Narrow-Shallow-Low-Light Trees with and without Steiner Points

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Abstract. We show that for every set $\mathcal{S}$ of n points in the plane and a designated point $rt \in \mathcal{S}$, there exists a tree T that has small maximum degree, depth and weight. Moreover, for every point $v \in \mathcal{S}$, the distance between rt and v in T is within a factor of $(1+\epsilon)$ close to their Euclidean distance $\|rt, v\|$. We call these trees *narrow-shallow-low-light* (NSLLTs). We demonstrate that our construction achieves optimal (up to constant factors) tradeoffs between *all* parameters of NSLLTs. Our construction extends to point sets in $\mathbb{R}^d$, for an arbitrarily large constant d. The running time of our construction is $O(n \cdot \log n)$.

We also study this problem in *general metric spaces*, and show that NSLLTs with small maximum degree, depth and weight can always be constructed if one is willing to compromise the root-distortion. On the other hand, we show that the increased root-distortion is inevitable, even if the point set $\mathcal{S}$ resides in a Euclidean space of dimension $\Theta(\log n)$.

On the bright side, we show that if one is allowed to use Steiner points then it is possible to achieve root-distortion $(1+\epsilon)$ together with small maximum degree, depth and weight for *general metric spaces.*

Finally, we establish some lower bounds on the power of Steiner points in the context of Euclidean spanning trees and spanners.

1 Introduction

Euclidean Spaces. Given a set $\mathcal{S}$ of n points in the plane and a designated root vertex rt, we want to construct a spanning tree T for $\mathcal{S}$ rooted at rt that enjoys a number of useful properties. First, we want T to be *light*, that is, to be not much heavier than the minimum spanning tree of $\mathcal{S}$ (denoted $MST(\mathcal{S})$). Second, we want it to be *low*, i.e., to have a small depth[1]. Third, we want T to be *shallow*, meaning that for every vertex v in T, the distance $dist_T(rt, v)$ between rt and v in T should not be much greater than the Euclidean distance $\|rt, v\|$. (The maximum ratio $\max\left\{\frac{dist_T(rt,v)}{\|rt,v\|} : v \in \mathcal{S}\right\}$ will be called the *root-stretch* or

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[1] The *depth* of a rooted tree (T, rt), denoted $h(T)$, is the maximum number of hops in a path connecting the root rt with a leaf z of T.

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root-distortion of T.) Fourth, the tree T should be *narrow*, that is, to have a small maximum degree.

Each of these requirements has a natural network-design analogue. The weight of T corresponds to the total cost of building and maintaining the network. The depth and the root-stretch of the tree correspond to communication delays experienced by network end-users. The maximum degree of T corresponds to the load experienced by the relay stations or network routers. Finally, the tree structure of the designed network may be necessary for some applications. In other applications which can be executed in a network that contains cycles, having a cycle-free network may still be very advantageous. Consequently, the problem of designing trees that enjoy all these properties is a basic problem in the area of *geometric network design.* Similar problems arise in the context of the VLSI design [1,6,7], telecommunications and distributed computing [3,4], road network design and medical imaging [9].

Clearly some of these requirements come at the expense of others, and there are inherent tradeoffs between the different parameters. In a seminal STOC'95 paper on Euclidean spanners, Arya et al. [2] have shown that for every set $\mathcal{S}$ of n points in the plane (or even in $\mathbb{R}^d$) there exists a rooted spanning tree (T, rt) with depth $O(\log n)$, constant maximum degree, and an arbitrarily small root-stretch at most $(1+\epsilon)$. However, their trees (called *single-sink spanners*) may have a large weight of $\Omega(n) \cdot w(MST(\mathcal{S}))$. Recently, Dinitz et al. [8] devised a construction that enjoys a small weight (i.e., $O(\log n) \cdot w(MST(\mathcal{S})))$, small depth (i.e., $O(\log n)$) and an arbitrarily small root-stretch at most $(1+\epsilon)$. However, the resulting trees may have vertices of arbitrarily large degree. (The construction of [8] applies to general metric spaces.) In this paper we fill in the gap and devise a single construction that combines all the useful properties of the constructions of [2] and [8]. Specifically, we show that for every n-point set $\mathcal{S}$, a point $rt \in \mathcal{S}$ and parameters ℓ and ϵ, $\ell = O(\log n)$, $\epsilon > 0$, there exists a rooted spanning tree (T, rt) with weight $O(\ell) \cdot w(MST(\mathcal{S}))$ ("light"), depth $O(\ell \cdot n^{1/\ell})$ ("low"), constant maximum degree ("narrow") and root-stretch at most $(1+\epsilon)$ ("shallow"). There also exists a rooted spanning tree (T', rt) with weight $O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$, depth $O(\ell)$, maximum degree $O(n^{1/\ell})$ and root-stretch at most $(1+\epsilon)$. Moreover, both these trees can be constructed in $O(n \cdot \log n)$ time.

Our results generalize and improve both previous constructions of single-sink spanners [2,8]. Specifically, substituting $\ell = O(\log n)$ in our results we obtain a construction of trees that enjoy all properties of the construction of Arya et al. [2], and, *in addition*, have small weight (specifically, $O(\log n) \cdot w(MST(\mathcal{S}))$. Also, our trees enjoy the same optimal combination between the weight and depth as the trees of Dinitz et al. [8] do, and, *in addition*, enjoy optimal maximum degree[2]. Similarly to the construction of Arya et al. [2], our construction extends to point sets $\mathcal{S} \subseteq \mathbb{R}^d$, for any constant dimension $d \geq 2$. The running time of the extended construction remains $O(n \cdot \log n)$.

[2] The optimality of our tradeoff between weight and depth in the entire range of parameters follows from lower bounds of [8]. The optimality of our tradeoff between depth and maximum degree, again in the entire range of parameters, is obvious.

General Metric Spaces. We also study the problem of constructing trees that satisfy all the aforementioned four properties (henceforth, *narrow-shallow-low-light* trees, or shortly, NSLLTs) in *general* metric spaces. We generalize the results of Dinitz et al. [8], and demonstrate that one can trade maximum degree for root-stretch. Specifically, we show that for every n-point metric space M, a point $rt \in M$ and an integer $\ell = O(\log n)$, there exists a rooted spanning tree (T, rt) with weight $O(\ell) \cdot w(MST(M))$, depth $O(\ell \cdot n^{1/\ell})$, constant maximum degree and root-stretch $O(\log n)$. There also exists a rooted spanning tree (T', rt) with weight $O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$, depth $O(\ell)$, maximum degree $O(n^{1/\ell})$ and root-stretch $O(\ell)$. In other words, these constructions achieve the optimal tradeoff between the weight and depth, *together with the optimal maximum degree*, at the expense of having root-stretch of $O(\log n)$ and $O(\ell)$, respectively. In addition, we show that this increase in root-stretch is inevitable as long as one considers general (rather than low-dimensional Euclidean) metric spaces. Specifically, we show that our tradeoff between the maximum degree D and root-stretch $O(\frac{\log n}{\log D})$ cannot be improved even if M is a set of n points in Euclidean space of dimension $d = \Omega(\log n)$. We also extend this lower bound and show that in any dimension $d = O(\log n)$, the root-stretch is at least $\Omega(\frac{d}{\log D})$.

On the bright side, we show that this inherent tradeoff between the maximum degree and root-stretch is only valid when considering *spanning trees*. The situation changes drastically if one is allowed to add *Steiner points*, that is, points that do not belong to the original point set of M. In this case the root-stretch can be improved all the way down to $(1+\epsilon)$, without increasing any of the other three parameters! We also show that our lower bounds on the tradeoff between the weight and depth apply to trees that may include Steiner points (henceforth, *Steiner trees*). Consequently, similarly to the case of spanning trees, our tradeoffs between the four involved parameters are optimal with respect to Steiner trees as well.

All our constructions for general metric spaces can be implemented in time $O(n^2)$, which is linear in the size of the input. If an MST, or a constant approximation of an MST, is given as a part of the input, then our constructions can be implemented in time $O(SORT(n)) = O(n \cdot \log n)$, where $SORT(n)$ is the time required to sort n distances. Moreover, if our metric space M is the induced metric of graph G with m edges, and G is given as a part of the input, then our constructions can be implemented in $O(m + n \cdot \log n)$ time.

Lower Bounds for Euclidean Spanners. We have proved two lower bounds on the tradeoffs between different parameters of NSLLTs. These lower bounds were mentioned above. Both these lower bounds have implications for Euclidean spanners. Next, we discuss these implications.

Our lower bound on the tradeoff between the weight and depth parameters of Steiner trees implies directly a lower bound on the tradeoff between these parameters for Euclidean Steiner spanners. Specifically, Dinitz et al. [8] considered the 1-dimensional Euclidean space ϑ_n with n points $1, 2, \ldots, n$ on the x-axis, and showed that any spanning tree of ϑ_n with depth $o(\log n)$ has weight $\omega(n \cdot \log n)$, and vice versa. This result implies that no construction of Euclidean spanners may guarantee hop-diameter[3] $O(\log n)$ and lightness[4] $o(\log n)$, and vice versa. Consequently, the construction of Arya et al. [2] of Euclidean spanners with weight and hop-diameter $O(\log n)$ is optimal. However, the lower bound of [8] does not preclude the existence of *Steiner* spanners with hop-diameter $O(\log n)$ and lightness $o(\log n)$, or vice versa. In the current paper we show that Steiner points do not help in this context, and thus the construction of Arya et al. [2] cannot be improved even if one allows the spanner to use (arbitrarily many) Steiner points.

Our lower bound on the tradeoff between the maximum degree D and root-stretch $\Omega(\frac{d}{\log D})$ of spanning trees for point sets in $\mathbb{R}^d$, $d = O(\log n)$, implies that if $d = \omega(1)$ is super-constant, then either the maximum degree or the root-stretch is super-constant as well. Hence no construction of Euclidean spanners for a super-constant dimension can possibly achieve simultaneously constant maximum degree and stretch. On the other hand, for any constant dimension d, Arya et al. [2] have built spanners with stretch at most $(1+\epsilon)$ for arbitrarily small $\epsilon > 0$ and constant maximum degree D. Hence our lower bound implies that this result of Arya et al. [2] cannot be extended to super-constant dimension.

Proof Overview. Both our Euclidean and general constructions of NSLLTs are based on the following insight. To construct a tree that enjoys the optimal combination of all four parameters, one can construct two different trees each of which is good with respect to only three out of the four parameters, and combine them into a single tree. Specifically, we start with constructing trees that achieve small maximum degree, root-stretch and depth, henceforth *narrow-shallow-low trees* (NSLoTs). Then we consider the *shallow-low-light trees* (SLLTs) of [8], and observe that in these trees all vertices but the root rt necessarily have small degree. To reduce the degree of the root we manipulate with the star subtree Z rooted at the root of the SLLT T. The vertex set $V(Z)$ of the subtree Z contains the root and all its children $c_1, c_2, \ldots, c_q$, and the edge set of Z is the set $\{(rt, c_1), (rt, c_2), \ldots, (rt, c_q)\}$. Then we construct an NSLoT $\tilde{T}$ for the point set $V(Z)$. Finally, we remove the star Z from the SLLT T, and replace it with the NSLoT $\tilde{T}$. We show that the resulting tree $\widehat{T}$ is an NSLLT, i.e., enjoys *all the four desired properties.* (See Fig. 2 in Sect. 3 for an illustration.)

Our Euclidean construction of NSLoTs is based on the construction of Arya et al. [2], which was, in turn, inspired by the work of Ruppert and Seidel [21]. However, it provides a general tradeoff between the maximum degree D and the depth $O(\log_D n)$, while in the construction of [2] the maximum degree is $O(1)$ and the depth is $O(\log n)$. (Moreover, for $D = \omega(1)$, the running time of our construction is $O(n \cdot \log_D n)$, which is better than the running time $O(n \cdot \log n)$ in [2].) This extension is not difficult, and we provide it for completeness.

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[3] *Hop-diameter* or *unweighted diameter* of a possibly weighted graph $G = (V, E, w)$ is the maximum unweighted distance between a pair of vertices in G.

[4] *Lightness* of a spanning subgraph $G' = (V, E, w)$ of the complete Euclidean graph on the point set V is the ratio between $w(G') = \sum_{e \in E} w(e)$ and $w(MST(V))$.

In Sect. 3 we show that the tradeoff of [8] between the hop-diameter and lightness of Euclidean spanners cannot be improved by using Steiner points. To this end we demonstrate that any Steiner tree can be "cleaned" from Steiner points, while increasing the depth and lightness by only a small factor. This result is reminiscent of the work by Gupta [13] that shows that as far as *maximum* stretch and lightness are concerned, one can do without Steiner points. However, our argument is substantially different from that of [13], since, in particular, the hop-diameter parameter exhibits a different behavior than the maximum stretch.

Related Work. Euclidean spanners are being subject of ongoing intensive research since the mid-eighties. See the recent book by Narasimhan and Smid [19] for an excellent survey on this subject. Euclidean single-sink spanners were studied by Arya et al. [2]; see also [19], Chapter 4.2. Lukovszki [16,15] devised fault-tolerant constructions of single-sink spanners. Single-sink spanners were also used in maintenance algorithms for wireless networks [12,17]. Farshi and Gudmundsson [10] conducted an experimental study of single-sink spanners.

Trees that have small weight and guarantee root-stretch at most $(1 + \epsilon)$, but do not necessarily have small depth or small maximum degree, are called *shallow-light trees* (henceforth SLTs). SLTs were studied by a number of authors, including Awerbuch et al. [3,4], Khuller et al. [14], Alpert et al. [1] and Cong et al. [6,7]. Salowe et al. [22] studied trees that combine small weight with small "bottleneck" size; see [22] for further details. Papadimitriou and Vazirani [20], Monma and Suri [18], Fekete et al. [11] and Chan [5] devised constructions of light trees with small maximum degree for low-dimensional Euclidean point sets. See also the survey of Eppstein [9] for other references to works that study geometric spanning trees.

The Structure of the Paper. In Sect. 2 we describe our constructions of NSLoTs, and prove lower bounds on the tradeoff between the maximum degree and root-stretch. In Sect. 3 we employ our constructions of NSLoTs from Sect. 2 to devise constructions of NSLLTs, and derive our lower bounds for Euclidean Steiner spanners. Due to space limitations, some proofs are omitted from this extended abstract.

Preliminaries. An *n-point metric space* $M = (V, dist)$ can be viewed as the complete graph $G(M) = (V, \binom{V}{2}, dist)$ in which for every pair of points $x, y \in V$, the weight of the edge $e = (x, y)$ in $G(M)$ is defined by $w(x, y) = dist(x, y)$.

For a rooted tree (T, rt) and a vertex v in T, the *level* of v in T is the hop-distance between the root rt of T and v in T. Denote by $deg(T, v)$ the degree of a vertex v in T and define $\Delta(T) = \max\{deg(T, v) : v \in V\}$. For any two vertices $u, v \in V(T)$, their weighted distance in T is denoted by $dist_T(u, v)$. For a positive integer D, a rooted tree in which every vertex has at most D children is called a *D-ary tree.*

A tree T is called a *Steiner tree* of a metric space $M = (V, dist)$ if it spans a superset of V and if for any pair of points $v, v \in V$, $dist_T(u, v) \geq dist(u, v)$. Let T be either a spanning or a Steiner tree of M rooted at an arbitrary designated vertex rt. We define the *stretch* between two vertices u and v in V to be $\zeta_T(u,v) = \frac{dist_T(u,v)}{dist(u,v)}$, and the *root-stretch* of (T, rt) to be $\varrho(T, rt) = \max\{\zeta_T(rt, v) : v \in V\}$.

For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

2 Narrow-Shallow-Low Trees (NSLoTs)

Upper Bounds. In this section we devise constructions of trees that have small maximum degree, depth and root-stretch, but may be quite heavy. On the other hand, we do require their weight to be bounded by $O(\sum_{v \in M} dist(rt, v))$, where rt is the designated root vertex. We denote the quantity $\sum_{v \in M} dist(rt, v))$ by $W^*(M, rt)$. The following statement summarizes the properties of our construction of NSLoTs for general metric spaces.

Proposition 1. *For any n-point metric space $M = (V, dist)$, an arbitrary designated point rt and a positive integer $2 \leq D \leq n-1$, there exists a D-ary rooted spanning tree (T, rt) of M with depth at most $\lceil \log_D n \rceil$, root-stretch at most $2 \cdot \lceil \log_D n \rceil$ and weight at most $2 \cdot W^*(M, rt)$.*

Proof. Let $V = (rt = v_0, v_1, \ldots, v_{n-1})$. Without loss of generality assume that the n points $rt = v_0, v_1, \ldots, v_{n-1}$ are ordered by their distance from rt, i.e., $0 = dist(rt, v_0) \leq dist(rt, v_1) \leq \ldots \leq dist(rt, v_{n-1})$. Next, we construct a rooted tree (T, rt) that satisfies the required conditions. The D points $v_1, v_2, \ldots, v_D$ become the children of $rt = v_0$ in T, the next D points $v_{D+1}, v_{D+2}, \ldots, v_{2 \cdot D}$ become the children of v_1, the next D points $v_{2 \cdot D+1}, v_{2 \cdot D+2}, \ldots, v_{3 \cdot D}$ become the children of v_2, and so on. Generally, the point v_i becomes the child of point $v_{\lceil \frac{i}{D} \rceil - 1}$ in T, for each $i \in [n-1]$. (See Fig. 1.a for an illustration.)

Lemma 1. *(1). For any pair of vertices v and w, such that v is an ancestor of w in T, $dist(rt, v) \leq dist(rt, w)$. (2) (T, rt) is a D-ary rooted spanning tree of M. (3) The depth $h(T)$ of the rooted tree (T, rt) is no greater than $\lceil \log_D n \rceil$.*

The first assertion of Lemma 1 and triangle inequality imply that $w(T) \leq 2 \cdot W^*(M, rt)$. The next lemma provides an upper bound on the root-stretch of the constructed tree.

Lemma 2. *For a vertex v of level i in T, $dist_T(rt, v) \leq (2 \cdot i - 1) \cdot dist(rt, v)$.*

The third assertion of Lemma 1 and Lemma 2 imply that the root-stretch of (T, rt) is at most $2 \cdot \lceil \log_D n \rceil$, which concludes the proof of Proposition 1. □

Next, we describe a construction of Steiner NSLoTs for general metric spaces.

Proposition 2. *For any n-point metric space $M = (V, dist)$, an arbitrary designated point rt, a positive integer $2 \leq D \leq n-1$ and a number $0 < \epsilon' < 1$, there exists a D-ary rooted Steiner tree (T, rt) of M with $O(n/D)$ Steiner points, depth at most $\lceil \log_D n \rceil$, root-stretch at most $(1+\epsilon')$ and weight at most $(1+\epsilon') \cdot W^*(M, rt)$.*

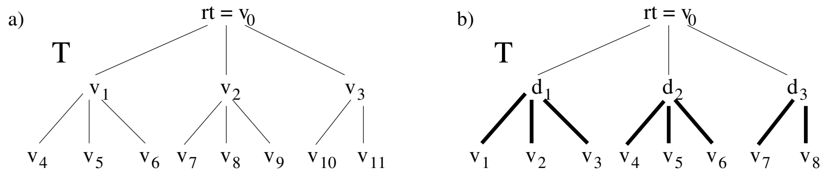

Fig. 1. a) An NSLoT for a 12-point metric space. b) A Steiner NSLoT for a 9-point metric space. The Steiner points are d_1, d_2 and d_3, and the required points are $rt = v_0, v_1, \ldots, v_8$. Edges of weight ϵ are depicted by thin lines. Edges of greater weight (specifically, Edges $(v_i, \pi(v_i))$ of weight $dist_M(rt, v_i)$ are depicted by thick lines.

Proof. Suppose first that $n-1$ is an integer power of D. We form the full D-ary tree T rooted at rt, whose $n-1$ leaves are the $n-1$ points of $V \setminus \{rt\}$. The remaining $\frac{n-2}{D-1}$ vertices of T (excluding rt) are Steiner points. The weight assignment for edges of T is set as follows. For each point $v \in V \setminus \{rt\}$, the weight of the edge $(v, \pi(v))$ that connects it to its parent in T is set as $dist_M(rt, v)$. All other edge weights are set as 0. If one prefers to avoid using weights 0, one can use an arbitrarily small number $\epsilon = \frac{\epsilon'}{2n} \cdot w_{min}$, where w_{min} is the minimum distance between a pair of points in M. It is easy to see that the resulting tree is a D-ary Steiner NSLoT with maximum degree D, depth $\log_D(n-1)$, root-stretch at most $(1+\epsilon')$, and weight at most $(1+\epsilon') \cdot W^*(M, rt)$. This construction generalizes in the obvious way to the case where $n-1$ is not an integer power of D, with the tree depth becoming $\lceil \log_D(n-1) \rceil$. (See Fig. 1.b for an illustration.) □

Next, we show that for point sets in the plane one can construct NSLoTs with significantly smaller root-stretch, without increasing any of the other parameters. The extension of our construction to higher constant dimensions is omitted due to space limitations.

Proposition 3. *Let $k \geq 9$ and $\theta = 2\pi/k$. For any set V of n points in the plane, an arbitrary designated point rt and and a positive integer $2 \leq D \leq n$, there exists a $(2D+k)$-ary rooted Euclidean spanning tree (T_θ, rt) for V with depth at most $\log_D n$, root-stretch at most $\frac{1}{\cos\theta - \sin\theta}$ and weight at most $\frac{1}{\cos\theta - \sin\theta} \cdot W^*(V, rt)$. Moreover, T_θ can be constructed in $O(n \cdot \log_D n)$ time.*

Remark: For large k, $\frac{1}{\cos\theta - \sin\theta} = 1 + O(\theta)$. Hence we get a tree with maximum degree $O(D + \theta^{-1})$, depth at most $\log_D n$, root-stretch $1 + O(\theta)$ and weight $O(W^*(V, rt))$.

Proof. For any $D \geq \frac{n-10}{2}$, the star graph rooted at rt satisfies the conditions of the proposition. We henceforth assume that $D < \frac{n-10}{2}$.

If we rotate the positive x-axis by angles $i \cdot \theta$, $0 \leq i < k$, then we get k rays. Each pair of successive rays defines a *cone* that spans an angle of θ and whose apex is at the origin. Denote by $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ the collection of the resulting k cones. For a cone C_i of $\mathcal{C}$ and a point p in the plane, let $C_i(p) = C_i + p = \{x + p : x \in C_i\}$ be the cone obtained from C_i by *translating* it such that its apex is at p, and define $\mathcal{C}(p) = \{C_1(p), C_2(p), \ldots, C_k(p)\}$. We denote by $V_i(p) = V \cap C_i(p)$ the subset of V contained in a cone $C_i(p)$ of $\mathcal{C}(p)$. Note that the collection $\{V_1(p), V_2(p), \ldots, V_k(p)\}$ is a partition of $V \setminus \{p\}$. For each $i \in [k]$, we define $n_i = |V_i(p)|$. Let $\mathcal{P}(p)$ be the collection obtained from $\{V_1(p), V_2(p), \ldots, V_k(p)\}$ by partitioning each set $V_i(p)$ in it (arbitrarily) into $\left\lceil \frac{n_i}{\lfloor n/D \rfloor} \right\rceil$ subsets of size at most $\lfloor n/D \rfloor$ each.

Claim. For any point set V and any point p in the plane, $|\mathcal{P}(p)| \leq 2D + k$.

The tree $T = T_\theta$ is constructed in the following way. First, a partition $\mathcal{P}(rt) = \{P_1(rt), P_2(rt), \ldots, P_m(rt)\}$ of $V \setminus \{rt\}$ is computed, where $m = |\mathcal{P}(rt)| \leq 2D + k$. For each $i \in [m]$, let $rt(i)$ be the point in $P_i(rt)$ whose orthogonal projection onto the bisector of the cone in $\mathcal{C}(rt)$ that contains it is closest to rt. For each $i \in [m]$, $rt(i)$ is set to be a child of rt, and a rooted tree $(T_i, rt(i))$ for the subset $P_i(rt)$ is constructed recursively. The recursion stops if a subset has size one.

Note that T is a $(2D + k)$-ary spanning tree of M rooted at rt, and its depth is at most $\log_D n$. Using arguments from [2] and [19], we show that T can be constructed in $O(n \cdot \log_D n)$ time, and that the root-stretch of T is at most $\frac{1}{\cos\theta - \sin\theta}$. As a corollary, we get that $w(T) \leq \frac{1}{\cos\theta - \sin\theta} \cdot W^*(V, rt)$. □

Lower Bounds. The next statement implies that the upper bound given in Proposition 1 is tight up to constant factors. In particular, it shows that the tradeoff D versus $O(\frac{\log n}{\log D})$ between the maximum degree and root-stretch established there cannot be improved even for Euclidean spaces of dimension $\Theta(\log n)$.

Proposition 4. *There exists a set V of n points in $\mathbb{R}^{O(\log n)}$, such that for any integer $2 \leq D \leq n - 1$ and any point $v \in V$, every D-ary spanning tree T of V rooted at $rt = v$ has depth at least $\lfloor \log_D n \rfloor$, weight at least $\Omega(W^*(V, rt))$, and root-stretch at least $\Omega(\log_D n)$.*

Proposition 4 should be compared with Proposition 3. Specifically, as long as the dimension d is constant, one can obtain NSLoTs with root-stretch at most $(1 + \epsilon)$, while for $d = \Omega(\log n)$ it is no longer possible. The next statement extends the lower bound on the tradeoff between the maximum degree D and root-stretch $\Omega(\frac{\log n}{\log D})$ established in Proposition 4 to any dimension $d = O(\log n)$. In particular, it shows that whenever $d = \omega(1)$ is super-constant, it is no longer possible to achieve simultaneously constant maximum degree and root-stretch.

Proposition 5. *For any parameter $d \leq \log n$, there exists a set $\tilde{V}$ of n points in $\mathbb{R}^{O(d)}$, such that for any integer $2 \leq D \leq n - 1$ and any point $v \in \tilde{V}$, every D-ary spanning tree $\tilde{T}$ of $\tilde{V}$ rooted at $rt = v$ has root-stretch at least $\Omega(\frac{d}{\log D})$.*

3 Narrow-Shallow-Low-Light Trees

In this section we present a general technique for constructing NSLLTs out of NSLoTs. Then we employ this technique in conjunction with the NSLoTs

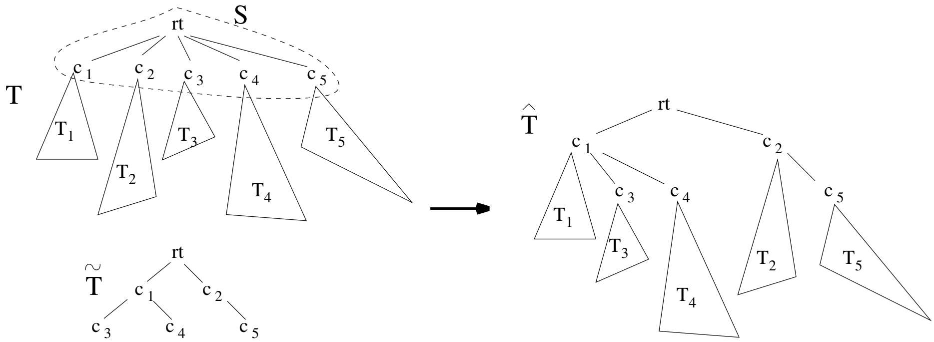

Fig. 2. The root rt of the SLLT T may have a large degree. The star subtree Z is replaced by the NSLoT $\tilde{T}$ to obtain the NSLLT $\widehat{T}$.

construction from Sect. 2 to obtain our constructions of NSLLTs, which exhibit optimal tradeoffs between all four parameters.

Consider an n-point metric space M, and let T be a spanning tree for M rooted at some designated point $rt \in M$. Next, we argue that by using NSLoTs one can significantly reduce the degree of rt, while only slightly increasing other parameters of T. Let Z be the star subtree of T rooted at rt. In other words, the vertex set of Z is $V(Z) = \{rt, c_1, c_2, \ldots, c_q\}$, where $c_1, c_2, \ldots, c_q$ are the children of rt in T. Also, the weights of edges (rt, c_i) agree in T and Z, for all indices $i \in [q]$. Let $\tilde{T}$ be some spanning tree rooted at rt for the metric space M_Z induced by the points in $V(Z)$. (Observe that $w(Z) = W^*(M_Z, rt) \le w(T)$.) Finally, let $\widehat{T}$ be the tree obtained from T by replacing the star Z with the tree $\tilde{T}$. (See Fig. 2 for an illustration.) For a tree τ, let $\lambda(\tau) = \max\{deg(\tau, v) : v \in V, v \neq rt\}$ be the degree of a non-root vertex in τ.

The properties of the resulting tree are summarized in the following statement.

Proposition 6. *(1)* $h(\widehat{T}) \le h(T) - 1 + h(\tilde{T})$, *(2)* $w(\widehat{T}) = w(T) - w(Z) + w(\tilde{T})$, *(3)* $\lambda(\widehat{T}) \le \lambda(T) + \lambda(\tilde{T})$, *(4)* $deg(\widehat{T}, rt) = deg(\tilde{T}, rt)$, *(5)* $\varrho(\widehat{T}, rt) \le \varrho(\tilde{T}, rt) \cdot \varrho(T, rt)$.

Remark: This statement remains valid if $\tilde{T}$ is a Steiner NSLoT of M_Z.
Dinitz et al. [8] devised two constructions of SLLTs for general metric spaces. For a metric space M, a point $rt \in M$ and an integer $\ell = O(\log n)$, the first construction provides a rooted SLLT (T, rt) with depth $h(T) = O(\ell)$, weight $w(T) = O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$ and root-stretch $\varrho(T, rt) \le 1 + \epsilon$. Moreover, all vertices of T except its root rt have optimal degree $O(n^{1/\ell})$. The degree of the root may, however, be arbitrarily large. The second construction provides an SLLT T' with depth $h(T') = O(\ell \cdot n^{1/\ell})$, weight $w(T') = O(\ell) \cdot w(MST(M))$, and root-stretch $\varrho(T', rt) \le 1 + \epsilon$. Similarly to the first construction, all vertices of T' but the root rt have optimal degree $O(1)$, and the root may have arbitrarily large degree.

Next, we reduce the root-degree in the first construction. Reducing the root-degree of the second construction is done similarly. Let Z be the star subtree of

T rooted at rt, and let $\tilde{T}$ be an NSLoT for the $(q+1)$-point metric space M_Z. To construct an NSLLT $\widehat{T}$ out of T and $\tilde{T}$, we replace the star Z by $\tilde{T}$.

Specifically, if M is a set of n points in the plane, then our construction of NSLoTs (Proposition 3) provides a rooted NSLoT $(\tilde{T}, rt)$ for M_Z with depth $h(\tilde{T}) = O(\ell)$, $\Delta(\tilde{T}) = O((q+1)^{1/\ell}) = O(n^{1/\ell})$, weight $w(\tilde{T}) = O(W^*(M_Z, rt))$, and root-stretch $\varrho(\tilde{T}, rt) \leq (1+\epsilon)$. By Proposition 6, replacing the star Z of T with $\tilde{T}$ produces a rooted NSLLT $(\widehat{T}, rt)$ for M with depth $h(\widehat{T}) \leq h(T) - 1 + h(\tilde{T}) = O(\ell)$, weight $w(\widehat{T}) = w(T) - w(Z) + w(\tilde{T}) = w(T) + O(W^*(M_Z, rt)) = O(w(T)) = O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$, $\Delta(\widehat{T}) = O(n^{1/\ell})$, and root-stretch $\varrho(\widehat{T}, rt) \leq (1+\epsilon)^2 = 1 + O(\epsilon)$. The SLLTs of [8] for Euclidean spaces can be constructed in $O(n \cdot \log n)$ time. By Proposition 3, $\tilde{T}$ can be constructed in $O(n \cdot \log n)$ time. Hence the overall time required to construct $\widehat{T}$ is $O(n \cdot \log n)$. This tradeoff extends to the complementary range of depth $h(\widehat{T}) = \Omega(\log n)$. This argument easily generalizes to point sets in $\mathbb{R}^d$, for any constant $d \geq 2$.

Theorem 1. *Let $d \geq 2$ be an integer constant. For a set M of n points in $\mathbb{R}^d$, an integer $\ell = O(\log n)$, and $\epsilon > 0$, there exists a spanning tree with depth $O(\ell)$, lightness $O(\ell \cdot n^{1/\ell})$, maximum degree $O(n^{1/\ell})$, and root-stretch at most $(1+\epsilon)$. In addition, there exists a spanning tree with depth $O(\ell \cdot n^{1/\ell})$, lightness $O(\ell)$, constant maximum degree, and root-stretch at most $(1+\epsilon)$. Both trees can be constructed in $O(n \cdot \log n)$ time.*

Our construction of NSLoTs for general metric spaces (Proposition 1) provides a rooted NSLoT $(\tilde{T}, rt)$ for M_Z with depth $h(\tilde{T}) = O(\ell)$, $\Delta(\tilde{T}) = O((q+1)^{1/\ell}) = O(n^{1/\ell})$, weight $w(\tilde{T}) = O(W^*(M_Z, rt))$ and root-stretch $\varrho(\tilde{T}, rt) = O(\ell)$. By Proposition 6, replacing the star Z of T with $\tilde{T}$ produces a rooted NSLLT $(\widehat{T}, rt)$ for M with depth $h(\widehat{T}) \leq h(T) - 1 + h(\tilde{T}) = O(\ell)$, weight $w(\widehat{T}) = w(T) - w(Z) + w(\tilde{T}) = w(T) + O(W^*(M_Z, rt)) = O(w(T)) = O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$, $\Delta(\widehat{T}) = O(n^{1/\ell})$ and root-stretch $\varrho(\widehat{T}, rt) \leq O(\ell) \cdot (1+\epsilon) = O(\ell)$. The SLLTs of [8] for general metric spaces can be constructed in $O(n^2)$ time. Clearly $\tilde{T}$ can be constructed in $O(n^2)$ time, and so the overall time required to construct $\widehat{T}$ is $O(n^2)$. This tradeoff extends to the complementary range of depth $h(\widehat{T}) = \Omega(\log n)$.

Theorem 2. *For a general n-point metric space M, and an integer $\ell = O(\log n)$, there exists a spanning tree with depth $O(\ell)$, lightness $O(\ell \cdot n^{1/\ell})$, maximum degree $O(n^{1/\ell})$, and root-stretch $O(\ell)$. In addition, there exists a spanning tree with depth $O(\ell \cdot n^{1/\ell})$, lightness $O(\ell)$, constant maximum degree, and root-stretch $O(\log n)$. Both trees can be constructed in $O(n^2)$ time.*

Similarly, using our construction of Steiner NSLoTs for general metric spaces (Proposition 2), we construct in $O(n^2)$ time a Steiner rooted NSLLT $(\widehat{T}, rt)$ for M with depth $h(\widehat{T}) = O(\ell)$, weight $w(\widehat{T}) = O(\ell \cdot n^{1/\ell}) \cdot w(MST(M))$, $\Delta(\widehat{T}) = O(n^{1/\ell})$ and root-stretch $\varrho(\widehat{T}, rt) = 1 + O(\epsilon)$. This tradeoff extends to the complementary range of depth $h(\widehat{T}) = \Omega(\log n)$.

Theorem 3. *For a general n-point metric space M, an integer $\ell = O(\log n)$, and $\epsilon > 0$, there exists a Steiner tree with depth $O(\ell)$, lightness $O(\ell \cdot n^{1/\ell})$, maximum degree $O(n^{1/\ell})$, and root-stretch at most $(1+\epsilon)$. In addition, there exists a Steiner tree with depth $O(\ell \cdot n^{1/\ell})$, lightness $O(\ell)$, constant maximum degree, and root-stretch at most $(1+\epsilon)$.*

Finally, we extend the lower bounds of Dinitz et al. [8] to Steiner trees.

Theorem 4. *For any metric space M and any Steiner rooted tree (T', rt') of M, there exists a rooted tree (T, rt) spanning only $V(M)$, with depth no greater than that of T' (i.e., $h(T) \leq h(T')$), weight at most twice the weight of T' (i.e., $w(T) \leq 2 \cdot w(T')$), and which also* dominates *T' in the following sense: for any two points u, v in $V(M)$, $dist_T(u,v) \geq dist_{T'}(u,v)$. Moreover, T can be constructed in $O(n)$ time.*

Dinitz et al. [8] analyzed the 1-dimensional metric space ϑ_n with n points $1, 2, \ldots, n$ on the x-axis and have shown that for any parameter $\ell = O(\log n)$, any spanning tree for ϑ_n that has depth $h(T) = O(\ell)$ has weight $w(T) = \Omega(\ell \cdot n^{1+1/\ell})$, and vice versa, i.e., if $w(T) = O(\ell \cdot n)$, then $h(T) = \Omega(\ell \cdot n^{1/\ell})$. Theorem 4 enables us to extend this lower bound to Steiner trees.

Corollary 1. *For a positive integer $\ell = O(\log n)$, any Steiner tree T for ϑ_n that has depth $O(\ell)$ satisfies $w(T) = \Omega(\ell \cdot n^{1+1/\ell}) = \Omega(\ell \cdot n^{1/\ell}) \cdot w(MST(\vartheta_n))$. Also, any Steiner tree T for ϑ_n that has weight $O(\ell \cdot n) = O(\ell) \cdot w(MST(\vartheta_n))$ satisfies $h(T) = \Omega(\ell \cdot n^{1/\ell})$.*

In particular, Corollary 1 implies that any Steiner tree T for ϑ_n has either depth $\Omega(\log n)$ or weight $\Omega(n \cdot \log n) = \Omega(\log n) \cdot w(MST(\vartheta_n))$. On the other hand, Arya et al. [2] devised a construction of Euclidean $(1+\epsilon)$-spanners with both hop-diameter and lightness (the ratio between the weight and the weight of the MST) at most $O(\log n)$. Corollary 1 implies that the result of [2] cannot be improved even if one allows the spanner to use Steiner points.

Corollary 2. *Any Euclidean (possibly Steiner) spanner for ϑ_n that guarantees hop-diameter $o(\log n)$ has lightness $\omega(\log n)$, and vice versa.*

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