf}}(\emptyset)$ may properly provide helpful suggestions for conflict management on selecting appropriate combination rules in some typical situations.

- If $\text{dif}_{\mathcal{F}}(m_1, m_2) = \mathcal{F}_1 \cup \mathcal{F}_2$ and $A \cap B = \emptyset$ for any $A \in \mathcal{F}_{m_1}$ and $B \in \mathcal{F}_{m_2}$, we have $m_{\oplus}^{\text{conf}}(\emptyset) = 1$ and two sources are fully conflict. In this case a discounting and then combination strategy should be applied, where different attitudes may suggest different combination rules for use.
- If $\text{dif}_{\mathcal{F}}(m_1, m_2) = \emptyset$ and $d(m_1, m_2) > 0$, we have $m_{\oplus}^{\text{conf}}(\emptyset) = 0$ and two sources qualitatively are not in conflict but having different beliefs attributed to focal elements. In this situation, a compromise attitude may suggest to use the *trade-off* rule [10], or its special case of averaging operator.
- If $\text{dif}_{\mathcal{F}}(m_1, m_2) \neq \emptyset$, then we have $d(m_1, m_2) > 0$. In this situation, if $m_{\oplus}(\Theta) = m_1(\Theta) m_2(\Theta) > 0$ two sources may provide complementary information each other as in the case of Example 3 above, and then Dempster's rule can be applied. If $m_{\oplus}(\Theta) = 0$, two sources may be in a partial conflict

and then depending on value $m_{\oplus}^{\mathrm{conf}}(\emptyset)$ whether it is tolerated and information on meta-belief is available or not, one may apply discounting and then combination strategy or a disjunctive combination rule.

The issue of justifying whether two bodies of evidence are in conflict plays an especially important role in selecting alternative combination rules [36]. Thus, identification of conflict should be analyzed as carefully as possible. An accurate determination of the origin of conflicts can also help to manage them properly. In addition, it is our opinion that justifying whether two bodies of evidence are in complementary each other, which has been ignored so far, also have some impact in the mentioned selection problem and should be incorporated into the conflict analysis. Intuitively, the role of value $m_{\oplus}(\Theta)$ may play for this purpose in a somehow similar fashion to that of $m_{\oplus}^{\mathrm{conf}}(\emptyset)$ for conflict analysis as roughly mentioned above; however, this is not a main topic of this paper.

4 Discounting and Combination Scheme

Previously, a common explanation for counterintuitive results yielded by applying Dempster's rule of combination is that possible conflicts between different sources of evidence are mismanaged by Dempster's rule, and this explanation has motivated for developing alternatives combination rules, which are mainly different in the way of managing possible conflicts [24]. Unfortunately, these alternatives are generally not associative, e.g. [10,21,39], and thus making them difficult to be applied in practice. In [23], Haenni also presented a critical note on the increasing number of possible combination rules.

Once possible conflicts have been identified, we may naturally wonder about the reliability of different sources of evidence being combined. If a meta-belief of the sources is available, we can first use the discounting operator for BPAs envolved and then apply Dempster's rule to discounted BPAs for combining them. The idea of using the discounting operator to resolve conflict has already been suggested in, i.e., [23,30,36]. However, in practice such a beta-belief is not always available, particularly in situations of applying the Dempster-Shafer theory to, for instance, information fusion in pattern recognition (see, e.g., [2,3,19,29,38]).

According to Smets' two-level view of evidence [34], to make decisions based on evidence, beliefs encoding evidence must be transformed into probabilities using the so-called pignistic transformation. Guided by this view, we propose to discount a BPA involving in combination based upon how sure in its decision when it is used alone for decision making. More particularly, we provide a method for defining discount rates of BPAs being combined using the entropy of their corresponding pignistic probability functions.

Let m_1 and m_2 be two BPAs on the frame Θ and $BetP_{m_1}$ and $BetP_{m_2}$ be pignistic probability functions of m_1 and m_2, respectively. For $i = 1, 2$, we denote

$$H(m_i) = -\sum_{\theta \in \Theta} BetP_{m_i}(\theta) \log_2(BetP_{m_i}(\theta))$$

the Shannon entropy expression of pignistic probability distribution $BetP_{m_i}$. This measure has been used in Jousselme *et al.* [14] as an *ambiguity measure* of belief functions.

Clearly, $H(m_i) \in [0, \log_2(|\Theta|)]$. We now define the *discount rate* of BPA m_i $(i = 1, 2)$, denoted by $\delta(m_i)$, as follows

$$\delta(m_i) = \frac{H(m_i)}{\log_2(|\Theta|)} \tag{12}$$

That is, the higher uncertainty (in its decision) a source of evidence is, the higher discount rate it is applied. Once discount rates have been defined, the discounting and combination strategy applied to two BPAs m_1 and m_2 can be generally formulated in the following form

$$m_{\oplus} = m_1^{(1-\delta(m_1))} \oplus m_2^{(1-\delta(m_2))} \tag{13}$$

where $\oplus$ is a combination operator in general and $m_i^{(1-\delta(m_i))}$ is the discounted BPA obtaining from m_i after discounting at a rate of $\delta(m_i)$ [refer to (11)-(12)].

It is of interest to note that if, for example, $\delta(m_1) = 1$, i.e. $BetP_{m_1}$ is the uniform distribution on Θ or m_1 is at the most uncertain in its decision, $m_1^{(1-\delta(m_1))}$ becomes a vacuous BPA and then plays no role in combination if Dempster's rule is applied. In other words, a decision made using the combined evidence represented by m then depends on the second source of evidence represented by m_2 only.

As for illustration, this discounting and combination strategy will be applied for combining multiple classifiers in the following section. Here Dempster's rule and averaging operator are used for combination. Thanks to its associativity, we can develop an efficient algorithm for combining multiple classifiers with Dempster's rule, where soft decisions by individual classifiers typically are represented in forms of probability distributions over the set of possible classes. Also, although simple in computation, averaging is suggested as providing a good solution to balance multiple evidence [27].

5 An Illustrative Application

Applying the D-S theory to classifier combination has received attention since early 1990s, e.g., [2,3,29,38]. In these methods, it is usually assumed that the involved individual classifiers provide fully reliable sources of information for identifying the label of a particular input pattern, i.e. discounting operator plays no role there. In this section, we present an illustration for applying the discounting and combination scheme discussed above to ensemble learning for the problem of word sense disambiguation (WSD) [12], which has received much interest and concern since the 1950s and is still one of the most challenging tasks in NLP.

Actually, Le *et al.* [19] recently have attempted to apply the D-S theory for weighted combination of classifiers for WSD, in which the weighting is also modeled by the discounting operator. However, their method of defining discounting

factors for individual classifiers is based on the strength of individual classifiers, which is determined by testing them on a designed sample data set and therefore does not be influenced by an input pattern under classification. Here, in the context of classification problem, the discounting method discussed above in this paper provides a new way of adaptively weighting individual classifiers based on ambiguity measures associated with their outputs corresponding to a particular pattern under consideration.

5.1 WSD

Roughly speaking, WSD is the task of associating a given word in a text or discourse with an appropriate sense among numerous possible senses of that word. This is an "intermediate task" which necessarily accomplishes most natural language processing tasks such as grammatical analysis and lexicography in linguistic studies, or machine translation, man-machine communication, message understanding in language understanding applications [12].

During the last two decades, many machine learning techniques and algorithms have been applied for WSD, including Naive Bayesian (NB) model, decision trees, exemplar-based model, support vector machines (SVM), maximum entropy models (MEM), etc. [1,20]. On the other hand, as observed in studies of classification systems, the set of patterns misclassified by different learning algorithms would not necessarily overlap [25]. This means that different classifiers may potentially offer complementary information about patterns to be classified. This observation highly motivated the interest in combining classifiers to build an ensemble classifier which would improve the performance of the individual classifiers. Particularly, classifier combination for WSD has been received considerable attention recently from the community as well.

5.2 Individual Classifiers in Combination

To build individual classifiers for combination, we use three well-known statistical learning methods including the Naive Bayes (NB), Maximum Entropy Model (MEM), and Support Vector Machines (SVM). The selection of these learning methods is basically guided by the direct use of output results for defining BPAs in the present work. Clearly, the first two classifiers produce classified outputs which are probabilistic in nature. Although a standard SVM classifier does not provide such probabilistic outputs, the issue of mapping SVM outputs into probabilities has been studied [28] and recently become popular for applications requiring posterior class probabilities [3,22]. We have used the library implemented for maximum entropy classification available at [37] for building the MEM classifier, whilst the SVM classifier is built based upon LIBSVM implemented by Chang and Lin [4], which has the ability to deal with the multiclass classification problem and output classified results as posterior class probabilities.

Due to the limitation of page number, the technical detail of these methods as well as the discounting and combination strategy applied to them is omitted here (see [18] for the detail). Informally, the output of individual classifiers is used to define corresponding BPAs. Then we apply the discounting and

combination strategy discussed in Section 4 to these BPAs and the final decision is made based on the resulted BPA. Two combination rules are applied in this application, namely Dempster's rule of combination and averaging. Accordingly, we develop two algorithms corresponding to these combination rules, namely *discounting-and-orthogonal sum combination algorithm* and *discounting-and-averaging combination algorithm*, respectively.

5.3 Experimental Results

Test Data. As for evaluation of exercises in automatic WSD, three corpora so-called Senseval-1, Senseval-2 and Senseval-3 were built during three corresponding workshops held in 1998, 2001, and 2004 respectively. Here, the developed combination algorithms will be tested on English lexical samples of Senseval-2 and Senseval-3. Currently, these two datasets are widely used in current WSD studies. The detail of these data sets can be referred to Kilgarriff [15] for Senseval-2 and to Mihalcea *et al.* [17] for Senseval-3.

Like Le *et al.* [19], we use the evaluation method proposed by Melamed and Resnik in [16], which provides a scoring method for exact matches to fine-grained senses as well as one for partial matches at a more coarse-grained level. Also, like most related studies, the fine-grained score is computed in the following experiments.

Results. Table 1 below provides the experimental results obtained by three individual classifiers and two combination algorithms developed, where DCA_1 and DCA_2 stand for the discounting-and-orthogonal sum combination algorithm and the discounting-and-averaging combination algorithm, respectively. The obtained results show that combined classifiers always outperform individual classifiers participating in the corresponding combination. It is of interest to see that the results yielded by the discounting-and-averaging combination algorithm (i.e., DCA_2) are comparable or even better than that given by the discounting-and-orthogonal sum combination algorithm (i.e., DCA_1), while the former is computational more simple than the latter. Although the averaging operation was actually mentioned briefly by Shafer [32] for combining belief functions, it has been almost completely ignored in the studies of information fusion and particularly classifier combination with D-S theory. Interestingly, Shafer [32] did show that discounting in fact turns combination into averaging when all the information sources being combined are highly conflicting and have been sufficiently discounted. This might, intuitively, provide an interpretation for a good performance of DCA_2.

To have a comparative view of obtained results, Table 2 provides comparative results of the developed algorithms with previous studies, namely the best systems in the contests for the English lexical sample tasks of Senseval-2 [15], Senseval-3 [17], and the method developed by Le *et al.* [19]. The best system of Senseval-2 contest also used a combination technique: the output of subsystems (classifiers) which were built based on different machine learning algorithms

Table 1. Experimental results

%	Individual			Combination	
	NB	MEM	SVM	DCA_1	DCA_2
Senseval-2	65.6	65.5	63.5	66.3	66.5
Senseval-3	72.9	72.0	72.5	73.3	73.3

Table 2. A comparative result

%	Best systems	Le [19]	DCA_1	DCA_2
Senseval-2	64.2	64.7	66.3	66.5
Senseval-3	72.9	72.4	73.3	73.3

were merged by using weighted and threshold-based voting and score combination [40]. The best system of Senseval-3 contest used the Regularized Least Square Classification (RLSC) algorithm with a correction of the a priori frequencies (for more details, see [11]). This comparative result shows that both developed combination algorithms deriving from the discounting and combination scheme yield an improvement in overall accuracy compared to previous work for WSD in the tests with Senseval-2 and Senseval-3.

6 Conclusions

In this paper, we have introduced a difference measure of two bodies of evidence serving as a basis for conflict analysis in Dempster-Shafer theory. We argued that the combined mass allocated to the empty set should be divided into two parts, one part represents the mass of uncommitted belief as a result of combination whilst the other reflects the conflict. Interestingly, this analysis might help to solve the question of the validity of Dempster's rule by the fact that two identical probability measures are always conflicting. We have also proposed the use of the discounting operator together with the combination operator for resolving conflict when combining evidence, in which an entropy-based method for defining discounting factors was introduced. As for illustrating the applicability of the proposed discounting and combination scheme, we have also provided an experimental study in combining multiple classifiers for WSD which produces better results in comparison to previous related studies.

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Evaluation Based on Pessimistic Efficiency in Interval DEA

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Abstract. In *Interval DEA* (Data Envelopment Analysis), *efficiency interval* has been proposed and its bounds are obtained from the optimistic and *pessimistic* viewpoints, respectively. Intervals are suitable to represent uncertainty of the given input-output data and decision makers' intuitive evaluations. Although the intervals give elements a partial order relation, it is sometimes complex, especially in case of many elements. The efficiency measurement combining optimistic and pessimistic efficiencies in Interval DEA is proposed. They are compared from the view that both of them represent the difference of the analyzed DMU (Decision Making Unit) from the most efficient one. The proposed efficiency measurement is mainly determined by the pessimistic efficiency. The optimistic one is considered if it is inadequate comparing to the pessimistic one. Such a pessimistic efficiency based evaluation is more similar to our natural evaluation and DMUs are *arranged* as a linear order.

Keywords: Interval DEA, efficiency interval, pessimistic, arrangement.

1 Introduction

In DEA (Data Envelopment Analysis), the maximum relative ratio of the weighted sum of outputs to that of inputs is regarded as the efficiency of DMU (Decision Making Unit) [1]. Because of the maximum ratio, it is an evaluation from the optimistic viewpoint for the analyzed DMU. In Interval DEA, efficiency is denoted as an interval in order to reflect various evaluation viewpoints [2]. All the possible relative evaluations for a DMU are included in the efficiency interval. The upper bound of efficiency interval is obtained from the optimistic viewpoint for the analyzed DMU relatively to others. It is the same as the conventional DEA [1]. On the other hand, the lower bound is obtained from the pessimistic viewpoint.

With the obtained efficiency intervals, the partial order relation of DMUs is obtained based on interval order relation [3]. The partial order relation, comparing to the linear one, is suitable to represent uncertainty in real situations. It enables a pair of elements to be unknown relation, that is, these elements are not related to each other. It can be illustrated as a kind of diagram and it helps us to recognize the relative relations of elements intuitively. However, the relative relations of many DMUs, for instance 20 DMUs, are not easy to understand, if it

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is illustrated. In case of evaluating elements, the partial order relation sometimes is not clear enough and does not give expected information to decision makers.

As for ranking of DMUs based on DEA, several methods have been proposed and compared in [4]. The conventional DEA rates more than one DMU as efficient, that is, their efficiencies are equal. Some methods focus on distinguishing among those DMUs. In [5], cross-efficiency which is based on peer-appraisal concept is used as a complement or alternative to simple efficiency which is based on self-appraisal. The average cross-efficiencies helps to distinguish DMUs whose simple efficiencies are 1 and to establish meaningful ranking. In [6] [7], efficient DMUs are ranked by the importance as a benchmark for inefficient DMUs. In [8], by removing constraint on the analyzed DMU, the supper efficiency which enables to be greater than 1 is proposed. The supper efficiency is also defined by using slacks-base measure in [9]. The sensitivity analysis has been done in [10] where the necessary and sufficient conditions for preserving efficiency when data changes are made for all DMUs are developed. It provides what-if tool and measures efficiency of DMU from the view. The method applying for rank voting data has been proposed in [11] [12]. The assurance region side constraints are used and the gap between the weights which should be greater for higher places than for lower ones is maximized in order to discriminate efficient DMUs. In this paper, DMUs are arranged with efficiency intervals in Interval DEA, differently to the above methods based on the conventional DEA.

In the conventional DEA [1] [13], the optimistic efficiencies are obtained and DMUs can be arranged as a linear order based on them. In this paper, DMUs are arranged based on the pessimistic efficiency, which has been proposed to compose efficiency interval in Interval DEA [2]. The pessimistic efficiency becomes small if the DMU has any unique input and/or output. In this sense, the DMU whose pessimistic efficiency is small has some weak points, though its optimistic efficiency is possible to be great. In the pessimistic efficiency sense, DMU which has the balanced input and output data can be rated as efficient. Since the pessimistic efficiency represents the efficiency at least guaranteed in any undesirable scenario, the evaluation based on them fits our natural sense, especially in case of risk-aversion situations. Therefore, the pessimistic efficiency alone can play an important and useful role in arranging DMUs. However, it is not a good idea to ignore the optimistic efficiency. From this view, DMUs are arranged mainly from their pessimistic efficiencies considering the optimistic ones if they are apparently inadequate. In order to compare the optimistic and pessimistic efficiencies, they are normalized, respectively. Originally the efficiency interval represents the possible efficiency by assuming production possibility set consisting of the existing DMUs so that its bounds, the optimistic and pessimistic efficiencies, are normalized by the common set. In this paper, the meanings of the optimistic and pessimistic efficiencies are reconsidered. They are dealt with individually and the optimistic or pessimistic efficiencies of all DMUs are compared one another. Then, it shows one of the ways to give DMUs a linear order based on the efficiency interval.

2 Interval DEA

DEA is a non-parametric technique for measuring the efficiency of DMUs with common input and output terms [1]. In DEA, the ratio of weighted sum of outputs to that of inputs is assumed to be the efficiency of each DMU. In the conventional DEA, the input and output weights are variables that are determined so as to maximize the analyzed DMU's ratio from its optimistic viewpoint subject to the condition concerning every DMU [1,13]. However, the relative efficiency can be obtained from various viewpoints. The efficiency from the pessimistic viewpoint has been already formulated in [2]. Then, an interval consists of the optimistic and pessimistic efficiencies and it is called efficiency interval. The problems to obtain the optimistic and pessimistic evaluations are formulated as the following way. The relative efficiency of the analyzed unit DMU_o is denoted as

$$\theta_o = \frac{\frac{\boldsymbol{u}^t \boldsymbol{y}_o}{\boldsymbol{v}^t \boldsymbol{x}_o}}{\max_j \left(\frac{\boldsymbol{u}^t \boldsymbol{y}_j}{\boldsymbol{v}^t \boldsymbol{x}_j} \right)} \tag{1}$$

where $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ are the given input and output vector of DMU_j whose elements are all positive, and $\boldsymbol{v}$ and $\boldsymbol{u}$ are the weight variables. The numbers of input and output items and DMUs are m, k and n, respectively. The ratio of weighted sum of output data to that of input data for DMU_o is compared to the maximum ratio of all DMUs. Corresponding to the denominator, the production possibility set is defined as follows.

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\}$$

It assures that the smaller outputs and/or greater inputs are possible.

By maximizing or minimizing the relative efficiency (1) with respect to the weight variables, it is approximated by two kinds of values, θ_o^* and θ_{o*}. They are the extreme values of the relative efficiency from the optimistic and pessimistic viewpoints for DMU_o.

The problem to obtain θ_o^* is formulated as follows.

$$\begin{aligned} \theta_o^* = \max_{\boldsymbol{u}, \boldsymbol{v}} & \frac{\frac{\boldsymbol{u}^t \boldsymbol{y}_o}{\boldsymbol{v}^t \boldsymbol{x}_o}}{\max_j \left(\frac{\boldsymbol{u}^t \boldsymbol{y}_j}{\boldsymbol{v}^t \boldsymbol{x}_j} \right)} \\ \text{s.t. } & \boldsymbol{u} \geq \boldsymbol{0}, \quad \boldsymbol{v} \geq \boldsymbol{0} \end{aligned} \tag{2}$$

The weight variables are determined to maximize the relative efficiency θ_o. θ_o^* focuses on the superior items for DMU_o and it is called optimistic evaluation. The fractional programming problem (2) is reduced to the following LP problem which is the same as CCR model usually used in the conventional DEA [1,2,13].

$$\begin{aligned} \theta_o^* = \max_{\boldsymbol{u}, \boldsymbol{v}} & \boldsymbol{u}^t \boldsymbol{y}_o \\ \text{s.t. } & \boldsymbol{v}^t \boldsymbol{x}_o = 1 \\ & \boldsymbol{v}^t \boldsymbol{x}_j - \boldsymbol{u}^t \boldsymbol{y}_j \geq 0 \quad \forall j \\ & \boldsymbol{u} \geq \boldsymbol{0} \\ & \boldsymbol{v} \geq \boldsymbol{0} \end{aligned} \tag{3}$$

On the other hand θ_{o*} is obtained by the following problem.

$$\theta_{o*} = \min_{\boldsymbol{u},\boldsymbol{v}} \frac{\frac{\boldsymbol{u}^t \boldsymbol{y}_o}{\boldsymbol{v}^t \boldsymbol{x}_o}}{\max_j \left(\frac{\boldsymbol{u}^t \boldsymbol{y}_j}{\boldsymbol{v}^t \boldsymbol{x}_j} \right)} \quad \text{s.t. } \boldsymbol{u} \geq \boldsymbol{0}, \quad \boldsymbol{v} \geq \boldsymbol{0} \tag{4}$$

θ_{o*} focuses on the inferior items for DMU_o and it is called pessimistic evaluation. The optimal value of (4) is the same as the following equation.

$$\theta_{o*} = \min_{p,r} \frac{\frac{y_{op}}{x_{or}}}{\max_j \frac{y_{jp}}{x_{jr}}} \tag{5}$$

The possible efficiency of DMU_o is denoted as efficiency interval, $\Theta_o = [\theta_{o*}, \theta_o^*]$. Comparing efficiency intervals of two DMUs, the DMU whose efficiency interval is greater than the other is evaluated as more efficient. The uncertainty in human intuitive evaluations is reflected by various viewpoints from which DMUs are evaluated and denoted as width of an interval. The obtained efficiencies depend on the evaluation viewpoints and such various possible efficiencies are all included in the efficiency interval. Since generally intervals give more information than crisp values, the efficiency interval is suitable to represent uncertain situations in real problems.

3 Arrangement of DMUs

With efficiencies in the conventional DEA, which are the optimistic efficiencies in Interval DEA, DMUs are arranged as a linear order. There are often several DMUs whose efficiencies are equal so that the differences among them can not be recognized. With efficiency intervals, DMUs are arranged based on the interval order relation is defined as follows in [3].

Definition 1. *For two intervals* $A = [\underline{a}, \overline{a}]$ *and* $B = [\underline{b}, \overline{b}]$, $A \succ B$ *holds if and only if* $\underline{b} \leq \underline{a}$ *and* $\overline{b} \leq \overline{a}$.

In case of efficiency intervals, the DMU whose both optimistic and pessimistic efficiencies are greater than those of the other is rated as more efficient. Otherwise, their order is unknown. The partial order relation is suitable to represent the situation as it is, since the width of an interval can reflect uncertainty of the efficiency. On the other hand, it is sometimes difficult for decision makers to recognize the relative relations of DMUs intuitively and visually if there are many unknown relations and many DMUs.

This paper proposes the method to arrange DMUs mainly based on their pessimistic efficiencies as well as considering the optimistic efficiency to some extent. Figure 1 shows efficiency intervals consisting of the simple 4 combinations of the optimistic and pessimistic efficiencies.

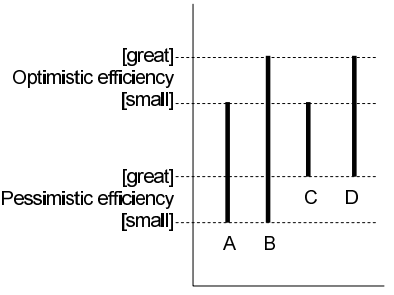

Fig. 1. Four examples of efficiency intervals

It is apparent that efficiency intervals of A and D are the least and most, respectively. It is not clear which is more efficient, B or C. The basic concept used in this paper is to rate C better than B, since the pessimistic efficiency of C is greater than that of B. It should be noted that it depends on the degree how great or small the optimistic or pessimistic efficiency is. The pessimistic efficiency gives a kind of strict evaluation and often fits our natural evaluation sense, especially in case of risk aversion problems. It is helpful to know the efficiency under the least desirable scenario and to evaluate DMU with it. In daily life, when we expect to feel secured, we make a decision by thinking pessimistically. The evaluation of DMUs based on only the pessimistic efficiencies is acceptable and reasonable. Then, focusing on the pessimistic efficiency, which is the lower bound of efficiency interval, a linear order of DMUs is found. However, it is not a good idea to ignore the optimistic efficiencies. In the following, the optimistic and pessimistic efficiencies are dealt with individually and compared, respectively.

The pessimistic efficiency represents the insufficiency of efficiency comparing to the ideal input-output which is peculiar to DMU_o and it seldom equals to 1. All DMUs have some insufficiency comparing to the ideal respective pseudo-DMU. It measures how far DMU_o is possible to be from the least insufficient DMU, which is the most efficient one in the pessimistic efficiency sense. In the similar way, the optimistic efficiency measures how close DMU_o is possible to be from the most efficient DMU. In order to compare pessimistic and optimistic efficiencies, they are normalized so as to make their maximum one, respectively, as follows.

$$\theta_o^{*\prime} = \frac{\theta_o^*}{\max_j \theta_j^*} = \theta_o^* \quad \theta'_{o*} = \frac{\theta_{o*}}{\max_j \theta_{j*}} \tag{6}$$

As for the optimistic efficiency, the possibility to be efficient comparing to others is considered as highly as possible. Then, their maximum is apparently 1 so that the optimistic one can be used as it is.

In the sense of measuring the difference from the efficient DMU, the optimistic efficiency and the normalized pessimistic efficiency are compared. Then, the new efficiency measurement of DMU_o is defined as follows.

$$\theta''_{o*} = \min\{\theta_o^*, \theta'_{o*}\} \tag{7}$$

The minimum represents how different DMU_o is from the most efficient DMU in any sense. It often equals to the normalized pessimistic one so that DMUs are arranged mainly based on the pessimistic efficiencies. While, such a DMU as its optimistic one is not enough comparing to its pessimistic one is evaluated by the smaller efficiency, that is, its optimistic one. This efficiency measurement excludes that the DMU, which is not efficient enough from the optimistic viewpoint, is highly evaluated even if its pessimistic efficiency is adequate.

The concept of defining (7) is similar to how the lower bound of interval efficiency is obtained by (5). The defined efficiency measurement is obtained by solving (5) assuming the upper and lower bounds of efficient interval as two outputs and 1 as one input for all DMUs.

4 Numerical Example

The numerical example consists of 15 DMUs with common one input and three outputs. In order to illustrate their relative relations, the efficiency measurements based on the ratio of the weighted sum of outputs to that of inputs are obtained. Table 1 shows the given input-output data and the efficiency intervals determined by (3) and (5). With the obtained efficiency intervals, the partial order relation of DMUs is illustrated on the left of Figure 2. It shows several unknown relations such as (D,O), (M,C) and (M,E). The more the number of DMUs increases, the more complex the diagram becomes. The efficiency interval includes evaluations from all the possible viewpoints and the partial order relation is suitable to represent uncertain situation as it is. By the proposed efficiency measurement, the partial order relation is reduced to a linear order relation.

Table 1. 1-input and 3-output data and efficiency intervals

DMU	Input1	Output 1	Output 2	Output 3	Efficiency interval	Efficiency measurement
A	59.08	18.23	3.22	49.25	[0.700,0.951]	0.805
B	67.09	25.50	2.12	44.47	[0.551,0.862]	0.634
C	59.61	25.21	2.76	47.97	[0.794,0.979]	0.913
D	60.71	20.94	3.03	47.92	[0.779,0.898]	0.896
E	60.00	25.10	2.99	49.00	[0.805,0.998]	0.927
F	72.00	27.60	4.00	73.00	[0.869,1.000]	1.000
G	64.02	28.23	2.92	55.88	[0.795,1.000]	0.915
H	70.77	26.10	4.06	68.21	[0.836,1.000]	0.962
I	61.71	26.41	2.90	51.98	[0.819,0.991]	0.942
J	67.61	25.50	3.78	65.78	[0.855,0.996]	0.984
K	64.26	25.51	3.05	58.21	[0.826,0.962]	0.950
L	67.99	27.92	2.94	60.93	[0.754,0.974]	0.867
M	65.31	23.59	3.14	54.12	[0.817,0.903]	0.903
N	51.89	21.00	2.74	45.31	[0.861,1.000]	0.991
O	65.00	23.70	3.03	52.00	[0.789,0.895]	0.895

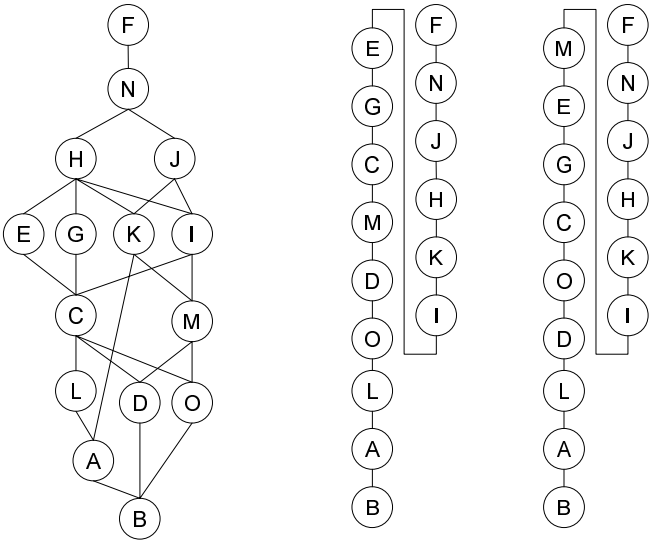

Fig. 2. Partial and linear order relations

The pessimistic efficiencies are normalized by (6) and compared with the optimistic ones. Then, the efficiency measurements are obtained by (7) and shown at the right column in Table 1. The center and the right of Figure 2 show the linear order relations with the proposed (7) and with the pessimistic efficiency (5), respectively. By (7), D is rated as more efficient than O and M is rated as less efficient than C and E. While, by (5), their order relations are reverse. The linear order relation at the center of Figure 2 by (7) is more similar to the partial order relation at its left than the linear one by (5). In order to modify the partial order relation into linear one, the proposed efficiency measurement is useful. Two DMUs, M and O, are rated as less efficient by the proposed efficiency measurements than by the conventional pessimistic efficiencies. Their efficiency measurements equal to their upper bounds of efficiency intervals. That is, although their pessimistic efficiencies are good enough, their optimistic efficiencies are poor comparing to DMUs with the similar pessimistic efficiencies. Their optimistic efficiencies are expected to be more relatively to the other DMUs. In this way, the proposed efficiency measurement is almost the same as the pessimistic efficiency. It could have an advantage since it considers the optimistic efficiency when it is needed.

5 Conclusion

In this paper, a new efficiency measurement based on the efficiency interval in Interval DEA has been proposed. The pessimistic and optimistic efficiencies, which are the upper and lower bounds of efficiency interval, are dealt with individually

and the pessimistic one is focused on. Since the pessimistic efficiency is obtained under the possible least desirable scenario, it is acceptable for decision makers to rate DMUs based on it. In order to take not only pessimistic efficiency but also optimistic one into account to some extent, the pessimistic efficiencies are normalized. Then, the new efficiency measurement is defined as the minimum of the optimistic and pessimistic efficiencies. It usually equals to the normalized pessimistic efficiency, except that the optimistic efficiency is apparently inadequate. In that case, it equals to the optimistic efficiency. Generally the greater pessimistic efficiency is, the better the DMU is evaluated. The proposed efficiency measurement gives a linear order relation considering efficiency intervals which reflect uncertainty of evaluations. The simple relative relation of DMUs helps decision makers to evaluate them.

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Stochastic Facility Construction Problem with Preference of Candidate Sites

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Abstract. This paper considers a facility construction problem in a rectangular urban area with some barriers and rectilinear distance. There exist some demand points and possible construction sites with preference. A random construction cost according to a normal distribution. The probability that the cost becomes below the budget should not be below the fixed level. One objective is that the budget should be minimized under the condition demand points are covered by at least one of facilities constructed within a certain critical distance. Another is that the minimal preference among constructed sites should be maximized. The other is to maximize minimal satisfaction degree with respect to critical distances among all demand points. We formulate our problem as a three criteria problem with a chance constraint. Since usually there exists no solution optimizing three objectives at a time, we seek some non-dominated solutions after the definition of non-domination.

Keywords: Facility construction, Random construction cost, Urban area, Preference of possible site, Barriers, Budget constraint, satisfaction of critical distance, Non-dominated solution.

1 Introduction

There are huge amount of papers about facility location problem (so called Weber problem) after Weber has published his paper [11]. However recently there are many developments from the usual facility location problem. One of them is combination of a supply chain management and it is surveyed in [9]. As another one, we consider the new facility planning problem under the stochastic construction cost which is the extension of the usual facility construction problem covering demands by at least one facility in the sense that each demand is within the critical distance from at least one facility where each critical distance is also a decision variable maximizing minimal satisfaction degree among all demand points. Related studies are surveyed in the paper [4]. In order to plan the facility construction, we should consider many factors

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such as the total cost, suitable construction sites, service to citizens. That is, budget for construction is sufficient though it should be minimized and it may change. Distance to the nearest facility is close as enough so that citizens can receive satisfied service. Anyway, since once facilities were constructed, we cannot destroy them nor stop the service easily. Therefore we should make a model, analyze the result and estimate the future situation.

First we formulate the problem and after the definition of non-domination, we propose a procedure finding some non-dominated solutions. Finally we summarize the results and discuss the further research problem.

2 Problem Formulation

We consider the following problem:

(1) There are m demand points
$D_i = (a_i, b_i),\ i = 1, 2, ..., m$ and n possible facility construction sites
$FP_j = (p_j, q_j),\ j = 1, 2, ..., n$ in an urban area
$X = \{(x, y) \mid 0 \le x \le p_0,\ 0 \le y \le q_0\}$ with some rectangular barriers

(2) $\mathbf{B}_k = \{(x, y) \mid B_k^1 < x < B_k^2,\ B_k^3 < y < B_k^4\}, k = 1, 2, ..., s$. Barrier means we cannot pass it inside and so in some case we must make a detour. We adopt rectilinear distance which is used often in an urban area. For each possible site FP_j, suitability of the construction of the facility is attached and this is denoted by the preference μ_j $(0 < \mu_j \le 1)$ (This idea is for the construction side and appeared in our related paper [7]). Facility construction cost at each possible site is an independent normal distributed random variables with mean m_j and variance σ_j^2 and the total cost of the construction should be below the budget F with a probability not less than the prescribed level $\alpha > 0.5$. F is a decision variable.

(3) One objective is that the budget F should be minimized under the condition demand points are covered by at least one of facilities constructed within the distance determined a priori. Another is that the minimal preference among constructed sites should be maximized. The other is minimal satisfaction degree among all demand points with respect to covering critical distance should be maximized. That is, following setting: If demand point i is covered by some facility constructed at possible site j (this means critical distance $d \ge d_{ij}$, distance between demand point i and possible site j) then its satisfaction degree is $\bar{\mu}_{ij}$ $(0 \le \bar{\mu}_{ij} \le 1)$. So $\min_{i,j}\{\bar{\mu}_{ij} \mid d_{ij} \le d,\ x_j = 1\}$ should be maximized where

we assume that $\bar{\mu}_{ij} \ge \bar{\mu}_{i'j'}$ holds when $d_{ij} \le d_{i'j'}$.

Under above setting, we have a following bi-criteria problem $\bar{P}$.

$\bar{P}$: Minimize F
Maximize $\min_j\{\mu_j \mid x_j = 1\}$

Maximize $\min\{\bar{\mu}_{ij} \mid j \in I_i(d),\ x_j = 1,\ i = 1,2,...,m\}$

subject to

$$\Pr\{\sum_{j=1}^{n} c_j x_j \le F\} \ge \alpha$$

$$\sum_{j \in I_i(d)} x_j \ge 1,\ i = 1,2,..,m,\ x_j = 0,1,\ j = 1,2,...,n,$$

where

$I_i(d) = \{j \mid d_{ij} \le d\}$, $i = 1,2,...,m$, $\alpha > 0.5$ and d_{ij} is the distance from the possible candidate site j to the demand point i considering barriers (how to calculate the distance is explained below). d is the critical distance in order to cover the demand points and this is also an adjustable decision variable and positive since covering with shorter distance is better.

Since

$$\Pr\{\sum_{j=1}^{n} c_j x_j \le F\} \ge \alpha \Leftrightarrow \Pr\left\{\frac{\sum_{j=1}^{n}(c_j - m_j)x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}} \le \frac{F - \sum_{j=1}^{n} m_j x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}}\right\} \ge \alpha,$$

$$\frac{\sum_{j=1}^{n}(c_j - m_j)x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}}$$

is the standard normal distributed random variable and $x_j^2 = x_j$ due to $x_j = 0 or 1$, this chance constraint is transformed as follows: $F \ge \sum_{j=1}^{n} m_j x_j + K_\alpha \sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}$ where K_α is the α percentile point of the cumulative distribution function of the standard normal distribution and positive since $\alpha > 0.5$. Since F should be minimized, $\bar{P}$ is transformed into the following deterministic equivalent problem P.

P: Minimize $\sum_{j=1}^{n} m_j x_j + K_\alpha \sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}$

Maximize $\min_j\{\mu_j \mid x_j = 1\}$

Maximize $\min\{\bar{\mu}_{ij} \mid j \in I_i(d),\ x_j = 1,\ i = 1,2,...,m\}$

subject to $\sum_{j \in I_i(d)} x_j \ge 1,\ i = 1,2,..,m,$,

$x_j = 0 or 1,\ j = 1,2,...,n.$

(How to calculate the distance d_{ij} and construct $I_i(d)$)

First note that rectilinear distance between visible two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ is $|x_1 - x_2| + |y_1 - y_2|$ where visible means that no tour is needed and in an invisible case, deour is needed in order to calculate distance between them.

Fig. 1. Invisible pair and distance calculation (Rectangular area is indicated by dotted line is a barrier)

Any way, as shown in Figure 1, distance is calculated via vertices of the barrier. So we first construct the network *N(V,A)* given as follows:

$$V = \{D_1, D_2, \cdots, D_m,\ (B_1^1, B_1^3), (B_1^1, B_1^4), (B_1^2, B_1^3), (B_1^2, B_1^4), \cdots,$$
$$(B_i^1, B_i^3), (B_i^1, B_i^4), (B_i^2, B_i^3), (B_i^2, B_i^4), \cdots, (B_s^1, B_s^4), (B_s^2, B_s^3), (B_s^2, B_s^4),$$
$$FP_1, FP_2, \cdots, FP_n\}\ (= \{v_1, v_2, \cdots, v_n, v_{n+1}, \cdots, v_{n+4s}, v_{n+4s+1}, \cdots, v_{n+4s+n}\})$$

and *E* consists of edges corresponding to visible pairs between two vertices in *V*. For each edge, rectilinear distance between corresponding vertices is attached. So using some algorithm (for example [8]) for finding a shortest path for each pair of vertices, we can calculate a distance d_{ij} from demand point i and candidate site j.

Based on these distances, we construct each set I_i checking whether each distance is farther than d or not. *P* is a three-criteria problem and so usually there exists no solution optimizing all objectives at a time and so we seek some non-dominated solutions after definition of non-domination.

(Non-dominated solution)

For two solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$, if

$$\sum_{j=1}^{n} m_j x_j^{(1)} + K_\alpha \sqrt{\sum_{j-1}^{n} \sigma_j^2 x_j^{(1)}} \le \sum_{j=1}^{n} m_j x_j^{(2)} + K_\alpha \sqrt{\sum_{j-1}^{n} \sigma_j^2 x_j^{(2)}},$$

$$\min\{\mu_j \mid x_j^{(1)} = 1\} \ge \min\{\mu_j \mid x_j^{(2)} = 1\}$$

and

$$\min\{\bar{\mu}_{ij} \mid j \in I_i(d),\ x_j^{(1)} = 1,\ i = 1, 2, ..., m\} \ge$$
$$\min\{\ \bar{\mu}_{ij} \mid j \in I_i(d),\ x_j^{(2)} = 1,\ i = 1, 2, ..., m\}$$

and at least one inequality holds without equality, then we call $\mathbf{x}^{(1)}$ dominates $\mathbf{x}^{(2)}$. For a solution $\mathbf{x}$, if there no solution that dominates $\mathbf{x}$, then we call $\mathbf{x}$ non-dominated solution.

3 Solution Procedure

In order to find non-dominated solutions, as for d, we consider $d = d_{ij},\ i = 1,2,...,m,\ j = 1,2,...,n$ only. So sorting $d_{ij},\ i = 1,2,...,m,\ j = 1,2,...,n$ and let the result be as follows: $0 < d_1 < d_2 < \cdots < d_\ell$ where ℓ is the number of different d_{ij} and corresponding $\overline{\mu}_{ij}$ also denoted by $\mu^{(1)} > \mu^{(2)} > \cdots > \mu^{(\ell)} > 0$. Then we introduce the following subproblem $P(t),\ t = 1,2,...,\ell$ with fixed critical distance d_t.

$$P(t): \text{Minimize} \sum_{j=1}^{n} m_j x_j + K_\alpha \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j}$$

$$\text{Maximize} \min_j \{\mu_j \mid x_j = 1\}$$

$$\text{subject to} \sum_{j \in I_i(d_t)} x_j \geq 1,\ i = 1,2,..,m,$$

$$x_j = 0 \text{ or } 1,\ j \in \bigcup_{i=1,2,...,m} I_i(d_t).$$

Each $P(t),\ t = 1,2,...,\ell$ is a bicriteria problem and so we define non-dominated solution also.

(Non-dominated solution of $P(t)$)

For two feasible solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ of $P(t)$, if

$$\sum_{j=1}^{n} m_j x_j^{(1)} + K_\alpha \sqrt{\sum_{j-1}^{n} \sigma_j^2 x_j^{(1)}} \leq \sum_{j=1}^{n} m_j x_j^{(2)} + K_\alpha \sqrt{\sum_{j-1}^{n} \sigma_j^2 x_j^{(2)}}$$

$$\min\{\mu_j \mid x_j^{(1)} = 1\} \geq \min\{\mu_j \mid x_j^{(2)} = 1\}$$

and at least one inequality holds without equality, then we call $\mathbf{x}^{(1)}$ dominates $\mathbf{x}^{(2)}$.

For a feasible solution $\mathbf{x}$ of $P(t)$, if there no feasible solution that dominates $\mathbf{x}$, then we call $\mathbf{x}$ non-dominated solution.

Note that as for non-dominated solutions of

Note that *a*s for non-dominated solutions of $P(t)$, the minimum $\overline{\mu}_{ij}$ with respect to the settled facility is not less than $\mu^{(t)}$. For finding non-dominated solution of $P(t)$, we first sort $\mu_1, \mu_2, \cdots, \mu_n$ and let the result be $\mu^{(1)} \leq \mu^{(2)} \leq \cdots \leq \mu^{(q)}$ where q is

the number of different μ_j. Next we define $I_i^{(k)}(t) = \{ j \mid d_{ij} \le d_t, \mu_j \ge \mu^{(k)} \}$ and $I^{(k)}(t) = I_1^{(k)}(t) \cup I_2^{(k)}(t) \cup \cdots \cup I_m^{(k)}(t)$. Then we introduce the following subsidiary problem $P^{(k)}(t)$, $k = 1, 2, ..., q$ of $P(t)$

$$P^{(k)}(t): \text{Minimize} \sum_{j \in I^{(k)}(t)} m_j x_j + K_\alpha \sqrt{\sum_{j \in I^{(k)}(t)} \sigma_j^2 x_j}$$

$$\text{subject to} \sum_{j \in I_i^{(k)}(t)} x_j \ge 1,\ i = 1, 2, .., m,\ x_j = 0 \, or \, 1,\ j \in I^{(k)}(t)$$

In order to solve $P^{(k)}(t)$, $k = 1, 2, ..., q$, we further introduce the following auxiliary, problem $P_R^{(k)}(t)$ *of* $P^{(k)}(t)$ with positive parameter R.

$$P_R^{(k)}(t): \text{Minimize} R \sum_{j \in I^{(k)}(t)} m_j x_j + \frac{1}{2} K_\alpha \sum_{j \in I^{(k)}(t)} \sigma_j^2 x_j \left(= \sum_{j \in I^{(k)}(t)} (R m_j + \frac{1}{2} \sigma_j^2) x_j \right)$$

$$\text{subject to} \sum_{j \in I_i^{(k)}(t)} x_j \ge 1,\ i = 1, 2, .., m,\ x_j = 0 \, or \, 1,\ j \in I^{(k)}(t)$$

This problem is an usual set covering problem and so we can solve it by using some set covering algorithm such as [1], [2], [4] ([10] has given a heuristic method and nice survey about set covering problem). As is shown in [6], a following useful relation holds between $P^{(k)}(t)$ and
$P_R^{(k)}(t)$(Proof is very same as a stochastic spanning tree version in [6].)

Theorem 1. Let $\mathbf{x}^R(k)$ and $\mathbf{x}^{(k)}$ be an optimal solution of $P_R^{(k)}(t)$ and that of $P^{(k)}(t)$ respectively. If

$R_k^* = \sqrt{\sum_{j \in I^{(k)}(t)} \sigma_j^2 x_j^{R^*}(k)}$, then $\mathbf{x}^{R^*}(k)$ is an optimal solution of $P^{(k)}(t)$. Let

$i_m(k) = \min\{\sigma_j^2 \mid j \in I_i^{(k)}(t)\},\ i_M(k) = \max\{\sigma_j^2 \mid j \in I_i^{(k)}(t)\}$

and $R_m^{(k)} = \sqrt{\sum_{i=1}^{m} \sigma_{i_m(k)}^2}$, $R_M^{(k)} = \sqrt{\sum_{i=1}^{m} \sigma_{i_M(k)}^2}$.

Then it is easy to see $R_m^{(k)} \le R_k^* \le R_M^{(k)}$. We divide the interval $[R_m^{(k)},\ R_M^{(k)}]$ by the suitable sizes into sub-intervals $[R_\ell^{(k)},\ R_{\ell+1}^{(k)}]$, $\ell = 0, 1, 2, ..., \ell(k)$ where $R_0^{(k)} = R_m^{(k)}$, $R_{\ell(k)+1}^{(k)} = R_M^{(k)}$. Then R is set to the center value $\bar{R}_\ell^{(k)}$ of each sub-interval $[R_\ell^{(k)},\ R_{\ell+1}^{(k)}]$ and solve $P_{\bar{R}_\ell^{(k)}}^{(k)}$ to find an optimal solution $\mathbf{x}_{(\ell)}^{(k)}$ of this auxiliary problem and calculate corresponding value of objective function for $P^{(k)}(t)$

$$F_\ell^{(k)} = \sum_{j \in I^{(k)}(t)} m_j \overline{x}_{(\ell)j}^{(k)} + K_\alpha \sqrt{\sum_{j \in I^{(k)}(t)} \sigma_j^2 \overline{x}_{(\ell)j}^{(k)}} \cdot$$

Then compare $F_\ell^{(k)}$ and choose the best one as an optimal solution of $P^{(k)}(t)$.

Now we have the following main algorithm for obtaining some non-dominated solutions.

[Algorithm for $P(t)$]

Step 0: Calculate $I_1(d_t), \cdots, I_m(d_t)$, $\mu^{(1)}, \cdots, \mu^{(q)}$ and set $DM(t) = \phi$, $k = 1$.

Go to Step 1.

Step 1: Solve $P^{(k)}(t)$ and obtain an optimal solution $\mathbf{x}^{(k)}$. If there exists no solution that dominates $\mathbf{x}^{(k)}$, then $DM(t) = DM(t) \cup \{\mathbf{x}^{(k)}\}$ and go to Step 2. Otherwise go to Step 2 directly.

Step 2: Set $k = k' + 1$ where k' is the index such that $\mu^{(k')} = \min\{\mu_j \mid x_j^{(k)} = 1\}$. If $k = q + 1$, then terminate with the set of non-dominated solutions *DM*. Otherwise return to Step 1.

Solving $P(1), P(2), \cdots, P(\ell)$ explicitly or implicitly and set non-dominated set of our problem as $DM = DM(1) \cup DM(2) \cup \cdots \cup DM(\ell)$. The above algorithm solves the set covering problem many times and the set covering problem is NP-complete([5]). Therefore it is not easy to solve our problem and its complexity is not polynomial. But we should endeavor to reduce the number of the set covering problem needed to solve.

4 Conclusion

We considered a facility construction problem with stochastic construction cost and have proposed a solution procedure to find some non-dominated solutions. But our procedure is straightforward and not enough efficient and so refinement is needed. Especially in some case, we need not solve all sub-problems $P(1), P(2), \cdots, P(\ell)$ from scratch. Role of changing critical distance d is that facility or facilities must be located as near as possible and so we considered consider satisfaction degree of service with respect to demand i and facility j. The other is we must change the distance measure depending on the importance of demand points or their request of service quantity such population.

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A Consensus Reaching Model for Web 2.0 Communities

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Abstract. Web 2.0 Communities allow large amounts of users to interact with each others. In fact, new Web 2.0 technologies allow to share resources and information in an easy and timely manner, allowing real time communication among persons all over the world. However, as Web 2.0 Communities are a quite recent phenomenon with its own characteristics and particularities, there is still a necessity of developing new tools that allow to reach decisions with a high enough consensus level among their users. In this contribution we present a new consensus reaching model designed to incorporate the benefits that a Web 2.0 Community offers (rich and diverse knowledge due to a large number of users, real-time communication...) and that tries to minimize the main problems that this kind of organization presents (low and intermittent participation rates, difficulty of establishing trust relations and so on).

1 Introduction

In the last years, the World Wide Web has allowed the creation of many different services in which users from all over the world can join, interact and produce new contents and resources. One of the most recent trends, the so called *Web 2.0*, which comprises a set of different web developement and design techniques, allows the easy communication, information sharing, interoperatbility and collaboration in this new virtual environment. Web 2.0 Communities, that can take different forms as Internet forums, groups of blogs, social network services and so on, provide a plataform in which, users can collectively contribute to a Web presence and generate massive content behind their virtual collaboration [1]. In fact, Web 2.0 represents a paradigm shift in how people use the web as nowadays, everyone can actively contribute content online.

Among the different activities that the users of Web Communities usually perform we can cite:

- **Generate online contents and documents**, which is greatly beneficiated with the great diversity and knowledge of the involved people. One of the clearest examples of this kind of collaboration success is Wikipedia [2], where millions of articles have been produced by its web community in dozens of different languages.

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- **Provide recommendations** about different products and services. Usual recommender systems are increasing their power and accuracy by exploiting their user bases and the explict and implicit knowledge that they produce [3]. A clear example of recommeder systems success, which exploits its users community knowledge to provide personalized recommendations, is the Amazon online store [4].
- **Make decisions** about particular problems. Group Decision Making (GDM) is a typical human activity which consists on selecting the best alternative from a feasible set according to a group of individuals. Thus, the main goal of any GDM process is to identify the best alternative according to some established criteria, and it is normally assumed that the experts have a common interest in obtaining a final solution for the problem. Examples of typical GDM processes are to vote in an election, to choose a place for a meeting or to select the model of laptop that a firm will buy to its employees. Usual simple group decision making schemes, as referendum or voting systems are now widely stablished in the Web. For example, services like PollDaddy [5] and BallotBin [6] allow to create online surveys and polls where users can vote about the best alternative to choose for a given decision problem.

There have been several efforts in the specialized literature to create different models to correctly address and solve GDM situations. Some of them make use of fuzzy theory as it is a good tool to model and deal with vague or imprecise opinions (which is a quite common situation in any GDM process) [7,8]. Many of those models are usually focused on solving GDM situations in which a particular issue or difficulty is present. For example, there have been models that allow to use linguistic assessments instead of numerical ones, thus making it easier for the experts to express their preferences about the alternatives [9]. Other models allow experts to use multiple preference structures (and even multi-granular linguistic information) [10,11] and other different approaches deal with incomplete information situations if experts are not able to provide all their preferences when solving a GDM problem [12] or when a consensus process is carried out [13].

Moreover, usual GDM models have been complemented with consensus schemes that allow users to interact until there is a certain degree of agreement on the selected solution [14,15]. This consensus models allow not only to provide better solutions to decision problems, but also to increase the users satisfaction with the decision process as all the opinions are reconsidered to achieve a high enough level of consensus.

However, those approaches are not usually well suited to be used by Web Communities due to some of their inherent properties. For example, dynamic situations in which some of the parameters of the problem, as the set of experts, the set of alternatives and even the set of criteria to select the solutions change, have not been modeled. This kind of situations are quite common in other environments: in [16] the problem of managing time-dependent preferences (that is preferences expressed at different periods) is presented; the problem of dealing with dynamic real-time information to choose the best routes is shown in [17], and a practical example about resource managment where the criteria to make decisions (climate) changes over time can be found in [18]. Thus, it is important to develop new models that take into account this kinds of dynamical situations to solve realistic GDM problems [19].

For the particular case of Web Communities, dynamic situations in which the group of experts vary over time are quite common: a new expert could incorporate to the process, some experts could leave it or a large group of experts could be simplified in order to minimize communications and to ease the computation of solutions. This behaviour is usually found in democratic systems where the individuals delegate into a smaller group of experts to make decisions (it is usually not possible to involve everyone in each decision). There have been some efforts to model this kind of situations. For example, in [20] a recursive procedure to select a qualified subgroups of individuals taking into account their own opinions about the group is presented. However, there is still a big necessity of creating new consensus models that suit Web Communities characterstics appropriately.

In this paper we present a consensus model which has been designed taking into account the characteristics of Web 2.0 Communities. In particular, it has been designed considering that the number of users of this kind of communities is usually large [21]. For example, online music communities usually gather hundreds or even thousands of individuals that share an interest about particular bands or music genres. To reach a consensual decision with such a large user base is not an easy task because, for example, not every member of the community is willing to participate and contribute to solve the problem [22]. In addition, this model allows dynamic sets of users, that is, the users set to solve the decision problem may change in time. Moreover, by means of a delegation scheme (based on a particular kind of trust network [23]) we may achieve an important simplification in the obtaining of a proper consensus level. Finally, a trust checking procedure allows to avoid some of the problems that the delegation scheme could introduce in the consensus reaching model.

To do so, the paper is set as follows: in section 2 we present our preliminaries, that is, some of the most important characteristics of Web 2.0 Communities and the basic concepts that we use in our paper. In section 3 we introduce the new consensus model that helps to obtain consensual decisions in Web 2.0 Communities. Finally, in section 4 we point out our conclusions.

2 Preliminaries

2.1 Web 2.0 Communities

New Web 2.0 technologies have provided a new framework in which virtual communities can be created in order to collaborate, communicate, share information and resources and so on. This very recent kind of communities allows people from all over the globe to meet other individuals which share some of their interests. Apart from the obvious advantage of meeting new people with similar interests, Web Communities present some characteristics that make them different from other more usual kinds of organizations. In the following we discuss some of those characteristics and how they can affect in the particular case of GDM situations:

– **Large user base:** Web Communities usually have a large user base [21] (it is easy to find web communities with thousands of users). This characteristic can be seen

from a double perspective. On the one hand, the total knowledge (explicit or implicit) that a large user base implies is usually greater and more diverse than in a small community. This can be seen as a clear advantage: taking decisions is usually better performed when there is a rich knowledge on the evaluated subject. On the other hand, managing a large and diverse amount of opinions in order to extract and use that knowledge might be a difficult task.

- **Low participation and contribution rates:** Although many Web Communities have a quite large user base, many of those users do not directly participate in the community activities. Moreover, encouraging them to do so can be difficult [22]. Many of the users of a web community are mere spectators which make use of the produced resources but that does not (and is not willing to) contribute themselves with additional resources. This can be a serious issue when making decisions if only a small subset of the users contribute to a decision and it does not reflect the overall opinion of the community.
- **Real time communication:** The technologies that support Web Communities allow near real time communication among its members. This fact let us create models that in traditional scenarios would be quite impractical. For example, in a referendum, it is not easy at all to make a second round if there has been a problem in the first one due to the high amount of resources that it requires.
- **Intermittent contributions:** Partially due to the fast communication possibilities and due to a very diverse involvement of the different members, it is a common issue that some of them might not be able to collaborate during a whole decision process, but only in part of it. This phenomenon is well known in web communities: new members are continuously incorporated to the community and existing users leave it or temporarily cease in their contributions.
- **Difficulty of establishing trust relations:** As the main communication schemes in Web Communities use electronic devices and, in the majority of the cases, the members of the community do not know each other personally, it might be difficult to trust in the other members to, for example, delegate votes. This fact implies that it might be necessary to implement control mechanisms to avoid a malicious user taking advantage of others.

2.2 Consensus Models with Fuzzy Preferences

Usual GDM models follow a scheme in which two phases are differentiated (see figure 1): the first one consists in a *consensus process* in which the users (that we will call *experts* in the following), discuss about the alternatives and express their preferences about them using a particular preference representation format. A special individual (the moderator) checks the different opinions and confirms if there is enough consensus among all the experts. If there is not enough consensus, the moderator urges the experts to re-discuss about the alternatives and to provide a new set of opinions to improve the consensus level in a new consensus round. Once the desired consensus have been reached (or a maximum number of consensus rounds has been reached) the second phase (the *selection process*) starts and the best solution is obtained by agreggating the last opinions from the experts and applying an exploitation step which identifies the best alternative from the agreggated information.

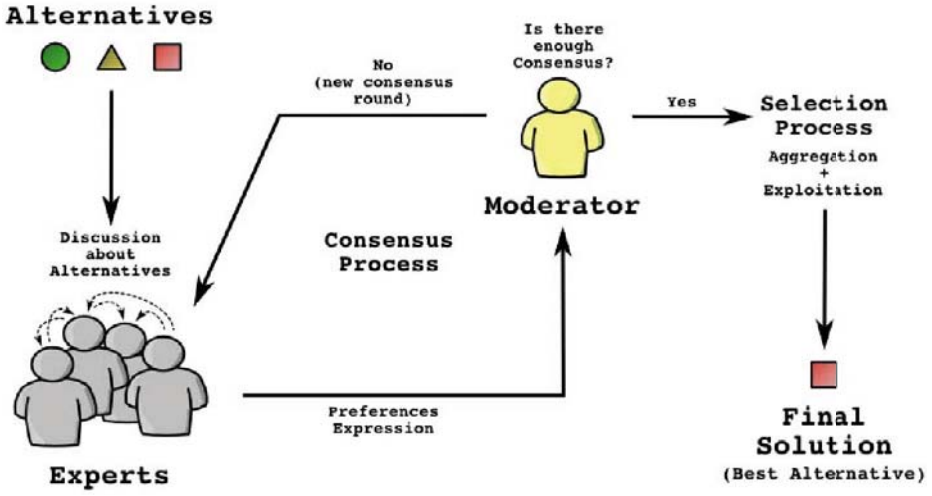

Fig. 1. Typical scheme of GDM models

In this paper we center our attention only in the consensus process, where the experts are supposed to narrow their different opinions about the alternatives to obtain a final solution with a high level of consensus. In the consensus model that we propose, the experts $E = \{e^1, \ldots, e^m\}$ will provide their preferences about the set of alternatives $X = \{x_1, \ldots, x_n\}$ in form of fuzzy preference relations [10]:

Definition: A fuzzy preference relation P^h given by expert e^h on a set of alternatives X is a fuzzy set on the product set $X \times X$, which is characterized by a membership function $\mu_P^h : X \times X \rightarrow [0, 1]$.

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P^h = (p_{ij}^h)$, being $p_{ij}^h = \mu_P^h(x_i, x_j)\ \forall i, j \in \{1, \ldots, n\}$ interpreted as the preference degree or intensity of the alternative x_i over x_j: $p_{ij}^h = 1/2$ indicates indifference between x_i and x_j ($x_i \sim x_j$), $p_{ij}^h = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij}^h > 1/2$ indicates that x_i is preferred to x_j ($x_i \succ x_j$). Based on this interpretation, we have that $p_{ii}^h = 1/2\ \forall i \in \{1, \ldots, n\}$ ($x_i \sim x_i$).

3 A Consensus Model for Web 2.0 Communities

In this section we present a new Consensus model that can be applied in Web 2.0 Communities. It takes into account the different characteristics of this kind of communities (see section 2.1) in order to increase the consensus level of the users when making a decision on a set of alternatives. One interesting property of our model is that it does not require the existance of a moderator. Its operation includes several different steps that are repeated in each consensus round:

1. **First step:** First preferences expression, computation of similar opinions and first global opinion and feedback.

2. **Second step:** Delegation (or change of preferences) and computation of consensus measures
3. **Third step:** Consensus and trust checks

In figure 2 we have depicted the main steps of the model and in the following we describe them more detail.

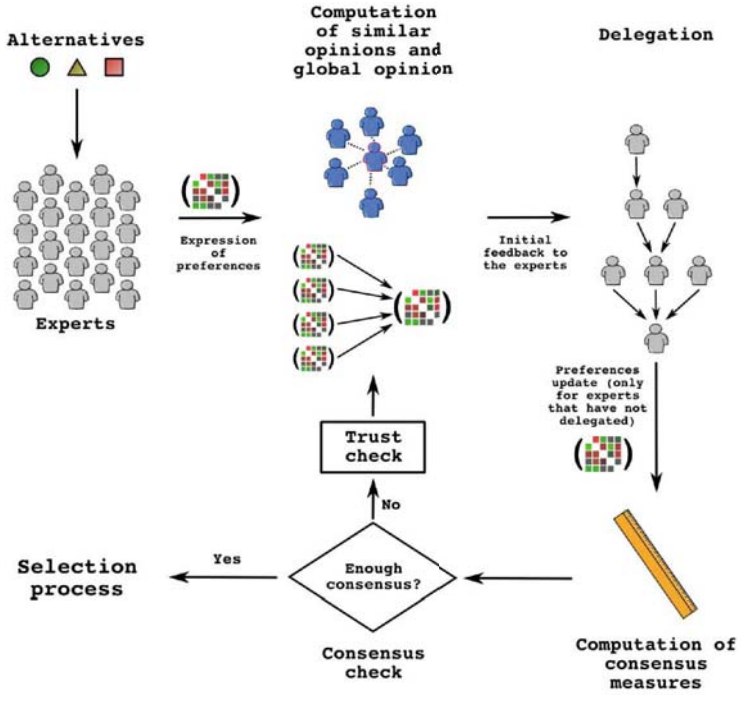

Fig. 2. Scheme of the presented consensus model

3.1 First Step: First Preferences Expression, Computation of Similar Opinions and First Global Opinion and Feedback

In this first step the different alternatives in the problem are presented to the experts (note than in figure 2 we have represented only a small amount of experts, but when applied to a Web 2.0 Community the number of users will usually by larger). Once they know the feasible alternatives, each expert $e^h \in E$ is asked to provide a fuzzy preference relation P^h that represent his opinions about the alternatives. Although every single member of the community has the oportunity of expressing his preferences about the alternatives, as we have previously mentioned, only a subset of those experts $\tilde{E}$ will really provide preference relations. We will note $\tilde{e}^h$ to the experts that have provided a preference relation. It is important to note that if an expert at this stage does not provide a preference relation the model will still allow him to contribute in the consensus process in a later stage. Once a certain amount of time has passed (to allow a

sufficient number of preferences to be provided) we compute the distance among each pair of experts $\tilde{e}^h$ and $\tilde{e}^g$ in the following way:

$$d^{hg} = d^{gh} = \sqrt{\sum_{i=1}\sum_{\substack{j=1\\ j\neq i}}(p_{ij}^h - p_{ij}^g)^2}$$

This distances will be used to provide information to each expert about the experts that share a similar opinion about of the alternatives. In fact, for each $\tilde{e}^h \in \tilde{E}$ we define his *set of neighbours* as

$$N^h = \{\tilde{e}^{\beta_1}, \ldots, \tilde{e}^{\beta_{nn}}\}$$

where nn is the number of neighbours that each expert will be presented (this parameter is defined prior to the start of the consensus process) and e^{β_i} is the i-th nearest expert to $\tilde{e}^h$ (with lowest $d^{h\beta_i}$).

In this step we also compute the current global preference as an aggregation of all the provided preference relations. To do so, we will apply a simple arithmetic average to compute it, as at this point the preferences expressed by all the experts are considered to have the same weight:

$$p_{ij}^c = \frac{\sum_{\tilde{e}^h \in \tilde{E}} p_{ij}^h}{\#\tilde{E}}$$

Once the distances among experts, the neighbours of each expert and the global preference relation have been computed, this information will be presented to the experts. After receiving this feedback, an expert will know if his opinions are very different to the current global preferences and he will also know which are the experts that share similar opinions. Apart from just his neighbour list, an expert is also able to check the particular preference relations that his neighbours have introduced in order to really check the preferences expressed by his neighbourhood.

3.2 Second Step: Delegation (or Change of Preferences) and Computation of Consensus Measures

In this second step the model incorporates a delegation scheme in which experts may choose to delegate into other experts (typically experts from their neighbourhood, with similar opinions). To allow that, we define $t^h \in \{1, \ldots, m\} \cup \emptyset$ as the expert in which $\tilde{e}^h$ delegates. Note that as experts may choose not to delegate, it is possible to have $t^h = \emptyset$. Thus, in this phase each expert that thinks that he will not be able to continue in the consensus process, instead of just leaving the process, can choose another expert and delegate on him. When an expert delegates on another expert, he will not be required to update his preferences to improve the consensus level.

In figure 3 we have depicted a group of experts in which some of them have delegated into other experts. Note that the two experts on the right have not delegated in any other expert and have neither been chosen by other experts to delegate in them. In the figure we can also see how some experts have delegated over experts that have already delegated in another one, thus creating a tree structure. This tree structure conforms a kind of trust

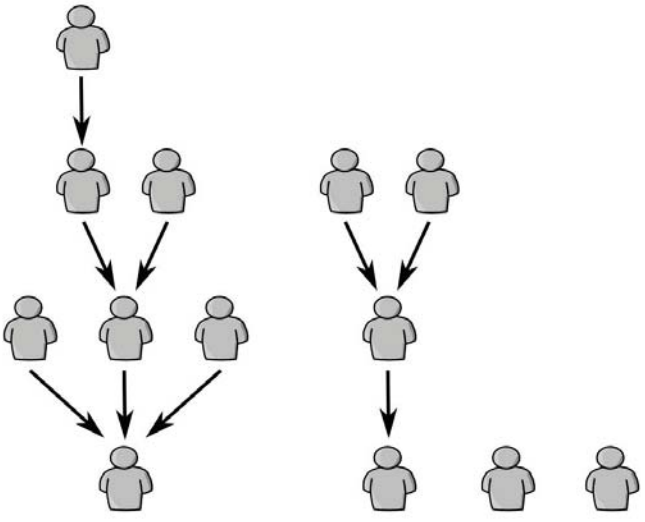

Fig. 3. Example delegation scheme

network in which some transitivity conditions are applied: if an expert $\tilde{e}^h$ delegates in an expert $\tilde{e}^k$ and $\tilde{e}^k$ delegates in $\tilde{e}^j$ the situation would be similar as if both $\tilde{e}^h$ and $\tilde{e}^k$ would have delegated in $\tilde{e}^j$. Note that the model should avoid cicles in the trust network. If an expert ask to delegate in another one and this delegation would produce a cicle in the trust network the system should alert him about this situation and ask him to reconsider his decision by delegating over a different expert or simply by not delegating.

Once a certain amount of time have passed (enough time for the experts to decide if they wanted to delegate or not), the system will compute a trust weight τ^h for every expert according to the trust network. Initially all the experts in $\tilde{E}$ have a $\tau^h = 1$. The system should then check every t^h and if $t^h \neq \emptyset$ it will follow the chain of delegations until it founds an expert $\tilde{e}^k$ which has not delegated. Then, the trust weights will be updated: $\tau^k = \tau^k + 1$ and $\tau^h = 0$.

This delegation mechanism provides several advantages to the model: first of all, it allows experts not to provide their preferences in every consensus round. If an expert delegates in another one, he will not have to update his preferences but, in a certain way (through the delegate), his opinion will still influence the consensus state. Thus, the consensus rounds may be carried out faster as only a subset of experts will have to change their preferences. Moreover, the computations will also be reduced as the system will not have to deal with a large amount of preference relations.

Once the trust weights have been computed the system will ask the remaining experts to update their fuzzy preference relations P^h in order to achieve a greater level of consensus. This experts will conform the new $\tilde{E}$ subset. Once the updated preferences have been given we can compute some consensus degrees. To do so, we firstly define for each pair of experts $(\tilde{e}^h, \tilde{e}^l)$ $(h < l)$ of the new $\tilde{E}$ a similary matrix $SM^{hl} = \left(sm_{ik}^{hl}\right)$ where

$$sm_{ik}^{hl} = \tau^h \cdot \tau^l \cdot (1 - |p_{ik}^h - p_{ik}^l|)$$

Then, a collective similarity matrix, $SM = (sm_{ik})$ is obtained by aggregating all the $(\#\tilde{E} - 1) \times (\#\tilde{E} - 2)$ similarity matrices using following expression:

$$sm_{ik} = \frac{\sum_{h,l \in \tilde{E} | h<l} sm_{ik}^{hl}}{T \cdot (T-1)/2}$$

where $T = \sum_{i=1}^{m} \tau^i$.

Once the similarity matrices are computed we proceed to obtain the consensus degrees at the three different levels:

L. 1. *Consensus degree on pairs of alternatives*. The consensus degree on a pair of alternatives (x_i, x_k), denoted cop_{ik}, is defined to measure the consensus degree amongst all the experts on that pair of alternatives:

$$cop_{ik} = sm_{ik}$$

L. 2. *Consensus degree on alternatives*. The consensus degree on alternative x_i, denoted ca_i, is defined to measure the consensus degree amongst all the experts on that alternative:

$$ca_i = \frac{\sum_{k=1;k\neq i}^{n} (cop_{ik} + cop_{ki})}{2(n-1)}$$

L. 3. *Consensus degree on the relation*. The consensus degree on the relation, denoted CR, is defined to measure the global consensus degree amongst all the experts' opinions:

$$CR = \frac{\sum_{i=1}^{n} ca_i}{n}$$

3.3 Third Step: Consensus and Trust Checks

In the end of each consensus round we must check the current consensus state. If it is considered a high enough consensus value the consensus process would finish and a selection process would be applied to obtain the final solution for the decision problem. To do so, we check if $CR > \gamma$, being γ a threshold value fixed prior to the beginning of the GDM process. In the case that the level of consensus is not high enough we would continue with the trust check that is described in the following. Note that in real applications it might be desirable to include a $maximumRounds$ parameter to control the maximum consensus rounds that can be executed in order to avoid stagnation.

The trust check is introduced to avoid some of the problems that can be derived to one of the characteristics of Web Communities: the difficulty of stablishing real trust relations. It is not difficult to imagine an scenario where some experts delegate into another that shares a common point of view on the decision that has to be made and in a certain consensus round, this expert decides to drastically change his preferences, probably not reflecting the other experts opinions anymore. To avoid this kind of situations the trust check will compare the last preference relation expressed by expert $\tilde{e}^h$ with

the last preference relations of the experts that delegated in him (direct or indirectly). This comparison can be made by applying a distance operator (as the euclidean or cosine distances) over the preference relations. If this distance is greater than a certain stablished threshold, the expert that delegated in $\tilde{e}^h$ would be informed with a special message to warn him about this problematic situation and thus allowing him to take a different course of action in the next consensus round if apropriate.

At this point a new consensus round begins. In this new round the current global preference will not be computed as a simple arithmetic mean but as a weighted mean of the preferences expresed by the experts in $\tilde{E}$. The weights to be used in this aggregation operation are the trust weights τ^h:

$$p_{ij}^c = \frac{\sum_{\tilde{e}^h \in \tilde{E}} \tau^h \cdot p_{ij}^h}{T}$$

We would like to note that in each new consensus round all the members of the Web Community can participate, independently of what they did in the previous rounds. For example, an expert that delegated in a previous consensus round may decide not to continue delegating (maybe because the trust check mechanism has warned him that the expert in which he delegated has drastically changed his preferences) and thus to provide again a new fuzzy preference relation or to delegate in a different individual; an expert which had not delegated in any of the previous rounds might decide to delegate in the current consensus round or even an expert which has not participated until this moment in the consensus process (he did not provide any preference relation in the first step of the model) could join the process by providing his initial preferences.

4 Conclusions

In this contribution we have presented a novel consensus model which has been specially designed to be applied in Web 2.0 Communities. Particularly, it was designed to be able to manage a large users base by means of a delegation scheme based in a particular kind of trust network that simplifies the computations and the time needed to obtain the users preferences. Moreover, this delegation scheme also solves the intermittent contrbutions problem which is present in almost any online community (that is, many of the users will not continuosly collaborate but will do it from time to time).

In addition, the model allows to incoporate new experts to the consensus process, that is, the model is able to handle some of the dynamic properties that real Web Communities have.

It is worth to note that the model can be applied to this new virtual environments due to the fast communication that web technologies offer to their users.

Finally, the model incorporates a trust check mechanism that allow to detect some abnormal situations in which an expert may try to take advantage of others by drastically changing his opinion and benefiting from the trust that the other experts might have deposited in him in previous consensus rounds.

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Refinement Properties in Agglomerative Hierarchical Clustering

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Abstract. Refinement properties that means each cluster of a method is included in another cluster of another method in agglomerative clustering which was proposed by Miyamoto are further studied. Although we have simple conditions so that a method generates refinements of clusters of the single linkage method, whether or not generalizations hold when the single linkage is not used is unknown. Here three conditions for refinement properties for the single linkage are shown, while three counterexamples are shown for the average linkage and the complete linkage, which show the theory of refinements is far from trivial and future works are needed.

Keywords: Agglomerative clustering; refinement; single linkage; complete linkage, average linkage.

1 Introduction

Agglomerative hierarchical clustering is well-known old method and effectively used in a variety of problems in sciences and engineering [1,2]. Recently researchers noticed there are rooms for theoretical studies in this old method. In this paper refinement properties proposed by the author [3,4] which is still unnoticed by many researchers is discussed and new results are shown. Refinement which will be strictly defined below implies that each cluster obtained by a linkage method is included in another cluster by another linkage method. Note that the two linkage methods are those used in agglomerative clustering and the two clusters are obtained at the same threshold level in the two dendrograms.

Although it has been shown that if a simple relation between a method and the single linkage holds, then the former method generates refinements of clusters of the single linkage method [4]. However, whether or not an analogous property of refinement holds without using the single linkage is unknown.

Here three theoretical properties for refinement using the single linkage are proved, and moreover three counterexamples are also shown that mean analogous properties do not hold for the average linkage and the complete linkage, which shows this theory is far from trivial.

We thus show that there are many theoretical problems and possibility of developments for further study in the method of agglomerative clustering.

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2 Agglomerative Hierarchical Clustering

Let the set of objects for clustering be

$$X = \{o_1, \ldots, o_n\}$$

Generally a cluster denoted by G_i is a subset of X. The family of clusters is denoted by

$$\mathcal{G} = \{G_1, G_2, \ldots, G_K\}$$

where the clusters form a crisp partition of X:

$$\bigcup_{i=1}^{K} G_i = X, \qquad G_i \cap G_j = \emptyset \quad (i \neq j). \tag{1}$$

Agglomerative clustering uses a dissimilarity measure, frequently called a distance, between two clusters $d(G, G')$ $(G, G' \in \mathcal{G})$ which also is called an inter-cluster dissimilarity. Sometimes a similarity measure $s(G, G')$ is used. The difference between a dissimilarity measure and a similarity measure is that a smaller $d(G, G')$ means G and G' are more similar, whereas a smaller $s(G, G')$ implies G and G' are less similar. We discuss $d(G, G')$ in this paper. In the following we sometimes write $d_{\mathsf{method}}(G, G')$ when a method of agglomerative clustering is assumed. More specifically, we use SL (single linkage), CL (complete linkage), or AL (average linkage) as a method. Moreover we write $d^{(i)}(G, G')$ $(i = 1, 2)$ when two different dissimilarities are assumed to the same pair of clusters.

Let us first describe a general procedure of agglomerative clustering [3,4].

AHC Algorithm (Agglomerative Hierarchical Clustering):

AHC1: Assume that initial clusters are given by
$\mathcal{G} = \{\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_N\}$ $(\hat{G}_j = \{o_j\} \subset X)$.
Set $K = N$ (K is the number of clusters) and
$G_i = \hat{G}_i$ $(i = 1, \ldots, K)$.
Calculate $d(G, G')$ for all pairs $G, G' \in \mathcal{G}$.

AHC2: Search the pair of minimum dissimilarity:

$$(G_p, G_q) = \arg \min_{G, G' \in \mathcal{G}} d(G, G'). \tag{2}$$

and let

$$m_K = d(G_p, G_q) = \min_{G, G' \in \mathcal{G}} d(G, G'). \tag{3}$$

Merge: $G_r = G_p \cup G_q$.
Add G_r to $\mathcal{G}$ and delete G_p, G_q from $\mathcal{G}$.
$K = K - 1$.
if $K = 1$ then stop and output the dendrogram.

AHC3: Update dissimilarity $d(G_r, G'')$ for all $G'' \in \mathcal{G}$.
Go to **AHC2**.

End AHC.

In **AHC**, how to construct a dendrogram is omitted (see e.g., [3,4]).

We focus on the single linkage, the complete linkage, and the average linkage as mentioned above. In these methods we do not care about how the initial dissimilarity measure is defined; we simply assume $d(o_i, o_j)$ is given in some way. Note that a class of inter-cluster dissimilarities is given as follows.

- the single linkage (SL): $d_{SL}(G, G') = \min_{o \in G, o' \in G'} d(o, o')$
- the complete linkage (CL): $d_{CL}(G, G') = \max_{o \in G, o' \in G'} d(o, o')$
- the average linkage (AL): $d_{AL}(G, G') = \frac{1}{|G||G'|} \sum_{o \in G, o' \in G'} d(o, o')$

The basic definition of $d(G_r, G'')$ in **AHC3** use one of the above definitions of the inter-cluster dissimilarity for SL, CL, or AL.

An important issue in agglomerative clustering is efficient updating of a dissimilarity measure. The single linkage, the complete linkage and the average linkage respectively use

$$d_{SL}(G_r, G'') = \min\{d_{SL}(G_p, G''), d_{SL}(G_q, G'')\} \tag{4}$$

$$d_{CL}(G_r, G'') = \max\{d_{CL}(G_p, G''), d_{CL}(G_q, G'')\} \tag{5}$$

$$d_{AL}(G_r, G'') = \frac{|G_p| d_{AL}(G_p, G'') + |G_q| d_{AL}(G_q, G'')}{|G_p| + |G_q|}. \tag{6}$$

3 Refinement Properties

A theory of refinement between a method of agglomerative clustering and the single linkage has been proposed by Miyamoto [3,4]. We first introduce a cut of hierarchical clusters for this purpose. Notice that the merging levels of clusters are $m_N, \ldots, m_2$. For SL, CL, and AL, we have the monotonicity

$$m_N \leq m_{N-1} \leq \cdots \leq m_2 \tag{7}$$

for any set of objects and any dissimilarity measure, as those methods have no reversals in the dendrograms [3,4].

For a method having the monotonicity condition (7), we can define an arbitrary cut

$$\mathcal{G}^\lambda = \{G_1^\lambda, \ldots, G_K^\lambda\}, \quad -\infty < \lambda < +\infty$$

from the family of hierarchical clusters:

(i) $\mathcal{G}^\lambda = \{\{o_1\}, \ldots, \{o_N\}\}, \quad \lambda < m_N.$
(ii) $\mathcal{G}^\lambda = \{G_1, \ldots, G_i\}, \quad m_{i+1} \leq \lambda < m_i, \quad \text{for } i = 2, \ldots, N-1.$
(iii) $\mathcal{G}^\lambda = \{X\}, \quad \lambda \geq m_2.$

In short, $\mathcal{G}^\lambda$ is the set of clusters when we cut the dendrogram at the level λ.

The single linkage can be taken as the standard among various methods of agglomerative clustering, as several researchers suggested [1,2,3,4]. A cluster at the level λ by the single linkage is denoted by

$$\mathcal{G}_{SL}^\lambda = \{G_{SL}^\lambda, \ldots, {G''}_{SL}^\lambda\}.$$

We next define a refinement between two methods of clustering.

Definition 1. *Let A and B be two methods of agglomerative clustering for which (7) hold. Clusters at the level λ by A and B are respectively denoted by*

$$\mathcal{G}_A^\lambda = \{G_A^\lambda, \ldots, {G''}_A^\lambda\}.$$

and

$$\mathcal{G}_B^\lambda = \{G_B^\lambda, \ldots, {G''}_B^\lambda\}.$$

We say clusters of B are refinements *of those of A if and only if*

$$\forall G_A^\lambda \in \mathcal{G}_A^\lambda, \quad \exists {G'}_B^\lambda \in \mathcal{G}_B^\lambda \text{ such that } {G'}_B^\lambda \subseteq G_A^\lambda \tag{8}$$

for all $-\infty < \lambda < +\infty$. If this condition (8) holds, we write $\mathcal{G}_B \triangleleft \mathcal{G}_A$, or more simply $B \triangleleft A$.

We now have the following.

Proposition 1. *([3,4]). Let $\mathcal{G}$ be an arbitrarily given partition and A be a method of agglomerative clustering, or in other words, a formula to calculate $d(G_r, G'')$ in* **AHC3**.

A sufficient condition for $A \triangleleft SL$ is

$$d_A(G \cup G', G'') \geq \min\{d_A(G, G''), d_A(G', G'')\} \tag{9}$$

for all $G, G', G'' \in \mathcal{G}$.

The proof is shown in [3,4] and is omitted here. It should however be noticed that the important point in Proposition 1 is that $\mathcal{G}$ need not be a set of clusters actually generated from a given data.

We moreover have the second proposition of which the proof is immediate from Proposition 1 and omitted.

Proposition 2. *Let $\mathcal{G}$ be an arbitrarily given partition and A be a method of agglomerative clustering, or in other words, a formula to calculate $d(G_r, G'')$ in* **AHC3**. *A sufficient condition for $A \triangleleft SL$ is*

$$d_A(G, G') \geq d_{SL}(G, G'') \tag{10}$$

for all $G, G' \in \mathcal{G}$.

We have the next proposition from Propositions 1 and 2.

Proposition 3.

$$AL \lhd SL, \tag{11}$$
$$CL \lhd SL. \tag{12}$$

The proof is easy by noting the updating formulas for CL and AL satisfy (9).

We proceed to see two more refinement properties that hold for SL.

Proposition 4. *Assume that two dissimilarity measures $d^{(1)}$ and $d^{(2)}$ are given for set X that satisfies*

$$d^{(1)}(o, o') \leq d^{(2)}(o, o'). \tag{13}$$

This property implies

$$d^{(1)}_{\mathsf{method}}(G, G') \leq d^{(2)}_{\mathsf{method}}(G, G'). \tag{14}$$

Let $\mathcal{G}^{(i)}_{\mathsf{method}}$ be a cluster using $d^{(i)}_{\mathsf{method}}$ $(i = 1, 2)$. We then have

$$\mathcal{G}^{(2)}_{SL} \lhd \mathcal{G}^{(1)}_{SL}.$$

Proof. It is immediate to see that (13) implies (14) and the proof for this part is omitted.

Let us next suppose that λ is the first level at which the refinement property is broken, i.e., we find $G^{(1)}$, $G'^{(1)}$, $\tilde{G}^{(2)}$, $\tilde{G}'^{(2)}$ such that $\tilde{G}^{(2)} \subseteq G^{(1)}$, $\tilde{G}'^{(2)} \subseteq G'^{(1)}$; $\tilde{G}^{(2)}$ and $\tilde{G}'^{(2)}$ are connected while $G^{(1)}$ and $G'^{(1)}$ not connected. Then, there are $\bar{o} \in \tilde{G}^{(2)}, \bar{o}' \in \tilde{G}'^{(2)}$ such that

$$d^{(2)}(\tilde{G}^{(2)}, \tilde{G}'^{(2)}) = \lambda = \min_{o \in \tilde{G}^{(2)}, o' \in \tilde{G}'^{(2)}} d^{(2)}(o, o') = d^{(2)}(\bar{o}, \bar{o}').$$

Since $d^{(1)}(\bar{o}, \bar{o}') \leq d^{(2)}(\bar{o}, \bar{o}')$ and $\bar{o} \in \tilde{G}^{(1)}, \bar{o}' \in \tilde{G}'^{(1)}$, this means that $G^{(1)}$ and $G'^{(1)}$ are already merged at λ, which contradicts the assumption. □

We now consider third problem to handle clustering of a subset of X. Let X' be a proper subset of X $(X' \subset X)$ and clusters formed for X' and X using a method are respectively denoted by $\mathcal{G}_{\mathsf{method}}[X']$ and $\mathcal{G}_{\mathsf{method}}[X]$. We have

Proposition 5.

$$\mathcal{G}_{SL}[X'] \lhd \mathcal{G}_{SL}[X].$$

Proof. Let us suppose that λ is the first level at which the refinement property is broken, i.e., we find $\bar{G}[X']$, $\bar{G}'[X']$, $G[X]$, $G'[X]$ such that $\bar{G}[X'] \subseteq G[X]$, $\bar{G}'[X'] \subseteq G'[X]$; $\bar{G}[X']$ and $\bar{G}'[X']$ are connected while $G[X]$ and $G'[X]$ not connected. Then, there are $\bar{o} \in \bar{G}[X'], \bar{o}' \in \bar{G}'[X']$ such that $d(o, o') = \lambda$.

Since $o \in G[X]$, $o' \in G'[X]$, this means that $G[X]$ and $G'[X]$ are already merged at λ, which contradicts the assumption. □

4 Counterexamples for Refinements

Propositions 3 – 5 show refinement properties for the single linkage, but Proposition 3 refers also to the complete linkage and average linkage.

We hence have a problem: *are similar results valid for the average linkage and the complete linkage?* More specifically, we consider the following questions.

1. Does $AL \triangleleft CL$ hold?
2. Does $\mathcal{G}^{(2)}_{\mathsf{method}} \triangleleft \mathcal{G}^{(1)}_{\mathsf{method}}$ hold when method refers to AL or CL?
3. Does $\mathcal{G}_{\mathsf{method}}[X'] \triangleleft \mathcal{G}_{\mathsf{method}}[X]$ hold when method refers to AL or CL?

In the following we answer these questions negatively, i.e., we have counterexamples.

Example 1. Seven objects $\{1, 2, \ldots, 7\}$ in the left side of Figure 1 are considered in which the objects are represented by the numbers in parentheses. The numbers without parentheses are measures of dissimilarity. The lines show the relation with the values of dissimilarity but there is no difference of contents between a solid line and a dotted line, in other words, the two types of lines are merely for the ease of illustration. The dotted ovals show that the values of dissimilarity (indicated by the arrows) are the same for the lines inside the ovals. The value of 0.99 is for the three relations inside each circles of solid curves. The values of dissimilarity are as follows. We assume $d(i, i) = 0$ and $d(i, j) = d(j, i)$ as usual.

$$
\begin{aligned}
d(i,j) &= 0.99, \quad i,j = 1,2,3;\\
d(k,\ell) &= 0.99, \quad k,\ell = 4,5,6;\\
d(i,k) &= 1.1, \quad i = 2,3,\ k = 4,5,6;\\
d(j,\ell) &= 1.1, \quad j = 1,2,3,\ \ell = 5,6;\\
d(1,4) &= 2.0;\\
d(i,7) &= 1.9, \quad i = 1,2,3;\\
d(k,7) &= 2.3, \quad k = 4,5,6.
\end{aligned}
$$

The two circles of the solid curves show two clusters named G_1 and G_2. Notice that they are first formed by both AL and CL applied to this example.

We show that $CL \triangleleft AL$ does not hold for this example. By applying **AHC**, we have dendrograms by AL and CL in the right side of Figure 1, where dendrogram by AL is shown by the solid lines and that by CL is by broken lines. For G_1 and G_2, the both methods generate the same clusters shown by the solid lines, and hence we have no confusion even when the solid and broken lines meet.

Now, let us take $1.9 \leq \lambda < 2.1$. We then have clusters $\{1, 2, 3, 4, 5, 6\}, \{7\}$ by AL, while CL has $\{1, 2, 3, 7\}, \{4, 5, 6\}$. Thus clusters by CL is not the refinement of those by AL.

Example 2. Assume that four objects a, b, c, d have two attributes A_1, A_2 and the objects are represented by pairs $a = (a_1, a_2)$, $b = (b_1.b_2)$, $c = (c_1, c_2)$,

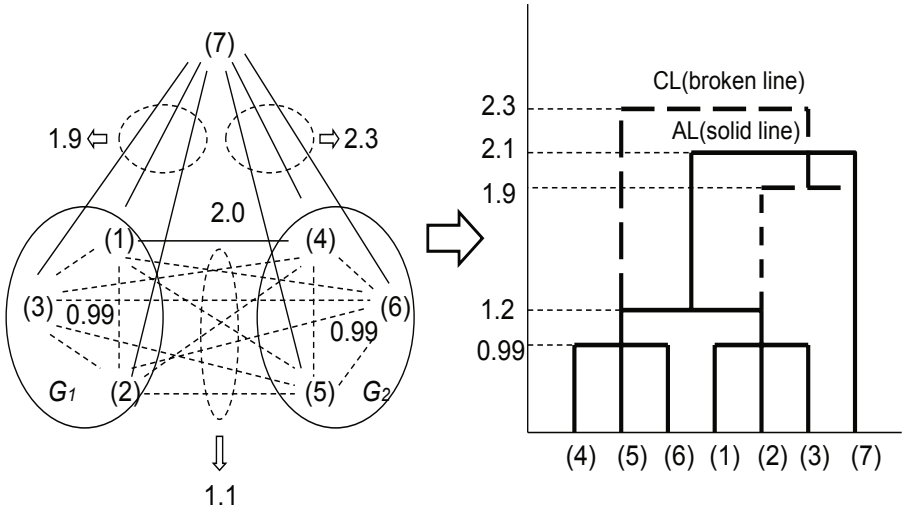

Fig. 1. A counterexample which shows $CL \triangleleft AL$ does not hold

and $d = (d_1, d_2)$. Dissimilarity measures d_{A_1} and d_{A_2} are given for each attribute:

$$\begin{array}{lll} d_{A_1}(a_1, b_1) = 1, & d_{A_1}(a_1, c_1) = 2 + \epsilon, & d_{A_1}(a_1, d_1) = 3 + \epsilon, \\ d_{A_1}(b_1, c_1) = 1 + \epsilon, & d_{A_1}(b_1, d_1) = 2 + \epsilon, & d_{A_1}(c_1, d_1) = 1, \\ d_{A_2}(a_2, b_2) = \delta, & d_{A_2}(a_2, c_2) = 2\delta, & d_{A_2}(a_2, d_2) = 2\delta + L, \\ d_{A_2}(b_2, c_2) = \delta, & d_{A_2}(b_2, d_2) = \delta + L, & d_{A_2}(c_2, d_2) = L\ . \end{array}$$

where ϵ and δ are small positive numbers, while L is a large positive number. We then assume $d^{(1)}(x, y) = d_{A_1}(x, y)$ and $d^{(2)}(x, y) = d_{A_1}(x, y) + d_{A_2}(x, y)$.

Let us observe the result from CL. Using $d^{(1)}$, CL first produces clusters $G = \{a, b\}$ and $G' = \{c, d\}$ at $\lambda = 1$. G and G' are merged at $\lambda = 3 + \epsilon$. When $d^{(2)}$ is used, CL first produces $G = \{a, b\}$ at $\lambda = 1 + \delta$. Noting L is large, we see that G and $\{c\}$ are merged at $\lambda = 2 + 2\delta + \epsilon$. If δ is sufficiently small, $\mathcal{G}^{(2)}_{CL} \triangleleft \mathcal{G}^{(1)}_{CL}$ does not hold.

Let us consider AL. Using $d^{(1)}$, CL first produces clusters $G = \{a, b\}$ and $G' = \{c, d\}$ at $\lambda = 1$. G and G' are then merged at $\lambda = \frac{1}{3}(5 + 2\epsilon)$. When $d^{(2)}$ is used, CL first produces $G = \{a, b\}$ at $\lambda = 1 + \delta$. Then G and $\{c\}$ are merged at $\lambda = \frac{1}{3}(4 + 4\delta + 2\epsilon)$. If δ is sufficiently small, $\mathcal{G}^{(2)}_{AL} \triangleleft \mathcal{G}^{(1)}_{AL}$ does not hold.

Example 3. We have a very simple counterexample. Consider three points $X = \{a, b, c\}$ on the line in the upper part of Figure 2, where we assume the dissimilarity is given by the segment length between two points on the line. Suppose a is removed from X: we have $X' = \{b, c\}$. In the lower part we show four dendrograms: In the left side two dendrograms by CL are superimposed. Solid lines show a dendrogram for X, while broken lines are for X'. In the right side we show two dendrograms by AL, one for X using solid lines and another for X' using broken lines.

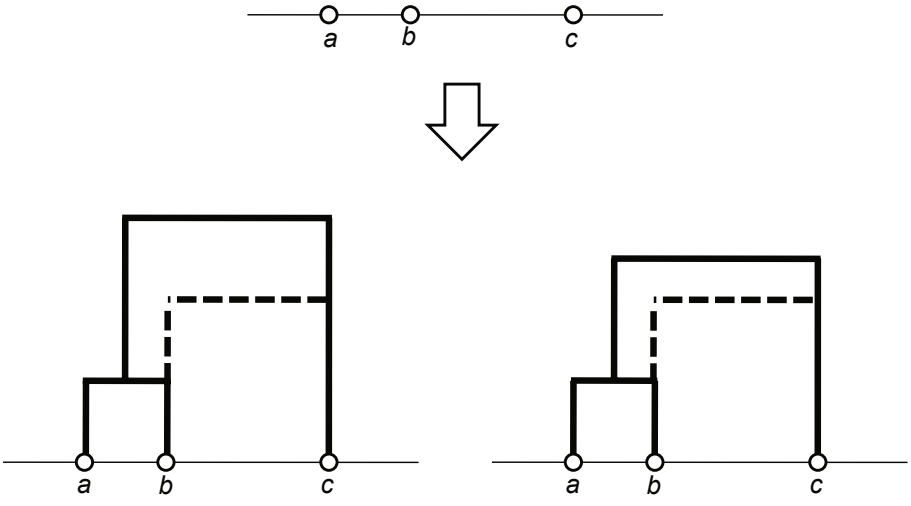

Fig. 2. Three points on a line are shown in the upper part and dendrograms by CL on the left side and those by AL on the right side are shown in the lower part. Solid lines show dendrograms for $X = \{a, b, c\}$, while broken lines show those for $X' = \{b, c\}$.

These two dendrograms imply neither $\mathcal{G}_{CL}[X'] \triangleleft \mathcal{G}_{CL}[X]$ nor $\mathcal{G}_{AL}[X'] \triangleleft \mathcal{G}_{AL}[X]$ holds.

5 Conclusion

We have discussed the concept of refinement originally proposed by the author [3] and have proved that three refinement properties hold for the single linkage, while the counterexamples have shown that they do not hold between the complete linkage and average linkage.

Note that Propositions 4 and 5 are related to the selection of columns and rows in clustering of information systems [6]. The present methods can also be extended to the selection of columns and rows in a generalized agglomerative clustering [5]. These problems thus should be considered as a future study.

It is well-known that the single linkage, complete linkage, and average linkage can be applied to an arbitrary dissimilarity measures, while another well-known methods of the centroid method and the Ward method can only be applied to the squared Euclidean distance. Another theoretical problem is whether or not analogous properties of refinement holds for the centroid method and the Ward method, or a counterexample as above exists.

Although many researchers still consider that studies of algorithms in agglomerative clustering have finished many years ago, there are rooms for further theoretical studies, as this paper shows a glimpse of a future theory of agglomerative hierarchical clustering, and such a theoretical consideration will give a deeper insight to agglomerative algorithms. New theoretical results by researchers are thus expected.

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Some Pairwise Constrained Semi-Supervised Fuzzy *c*-Means Clustering Algorithms

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Abstract. In this paper, some semi-supervised clustering methods are proposed with two types of pair constraints: two data have to be together in the same cluster, and two data have to be in different clusters, which are classified into two types: one is based on the standard fuzzy *c*-means algorithm and the other is on the entropy regularized one. First, the standard fuzzy *c*-means and the entropy regularized one are introduced. Second, a pairwise constrained semi-supervised fuzzy *c* means are introduced, which is derived from pairwise constrained competitive agglomeration. Third, some new optimization problem are proposed, which are derived from adding new loss function of memberships to the original optimization problem, respectively. Last, an iterative algorithm is proposed by solving the optimization problem.

Keywords: Pairwise Constraints, Semi-Supervised Clustering, Fuzzy *c*-Means.

1 Introduction

Fuzzy *c*-means (FCM) [1] is one of the well-known fuzzy clusterings and many FCM variants have been proposed after FCM. In these variants, FCM algorithm based on the concept of regularization by entropy has been proposed by one of the authors [2]. This algorithm is called entropy regularized FCM (eFCM) and is discussed not only for its usefulness but also for its mathematical relations with other techniques.

In real case of clustering, there is not a supply of only unlabeled data but also labeled data. In semi-supervised clustering, some labeled data are used along with the unlabeled data to obtain a better clustering, such as quickly convergence speed and higher accuracy. Semi-supervised fuzzy *c*-means algorithm by Pedrycz [3] is the representative of semi-supervised fuzzy clustering methods and many similar methods have also proposed [4], [5], [6].

On the other hand, COP-K-Means [7] is a semi-supervised variant of K-Means, where there are two types of constraints, *must-link* (two data have to be together in the same cluster) and *cannot-link* (two data have to be in different clusters), which are used in the clustering process to generate a partition that satisfies all

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the given constraints. By making the use of *must-link* and *cannot-link*, Competitive Agglomeration (CA) [8], which is one of fuzzy clustering algorithms based on FCM, has been extended into Pairwise Constrained Competitive Agglomeration (PCCA) [9]. But PCCA consider the case of the fuzzifier parameter $m = 2$. Here we consider the fuzzy clustering method where the feature progressively reducing the number of clusters is eliminated from PCCA, and we call this method Pairwise Constrained standard Fuzzy c-Means (PCsFCM).

In this paper, we propose new some semi-supervised fuzzy c-means algorithms with *must-link* and *cannot-link*. Some of them are based on the standard FCM (sFCM) and are no longer limited to fuzzifier parameter $m = 2$, by introducing new optimization problems. One of them is based on eFCM using the similar optimization problem to PCsFCM.

The contents of this paper are the followings. In the second section, we define some notations and introduce sFCM, eFCM and PCsFCM, which is derived from PCCA proposed by Grira, Crucianu and Boujemaa [9]. In the third section, we discuss the loss function describing *must-link* and *cannot-link* and propose new some types of PCsFCM. In the last section, we conclude this paper.

2 Preliminaries

In this section, we define some notations and introduce two types of fuzzy c-means(FCM), which are the standard type (sFCM) and the entropy regularized type (eFCM), and also introduce a semi-supervised standard fuzzy c-means, which is derived from the pairwise constrained competitive agglomeration proposed by Grira, Crucianu and Boujemaa [9].

In the first subsection, we define some notations which are the data for clustering, the membership by which the each data belongs to the each cluster, and the cluster centers. In the second subsection, we introduce sFCM and eFCM. In the third subsection, we introduce a semi-supervised sFCM.

2.1 Notations

In this subsection, we define some notations which are the data for clustering, the membership by which the each data belongs to the each cluster, and the cluster centers.

The data set $x = \{x_i \mid x_i \in \mathbb{R}^p, i \in \{1, \ldots, N\}\}$ is given. The membership by which x_i belongs to the j-th cluster is denoted by $u_{i,j}$ $(i \in \{1, \cdots, N\}, j \in \{1, \cdots, C\})$ and the set of $u_{i,j}$ is denoted by $u \in \mathbb{R}^{N \times C}$ called the partition matrix. The constraint for u is

$$\sum_{j=1}^{C} u_{i,j} = 1 \quad (0 \leq u_{i,j} \leq 1). \tag{1}$$

The cluster center set is denoted by $v = \{v_j \mid v_j \in \mathbb{R}^p, j \in \{1, \ldots, C\}\}$.

2.2 FCM

In this subsection, we introduce two types of FCM, which are the standard FCM (sFCM) [1] and the entropy regularized FCM (eFCM) [2].

sFCM is the algorithm obtained by solving the following optimization problem:

$$\underset{u,v}{\text{minimize}}\, J_{\text{sFCM}}(u,v) \quad \text{subject to} \sum_{j=1}^{C} u_{i,j} = 1, \tag{2}$$

where

$$J_{\text{sFCM}}(u,v) = \sum_{i=1}^{N}\sum_{j=1}^{C} u_{i,j}^{m}\|x_i - v_j\|^2. \tag{3}$$

The parameter m is the fuzzifier satisfying $m > 1$. In this paper, $\|\cdot\|^2$ stands for square of Euclidean norm:

$$\|x_i - v_j\|^2 = \sum_{k=1}^{p}(x_i - v_j)^2. \tag{4}$$

The optimal solutions u and v are obtained by the following algorithm.

Algorithm 1 (sFCM)

Step 1. *Give the number of cluster C and the value m. Set the initial cluster centers v.*

Step 2. *Calculate u such that*

$$u_{i,j} = \left(\sum_{k=1}^{C}\left(\frac{d_{i,j}}{d_{i,k}}\right)^{\frac{1}{m-1}}\right)^{-1}, \quad \textit{where } d_{i,j} = \|x_i - v_j\|^2. \tag{5}$$

Step 3. *Calculate v such that*

$$v_j = \left(\sum_{i=1}^{N} u_{i,j}^{m} x_i\right) / \left(\sum_{i=1}^{N} u_{i,j}^{m}.\right) \tag{6}$$

Step 4. *Check the stopping criterion for (u,v). If the criterion is not satisfied, go back to* Step 2.

eFCM is the algorithm obtained by solving the following optimization problem:

$$\underset{u,v}{\text{minimize}}\, J_{\text{eFCM}}(u,v) \text{ subject to} \sum_{j=1}^{C} u_{i,j} = 1, \tag{7}$$

where

$$J_{\text{eFCM}}(u,v) = \sum_{i=1}^{N}\sum_{j=1}^{C} u_{i,j}\|x_i - v_j\|^2 + \lambda^{-1}\sum_{i=1}^{N}\sum_{j=1}^{C} u_{i,j}\log(u_{i,j}). \tag{8}$$

The second term of the right-hand side in Eq. (8) is for regularization by entropy. The parameter λ is the fuzzifier satisfying $\lambda > 0$. The optimal solutions u and v are obtained by the following algorithm.

Algorithm 2 (eFCM)

Step 1. *Give the number of cluster C and the value λ. Set the initial cluster centers v.*

Step 2. *Calculate u such that*

$$u_{i,j} = \exp(-\lambda d_{i,j}) / \sum_{k=1}^{C} \exp(-\lambda d_{i,k}), \text{ where } d_{i,j} = \|x_i - v_j\|^2. \tag{9}$$

Step 3. *Calculate v such that*

$$v_j = \left(\sum_{i=1}^{N} u_{i,j} x_i \right) / \left(\sum_{i=1}^{N} u_{i,j} \right). \tag{10}$$

Step 4. *Check the stopping criterion for (u, v). If the criterion is not satisfied, go back to* Step 2.

2.3 Pairwise Constrained sFCM with $m = 2$

In this subsection, we introduce a pairwise constrained standard fuzzy c-means (PCsFCM_0), which is derived from the pairwise constrained competitive agglomeration proposed by Grira, Crucianu and Boujemaa [9] by omitting the term for progressively reducing the number of clusters from the original objective function. 0 of PCsFCM_0 is just the number which means that this method is the basis of our proposed methods. We will call our proposed methods PCsFCM_1–PCsFCM_3 and PCeFCM.

Let $\mathcal{M}$ be the set of must-link pairs such that $(x_i, x_{\tilde{i}}) \in \mathcal{M}$ implies x_i and $x_{\tilde{i}}$ should be assigned to the same cluster, and $\mathcal{C}$ be the set of cannot-link pairs such that $(x_i, x_{\tilde{i}}) \in \mathcal{C}$ implies x_i and $x_{\tilde{i}}$ should be assigned to different clusters.

PCsFCM_0 is the algorithm obtained by solving the following optimization problem:

$$\underset{u,v}{\text{minimize}}\, J_{\mathrm{PCsFCM}_0}(u, v) \text{ subject to } \sum_{j=1}^{C} u_{i,j} = 1, \tag{11}$$

where

$$J_{\mathrm{PCsFCM}_2}(u, v) = \sum_{i=1}^{N} \sum_{j=1}^{C} u_{i,j}^2 d_{i,j} + \sum_{i=1}^{N} \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{j=1}^{C} \sum_{\substack{\tilde{j}=1 \\ \tilde{j} \neq j}}^{C} u_{i,j} u_{\tilde{i},\tilde{j}} + \sum_{i=1}^{N} \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \beta_{i,\tilde{i}} \sum_{j=1}^{C} u_{i,j} u_{\tilde{i},j} \tag{12}$$

The second term of Eq. (12) is composed of the cost of not respecting the pairwise *must-link* constraints. The third term of Eq. (12) is composed of the cost of not respecting the pairwise *cannot-link* constraints. The penalty corresponding to the presence of two such points in different clusters or in the same clusters is weighted by their membership values. These second and the third terms are weighted by $\alpha_{i,\tilde{i}}$ and $\beta_{i,\tilde{i}}$, respectively, which are a way to specify the relative importance of the supervision. If $(x_i, x_{\tilde{i}}) \notin \mathcal{M}$, then $\alpha_{i,\tilde{i}} = 0$, and if $(x_i, x_{\tilde{i}}) \notin \mathcal{C}$, then $\beta_{i,\tilde{i}} = 0$. The optimal solutions u and v are obtained by the following algorithm.

Algorithm 3

Step 1. *Give the number of cluster C and the parameters $\alpha_{i,\tilde{i}}$ and $\beta_{i,\tilde{i}}$. Set the initial cluster centers v.*

Step 2. *Calculate u such that*

$$u_{i,j} = \frac{1 + \sum_{k=1}^{C} \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},k}}{2d_{i,k}}}{\sum_{k=1}^{C} \frac{d_{i,j}}{d_{i,k}}} - \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j}}{2d_{i,j}} \tag{13}$$

where

$$d_{i,j} = \|x_i - v_j\|^2. \tag{14}$$

Step 3. *Calculate v as Eq.(6).*

Step 4. *Check the stopping criterion for (u, v). If the criterion is not satisfied, go back to* Step 2.

3 Some Pairwise Constrained Semi-Supervised Fuzzy *c*-Means Clustering Algorithms

In this section, we propose some pairwise constrained semi-supervised fuzzy c-means clustering algorithms, which are classified into two types: one is based on the standard fuzzy c means (sFCM) and the other is based on the entropy regularized one (eFCM). First, we propose some loss functions of not respecting the pairwise *must-link* constraints and the cost of not respecting the pairwise *cannot-link* constraints, where one is generalized from PCsFCM$_0$. Second, we propose some pairwise constrained semi-supervised fuzzy c-means algorithms by solving the optimization problem by adding each loss function to the original optimization problem of sFCM. Third, we do based on eFCM.

3.1 Loss Functions for Pairwise Constraints

In this subsection, we propose two loss functions of not respecting the pairwise *must-link* constraints and the cost of not respecting the pairwise *cannot-link* constraints.

Grira, Crucianu and Boujemaa [9] proposed the following loss function

$$\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{j=1}^{C}\sum_{\tilde{j}=j+1}^{C}u_{i,j}u_{\tilde{i},\tilde{j}}+\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}\sum_{j=1}^{C}u_{i,j}u_{\tilde{i},j}. \tag{15}$$

Loss functions for *must-link* must be less if $u_{i,j}$ is close to $u_{\tilde{i},j}$ and should be convex, while loss functions for *cannot-link* must be less if $u_{i,j}$ is far from $u_{\tilde{i},j}$ and should be convex. Here, we propose the following two loss functions

$$P_{1,\nu}(u)=\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{j=1}^{C}\sum_{\tilde{j}=j+1}^{C}u_{i,j}^{\nu}u_{\tilde{i},\tilde{j}}^{\nu}+\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}\sum_{j=1}^{C}u_{i,j}^{\nu}u_{\tilde{i},j}^{\nu} \tag{16}$$

$$P_{2,\nu}(u)=\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{j=1}^{C}|u_{i,j}-u_{\tilde{i},j}|^{\nu}+\sum_{i=1}^{N}\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}\sum_{j=1}^{C}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq j}}^{C}|u_{i,j}-u_{\tilde{i},\tilde{j}}|^{\nu} \tag{17}$$

$P_{1,1}(u)$ coincides with one proposed by Grira, Crucianu and Boujemaa [9]. All parts for *must-link* in $P_{1,\nu}$ and $P_{2,\nu}$ are less if $u_{i,j}$ is close to $u_{\tilde{i},j}$ and are convex, while all parts for *cannot-link* in $P_{1,\nu}$ and $P_{2,\nu}$ are less if $u_{i,j}$ is far from $u_{\tilde{i},j}$ and are convex. While $P_{1,\nu}$ is derived from the idea of correlation between memberships of other data, $P_{2,\nu}$ is from distance between memberships of other data. Correlation is one of similarity and distance is one of dissimilarity, so the forms of summations for j, $\tilde{j}$ in $P_{1,\nu}$ and $P_{2,\nu}$ are used in the opposite way from each other.

3.2 Some Pair Constrained Semi-Supervised sFCM

In this subsection, we propose three pairwise constrained semi-supervised standard fuzzy c-means clustering algorithms, we call PCsFCM_1–PCsFCM_3, based on the standard fuzzy c means (sFCM). The first one is obtained by solving the optimization problem by the loss function $P_{1,1}$ to the original optimization problem of sFCM, J_{sFCM}, i.e. Eq.(3). The second one is by solving the optimization problem whose objective function derived from adding the loss function $P_{1,m}$ to J_{sFCM}. The last one is for the objective function by adding $P_{2,2}$ to J_{sFCM} with the fuzzifier parameter $m=2$.

Case of Objective Function $J_{\mathrm{sFCM}}+P_{1,1}$. We consider the following optimization problem

$$\underset{u,v}{\mathrm{minimize}}(u,v)J_{\mathrm{PCsFCM}_1}\ \text{subject to}\ \sum_{j=1}^{C}u_{i,j}=1, \tag{18}$$

where $J_{\mathrm{PCsFCM}_1} = J_{\mathrm{sFCM}} + P_{1,1}$. The Lagrange function L_{PCsFCM_1} is described as $L_{\mathrm{PCsFCM}_1} = L_{\mathrm{sFCM}} + P_{1,1}$. Karush-Kuhn-Tucker conditions are described as

$$\frac{\partial L_{\mathrm{PCsFCM}_1}}{\partial u_{i,j}} = 0, \tag{19}$$

$$\frac{\partial L_{\mathrm{PCsFCM}_1}}{\partial \gamma_i} = 0, \tag{20}$$

$$\frac{\partial L_{\mathrm{PCsFCM}_1}}{\partial v_j} = 0. \tag{21}$$

KKT condition (21) implies that the optimal solution of v_j is the same as Eq. (6). KKT condition (19) implies that

$$m u_{i,j}^{m-1} d_{i,j} + \gamma_i + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j} \neq j}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j} = 0, \tag{22}$$

from which we have

$$u_{i,j} = \left(- \frac{\gamma_i + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j} \neq j}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j}}{m d_{i,j}} \right)^{1/(m-1)}. \tag{23}$$

Considering this equation with KKT condition (20), γ_i must satisfy that

$$\sum_{k=1}^{C} \left(- \frac{\gamma_i + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j} \neq k}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i} \neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},k}}{m d_{i,k}} \right)^{1/(m-1)} = 1 \tag{24}$$

Because it is difficult to solve this equation analytically, we must obtain $u_{i,j}$ using some numerical method. But we can use relatively simple numerical method, e.g. bisection method, since the equation is monotone for γ_i.

Case of Objective Function $J_{\mathrm{sFCM}} + P_{1,m}$. We consider the following optimization problem

$$\underset{u,v}{\mathrm{minimize}} (u, v) J_{\mathrm{PCsFCM}_2} \text{ subject to } \sum_{j=1}^{C} u_{i,j} = 1, \tag{25}$$

where $J_{\mathrm{PCsFCM}_2} = J_{\mathrm{sFCM}} + P_{1,m}$. The Lagrange function L_{PCsFCM_2} is described as $L_{\mathrm{PCsFCM}_2} = L_{\mathrm{sFCM}} + P_{1,m}$. Karush-Kuhn-Tucker conditions are described as

$$\frac{\partial L_{\mathrm{PCsFCM}_2}}{\partial u_{i,j}} = 0, \tag{26}$$

$$\frac{\partial L_{\mathrm{PCsFCM}_2}}{\partial \gamma_i} = 0, \tag{27}$$

$$\frac{\partial L_{\mathrm{PCsFCM}_2}}{\partial v_j} = 0. \tag{28}$$

KKT condition (28) implies that the optimal solution of v_j is the same as Eq. (6). KKT condition (26) implies that

$$mu_{i,j}^{m-1}d_{i,j} + \gamma_i + m\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq j}}^{C}u_{i,j}^{m-1}u_{\tilde{i},\tilde{j}}^{m} + m\sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{i,j}^{m-1}u_{\tilde{i},j}^{m} = 0, \tag{29}$$

from which we have

$$u_{i,j} = \left(-\frac{\gamma_i}{m(d_{i,j} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq j}}^{C}u_{\tilde{i},\tilde{j}}^{m} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{\tilde{i},j}^{m})}\right)^{1/(m-1)}. \tag{30}$$

Considering this equation with KKT condition (27), γ_i must satisfy that

$$\left(-\frac{\gamma_i}{m}\right)^{1/(m-1)}\sum_{k=1}^{C}\left(\frac{1}{d_{i,k} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq k}}^{C}u_{\tilde{i},\tilde{j}}^{m} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{\tilde{i},k}^{m}}\right)^{1/(m-1)} = 1. \tag{31}$$

This equation is re-described as

$$\left(-\frac{\gamma_i}{m}\right)^{1/(m-1)} = 1/\sum_{k=1}^{C}\left(\frac{1}{d_{i,k} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq k}}^{C}u_{\tilde{i},\tilde{j}}^{m} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{\tilde{i},k}^{m}}\right)^{1/(m-1)}, \tag{32}$$

hence we have the optimal solution of $u_{i,j}$ as

$$u_{i,j} = 1/\sum_{k=1}^{C}\frac{d_{i,j} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq j}}^{C}u_{\tilde{i},\tilde{j}}^{m} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{\tilde{i},j}^{m}}{d_{i,k} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\alpha_{i,\tilde{i}}\sum_{\substack{\tilde{j}=1\\ \tilde{j}\neq k}}^{C}u_{\tilde{i},\tilde{j}}^{m} + \sum_{\substack{\tilde{i}=1\\ \tilde{i}\neq i}}^{N}\beta_{i,\tilde{i}}u_{\tilde{i},k}^{m}}. \tag{33}$$

Remark that this equation is completely solved by $u_{i,j}$ because the subscripts of both sides are different with each other.

Case of Objective Function $J_{\mathbf{sFCM:}m=2} + P_{2,2}$. We consider the following optimization problem

$$\underset{u,v}{\text{minimize}}(u,v)J_{\text{PCsFCM}_3} \text{ subject to } \sum_{j=1}^{C} u_{i,j} = 1, \tag{34}$$

where $J_{\text{PCsFCM}_3} = J_{\text{sFCM:}m=2} + P_{2,2}$. The Lagrange function L_{PCsFCM_3} is described as $L_{\text{PCsFCM}_3} = L_{\text{sFCM:}m=2} + P_{2,2}$. Karush-Kuhn-Tucker conditions are described as

$$\frac{\partial L_{\text{PCsFCM}_3}}{\partial u_{i,j}} = 0, \tag{35}$$

$$\frac{\partial L_{\text{PCsFCM}_3}}{\partial \gamma_i} = 0, \tag{36}$$

$$\frac{\partial L_{\text{PCsFCM}_3}}{\partial v_j} = 0. \tag{37}$$

KKT condition (37) implies that the optimal solution of v_j is the same as Eq. (6). KKT condition (35) implies that

$$2u_{i,j}d_{i,j} + \gamma_i + 2\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}}(u_{i,j} - u_{\tilde{i},j}) + 2\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} (u_{i,j} - u_{\tilde{i},\tilde{j}}) = 0, \tag{38}$$

from which we have

$$u_{i,j} = \frac{-\gamma_i}{2d_{i,j} + 2\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + 2C\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}} + \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} u_{\tilde{i},j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},\tilde{j}}}{d_{i,j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}. \tag{39}$$

Considering this equation with KKT condition (36), γ_i must satisfy that

$$-\gamma_i \sum_{k=1}^{C} \frac{1}{2d_{i,k} + 2\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + 2C\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}} + \sum_{k=1}^{C} \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} u_{\tilde{i},k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},\tilde{j}}}{d_{i,k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}} = 1. \tag{40}$$

This equation is re-described as

$$-\gamma_i = \frac{1 - \sum_{k=1}^{C} \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} u_{\tilde{i},k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},\tilde{j}}}{d_{i,k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}}{\sum_{k=1}^{C} \frac{1}{2d_{i,k} + 2\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + 2C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}}, \tag{41}$$

hence we have the optimal solution of $u_{i,j}$ as

$$u_{i,j} = \frac{1 - \sum_{k=1}^{C} \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} u_{\tilde{i},k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},k}}{d_{i,k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}}{d_{i,j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}} + \frac{\sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} u_{\tilde{i},j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},j}}{d_{i,j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}}{\sum_{k=1}^{C} \frac{d_{i,j} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}{d_{i,k} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} + C \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}}}}. \tag{42}$$

Remark that this equation is completely solved by $u_{i,j}$ because the subscripts of both sides are different with each other.

Algorithms. From the above discussion, we propose the following iterative algorithm:

Algorithm 4 (PCsFCM$_1$, PCsFCM$_2$, PCsFCM$_3$)

Step 1. *Give the fuzzifier parameter m for* PCsFCM$_1$ *and* PCsFCM$_2$. *Give the number of cluster C, and the weight parameter $\alpha_{i,\tilde{i}}$, $\beta_{i,\tilde{i}}$. Set the initial cluster centers v and the initial membership u.*

Step 2. *Calculate u as Eq. (23) for* PCsFCM$_1$, *Eq. (33) for* PCsFCM$_2$ *and Eq. (42) for* PCsFCM$_3$.

Step 3 *Calculate v as Eq. (6).*

Step 4 *Check the stopping criterion for (u, v). If the criterion is not satisfied, go back to* Step 2.

3.3 Pair Constrained Semi-Supervised eFCM

In this subsection, we propose a pairwise constrained semi-supervised entropy regularized fuzzy c-means clustering algorithms, we call PCeFCM, based on the entropy regularized fuzzy c means (eFCM). This is obtained by solving the optimization problem by the loss function $P_{1,1}$ to the original optimization problem of eFCM, J_{eFCM}, i.e. Eq. (8). We consider the following optimization problem

$$\underset{u,v}{\text{minimize}}(u,v)J_{\mathrm{PCeFCM}} \text{ subject to } \sum_{j=1}^{C} u_{i,j} = 1, \tag{43}$$

where $J_{\mathrm{PCeFCM}} = J_{\mathrm{eFCM}} + P_{1,1}$. The Lagrange function L_{PCeFCM} is described as $L_{\mathrm{PCeFCM}} = L_{\mathrm{eFCM}} + P_{1,1}$. Karush-Kuhn-Tucker conditions are described as

$$\frac{\partial L_{\mathrm{PCeFCM}}}{\partial u_{i,j}} = 0, \tag{44}$$

$$\frac{\partial L_{\mathrm{PCeFCM}}}{\partial \gamma_i} = 0, \tag{45}$$

$$\frac{\partial L_{\mathrm{PCeFCM}}}{\partial v_j} = 0. \tag{46}$$

KKT condition (46) implies that the optimal solution of v_j is the same as Eq. (10). KKT condition (44) implies that

$$d_{i,j} + \lambda^{-1}(\log(u_{i,j}) + 1) + \gamma_i + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},\tilde{j}} + \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j} = 0, \tag{47}$$

from which we have

$$u_{i,j} = \exp\left(-\lambda\gamma_i - 1 - \lambda d_{i,j} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},\tilde{j}} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j} \right). \tag{48}$$

Considering this equation with KKT condition (36), γ_i must satisfy that

$$\exp(-\lambda\gamma_i - 1) = 1 / \sum_{k=1}^{C} \exp\left(-\lambda d_{i,k} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},\tilde{j}} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},k} \right), \tag{49}$$

from which we have the optimal solution of $u_{i,j}$ as

$$u_{i,j} = \frac{\exp\left(-\lambda d_{i,j} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq j}}^{C} u_{\tilde{i},\tilde{j}} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},j} \right)}{\sum_{k=1}^{C} \exp\left(-\lambda d_{i,k} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \alpha_{i,\tilde{i}} \sum_{\substack{\tilde{j}=1 \\ \tilde{j}\neq k}}^{C} u_{\tilde{i},\tilde{j}} - \lambda \sum_{\substack{\tilde{i}=1 \\ \tilde{i}\neq i}}^{N} \beta_{i,\tilde{i}} u_{\tilde{i},k} \right)}. \tag{50}$$

Remark that this equation is solved by $u_{i,j}$ completely because the subscripts of both sides are different with each other.

From the above discussion, we propose the following iterative algorithm:

Algorithm 5 (PCeFCM)

Step 1. *Give the number of cluster C, the fuzzifier parameter λ and the weight parameter $\alpha_{i,\tilde{i}}$, $\beta_{i,\tilde{i}}$. Set the initial cluster centers v and the initial membership u.*
Step 2. *Calculate u as Eq. (50).*
Step 3. *Calculate v as Eq. (10).*
Step 4. *Check the stopping criterion for (u, v). If the criterion is not satisfied, go back to* Step 2.

4 Constructing Dissimilarity Matrix, Kernel Gram Matrix and Kernel Function for Must-Link

In this section, we construct dissimilarity matrix $d^* \in \mathbb{R}^{n\times n}$ and kernel function $K^* : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ affected by *must-link* — two data have to be together in the same cluster — in order to apply to fuzzy relational clustering method and to kernel fuzzy clustering method, respectively.

The easiest idea of constructing dissimilarity matrix d^* affected by *must-link* that x_i and $x_{\tilde{i}}$ have to be together in the same cluster, is replacing the corresponding element $d_{i,\tilde{i}}$ by a certain small positive value. With such the dissimilarity matrix d^*, fuzzy relational clustering algorithm will produce a result.

If x_i and $x_{\tilde{i}}$ have to be together in the same cluster, other data close to x_i and $x_{\tilde{i}}$ (x_i and other data close to $x_{\tilde{i}}$) also should be together in the same cluster. Based on this idea, we construct dissimilarity matrix as below. First, make the graph whose i-th node is correspond to x_i and whose edge connecting i-th and $\tilde{i}$-th nodes has the value $d_{i,\tilde{i}}$. Second, replace the value of $(i, \tilde{i})$-edge by a certain small nonnegative value if $(i, \tilde{i})$ is an element of *must-link*. Third, replace the value of $(i, \tilde{i})$-edge by the minimum adding each edge in $(i, \tilde{i})$-path, which can be achieved by dynamic programming. Last, adopt the value of $(i, \tilde{i})$-edge as $d^*_{i,\tilde{i}}$. With such the dissimilarity matrix d^*, fuzzy relational clustering algorithm produce a result affected not only by *must-link* but also by other data close to the data in *must-link*. From the dissimilarity matrix d^*, we can construct kernel gram matrix and apply to kernel fuzzy clustering methods. Kernel gram matrix must be positive semi-definite. Another matrix d^{**} by diagonal shift or eigen-value shift of d^* can be kernel gram matrix.

In order to construct fuzzy classification function, we need not only kernel gram matrix but also kernel function affected by *must-link*. Such kernel function based on Gaussian kernel can be obtained as below. First, insert a simple Gaussian kernel $K(x, y) = \exp(-\sigma\|x - y\|^2)$ to a list of functions $\mathcal{L}$ and adopt the kernel function as the maximal of $\mathcal{L}$,

$$K(\bar{x}, \bar{y}) = \max_{\kappa(x,y)\in\mathcal{L}} \kappa(\bar{x}, \bar{y}), \tag{51}$$

which is corresponding to original dissimilarity matrix d. Second, insert two functions $\exp(-\sigma_{i,\tilde{i}}(\|x-x_i\|^2+\|x_{\tilde{i}}-y\|^2+d^*_{i,\tilde{i}}))$ and $\exp(-\sigma_{i,\tilde{i}}(\|x-x_{\tilde{i}}\|^2+\|x_i-y\|^2+d^*_{i,\tilde{i}}))$ to $\mathcal{L}$ if $(x_i, x_{\tilde{i}})$ is an element of *must-link*, and adopt the kernel function as the maximal of $\mathcal{L}$. This updated kernel function is corresponding to the second step in the previous paragraph and remark that $K(x_i, x_{\tilde{i}}) = d^*_{i,\tilde{i}}$. Third, with d^* obtained through the third step in the previous step, insert two functions $\exp(-\sigma_{i,\tilde{i}}(\|x-x_i\|^2+\|x_{\tilde{i}}-y\|^2+d^*_{i,\tilde{i}}))$ and $\exp(-\sigma_{i,\tilde{i}}(\|x-x_{\tilde{i}}\|^2+\|x_i-y\|^2+d^*_{i,\tilde{i}}))$ to $\mathcal{L}$ and adopt the kernel function as the maximal of $\mathcal{L}$. In order for such function to be kernel, the original function $\exp(-\sigma\|x-y\|^2)$ may be replaced by $a\exp(-\sigma\|x-y\|^2)$ with $a > 1$, where this replacement is corresponding to diagonal shift of $d^*_{i,\tilde{i}}$.

5 Conclusion

In this paper, we proposed some pair constrained semi-supervised fuzzy c-means algorithms. These algorithms are classified into two types: the first three algorithms (PCsFCM_1–PCsFCM_3) are based on the standard fuzzy c-means and the other (PCeFCM) is on the entropy regularized one. Any of these algorithms is obtained by solving the optimization problem where different loss function for *must-link* and *cannot-link* is added to the original one. The loss function for PCsFCM_1 is the same as the one proposed by [9] but the fuzzifier parameter is not longer limited to $m = 2$ but $m > 1$. The loss function for PCsFCM_2 is generalized from [9]. PCsFCM_1 and PCsFCM_2 can be executed with various values of fuzzifier parameter than [9]. The loss function for PCsFCM_3 is different from [9], by which another view of semi-supervised fuzzy clustering can be expected. The loss function for PCeFCM is the same as [9] but the original problem on which PCeFCM is based is not sFCM in [9] but eFCM.

As future works, we will compare the proposed methods with each other and with the already proposed method [9] theoretically and numerically.

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PCA-Guided k-Means with Variable Weighting and Its Application to Document Clustering

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Abstract. PCA-guided k-Means is a deterministic approach to k-Means clustering, in which cluster indicators are derived in a PCA-guided manner. This paper proposes a new approach to k-Means with variable selection by introducing variable weighting mechanism into PCA-guided k-Means. The relative responsibility of variables is estimated in a similar way with FCM clustering while the membership indicator is derived from a PCA-guided manner, in which the principal component scores are calculated by considering the responsibility weights of variables. So, the variables that have meaningful information for capturing cluster structures are emphasized in calculation of membership indicators. Numerical experiments including an application to document clustering demonstrate the characteristics of the proposed method.

1 Introduction

PCA-guided k-Means proposed by Ding and He [1] is a deterministic approach to k-Means clustering that is based on the close relation between Principal Component Analysis (PCA) and k-Means clustering. A relaxed cluster indicator vectors representing cluster memberships are calculated by a PCA-like manner, in which the indicator vectors are identified with the eigenvectors of a within-cluster (inner-product) similarity matrix, i.e., a continuous (relaxed) solution of the cluster membership indicators in k-Means is identified with principal components in PCA.

This paper proposes a new approach to k-Means with variable selection by introducing the variable weighting mechanism into PCA-guided k-Means. Huang *et al.* [2] proposed an automated variable weighting approach for k-Means clustering, in which the relative responsibility of variables is estimated in each iteration of k-Means process in a similar way with FCM clustering [3]. In this paper, a similar variable weighting mechanism is introduced into PCA-guided k-Means, in which variable weights are estimated in a similar way with FCM while the membership indicator is derived in a PCA-guided manner. So, the variables that have meaningful information for capturing cluster structures are emphasized and principal component scores are calculated by considering the responsibility weights of variables.

This paper includes two major contributions. First, a deterministic procedure for variable weighting in k-Means clustering is proposed based on a PCA-guided

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manner. Second, what the responsibility of variables in k-Means means is revealed from the comparative study with PCA for information summarization, i.e., it is shown that the weights for cluster structure clarification correspond to the degree of mutual dependencies among variables in information summarization.

The remaining parts of this paper are organized as follows: Section 2 briefly reviews the automated variable weighting technique in k-Means and the PCA-guided k-Means procedure. Section 3 proposes a new approach for variable weighting in PCA-guided k-Means process. In order to demonstrate the characteristics of the proposed method, numerical experiments including an application to document clustering are shown in Section 4. Section 5 presents the summary conclusion.

2 *k*-Means Type Clustering: Variable Weighting and PCA-Guided Process

Assume that we have n samples with m-dimensional observation $\boldsymbol{x}_i$, $i = 1, \ldots, n$, and the goal is to partition the samples into several clusters where samples belonging to same cluster are similar while samples belonging to different clusters are dissimilar.

k-Means [4] is a non-hierarchical prototype-based clustering method where prototypes (centroids) are used for representing clusters and the objective function is defined as the sum of within-cluster errors:

$$L_{km} = \sum_{k=1}^{K} \sum_{i \in G_k} ||\boldsymbol{x}_i - \boldsymbol{b}_k||^2, \tag{1}$$

where K is the pre-defined number of clusters and $\boldsymbol{b}_k$ is the representative prototype (centroid) of cluster G_k. The k-Means process is composed of two phases: prototype estimation and sample assignment, and the two phases are iterated until the solution is trapped in a local minimum.

2.1 Automated Variable Weighting in *k*-Means Clustering

For variable selection in the k-Means clustering, Huang *et al.* [2] introduced the memberships of variables that play a role for fuzzy partitioning of variables. The variable weights measure the importance of variables in clustering process. The modified objective function for k-Means with variable weighting (W-k-Means) is defined as:

$$L_{wkm} = \sum_{k=1}^{K} \sum_{i \in G_k} \sum_{j=1}^{m} w_j^{\beta} (x_{ij} - b_{kj})^2, \tag{2}$$

where w_j is the additional weight (membership) for variable j and β is the fuzzification parameter as is used in Fuzzy c-Means (FCM) clustering model [3].

Instead of the row-sum condition in FCM, in order to estimate the relative importance among variables, w_j is estimated under the following constraint:

$$\sum_{j=1}^{m} w_j = 1. \tag{3}$$

The clustering process including optimization of variable weights is composed of three phases: prototype estimation, sample assignment and weight estimation.

The updating formula for prototypes is still given as the within-cluster mean vector:

$$b_{kj} = \frac{1}{n_k} \sum_{i \in G_k} x_{ij}, \tag{4}$$

where n_k is the number of samples belonging to cluster G_k. On the other hand, variable weights are given as:

$$w_j = \left\{ \sum_{\ell=1}^{m} \left(\frac{d_j}{d_l} \right)^{\frac{1}{\beta-1}} \right\}^{-1}, \tag{5}$$

where

$$d_j = \sum_{k=1}^{K} \sum_{i \in G_k} (x_{ij} - b_{kj})^2. \tag{6}$$

2.2 PCA-Guided Process for *k*-Means Clustering

Ding and He [1] pointed out a close relation between PCA and k-Means clustering, and proposed an analytical (deterministic) way for k-Means clustering in a PCA-guided manner. The k-Means objective function of Eq.(1) can be re-defined by a centroid-less formulation as follows [5]:

$$L_{km} = \sum_{i=1}^{n} ||\boldsymbol{x}_i||^2 - \sum_{k=1}^{K} \frac{1}{n_k} \sum_{i,j \in G_k} \boldsymbol{x}_i^\top \boldsymbol{x}_j, \tag{7}$$

where n_k is the number of samples belonging to cluster G_k. $\top$ represents the transpose of a vector (or matrix). Here, the first term is a constant while the second term is the sum of within-cluster (inner product) similarities.

Ding and He showed that a relaxed solution for the membership indicator in the k-Means model is derived from the principal component scores in PCA. In their PCA-guided k-Means procedure, a rotated membership indicator matrix Q_K is estimated from the eigenvectors corresponding to the principal eigenvalues of the covariance matrix of the observed data matrix, i.e., a continuous solution for k-Means clustering is derived from a PCA-guided manner although the rotation matrix cannot be explicitly given.

In order to capture the cluster structure from the (continuous) solution, Ding and He applied a cluster connectivity analysis using a visual assessment approach based on distance sensitive ordering of samples (objects) [6] in the connectivity matrix that is derived from the (rotated) membership indicator matrix.

3 Variable Weighting in PCA-Guided k-Means Process

In this section, a new clustering model of PCA-guided k-Means with variable weighting is proposed by introducing variable weights into the PCA-guided objective function.

3.1 Introduction of Variable Weights into PCA-Guided Objective Function

In order to introduce variable weights into the PCA-guided objective function, the objective function of W-k-Means is re-defined in a centroid-less formulation. Considering Eq.(4), Eq.(2) is modified as:

$$\begin{aligned} L_{wkm} &= \sum_{k=1}^{K} \sum_{i \in G_k} \sum_{j=1}^{m} w_j^{\beta} \left(x_{ij} - \frac{1}{n_k} \sum_{\ell \in G_k} x_{\ell j} \right)^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} w_j^{\beta} x_{ij}^2 - \sum_{k=1}^{K} \frac{1}{n_k} \sum_{i,\ell \in G_k} \sum_{j=1}^{m} w_j^{\beta} x_{ij} x_{\ell j} \\ &= \sum_{i=1}^{n} \boldsymbol{x}_i^{\top} W^{\beta} \boldsymbol{x}_i - \sum_{k=1}^{K} \frac{1}{n_k} \sum_{i,\ell \in G_k} \boldsymbol{x}_i^{\top} W^{\beta} \boldsymbol{x}_{\ell}, \end{aligned} \tag{8}$$

where $W = \mathrm{diag}(w_1, \ldots, w_m)$.

With fixed weights w_j, the solution of (hard) k-Means clustering is represented by K non-negative indicator vectors $H_K = (\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K)$,

$$h_{ki} = \begin{cases} \frac{1}{n_k^{1/2}} & ; \text{if sample } i \text{ belongs to cluster } G_k. \\ 0 & ; \text{otherwise.} \end{cases}$$

where $H_K^{\top} H_K = I_K$ and I_K is the $K \times K$ unit matrix. Because of $\sum_{k=1}^{K} n_k^{1/2} h_{ki} = 1$, the indicator vectors have redundancies. In order to remove the redundancies and derive a unique solution, Ding and He [1] introduced a $K \times K$ orthogonal transformation $T = \{t_{ij}\}$,

$$Q_K = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_K) = H_K T \tag{9}$$

and set the last column of T as:

$$\boldsymbol{t}_K = \left(\sqrt{n_1/n}, \ldots, \sqrt{n_K/n} \right)^{\top}. \tag{10}$$

From the mutual orthogonality of $\boldsymbol{h}_k, k = 1, \ldots, K$ and $\boldsymbol{q}_K = (\sqrt{1/n}, \ldots, \sqrt{1/n})^{\top}$, we have the following relations:

$$Q_{K-1}^{\top} Q_{K-1} = I_{K-1}, \tag{11}$$

$$\textstyle\sum_{i=1}^{n} q_{ki} = 0, \quad k = 1, \ldots, K-1, \tag{12}$$

where $Q_{K-1} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{K-1})$ and $\boldsymbol{q}_k = (q_{k1}, \ldots, q_{kn})^\top$. Then, Eq.(8) can be written as

$$L_{wkm} = \sum_{i=1}^{n} \boldsymbol{x}_i^\top W^\beta \boldsymbol{x}_i - \frac{1}{n} \boldsymbol{e}^\top X^\top W^\beta X \boldsymbol{e} - \mathrm{Tr}(Q_{K-1}^\top X^\top W^\beta X Q_{K-1}), \quad (13)$$

where $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ and $\boldsymbol{e}$ is the n-dimensional vector whose elements are all 1.

Because the k-Means problem does not distinguish the original data $\boldsymbol{x}_i$ and the centered data $\boldsymbol{y}_i$, the above objective function can be replaced with

$$L_{wkm} = \sum_{i=1}^{n} \boldsymbol{y}_i W^\beta \boldsymbol{y}_i - \mathrm{Tr}(Q_{K-1}^\top Y^\top W^\beta Y Q_{K-1}), \quad (14)$$

where $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$ and $Y\boldsymbol{e} = \boldsymbol{0}$. The optimal solutions for Q_{K-1} are derived by maximizing $\mathrm{Tr}(Q_{K-1}^\top Y^\top W^\beta Y Q_{K-1})$, and continuous (relaxed) solutions are the eigenvectors corresponding to the $K-1$ largest eigenvalues of $Y^\top W^\beta Y$.

Next, the weights of variables are updated with the fixed membership indicator Q_K. In order to calculate the clustering criterion for variable weights w_j, Eq.(8) with normalized data is transformed as follows:

$$\begin{aligned} L_{wkm} &= \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^\beta y_{ij}^2 - \sum_{k=1}^{K} \frac{1}{n_k} \sum_{i,\ell \in G_k} \sum_{j=1}^{m} w_j^\beta y_{ij} y_{\ell j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^\beta y_{ij}^2 - \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{\ell=1}^{n} h_{ki} h_{k\ell} \sum_{j=1}^{m} w_j^\beta y_{ij} y_{\ell j}. \end{aligned} \quad (15)$$

From $H_K H_K^\top = Q_K T^\top T Q_K^\top = Q_K Q_K^\top$, we have $\sum_{k=1}^{K} h_{ki} h_{kj} = \sum_{k=1}^{K} q_{ki} q_{kj}$. Then, Eq.(15) is reformulated as:

$$\begin{aligned} L_{wkm} &= \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^\beta y_{ij}^2 - \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{\ell=1}^{n} q_{ki} q_{k\ell} \sum_{j=1}^{m} w_j^\beta y_{ij} y_{\ell j} \\ &= \sum_{j=1}^{m} w_j^\beta \left(\sum_{i=1}^{n} y_{ij}^2 - \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{\ell=1}^{n} q_{ki} q_{k\ell} y_{ij} y_{\ell j} \right). \end{aligned} \quad (16)$$

Then, variable weights are given as:

$$w_j = \left\{ \sum_{\ell=1}^{m} \left(\frac{d_j}{d_l} \right)^{\frac{1}{\beta-1}} \right\}^{-1}, \quad (17)$$

where

$$d_j = \sum_{i=1}^{n} y_{ij}^2 - \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{\ell=1}^{n} q_{ki} q_{k\ell} y_{ij} y_{\ell j}. \quad (18)$$

By the way, if we set all w_j to be 1, the proposed method is reduced to the conventional PCA-guided k-Means model.

3.2 Connection with Variable Selection in PCA

Although the goal of the proposed variable weighting process is to emphasize the variables that are most responsible for capturing cluster structures, the updating formula has a close connection with the variable selection model in PCA for information summarization. Honda *et al.* [7] proposed a variable selection model with the goal being to identify the variables that are most responsible for reduced-rank prediction, and extended the model to fuzzy clustering-based local PCA [8,9]. When we use the single cluster model, i.e., the conventional PCA, data summarization is performed as:

$$X \approx A^\top F + \boldsymbol{b}\boldsymbol{e}^\top, \tag{19}$$

where $F = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n)$ is the $(p \times n)$ score matrix whose i-th column $\boldsymbol{f}_i$ is the p-dimensional score vector for object i. $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^\top$ is the $(p \times m)$ principal component matrix whose ℓ-th row $\boldsymbol{a}_\ell$ is the m-dimensional principal component vector. $\boldsymbol{b}$ is the mean vector.

In [7], the goal of variable selection is to find variables that are useful for constructing association rules, i.e., revealing the mutual dependencies among variables, instead of reducing redundant information [10,11,12]. Then, when we use the standard fuzzification method used in FCM [3], the objective function for summarizing structural information in p-dimensional subspace is defined as:

$$\begin{aligned} L_{pcavs} &= \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^\beta \left(x_{ij} - \sum_{k=1}^{p} a_{kj} f_{ik} - b_j \right)^2 \\ &= \mathrm{tr}\left((X - A^\top F - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (X - A^\top F - \boldsymbol{b}\boldsymbol{e}^\top)\right) \\ &= \mathrm{tr}\left((X - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (X - \boldsymbol{b}\boldsymbol{e}^\top)\right) - 2\mathrm{tr}\left((X - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (A^\top F)\right) \\ &\quad + \mathrm{tr}\left(F^\top A W^\beta A^\top F\right). \end{aligned} \tag{20}$$

Although the solution was derived in an iterative optimization scheme in [7], the principal scores can be calculated in an analytical way with fixed variable weights w_j when we use the single cluster model. From the necessary condition for the optimality $\partial L / \partial F = O$, we have

$$A^\top F F^\top = (X - \boldsymbol{b}\boldsymbol{e}^\top) F^\top. \tag{21}$$

Under the constraint of $FF^\top = I$, Eq.(20) is reduced to

$$\begin{aligned} L_{pcavs} = &\ \mathrm{tr}\left((X - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (X - \boldsymbol{b}\boldsymbol{e}^\top)\right) \\ &- \mathrm{tr}\left(F(X - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (X - \boldsymbol{b}\boldsymbol{e}^\top) F^\top\right). \end{aligned} \tag{22}$$

So, F is calculated from the eigenvector corresponding to the principal eigenvalues of $(X - \boldsymbol{b}\boldsymbol{e}^\top)^\top W^\beta (X - \boldsymbol{b}\boldsymbol{e}^\top) = Y^\top W^\beta Y$. Note that the updating formula is same with that for the membership indicator in the proposed k-Means model although the special constraint of $\boldsymbol{q}_K = (\sqrt{1/n}, \ldots, \sqrt{1/n})^\top$ is used only in the k-Means model.

On the other hand, Eq.(22) implies that the objective function is also written as

$$L_{pcavs} = \sum_{i=1}^{n}\sum_{j=1}^{m} w_j^{\beta}(x_{ij}-b_j)^2 - \sum_{k=1}^{p}\sum_{\ell=1}^{n} w_j^{\beta} f_{ik} f_{\ell k}(x_{ij}-b_j)(x_{\ell j}-b_j)$$
$$= \sum_{j=1}^{m} w_j^{\beta}\left(\sum_{i=1}^{n} y_{ij}^2 - \sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{\ell=1}^{n} f_{ik} f_{\ell k} y_{ij} y_{\ell j}\right). \tag{23}$$

This equation indicates that the variable weights w_j in the proposed k-Means model are also given in a same formula with variable selection in PCA although the two methods are based on different purposes. The difference in the algorithmic aspect is the constraint on the last component ($\boldsymbol{q}_K$ in the k-Means model and $\tilde{\boldsymbol{f}}_p = (f_{1p}, \ldots, f_{np})$ in the PCA model).

By the way, when we use the centered data $\boldsymbol{y}_i$ instead of the original data $\boldsymbol{x}_i$, $\sum_{i=1}^{n} y_{ij} = 0$ and $\boldsymbol{q}_K = (\sqrt{1/n}, \ldots, \sqrt{1/n})^{\top}$ imply that the clustering criterion of Eq.(18) can be written as:

$$d_j = \sum_{i=1}^{n} y_{ij}^2 - \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{\ell=1}^{n} q_{ki} q_{k\ell} y_{ij} y_{\ell j}$$
$$= \sum_{i=1}^{n} y_{ij}^2 - \sum_{k=1}^{K-1}\sum_{i=1}^{n}\sum_{\ell=1}^{n} q_{ki} q_{k\ell} y_{ij} y_{\ell j}. \tag{24}$$

So, the variable weights w_j with K clusters in k-Means are equivalent to those in PCA with $p = K-1$ where the reduced membership indicator Q_{K-1} is equivalent to the principal component score matrix F, i.e., the additional elements of $\boldsymbol{q}_K$ is responsible only for reconstructing the membership indicator H_K after the PCA process.

4 Numerical Experiments

In this section, two experimental results are shown in order to demonstrate the characteristic features of the proposed method.

4.1 Artificial Data Set

In the first example, an artificially generated data set with 5-dimensional observation is used. The data set composed of 150 samples forms three compact masses (clusters) with 50 samples each in the 2-dimensional space spanned by the first two variables, i.e., $x_1 - x_2$ space, while the remaining three variables (x_3, x_4, x_5) are given by uniform random values drawn from the interval of $[0, 10]$. Figure 1 shows the 2-D plots in the $x_1 - x_2$ space and indicates that the centroids of the three masses are located in (0,0), (0,7) and (7,0), respectively. In this example, we can capture the cluster structure only when we consider the first 2 variables and the goal is to find the structure by emphasizing the 2 variables.

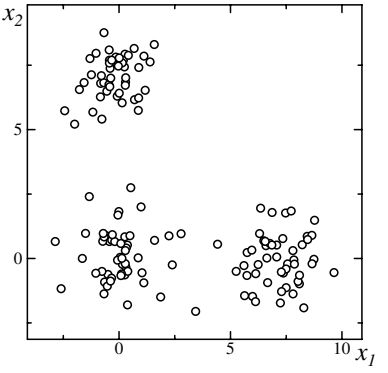
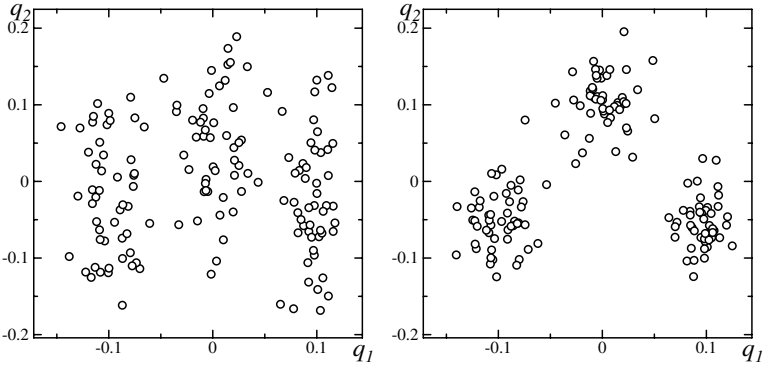


Fig. 1. $x1 - x2$ plots of artificial data set



(a) without variable weighting (b) with variable weighting

Fig. 2. Comparison of cluster indicator in artificial data set

Figure 2 compares the cluster indicators given by PCA-guided k-Means with/without variable weighting. In this experiment, cluster number k was set as $k = 3$ and the 150×3 cluster indicator matrix Q_3 was calculated by the two approaches: the conventional one without variable weighting and the proposed one with variable weighting. In the proposed method, fuzzifier β was set as $\beta = 2$. The 2-D plots show the first two scores (q_{i1}, q_{i2}) for sample i. Note that the third score q_{i3} is a constant value for all samples because of the constraint of Eq.(10). The figure implies that the conventional method without variable weighting (Fig. 2-(a)) failed to capture the cluster structure because it was shaded by three "noisy" variables (x_3, x_4, x_5), i.e., we cannot reconstruct the optimal cluster indicator by using any orthogonal transformation T. On the other hand, the proposed method could successfully capture the cluster structures and it is possible to reconstruct the optimal cluster indicator by estimating a certain orthogonal transformation T in the 3-D space. Table 1 shows the derived weights

Table 1. Variable weights for artificial data set

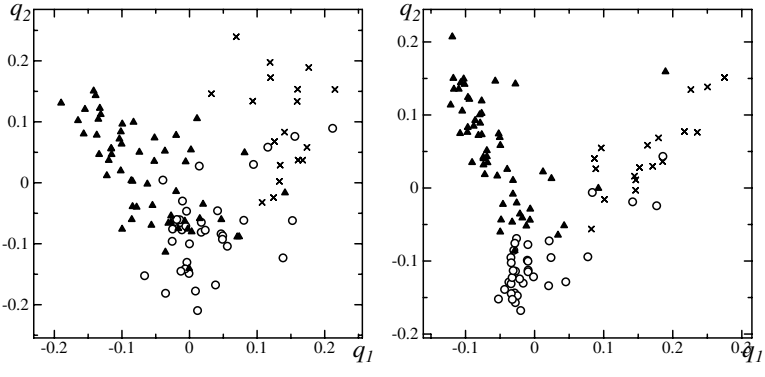
variable	x_1	x_2	x_3	x_4	x_5
weight	0.500	0.500	0.000	0.000	0.000

for the five variables and indicates that the proposed method could assign larger values for meaningful variables. In this way, the proposed method is useful for emphasizing the variables that have meaningful information for capturing cluster structures.

4.2 Application to Document Clustering

In the second example, a Japanese novel "Kokoro" written by Soseki Natsume, which can be downloaded from Aozora Bunko (http://www.aozora.gr.jp), is used. The novel is composed of 3 chapters (Sensei and I, My Parents and I, Sensei and His Testament) and the chapters include 36, 18, 56 sections, respectively. In this experiment, the sections were given as individual text documents (number of samples is $n = 110$) and the documents are partitioned into 3 clusters ($k = 3$) without the chapter information. So, the goal of the experiment is to reconstruct the chapter information from the word analysis avoiding the influences of "noisy" keywords. The text documents were preprocessed using a morphological analysis system software "Chasen" (http://chasen.naist.jp/hiki/ChaSen/), which segments Japanese text string into morphemes and tags those morphemes with their parts of speech and pronunciations, and the 83 most frequently used substantives and verbs (they were used more than 50 times in the novel) were given as attributes to be analyzed with their tf-idf weights.

The PCA-guided k-Means procedure was performed with/without variable weighting, in which $k = 3$. Figure 3 compares the cluster indicators. From the



(a) without variable weighting (b) with variable weighting

Fig. 3. Comparison of cluster indicator in "kokoro" (circle: chapter 1, times: chapter 2, triangle: chapter 3)

Table 2. Top 10 high responsibility keywords

keyword	haha	K	chichi	sensei	kare	mukau	ojohsan	shitsu	byouki	letter
weight	0.015	0.014	0.014	0.013	0.013	0.013	0.012	0.012	0.012	0.012

figure, we can see that the variable weighting mechanism played the role for clarifying the cluster boundaries although the conventional k-Means had severe overlap area in the chapter boundaries. So, the proposed variable weighting mechanism made it possible to reconstruct the clear cluster membership indicators by estimating a certain orthogonal transformation T. Table 2 shows the keywords having high membership degrees in the proposed method. The keywords are often used only in a certain chapter and are useful for characterize the chapters. The result implies that document clustering is a promising application area of the proposed method, in which keyword selection is performed for emphasizing the cluster characteristics. Especially, it is expected that the variable selection mechanism based on cluster structure clarification is useful for reducing the problem space (size) in document analysis with huge sets of documents although eigen decomposition of covariance matrix may be time consuming if too many keywords are used.

Here, it is worth to note that a large fuzzifier of "$\beta = 10$" was used in this experiment because $\beta = 2$, which is an often used value in FCM, gave larger membership degrees only for a few keywords. So, a relatively large fuzzifier may be valid in real applications with many variables. A study on the effect of "fuzzifier" remains in future work.

5 Conclusion

In this paper, the variable weighting mechanism was introduced into the PCA-guided k-Means procedure. The relative responsibility of variables is estimated in a similar way with FCM clustering while the membership indicator is derived from a PCA-guided manner, in which the principal component scores are calculated by considering the responsibility weights of variables. So, the variables that have meaningful information for capturing cluster structures are emphasized in calculation of membership indicators. Numerical experiments including an application to document clustering demonstrated the characteristics of the proposed method although a study on the effect of "fuzzifier" remains in future work. Another potential future work is to hybridization with kernel method that have been shown to be useful for capturing non-linear cluster boundaries in PCA-guided k-Means.

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Partial Symbol Ordering Distance

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Abstract. Nowadays sequences of symbols are becoming more important, as they are the standard format for representing information in a large variety of domains such as ontologies, sequential patterns or non numerical attributes in databases. Therefore, the development of new distances for this kind of data is a crucial need. Recently, many similarity functions have been proposed for managing sequences of symbols; however, such functions do not always hold the triangular inequality. This property is a mandatory requirement in many data mining algorithms like clustering or k-nearest neighbors algorithms, where the presence of a metric space is a must. In this paper, we propose a new distance for sequences of (non-repeated) symbols based on the partial distances between the positions of the common symbols. We prove that this Partial Symbol Ordering distance satisfies the triangular inequality property, and we finally describe a set of experiments supporting that the new distance outperforms the Edit distance in those scenarios where sequence similarity is related to the positions occupied by the symbols.

Keywords: Sequences of Symbols, Distances, Triangular Inequality.

1 Introduction

Sequences of symbols are a well-known kind of data representation as they are widely used in databases for representing many types of non numerical attributes, such as names or addresses. However, nowadays, sequences of symbols are gaining more an more attention in many other communities because they are a natural way to represent data in a large variety of domains, such as gene information [9], vehicular tracking [15] or sequential patterns [1].

For this reason, there are a lot of work for computing similarities among sequences of symbols [5,7,8,11,14]. However, only the Hamming [8] and the Levenstein (Edit) distance [11] satisfy the triangular inequality ($d(A,B) \leq d(A,C) + d(B,C)$, where A, B, C are sequences of symbols). Therefore, these distances are the most used and

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studied in applications where a metric space is required. Examples of such applications are clustering [10], k-nearest neighbors algorithms [3] or gene sequential pattern mining [13] or record linkage [16].

As one can imagine, those scenarios have different requirements for computing the similarity between two sequences. On the one hand, in the record linkage scenario, an insertion, deletion or update of a symbol in a sequence can be considered as a typo, whereas a swap between two non-contiguous symbols has a lower probability of being a typo. On the other hand, in the sequential patterns mining scenario in general, and in the gene sequential patterns mining scenario in particular, a far swap between two symbols could be considered as a measurement error whilst an insertion, deletion or update of a symbol could not. Since the Edit distance measures the minimum number of operations (*i.e.* insertion, deletion, or update of a single symbol) needed to transform one sequence into the other, and the Hamming distance computes the number of positions for which the corresponding symbols are different, their application in this latter scenario is unnatural.

In this paper, we define a new distance for sequences of non-repeated symbols that we call *Partial Symbol Ordering distance*, especially suitable for scenarios where swaps are considered as the key-point for sequences comparison. This distance is based on computing the distances between the positions of the common symbols of two sequences. We prove that this is actually a distance, because the triangular inequality holds. We then describe some record linkage experiments showing that the new distance achieves a similar performance as the Edit distance in the classical string matching problem. Note that, in this scenario, insertions, deletions and updates have a larger probability to be a typo. Therefore, Edit distance should have some advantage compared to our new distance. Finally, we describe an alternative set of experiments simulating the scenario where swaps are considered as the key-element in the sequence of symbols comparison. In this second scenario, the performance of Partial Symbol Ordering distance is better than the one obtained with the Edit distance. In both experiments we have disregarded the Hamming distance because such distance only works when the sequences to be compared have exactly the same length, which is not the normal case in record linkage.

The rest of this paper is organized as follows. Firstly, in Section 2 we introduce some basic concepts about similarity functions and distances. Then, in Section 3 we define the Partial Symbol Ordering distance and we provide a proof for the triangular inequality condition. Experiments are described in Section 4, highlighting the relevance of the new distance. Finally, Section 5 draws some conclusions and describes some lines for future work.

2 Basic Notions

From a formal point of view, a distance function d over two sequences of symbols A and B has to fulfill the following conditions:

1. Symmetry: $d(A, B) = d(B, A)$
2. Positivity: $d(A, B) \geq 0$ for all sequences A, B
3. Reflexivity: $d(A, A) = 0$
4. Triangular Inequality: $d(A, B) \leq d(A, C) + d(B, C)$ for all sequences A, B, C

When a function does not fulfill the triangular inequality, it is not considered a distance, but a similarity measure. In the literature we can find a large number of similarity measures. For instance, in [7] the *Ordered-based Sequence Similarity* (OSS) measure was defined based on the comparison of the common symbols in the two sequences and the positions where they appear; this measure is not a distance because it does not satisfy the triangular inequality. In [14] a flexible similarity measure was presented, which is computed by comparing several aspects of the sequences and aggregating them by means of a weighted mean. The weighting vector is a user parameter. As in the former work, no proof is given for the triangular inequality.

Only few distance functions for sequences of symbols have been defined. Hamming distance [8] and Edit (Levenstein) distance [11] are the two mainly reported ones [5]. The descriptions of both distances are the following ones:

- **Hamming Distance** [8] between two sequences of symbols of equal length is the number of positions for which the corresponding symbols are different. In other words, it measures the minimum number of substitutions required to change one into the other, or the number of errors that transformed one sequence into the other. For example, the Hamming distance between 'toned' and 'roses' is equal to 3.
- **Edit Distance** [11] between two sequences of symbols is given by the minimum number of operations needed to transform one string into the other, where an operation is an insertion, deletion, or update of a single symbol. A generalization of the Edit distance is the Damerau-Edit distance [4] , which allows the transposition of two symbols as an operation. For instance, the Edit distance between 'kitten' and 'sitting' is equal to 3 (the update of 's' for 'k' and 'i' for 'e' and the insertion of 'g' at the end).

Hamming distance satisfies all the conditions presented below but its application is limited to sequences with the same length. On the contrary, Edit (Levenstein) distance can deal with sequences of different lengths; however, it gives less importance to the suppression or insertion of a non common symbol than to a swap of two symbols (because, in this latter case, two update operations are counted). The more general Damerau-Edit distance solves this drawback including the transposition (swap) in the set of basic operations. Unfortunately, the resulting distance does not hold the triangular inequality and therefore is not a distance.

3 Partial Symbol Ordering Distance

In this section we define the new Partial Symbol Ordering distance, and we prove that it is actually a distance.

Given two sequences of non-repeated symbols $A = (a_1, \ldots, a_{n_A})$ and $B = (b_1, \ldots, b_{n_B})$, that is, $a_i \neq a_j$ and $b_i \neq b_j$ for all $i \neq j$, we will sum a 1 for each symbol which is only in one of the two sequences. The symbols which are in both A and B will be considered only once. For simplicity, we will use set notation for the sequences of symbols A and B. The definition of the new distance is therefore:

$$d(A, B) = |A - B| + |B - A| + \sum_{x \in A \cap B} d(x, A, B),$$

where $d(x, A, B) = \frac{|x_A - x_B|}{n_{AB}}$, denoting by n_{AB} the total amount of different symbols in $A \cup B$, and by x_A the position that symbol x occupies in A, i.e. $x_A = i \Leftrightarrow a_i = x$.

It is immediate to check that this function satisfies the properties symmetry, positivity and reflexivity. Let us show that it also satisfies the triangular inequality property. Let A, B, C be three arbitrary sequences of symbols: $A = (a_1, \ldots, a_{n_A})$, $B = (b_1, \ldots, b_{n_B})$ and $C = (c_1, \ldots, c_{n_C})$. We want to prove that $d(A, B) \leq d(A, C) + d(B, C)$.

For each symbol $x \in A \cup B$, we have one of the three following cases:

1. $x \in A, x \notin B$, then the contribution of x to $d(A, B)$ is exactly 1. We have either $x \in C$, which implies the contribution of x to $d(B, C)$ is exactly 1, or $x \notin C$, which implies the contribution of x to $d(A, C)$ is exactly 1. In both cases, the contribution of x to $d(A, C) + d(B, C)$ is greater or equal than the contribution of x to $d(A, B)$.
2. $x \in B, x \notin A$, symmetric case.
3. $x \in A \cap B$. In this case, we have that the contribution of x to $d(A, B)$ is $d(x, A, B) = \frac{|x_A - x_B|}{n_{AB}} \leq 1$. If $x \notin C$, then the contribution of x to $d(A, C) + d(B, C)$ is 2. If $x \in C$, we have $x \in A \cap B \cap C$, and we can write

$$d(x, A, B) = \frac{|x_A - x_B|}{n_{AB}} = \frac{|x_A - x_C + x_C - x_B|}{n_{AB}} \leq \frac{|x_A - x_C|}{n_{AB}} + \frac{|x_C - x_B|}{n_{AB}}.$$

Now we can consider two different cases. The first one is when $C \subset A \cup B$. In this case, we have $n_{AC} \leq n_{AB}$ and $n_{BC} \leq n_{AB}$, and so the above value $d(x, A, B)$ is less or equal than

$$\leq \frac{|x_A - x_C|}{n_{AC}} + \frac{|x_C - x_B|}{n_{BC}} = d(x, A, C) + d(x, B, C).$$

Now for the second case, where there are symbols in C which are not in $A \cup B$, let $k = |C - (A \cup B)|$. Now we have the bounds $n_{AC} \leq n_{AB} + k$ and $n_{BC} \leq n_{AB} + k$. Note that these k symbols will not contribute to the value $d(A, B)$, but will contribute with $2k$ to the value $d(A, C) + d(B, C)$.

Let us go back to our situation where $x \in A \cap B \cap C$, we have

$$d(x, A, B) \leq \frac{|x_A - x_C|}{n_{AB}} + \frac{|x_C - x_B|}{n_{AB}} \leq \frac{|x_A - x_C|}{n_{AC} - k} + \frac{|x_C - x_B|}{n_{BC} - k}.$$

Now we can use the fact that $\frac{a}{b-k} = \frac{a}{b} + \frac{ak}{b(b-k)}$ and so the last inequality becomes

$$\begin{aligned} d(x, A, B) &\leq \frac{|x_A - x_C|}{n_{AC}} + \frac{|x_A - x_C| \cdot k}{n_{AC}(n_{AC} - k)} + \frac{|x_C - x_B|}{n_{BC}} + \frac{|x_C - x_B| \cdot k}{n_{BC}(n_{BC} - k)} \leq \\ &\leq \frac{|x_A - x_C|}{n_{AC}} + \frac{|x_C - x_B|}{n_{BC}} + \frac{k}{n_{AC} - k} + \frac{k}{n_{BC} - k} \\ &= d(x, A, C) + d(x, B, C) + \frac{k}{n_{AC} - k} + \frac{k}{n_{BC} - k}. \end{aligned}$$

Here we have used that $|x_A - x_C| \leq n_{AC}$ and $|x_C - x_B| \leq n_{BC}$.

If we consider all the symbols $x \in A \cap B \cap C$, together, we have

$$\sum_{x \in A \cap B \cap C} d(x, A, B) \leq \left(\sum_{x \in A \cap B \cap C} d(x, A, C) + \sum_{x \in A \cap B \cap C} d(x, B, C) \right) + \\ + |A \cap B \cap C| \cdot \left(\frac{k}{n_{AC} - k} + \frac{k}{n_{BC} - k} \right)$$

But we can now use the fact that $k + |A \cap B \cap C| \leq n_C \leq n_{AC}$ (and the same happens with n_{BC}, of course) which implies that the last part of the expression above is less or equal than $2k$. Recall that the k symbols which are in $C - (A \cup B)$ contribute with $2k$ to the value $d(A, C) + d(B, C)$.

Summing up, if we consider the symbols $x \in A \cap B \cap C$, their contribution to $d(A, B)$ is less or equal than the contribution of these symbols to $d(A, C)+d(B, C)$ plus the contribution of the symbols in $C - (A \cup B)$ to $d(A, C) + d(B, C)$.

Putting all the pieces together, we finally have that $d(A, B) \leq d(A, C) + d(B, C)$ always holds, as desired.

4 Experiments

In this section, we describe the experiments that we have carried out to test the performance of Partial Symbol Ordering distance. We have focused on the record linkage scenario

4.1 Dataset Generation

In order to create a realistic record linkage scenario, we have used names and surnames extracted from a frequency dictionary containing 1,564 names and 13,068 surnames obtained from the Catalan Official Statistics Institute (IDESCAT) [2]. We have generated 100 different databases, each containing 20 different full names. After that, we have added some noise to the synthetic names creating 5 duplicated full names inside the database for each one of the synthetic full names. Duplicates have been created in two different ways to simulate the following two different scenarios:

- **String matching.** Duplicates are created adding insertions, deletions and updates to the original synthetic names. Different cases have been studied, by considering different amounts of typos in the duplicates: $5, 10, 15, \ldots, 50$.
- **Sequential patterns matching.** Duplicates are created swapping values of the original synthetic names. As before, different scenarios have been considered changing the amount of swaps ($5, 10, 15, \ldots, 50$ swaps). Here, we are interested in simulating the measurement differences of a set of sensors. This is quite common, for instance, in gene sequential patterns research where values are extracted for different patients using different DNA chips.

4.2 Measures

For each experiment performed in this paper, we have analyzed the quality of the results using two typical classifier performance measures: ROC graphs and AUC measure.

ROC Graphs. A ROC (Receiver Operating Characteristics) graph [6] is a technique for visualizing, organizing and selecting classifiers based on their performance. ROC graphs have long been used in signal detection theory to depict the trade-off between hit rate (also called true positive rate or recall) and false positive rate of classifiers. Usually, ROC graphs plot the hit rate value on the Y axis and false positive rate on the X axis. In this way, ROC graphs depict the relative trade-o? between benefits (true positives) and costs (false positives).

In the experiments presented in Section 4.3, we will see the record linkage process as a discrete classifier, *i.e.* a classifier that outputs only a class label. Such classifiers produce an (FP rate,TP rate) pair, which corresponds to a single point in the ROC space. The closer the point to the (0,1) vantage point, the better the classifier.

The True Positive rate of a classifier is estimated as:

$$TP\ rate \approx \frac{positives\ correctly\ classified}{total\ positives}$$

The False Positive rate of a classifier is:

$$FP\ rate \approx \frac{negatives\ incorrectly\ classified}{total\ negatives}$$

Algorithm 1. AUC Measure

Data: L: the set of test instances; $f(i)$: the probabilistic classifier estimation that instance i is positive; N: Number of instances; P: number of positive instances.
Result: R: an increasing list of ROC points, starting at (0,0) and finishing at (1,1).

```
1  begin
2      L_sorted = L sorted decreasing by f scores
3      FP = 0, TP = 0, R = ⟨⟩, f_prev = −∞
4      for i ∈ L_sorted do
5          if f(i) ≠ f_prev then
6              AddPoint((FP/N, TP/P), R)
7              f_prev=f(i)
8          if i is a positive example then TP=TP+1
9          else FP = FP+1
10     AddPoint((FP/N, TP/P), R)
11 end
```

AUC Measure. ROC curve or AUC (Area Under the Curve) [12] is a two-dimensional depiction of classifier performance. It shows the ability of a classifier to rank the positive instances relative to the negative ones. Since the AUC is a portion of the area of the unit square, its value is always between 0 and 1.0. However, as a random classifier produces the diagonal line between (0, 0) and (1, 1), which has an area equal to 0.5, no realistic classifier should have an AUC less than 0.5.

AUC measure has an important statistical property: the AUC measure of a classifier is equivalent to the probability that the classifier will rank a randomly chosen positive

instance higher than a randomly chosen negative instance. Many theoretical and practical algorithms have been defined to compute the AUC curve and its area value. Since in this work we are interested in evaluating the performance of record linkage methods in a practical way, we have compute the AUC curve using the Algorithm 1.

4.3 Results Analysis

Figures 1 and 2 show the ROC Graphs obtained by the Edit and Partial Symbol Ordering distances for the string matching and the sequential patterns matching scenarios, respectively. Recall that ROC points have been computed as the average of 100 different executions. As we can observe in Figure 1, both distances obtain very similar results in this scenario, but, in general, Edit distance performs slightly better than Partial Symbol Ordering distance. Note that this is the perfect scenario for the Edit distance because insertions, deletions and updates are considered as typos.

However, in Figure 2, where the sequential patterns matching scenario is considered, the Partial Symbol Ordering distance achieves a perfect classification, independently of the number of swaps, whereas the performance of the Edit distance decreases when the number of swaps increases. The good performance of the new distance is understandable because the only typos considered here are swaps.

Figures 3 and 4 present the AUC measure computed in both scenarios with two different parameterizations, (a) 25 typos / swaps and (b) 50 typos / swaps. In Figure 3 (a) we can observe that the area covered by the Edit distance is larger than the covered by Partial Symbol Ordering distance. But, in the second configuration of the record linkage scenario (Figure 3 (b)), the new Partial Symbol Ordering distance outperforms in some parts of the graph the results achieved by the Edit distance.

Figure 4 depicts the AUC values obtained in the sequential patterns scenario. Here, we can observe that in both cases the Partial Symbol Ordering distance outperforms the results achieved by the Edit distance. Moreover, the results obtained by the Partial Symbol Ordering distance are the expected ones for a perfect classifier.

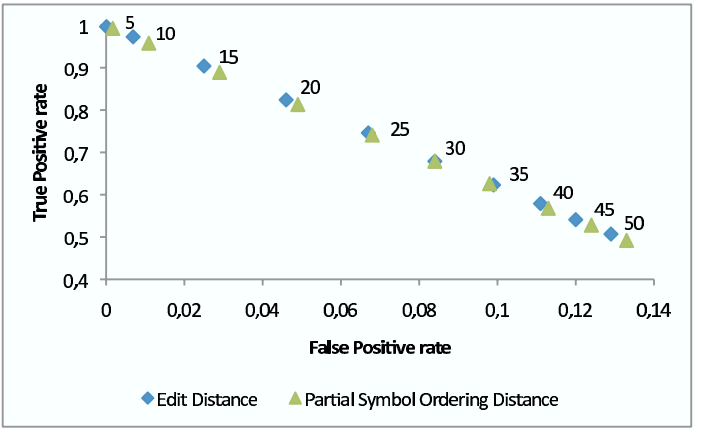


Fig. 1. ROC graph for the string matching scenario

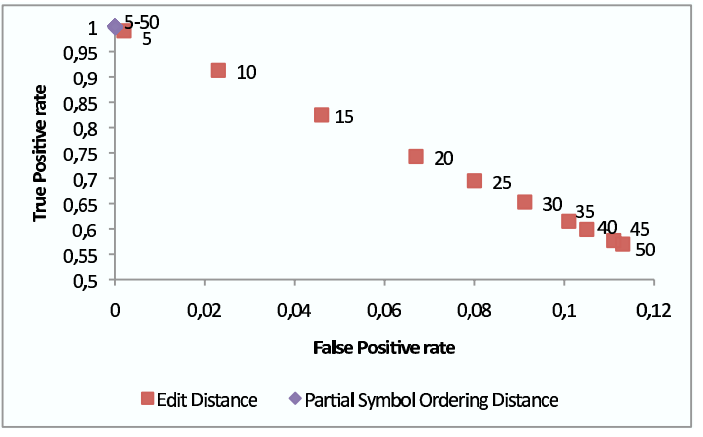
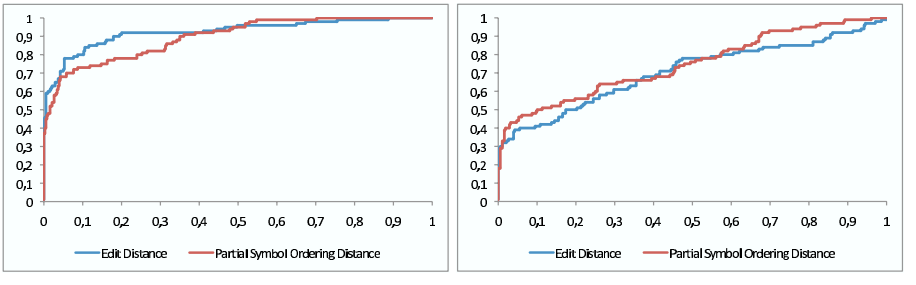
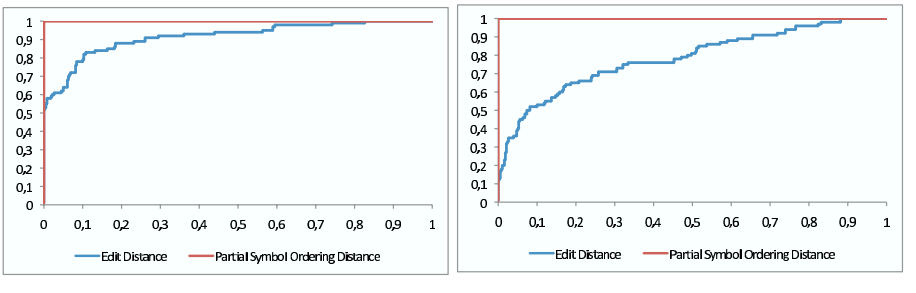


Fig. 2. ROC graph for the sequential patterns matching scenario



(a) 25 typos (b) 50 typos

Fig. 3. AUC measure for the string matching scenario



(a) 25 swaps (b) 50 swaps

Fig. 4. AUC measure for the sequential patterns matching scenario

5 Conclusions and Future Work

In this paper, we have defined a new distance for sequences of non-repeated symbols based on the difference between the positions occupied by common symbols. We have proved that the new Partial Symbol Ordering distance is really a distance, in particular by showing that the triangular inequality holds. We have then presented some experiments to argue that the new distance achieves better performance than the Edit distance when sequence similarity is related to the order of the symbols in a sequence. Finally we have discussed other experiments on the string matching scenario (where similarity is more related to insertions and deletions of symbols), which show that the Partial Symbol Ordering distance achieves similar results than the Edit distance in that case.

The drawback of the new distance is that it is valid only for the scenario where sequences do not have repeated symbols (for example, genes sequential patterns). As future work, we would like to extend this distance so that it works also for sequences of symbols admitting repetitions. This would open the door to the application of these new Partial Symbol Ordering distances in other scenarios like ontology or location routing matching.

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Situation Recognition and Hypothesis Management Using Petri Nets

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Abstract. Situation recognition – the task of tracking states and identifying situations – is a problem that is important to look into for aiding decision makers in achieving enhanced situation awareness. The purpose of situation recognition is, in contrast to producing more data and information, to aid decision makers in focusing on information that is important for them, i.e. to detect potentially interesting situations. In this paper we explore the applicability of a Petri net based approach for modeling and recognizing situations, as well as for managing the hypothesis space coupled to matching situation templates with the present stream of data.

Keywords: Situation recognition, information fusion, petri nets, hypothesis management, multi-agent activity recognition, situation assessment.

1 Introduction

The amount of data is growing rapidly in many domains and the purpose of techniques for information fusion is to aid decision makers in synthesizing this data into meaningful information of higher value. Information fusion is typically discussed in light of the Joint Directors of Laboratories (JDL) model of information fusion [1,2], in which object assessment typically is concerned with estimating discrete objects and their properties, situation assessment is about estimating relational information between objects, and impact assessment is concerned with estimating threats or impacts on the goals of the friendly organization.

Techniques for object assessment can according to Lambert [3] be considered quite mature today. Higher-level information fusion, situation and impact assessment, have however received much less effort [4] and is in need of more attention. Even if we eventually do have capabilities for also inferring impacts and relational information, the workload on operators is not likely to decrease. It might rather be the opposite, since they also need to consider relations in addition to objects, i.e. more data. This can become a problem since more data typically does not equal more information [5]. This is where situation recognition becomes important, as its purpose is to aid decision makers in finding and focusing on situations that are interesting for them.

The situation recognition problem involves finding a subset of facts or properties, which matches a priori knowledge of typically interesting situations, in a flow of data and information. In contrast to classical pattern recognition problems however, the

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situation recognition problem involves finding complex patterns of relations that are causally related to each other, and which cannot easily be modeled sequentially.

The situation recognition problem can be cast in terms of searching through a state space. In this view, the state space that we model consists of all relations over all objects, and it can be said to capture an abstract process of our observable universe. In [6] we define a state sequence, describing the evolution of an abstract process of our observable universe, as a vector of states $C = \langle s_1, \ldots, s_w \rangle$, where each state s_i is a set of facts $s_i = \{f_1, \ldots, f_u\}$. Each fact f_j is valid for a specific interval of time and consist of a relation between a number of objects. Formally, a fact is defined as $f_j: r_k(o_1, \ldots, o_q) = \{True, False\} | r_k \in R, o_l \in O$, where R is a set of relations and O is a set of objects. Situation recognition can now be formulated as a search for specific patterns in such a state sequence.

It can be problematic to define patterns exactly of what we are looking for, and we do generally not know in advance which objects we will observe. Instead, we must resort to more general patterns of activity, templates. The purpose of a template is to capture the most essential characteristics of a situation, and it imposes constraints on the domain. A template can be defined as a set of constraints that are partially ordered with respect to time. In [6] we define two types of constraints: (1) constraints that model relations between objects, and (2) constraints that impose a temporal ordering between constraints. The first type uses relations in R, and the second uses temporal relations from a set $TR = \{tr_1, .., tr_m\}$, in which each tr_i implies an ordering between two constraints, such as before and after, c.f. [7,8] for more information on temporal relations. A template can now be defined as $T = (C, X)$, where $X = \{x_1, \ldots, x_n\}$ is a set of variables for objects, and where $C = \{c_1, \ldots, c_b\}$ is a set of constraints [6].

1.1 Problem

Given a state sequence of an abstract process describing our observable universe, the situation recognition problem consists of searching through the state sequence in order to find all possible instantiations of a template. An alternative to performing an extensive search for finding a pattern in a set of data, is to do it the other way around, and feed the data through representations that model situations, i.e. to use some form of state transition technique for modelling and recognition.

In this paper we explore the applicability of an approach based on Petri nets, which builds upon previous work by [9,10,11,12], in which Petri nets are used for recognizing complex patterns of activity. Although used differently in the different approaches, there are at least two common problems which need to be addressed. First, there is a role assignment issue. Which object in the input stream corresponds to which variable in a template? Do we defer this problem to some external process, or is it possible to also solve the role assignment problem in the Petri net itself? Secondly, the size of the state space that we search grows exponentially in the number of objects and relations [6], and there can in the input stream be very many different instantiations of a pattern that we are interested in recognizing. There is a need for managing this hypothesis space of possible matches. Do we defer the management of the hypothesis space coupled to matching to some external process, or can we manage this hypothesis space in the Petri net?

We thus have two research questions in this paper: (1) can we manage the hypothesis space coupled to recognizing a situation within a Petri net, and (2) can we implicitly solve the role assignment problem within a Petri net?

1.2 Related Work

In their work on situation recognition for environment surveillance, Dousson et al. [13] address the situation recognition problem as an online search for specific patterns of activity in an event stream. The basis of their approach is propositional reified logic, and a situation model is a set of event patterns and a set of temporal constraints, which describe what they are interested in recognizing. The problem is that of identifying subsets of an event stream that matches a situation model. The technique they use for managing the number of potential instantiations of situation models is based on temporal constraint propagation [13].

Meyer-Delius et al. [14] use HMMs for probabilistically recognizing situations in connection to their work on intelligent driving assistants. Dynamic Bayesian networks are used to estimate a system state, and on top of this, relational logic is used to create an abstract system description. HMMs are used to describe situations consisting of sequences of relations. Candidate HMMs are instantiated and tracked as hypotheses over time, and Bayes factors are used for determining which hypotheses to keep.

In their work on automated scene recognition, Castel et al. [9] use Petri nets for modelling plan and activity prototypes. Given a set of observed objects, plan prototypes are instantiated and possible explanations of what is going on can be derived. New plan prototypes are instantiated when the set of currently instantiated prototypes cannot explain the currently detected objects and events. A Petri net is thus instantiated for each possible explanation that is considered for a given set of objects.

In their work on complex event recognition in video surveillance, Ghanem et al. [10] extends the work of [9] by representing each instance of simple events as tokens in a Petri net, instead of instantiating a new Petri net for each instance. Furthermore, Ghanem et al. [10] also propose extensions to the Petri net formalism for using it in a recognition context: (1) conditional transitions – additional conditions can be added to transitions, which must be true in order for a transition to fire and (2) hierarchical transitions – simpler Petri nets can be used as building blocks in more complex nets, thus simplifying the graphical representation.

In their work on multi agent activity recognition for recognizing different plays in basketball games, Perše et al [11] use Petri nets for modelling and recognizing play tactics. Perše et al. [11] extend the work of [10,12] by proposing an automatic procedure for building Petri nets from activity templates. Furthermore, the basic Petri net concept is also extended by attaching a property to tokens, which represents the degree of similarity between observed data and evaluated tactics [11].

1.3 Outline

The rest of this paper is organized as follows. In section two we provide a background covering the basics of Petri Nets and how they can be used for recognition. In section three we present an approach for representing and recognizing situations with Petri nets, which also models the hypothesis space and which implicitly takes care of the

role assignment problem. In section four we provide initial experimental results when applying the approach on a simulated pick-pocket scenario. Section five concludes the paper and outlines our future work.

2 Petri Nets

Petri nets are according to Murata [15], a graphical and mathematical tool suitable for modeling and studying systems that are characterized as being concurrent, distributed, asynchronous, nondeterministic, or stochastic. Sowa [16] argues that Petri nets are good when representing causes and effects and when simulating processes and casual dependencies, and that their major strength is their ability of representing parallel and concurrent processes.

Murata [15] describes a Petri net as a directed, weighted, bipartite graph in which nodes are either places or transitions, and where arcs either lead from places to transitions or from transitions to places. Arcs from places to transitions are referred to as input arcs and arcs from transitions as output arcs. A Petri net can according to Murata [15] be defined as a 5-tuple $PN = (P, T, F, W, M_0)$, where $P = \{p_1, \dots, p_m\}$ is a finite set of places, $T = \{t_1, \dots, t_n\}$ is a finite set of transitions, $F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs, $W: F \to \{1,2, \dots\}$ is a weight function, and $M_0: P \to \{0,1,2, \dots\}$ is an initial marking of the net such that $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

In a graphical representation, places are represented by circles and correspond to states in finite state automata, and transitions are represented by vertical bars and correspond to events in flow charts [16]. Tokens are represented as dots inside places and correspond to processes. The state of a Petri net is called a marking and consists of the number of tokens located at each place in the net [15]. When a transition is enabled, the number of tokens represented on each input arc is removed from each corresponding input place, and the number of tokens represented on each output arc is inserted to each output place [15]. Places can be seen as conditions for transitions, which can only fire when each input place contains the number of tokens that are required [15]. Figure 1 illustrates a Petri net for a chemical reaction resulting in water.

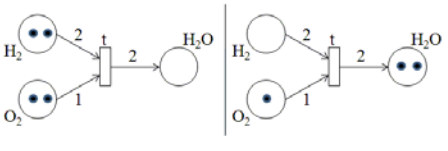

Fig. 1. Example of a Petri net that describes the chemical reaction $\boldsymbol{2H_2 + O_2 \to 2H_2O}$ (adapted from [15])

An extension to Petri nets is called coloured Petri nets, in which tokens are allowed to contain additional information, such as for example color or similar properties of interest in a domain. Even though Petri nets are not commonly associated with recognition, Castel et al. [9] argue that Petri nets are suitable for this task, since they allow sequencing, parallelism, and synchronization to be easily represented and visualized. This can be important in order for decision makers to more easily understand recognized situations.

2.1 Petri Nets for Recognition

According to Lavee et al. [12] there are mainly two approaches for using Petri nets to recognize complex patterns. The first approach is called object Petri nets, and in these, tokens correspond to detected objects and places represent particular states for objects. The second approach is called a plan Petri net, in which each place corresponds to a sub event, and a token in a place denotes the occurrence of that particular event. Lavee et al. [12] argue that object Petri nets provide a more robust model for recognition. An example of an object Petri net is illustrated in Figure 2.

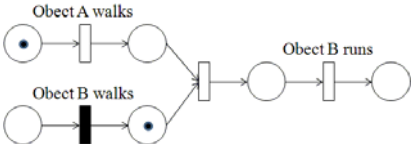

Fig. 2. Illustration of a Petri net that describes a template for the behaviour of two people (A and B) that walk, after which the second of them (B) starts to run (adapted from [11])

In Figure 2, a Petri net is used to model the causal relations between two objects and events in a template describing a behaviour in which the two objects first walk, after which one of them starts to run. As discussed in the related works section, Perše et al. [11] also provides an approach for accounting for missed events and events that do not occur in an expected interval. This is done by attaching a penalty to each token, which represents how well the template has been matched.

3 Hypothesis Management and Recognition Using Petri Nets

In this section we propose extensions to the approaches used by Ghanem et al. [10], Lavee et al. [12], and Perše et al. [11], which also allows us to manage the hypothesis space when using Petri nets for recognizing situations, as well as for automatically managing role the role assignment problem. First, however, let us discuss the hypothesis space of matching a flow of events with a template.

An event represents a change of a state for some process, and at its lowest level, this would mean that the truth value of a relation has changed. Thus, each change in truth value of a relation would raise an event. These events flow through the situation recognition system, and it is these events that we propose to use as the basic building blocks in our suggested approach. Consider a hypothesis space $H = \{h_i\}$, where each hypothesis describes a specific mapping between a template and a set of events. Each new event can be combined with every existing hypothesis, as well as resulting in a new hypothesis. Adding an event would thus result in $|H| + 1$ new hypotheses. It is the management of this hypothesis space that needs to be taken care of.

3.1 Representation

Similar to [10,11,12], *transitions* can represents types of events that we are interested in. For example, $close(x_1, x_2)$ could be a specific transition in a Petri net. Moreover,

we also exploit the conditional transition construct introduced by Ghanem et al. [10], in which a transition only is allowed to fire when both its input place restrictions are fulfilled, as well as an optional condition. We suggest that *tokens* can be used for representing individual hypotheses that we are tracking. Each token would thus contain a template, in which some constraints possibly have been assigned and in which some of the object variables possibly have been bound to real objects. We thus adopt the ideas of coloured Petri nets by attaching a hypothesis as additional information to each token. *Places* represent partial stages of the matching between a specific Petri net and the history of events. Tokens in places describe hypotheses that have reached a certain stage in the matching procedure. Finally, we also suggest that *output arcs* can be assigned a formula, which model how tokens should be created from a transition that fire. In this way, hypotheses can be kept for later matching even though specific conditions are met. Figure 3 gives an illustration.

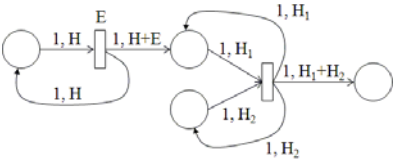

Fig. 3. Illustration of how arcs are assigned formulas to enable hypotheses to be stored

3.2 Event Processing

Each new event that we infer from data would be assigned to each transition for evaluation. In case all input place restrictions together with the optional conditions are fulfilled (same event type and value), each unique pair of input tokens would be combined with each other according to the formula on each respective output arc. When combining a hypothesis with an event, we would need to assert that the specific objects that the event concerns do not stand in conflict with the variable bindings and already assigned constraints in the hypothesis. A similar procedure is necessary when combining two hypotheses, each bound constraint and variable in each of the two hypotheses must not stand in conflict with each other. If the two hypotheses to be combined do not stand in conflict, the resulting hypothesis is the union of the two hypotheses with respect to object variable bindings and constraint assignments.

An algorithm for updating the configuration (marking) of the Petri net, for each event that is received, consists of four distinct steps. The first step is to derive a list of output tokens at each transition, based on the new event. The second step concerns inserting derived tokens to the output places of each activated transition. However, in order to also allow for missed detections, we suggest propagating tokens through the net, thus allowing for partial matching. Each token would thus pass through each consecutive transition, but instead of registering an event constraint as being fulfilled, a missed detection would be recorded. The hypothesis space is thus modelled in the Petri net. The third step concerns updating transitions that do not have a condition attached to them. By propagating hypotheses, the hypothesis space grows very quickly and as a fourth step, we need to prune the Petri net and remove low ranked hypotheses. In the pruning step, each place is inspected to remove hypotheses. As an

initial similarity measure, we suggest using a simplistic measure that consists of two numbers: (1) percentage of fulfilled constraints, and (2) percent of matching completed. These numbers can be used to discard hypotheses and in this manner keep the hypothesis space of manageable size. In addition to this, we also suggest pruning hypotheses that are too old, i.e. hypotheses that have not been updated within a specific period of time. This time, naturally, depend on the situation being modelled.

To initialize the Petri net, an empty hypothesis is inserted to all places which do not have any arcs towards them, i.e. the initial marking of the net contains an empty hypothesis at each such place. This result in that the role assignment problem is handled implicitly in the Petri net, since each object potentially can be bound to any object variable. This depends on the details of the Petri net and on the detected events.

4 Simulation and Results

In order to carry out some initial investigations on the suggested approach of using Petri nets for situation recognition whilst also managing the hypothesis space, we have constructed a simulated pick-pocket scenario. This scenario has been implemented in a simulator previously presented in [17].

4.1 Experimental Setup

The scenario takes place in a shopping zone, where a number of pedestrians move around, pass through, or move between shops. The pedestrians are intended to model some form of normal behaviour in which we would like to be able to recognize a pick-pocket situation. A pick-pocket situation consists of a thief moving towards a victim until close enough to pick the pocket, after which the thief moves towards an accomplice to hand over the stolen goods. The situation has previously been presented in [6]. The setup of the environment is illustrated in Figure 4.

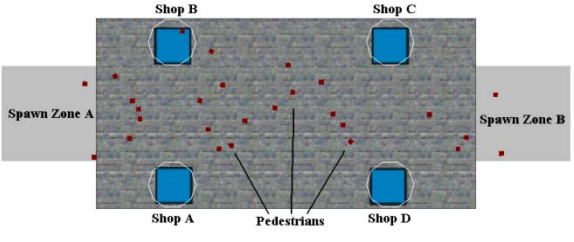

Fig. 4. Illustration of the setup of the scenario

As can be seen in Figure 4 there are two spawn zones in which pedestrians are created according to some probability (from a uniform distribution) every five seconds. Some pedestrians will move straight on through the environment towards a random location in the opposite spawn zone (effectively becomes a de-spawn zone for this pedestrian), and some go directly to a shop and then return to whence they came from. A third type of pedestrian, whom we call a happy shopper, starts by moving towards the opposite zone. However, this pedestrian can be affected by ads sent from each shop

every five seconds. Each shopper keeps track of an attraction value for going towards each shop, and when the attraction reaches a certain value, the pedestrian sets of to the most attractive store. This continues until a will to shop variable reaches zero.

Pick pockets will also be spawned randomly in the two spawn zones. If there are currently no pick-pockets in the scenario, then there is a 30% chance of creating two pick-pockets every five seconds. When created, the pick-pockets will move towards the centre of the scenario. After 5-15 seconds one of the thieves will start looking for a victim (each pedestrian has a 40% chance of being a likely victim, and this is determined when the pedestrian is created). When a victim has been found, the primary thief moves to intercept the pedestrian, and when close enough (their bounding volumes in the simulated world overlap), the thieves moves to intercept each other. When the thieves are close enough, the hand over is considered complete and they leave the scenario. It should be mentioned that all pedestrians effectively tries to avoid colliding with other objects according to a certain probability.

Object statistics. The scenario has been modelled with three different probabilities for creating pedestrians. In scenario 1, there is a 10% chance of creating one pedestrian and a 10% chance of creating two pedestrians, in each of the two spawn zones every five seconds. In scenario 2, there is a 35% chance of creating one pedestrian and a 10% chance of creating two pedestrians, in each of the spawn zones every five seconds. In scenario 3, there is an 80% chance of creating one pedestrian and a 10% chance of creating two pedestrians, every five seconds in each of the spawn zones. The three scenarios have been simulated 30 times, thus creating a total of 90 different data files. Each simulation has a length of 10 minutes. Table 1 shows some statistics for the number of objects in each of the scenarios.

Table 1. Statistics for the total number of objects created in the three scenarios and statistics for the number of active objects at each instant

	Scenario	Mean	Min	Max	σ
Total objects	1	194	165	237	14
Active objects	1	15.09	1.57	17.20	3.00
Total objects	2	347	310	386	20
Active objects	2	27.43	2.00	31.44	5.69
Total objects	3	612	597	635	9
Active objects	3	49.35	2.76	55.40	10.59

4.2 Extraction of Events

The simulator provides “ground-truth” data, i.e. exact locations for all objects in the universe of interest, at a frequency of 4 Hz. For each data set, the object-level data is fed into a recognition platform, which as a first step tries to extract relevant relations from the track data. In our previous work [6] we have argued for modelling the pick-pocket situation through the use of two different relations, close and approach. In this paper we also introduce a third relation, intercept.

Close is defined with an activation distance and a deactivation distance. If the distance between two objects x and y becomes less than the activation distance, a $close(x, y) = true$ event is fired and the relation is considered to hold between the

two objects. If the relation holds, a $close(x, y) = false$ event is fired when the distance becomes larger than the deactivation distance. The close relation is symmetrical, e.g. $close(x, y) = close(y, x)$. In the experiments, the activation distance has been set to 2 m and the deactivation distance has been set to 3 m.

Approach is defined as an object approaching another object and it is an asymmetrical relation, e.g. $approach(x, y) \nrightarrow approach(y, x)$. In order for $approach(x, y)$ to be considered to hold, the distance from x to y must decrease between two consecutive updates of x. Furthermore, the angle between the direction of x and its relative heading to y must be smaller than an activation angle. If these conditions are true, an $approach(x, y) = true$ event is fired. In case the relations holds, an $approach(x, y) = false$ event is fired if either the distance between x and y does not decrease between two consecutive updates, or if the angle becomes larger than a deactivation limit. The activation angle has in the experiments been set to 3 degrees and the deactivation angle to 5 degrees. Furthermore, $approach(x, y)$ is for object x only considered for each object y within a circle with a radius of 30 meters.

Intercept is defined as two objects being on a collision course towards each other, and it is derived using the closest point of approach metric (CPA). An activation limit and a deactivation limit have been used to provide stability. An $intercept(x, y) = true$ event is fired if the CPA becomes smaller than the activation distance and if the distance between x and y decreases between two consecutive updates of x. Thus, $intercept(x, y)$ is also an asymmetrical relation. An $intercept(x, y) = false$ event is fired if the CPA becomes larger than the deactivation distance, or if the distance between x and y does not decrease between two consecutive updates of x. The activation and deactivation limits have been set to 1 and 3 meters, respectively. Similarly to approach, $intercept(x, y)$ is for object x only considered for each object y within a bounding circle with a radius of 30 meters.

Event statistics. The definitions of the relations close, approach and intercept have been applied to the 30 data sets in each of the 3 scenarios that have been created, in order to extract all occasions on which the three relations are activated, i.e. when the relation they describe becomes true after previously being false. Table 2 presents some statistics for the events that have been extracted.

Table 2. Statistics for events, describing the activation of relations, extracted from the 30 data sets for each of the 3 scenarios. In the table, $\boldsymbol{\sigma}$ is calculated as the sample standard deviation.

Scenario	Event	Mean	Min	Max	σ
1	Approach	1624.7	992	2549	332.6
1	Close	104.2	75	145	15.1
1	Intercept	904.7	569	1405	183.1
1	All events	2633.6	1640	4099	525.1
2	Approach	6545.5	4640	8727	997.4
2	Close	344.2	256	437	46.0
2	Intercept	3690.7	2551	4956	590.8
2	All events	10580.3	7476	14103	1626.0
3	Approach	26843.5	22360	30972	2007.3
3	Close	1180.9	1000	1349	84.1
3	Intercept	15749.5	13271	17947	1204.9
3	All events	43773.9	36662	50268	3277.9

4.3 Modelling the Pick-Pocket Situation with Petri Nets

In our previous work [6] we have modelled a template for the pick-pocket situation in accordance with the definition of a template that was presented in the introduction, $T = (X, C)$. The template for the pick-pocket situation is thus defined by specifying X and C. An inspection of the pick-pocket situation reveals that it contains a primary thief, a victim, and a secondary thief (accomplice), and therefore, $X = \{x, y, z\}$. The constraints of the template for a pick-pocket situation can be defined as $C = \{c_1, \ldots, c_{10}\}$, where

$$\begin{array}{lll} c_1 \leftarrow approach(x,y) & c_2 \leftarrow close(x,y) & c_3 \leftarrow intercept(x,z) \\ c_4 \leftarrow intercept(z,x) & c_5 \leftarrow close(x,z) & c_6 \leftarrow before(c_1,c_2) \\ c_7 \leftarrow before(c_2,c_3) & c_8 \leftarrow before(c_2,c_4) & c_{\{9,10\}} \leftarrow before\left(c_{\{3,4\}},c_5\right) \end{array} . \quad (1)$$

The present paper however concerns the usage of Petri nets for representing situation templates, and thus, we define a Petri net in accordance with equation 1. This is illustrated in Figure 5.

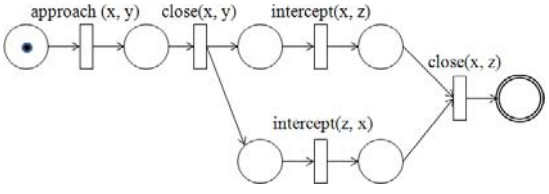

Fig. 5. Illustration of how the pick-pocket situation has been modelled with a Petri Net. Tokens in the net will contain a candidate hypothesis in accordance with equation 1.

4.4 Recognition Results

In the experiments, all events that are generated have been used as input to the Petri net presented in Figure 5. At this point, we have only considered perfect matching, and the hypothesis propagation step of the algorithm has thus been turned off. A summary of the results from each of the scenarios is presented in Table 3.

As can be seen in Table 3, the recall is quite good. We find 89%, 86% and 89% of the modelled pick-pocket situations. However, the precision is quite low, which indicates that we recognize very many situations as pick-pocket situations although they are not considered to be pick-pocket situations in the simulation. This indicates that the specific Petri net and template that has been used, is not restrictive enough for finding the modelled behaviour. It should be mentioned however, that the number of events generated is very high, and as the number of events increase, the chance of the modelled behaviour to randomly occur increase as well.

Table 3. Results when using the previously described Petri net on the events that have been extracted from each of the three scenarios

	Scenario	Mean	Min	Max	σ
Situations recognized	1	19.1	12	30	4.7
Pick-pocket situations occurred	1	7.2	6	9	0.8
Pick-pocket situations recognized	1	6.4	4	9	1.2
Recall	1	0.89	0.66	1.00	0.11
Precision	1	0.35	0.19	0.58	0.10
Situations recognized	2	80.9	32	134	21.8
Pick-pocket situations occurred	2	7.7	5	9	1.1
Pick-pocket situations recognized	2	6.6	4	9	1.3
Recall	2	0.86	0.50	1.00	0.12
Precision	2	0.09	0.04	0.19	0.04
Situations recognized	3	645.7	386	895	112.5
Pick-pocket situations occurred	3	8	7	10	0.7
Pick-pocket situations recognized	3	7.13	5	9	1.11
Recall	3	0.89	0.63	1.00	0.12
Precision	3	0.01	0.01	0.02	0.003

5 Conclusions

Situation recognition is an important problem to look into for aiding decision makers in achieving enhanced situation awareness. In this paper we have proposed a novel approach for using Petri nets for representing templates as well as for managing the hypothesis space. This approach extends previous work of Ghanem et al. [10], Lavee et al. [12], and Perše et al. [11]. Our research questions in this paper concerned the possibility of managing the hypothesis space related to matching from within a Petri net, and the possibility of implicitly handling the role assignment problem within the Petri net. We have in the paper presented an approach that does this.

Our initial results show that we have can find the modelled behaviour; however, we also have a quite high false alarm rate. This indicates that the specific nets that have been used are not restrictive enough. It should also be mentioned that the number of events that was generated in the densest scenario was more than 4300 in average per minute. This was not a problem for the Petri net to handle in real-time (including event extraction).

Our future work will take three different paths: (1) we will evaluate the suggested approach more thoroughly, (2) we will compare the Petri net approach with an approach based on logic, and (3) we will investigate how to incorporate learning in order to adapt templates with respect to data.

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A Hybrid Algorithm Based on Tabu Search and Ant Colony Optimization for k-Minimum Spanning Tree Problems

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Abstract. This paper considers an efficient approximate algorithm for solving k-minimum spanning tree problems which is one of the combinatorial optimization in networks. A new hybrid algorithm based on tabu search and ant colony optimization is provided. Results of numerical experiments show that the proposed method updates some of the best known values and that the proposed method provides a relatively better performance with solution accuracy over existing algorithms.

Keywords: k-minimum spanning tree, tabu search, ant colony optimization, hybrid algorithm, approximate solution.

1 Introduction

A k-Minimum Spanning Tree (k-MST) problem is one of combinatorial optimization problems formulated in networks, and the objective of the problem is to find a subtree with exactly k edges, called k-subtree, such that the sum of the weights attached to edges is minimal. The k-MST problem is a generalized version of minimum spanning tree (MST) problems; when $k = |V| - 1$ where V is a cardinality of vertices in a graph, the k-MST problem corresponds to the MST problem. The wide varieties of decision making problems in the real world can be formulated as k-MST problems, e.g. telecommunications [10], facility layout [8], open pit mining [16], oil-field leasing [12], matrix decomposition [2,3] and quorum-cast routing [4].

The k-MST problem was firstly introduced by Hamacher *et al.* [12] in 1991. Since the k-MST problem is NP-hard [7,15], it is difficult to solve large-scale problems within a practically acceptable time. Therefore, it is very important to construct approximate solution methods which quickly find a near optimal solution. Blum and Blesa [1] proposed several approximate solution methods including metaheuristics such as evolutionary computation, ant colony optimization and tabu search.

In this paper, we propose a new hybrid approximate solution algorithm based on tabu search and ant colony optimization. In order to demonstrate efficiency of the proposed solution method, we compare the performances of the proposed method with those of existing algorithms.

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2 Problem Formulation and Existing Solution Methods

Given that a graph $G = (V, E)$ where V is a set of vertices and E is a set of edges, k-subtree T_k is defined as

$$T_k \in G,\ k \leq |V| - 1.$$

Then a k-MST problem is formulated as

$$\begin{aligned} &\text{minimize} \sum_{e \in E(T_k)} w(e) \\ &\text{subject to } T_k \in \mathcal{T}_k, \end{aligned}$$

where $\mathcal{T}_k$ is the set of all possible k-subtrees T_k in G, $E(T_k)$ denotes the edges of T_k and $w(e)$ is a weight attached to an edge e. The above problem is to seek a k-subtree with the minimum sum of weights, called k-MST. If the problem size is small, the problem can be easily solved by finding an optimal solution after enumerating all possible k-subtrees in a given graph. If the size of problem is not so large, it can be solved by some exact solution algorithm such as a branch and bound method [4] and a branch and cut algorithm [9].

However, it has been shown that the k-MST problem is NP-hard even if the edge weight is in {1,2,3} for all edges, or if a graph is fully connected. The problem is also NP-hard for planar graphs and for points in the plane [15]. Therefore, it is important to construct not only exact solution methods but also efficient approximate solution methods.

Blum and Blesa [1] proposed three approximate solution algorithms for k-minimum spanning tree problems which are based on evolutionary computation, tabu search and ant colony optimization. They compared their performances through benchmark instances [14] and showed that an ant colony optimization approach is the best for relatively small ks, whereas a tabu search approach has an advantage for large ks with respect to solution accuracy.

In this paper, we propose an efficient hybrid algorithm based on tabu search and ant colony optimization by combining the desirable features of both algorithms.

3 Proposed Algorithm

The outline of the proposed algorithm is as follows:

Step 1 (Generation of an initial solution). For a node selected at random, the application of Prim method is continued until a k-subtree is constructed. Let the obtained k-subtree be an initial solution and the current solution T_k^{cur}.

Step 2 (Initialization of parameters). Initialize the tabu lists and the values of parameters such as tabu tenure tl_{ten} and aspiration criterion levels.

Step 3 (Tabu search-based local search). Search the neighborhood based tabu search, and store a set of local minimum solutions. If the current tabu tenure tl_{ten} is greater than tt_{max}, go to Step 4. Otherwise, return to Step 2.

Step 4 (Ant colony optimization-based intensification procedure). Explore the promising region intensively based on ant colony optimization.

Step 5 (Terminal condition). If the current computational time is greater than $TimeLimit$, terminate the algorithm. Otherwise, return to Step 2.

Let T_k^{cur}, T_k^{gb} and T_k^{lb} be the current solution, the best found solution and local optimum solution, respectively. Then, we describe the details on the procedures in Steps 3 and 4.

3.1 Tabu Search-Based Local Search

For a set $V(T_k)$ of nodes included in k-subtree T_k, we define

$$V_{NH}(T_k) := \{v | \{v, v'\} \in E(G), v \notin V(T_k), v' \in V(T_k)\}.$$

Let T_k^{NH} be a local minimum solution of k-subtree obtained by adding $v_{in} \in V_{NH}(T_k)$ to T_k and deleting $v_{out} \in V(T_k)$. Then, the neighborhood of T_k denoted by $NH(T_k)$ is defined as a whole set of possible T_k^{NH} in G.

In the proposed local search algorithm, the next solution through transition is selected as the k-subtree that has the best objective function value of all solutions $T_k^{NH} \in N(T_k^{cur})$ as follows:

$$T_k^{NH_{best}} := \arg \min_{T_k^{NH} \in NH(T_k^{cur})} \{f(T_k^{NH})\}.$$

In order to avoid cycling among a set of some solutions, we use two tabu lists *InList* and *OutList*, which keep the induces of removed edges and added edges, respectively. A tabu tenure, denoted by θ, is a period for which it forbids edges in the tabu lists from deleting or adding. In details, at the beginning, we set an initial value of the tabu tenure tl_{ten} to tt_{min} which is the minimum tabu tenure defined as

$$tt_{min} := \min \left\{ \left\lfloor \frac{|V|}{20} \right\rfloor, \frac{|V| - k}{4}, \frac{k}{4} \right\}.$$

Let nic_{int} be the period of the best found solution T_k^{gb} not being updated. If $nic_{int} > nic_{max}$, then tabu tenure is updated as $tl_{ten} \leftarrow tl_{ten} + tt_{inc}$, where

$$nic_{max} := \max\{tt_{inc}, 100\}, \qquad tt_{inc} := \left\lfloor \frac{tt_{max} - tt_{min}}{10} \right\rfloor + 1.$$

If the current tabu tenure tl_{ten} is greater than tt_{max} defined as

$$tt_{max} := \left\lfloor \frac{|V|}{5} \right\rfloor,$$

the local search algorithm is terminated, and intensification strategy based on ant colony optimization is performed.

When checking whether the transition from the current solution to some solution in T_k^{NH} is acceptable, if an edge e in $InList$ or $OutList$, which is related to the transition as the added edge or deleted edge, satisfies the condition $\gamma_e > f(T_k^{NH})$, then the transition is permitted. The parameter γ_e called *aspiration criterion level* is given to all of edges and is initially set to

$$\gamma_e = \begin{cases} f(T_k^{cur}), & e \in E(T_k^{cur}) \\ \infty, & e \notin E(T_k^{cur}). \end{cases} \tag{1}$$

In each explored solution T_k, γ_e is updated as $\gamma_e \leftarrow f(T_k)$ for every $e \in E(T_k)$.

The following is the details on the proposed local search algorithm.

[Tabu search-based local search algorithm]

Step 1(Initialization of the list of a deleted node). Let $V_{in} \leftarrow V_{NH}(T_k^{cur})$.

Step 2(Decision of a deleted node). If $V_{in} = \emptyset$, terminate the algorithm. Otherwise, go to Step 2-1.

Step 2-1. Find

$$v_{in} := \arg\min_{v \in V_{in}} \left\{ \frac{\sum_{v' \in V(T_k^{cur})} w(e)}{d(v)} \middle| e = (v, v') \right\}$$

and set $V_{in} \leftarrow V_{in} \backslash v_{in}$, where $d(v)$ is the number of edges existing between $v \in V$ and T_k^{cur}. Go to Step 2-2.

Step 2-2. Find $E_{in_1} := \{(v, v_{in}) \mid v \in V(T_k^{cur})\}$ (see Fig. 1) and $e_{\min_1} := \arg\min_{e \in E_{in_1}} \{w(e)\}$. Set $T_{k+1}^{NH} \leftarrow (V(T_k^{cur}) \cup v_{in}, E(T_k^{cur}) \cup e_{min_1})$ and $E_{in_1} \leftarrow E_{in_1} \backslash e_{\min_1}$ (see Fig. 2), and go to Step 2-3.

Step 2-3. Find $e_{\min_1} := \arg\min_{e \in E_{in_1}} \{w(e)\}$, and set $T_{k+1}^{NH} \leftarrow T_{k+1}^{NH} \cup e_{\min_1}$ and $E_{in_1} \leftarrow E_{in_1} \backslash e_{\min_1}$ (see Fig. 3). Go to Step 2-4.

Step 2-4. For a set E_{loop} of edges which compose a loop in Step 2-3, find $e_{\max} := \arg\max_{e \in E_{loop}} \{w(e)\}$ and set $T_{k+1}^{NH} \leftarrow T_{k+1}^{NH} \backslash e_{\max}$ (see Fig. 3)D

Step 2-5. If $E_{in_1} = \{\emptyset\}$, then set $V_{out} \leftarrow V(T_k^{cur})$ and go to Step 3. Otherwise, return to Step 2-4.

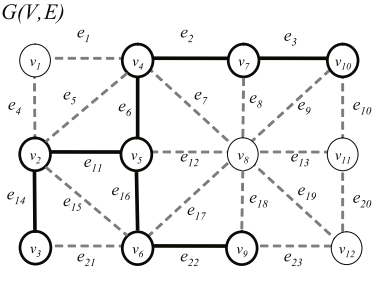

Fig. 1. Current solution (when $v_{in} = v_8$ and $E_{in_1} = \{e_7, e_8, e_9, e_{12}, e_{17}, e_{18}\}$)

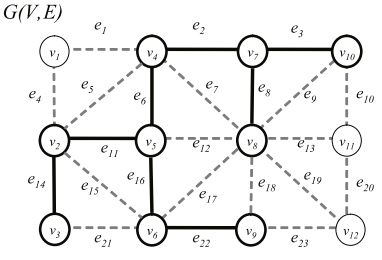

Fig. 2. $k+1$-subtree T_{k+1}^{NH} (e_8 is added to the current solution)

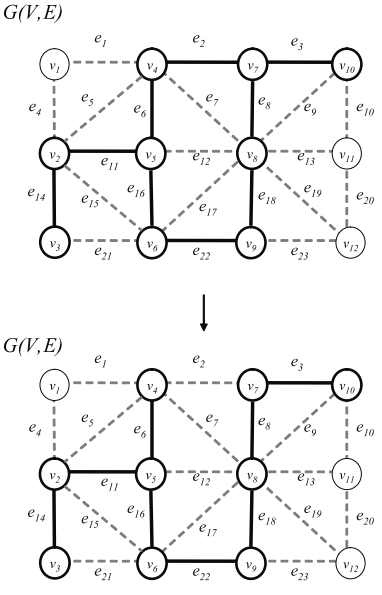

Fig. 3. Improvement of $k+1$-subtree T_{k+1}^{NH} (e_{18} is added, and then e_2 is deleted so that a new $k+1$-subtree T_{k+1}^{NH} is constructed)

Step 3(Decision of an added node for constructing T_k^{NH}). If $V_{out} = \{\emptyset\}$, then return to Step 2. Otherwise, go to Step 3-1.

Step 3-1. Find

$$v_{out} := \arg \max_{v \in V_{out}} \left\{ \frac{\sum_{v' \in V(T_k^{cur})} w(e)}{d(v)} \middle| e = (v, v') \right\}$$

and set $V_{out} \leftarrow V_{out} \backslash v_{out}$. Go to Step 3-2.

Step 3-2. Find $e_{\min}^{out} := \arg\min_{e \in \{(v_{out}, v')\}} \{w(e) \mid v' \in T_k^{cur}\}$. If $f(T_k^{NH_{best}}) < \left(\sum_{e \in E(T_{k+1}^{NH})} w(e)\right) - w(e_{\min}^{out})$ for $e_{\min}^{out}$, then return to Step 3. Otherwise, go to Step 3-3.

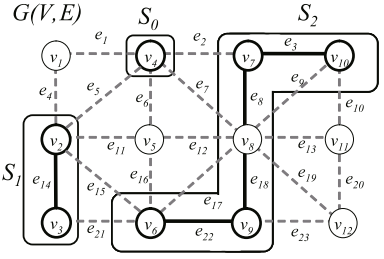

Fig. 4. Set of super-nodes

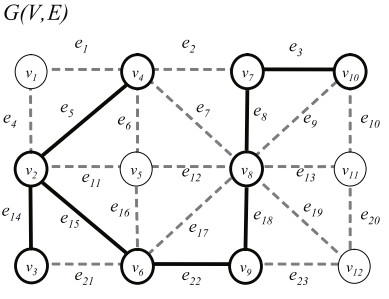

Fig. 5. Solution T_k^{NH} in neighborhood $NH(T_k)$

Step 3-3. For a set of super-node S_r, $r = 0, 1, 2 \cdots$, each of which is a connected component obtained by deleting v_{in} from T_k^{NH}, find $E_{in_2} := \{(v_i, v_j) \mid v_i \in S_k, v_j \in S_l, k \neq l\}$ (see Fig. 4). Go to Step 3-4.

Step 3-4. Find $e_{\min_2} := \arg\min_{e \in E_{in_2}} \{w(e)\}$. If $f(T_k^{NH_{best}}) < w(e_{\min_2}) + \sum_{e \in E(T_k^{NH})} w(e)$ for $e_{\min_2}$, then return to Step 3. Otherwise, go to Step 3-5.

Step 3-5. If there is no loop in $e_{\min_2} \cup T_k^{NH}$, then set $E(T_k^{NH}) \leftarrow E(T_k^{NH}) \cup e_{\min_2}$ and $E_{in_2} \leftarrow E_{in_2} \backslash e_{\min_2}$. Otherwise, set $E_{in_2} \leftarrow E_{in_2} \backslash e_{\min_2}$. Go to Step 3-6.

Step 3-6. If T_k^{NH} is a tree (see Fig. 5)Cthen set $f(T_k^{NH_{best}}) \leftarrow f(T_k^{cur})$ and return to Step 3. Otherwise, return to Step 3-4.

3.2 Ant Colony Optimization-Based Intensification Procedure

In this section, we describe the details on the ant colony optimization-based intensification procedure performed in Step 4.

Ant Colony Optimization (ACO) [5,6] is a metaheuristic approach for solving hard combinatorial optimization problems. This basic behavior is the basis for a cooperative interaction which leads to the emergence of shortest paths by depositing a substance called *pheromone* on the ground so as to minimize the length of the path between nest and food source.

In this paper, we propose an intensification algorithm based on ant colony optimization by extending the Blum-Blesa algorithm [1]. One of the characteristics in the proposed algorithm is that our algorithm deposits pheromone on the edges selected in the local optimal solutions which were obtained by the tabu search-based local algorithm. This procedure allows the proposed hybrid algorithm to intensively explore the promising region.

[**Intensification algorithm based on ant colony optimization**]

Step 1 (Setting of learning rate). Set the learning rate of each solution in E_{lb} to the value defined by

$$\rho = \begin{cases} 0.15, & cf < 0.7 \\ 0.1, & 0.7 \leq cf \leq 0.95 \\ 0.05, & cf > 0.95, \end{cases}$$

where cf is a convergence factor defined by

$$cf \leftarrow \frac{\sum_{e \in E_{lb}} \tau_e}{|E_{lb}| \cdot \tau_{max}}.$$

Step 2 (Update of pheromone). Update the amount of pheromone assigned to each edge e as follows:

$$\tau_e = f_{mmas}(\tau_e + \rho(\delta_e - \tau_e))$$

where

$$f_{mmas}(x) = \begin{cases} \tau_{min}, & x < \tau_{min} \\ x, & \tau_{min} \leq x \leq \tau_{max} \\ \tau_{max}, & x > \tau_{max} \end{cases}, \quad \delta_e = \begin{cases} 1, & e \in E_{lb} \\ 0, & e \notin E_{lb}. \end{cases}$$

Step 3 (Generation of a k-subtree). Replace the weight attached to each edge e in G by $w_d(e)$ defined as

$$w_d(e) \leftarrow \frac{w(e)}{\tau_e}.$$

Stating from a randomly selected node, a k-subtree T_k is constructed by applying the Prim method. After that, replace $w_d(e)$ by the original weight $w(e)$ and construct a k-subtree T_k^{cur} by applying the Prim method again to the subgraph whose nodes and edges are $V(T_k)$ and $E(T_k)$, respectively.

In this paper, we set the initial values of τ_e, the values of τ_{min} and τ_{max} to 0.5, 0.001 and 0.999, respectively.

4 Numerical Experiments

In order to compare the performances of our method with two of existing solution algorithms proposed by Blum and Blesa [1]. We use C as the programming language and compiled all software with C-Compiler: Microsoft Visual C++ 7.1. All the metaheuristic approaches were tested on a PC with Celeron 3.06GHz CPU and RAM 1GB under Microsoft Windows XP.

Tables 1-4 and 5 show the results for several existing instances [14] and our new instances, respectively. Bold face means that it is the best obtained value among the three algorithms to be compared. In Tables 1-4, BNV denotes the best known values which have been obtained by Blum and Blesa through their

Table 1. Grid graph [14]

Graph	k	BNV		Objective function values		
				HybridK	TSB	ACOB
$\|N\| = 225$	40	695	best	**695**	696	**695**
$\|E\| = 400$			mean	**695**	696	695.4
$\bar{d}(v) = 3.55$			worst	**695**	696	696
(bb45x5_1.gg)	80	*1552	best	**1552**	1579	1572
		(1568)	mean	**1565.1**	1592.7	1581.2
			worst	**1572**	1615	1593
	120	*2444	best	**2444**	2546	2457
		(2450)	mean	**2457.9**	2558.5	2520.3
			worst	**2465**	2575	2601
	160	*3688	best	**3688**	3724	3700
		(3702)	mean	**3688**	3724.9	3704.7
			worst	**3688**	3729	3720
	200	5461	best	**5461**	5462	**5461**
			mean	**5461**	5462.4	5469
			worst	**5461**	5463	5485
$\|N\| = 225$	40	654	best	**654**	**654**	**654**
$\|E\| = 400$			mean	**654**	**654**	**654**
$\bar{d}(v) = 3.55$			worst	**654**	**654**	**654**
(bb45x_5_2.gg)	80	1617	best	**1617**	**1617**	**1617**
			mean	1619.1	**1617.1**	1626.9
			worst	1620	**1619**	1659
	120	*2632	best	**2632**	2651	2637
		(2633)	mean	**2641.3**	2677.9	2664.6
			worst	**2648**	2719	2706
	160	3757	best	**3757**	3815	**3757**
			mean	**3764.3**	3815.0	3797.6
			worst	**3779**	3815	3846
	200	5262	best	**5262**	**5262**	**5262**
			mean	**5262**	5268.6	5272
			worst	**5262**	5296	5288

Table 2. Regular graph [14]

graph	k	BNV		Objective function values HybridK	TSB	ACOB
$\|N\| = 1000$	200	3308	best	3393	3438	**3312**
$\|E\| = 2000$			mean	3453.1	3461.4	**3344.1**
$d(v) = 4$			worst	3517	3517	**3379**
(1000-4-01.g)	400	7581	best	**7659**	7712	7661
			mean	7764	7780.2	**7703**
			worst	7819	7825	**7751**
	600	12708	best	**12785**	12801	12989
			mean	12836.6	**12821.8**	13115.6
			worst	13048	**12869**	13199
	800	19023	best	19099	**19093**	19581
			mean	**19101.1**	19112.6	19718.7
			worst	**19128**	19135	19846
	900	22827	best	**22827**	22843	23487
			mean	**22827**	22859.2	23643
			worst	**22827**	22886	23739
$\|N\| = 1000$	200	3620	best	3667	3692	**3632**
$\|E\| = 2000$			mean	3697.5	3722.0	**3670.1**
$d(v) = 4$			worst	3738	3751	**3710**
(g400-4-05.g)	400	8206	best	**8323**	8358	8376
			mean	**8357.1**	8385.6	8408.3
			worst	**8424**	8415	8442
	600	13584	best	13807	**13735**	14085
			mean	13824.3	**13759.4**	14164.5
			worst	13900	**13820**	14235
	800	20076	best	**20110**	20130	20661
			mean	**20129.9**	20142.9	20811.3
			worst	**20143**	20155	20940
	900	24029	best	**24035**	24044	24782
			mean	**24035**	24052.6	24916
			worst	**24035**	24064	25037

tremendous experiment for several months. The values with $*$ denotes new best known values that are updated by the proposed algorithm. HybridK, TSB and ACOB represent the proposed algorithm, tabu search algorithm [1] and ant colony optimization algorithm [1] by Blum and Blesa. We executed each method for 30 runs under the condition that $TimeLimit = 300(s)$ and computed the *best*, *mean* and *worst* objective function values for each method. We describe '−' in the tables when the algorithms do not derive solutions within the given time limit.

Tables 1-4 show that the performance of the proposed method is better than those of the existing algorithms by Blum and Blesa, especially in the case of high cardinality k and high degree graphs, whereas the performance of the ant colony optimization algorithm by Blum and Blesa is the best for low cardinality

Table 3. Instances constructed from Steiner tree problems [14]

graph	k	BNV		Objective function values HybridK	TSB	ACOB
$\|N\| = 1000$	200	1018	best	**1034**	1036	1036
$\|E\| = 5000$			mean	1048.6	1047.3	**1045.9**
$\bar{d}(v) = 10.0$			worst	1063	**1056**	**1056**
(steind15.g)	400	2446	best	**2469**	2493	2665
			mean	**2480.7**	2502.5	2806.6
			worst	**2492**	2524	2928
	600	4420	best	**4426**	4442	5028
			mean	**4433**	4454.6	5398.4
			worst	**4451**	4490	5602
	800	7236	best	**7236**	7252	8457
			mean	**7236.9**	7272.8	8839.6
			worst	**7237**	7308	9006
	900	9248	best	**9256**	9283	10873
			mean	**9256**	9294.2	11166.3
			worst	**9256**	9304	11423

Table 4. Instances constructed from graph coloring problems [14]

graph	k	BNV		Objective function values HybridK	TSB	ACOB
$\|N\| = 450$	90	135	best	**135**	**135**	**135**
$\|E\| = 8168$			mean	**135.1**	135.3	135.7
$\bar{d}(v) = 36.30$			worst	137	**136**	137
(le450_15a.g)	180	336	best	**336**	337	352
			mean	**337**	337.1	374.4
			worst	**337**	338	419
	270	630	best	**630**	630	696
			mean	**630.1**	630.3	839
			worst	**631**	633	913
	360	1060	best	**1060**	**1060**	1267
			mean	**1060**	1064.1	1461.2
			worst	**1060**	1118	1566
	405	1388	best	**1388**	**1388**	1767
			mean	**1388**	1391.1	1888.7
			worst	**1388**	1392	2015

k. Table 5 shows that the proposed method is the best for instances of graphs with higher degrees than existing ones. It should be stressed here that as shown in Table 1, the proposed method updates some of best known values despite very short computational time limit (300s), while the time limits in the experiments by Blum and Blesa are fairly large, at most several hours. From these results, we can occlude that the proposed algorithm is considerably promising for solving k-minimum spanning tree problems.

Table 5. New instances

graph	k		Objective function values		
			HybridK	TSB	ACOB
$\|N\| = 500$	100	best	**1943**	1954	**1943**
$\|E\| = 15000$		mean	**1950.5**	1990.9	2022.3
$\bar{d}(v) = 60$		worst	**1966**	2023	2241
	200	best	**5037**	5063	5517
		mean	**5047.3**	5080.4	7444.4
		worst	**5066**	5221	9859
	300	best	**9758**	9821	-
		mean	**9769.6**	9922.6	-
		worst	**9795**	11696	-
	400	best	**16351**	16373	-
		mean	**16363.8**	16488	-
		worst	**16368**	17953	-
	450	best	**20929**	20934	-
		mean	**20929**	20945.2	-
		worst	**20929**	20992	-
$\|N\| = 500$	100	best	**1294**	1319	1398
$\|E\| = 30000$		mean	**1303.7**	1352.8	1743.5
$\bar{d}(v) = 120$		worst	**1321**	1385	2479
	200	best	**3064**	3150	4013
		mean	**3097.1**	3934.4	6861.4
		worst	**3127**	6032	9623
	300	best	**5312**	5380	-
		mean	**5312.5**	6471.9	-
		worst	**5318**	8308	-
	400	best	**8582**	8586	-
		mean	**8582**	9540	-
		worst	**8582**	11485	-
	450	best	**10881**	10882	-
		mean	**10881**	11300.4	-
		worst	**10881**	13570	-

5 Conclusion

In this paper, we have proposed a new hybrid approximate solution algorithm for k-MST problems and compared the performance of the proposed method with those of existing methods through numerical experiments for several benchmark instances. It has been shown that the proposed method updates some of the best known values and that has provided better performances than the existing methods. We will execute more experiments to clarify the efficiency of the proposed algorithm as well as its advantages and disadvantages over the existing algorithms.

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Dynamic Neighborhood Selection for Nonlinear Dimensionality Reduction

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Abstract. Neighborhood construction is a necessary and important step in nonlinear dimensionality reduction algorithm. In this paper, we first summarize the two principles for neighborhood construction via analyzing existing nonlinear dimensionality reduction algorithms: 1) data points in the same neighborhood should approximately lie on a low dimensional linear subspace; and 2) each neighborhood should be as large as possible. Then a dynamic neighborhood selection algorithm based on this two principles is proposed in this paper. The proposed method exploits PCA technique to measure the linearity of a finite points set. Moreover, for isometric embedding, we present an improved method of constructing neighborhood graph, which can improve the accuracy of geodesic distance estimation. Experiments on both synthetic data sets and real data sets show that our method can construct neighborhood according to local curvature of data manifold and then improve the performance of most manifold algorithms, such as ISOMAP and LLE.

Keywords: neighborhood construction, manifold learning, local linearity, geodesic distance.

1 Introduction

Manifold learning is an effective technology for nonlinear dimensionality reduction which is introduced to overcome the curse of dimensionality when dealing with high dimensional data. All of the existing manifold learning algorithms can be unified into the following framework which consists of three steps[3,4,5,6,7]:

1. Computing neighborhood of each point in original space;
2. Constructing a special matrix based on the neighborhoods;
3. Calculating spectral embedding using eigenvalue decomposition of that matrix.

From this framework, one can see that neighborhood construction is a necessary and key step in manifold learning algorithm. All existing manifold learning algorithm construct neighborhoods by k-NN algorithm or ε-neighborhood which serves points in ball with ε radius as neighbors. Therefore, the parameters k and

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ε which should be specified beforehand in these two algorithms play an important role in produce a faithful low-dimensional embedding. If the neighborhood parameter is too large, the local neighborhoods will include points in different patches of data manifold, and result in fatal errors in the final embedding. Too small neighborhood parameter, however, may divide the manifold into several disconnected sub-manifolds, which also are not the expected embedding[8]. However, most of existing manifold learning algorithms subjectively specify these two parameters manually. What's more, these two neighborhood construction methods can't guarantee a connected Neighborhood Graph(hereafter NG for short), which is essential requisition in most of manifold learning algorithms.

Therefore, there are some researches concentrated on improving neighborhood selection approach. Since the NG constructed by k-NN and ε-neighborhood can't be guaranteed to be a connected one, Li Yang used k-edge-connected and k-connected spanning subgraph of complete Euclidean graph of all input data points to construct connected[9,10,11,12]. Another improved method is to determine a optimal neighborhood parameters k or ε by some heuristics. However, up to now, only some weak heuristics have been developed. For example, the residual variance, which is originally proposed to evaluate the quality of mapping between the input data and output data in ISOMAP algorithm, is exploited to automatically determine the optimal neighborhood parameter[13]. In fact, the residual variance is not reasonable to measure the quality of neighborhood parameter. In addition, this type of method takes the fixed neighborhood size for all data points, obviously it is not suitable for manifold whose curvature varies sharply and the unevenly sampled manifolds, which are both the real cases in many real applications. In context of NG with "short circuit" edge, C. Shao and Xia improve the NG via deleting the "short circuit" edge from the NG[14,15,16]. They serve edges that pass through low-density area as "short circuit" edges. However, sometimes this is not the real case and the true "short circuit" edges are always hard to distinguish.

Due to the drawbacks of existing neighborhood selection approaches, this paper concentrates on constructing neighborhood for manifold learning. In ISOMAP algorithm, NG is only used to estimate the geodesic distance of all pairwise points, and the algorithm uses the Euclidean distances as the estimation of geodesic distance between a point and its neighbors. So it is obvious that if all data points in the same neighborhood are in d-dimensional(assume d is the dimensionality of the data manifold) linear subspace, the ISOMAP will get more accurate geodesic distance estimation than estimation in other case, this will then further result in more faithful embedding. In another representative work LLE algorithm based on geometric intuition that data manifold is locally linear, it reconstructs every points by the linear combinations of neighbors. So in these two algorithms linearity can be the requisition for neighborhood. Moreover, many other manifold learning algorithms require that data points in each neighborhood are in or close to d-dimensional linear subspace. For example, in the LTSA algorithm[7], each neighborhood is approximated by the local tangent space, therefore, linearity is also a natural requisition. Meanwhile, in order to

construct a connected NG, each neighborhood should contain points as more as possible. Then we can summarize the principles of constructing neighborhoods as follows:

1. All data points in each neighborhood should be or approximate in a d-dimensional linear subspace.
2. The number of data points in each neighborhood should be as large as possible.

Based on the above two principles, this paper exploits PCA[1] to evaluate the linearity of a finite data set and proposes an Dynamic Neighborhood Selection approach(DNS) and a modified method to construct NG. In RML algorithm[17] the neighborhood selection algorithm obtains the neighborhood through deleting "invisible" edges and "unsafe" edges. Even though it can obtain good performance in RML algorithm, it is inclined to obtain small size neighborhood and not suitable for other manifold learning algorithms.

2 Measure Linearity of Finite Data Points Set

PCA is a classical subspace method for image modeling and has widespread applications in dimensionality reduction and image recognition. Given a data matrix $X = [x_1, x_2, \cdots, x_n]$, where column vector $x_i \in R^D$ represents sample data. The goal of the PCA is to project X into a d-dimensional ($d < D$) linear subspace such that the projected data points are as close as possible to the original data points. This can be formalized as follows:

$$\min \varepsilon = \min \sum_{j=1}^{n} \|\hat{x}_j - V \cdot (V^T \cdot \hat{x}_j)\|^2 \tag{1}$$

where V is a $D \times d$ matrix and $V^T V = I$. Denote the eigenvalues of the covariance matrix XX^T by $\lambda_i (0 < i \leq D)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D \geq 0$, and the corresponding eigenvectors are v_i. So the optimal solution of equation (4) is given by $V = [v_1, v_2, \cdots, v_d]$. Substituting the optimal solution into the equation(4), then it can be further rewritten as:

$$\min \sum_{i=1}^{n} \|x_i - V \cdot (V^T x_i)\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - \sum_{i=1}^{n} \|V^T x_i\|^2 \tag{2}$$

$$= \sum_{i=1}^{n} \|x_i\|^2 - \sum_{j=1}^{d} \sum_{i=1}^{n} (v_j^T x_i)^2 = \mathrm{tr}(X^T X) - \sum_{j=1}^{d} \lambda_j \tag{3}$$

$$= \sum_{j=1}^{D} \lambda_j - \sum_{j=1}^{d} \lambda_j = \sum_{j=d+1}^{D} \lambda_j \tag{4}$$

Now our task is to evaluate how close data points in X to a d-dimensional linear subspace for a given dimensionality d(in manifold learning, the data manifold dimensionality d is always known as a priori). The formula (4) gives us

a good heuristic. If the data set X is really in a d-dimensional linear subspace, then $\lambda_i(d < i \leq D) = 0$ and the minimum of (4) is 0, and intuitionally the closer the data points set X to a d-dimensional linear subspace, the less the minimum.

Although the projection error $\sum_{j=d+1}^{D} \lambda_j$ be capable of evaluating how close X to a d-dimensional linear subspace, it ignores their own scatter of X and varies as different concrete problems. Therefore, it is necessary for us to adopt a normalized term of this absolute projection error by taking into account the scatter of X. Then the term $1 - \frac{\sum_{j=d+1}^{D} \lambda_j}{\sum_{j=1}^{D} \lambda_j} = \frac{\sum_{j=1}^{d} \lambda_j}{\sum_{j=1}^{D} \lambda_j}$ is naturally serve as the measurement to evaluate the linearity of X. Given a tolerable threshold η , if $\frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{D} \lambda_i} \geq \eta$ then data points in X can be seen in a d-dimensional linear subspace.

3 Dynamic Neighborhood Selection

As analysis in section 1, assume the dimensionality of the data manifold d is known as a priori knowledge, a good neighborhood selection algorithm should follow two principles:

1. Data points in the same neighborhood should lie on or close to a d-dimension linear subspace.
2. The number of data points in the neighborhood should be as large as possible.

DNS method with contraction strategy

input: high-dimensional data set X, the manifold dimension d,
the initial neighborhood size K, the tolerable threshold η.
output: the neighborhood of each point $\mathcal{N}(x_i)$.

Initialize: for each x_i, add its K nearest neighbors to $\mathcal{N}(x_i)$
Contraction: for each $\mathcal{N}(x_i)$,
assume $\|x_i - x_{i_1}\| \leq \|x_i - x_{i_2}\| \leq \cdots \leq \|x_i - x_{i_K}\|$, set $k = K$
1. let matrix $X_i^{(k)}$ consist of the vector $x_{i_j}(1 \leq j \leq k)$,
and $\bar{X}_i^{(k)} = X_i^{(k)}(I - (1/k)ee^T)$[1]
2. compute eigenvalues $\lambda_l(0 < l \leq D)$ of covariance matrix of $\bar{X}_i^{(k)} \bar{X}_i^{(k)T}$
3. if $\frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{D} \lambda_i} \leq \eta$, let $k = k - 1$, go to step 1;
else let $\mathcal{N}(x_i) = \{x_{i_j} : 1 \leq j \leq k\}$, return.

Fig.1. DNS method with contraction strategy

Based on PCA linearity measurement of finite data points set and the above two principles, there are two strategies for us to dynamically construct neighborhoods: contraction and expansion. For contraction strategy, we first construct neighborhood for each point with a large neighborhood size, then iteratively delete some data points until the rest of data points are in a d-dimensional linear subspace according to a given tolerable threshold η. For expansion strategy, we

[1] e is a column vector of all 1's with suitable dimensionality.

first construct neighborhood of each point with a small neighborhood size, then expand each neighborhood via adding some new points into it while preserving linearity. Fig.1 and fig.2 describes the pseudo-code of our dynamic neighborhood selection method with contraction strategy and expansion strategy respectively.

There is an issue that we should emphasize for expansion strategy. In the iterative process, as a matter of factor, there is no need to recompute eigenvalues of covariance matrix of sample points $\mathcal{N}(x_i) \cup x_{i_j}^{(k)}$, which is time-consuming and costly, there are many Incremental updating techniques can be used, such as Incremental Principal Component Analysis(IPCA) and Incremental SVD techniques, which can quickly compute the eigenvalues and eigenvectors based on previous eigenvalues and eigenvectors when a new sample is arriving. In our implementation, LET-IPCA[18] is adopted.

DNS method with expansion strategy

input: high-dimensional data set X, the manifold dimension d,
the scope for candidate neighbors m, the tolerable threshold η.
output: the neighborhood of each point $\mathcal{N}(x_i)$.

Initialize: for each x_i, add its $d+1$ nearest neighbors to $\mathcal{N}(x_i)$
Expansion: for each $\mathcal{N}(x_i)$
1. $\forall x_{i_j} \in \mathcal{N}(x_i)$ that has not been processed
2. $\cdots$ for k-th neighbor $x_{i_j}^{(k)} (k = 1, 2, \cdots, m)$ of x_{i_j}
3. $\cdots\cdots$ if $x_{i_j}^{(k)} \notin \mathcal{N}(x_i)$
4. $\cdots\cdots\cdots$ compute eigenvalues $\lambda_l (0 < l \leq D)$ of covariance matrix of samples $\mathcal{N}(x_i) \cup x_{i_j}^{(k)}$
5. $\cdots\cdots\cdots$ if $\frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{D} \lambda_i} \geq \eta$, add $x_{i_j}^{(k)}$ into $\mathcal{N}(x_i)$,
6. $\cdots x_{i_j}$ has been processed.

Fig.2. DNS method with expansion strategy

4 Constructing Neighborhood Graph for Isometric Embedding

In ISOMAP algorithm the geodesic distance is estimated by the shortest path distance in the NG. And in this NG there is an edge between point x_i and x_j if and only if $x_j \in \mathcal{N}(x_i)$ or $x_i \in \mathcal{N}(x_j)$, the length w_{ij} of the edge (x_i, x_j) is the Euclidean distance $d_E(x_i, x_j)$ between x_i and x_j in the input space. Note that each neighborhood constructed by our DNS method lies on or close to a d-dimensional linear subspace, for any two points in the same neighborhood, even though there is no edge between them(no point is in the neighborhood of the other), the direct Euclidean distance between them in input space is a more accurate geodesic distance estimate than others because of the linearity of the neighborhood. See an illustration example in fig.3.

Therefore we improve the NG construction method used in ISOMAP as follows: connect any two points in the same neighborhood. Then the constructed

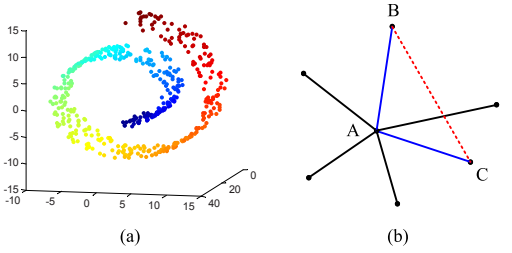

Fig. 3. An example to illustrate the superiority of the improved NG construction method. (a). 500 data points sampled from 2-dimensinal Swiss Roll manifold. (b). a representative neighborhood constructed by DNS when dealing with the 500 sampled data points in panel (a). All the points in panel b are in the neighborhood of point A. Suppose point B and C are not in the neighborhood of each other, and the shortest path between them is B-A-C. In original ISOMAP algorithm the estimation of the geometric distance between B and C is $d_E(B,A) + d_E(A,C)$, however, in the case that all points are in a plane, obviously $d_E(B,C)$ is a direct and still a more accurate estimation than $d_E(B,A) + d_E(A,C)$. Then we add a new edge connecting B and C in the NG.

NG is used to estimate the geodesic distance as the way in ISOMAP. As a matter of fact, if the entire input data set is in a d-dimensional linear subspace, then all the data points are in one neighborhood, then the nonlinear dimensionality reduction degenerates into linear method, such as ISOMAP, which degenerates into MDS[2].

5 Experiment Results

We conducted many experiments on both synthetic data set and real world data sets, all the experimental results show that our DNS method can dynamically construct neighborhood according to the manifold curvature and significantly improve the performance of most manifold learning algorithms such as ISOMAP and LLE. Moreover our method is not sensitive to the parameters K and m as long as they are comparatively large numbers, and the tolerable threshold η can be easily specified manually. In the following, we report results of our method with expansion strategy, and our experiments show that for contraction strategy it can also produce the same results. All the experiments showed below we set $m = 12$, $\eta = 0.95$ if they are not specified. And we call Isomap and LLE algorithm used our DNS method DNS-Isomap and DNS-LLE respectively.

5.1 Z-Shape Curve Manifold

First, we conduct experiments on data set uniformly sampled from a Z-shape curve manifold. The fig.4 shows the neighborhoods of two representative points constructed by two algorithms on the manifold. One point A is in the region

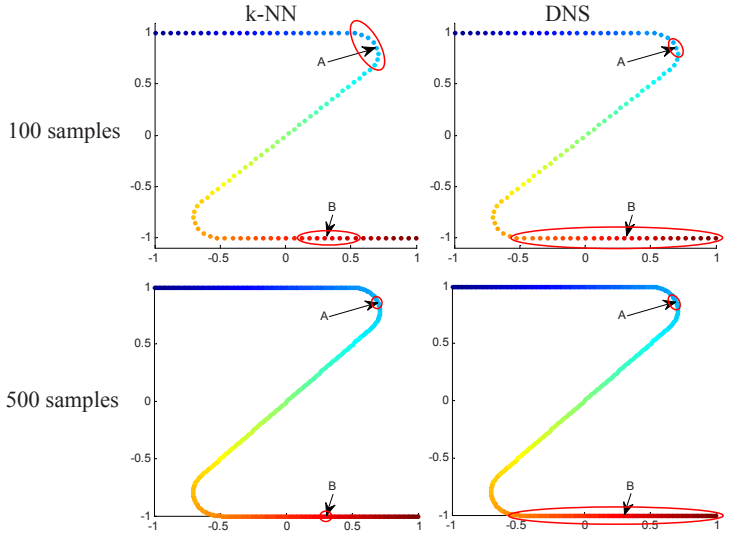

Fig. 4. The neighborhoods of A and B constructed by k-NN algorithm($k = 7$) and DNS. All the points in a red ellipse make up of a neighborhood respectively.

where the manifold curvature is very large; the other point B is in the region where the curvature is very small.

Our DNS method can dynamically construct the neighborhood according to the curvature of the manifold, this can be seen from the A's and B's neighborhood constructed by our method. The curvature of the manifold at point A is very large, therefore the number of points in A's neighborhood constructed by DNS is small whereas B's neighborhood contains much more points because of the small curvature of the manifold at B. Comparing the neighborhoods constructed by k-NN algorithm, we can see that k-NN has no such an ability.

Moreover, comparing the neighborhoods constructed under different sample size, A's neighborhoods constructed by our method always cover the same region on the manifold in different sample size, the same is point B. However, region covered by neighborhood constructed by k-NN varies as the sample size. This phenomenon implies that DNS constructs neighborhoods only according to the intrinsic structure of the manifold(curvatures in different region) and this construct process is little influenced by the sample size. We think this property is just what a superior neighborhood selection algorithm should have. However, the k-NN has no such a property either.

Then we apply LLE and DNS-LLE on 100 uniformly sampled data points, the unfolding results of these two algorithms and the real unfolding are shown on fig.5. One can see that data points in the unfolding of LLE are no longer uniformly distributed, and the unfolding result of DNS-LLE algorithm is nearly the same as the real unfolding of the sampled data points, this demonstrates that our DNS algorithm can improve the performance of LLE significantly.

Fig. 5. The unfolding results of LLE and DNS-LLE and the real unfolding of 100 data points uniformly sampled from z-shape curve

5.2 Swiss Roll Data Set

We run Isomap and DNS-isomap algorithms on 1000 data points randomly respectively sampled from Swiss Roll manifold and Swiss Roll with hole manifold. The 2-D embeddings are shown in fig.6. From the result we can see that DNS-isomap algorithm can produce more faithful embedding than Isomap algorithm. In particular, for swiss roll with hole manifold, Isomap amplifies the hole, while DNS-isomap can preserve the shape of the hole more faithfully.

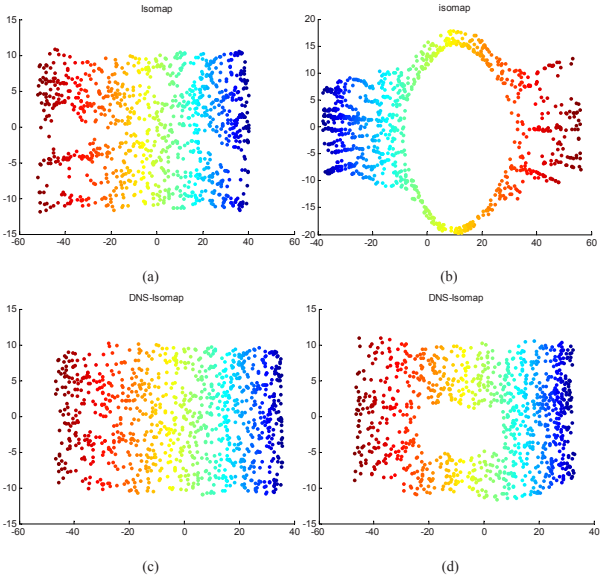

Fig. 6. Results of algorithm run on 1000 samples

We exploit the residual variance proposed in paper[3] and geodesic estimate error to quantitatively evaluate the performance of our method. Denote the true geodesic distance matrix by D_g and its estimate by $\hat{D}_g$, then the geodesic estimate error can be computed by the following formula:

$$error = \frac{\|D_g - \hat{D}_g\|_F}{n^2}$$

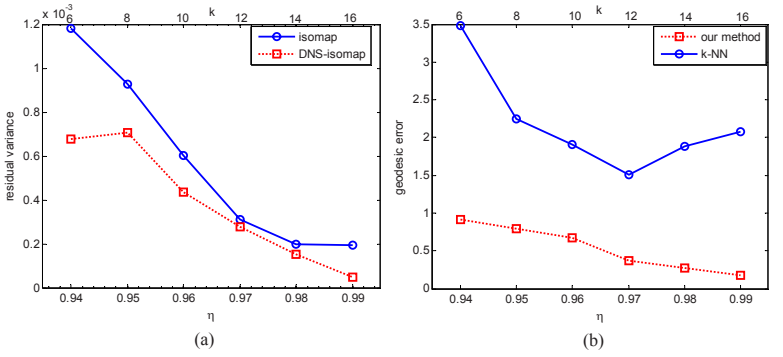

Fig. 7. Residual variance and geodesic estimate error of two algorithms under different parameters on 1000 data points randomly sampled from Swiss Roll manifold

where $\|\cdot\|_F$ is the Frobenius-norm of matrix and n is the number of examples. In our experiment, the Swiss Roll data set is generated by the following matlab code:

```
t=3*pi/2*(1+2*rand(n,1));h=21*rand(n,1);
X=[t.*cos(t) h t.*sin(t)];
```

So for any two points x_i and x_j, the true geodesic distance between them on data manifold can be calculated as follows:

$$d_g(x_i, x_j) = \sqrt{(h_i - h_j)^2 + \frac{1}{4}\left(t_i\sqrt{t_i^2+1} + \ln\left|t_i + \sqrt{t_i^2+1}\right| - t_j\sqrt{t_j^2+1} - \ln\left|t_j + \sqrt{t_j^2+1}\right|\right)^2}$$

then we can easily obtain the true geodesic distance matrix D_g. Fig.7 plots the residual variance and geodesic estimate error of Isomap and DNS-isomap under different parameters k and η when they obtain the 2-D embedding of 1000 data points randomly sampled from Swiss roll manifold. One can see that the residual variance and geodesic estimate error DNS-isomap are smaller than those of Isomap algorithm. So we can conclude that our DNS method and the improved NG construction method can significantly improve the performance of Isomap algorithm.

5.3 Rendered Face Data Set

To further evaluate the performance of our DNS method on real world data set, we use Isomap algorithm and DNS-isomap to obtain 3-D embedding of Rendered face data set[2]. This data set consists of 698 face images with 64∗64 pixels collected under different poses and light conditions. Each face image is represented as a 4096-dimensional vector. Denote the matrix that consists of

[2] http://isomap.stanford.edu

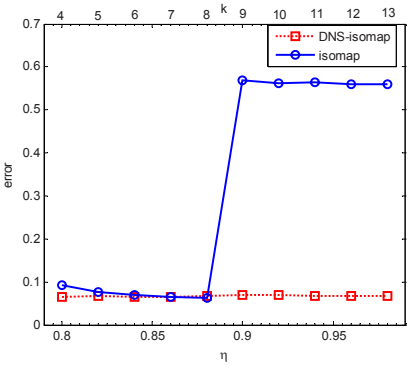

Fig. 8. Reconstruction error of two algorithms on Rendered face data set under different parameters

light parameters and poses parameters by P, let $\bar{P} = P - (1/n)Pee^T$, then we use the following relative reconstruction error to evaluate the performance:

$$error = \frac{\min\limits_{L \in R^{3\times 3}} \|\bar{P} - LY\|_F}{\|\bar{P}\|_F} \tag{5}$$

where Y is the embedding coordinates obtained by algorithm. The less this relative reconstruction error is, the better the algorithm recovers the intrinsic parameters of this face data set. Fig.8 plots the relative reconstruction error of the 3-D coordinates computed by isomap and DNS-isomap under different parameters k and η. One can clearly see that the relative reconstruct error of Isomap algorithm is sensitive to the parameter k, and DNS-isomap can get smaller relative reconstruction error, and the relative reconstruction error of DNS-isomap is not sensitive to the parameter η. This all demonstrates the superiority of our DNS method and improved NG construction method.

6 Conclusion

This paper concentrates on the neighborhood selection problem in manifold learning. Existing neighborhood selection algorithms are not so satisfactory. We propose an dynamic neighborhood selection algorithm based on local linearity and improves the NG construction method originally used in ISOMAP algorithm. Our method adopts local linearity as the criterion of neighborhood. So it can dynamically construct the neighborhood of each point only according to data manifold curvature at that point. Meanwhile, the parameters can be specified easily because of their independence of the concrete input data set, such as the size of the input data. Experiments on both synthetic data set and real world data set demonstrate effectiveness of our method. There are, however, still some problems we have to solve in our DNS method. DNS adopts the PCA to measure the linearity of finite data set, so sometimes over fitting can be happened. In the future, we will try to solve the over fitting problem.

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A Consistency-Constrained Feature Selection Algorithm with the Steepest Descent Method

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Abstract. This paper proposes a new consistency-based feature selection algorithm, which presents a new balance to the fundamental tradeoff between the quality of outputs of feature selection algorithms and their efficiency. Consistency represents the extent of corrective relevance of features to classification, and hence, consistency-based feature selection algorithms such as INTERACT, LCC and CCC can select relevant features more correctly by taking interaction among features into account. INTERACT and LCC are fast by employing the linear search strategy. By contrast, CCC is slow, since it is based on the complete search strategy, but can output feature subsets of higher quality. The algorithm that we propose in this paper, on the other hand, takes the steepest descent method as the search strategy. Consequently, it can find better solutions than INTERACT and LCC, and simultaneously restrains the increase in computational complexity within a reasonable level: it evaluates $(|\mathcal{F}|+|\tilde{\mathcal{F}}|)(|\mathcal{F}|-|\tilde{\mathcal{F}}|+1)/2$ feature subsets to output $\tilde{\mathcal{F}}$. We prove effectiveness of the new algorithm through experiments.

1 Introduction

Feature selection plays an indispensable role in preprocessing high dimensional datasets of various data mining problems. In machine learning problems, data are represented as vectors $(f_1, \ldots, f_L, c)$: f_i's are values for features F_i's, and c is a class label. The objective of classifier algorithms is to guess a class label c given a feature vector $(f_1, \ldots, f_L)$, and large L definitely does harm to the performance of the classifier algorithms in terms of accuracy and efficiency. Feature selection, therefore, aims to reduce L by eliminating those features that are irrelevant to classification. In a broad sense, feature selection algorithms take either the *filter* approach or the *wrapper* approach. The filter approach takes advantage of statistical properties intrinsic to datasets, and aims to extract relevant features that are effective to arbitrary classifier algorithms. By contrast, the wrapper approach aims to select relevant features so as to improve performance of particular focused classifier algorithms, by involving the classifier algorithms in the selection process. In this paper, we focus on the filter approach.

Many filter-type feature selection algorithms in the literature evaluate relevance to classification per individual feature, and select features with higher evaluation. For the evaluation, they take advantage of information-theoretic

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measures such as *mutual information* $\mathrm{I}(\mathrm{F};\mathrm{C})$ ([1]) and *symmetric uncertainty* $\mathrm{SU}(\mathrm{F};\mathrm{C})$ ([2,3,4]).

$$\mathrm{I}(\mathrm{F};\mathrm{C}) = \sum_{f,c} \Pr[\mathrm{F}=f,\mathrm{C}=c] \ln \frac{\Pr[\mathrm{F}=f,\mathrm{C}=c]}{\Pr[\mathrm{F}=f]\cdot\Pr[\mathrm{C}=c]}$$

$$\mathrm{SU}(\mathrm{F};\mathrm{C}) = 2\cdot\frac{\mathrm{I}(\mathrm{F};\mathrm{C})}{\mathrm{H}(\mathrm{F})+\mathrm{H}(\mathrm{C})}$$

This method, however, ignores interaction among relevant features.

For example, let $\mathrm{F}_1, \mathrm{F}_2, \mathrm{G}_1$ and G_2 be binary features such that $\Pr[\mathrm{F}_1=f_1, \mathrm{F}_2=f_2, \mathrm{G}_1=g_1, \mathrm{G}_2=g_2]$ for $f_1, f_2, g_1, g_2 \in \{0,1\}$ are defined as depicted in the right. When we let $\mathrm{C} = \mathrm{F}_1 \oplus \mathrm{F}_2$, $\mathrm{I}(\mathrm{F}_i;\mathrm{C}) = \mathrm{SU}(\mathrm{F}_i;\mathrm{C}) = 0$ hold. Thus, F_1 and F_2 interact each other to determine C, but each has no relevance to C. On the other hand, the relevance of G_i to C is calculated as

	g_1	0	0	1	1	
f_1	$f_2 \backslash g_2$	0	1	0	1	Sum
0	0	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{4}$	$\frac{1}{4}$
0	1	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{4}$	$\frac{1}{4}$
1	0	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{4}$	$\frac{1}{4}$
1	1	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}+\frac{\varepsilon}{12}$	$\frac{1}{16}-\frac{\varepsilon}{4}$	$\frac{1}{4}$
C = 0		$\frac{1}{8}+\frac{\varepsilon}{6}$	$\frac{1}{8}+\frac{\varepsilon}{6}$	$\frac{1}{8}+\frac{\varepsilon}{6}$	$\frac{1}{8}-\frac{\varepsilon}{2}$	$\frac{1}{2}$
C = 1		$\frac{1}{8}-\frac{\varepsilon}{6}$	$\frac{1}{8}-\frac{\varepsilon}{6}$	$\frac{1}{8}-\frac{\varepsilon}{6}$	$\frac{1}{8}+\frac{\varepsilon}{2}$	$\frac{1}{2}$
Sum		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

$$\mathrm{I}(\mathrm{G}_i;\mathrm{C}) = \frac{3+4\varepsilon}{6}\cdot\ln\frac{3+4\varepsilon}{3} + \frac{3-4\varepsilon}{6}\cdot\ln\frac{3-4\varepsilon}{3} > 0,$$

$$\mathrm{SU}(\mathrm{G}_i;\mathrm{C}) = 2\cdot\frac{\mathrm{I}(\mathrm{G}_i;\mathrm{C})}{\mathrm{H}(\mathrm{G}_i)+\mathrm{H}(\mathrm{C})} = \frac{\mathrm{I}(\mathrm{G}_i;\mathrm{C})}{\ln 2} > 0.$$

Hence, those feature selection algorithms that evaluate relevance per individual feature definitely select $(\mathrm{G}_1, \mathrm{G}_2)$ rather than $(\mathrm{F}_1, \mathrm{F}_2)$, and the predictive accuracy of a classifier receiving $(\mathrm{G}_1, \mathrm{G}_2)$ cannot exceed $3\cdot\left(\frac{1}{8}+\frac{\varepsilon}{6}\right)+\left(\frac{1}{8}+\frac{\varepsilon}{2}\right) = \frac{1}{2}+\varepsilon$.

The consistency-based approach is a solution to this problem (*e.g.* [5]). Let $\mathcal{E}$ be a finite set of *samples* (*training data, examples*) with respect to a feature set $\mathcal{F} = \{\mathrm{F}_1, \ldots, \mathrm{F}_L\}$ and a class variable C.

Definition 1. *A subset $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$ is said to be* $\mathcal{E}$-consistent*, when $c = c'$ holds for all $(f_1, \ldots, f_L, c), (f'_1, \ldots, f'_L, c') \in \mathcal{E}$ such that $f_i = f'_i$ for $\forall \mathrm{F}_i \in \tilde{\mathcal{F}}$. We may simply say that $\tilde{\mathcal{F}}$ is consistent to* C *or* $\tilde{\mathcal{F}}$ determines C.

To the best of our knowledge, FOCUS ([6]) is the first instance of the consistency-based filter algorithm. On input of $\mathcal{F}$ and $\mathcal{E}$, FOCUS selects the smallest $\mathcal{E}$-consistent subset $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$. When applied to the aforesaid example, FOCUS correctly selects $\{\mathrm{F}_1, \mathrm{F}_2\}$. A problem of FOCUS is that, if the entire feature set $\mathcal{F}$ is not $\mathcal{E}$-consistent due to noises in $\mathcal{E}$, FOCUS cannot help failing. Zhao and Liu ([7]) approached this problem by proposing *inconsistency rate*, a measure to evaluate the extent to which $\tilde{\mathcal{F}}$ is consistent to C, and INTERACT, a feature selection algorithm that evaluates the inconsistency rate.

Shin and Xu ([8]) have revealed that INTERACT has a theoretical deficiency in its design, and have proved it through experiments with multiple synthetic and real datasets. Moreover, in order to fix the deficiency of INTERACT, they have proposed two new algorithms, referred to as the *linear consistency-constrained* (LCC) algorithm and the *complete consistency-constrained* (CCC) algorithm. Table 1 briefly explains LCC and CCC, where μ denotes an arbitrary consistency measure such that it meets the conditions of determinacy and monotonicity ($\mu(\mathcal{F};\mathrm{C})$ represents the measurement of consistency of a feature set $\mathcal{F}$ to a class variable C according to μ):

Determinacy: $\mu(\mathcal{F};\mathrm{C}) = 0$, if, and only if, $\mathcal{F}$ determines C; and
Monotonicity: $\mu(\mathcal{F};\mathrm{C}) \leq \mu(\mathcal{G};\mathrm{C})$ for $\mathcal{F} \supsetneqq \mathcal{G}$.

Intuitively speaking, μ is defined so that, the more consistent to C is $\mathcal{F}$, the smaller is $\mu(\mathcal{F};\mathrm{C})$. On input of $\mathcal{F}$ and $\delta > 0$, LCC and CCC evaluate $\mu(\tilde{\mathcal{F}};\mathrm{C})$ for $\tilde{\mathcal{F}}$ in their search range, and finally output $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$ with $\mu(\tilde{\mathcal{F}};\mathrm{C}) \leq \delta$.

Table 1. LCC and CCC

Algorithm	Search	Evaluation of μ	Properties of the output $\tilde{\mathcal{F}}$
LCC	Linear	$\mid \mathcal{F} \mid$ times	$\tilde{\mathcal{F}}$ meets $\mu(\tilde{\mathcal{F}};\mathrm{C}) \leq \delta$, and is minimal in the sense $\mu(\mathcal{G};\mathrm{C}) > \delta$ for any $\mathcal{G} \subsetneqq \tilde{\mathcal{F}}$.
CCC	Complete	$\leq 2^{\mid \mathcal{F} \mid}$ times	$\tilde{\mathcal{F}}$ is the smallest in size of all of the feature subsets $\mathcal{G} \subseteqq \mathcal{F}$ such that $\mu(\mathcal{G};\mathrm{C}) \leq \delta$.

In general, the outputs $\tilde{\mathcal{F}}$ of a feature selection algorithm are evaluated by their size $\mid \tilde{\mathcal{F}} \mid$. Hence, CCC has more chances to output better results than LCC. This is due to the difference in their search strategies: CCC employs the complete search method, and hence, the search range is the entire space of the subsets of $\mathcal{F}$, while LCC evaluates only $\mid \mathcal{F} \mid$ subsets. On the other side of the coin, this obviously implies that LCC is superior to CCC in terms of efficiency. In fact, Shin and Xu ([8]) reported that CCC had failed to return answers for larger datasets in their experiments. Thus, there is a fundamental tradeoff between the quality of outputs of feature selection algorithms and their efficiency.

In this regard, this paper aims to propose a new feature selection algorithm, which presents a new balance to the aforementioned tradeoff. This paper is organized as follows.

1. Section 2 is devoted to introduction of the works in the literature relating to our work. We first see the notion of inconsistency rate, and show that, when generalized, it is equivalent to the well-known notion of the Bayes risk. In addition, we derive two more measures of consistency from well-known information theoretic indices. Finally, we review three consistency-based feature selection algorithms INTERACT, LCC and CCC.
2. In Section 3, we propose a new feature selection algorithm, which employs the method of steepest descent.
3. In Section 4, we prove effectiveness of our algorithm through experiments.

2 Consistency Measures and Consistency-Based Algorithms

2.1 Inconsistency Rate and Bayes Risk

Zhao and Liu ([7]) introduced *inconsistency rate* for finite datasets as follows.

Let $\mathcal{E}$ be a finite dataset with a feature set $\mathcal{F} = \{\mathrm{F}_1, \ldots, \mathrm{F}_L\}$ and a class variable C. First, an *inconsistent-instances set* $\tilde{\mathcal{E}} \subseteqq \mathcal{E}$ is defined so that, if $(f_1, \ldots, f_L, c)$ and $(f'_1, \ldots f'_L, c')$ are in $\tilde{\mathcal{E}}$, $f_i = f'_i$ holds for $i = 1, \ldots, L$ ($c = c'$ is not required). Secondly, an inconsistent-instances set $\tilde{\mathcal{E}}$ is partitioned into subsets $S_1, \ldots, S_n$ by the class labels $c_1, \ldots, c_n$. When the size $\mid S \mid$ of a set S is Then, the *inconsistency count* of $\tilde{\mathcal{E}}$ is defined by

$$inconsistencyCount(\tilde{\mathcal{E}}) = \mid \tilde{\mathcal{E}} \mid - \max_{1 \leq i \leq n} \mid S_i \mid .$$

Finally, letting $\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_p$ be the decomposition of $\mathcal{E}$ into maximal inconsistency-instances sets, $\mathrm{ICR}(\mathcal{E})$ is defined by

$$\mathrm{ICR}(\mathcal{E}) = \frac{\sum_{1 \leq i \leq p} inconsistencyCount(\tilde{\mathcal{E}}_i)}{\mid \mathcal{E} \mid}.$$

The definition can be naturally generalized so that it applies to arbitrary probability distributions. When $\mathcal{F}^*$ is the random variable that represents the joint distribution of $\mathrm{F}_1, \ldots, \mathrm{F}_L$, we define

$$\mathrm{ICR}(\mathcal{F}; \mathrm{C}) = 1 - \sum_{f \in \mathsf{R}(\mathcal{F}^*)} \max_{c \in \mathsf{R}(\mathrm{C})} \Pr[\mathcal{F}^* = f, \mathrm{C} = c],$$

where $\mathsf{R}(\mathcal{F}^*)$ and $\mathsf{R}(\mathrm{C})$ denote the ranges of $\mathcal{F}^*$ and the class variable C. It is obvious that, when calculated over a finite $\mathcal{E}$, $\mathrm{ICR}(\mathcal{F}; \mathrm{C})$ is identical to $\mathrm{ICR}(\mathcal{E})$, and $\mathrm{ICR}(\mathcal{F}; \mathrm{C})$ is equivalent to the notion of Bayes risk. In this paper, however, we use the term "inconsistency rate" in order to emphasize its application to the feature selection problem. Inconsistency rate has the following properties.

Proposition 1.

(1) $\mathrm{ICR}(\mathcal{F}; \mathrm{C}) = 0$, *if, and only if,* $\mathcal{F}$ *determines* C.
(2) $\mathrm{ICR}(\mathcal{F}; \mathrm{C}) \leq \mathrm{ICR}(\mathcal{G}; \mathrm{C})$, *if* $\mathcal{F} \supset \mathcal{G}$.
(3) $\mathrm{ICR}(\mathcal{F}; \mathrm{C}) \leq \frac{n-1}{n}$, *if* C *takes* n *class labels.*

2.2 Other Consistency Measures

Shin and Xu ([8]) have introduced two measures of consistency of feature sets, which meet the determinacy and monotonicity conditions (Section 1). The measures are derived from two well-known information theoretic indices, that is, mutual information and symmetric uncertainty, as follows.

$$\mathrm{H}(\mathrm{C} \mid \mathcal{F}^*) = \mathrm{H}(\mathrm{C}) - \mathrm{I}(\mathcal{F}^*; \mathrm{C}),$$

$$\overline{\mathrm{SU}}(\mathcal{F}; \mathrm{C}) = \frac{2 \cdot \mathrm{H}(\mathrm{C})}{\mathrm{H}(\mathcal{F}^*) + \mathrm{H}(\mathrm{C})} - \mathrm{SU}(\mathcal{F}^*; \mathrm{C}) \frac{2 \cdot \mathrm{H}(\mathrm{C} \mid \mathcal{F}^*)}{\mathrm{H}(\mathcal{F}^*) + \mathrm{H}(\mathrm{C})}$$

LCC and CCC are designed so that they can evaluate any given consistency measure $\mu(\mathcal{F}; \mathrm{C})$, as far as it meets the determinacy and monotonicity conditions. Also, INTERACT can be generalized so that it evaluates consistency of feature sets by means of $\mu(\mathcal{F}; \mathrm{C})$, since it relies on only the determinacy and monotonicity properties of inconsistency rate.

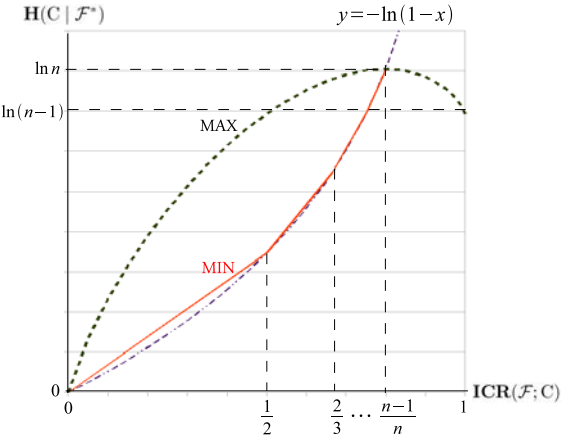

Fig. 1. $\mathrm{ICR}(\mathcal{F}; \mathrm{C})$ and $\mathrm{H}(\mathrm{C} \mid \mathcal{F}^*)$

Theorem 1 exhibits quantitative correlation between $\mathrm{H}(\mathrm{C} \mid \mathcal{F}^*)$ and $\mathrm{ICR}(\mathcal{F}; \mathrm{C})$ (Fig. 1).

Theorem 1 ([8]). *Let $\mathcal{F}$ and C be a feature set and a class variable such that the ranges of $\mathcal{F}^*$ and C are $\mathsf{R}(\mathcal{F}^*) = \{f_1, \ldots, f_m\}$ and $\mathsf{R}(\mathrm{C}) = \{c_1, \ldots, c_n\}$ for $m \geq 2$ and $n \geq 2$. For r such that $\mathrm{ICR}(\mathcal{F}; \mathrm{C}) = 1 - r$, the following formulas give the maximum and the minimum of $\mathrm{H}(\mathrm{C} \mid \mathcal{F}^*)$.*

$$\max_{\mathrm{ICR}(\mathcal{F};\mathrm{C})=1-r} \mathrm{H}(\mathrm{C} \mid \mathcal{F}^*) = -r \ln r - (1-r) \ln \frac{1-r}{n-1}$$

$$\min_{\mathrm{ICR}(\mathcal{F};\mathrm{C})=1-r} \mathrm{H}(\mathrm{C} \mid \mathcal{F}^*) = \left(\left\lfloor \frac{1}{r} \right\rfloor + 1\right)\left(1 - r\left\lfloor \frac{1}{r} \right\rfloor\right) \ln\left(\left\lfloor \frac{1}{r} \right\rfloor + 1\right) + \left\lfloor \frac{1}{r} \right\rfloor \left(r\left(\left\lfloor \frac{1}{r} \right\rfloor + 1\right) - 1\right) \ln \left\lfloor \frac{1}{r} \right\rfloor$$

2.3 INTERACT

In designing INTERACT, Zhao and Liu ([7]) defined *consistency contribution* by $\mathrm{CC}(\mathrm{F}, \tilde{\mathcal{F}}) = \mathrm{ICR}(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C}) - \mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ to evaluate the contribution of an individual feature $\mathrm{F} \in \tilde{\mathcal{F}}$ to $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$. INTERACT sets $\tilde{\mathcal{F}} = \mathcal{F}$ as the initial value, examines a feature $\mathrm{F} \in \mathcal{F}$ one by one in the incremental order of $\mathrm{SU}(\mathrm{F}; \mathrm{C})$, and eliminates F from $\tilde{\mathcal{F}}$, if $\mathrm{CC}(\mathrm{F}, \tilde{\mathcal{F}}) \leq \delta$ for a given threshold δ (Fig. 2).

Shin and Xu ([8]) presented a problem of the design of INTERACT: Even if the threshold δ is set small, $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ for its output $\tilde{\mathcal{F}}$ can be large. Let $\tilde{\mathcal{F}}_0 \supsetneqq \tilde{\mathcal{F}}_1 \supsetneqq \cdots \supsetneqq \tilde{\mathcal{F}}_L$ be a history of the values of the variable $\tilde{\mathcal{F}}$ managed by

Algorithm: INTERACT ([7])
INPUT: A feature set $\mathcal{F}$, an example set $\mathcal{E}$, a threshold δ
OUTPUT: A feature subset $\tilde{\mathcal{F}}$
STEPS:
Let $\tilde{\mathcal{F}} = \mathcal{F}$.
Order the features F in $\tilde{\mathcal{F}}$ in incremental order of SU(F; C).
For each F $\in \tilde{\mathcal{F}}$ from the first to the end.
If CC(F, $\tilde{\mathcal{F}}$) $\leq \delta$, let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}} \setminus \{\mathrm{F}\}$.
End For.

Fig. 2. The algorithm of INTERACT

INTERACT. Although ICR($\tilde{\mathcal{F}}_{k+1}$; C) - ICR($\tilde{\mathcal{F}}_k$ C) $\leq \delta$, ICR($\tilde{\mathcal{F}}_L$; C) can be large even for small δ. In fact, Shin and Xu showed an example of a type of probability distribution, for which INTERACT can only output the empty set.

2.4 LCC and CCC

In order to fix the aforesaid deficiency of INTERACT, Shin and Xu ([8]) have introduced two feature selection algorithms, namely, Linear Consistency-Constrained (LCC) and Complete Consistency-Constrained (CCC) algorithms.

Both take a measurement function μ as an input, and outputs a feature subset $\tilde{\mathcal{F}}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$ for a threshold δ.

Algorithm: Linear CC (LCC) ([8])
INPUT: A measurement function μ, an ordered feature set $\mathcal{F}$, an example set $\mathcal{E}$, a threshold δ
OUTPUT: A minimal subset $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$.
STEPS:
Let $\tilde{\mathcal{F}} = \mathcal{F}$.
If $\mu(\tilde{\mathcal{F}}; \mathrm{C}) > \delta$, abort.
For each F $\in \mathcal{F}$ from the first to the end.
If $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$, let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}} \setminus \{\mathrm{F}\}$.
End For.

Fig. 3. The algorithm of LCC

LCC receives an ordered feature set $\mathcal{F}$ (*e.g.* ordered in the incremental order of SU(F; C)). It first sets $\tilde{\mathcal{F}} = \mathcal{F}$, examines F in the given order, and eliminates F from $\tilde{\mathcal{F}}$, if $\mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C}) \leq \delta$. The resultant $\tilde{\mathcal{F}}$ is minimal in the sense that no $\mathcal{G} \subsetneqq \tilde{\mathcal{F}}$ meets $\mu(\mathcal{G}; \mathrm{C}) \leq \delta$ (Fig. 3).

By contrast, CCC first sets the search range to the space of the singleton feature subsets (*i.e.*, $\{\tilde{\mathcal{F}} \subseteqq \mathcal{F} \mid\mid \tilde{\mathcal{F}} \mid= 1\}$), and expands the range by increasing the size of subsets until it finds $\tilde{\mathcal{F}}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$. Thus, the output $\tilde{\mathcal{F}}$ of CCC has the smallest size among $\mathcal{G} \subseteqq \mathcal{F}$ such that $\mu(\mathcal{G}; \mathrm{C}) \leq \delta$.

LCC evaluates $\mid \mathcal{F} \mid$ different subsets, while CCC does $2^{|\mathcal{F}|}$ subsets at most.

3 Steepest-Descent Consistency-Constrained Algorithm

In this section, we introduce a new algorithm for the consistency-constrained feature selection. We first introduce the notion of the consistency-constrained feature selection in Section 3.1, and present the algorithm in Section 3.2. The computational complexity of the algorithm is examined in Section 3.3.

3.1 Consistency-Constrained Feature Selection

The notion of the consistency-constrained feature selection is defined as follows.

A consistency-constrained feature selection algorithm takes a function μ to measure consistency of subsets of the entire feature sets $\mathcal{F}$ and a threshold δ as inputs, and outputs a feature subset $\tilde{\mathcal{F}}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$. An additional requirement for the output $\tilde{\mathcal{F}}$ is that it is minimal in the sense that $\mu(\mathcal{G}; \mathrm{C}) > \delta$ holds for any $\mathcal{G} \subsetneqq \tilde{\mathcal{F}}$. $\mu(\tilde{\mathcal{F}}; \mathrm{C})$ is evaluated over the finite dataset $\mathcal{E}$ given as input as well (Fig. 4).

Algorithm:	Consistency-Constrained (CC) Feature Selection
INPUT:	A consistency measure function μ, a feature set $\mathcal{F}$, an example set $\mathcal{E}$, a threshold δ
OUTPUT:	A minimal subset $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$.

Fig. 4. Consistency-Constrained Feature Selection

3.2 The Algorithm

The algorithm to be introduced here, namely, the *Steepest-Descent* Consistency-Constrained (SDCC) algorithm, aims at exhibiting a new balance of the fundamental tradeoff between the quality of outputs and the efficiency of execution.

Although CCC can return outputs of high quality, it may execute evaluation of $\mu(\tilde{\mathcal{F}}; \mathrm{C})$ in $2^{|\mathcal{F}|}$ times at worst. It is slow even on average, and, in fact, Shin and Xu ([8]) have reported that it couldn't find answers for large datasets in their experiments. Thus, we should view that LCC exhibits a current state-of-the-art balance to the tradeoff. LCC is indeed as fast as INTERACT, and shows better performance than INTERACT in terms of quality of outputs.

The quality of outputs of LCC is, however, still subject to improvement. The SDCC feature selection algorithm employs the steepest descent method to pursue a new balance to the tradeoff.

SDCC first sets the variable $\tilde{\mathcal{F}}$ to the entire feature set $\mathcal{F}$, and then iterates the step of eliminating a single feature F from $\tilde{\mathcal{F}}$ while $\mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C}) \leq \delta$. The last $\tilde{\mathcal{F}}$ is the output of SDCC. For each step of elimination, it selects F to be eliminated by

Algorithm:	Steepest-Descent CC (SDCC)
INPUT:	A consistency measure function μ, a feature set $\mathcal{F}$, an example set $\mathcal{E}$, a threshold δ
OUTPUT:	A minimal subset $\tilde{\mathcal{F}} \subseteqq \mathcal{F}$ such that $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$.
STEPS:	
	Let $\tilde{\mathcal{F}} = \mathcal{F}$.
	If $\mu(\tilde{\mathcal{F}}; \mathrm{C}) > \delta$, abort.
	Repeat
	Take $\mathrm{F} \in \tilde{\mathcal{F}}$ with the smallest $\delta' = \mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C})$.
	If $\delta' \leq \delta$, let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}} \setminus \{\mathrm{F}\}$, else break.
	End Repeat.

Fig. 5. The algorithm of SDCC

$$\mathrm{F} = \operatorname*{argmin}_{\mathrm{F}\in\tilde{\mathcal{F}}} \mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C}).$$

Fig. 5 specifies the algorithm. When we plot $\tilde{\mathcal{F}}$'s on a plain of coordinates so that the point $(\mu(\tilde{\mathcal{F}}; \mathrm{C}), \mid \tilde{\mathcal{F}} \mid)$ represents $\tilde{\mathcal{F}}$, the descending gradient from $\tilde{\mathcal{F}}$ to $\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}$ is $\frac{1}{\mu(\tilde{\mathcal{F}}\setminus\{\mathrm{F}\};\mathrm{C})-\mu(\tilde{\mathcal{F}};\mathrm{C})} \in (0, \infty]$ (by the monotonicity condition, we have $\mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}; \mathrm{C}) \geq \mu(\tilde{\mathcal{F}}; \mathrm{C})$), and hence, SDCC selects the steepest descent from the current solution $\tilde{\mathcal{F}}$ to the next one $\tilde{\mathcal{F}} \setminus \{\mathrm{F}\}$.

We explain the design of SDCC using an example. We assume that the entire feature set $\mathcal{F}$ consists of four features, that is, $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4$. In the top chart of Fig. 6, a node represents a feature subset of $\mathcal{F}$ (we have $2^4 = 16$ nodes in total), while a link between two feature subsets means that SDCC can move from the feature subset placed above to the other placed below by eliminating a single feature. Thus, starting from $\tilde{\mathcal{F}} = \mathcal{F}$, SDLL keeps traveling downward along links, while the visited feature subset $\tilde{\mathcal{F}}$ meet the condition of $\mu(\tilde{\mathcal{F}}; \mathrm{C}) \leq \delta$.

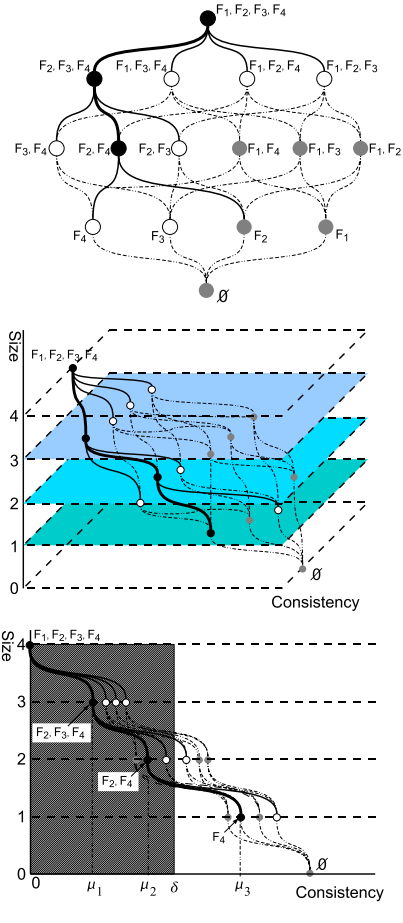

Fig. 6. Illustration of SDCC

Further, in the middle and bottom charts of Fig. 6, the graph of the top chart is mapped into the coordinate space whose x-, y- and z-axes represent the measurement $\mu(\tilde{\mathcal{F}}; \mathrm{C})$, the horizontal span of the graph in the above and the size $\mid \tilde{\mathcal{F}} \mid$, respectively. Also, the diagram just below represents its projection to the size-consistency plain.

We assume that $\tilde{\mathcal{F}}$ is consistent to C, (that is, $\mu(\mathcal{F}; \mathrm{C}) = 0$), and hence, $\mathcal{F}$ is plotted on the plain of $x = 0$. When invoked, SDCC stays at $\mathcal{F}$. Among the candidates of the next position, *i.e.* $\mathcal{F} \setminus \{\mathrm{F}_i\}$ for $i = 1, 2, 3, 4$, $\{\mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4\}$ exhibits the smallest consistency measurement $\mu_1 \leq \delta$, and equivalently, the corresponding descending gradient $\frac{1}{\mu_1}$ is the largest. Hence, SDCC moves to $\tilde{\mathcal{F}}_1 = \{\mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4\}$. Then, SDCC proceeds to evaluation of $\mu(\tilde{\mathcal{F}}_1 \setminus \{F_i\}; \mathrm{C})$ for $i = 2, 3, 4$, and determines to move to $\tilde{\mathcal{F}}_2 = \{\mathrm{F}_2, \mathrm{F}_4\}$ with the minimum measurement $\mu_2 \leq \delta$. The corresponding descending gradient of the move is $\frac{1}{\mu_2-\mu_1}$, which is the largest of the

calculated gradients. SDCC further evaluates $\mu(\tilde{\mathcal{F}}_2 \setminus \{\mathrm{F}_i\}; \mathrm{C})$ for $i = 2, 4$, but the minimum measurement μ_3 for $\{F_4\}$ exceeds δ. SDCC gives up to travel any more, and finally, outputs $\{\mathrm{F}_2, \mathrm{F}_4\}$.

In the diagrams, the black nodes and the thick lines represent the paths that SDCC has taken. On the other hand, the white nodes and the thin lines indicate the feature subsets whose consistency SDCC has evaluated. The other feature subsets are represented by the gray nodes and the dashed lines.

As depicted in the diagrams, and by the generic nature of the steepest step method, SDCC can be trapped by local optima, and does not necessarily reach the global optima.

3.3 Computational Complexity

Obviously, SDCC evaluates consistency of $\frac{|\mathcal{F}|(|\mathcal{F}|+1)}{2}$ feature subsets at most out of the possible $2^{|\mathcal{F}|}$ feature subsets of $\mathcal{F}$. More precisely, when SDCC has finally output $\tilde{\mathcal{F}}$, $\frac{(|\mathcal{F}|+|\tilde{\mathcal{F}}|)(|\mathcal{F}|-|\tilde{\mathcal{F}}|+1)}{2}$ feature subsets have been evaluated. This figure is greater than $|\ \mathcal{F}\ |$ for LCC, but is usually significantly smaller than $|\ \mathcal{F}\ | + {}_{|\mathcal{F}|}C_2 + \cdots + {}_{|\mathcal{F}|}C_{|\tilde{\mathcal{F}}|} + {}_{|\mathcal{F}|}C_{|\tilde{\mathcal{F}}|+1}$ for CCC.

4 Experimental Results

We prove effectiveness of the SDCC feature selection algorithm through experiments using the same datasets as used in [8].

In [8], the authors proved advantages of LCC over INTERACT using 40 synthetic datasets, each of which is of one of the four types described in Table 2, and four real datasets from the well-known UCI repository of machine learning databases ([9]) and the performance prediction challenge at IEEE WCCI ([10]).

Table 2. Types of synthetic datasets

Relevant features: F_i $(i = 1, \ldots, k)$
Irrelevant features: $\mathrm{G}_{i,j}$ $(i, j = 1, \ldots, \ell)$

Parameters	#1	#2	#3	#4
$k =$	5	5	5	10
$\ell =$	3	3	3	4
Size of dataset =	100	100	100	1000
$\Pr[\mathcal{F}^* = 0, \mathrm{C} = 0] =$	0.85	0.55	0.55	0.725
$\Pr[\mathcal{F}^* = 2^{i-1}, \mathrm{C} = 1] =$	0.03	0.09	$0.03i$	$0.005i$
$\Pr[\mathrm{G}_{i,j} = 0 \mid \mathrm{C} = 0] \approx$	$0.25i$	$0.25i$	$0.25i$	$0.2i$
$\Pr[\mathrm{G}_{i,j} = 0 \mid \mathrm{C} = 1] \approx$	$0.25j$	$0.25j$	$0.25j$	$0.2j$

Each type of synthetic datasets determines that an instance dataset shall include k relevant features F_i $(i = 1, \ldots, k)$, ℓ^2 irrelevant features $\mathrm{G}_{i,j}$ $(i, j = 1, \ldots, \ell)$ and a class variable C, which are all binary. The types are designed so as to meet the following conditions.

1. A feature subset $\mathcal{G}$ determines C, iff $\mathcal{G} \supseteqq \{\mathrm{F}_1, \ldots, \mathrm{F}_k\}$.
2. For any $\tilde{\mathcal{F}} \subsetneqq \{\mathrm{F}_1, \ldots, \mathrm{F}_k\}$, $\mu(\tilde{\mathcal{F}} \setminus \{\mathrm{F}_i\}; \mathrm{C}) - \mu(\tilde{\mathcal{F}}; \mathrm{C})$ is small.

Taking advantage of the first condition, we can evaluate quality of an output $\tilde{\mathcal{F}}$ of a feature selection algorithm with the following two measures.

Coverage rate R_C**:** The ratio of the number of $\mathrm{F}_i \in \tilde{\mathcal{F}}$ to k. Larger value indicates higher quality.

Positively false rate R_P**:** The ratio of the number of $\mathrm{G}_{i,j} \in \tilde{\mathcal{F}}$ to $\mid \tilde{\mathcal{F}} \mid$. Smaller value indicate higher quality.

On the other hand, the second condition makes the problem to find correct answers for the resultant datasets difficult. Hence, the datasets we used in the experiments are hard challenges to future selection algorithms.

Table 3 illustrates the real datasets that we used in the experiments. Two are small, and the others are relatively large. Different from the synthetic datasets, there exists no clear distinction between relevant and irrelevant features. Hence, neither R_C nor R_P can be used for evaluation of the experiments with the real datasets. In the experiments, we used the following measures instead of R_C and R_P.

Table 3. Real datasets

Dataset Name	Source	# of Examples	# of Features	# of Labels
Wine	[9]	178	13	3
Zoo	[9]	101	16	7
Kr-vs-Kp	[9]	3196	36	2
ADA	[10]	4147	48	2

Size $\mid \tilde{\mathcal{F}} \mid$**:** The number of the features in an output $\tilde{\mathcal{F}}$. Smaller size indicates higher quality.

Consistency measurement $\mu(\tilde{\mathcal{F}}; \mathrm{C})$**:** As the measure μ, we used $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$, $\mathrm{H}(\mathrm{C} \mid \tilde{\mathcal{F}}^*)$ and $\overline{\mathrm{SU}}(\tilde{\mathcal{F}}; \mathrm{C})$. Smaller value indicates better quality.

As the monotonicity condition implies, the measures of $\mid \tilde{\mathcal{F}} \mid$ and $\mu(\tilde{\mathcal{F}}; \mathrm{C})$ are in relation of a loose tradeoff. Nevertheless, if an algorithm exhibits better values for both of the measures than another, or if they are comparable in one of the measures, we can compare the algorithms according to the measures.

We examined three feature selection algorithms, namely LCC, CCC and SDCC (comparison between INTERACT and LCC is given in [8]) in combination with the consistency measures of $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$, $\mathrm{H}(\mathrm{C} \mid \tilde{\mathcal{F}}^*)$ and $\overline{\mathrm{SU}}(\tilde{\mathcal{F}}; \mathrm{C})$. We first remark the following, which we have found through the experiments.

1. CCC could be applied only to the smaller datasets of the synthetic datasets of Type 1 to 3 and the real datasets of Wine and Zoo. For these datasets, the quality of the outputs of CCC was comparable with that of SDCC.
2. There was observed no meaningful difference among the results of the experiments with $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$, $\mathrm{H}(\mathrm{C} \mid \tilde{\mathcal{F}}^*)$ and $\overline{\mathrm{SU}}(\tilde{\mathcal{F}}; \mathrm{C})$.
3. SDCC showed clear advantages over LCC (and therefore over INTERACT) for the synthetic datasets of Type 4 and the real datasets of Wine and ADA. For the other datasets, SDCC and LCC were comparable with each other.

In the remainder of this section, based on the aforementioned finding, we verify the advantages of SDCC over LCC by seeing the results of the experiments with the measure of ICR($\tilde{\mathcal{F}}$; C) and with the synthetic datasets of Type 4 and the real datasets of Wine and ADA. In the graphs below, the solid lines with the triangular dots represent SDCC, while the dashed lines with the square dots do LCC. Just for the reader's reference, the results of INTERACT are plotted in the same graphs by the dashed dotted lines and the diamond-shaped dots.

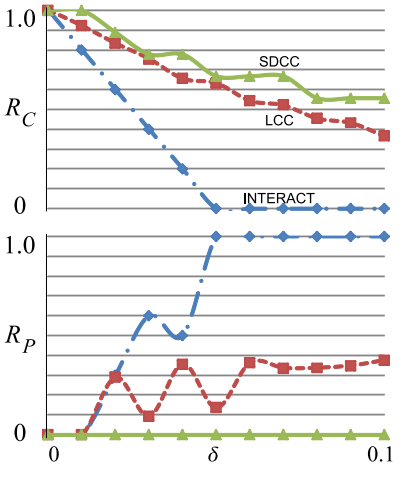

Fig. 7. The results for Type 4

Fig. 7 shows the results of the experiments with the 10 synthetic datasets of Type 4. We evaluated the R_C and R_P value changing the threshold δ from 0 to 0.1 with 0.01 increments in between for each dataset, and plotted the average over the 10 datasets.

The R_C curve for SDCC descends slightly slower than that for LCC, as the threshold δ increases. On the other hand, the R_P value for SDCC consistently remains 0 over the interval $[0, 0.1]$ of δ, while the value for LCC holistically increases. This means that SDCC and LCC dropped more relevant features as δ increases, but the number of the relevant features that SDCC dropped was consistently smaller than LCC. Furthermore, SDCC selected no irrelevant features at all, while the outputs of

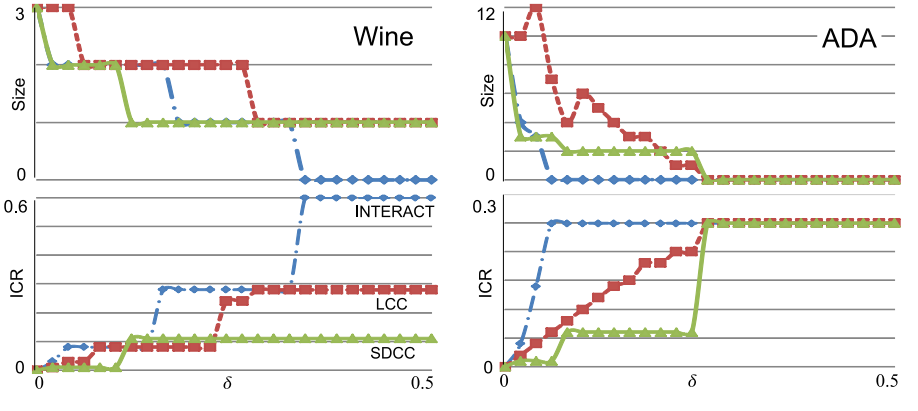

Fig. 8. Results for Wine and ADA

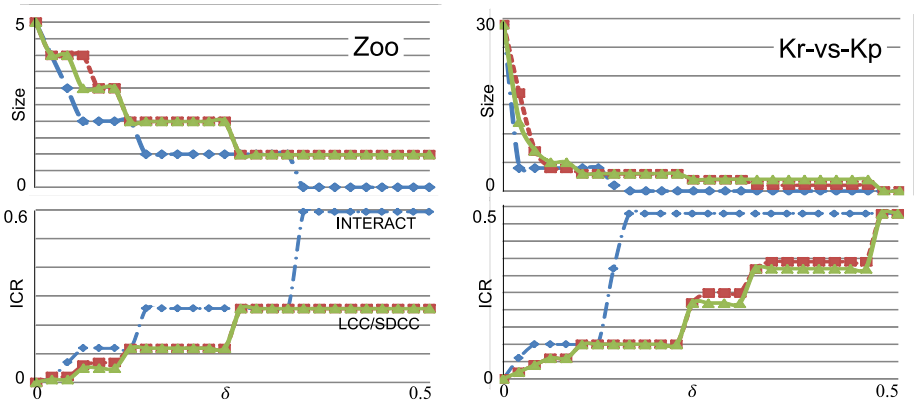

Fig. 9. Results for Zoo and Kr-vs-Kp

LCC included irrelevant features up to 38%. By contrast, INTERACT could output only irrelevant features for $\delta \geq 0.5$.

On the other hand, Figure 8 depicts the results of the experiments with Wine and ADA, which also exhibit advantages of SDCC over LCC.

For each examined δ with Wine, one of the following is observed: the size $|\ \tilde{\mathcal{F}}\ |$ is smaller for SDCC than for LCC, and the measurement $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ is comparable between SDCC and LCC; or $|\ \tilde{\mathcal{F}}\ |$ is the same for SDCC as for LCC, but $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ is smaller for SDCC than LCC.

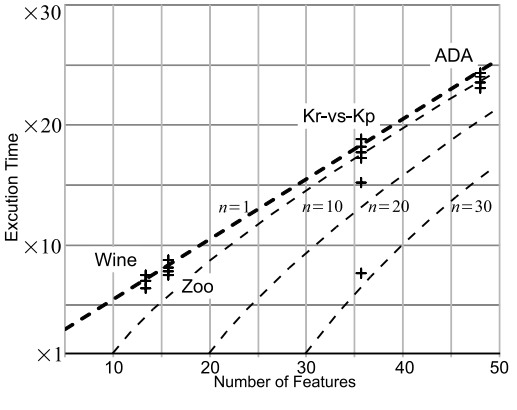

Fig. 10. Efficiency

For ADA, SDCC and LCC could not help outputting the empty set for $\delta \geq 0.26$, since $\mathrm{ICR}(\emptyset; \mathrm{C}) = 0.25 < 0.26$. For $\delta \leq 0.20$ both of the size $|\ \tilde{\mathcal{F}}\ |$ and the measurement $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ are smaller for SDCC than for LCC. In the case of $\delta = 0.22$ and 0.24, LCC and SDCC output a singleton feature set and a set with two features. Hence, we should understand that the outputs of LCC are better than those of SDCC, but the difference is not significant. By contrast, $\mathrm{ICR}(\tilde{\mathcal{F}}; \mathrm{C})$ is significantly smaller for SDCC than for LCC. Thus, we should conclude that the outputs of SDCC are better than those of LCC.

For the reader's reference, Figure 9 shows the results of the experiments with Zoo and Kr-vs-Kp. It can be observed that the results for SDCC and CCC were comparable with each other.

Finally, we see the experimental results on efficiency. Fig. 10 depicts comparison in the execution time between SDCC and LCC. The x-axis represents the total number of features $\mid \mathcal{F} \mid$ of the experimented datasets, and the y-axis represents the ratio of the execution time of SDCC to that of LCC for the same datasets.

In theory, LCC executes evaluation of the consistency measure μ in $\mid \mathcal{F} \mid$ times, while SDCC evaluates μ in $\frac{(|\mathcal{F}|+|\tilde{\mathcal{F}}|)(|\mathcal{F}|-|\tilde{\mathcal{F}}|+1)}{2}$ times for the output $\tilde{\mathcal{F}}$. The curves in Fig. 10 represent the theoretical estimation of the ratio, and are determined by

$$y = \frac{(x+n)(x-n+1)}{2x}$$

for $n = 1, 10, 20$ and 30. The $+$ dots in the graph show the results of the experiments with the real datasets of Wine, Zoo, Kr-vs-Kp and ADA. Since each dataset was experimented with in 26 times changing δ, more than one dots are plotted for each dataset. From the diagram, we see that the experimental results follow the theoretical estimation. This also indicates that the time for evaluating μ is dominant to the total execution time of SDCC and LCC.

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A Heuristic Algorithm for Attribute Reduction Based on Discernibility and Equivalence by Attributes

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Abstract. In this paper, we consider a heuristic method to partially calculate relative reducts with better evaluation by the evaluation criterion proposed by the authors. By considering discernibility and equivalence of elements with respect to values of condition attributes that appear in relative reducts, we introduce an evaluation criterion of condition attributes, and consider a heuristic method for calculating a relative reduct with better evaluation.

1 Introduction

Pawlak's rough set theory [7,9] provides a theoretical framework of data mining from categorical data. In this viewpoint, generating relative reducts and decision rules from a given decision table have been central topics, and there are various studies about this topics. Skowron and Rauszer [12] have proposed an algorithm to calculate all relative reducts by using the concept of the discernibility matrix. However, they have also proved that computational complexity of calculation of all relative reducts in the given decision table is NP-hard [12]. Thus, there have been many proposals of approximate algorithms to partially calculate relative reducts instead of calculating all relative reducts [1,2,3,10,13,15,16].

In this paper, we introduce an evaluation criterion of relative reducts based on discernibility and equivalence by condition attributes. Moreover, we also consider a heuristic method to calculate a relative reduct with better evaluation by the proposed evaluation criterion. Note that this paper is a revised and extended version of the authors' manuscripts [4,6].

2 Rough Sets

In this section, we review the rough set theory, in particular, decision tables, relative reducts, and discernibility matrices. Note that contents of this section are based on [5,11].

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2.1 Decision Tables and Lower and Upper Approximations

Generally, subjects of data analysis by rough sets are illustrated by decision tables. Formally, a decision table is the following quadruple:

$$DT = (U, C \cup D, V, \rho), \tag{1}$$

where U is a finite and non-empty set of elements, C and D are finite and non-empty sets of condition attributes and decision attributes such that $C \cap D = \emptyset$, respectively, V is a set of all values of attributes $a \in C \cup D$, and $\rho : U \times (C \cup D) \rightarrow V$ is a function which assigns a value $\rho(x, a) \in V$ at the attribute $a \in C \cup D$ to the element $x \in U$.

Classification of elements in decision tables are done based on indiscernibility relations. For any set of attributes $A \subseteq C \cup D$, the indiscernibility relations R_A is the following binary relation on U:

$$R_A = \{(x, y) \mid \rho(x, a) = \rho(y, a), \forall a \in A\}. \tag{2}$$

If a pair (x, y) is in R_A, then two elements x and y are indiscernible with respect to all attributes in A. It is well-known that any indiscernibility relation is an equivalence relation, and equivalence classes by an equivalence relation consists of a partition on the domain of the equivalence relation. In particular, the indiscernibility relation R_D based on the set of decision attributes D provides a partition $\mathcal{D} = \{D_1, \cdots, D_k\}$, and each element $D_i \in \mathcal{D}$ is called a decision class.

Classifying elements with respect to condition attributes provides approximation of decision classes. Formally, for any set $A \subseteq C$ of condition attributes and any decision class $D_i \in \mathcal{D}$, we let

$$\underline{A}(D_i) = \{x \in U \mid [x]_A \subseteq D_i\}, \tag{3}$$
$$\overline{A}(D_i) = \{x \in U \mid [x]_A \cap D_i \neq \emptyset\}, \tag{4}$$

where the set $[x]_A$ is the equivalence class of x by the indiscernibility relation R_A. The set $\underline{A}(D_i)$ and the set $\overline{A}(D_i)$ are called the lower approximation and the upper approximation of the decision class D_i with respect to A, respectively. Note that the lower approximation $\underline{A}(D_i)$ illustrates the set of elements which are correctly classified to the decision class D_i by checking all attributes in A.

Example 1. Table 1 illustrates an decision table we use in this paper, and consists of the following elements: $U = \{x_1, \cdots, x_6\}$, $C = \{c_1, \cdots, c_6\}$, $D = \{d\}$, $V = \{0, 1, 2\}$, and the function $\rho : U \times (C \cup D) \rightarrow V$ illustrates values of elements at attributes such that $\rho(x1, c1) = 1$. The set of decision attributes D provides the following three decision classes; $D_1 = \{x_1, x_2, x_5\}$, $D_2 = \{x_3, x_4\}$ and $D_3 = \{x_6\}$.

2.2 Relative Reducts

By checking values of all condition attributes, we can classify all discernible elements in a given decision table to those correct decision classes. However, not

Table 1. Decision table

U	c_1	c_2	c_3	c_4	c_5	c_6	d
x_1	1	1	1	1	1	2	1
x_2	2	2	1	1	1	2	1
x_3	2	3	2	1	2	1	2
x_4	2	2	2	2	2	1	2
x_5	2	2	3	1	1	2	1
x_6	1	2	1	1	2	2	3

all condition attributes may need to be checked in the sense that some condition attributes are essential to classify, and the other attributes are redundant. A minimal set of condition attributes to classify all discernible elements to correct decision classes is called a relative reduct of the decision table.

To introduce the concept of relative reducts, for any subset $X \subseteq C$ of condition attributes in a decision table DT, we let

$$POS_X(\mathcal{D}) = \bigcup_{D_i \in \mathcal{D}} \underline{X}(D_i). \tag{5}$$

The set $POS_X(\mathcal{D})$ is called the positive region of $\mathcal{D}$ by X. All elements $x \in POS_X(\mathcal{D})$ are classified to correct decision classes by checking all attributes in X. In particular, the set $POS_C(\mathcal{D})$ is the set of all discernible elements in DT.

Here, we define relative reducts formally. A set $A \subseteq C$ is called a relative reduct of the decision table DT if the set A satisfies the following conditions:

1. $\mathrm{POS}_A(\mathcal{D}) = \mathrm{POS}_C(\mathcal{D})$.
2. $\mathrm{POS}_B(\mathcal{D}) \neq \mathrm{POS}_C(\mathcal{D})$ for any proper subset $B \subset A$.

Note that, in general, there are plural relative reducts in a decision table. Common part of all relative reducts are called the core of the decision table.

For example, there are the following three relative reducts in Table 1: $\{c_3, c_5\}$, $\{c_5, c_6\}$, and $\{c_2, c_4, c_5\}$. The condition attribute c_5 appears in all of the relative reducts in Table 1, and therefore the core of Table 1 is $\{c_5\}$.

2.3 Discernibility Matrix

The discernibility matrix is one of the most popular methods to calculate all relative reducts in the decision table. Let DT be a decision table with $|U|$ elements, where $|U|$ is the cardinality of U. The discernibility matrix DM of DT is a symmetric $|U| \times |U|$ matrix whose element at i-th row and j-th column is the following set of condition attributes to discern between two elements x_i and x_j:

$$\delta_{ij} = \begin{cases} \{a \in C \mid \rho(x_i, a) \neq \rho(x_j, a)\}, & \exists d \in D \text{ s. t. } \rho(x_i, d) \neq \rho(x_j, d), \\ & \text{and } \{x_i, x_j\} \cap Pos_C(\mathcal{D}) \neq \emptyset. \\ \emptyset, & \text{otherwise.} \end{cases} \tag{6}$$

Each element $a \in \delta_{ij}$ represents that x_i and x_j are discernible by checking the value of a.

Table 2. The discernibility matrix of Table 1

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\emptyset$					
x_2	$\emptyset$	$\emptyset$				
x_3	$\{c_1, c_2, c_3, c_5, c_6\}$	$\{c_2, c_3, c_5, c_6\}$	$\emptyset$			
x_4	$\{c_1, c_2, c_3, c_4, c_6\}$	$\{c_3, c_4, c_6\}$	$\emptyset$	$\emptyset$		
x_5	$\emptyset$	$\emptyset$	$\{c_2, c_3, c_5, c_6\}$	$\{c_3, c_4, c_6\}$	$\emptyset$	
x_6	$\{c_1, c_2, c_5\}$	$\{c_5\}$	$\{c_2, c_3, c_6\}$	$\{c_3, c_4, c_5, c_6\}$	$\{c_3, c_5\}$	$\emptyset$

Using the discernibility matrix, we get all relative reducts of the decision table as follows:

1. Construct the following logical formula $L(\delta_{ij})$ from each non-empty set $\delta_{ij} = \{a_{k1}, \cdots, a_{kl}\}$ ($i > j$ and $l \geq 1$) in the discernibility matrix:
$$L(\delta_{ij}) : a_{k1} \vee \cdots \vee a_{kl}. \tag{7}$$
2. Construct a conjunctive normal form $\bigwedge_{i>j} L(\delta_{ij})$.
3. Transform the conjunctive normal form to the minimal disjunctive normal form:
$$\bigwedge_{i>j} L(\delta_{ij}) \equiv \bigvee_{p=1}^{s} \bigwedge_{q=1}^{t_p} a_{pq} \tag{8}$$
4. For each conjunction $a_{p1} \wedge \cdots \wedge a_{pt_p}$ ($1 \leq p \leq s$) in the minimal disjunctive normal form, construct a relative reduct $\{a_{p1}, \cdots, a_{pt_p}\}$.

Example 2. Table 2 illustrates the discernibility matrix of the decision table by Table 1. Each non-empty set that appears in the matrix represents the set of condition attributes that we should check to discern the corresponding objects. For example, the set $\delta_{65} = \{c_3, c_5\}$ represents that we can discern the objects x_6 and x_5 by either of values of these objects at the condition attribute c_3 and c_5. Note that we omit upper triangular components of the discernibility matrix in Table 2 because the discernibility matrix is symmetric by the definition. We construct a conjunctive normal form by connecting logical formulas based on non-empty elements in Table 2, and transform the conjunctive normal form to the minimal disjunctive normal form as follows;

$$\begin{aligned} &(c_1 \vee c_2 \vee c_3 \vee c_5 \vee c_6) \wedge (c_2 \vee c_3 \vee c_5 \vee c_6) \wedge \cdots \wedge (c_3 \vee c_5) \\ \equiv\ &(c_3 \wedge c_5) \vee (c_5 \wedge c_6) \vee (c_2 \wedge c_4 \wedge c_5). \end{aligned}$$

Consequently, from this minimal disjunctive normal form, we have the three relative reducts $\{c_3, c_5\}$, $\{c_5, c_6\}$, and $\{c_2, c_4, c_5\}$.

3 Evaluation of Relative Reducts Based on Classification Ability

In this section, we introduce an evaluation method of relative reducts based on classification ability of condition attributes.

We intend to evaluate relative reducts by classification ability of condition attributes that appear in relative reducts with respect to decision classes. Here, we consider that condition attributes that satisfy the following two conditions have high classification ability:

- Condition attributes that discern elements that belong to different decision classes as long as possible.
- Condition attributes that do not discern elements that belong to the same decision class as long as possible.

Following to this evaluation policy, we consider the following two sets, for each condition attribute $a \in C$:

$$Dis(a) = \left\{ (x_i, x_j) \in U \times U \middle| \begin{array}{l} \{x_i, x_j\} \cap Pos_C(\mathcal{D}) \neq \emptyset, \\ \rho(x_i, a) \neq \rho(x_j, a), \\ \rho(x_i, d) \neq \rho(x_j, d), \\ \exists d \in D, i > j \end{array} \right\}, \tag{9}$$

$$Indis(a) = \left\{ (x_i, x_j) \in U \times U \middle| \begin{array}{l} \{x_i, x_j\} \cap Pos_C(\mathcal{D}) \neq \emptyset, \\ \rho(x_i, a) = \rho(x_j, a), \\ \rho(x_i, d) = \rho(x_j, d), \\ \forall d \in D, i > j \end{array} \right\}. \tag{10}$$

The set $Dis(a)$ describes the set of ordered pair (x_i, x_j) such that x_i and x_j belong to different decision classes and discerned by the values of the attribute a. The set $Indis(a)$ describes the set of (x_i, x_j) such that x_i and x_j belong to the same decision classe and are not discerned by the values of a.

For each condition attribute $a \in C$, we introduce the following criterion for classification ability with respect to decision classes:

$$Eval(a) \stackrel{\text{def}}{=} |Dis(a)| + |Indis(a)|. \tag{11}$$

We consider that the higher the evaluation value $Eval(a)$, the higher classification ability the condition attribute a has. It is easily confirmed that the degree $|Dis(a)|$ is the number of the attribute a appeared in the discernibility matrix DM, thus we can calculate the degree $|Dis(a)|$ when we construct the discernibility matrix DM. We can also calculate the degree $|Indis(a)|$ simultaneously, thus we can evaluate all condition attributes during construction of the discernibility matrix.

Let $B \subseteq C$ be a relative reduct of a given decision table. The evaluation value $Eval(B)$ of the relative reduct is defined by

$$Eval(B) \stackrel{\text{def}}{=} \frac{1}{|B|} \sum_{c \in B} Eval(c). \tag{12}$$

Similar to the case of condition attributes, we consider that the higher the evaluation value $Eval(B)$, the higher classification ability the relative reduct B has.

Note that Yamaguchi [17] has proposed an improvement of Pawlak's attribute dependency [8] by using a similar policy used in this evaluation method. However, his formulation is different from this evaluation method, and not used for evaluation of relative reducts.

4 A Heuristic Algorithm for Calculating a Candidate of Relative Reduct with Better Evaluation

In this section, we propose a heuristic algorithm for calculating a relative reduct with higher evaluation value by the proposed evaluation criterion of relative reducts by (12).

For providing a candidate $S \subseteq C$ of relative reducts, we revise the current candidate S by adding a condition attribute with the highest evaluation value, and update evaluation values of condition attributes till the candidate has nonempty intersections with all nonempty elements in the discernibility matrix. As we have described in the previous section, the degree $|Dis(a)|$ used in (11) is the number of the attribute $a \in C$ appeared in the discernibility matrix and therefore we can update the degree $|Dis(a)|$ of each condition attributes a by revising the discernibility matrix. For updating the degree $|Indis(a)|$ of each a, we introduce a concept of equivalent matrix of a given decision table. Formally, an equivalent matrix EM of a given decision table $DT = (U, C \cup D, V, \rho)$ is a $|U| \times |U|$ matrix whose element at i-th row and j-th column is the following set ϵ_{ij} of condition attributes:

$$\epsilon_{ij} \stackrel{\text{def}}{=} \begin{cases} \{a \in C \mid \rho(x_i, a) = \rho(x_j, a)\}, & \{x_i, x_j\} \cap Pos_C(\mathcal{D}) \neq \emptyset, \text{and} \\ & \forall d \in D,\ \rho(x_i, d) = \rho(x_j, d). \\ \emptyset, & \text{otherwise.} \end{cases} \tag{13}$$

The nonempty element ϵ_{ij} is the set of condition attributes that x_i and x_j in the same decision class have the same value. From the definition of equivalent matrix, it is easily confirmed that $\epsilon_{ij} = \epsilon_{ji}$ and $\epsilon_{ii} = C$ for any $i, j \in \{1, \cdots, |U|\}$. Thus, similar to the case of discernibility matrix, the equivalent matrix is symmetric, and therefore it is enough to construct the upper (lower) triangular part of the equivalent matrix for actual computation. Table 3 represents the equivalent matrix of the decision table described by Table 2 as an example of equivalent matrix.

Using the discernibility matrix and the equivalent matrix, the definition of $Dis(a)$ and $Indis(a)$ are revised as follows:

$$Dis(a) = \left\{ (x_i, x_j) \in U \times U \middle| a \in \delta_{ij}, i > j \right\}, \tag{14}$$

$$Indis(a) = \left\{ (x_i, x_j) \in U \times U \middle| a \in \epsilon_{ij}, i > j \right\}. \tag{15}$$

Thus, we can calculate the evaluation value $Eval(a)$ for each condition attribute $a \in C$ based on the discernibility matrix and the equivalent matrix.

Table 3. The equivalent matrix of Table 1

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	C					
x_2	$\{c_3, c_4, c_5, c_6\}$	C				
x_3	$\emptyset$	$\emptyset$	C			
x_4	$\emptyset$	$\emptyset$	$\emptyset$	C		
x_5	$\{c_4, c_5, c_6\}$	$\{c_1, c_2, c_4, c_5, c_6\}$	$\emptyset$	$\emptyset$	C	
x_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	C

We propose the following heuristic algorithm to calculate a candidate of relative reducts with better evaluation:

Algorithm for Calculating a Candidate of Relative Reduct with Better Evaluation
Input: The set of condition attributes CCthe discernibility matrix DMC and the equivalent matrix EM.
Output: A candidate of relative reducts $S \subseteq C$.

1. $S := \bigcup\{\delta \in DM \mid |\delta| = 1\}$D
2. For each $\delta_{ij} \in DM$, if $S \cap \delta_{ij} \neq \emptyset$ holds, then we set $\delta_{ij} := \emptyset$. Similarly, for each $\epsilon_{ij} \in EM$, if $S \cap \epsilon_{ij} \neq \emptyset$ hold, then we set $\epsilon_{ij} := \emptyset$.
3. For each $c \in C$, calculate the evaluation value $Eval(c) = |Dis(c)| + |Indis(c)|$ by using DM and EM.
4. Add a condition attribute $c_h \in C$ with the highest evaluation value to the set S; $S := S \cup \{c_h\}$.
5. Remove c_h from C; $C := C - \{c_h\}$.
6. For all $\delta_{ij} \in DM$, if $S \cap \delta_{ij} \neq \emptyset$ holds, output S and quit; otherwise, go back to Step. 2.

In this algorithm, we construct the core of relative reducts at Step 1. by using the following property; a condition attribute $a \in C$ is in the core if and only if there is an element δ_{ij} in the discernibility matrix such that $\delta_{ij} = \{a\}$ [12]. In Step 2. and 3., we revise the discernibility matrix and the equivalent matrix for updating the current evaluation values of condition attributes. In Step 4, we revise the current candidate by adding a condition attribute with the currently highest evaluation value.

5 Experiments

We applied the proposed method to the following ten data sets in UCI machine learning repository [14]; audiology, lung-cancer, Monk1, Monk3, soybean-s, soybean-l, SPECT-test, SPECT-train, vote, and Zoo.

Table 4. Experiment results

Data set	Attributes	Instances	Reducts	Selected	Ranking
audiology	70	200	-	13	-
lung-cancer	57	32	-	5	-
Monk1	7	124	1	3	1
Monk3	7	122	1	4	1
soybean-s	36	47	-	2	-
soybean-l	36	307	-	9	-
SPECT-test	22	187	40	12	20
SPECT-train	22	80	26	11	5
vote	16	435	3	8	1
Zoo	17	101	33	5	1

Table 4 describes experiment results. In Table 4, the first column represents names of data sets. The second and third columns represent numbers of attributes and instances in each data set. The fourth column represents numbers of all relative reducts of each data set, and the fifth column represents numbers of attributes that appear in the output of the proposed method for each data set. Finally, the sixth column represents the ranking of the output among all relative reducts for each data set. The notation '-' in the fourth and sixth columns means that we could not calculate all relative reducts of these data sets, then we could not also evaluate the ranking of the output among all relative reducts.

As the experiment results show, in the cases of SPECT-train, vote, and zoo, the proposed method could generate better relative reducts with respect to the proposed evaluation criterion (12). In the other cases, the proposed method could provide a relative reduct, i. e. any output of the proposed method had no redundant condition attributes.

6 Conclusion

In this paper, we proposed an evaluation criterion of relative reducts based on classification ability of condition attributes that appear in relative reducts. Moreover, we considered a heuristic method for partial calculation of relative reducts with better evaluation by the proposed evaluation criterion. The proposed method was applied to ten data sets in UCI machine learning repository, and provided better outputs for almost data sets. More refinement and consideration of computational complexity of the proposed method, and experiments using more large data are future issues.

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Multiobjective Multiclass Soft-Margin Support Vector Machine and Its Solving Technique Based on Benson's Method

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Abstract. In this paper, we focus on the *all together* model, which is one of the support vector machine (SVM) using a piece-wise linear function for multiclass classification. We already proposed a multiobjective hard-margin SVM model as a new all together model for piecewise linearly separable data, which maximizes all of the geometric margins simultaneously for the generalization ability. In addition, we derived a single-objective convex problem and showed that a Pareto optimal solution for the proposed multiobjective SVM is obtained by solving single-objective problems. However, in the real-world classification problem the data are often piecewise linearly inseparable. Therefore, in this paper we extend the hard-margin SVM for the data by using penalty functions for the margin slack variables between outliers and the corresponding discriminant hyperplane. Those functions are incorporated into the objective functions. Moreover, we derive a single-objective second-order cone programming (SOCP) problem based on Benson's method and some techniques, and show that a Pareto optimal solution for the proposed soft-margin SVM is obtained by solving the SOCP iteratively. Furthermore through numerical experiments we verify that the proposed iterative method maximizes the geometric margins and constructs a classifier with a high generalization ability.

1 Introduction

The support vector machine (SVM) is one of the major machine learning methods for classification problems. Since it was originally proposed for binary-class problem, several kinds of extensions for multiclass classification have been investigated [3,7,8]. In this paper, we focus on the *all together* method which finds a discriminant function directly by solving an optimization problem by using all patterns, which is proposed in [4,6,12,13], where all patterns are classified into the corresponding classes by using a piece-wise linear function. This model is formulated as a single-objective quadratic optimization problem which maximizes the sum of margins between all of the pairs of classes, where the margin is defined as the distance between two normalized support hyperplanes parallel to the corresponding discriminant hyperplane. However, as we point out in

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[10,11], the margin is not always equal to the geometric margin which is defined as the minimal distance of patterns to the corresponding discriminant hyperplane, and the geometric margin can exactly indicate the relation between each pattern and the discriminant function. Therefore, in [11], we emphasized that maximizing the geometric margins is important for the generalization of multiclass classification, and proposed a hard-margin multiobjective SVM model which maximizes all of the geometric margins simultaneously. Moreover, we derived single-objective second-order cone programming (SOCP) problems, which are solvable convex programming problems, by using scalarization approaches for multiobjective optimization, ε-constraint and Benson's methods, and theoretically showed that the optimal solution of the derived models are Pareto optimal for the proposed multiobjective model. Moreover, we applied them to some examples to demonstrate that the proposed models can achieve maximization of the geometric margins and can obtain classifiers with high generalization ability.

In this paper, we consider the classification problem for piecewise linearly inseparable data, which are often appeared in the real-world problems. Thus, we extend the proposed hard-margin model into a soft-margin one by using a penalty function of the margin slack variables between outliers and the discriminant hyperplane. Those functions are incorporated into the objective functions. Then, we derive a single-objective SOCP problem based on Benson's method and some techniques, and show that a Pareto optimal solution of the proposed soft-margin SVM is obtained by solving the SOCP models iteratively. Finally we verify that the proposed method maximizes the geometric margins and minimizes the penalty functions in the sense of multiobjective optimization and compare the classification abilities of the proposed and the existing models.

2 Multiclass Classification

In this paper, we shall use the following notations for the orders of vectors x, $y \in \Re^n$:

$$
\begin{aligned}
&x \leqq y, \text{ if } x_i \leqq y_i, \ i = 1, \ldots, n. \\
&x \leq y, \text{ if } x_i \leqq y_i, \ i = 1, \ldots, n, \text{ and } x \neq y, \\
&x < y, \text{ if } x_i < y_i, \ i = 1, \ldots, n.
\end{aligned}
$$

Now, let us consider the following multiclass classification problem: For given data: $D = \{x^i, y_i\}, i = 1, \ldots, m$, where $x^i \in \Re^n$ is an input pattern and $y_i \in K := \{1, \ldots, k\}$ denotes the corresponding class, we construct a classifier which divides all patterns into the corresponding classes:

$$
f(x) = \arg\max_p \{w^{p\top} x + b^p\}.
$$

where $w^p \in \Re^n$ and b^p, $p \in K$ are decision variables and the linear function $w^{p\top} x + b^p$ indicates the degree of confidence when a point x is classified into class p. Then,

$$(w^p - w^q)^\top x + (b^p - b^q) = 0,\ q \neq p,\ p, q \in K, \tag{1}$$

is the discriminant hyperplane which distinguishes between classes p and q. Note that the representation of discriminant hyperplanes (1) is not unique. For any constants $t(>0)$, $s \in \Re$ and any vector $v \in \Re^n$, $(w^{1\top}, \ldots, w^{k\top})$, $(b^1, \ldots, b^k)$ and $(tw^{1\top} + v^\top, \ldots, tw^{k\top} + v^\top)$, $(tb^1 + s, \ldots, tb^k + s)$ are different representations of the same discriminant hyperplanes.

Now, in this section, we suppose that data D are piecewise linearly separable. Then, there exist an infinite number of discriminant functions to distinguish all classes correctly. In the multiclass classification, on the analogy of the binary SVM, the model maximizing $1/\|w^p - w^q\|$ for all pairs $\{p, q\}$, $q \neq p,\ p, q \in K$ was proposed in [4,6,12,13],

$$\text{(O)} \quad \begin{array}{ll} \min\limits_{w,b} & \dfrac{1}{2} \displaystyle\sum_{p=1}^{k} \sum_{q=1, q\neq p}^{k} \|w^p - w^q\|^2 \\ \text{s.t} & (w^p - w^q)^\top x^i + (b^p - b^q) \geqq 1,\ \ i \in I_p,\ \ q \neq p,\ \ p, q \in K, \end{array}$$

where I^p denotes an index set defined by $I^p := \{i \in \{1, \ldots, m\} \mid y^i = p\}$. This model is called the *all together* model. Here, note that $1/\|w^p - w^q\|$ denotes a half of the distance between two normalized support hyperplanes $(w^p - w^q)^\top x + (b^p - b^q) = 1$ or -1 parallel to the corresponding discriminant hyperplane (1), which is called the *functional margin* in this paper.

However, since the normalized support hyperplane often does not include support vectors, the functional margin obtained in model (O) is not necessarily equal to the *geometric margin* defined as the distance of the nearest pattern in a pair of classes $\{p, q\}$ to the corresponding discriminant hyperplane (1),

$$d^{\mathrm{g}}_{pq}(w, b) := \min \left\{ \min_{i \in I_p} \frac{|(w^p - w^q)^\top x^i + (b^p - b^q)|}{\|w^p - w^q\|}, \min_{i \in I_q} \frac{|(w^p - w^q)^\top x^i + (b^p - b^q)|}{\|w^p - w^q\|} \right\}, \quad q > p,\ \ p, q \in K,$$

which classifies all patterns in both classes correctly, as we pointed out in [10,11]. Thus, it cannot guarantee that margins obtained by maximizing $\|w^p - w^q\|$, $q \neq p \in K$ in the model (O) are equal to the corresponding geometric margins $d^{\mathrm{g}}_{pq}(w, b)$. Therefore, we proposed a hard-margin multiobjective SVM (M1) which maximizes all geometric margins simultaneously in [11]

$$\text{(M1)} \quad \begin{array}{ll} \max\limits_{w,b} & d(w, b) \\ \text{s.t.} & (w^p - w^q)^\top x^i + (b^p - b^q) \geqq 1,\ \ i \in I_p,\ \ q \neq p,\ \ p, q \in K, \end{array}$$

where $d(w, b)$ is defined by

$$d(w, b) = \left(d^{\mathrm{g}}_{12}(w, b), d^{\mathrm{g}}_{13}(w, b), \ldots, d^{\mathrm{g}}_{(k-1)k}(w, b) \right)^\top.$$

Moreover, since model (M1) is difficult to solve directly, we proposed the following model (M2) using a vector $\sigma \in \Re^{k(k-1)/2}$ and a function $\theta(w, \sigma)$:

$$
\begin{array}{ll}
 & \max\limits_{w,b,\sigma} \theta(w,\sigma) \\
\text{(M2)} & \text{s.t.} \quad (w^p - w^q)^\top x^i + (b^p - b^q) \geqq \sigma_{pq}, \quad i \in I_p, \quad q > p, \quad p, q \in K, \\
 & \qquad (w^q - w^p)^\top x^i + (b^q - b^p) \geqq \sigma_{pq}, \quad i \in I_q, \quad q > p, \quad p, q \in K, \\
 & \qquad \sigma_{pq} \geqq 1, \quad q > p, \quad p, q \in K,
\end{array}
$$

where $\theta(w, b)$ is defined by

$$\theta(w,\sigma) = \left(\theta_{12}(w,\sigma), \theta_{13}(w,\sigma), \ldots, \theta_{(k-1)k}(w,b,\sigma)\right)^\top,$$

and

$$\theta_{pq}(w,\sigma) = \frac{\sigma_{pq}}{\|w^p - w^q\|}, \quad q > p, \quad p, q \in K.$$

Suppose that there exist Pareto optimal solutions of (M2). Then, we showed that the optimal solutions of (M2) can be considered to be equivalent to those of (M1) as follows:

Theorem 1. *If* (w^*, b^*, σ^*) *is Pareto optimal for* (M2)*,* (w^*, b^*) *is Pareto optimal for* (M1)*. Conversely, if* (w^*, b^*) *is Pareto optimal for* (M1)*,* $(w^*, b^*, \sigma(w^*, b^*))$ *is Pareto optimal for* (M2)*, where an element of* σ *is defined by*

$$\sigma_{pq}(w,b) := \min\left\{ \min_{i \in I_p} |(w^p - w^q)^\top x^i + (b^p - b^q)|, \right.$$
$$\left. \min_{i \in I_q} |(w^q - w^p)^\top x^i + (b^q - b^p)| \right\}, \quad q > p, \quad p, q \in K.$$

In addition, we derived two kinds of single-objective optimization problems by scalarization approaches to multiobjective optimization, ε-constraint approach and Benson's method, and transform them into single-objective second-order cone programming (SOCP) problems which are solvable convex programming ones. Furthermore, we showed theoretically that Pareto optimal solutions of the multiobjective problem (SM2) can be obtained by solving SOCP models, and applied them to some examples to demonstrate that they can achieve maximization of the geometric margins [11].

However, in the real-world classification problem the data are often piecewise linearly inseparable. Thus, we extend the hard-margin SVM to a soft-margin one for the data and propose a SOCP model based on Benson's method to solve the soft-margin model in the next section.

3 Multiobjective Soft-Margin Model Maximizing Geometric Margins

In this section, we focus on the classification problem for piecewise linearly inseparable data. In the case, instead of the existing hard-margin model (O), the following model (SO) is often used:

(SO)

$$\begin{aligned}\min_{w,b} \quad & \frac{1}{2}\sum_{p=1}^{k}\sum_{q=1,q>p}^{k}\|w^p - w^q\|^2 + c_\xi \sum_{p=1}^{k}\sum_{q=1,q>p}^{k}\eta_{pq}(\xi) \\ \text{s.t.} \quad & (w^p - w^q)^\top x^i + (b^p - b^q) \geqq 1 - \xi_{pqi}, \ \ i \in I_p, \ \ q \neq p, \ \ p,q \in K, \\ & \xi_{pqi} \geqq 0, \ \ i \in I_p, \ \ q \neq p, \ \ p,q \in K,\end{aligned}$$

where ξ_{pqi} is a margin slack variable to relax the constraints, and $\eta_{pq}(\xi)$ is a penalty function defined by $\eta_{pq}(\xi) := \sum_{i\in I_p}\xi_{pqi} + \sum_{i\in I_q}\xi_{qpi}$.

Now, we extend the hard-margin model (M2) by using the above penalty function. These functions are incorporated into the objective functions $\theta_{pq}(w,\sigma)$ in the following model:

(SM2)

$$\begin{aligned}\max_{w,b,\sigma,\xi} \quad & \big(\theta_{12}(w,\sigma),\ldots,\theta_{(k-1)k}(w,\sigma), \ -\eta_{12}(\sigma,\xi),\ldots,-\eta_{(k-1)k}(\sigma,\xi)\big) \\ \text{s.t.} \quad & (w^p - w^q)^\top x^i + (b^p - b^q) \geqq \sigma_{pq} - \xi_{pqi}, \ \ i \in I_p, \ \ q > p, \ \ p,q \in K, \\ & (w^q - w^p)^\top x^i + (b^q - b^p) \geqq \sigma_{pq} - \xi_{qpi}, \ \ i \in I_q, \ \ q > p, \ \ p,q \in K, \\ & \sigma_{pq} \geqq 1, \ \ q > p, \ \ p,q \in K, \\ & w^p \neq w^q, \ \xi_{pqi} \geqq 0, \ \ i \in I_p, \ \ q \neq p, \ \ p,q \in K.\end{aligned}$$

We can see that (SM2) maximizes all geometric margins and minimizes all penalty functions, simultaneously. However, since (SM2) is complicate, we derive a single-objective convex optimization problem whose optimal solution is Pareto optimal for (SM2) in the next section.

4 SOCP Model Based on Benson's Method

In this section, we first consider the following single-objective problem which is derived from a scalarization approach to multiobjective optimization called Benson's method.

(SPmax-sum)

$$\begin{aligned}\max_{w,b,\sigma,\xi,l,m} \quad & \sum_{q\in K}\sum_{q>p\in K}(l_{pq} + m_{pq}) \\ \text{s.t.} \quad & l_{pq} \geqq 0, \ m_{pq} \geqq 0, \ \ q > p, \ \ p,q \in K, \\ & \frac{\sigma_{pq}}{\|w^p - w^q\|} - \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^q\|} = l_{pq}, \ \ q > p, \ \ p,q \in K, \\ & -\eta_{pq}(\xi) + \eta_{pq}(\xi) = m_{pq}, \ \ q > p, \ \ p,q \in K, \\ & (w^p - w^q)^\top x^i + (b^p - b^q) \geqq \sigma_{pq} - \xi_{pqi}, \ \ i \in I_p, \ \ q > p, \ \ p,q \in K, \\ & (w^q - w^p)^\top x^i + (b^q - b^p) \geqq \sigma_{pq} - \xi_{qpi}, \ \ i \in I_q, \ \ q > p, \ \ p,q \in K, \\ & \sigma_{pq} \geqq 1, \ \ q > p, \ \ p,q \in K, \\ & w^p \neq w^q, \ \xi_{pqi} \geqq 0, \ i \in I^p, \ q \neq p, \ p,q \in K,\end{aligned}$$

where $(\bar{w}, \bar{b}, \bar{\sigma}, \bar{\xi})$ is a feasible solution. This method improves $(\bar{w}, \bar{b}, \bar{\sigma})$ by maximizing the sum of nonnegative deviation variables l_{pq} and m_{pq}, which are defined by $l_{pq} = \theta_{pq}(w, b, \sigma) - \theta_{pq}(\bar{w}, \bar{b}, \bar{\sigma})$ and $m_{pq} = -\eta_{pq}(\xi) + \eta_{pq}(\bar{\xi})$, $q > p,\ p, q \in K$. Then, it is known that an optimal solution of (SPmax-sum) is Pareto optimal for (SM2) (cf. [5]).

However, (SPmax-sum) is still difficult to solve because of its fractional constraints. Thus, secondly, we consider the following solvable problem in which some constraints of (SPmax-sum), $\frac{\sigma_{pq}}{\|w^p - w^q\|} - \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^q\|} = l_{pq}$ and $\sigma_{pq} \geqq 1$, are replaced with $\sigma_{pq} - \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^q\|}\|w^p - w^q\| \geqq l_{pq}$ and $1 \leqq \sigma_{pq} \leqq c_{pq}$ by using constants c_{pq}, respectively.

(SP2max-sum)

$$
\begin{aligned}
\max_{w,b,\sigma,\xi,l,m} \ & \sum_{q \in K} \sum_{q > p \in K} (l_{pq} + m_{pq}) \\
\text{s.t.} \quad & l_{pq} \geqq 0,\ m_{pq} \geqq 0,\ \ q > p,\ \ p, q \in K, \\
& \sigma_{pq} - \|w^p - w^q\| \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^q\|} \geqq l_{pq},\ \ q > p,\ \ p, q \in K, \\
& -\eta_{pq}(\xi) + \eta_{pq}(\bar{\xi}) \geqq m_{pq},\ \ q > p,\ \ p, q \in K, \\
& (w^p - w^q)^\top x^i + (b^p - b^q) \geqq \sigma_{pq} - \xi_{pqi},\ \ i \in I_p,\ \ q > p,\ \ p, q \in K, \\
& (w^q - w^p)^\top x^i + (b^q - b^p) \geqq \sigma_{pq} - \xi_{qpi},\ \ i \in I_q,\ \ q > p,\ \ p, q \in K, \\
& 1 \leqq \sigma_{pq} \leqq c_{pq},\ \ q > p,\ \ p, q \in K, \\
& w^p \neq w^q,\ \xi_{pqi} \geqq 0,\ i \in I^p,\ q \neq p,\ p, q \in K.
\end{aligned}
$$

Here, note that even if the problem (SP2max-sum) does not include the constraint

$$w^p \neq w^q,\ q \neq p,\ q, p \in K, \tag{2}$$

for many feasible solutions $(\bar{w}, \bar{b}, \bar{\sigma})$ the constraint (2) holds. Then, the model (SP2max-sum) without the constraint (2) can be easily transformed into a second-order cone programming problem (SOCP). The SOCP is a convex programming problem having a linear objective function and linear and second-order cone constraints, which can be efficiently solved by a number of methods such as the primal-dual interior point method within the almost same time as a quadratic programming problem of the same size (cf. [1]). Moreover, several commercial and noncommercial solvers have been developed (e.g. [9]).

Next, let us consider what kind of solution we can obtain by solving (SP2max-sum). Here, by focusing on the constraints of (SP2max-sum) and (SM2), we can easily confirm that for any feasible solution $(w, b, \sigma, \xi, l, m)$ of (SP2max-sum), (w, b, σ, ξ) is always feasible for (SM2), while for a feasible solution (w, b, σ, ξ) of (SM2), σ does not necessarily satisfy the constraints of (SP2max-sum), $1 \leqq \sigma_{pq} \leqq c_{pq},\ q > p,\ \ p, q \in K$. Thus, let us consider the relation between feasible solutions of (SM2) and (SP2max-sum).

Now, we define $t(\sigma)$ by

$$t(\sigma) := \max\{1/\sigma_{pq} \mid q > p,\ p, q \in K\}.$$

and define $F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{M2})$ for a feasible solution $(\bar{w},\bar{b},\bar{\sigma},\bar{\xi})$ of (SM2) by

$$F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2}) := \{(w,b,\sigma,\xi) \in F(\mathrm{SM2}) \mid \theta_{pq}(\bar{w},\bar{\sigma}) \leqq \theta_{pq}(w,\sigma),\ -\eta_{pq}(\bar{\xi}) \leqq -\eta_{pq}(\xi),\ q<p,\ p,q \in K\},$$

where $F(\mathrm{SM2})$ denotes the set of all feasible solutions of (SM2).

Then, for any feasible solution (w,b,σ,ξ) for (SM2), $t(\sigma)$ is the minimal $t>0$ such that $(tw,tb,t\sigma,t\xi) \in F(\mathrm{SM2})$, and $\theta(t(\sigma)w,t(\sigma)\sigma) = \theta(w,\sigma)$. Next, we define $c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}$ by using $F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})$

$$c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}} := \sup\{t(\sigma)\sigma_{pq} \mid q>p,\ p,q \in K, (w,b,\sigma,\xi) \in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})\}.$$

Then, the relation between two feasible solutions is shown in the following lemma.

Lemma 1. *Suppose that* $(\bar{w},\bar{b},\bar{\sigma},\bar{\xi})$ *is an initial solution of* (P2max-sum). *If parameters* c_{pq} *in* (P2max-sum) *satisfy* $c_{pq} \geqq c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}$ *for any* $q>p,\ p,q \in K$, *then for any solution* $(w,b,\sigma,\xi) \in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})$, $(t(\sigma)w, t(\sigma)b,\ t(\sigma)\sigma,\ t(\sigma)\xi,\ l(w,\sigma),\ m(\xi))$ *is feasible for* (SP2max-sum), *where* $l(w,\sigma)$ *and* $m(\xi)$ *are defined by*

$$l_{pq}(w,\sigma) := t(\sigma)\sigma_{pq} - \|t(\sigma)(w^p-w^q)\|\frac{\bar{\sigma}_{pq}}{\|\bar{w}^p-\bar{w}^p\|},\ q>p,\ p,q \in K,$$

and $m(\xi) := t(\sigma)(\eta(\bar{\xi})-\eta(\xi))$, *respectively. Moreover, we have* $\theta(w,\sigma) = \theta(t(\sigma)w, t(\sigma)\sigma)$.

Proof. Since for any $(w,b,\sigma,\xi) \in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})$, $(\theta(w,\sigma), -\eta(\xi)) \geq (\theta(\bar{w},\bar{\sigma}), -\eta(\bar{\xi}))$, we have

$$\begin{aligned} l_{pq}(w,\sigma) &= \|t(\sigma)(w^p-w^q)\|\left(\frac{\sigma_{pq}}{\|(w^p-w^q)\|} - \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p-\bar{w}^p\|}\right) \\ &= \|t(\sigma)(w^p-w^q)\|\,(\theta_{pq}(w,\sigma) - \theta_{pq}(\bar{w},\bar{\sigma})) \geqq 0,\ q>p,\ p,q \in K, \end{aligned}$$

and $m_{pq}(\xi) \geqq 0\ \ q>p,\ p,q \in K$. From the definitions of $t(\sigma)$ and $c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}$, and the assumption of the lemma, we have $1 \leqq t(\sigma)\sigma_{pq} \leqq c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}} \leqq c_{pq}$, for any $q>p,\ p,q \in K$. In addition, since $(t(\sigma)w, t(\sigma)b, t(\sigma)\sigma, l(w,\sigma), m(\xi))$ satisfies other constraints of (SP2max-sum), it is feasible for (SP2max-sum). Moreover, from the definition of $\theta(w,b)$ we have $\theta(w,b) = \theta(t(\sigma)w, t(\sigma)b)$. □

In this paper, we assume that parameters c_{pq} in (SP2max-sum) satisfy $c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}} \leqq c_{pq}$, $q>p,\ p,q \in K$ for an initial solution $(\bar{w},\bar{b},\bar{\sigma},\bar{\xi})$. This assumption can be considered to hold by selecting a sufficiently large c_{pq}. Then, we can show the following theorems.

Theorem 2. *If the optimal value of* (SP2max-sum) *is* 0 *and its optimal solution is* $(w^*,b^*,\sigma^*,\xi^*,l^*,m^*)$, *then* (w^*,b^*,σ^*,ξ^*) *is Pareto optimal for* (SM2) *and* $\theta(w^*,\sigma^*) = \theta(\bar{w},\bar{\sigma})$. *Conversely, if* $(\bar{w},\bar{b},\bar{\sigma},\bar{\xi})$ *is Pareto optimal for* (SM2), *then the optimal value of* (SP2max-sum) *is* 0, *and* $(t(\bar{\sigma})\bar{w}, t(\bar{\sigma})\bar{b}, t(\bar{\sigma})\bar{\sigma}, 0, 0)$ *is optimal for* (P2max-sum).

Proof. First, we show that if the optimal value of (SP2max-sum) is 0 and its optimal solution is $(w^*, b^*, \sigma^*, \xi^*, l^*, m^*)$, then $(w^*, b^*, \sigma^*, \xi^*)$ is Pareto optimal for (SM2). Thus, assume that $(w^*, b^*, \sigma^*, \xi^*)$ is not Pareto optimal for (SM2). Then, since $(w^*, b^*, \sigma^*, \xi^*)$ is feasible for (SM2), there exists a feasible solution $(\hat{w}, \hat{b}, \hat{\sigma}, \hat{\xi})$ of (SM2) such that $(\theta(w^*, \sigma^*), -\eta(\xi^*)) \leq (\theta(\hat{w}, \hat{\sigma}), -\eta(\hat{\xi}))$. Now, let us define

$$\hat{l}_{pq} := t(\hat{\sigma})\hat{\sigma}_{pq} - \|t(\hat{\sigma})(\hat{w}^p - \hat{w}^q)\| \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^p\|}, \ q > p, \ p, q \in K,$$

and $\hat{m} := t(\hat{\sigma})(\eta(\bar{\xi}) - \eta(\hat{\xi}))$. Then, we have $\sum_{q \in K} \sum_{q > p \in K} (\hat{l}_{pq} + \hat{m}_{pq}) > 0$. In addition, $(t(\hat{\sigma})\hat{w}, t(\hat{\sigma})\hat{b}, t(\hat{\sigma})\hat{\sigma}, \hat{l}, \hat{m})$ is feasible for (SP2max-sum) from Lemma 1. These facts contradict that the optimal value of (SP2max-sum) is 0. Therefore, $(w^*, b^*, \sigma^*, \xi^*)$ is Pareto optimal for (SM2). Moreover, since $\sigma^*_{pq} - \frac{\bar{\sigma}_{pq}}{\|\bar{w}^p - \bar{w}^q\|} \|w^{p*} - w^{q*}\| = 0, \ p < q, \ p, q \in K$, we have $\theta(\bar{w}, \bar{\sigma}) = \theta(w^*, \sigma^*)$.

Next, we show that if $(\bar{w}, \bar{b}, \bar{\sigma}, \bar{\xi})$ is Pareto optimal for (SM2), then the optimal value of (SP2max-sum) is 0. Now, we assume that the optimal value is not 0. Then, there exist an optimal solution $(w^*, b^*, \sigma^*, \xi^*, l^*, m^*)$ for (SP2max-sum) such that $\sum_{q \in K} \sum_{q > p \in K} (l^*_{pq} + m^*_{pq}) > 0$. It yields that $\big(\theta(\bar{w}, \bar{\sigma}), -\eta(\bar{\xi})\big) \leq (\theta(w^*, \sigma^*), -\eta(\xi^*))$ and $(w^*, b^*, \sigma^*, \xi^*)$ is feasible for (SM2). Thus, the facts contradict the Pareto optimality of $(\bar{w}, \bar{b}, \bar{\sigma}, \bar{\xi})$ for (SM2). Therefore, the optimal value of (SP2max-sum) is 0. Then, we can easily show that $(t(\bar{\sigma})\bar{w}, t(\bar{\sigma})\bar{b}, t(\bar{\sigma})\bar{\sigma}, 0, 0)$ is optimal for (P2max-sum). □

Theorem 3. *Suppose that* $(w^*, b^*, \sigma^*, \xi^*, l^*, m^*)$ *is an optimal solution of* (SP2 max-sum). *If its optimal value* $\sum_{q \in K} \sum_{q > p \in K} (l^*_{pq} + m^*_{pq})$ *is greater than* 0, *then* $\big(\theta(\bar{w}, \bar{\sigma}), -\eta(\bar{\xi})\big) \leq (\theta(w^*, \sigma^*), -\eta(\xi^*))$.

Proof. From the assumption of theorem, we have $(l^*, m^*) \geq 0$. Since we have $\sigma^*_{pq} - \frac{\bar{\sigma}}{\|\bar{w}^p - \bar{w}^q\|} \|w^{p*} - w^{q*}\| = l^*_{pq}$ and $-\eta(\xi^*) + \eta(\bar{\xi}) = m^*_{pq}, \ q > p, \ p, q \in K$ from optimality of $(w^*, b^*, \sigma^*, \xi^*)$, we can derive the result of the theorem. □

Theorems 2 and 3 show that if an obtained optimal value for (SP2max-sum) is 0, the obtained solution is Pareto optimal for (SM2), and otherwise, the obtained solution $(w^*, b^*, \sigma^*, \xi^*)$ dominates the initial solution $(\bar{w}, \bar{b}, \bar{\sigma}, \bar{\xi})$. Furthermore, we propose the following iterative method of solving (SM2) by exploiting these properties of (SP2max-sum).

Iterative method based on Benson's method: IMB

Step 0. Set $\tau := 0$ and $(w^{(0)}, b^{(0)}, \sigma^{(0)}, \xi^{(0)}) = (\bar{w}, \bar{b}, \bar{\sigma}, \bar{\xi})$.

Step 1. Solve (SP2max-sum) using $(w^{(\tau)}, b^{(\tau)}, \sigma^{(\tau)}, \xi^{(\tau)})$ as an inial solution and $c^{(\tau)}_{pq} > 0, \ q > p \in K$, and obtain the optimal solution $(w^*, b^*, \sigma^*, \xi^*, \ l^*, \ m^*)$.

Step 2. Set $(w^{(\tau+1)}, b^{(\tau+1)}, \sigma^{(\tau+1)}, \xi^{(\tau+1)}, l^{(\tau+1)}, m^{(\tau+1)}) := (w^*, b^*, \sigma^*, \xi^*, \ l^*, \ m^*)$. If $\sum_{q \in K} \sum_{q > p \in K} \left(l^{(\tau+1)}_{pq} + m^{(\tau+1)}_{pq}\right) \leqq \delta$, then terminate. Otherwise, $\tau := \tau + 1$ and go to **Step 1**.

Here, δ is non-negative constant.

If $\sum_{q\in K}\sum_{q>p\in K}\left(l_{pq}^{(\tau+1)}+m_{pq}^{(\tau+1)}\right)=0$ holds at some τ, then IMB obtains a Pareto optimal solution for (SM2). Otherwise, the method may generate an infinite sequence $\{w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},\xi^{(\tau)}\}$, $\tau=0,\ldots$. Thus, let us consider the case where the condition $\sum_{q\in K}\sum_{q>p\in K}\left(l_{pq}^{(\tau)}+m_{pq}^{(\tau)}\right)=0$ does not hold at any τ.

Theorem 4. *Assume that* $c_{pq}^{(\tau)}$ *in* IMB *satisfies that* $c_{pq}^{(\tau)}\geqq c^{M}_{w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},\xi^{(\tau)}}$, $q>p\in K$ *for any* $\tau\geqq 0$, *and that* $\{(\theta(w,\sigma),-\eta(\xi))\mid(w,b,\sigma,\xi)\in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})\}$ *is bounded and closed. If* $\delta=0$ *and* $l^{(\tau)}\geq 0$ *for any* τ *in* IMB, *then a sequence* $\{\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)})\}$ *generated by* IMB *converges to a point* $\left(\theta(\hat{w},\hat{\sigma}),-\eta(\hat{\xi})\right)$ *such that* $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})\in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})$ *is Pareto optimal for* (SM2). *In addition,* $\left\{\sum_{q\in K}\sum_{q>p\in K}\left(l_{pq}^{(\tau)}+m_{pq}^{(\tau)}\right)\right\}$ *converges to 0.*

Proof. First, we show the convergence of the sequences $\{\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)})\}$ and $\left\{\sum_{q\in K}\sum_{q>p\in K}\left(l_{pq}^{(\tau)}+m_{pq}^{(\tau)}\right)\right\}$. Since (SP2max-sum) solved at iteration τ in IMB uses $(w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},\xi^{(\tau)})$ as an initial solution and the obtained optimal solutions is given by $(w^{(\tau+1)},b^{(\tau+1)},\sigma^{(\tau+1)},\xi^{(\tau+1)})$, we have $(\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)}))\leq(\theta(w^{(\tau+1)},\sigma^{(\tau+1)}),-\eta(\xi^{(\tau+1)}))$ from Theorem 3. In addition, the sequence $\{\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)})\}$ is monotone nondecreasing and included in $\{(\theta(w,\sigma),-\eta(\xi))\mid(w,b,\sigma,\xi)\in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})\}$, which is bounded and closed from the assumption of this theorem. Therefore, $\{\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)})\}$ converges to a point $(\theta(\hat{w},\hat{\sigma}),-\eta(\hat{\xi}))$ such that $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})\in F_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}(\mathrm{SM2})$.

Furthermore, since $\theta(\bar{w},\bar{\sigma})=\theta(w^{(0)},\sigma^{(0)})\leq\theta(w^{(\tau)},\sigma^{(\tau)})$ and $\sigma_{pq}^{(\tau)}\leqq c_{pq}^{(\tau)}$ from the feasibility of $(w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},l^{(\tau)})$ for (P2max-sum), we have

$$\|w^{p(\tau)}-w^{q(\tau)}\|\leqq\frac{\sigma_{pq}^{(\tau)}}{\theta_{pq}(\bar{w},\bar{\sigma})}\leqq\frac{\max_{\tau}c_{pq}^{(\tau)}}{\theta_{pq}(\bar{w},\bar{\sigma})},\ q>p,\ p,q\in K.$$

Thus, $\|w^{p(\tau)}-w^{q(\tau)}\|$ is bounded from above. At the same time, we have

$$\begin{aligned}l_{pq}^{(\tau)}&=\sigma^{(\tau)}-\|w^{p(\tau)}-w^{q(\tau)}\|\frac{\sigma_{pq}^{(\tau-1)}}{\|w^{p(\tau-1)}-w^{q(\tau-1)}\|}\\&=\|w^{p(\tau)}-w^{q(\tau)}\|\left(\theta_{pq}(w^{(\tau)},\theta^{(\tau)})-\theta_{pq}(w^{(\tau-1)},\theta^{(\tau-1)})\right),\\&\quad q<p,\ p,q\in K,\ \tau=1,\ldots,\end{aligned}$$

which, together with the upper boundedness of $\|w^{p(\tau)}-w^{q(\tau)}\|$ and the convergence of $\{\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)})\}$, yields that $\sum_{q\in K}\sum_{p<q\in K}l_{pq}^{(\tau)}\to 0$ as $\tau\to\infty$. Moreover, from the definition of $m^{(\tau)}$ we have that $\sum_{q\in K}\sum_{p<q\in K}m_{pq}^{(\tau)}\to 0$ as $\tau\to\infty$. Next, we show $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})$ is Pareto optimal for (M2). Assume that $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})$ is not Pareto optimal. Then, let us consider the problem (P2max-sum) using $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})$ as an initial solution and suppose that (w,b,σ,ξ,l,m) is

its optimal solution. Then, we have $\sum_{q\in K}\sum_{p<q\in K}(l_{pq}+m_{pq})>0$, $(l,m)\geq 0$ and

$$\sigma_{pq}-\|w^p-w^q\|\frac{\hat{\sigma}_{pq}}{\|\hat{w}^p-\hat{w}^q\|}=l_{pq},\quad q>p,\ p,q\in K.$$
$$-\eta_{pq}(\xi)+\eta_{pq}(\hat{\xi})=m_{pq},$$

Moreover, from Theorem 3 we have $(\theta(\hat{w},\hat{\sigma}),-\eta(\hat{\xi}))\geq(\theta(w^{(\tau)},\sigma^{(\tau)}),-\eta(\xi^{(\tau)}))$ Thus, we have

$$\sigma_{pq}-\|w^p-w^q\|\frac{\sigma_{pq}^{(\tau)}}{\|w^{p(\tau)}-w^{q(\tau)}\|}\geqq l_{pq},\quad \tau\geqq 0,\ q>p,\ p,q\in K,$$
$$-\eta_{pq}(\xi)+\eta_{pq}(\xi^{(\tau)})\geqq m_{pq},$$

which means (w,b,σ,ξ,l,m) satisfies the second constraints of (P2sum-max) using an initial solution $(w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},\xi^{(\tau)})$. In addition, since (w,b,σ,ξ,l,m) satisfies other constraints, it is feasible. Furthermore, since $\sum_{q\in K}\sum_{p<q\in K}\left(l_{pq}^{(\tau+1)}+m_{pq}^{(\tau+1)}\right)\to 0$ as $\tau\to\infty$, we have $\sum_{q\in K}\sum_{p<q\in K}\left(l_{pq}^{(\tau+1)}+m_{pq}^{(\tau+1)}\right)<\sum_{q\in K}\sum_{p<q\in K}(l_{pq}+m_{pq})$ for a sufficiently large τ. However, the result contradicts the fact that $\left(w^{(\tau+1)},\ b^{(\tau+1)},\sigma^{(\tau+1)}\right)$ is optimal for (P2sum-max) using $(w^{(\tau)},b^{(\tau)},\ \sigma^{(\tau)},\ \xi^{(\tau)})$ as an initial solution. Therefore, $(\hat{w},\hat{b},\hat{\sigma},\hat{\xi})$ is Pareto optimal for (M2). □

Theorem 4 means that if the constant δ is small positive, then IMB terminates within a finite number of iterations. Additionally if $\sum_{q\in K}\sum_{q>p\in K}\left(l_{pq}^{(\tau+1)}+m_{pq}^{(\tau+1)}\right)=0$, the obtained solution is Pareto optimal for (SM2). Otherwise, the obtained solution is approximately Pareto optimal. Here, we can use the same constant vector $\bar{c}$ such that $\bar{c}\geqq c^M_{\bar{w},\bar{b},\bar{\sigma},\bar{\xi}}$ as $c^{(\tau)}$ at each iteration τ because $\bar{c}\geqq c^M_{w^{(\tau)},b^{(\tau)},\sigma^{(\tau)},\xi^{(\tau)}}$ for any $\tau\geqq 0$.

In this section, we have shown that the proposed method IMB can obtain a Pareto optimal solution. In order to obtain various Pareto optimal solutions, we can extend (P2max-sum) by replacing the objective function with $\sum_{q\in K}\sum_{q>p\in K}(\omega_{pq}l_{pq}+\upsilon_{pq}m_{pq})$, where ω_{pq} and υ_{pq} are positive weights for each l_{pq} and m_{pq}, $q>p\in K$, respectively. In the next section, we apply the proposed model to a classification problem.

5 Numerical Examples

In this section, we report the results of numerical experiments where existing model (SO) and proposed model IMB were applied to a real-world problem Iris (4-dimensional three-class classification problem). 70 % of all data was used as training data and the remaining data was used as test data. We used optimization tools in MathWorks Matlab 7.0.1 and Mosek version 5.0 [2] to solve two models. We solved (SO) with varying $c_\xi\in\{0.01,0.05,0.1,0.5,1,5,10\}$, and used each

Table 1. Comparison of results obtained two models (Iris)

(SO)							
c_ξ	0.01	0.05	0.1	0.5	1	5	10
TR-rate	**74.29**	96.19	96.19	**100.00**	**100.00**	**100.00**	**100.00**
T-rate	**71.11**	**95.56**	91.11	93.33	**93.33**	93.33	**93.33**
θ_{12}	1.81	1.33	1.21	0.81	0.77	0.59	0.52
θ_{13}	1.41	0.70	0.57	0.79	0.83	0.72	0.69
θ_{23}	3.64	1.14	0.82	0.43	0.36	0.21	0.18
(SP2max-sum)							
TR-rate	69.52	**97.14**	**97.14**	**100.00**	**100.00**	**100.00**	**100.00**
T-rate	**71.11**	**95.56**	**95.56**	**95.56**	**93.33**	**95.56**	**93.33**
θ_{12}	1.81	1.33	1.21	0.87	0.80	0.80	0.77
θ_{13}	1.56	1.10	0.98	1.56	1.56	1.56	1.56
θ_{23}	4.08	1.14	0.82	0.43	0.36	0.21	0.18
No. of ite.	2	3	4	3	8	10	5

obtained solution as the initial solution for IMB, where c_{rs} is set to be 10 and the algorithm was terminated if the obtained optimal value is less than 10^{-4}.

We observed that all solutions obtained by the proposed model dominate ones by existing models. Table 1 shows a part of obtained results, where TR-rate and T-rate denote the training and the test correct classification rates, respectively, and "No. of ite." denotes the number of iterations required in IMB, namely, how many times problems (SP2max-sum) were solved. These results indicate that Pareto optimal solutions of (SM2) are obtained by obtained IMB, and that IMB requires a small number of iterations. Moreover, we can see that IMB can improve the generalization ability of classifiers obtained by (SO). At the same time, we can see that obtained solutions by IMB considerably depends on the initial solutions.

6 Conclusion

In this paper, we have focused on the *all together* model of the support vector machine (SVM) for multiclass classification, which uses a piece-wise linear function to construct a discriminant function. We have pointed out that maximizing geometric margins defined as the minimal distance of patterns to the corresponding discriminant hyperplane is important for the generalization, and that multiclass classification should be essentially formulated as a multiobjective optimization problem which maximizes all geometric margins. Then, we have proposed a multiobjective soft-margin SVM model which maximizes all geometric margins and minimizes penalty functions based on the distances between outliers and the discriminant hyperplane simultaneously. In order to solve the proposed soft-margin model, we have derived single-objective second-order cone programming (SOCP) problem by Benson's method and some techniques, which can be efficiently solved by several interior point methods. Moreover, we

have theoretically shown that a Pareto optimal solution of the multiobjective soft-margin SVM is obtained by solving SOCPs iteratively, and we have observed the high generalization ability of the obtained classifiers through some numerical experiments.

For further tasks, we should apply the proposed IMB to many kinds of classification problems to investigate its performance. Moreover, we need to estimate the classification ability of discriminant functions corresponding to a great number of Pareto optimal solutions of the proposed multiobjective SVM model, which can be obtained by solving the proposed SOCP model with weights ω and υ in objective functions or an initial solution varying.

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