Chapter 5
Ordinal Structure of the Signed Shifts

Shift transformations are a special case of a more general family: signed shift transformations—a sort of state-dependent shifts. The tent map is the simplest and perhaps most popular representative of the signed shifts. In this chapter we are going to show that most of the results on the ordinal structure of the shifts can be generalized to the signed shifts. By order isomorphy, these results apply also to more interesting cases, like the signed sawtooth maps.

5.1 Ordinal Patterns and the Tent Map

In this section we mimic the strategy used in the previous chapter, in order to get a handle on the ordinal patterns of the symmetric tent map. We will also address an issue pointed out in Fig. 1.7, namely, the interval structure of the sets P_π defining the allowed ordinal patterns of the logistic map.

5.1.1 A State-Dependent Shift Approach to the Tent Map

Just as some important dynamical properties of the sawtooth map E_N (like density of periodic points, sensitivity to initial conditions, topological transitivity, and the structure of its admissible and forbidden ordinal patterns) can be easily studied in the sequence space with the help of the relevant order isomorphisms, the same happens with the symmetric tent map. Remember from Sect. 1.1.3 that the symmetric tent map $\Lambda:[0,1] \to [0,1]$ is given by

$$\Lambda(x) = 1 - |1 - 2x| = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases} . \tag{5.1}$$

For $x \in [0,1]$, write

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)} = 0.x_0 x_1 \ldots x_n \ldots ,$$

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$x_n \in \{0, 1\}$. If $0 \leq x < 1/2$, then

$$\Lambda(x) = 2x = 0.x_1 x_2 \ldots x_{n+1} \ldots,$$

hence the action of Λ coincides with the action of the sawtooth map E_2. Otherwise, if $1/2 \leq x \leq 1$, then

$$\begin{aligned}\Lambda(x) &= 2 - 2x \equiv 1 - 2x \bmod 1 \\ &= 1 - 0.x_2 x_3 \ldots x_{n+1} \ldots\end{aligned}$$

Introducing the *dual bit*

$$x^* = 1 - x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases} \tag{5.2}$$

(thus, $(x^*)^* = x$), we have

$$\Lambda(x) = 0.x_1^* x_2^* \ldots x_{n+1}^* \ldots$$

because

$$0.x_1 x_2 \ldots x_{n+1} + \cdots + 0.x_1^* x_2^* \ldots x_{n+1}^* \cdots = 0.11 \ldots 1 \ldots = 1.$$

All in all,

$$\Lambda(0.x_0 x_1 \ldots x_n \ldots) = \begin{cases} 0.x_1 x_2 \ldots x_{n+1} \ldots & \text{if } x_0 = 0, \\ 0.x_1^* x_2^* \ldots x_{n+1}^* \ldots & \text{if } x_0 = 1. \end{cases} \tag{5.3}$$

Identify now the binary representation $0.\, x_0\, x_1 \ldots x_n \ldots$, $x_n \in \{0, 1\}$, of a number $x \in [0, 1]$, with the sequence

$$(x_0, x_1, \ldots, x_n, \ldots) \in \{0, 1\}^{\mathbb{N}_0},$$

via the map $\phi_2 {:} \{0, 1\}^{\mathbb{N}_0} \to [0, 1]$ defined as in (4.3) with $N = 2$. Then action (5.3) translates into the following zeroth-state-dependent shift on $\{0, 1\}^{\mathbb{N}_0}$:

$$\Sigma_{(+,-)}(x_0, x_1, \ldots, x_n, \ldots) = \begin{cases} (x_1, x_2, \ldots, x_{n+1}, \ldots) & \text{if } x_0 = 0 \\ (x_1^*, x_2^*, \ldots, x_{n+1}^*, \ldots) & \text{if } x_0 = 1 \end{cases} \tag{5.4}$$

(the subscripts $(+, -)$ will be explained later). Observe that if we write

$$\mathbf{x}^* = (x_0^*, x_1^*, \ldots, x_n^*, \ldots),$$

then

$$\Sigma_{(+,-)}(\mathbf{x}) = \begin{cases} \Sigma_2(\mathbf{x}) & \text{if } x_0 = 0, \\ \Sigma_2(\mathbf{x}^*) & \text{if } x_0 = 1, \end{cases}$$

where Σ_2 is the usual one-sided shift on sequences of two symbols.

A method of visualizing how the orbits of $\mathbf{x}$ are generated by $\Sigma_{(+,-)}$ is the following. Take as way of illustration

$$\mathbf{x} = (0,1,1,0,0,0,1,0,1,1,0,0,1,1,1,0,1,0,0,1,1,0,1,\ldots), \qquad (5.5)$$

so as

$$\begin{array}{lcllllllllllllcl}
\Sigma^1_{(+,-)}(\mathbf{x}) & = & (1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1\ldots) & = & \Sigma^1_2(\mathbf{x}) \\
\Sigma^2_{(+,-)}(\mathbf{x}) & = & (0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0\ldots) & = & \Sigma^2_2(\mathbf{x}^*) \\
\Sigma^3_{(+,-)}(\mathbf{x}) & = & (1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0\ldots) & = & \Sigma^3_2(\mathbf{x}^*) \\
\Sigma^4_{(+,-)}(\mathbf{x}) & = & (0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0\ldots) & = & \Sigma^4_2(\mathbf{x}) \\
\Sigma^5_{(+,-)}(\mathbf{x}) & = & (0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1\ldots) & = & \Sigma^5_2(\mathbf{x}) \\
\Sigma^6_{(+,-)}(\mathbf{x}) & = & (1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0\ldots) & = & \Sigma^6_2(\mathbf{x}) \\
\Sigma^7_{(+,-)}(\mathbf{x}) & = & (1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1\ldots) & = & \Sigma^7_2(\mathbf{x}^*) \\
\Sigma^8_{(+,-)}(\mathbf{x}) & = & (1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1\ldots) & = & \Sigma^8_2(\mathbf{x}) \\
\Sigma^9_{(+,-)}(\mathbf{x}) & = & (0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0\ldots) & = & \Sigma^9_2(\mathbf{x}^*) \\
\Sigma^{10}_{(+,-)}(\mathbf{x}) & = & (1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1\ldots) & = & \Sigma^{10}_2(\mathbf{x}^*)
\end{array}$$

etc., that is,

$$\Sigma^i_{(+,-)}(\mathbf{x}) = \begin{cases} \Sigma^i_2(\mathbf{x}) & \text{for } i = 0,1,4,5,6,8,\ldots, \\ \Sigma^i_2(\mathbf{x}^*) & \text{for } i = 2,3,7,9,10,\ldots. \end{cases}$$

Write now $\mathbf{x}^*$ directly under $\mathbf{x}$, and mark (for example, with an underline) the initial digit of $\Sigma^i_{(+,-)}(\mathbf{x}), i \geq 0$:

$i =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{x} =$	$\underline{0}$	$\underline{1}$	1	0	$\underline{0}$	$\underline{0}$	$\underline{1}$	0	$\underline{1}$	1	0	$\underline{0}$	$\underline{1}$
$\mathbf{x}^* =$	1	0	$\underline{0}$	$\underline{1}$	1	1	0	$\underline{1}$	0	$\underline{0}$	$\underline{1}$	1	0

(5.6)

That is, we set out from x_0, which is always underlined. If $x_0 = 0$, then go over to x_1 and underline it. If $x_0 = 1$, then go down to x_1^* and underline it. In general, if $\underline{x_i} = 0$ or $\underline{x_i^*} = 0$, go one step rightward on the same row and underline x_{i+1} or x_{i+1}^*, respectively. On the other hand, if $\underline{x_i} = 1$ or $\underline{x_i^*} = 1$, we go one step rightward on the other row and underline x_{i+1}^* or x_{i+1}, respectively. The L-pattern π defined by $\mathbf{x}$ can be found now by ordering all the sequences on the $\mathbf{x}$-row and $\mathbf{x}^*$-row starting with an underlined bit, for $0 \leq i \leq L-1$.

If $\mathbf{x}$ is sequence (5.5), then the ordinal L-patterns of $\mathbf{x}$ under $\Sigma_{(+,-)}$ are obtained by comparing the shifts $\Sigma^i(\mathbf{x})$ for $i = 0,1,4,5,6,8,\ldots$ with the shifts $\Sigma^j(\mathbf{x}^*)$ for $j \neq i$. In particular, $\mathbf{x}$ is of type

$$\pi = \langle 4,5,9,0,2;7,6,10,1,8,3\rangle \in \mathcal{S}_{11} \tag{5.7}$$

under the action of $\Sigma_{(+,-)}$.

Rather than deriving at this point the structure of the allowed ordinal patterns for $\Sigma_{(+,-)}$ (or the tent map Λ for this matter), which follows from the general results of the next section, let us prove here a particular property of the allowed patterns for $\Sigma_{(+,-)}$.

Lemma 6 *The subsequence* $n+2,\dots,n+1,\dots,n$ $(0 \le n \le L-3)$ *cannot appear in the entries of an allowed L-pattern for* $\Sigma_{(+,-)}$. *Thus, the allowed ordinal patterns of* $\Sigma_{(+,-)}$ *cannot contain decreasing subsequences of length* 3.

Proof We prove by contradiction that the order relation

$$\Sigma^2_{(+,-)}(\mathbf{x}) < \Sigma_{(+,-)}(\mathbf{x}) < \mathbf{x} \tag{5.8}$$

cannot hold true. If $x_0 = 0$ there is no way that $\Sigma_{(+,-)}(\mathbf{x}) \equiv \Sigma_2(\mathbf{x}) < \mathbf{x}$. Hence $\mathbf{x} = (1, x_1, x_2, \dots)$ and

$$\Sigma_{(+,-)}(\mathbf{x}) \equiv \Sigma_2(\mathbf{x}^*) = (x_1^*, x_2^*, \dots).$$

By the same token, if $x_1^* = 0$ there is no way that $\Sigma_{(+,-)}(\Sigma_{(+,-)}(\mathbf{x})) \equiv \Sigma^2_{(+,-)}(\mathbf{x}) < \Sigma_{(+,-)}(\mathbf{x})$. Hence

$$\mathbf{x} = (1,0,x_2,\dots), \quad \Sigma_{(+,-)}(\mathbf{x}) = (1, x_2^*, x_3^*, \dots), \quad \Sigma^2_{(+,-)}(\mathbf{x}) = (x_2, x_3, \dots).$$

From $\Sigma_{(+,-)}(\mathbf{x}) < \mathbf{x}$ it follows $x_2^* = 0$. In turn, from $\Sigma^2_{(+,-)}(\mathbf{x}) = (1, x_3, \dots) < \Sigma_{(+,-)}(\mathbf{x}) = (1, 0, x_3^*, \dots)$ it follows $x_3 = 0$. So far, we found that $\mathbf{x} = (1,0,1,0,x_4,\dots)$ (thus $\Sigma_{(+,-)}(\mathbf{x}) = (1,0,1,x_4^*,\dots)$ and $\Sigma^2_{(+,-)}(\mathbf{x}) = (1,0,x_4,\dots)$).

A straightforward induction along these lines yields

$$\mathbf{x} = (1,0,1,0,\dots,1,0,\dots) = ((1,0)^\infty),$$

which is the binary expansion of the rational number 2/3. Since $\Sigma^2_{(+,-)}(\mathbf{x}) = \Sigma_{(+,-)}(\mathbf{x}) = \mathbf{x}$ for this particular sequence (in other words, 2/3 is a fixed point of $\Sigma_{(+,-)}$), the statement follows by contradiction. □

Exercise 5 Prove, using representation (5.4) that the symmetric tent map has dense periodic points, sensitive dependence on initial conditions, and is topologically transitive.

5.1.2 The Interval Structure of the Sets P_π

The points in state space Ω defining an ordinal L-pattern π under the action of a map $f{:}\Omega \to \Omega$ build the set P_π, (3.4). The sets $P_\pi \neq \emptyset$, $\pi \in \mathcal{S}_L$, build in turn the set $\mathcal{P}_L$, which build a finite partition of Ω under the condition set by Proposition 2. In this section we examine the "topology" of $P_\pi \in \mathcal{P}_L$ for some one-dimensional interval maps. For continuous maps, those sets are clearly open sets (hence, an enumerable union of disjoint open intervals), but no further dissection can be made. For the sawtooth map family $x \mapsto Nx \bmod 1$, $N \geq 2$, it is easy to convince oneself that P_π consists of a single open or half-open interval for all admissible patterns $\pi \in \mathcal{S}_L$, $L \geq 2$ (see Figs. 3.2 and 3.3). For the logistic map, Figs. 1.5 and 1.6 show that all $P_\pi \in \mathcal{P}_L$ with $L = 2, 3$ consist of a single open interval, but from Fig. 1.7 it can be read that

$$\begin{aligned}
P_{\langle 0,3,1,2\rangle} &\approx (0.09549, 0.11698) \cup (0.18826, 0.25),\\
P_{\langle 2,0,3,1\rangle} &\approx (0.34549, 0.41318) \cup (0.61126, 0.65451),\\
P_{\langle 1,2,3,0\rangle} &\approx (0.93301, 0.95048) \cup (0.96985, 1).
\end{aligned}$$

We claim the following.

Proposition 6 *For the logistic map and the symmetric tent map, all $P_\pi \neq \emptyset$ consist of one or two components.*

As stated in Example 4 (1), the logistic map g and the symmetric tent map Λ are order isomorphic. Specifically, $g(\phi(x)) = \phi(\Lambda(x))$, where $\phi(x) = \sin^2(\frac{\pi}{2}x)$, $0 \leq x \leq 1$, so that

$$g^n(\phi(x)) = g^m(\phi(x)) \Leftrightarrow \phi(\Lambda^n(x)) = \phi(\Lambda^m(x)) \Leftrightarrow \Lambda^n(x) = \Lambda^m(x).$$

Thus, the curves $y = g^n(x)$ and $y = g^m(x)$ cross at x_0 if and only if the piecewise straight lines $y = \Lambda^n(x)$ and $y = \Lambda^m(x)$ cross at $\phi^{-1}(x_0)$. Moreover, the iterates of Λ have not only a simple graphical representation (triangular waves with frequencies increasing as powers of 2) but also a scaling property that makes Λ handier for the proof of Proposition 6:

$$\begin{aligned}
\Lambda^n(x) &= \Lambda^{n-1}(2x), & 0 \leq x \leq \tfrac{1}{2},\\
\Lambda^n(x) &= \Lambda^{n-1}(2(1-x)), & \tfrac{1}{2} \leq x \leq 1.
\end{aligned} \tag{5.9}$$

Therefore, the left-half part of the graphs $(x, \Lambda^0(x)), (x, \Lambda^1(x)), \ldots, (x, \Lambda^L(x))$ is a "squeezed" copy of the graphs $(x, \frac{x}{2}), (x, \Lambda^0(x)), \ldots, (x, \Lambda^{L-1}(x))$ on the interval $0 \leq x \leq \frac{1}{2}$; indeed, upon rescaling the X-axis by a factor $\frac{1}{2}$, we have $(x, \frac{x}{2}) \mapsto (\frac{x}{2}, \frac{x}{2})$ and $(x, \Lambda^l(x)) \mapsto (\frac{x}{2}, \Lambda^l(x)) = (\frac{x}{2}, \Lambda^{l+1}(\frac{x}{2}))$. The corresponding right-half parts require the squeezed copy of the graphs $(x, 1 - \frac{x}{2}), (x, \Lambda^0(x)), \ldots, (x, \Lambda^{L-1}(x))$ on $0 \leq x \leq \frac{1}{2}$ to be further mirrored with respect to the line $x = \frac{1}{2}$ (this is the transformation $(x, y) \mapsto (x, 1 - x)$); see Fig. 5.1 for further insights.

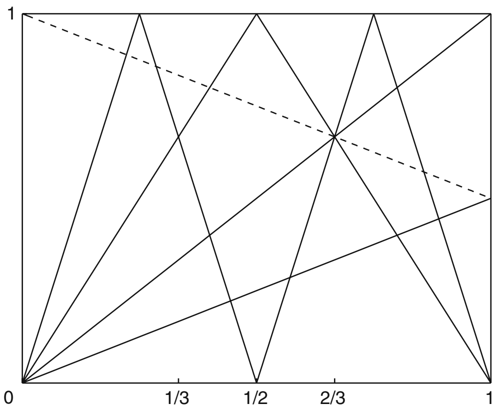

Fig. 5.1 If this figure is "opened" at the right side as a book put upside down, with the line $y = x/2$ only on the left page, the (*dashed*) line $y = 1-x/2$ only on the right page, and the triangular waves $y = \Lambda(x)$, $y = \Lambda^2(x)$ on both, and the resulting graph is shrunk by a factor 1/2 along the X-axis, then we get the graphs of $y = \Lambda^n(x)$, $0 \le n \le 3$. Alternatively, we can go from $\mathcal{P}_3^*$ to $\mathcal{P}_4^*$ just by going first rightward on the bottom page (containing $y = x/2$) of the closed book and then leftward on the top page (containing $y = 1 - x/2$)

Proof Proposition 6 follows from the considerations prior to Proposition 3 (remember the terminology mother and daughter intervals, here shortened to *mother* and *daughter*), together with the following facts.

The decomposition of a mother $P_{\pi_{\text{mother}}} \in \mathcal{P}_L$ into several daughters including two or more *twins* (disjoint subintervals with the same ordinal label) can only happen in intervals containing "vertex" or "bouncing-off" points x_v. As their name indicates, these points correspond to projections onto the X-axis of points at the bottom ($y = 0$) or at the ceiling ($y = 1$) of the unit square at which incoming (left) and outgoing (right) lines $y = \Lambda^l(x)$ meet, like $(\frac{1}{2}, 0)$ and $(\frac{1}{4}, 1)$ in Fig. 5.1. Possibly the most intuitive way to follow the growth of twins around vertex points uses the scaling property (5.9). If $0 < x_v < \frac{1}{2}$, consider the graphs of $y = \frac{x}{2}, y = \Lambda^0(x), \ldots, y = \Lambda^{L-1}(x)$ around $x = 2x_v$. If $2x_v \in P_{\langle \pi_0,\ldots,\pi_{L-1}\rangle}$, then the straight line $y = \frac{x}{2}$ generates (left to right) daughters of $P_{\pi_{\text{mother}}}$ (after squeezing) with labels $\pi_{\text{left}} = \langle \pi_0+1, \ldots, 0, \pi_k+1, \ldots, \pi_{L-1}+1\rangle$, $\pi_{\text{central}} = \langle \pi_0+1, \ldots, \pi_k+1, 0, \ldots, \pi_{L-1}+1\rangle$ and $\pi_{\text{right}} = \langle \pi_0+1, \ldots, 0, \pi_k+1, \ldots, \pi_{L-1}+1\rangle = \pi_{\text{left}}$, with $x_v \in P_{\pi_{\text{central}}} \in \mathcal{P}_{L+1}$. Here k depends on the number of lines meeting at $(x_v, 0)$; if $k = 0$ or $L-1$, then 0 is the first or last entry of the label, respectively. Hence, the set $P_{\pi_{\text{left}}} \cup P_{\pi_{\text{right}}} \in \mathcal{P}_{L+1}$ ($\pi_{\text{left}} = \pi_{\text{right}}$) consists of two disjoint interval components, one on each side of $P_{\pi_{\text{central}}}$. If, on the other hand, $\frac{1}{2} < x_v < 1$, consider the graphs of $y = 1 - \frac{x}{2}, y = \Lambda^0(x), \ldots, y = \Lambda^{L-1}(x)$ around $x = 2(1-x_v)$. If $2(1-x_v) \in P_{\langle \pi_0,\ldots,\pi_{L-1}\rangle}$, then the straight line $y = 1 - \frac{x}{2}$ generates daughters of $P_{\pi_{\text{mother}}}$ (after squeezing and mirroring) with labels $\pi_{\text{left}} = \langle \pi_0+1, \ldots, \pi_k+1, 0, \ldots, \pi_{L-1}+1\rangle$, $\pi_{\text{central}} = \langle \pi_0+1, \ldots, 0, \pi_k+1, \ldots, \pi_{L-1}+1\rangle$, and $\pi_{\text{right}} = \langle \pi_0+1, \ldots, \pi_k+1, 0, \ldots, \pi_{L-1}+1\rangle = \pi_{\text{left}}$. As before, $P_{\pi_{\text{left}}} \cup P_{\pi_{\text{right}}}$ consists of two disjoint interval components, one on each side of $P_{\pi_{\text{central}}}$. Finally, for

$x_v = \frac{1}{2}$ the first set of graphs produces π_{left} and π_{central}, while the second produces π_{central} and $\pi_{\text{right}} = \pi_{\text{left}}$, with $x_v \in P_{\pi_{\text{central}}}$. This mechanism repeats again and again over all generations. After the step $L \to L+1$, only the one-component daughters $P_{\pi_{\text{central}}}$, all of which contain some x_v, can in turn generate twins (two-component grand daughters); the corresponding two-component sisters $P_{\pi_{\text{left}}} \cup P_{\pi_{\text{right}}}$ cannot generate twins because they contain no vertex point. As a result, only one- or two-component intervals are possible, the latter forming a nested structure around some vertex points. From Fig. 5.1 it is clear that all such vertex points originate from $x = \frac{1}{2}, 1$ by squeezing and from $x = \frac{1}{4}$ by squeezing and mirroring. □

Exercise 6 Discuss the interval structure of the sets P_π for the map E_{-2}:$x \mapsto -2x$ mod 1.

5.2 Ordinal Patterns and the Signed Shifts

The results of Sect. 5.1.1 can be generalized to a particular case of piecewise linear maps. Partition the unit interval [0, 1] in $N \geq 2$ equal subintervals,

$$I_k = \left[\frac{k}{N}, \frac{k+1}{N}\right), \quad 0 \leq k \leq N-2 \quad \text{and} \quad I_{N-1} = \left[\frac{N-1}{N}, 1\right]$$

(other choices regarding the endpoints are of course possible), and raise over I_k a "/-lap" of slope $+N$,

$$f(x) = Nx - k,\ x \in I_k,$$

or a "\-lap" of slope $-N$,

$$f(y) = k + 1 - Nx,\ x \in I_k.$$

A map of the unit interval whose graph consists of /-laps and \-laps of slopes $\pm N$, respectively, over the intervals I_k, $0 \leq k \leq N-1$, will be called a *signed saw-tooth map*, the term "signed" referring to the fact that its laps can have positive or negative slope (see Fig. 5.2). We say that a signed sawtooth map f has *signature* $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$, where $\sigma_k \in \{+,-\}$, $0 \leq k \leq N-1$, to summarize that (the graph of) f has a /-lap over I_k whenever $\sigma_k = +$ and a \-lap whenever $\sigma_k = -$. In other words, the kth component of the signature gives the slope sign of the kth lap.

We have already met two important representatives of the signed sawtooth map family: the sawtooth map E_N: $x \mapsto Nx$ mod 1 ($\sigma = (+,\ldots,+)$) and the symmetric tent map Λ ($\sigma = (+,-)$).

Given a signature σ, define the *signed shift* Σ_σ:$\{0,\ldots,N-1\}^{\mathbb{N}_0} \to \{0,\ldots,N-1\}^{\mathbb{N}_0}$ as follows:

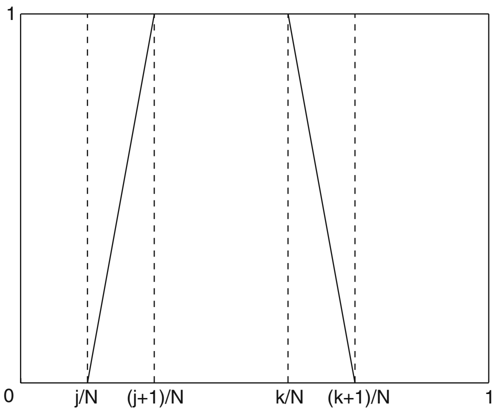

Fig. 5.2 The graph of a generic signed sawtooth map with slopes $\pm N$. The figure only depicts the jth lap, with positive slope, and the kth slope, with negative slope

$$\Sigma_\sigma(x_0, \ldots, x_n, \ldots) = \begin{cases} (x_1, \ldots, x_{n+1}, \ldots) & \text{if } x_0 = k, \sigma_k = +, \\ (N-1-x_1, \ldots, N-1-x_{n+1}, \ldots) & \text{if } x_0 = k, \sigma_k = -. \end{cases}$$

Therefore, if we define the *dual digit* of $k \in \{0, 1, \ldots, N-1\}$ as

$$k^* = N - 1 - k, \tag{5.10}$$

(thus $(k^*)^* = k$), then

$$\Sigma_\sigma(\mathbf{x}) = \begin{cases} \Sigma_N(\mathbf{x}) & \text{if } x_0 = k \text{ and } \sigma_k = +, \\ \Sigma_N(\mathbf{x}^*) & \text{if } x_0 = k \text{ and } \sigma_k = -, \end{cases} \tag{5.11}$$

where

$$\mathbf{x}^* = (x_0^*, \ldots, x_n^*, \ldots) = (N-1-x_0, \ldots, N-1-x_n, \ldots)$$

is the *dual sequence* to $\mathbf{x} = (x_0, \ldots, x_n, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$. In particular, if

$$N = 2\nu + 1,$$

then $\nu = (N-1)/2$ is "self-dual": $\nu^* = \nu$. Note that (5.10) generalizes the definition of dual bit, (5.2).

Important for us is that if f is a signed sawtooth map with signature σ, then f and Σ_σ are order isomorphic via the map ϕ_N:$\{0, 1, \ldots, N-1\}^{\mathbb{N}_0} \to [0, 1]$ defined in (4.3). Observe that $\phi_N(0^\infty) = 0$, $\phi_N(1^\infty) = 1$, and

$$\frac{k}{N} \le \phi_N(\mathbf{x}) \le \frac{k+1}{N} \quad \text{iff} \quad x_0 = k.$$

The technique described in Sect. 5.1 to keep track of the orbits of $\mathbf{x}$ under $\Sigma_{(+,-)}$ can be used for Σ_σ too. The number of symbols N goes in the definition of $\mathbf{x}^*$, while σ_k tells whether we have to jump from the current entry $\underline{x_i} = k$ to x^*_{i+1} or from the current entry $\underline{x^*_i} = k$ to x_{i+1} ($\sigma_k = -$), instead of remaining on the same line ($\sigma_k = +$), when underlining the entries of $\mathbf{x}$ in table (5.6).

Exercise 7 Check that

$$f(\phi_N(\mathbf{x})) = \phi_N(\Sigma_\sigma \mathbf{x}) = \begin{cases} \sum_{n=1}^{\infty} x_n N^{-n} & \text{if } x_0 = k \text{ and } \sigma_k = +, \\ 1 - \sum_{n=1}^{\infty} x_n N^{-n} & \text{if } x_0 = k \text{ and } \sigma_k = -. \end{cases}$$

We turn now to the ordinal patterns realized by a signed shift Σ_σ. Completely analogous to the case $\Sigma_{(+,\ldots,+)} \equiv \Sigma_N$, Chap. 4, the allowed ordinal patterns for Σ_σ can also be decomposed into s-blocks, (4.6), where now the s-block (4.7) contains the locations of the symbol $s \in \{0, \ldots, N-1\}$ in the segments $x_0^{L-1} := x_0, \ldots, x_{L-1}$ of $\mathbf{x}$ and $(x^*)_0^{L-1} := x_0^*, \ldots, x_{L-1}^*$ of $\mathbf{x}^*$, such that the zeroth component of $\Sigma_\sigma^i \mathbf{x}$, $0 \le i \le L-1$, is s (i.e., the locations of the symbol s which are underlined in the $\mathbf{x}$- or $\mathbf{x}^*$-row of table (5.6)). We shall presently see that each s-block consists basically of two kinds of subsequences: monotone ($\sigma_s = +$) or spiraling ($\sigma_s = -$), eventually intertwined by other subsequences of the same kind. Entries in an s-block not belonging to a subsequence will be referred to as solitary or single components or entries.

Theorem 4 *The non-empty blocks* $\pi_{k_0+\cdots+k_{s-1}}, \ldots, \pi_{k_0+\cdots+k_{s-1}+k_s-1}$, $0 \le s \le N-1$, *of* $\pi(\mathbf{x}) \in \mathcal{S}_L$ *fulfill the following basic restrictions:*

R*1 *If* $\sigma_s = +$, $0 < s < N-1$, *then the s-block is built by increasing subsequences,*

$$n, \ldots, n+1, \ldots, n+l-1 \tag{5.12}$$

($l \ge 2$) *and/or decreasing subsequences,*

$$n+l-1, \ldots, n+1, \ldots, n \tag{5.13}$$

($l \ge 2$) *and/or solitary components* ($l = 1$). *If* $\sigma_0 = +$, *then the* 0*-block consists of increasing subsequences (5.12) and/or solitary components. If* $\sigma_{N-1} = +$, *then the* $(N-1)$*-block consists of decreasing subsequences (5.13) and/or solitary components.*

R*2 *If* $\sigma_s = -$, $0 < s < N-1$, *then the s-block is built by even-length spiraling subsequences*

$$n+2l-2, \ldots, n+2, \ldots, n, \ldots, n+1, \ldots, n+3, \ldots, n+2l-1 \tag{5.14}$$

with the entry $n+2l$ *on an anterior block* (*if* $n+2l \le L-1$) *and/or the mirrored subsequences*

$$n+2l-1,\ldots,n+3,\ldots,n+1,\ldots,n,\ldots,n+2,\ldots,n+2l-2 \quad (5.15)$$

with the entry $n+2l$ *on a posterior block* ($if\ n+2l \leq L-1$) *and/or odd-length spiraling subsequences*

$$n+2l,\ldots,n+2l-2,\ldots,n+2,\ldots,n,\ldots,n+1,\ldots,n+3,\ldots,n+2l-1 \quad (5.16)$$

with the entry $n+2l+1$ *on a posterior block* ($if\ n+2l+1 \leq L-1$) *and/or the mirrored subsequences*

$$n+2l-1,\ldots,n+3,\ldots,n+1,\ldots,n,\ldots,n+2,\ldots,n+2l-2,\ldots,n+2l \quad (5.17)$$

with the entry $n+2l+1$ *on an anterior block* ($if\ n+2l+1 \leq L-1$) *and/or solitary components. If* $\sigma_0 = -$*, then the first block consists of spiraling subsequences of the form (5.15) and/or (5.16) and/or solitary components. If* $\sigma_{N-1} = -$*, then the last block consists of spiraling subsequences of the form (5.14) and/or (5.17) and/or solitary components.*

R*3 *If (i)* $\sigma_s = +$*, (ii) the entries* $m, n \leq L-2$ *belong to the s-block of* $\pi(\mathbf{x})$*, and (iii) m appears on the left of n, then* $m+1$ *appears also on the left of* $n+1$ *(not necessarily in the same block). If, on the other hand, (i)* $\sigma_s = -$*, (ii) the entries* $m, n \leq L-2$ *belong to the s-block of* $\pi(\mathbf{x})$*, and (iii) m appears on the left of n, then* $m+1$ *appears on the right of* $n+1$ *(not necessarily in the same block).*

Proof **R*1)** Let $s \in \{0, 1, \ldots, N-1\}$ and consider an s-run of length $l \geq 2$ in the segment x_0^{L-1} of $\mathbf{x}$:

$i =$	$\ldots$	n	$n+1$	$\ldots$	$n+l-1$	$n+l$	$\ldots$
$\mathbf{x} =$	$\ldots$	$\underline{s}$	$\underline{s}$	$\ldots$	$\underline{s}$	$\underline{r}$	$\ldots$
$\mathbf{x}^* =$	$\ldots$	$N-1-s$	$N-1-s$	$\ldots$	$N-1-s$	$N-1-r$	$\ldots$

where $r \in \{0, 1, \ldots, N-1\}$ and $r \neq s$. If (i) $s < N-1$ and (ii) $x_{n+l} = r > s$, then this s-run contributes the increasing subsequence

$$n,\ldots,n+1,\ldots,n+l-1 \quad (5.18)$$

to the s-block of $\pi(\mathbf{x})$. If, on the other hand, (i) $s > 0$ and (ii) $x_{n+l} = r < s$, then the s-run contributes the decreasing subsequence

$$n+l-1,\ldots,n+1,\ldots,n. \quad (5.19)$$

The "..." between the entries of these subsequences allow for entries eventually proceeding from other s-runs in $\mathbf{x}$ or $\mathbf{x}^*$ (see Example 7).

It follows that the 0-block can contain only increasing subsequences (and single entries not belonging to subsequences in the block), whereas the $(N-1)$-block can contain only decreasing subsequences (and single entries not belonging to subsequences in the block).

R*2) Consider an s-run of even length $2l$ in the segment x_0^{L-1} of $\mathbf{x}$. Thus,

$i =$	n	$n+1$	$\dots$	$n+2l-2$	$n+2l-1$	$n+2l$
$\mathbf{x} =$	$\underline{s}$	$N-1-1$	$\dots$	$\underline{s}$	$N-1-s$	$\underline{r}$
$\mathbf{x}^* =$	$N-1-s$	$\underline{s}$	$\dots$	$N-1-s$	$\underline{s}$	$N-1-r$

where $r \in \{0, 1, \dots, N-1\}$ and $r \neq s$. Therefore, if (i) $s > 0$ and (ii) $x_{n+2l-1} = N-1-s < x^*_{n+2l} = N-1-r$, i.e., $r < s$, then the s-block of $\pi(\mathbf{x})$ will contain the spiraling subsequence

$$n+2l-2, \dots, n+2, \dots, n, \dots, n+1, \dots, n+3, \dots, n+2l-1. \qquad (5.20)$$

Hence the entry $n+2l$ will appear in the r-block (provided $n+2l \leq L-1$), which precedes the s-block in $\pi(\mathbf{x})$ because $r < s$. If, on the other hand, (i) $s < L-1$ and (ii) $x_{n+2l-1} = N-1-s > x^*_{n+2l} = N-1-r$, i.e., $r > s$, then we obtain the mirrored, spiraling subsequence

$$n+2l-1, \dots, n+3, \dots, n+1, \dots, n, \dots, n+2, \dots, n+2l-2, \qquad (5.21)$$

with the symbol $n+2l$ in a posterior block (provided $n+2l \leq L-1$), namely, on the r-block.

Consider now an s-run of odd length $2l+1$ in the segment x_0^{L-1} of $\mathbf{x}$. Thus,

$i =$	n	$n+1$	$\dots$	$n+2l-1$	$n+2l$	$n+2l+1$
$\mathbf{x} =$	$\underline{s}$	$N-1-s$	$\dots$	$N-1-s$	$\underline{s}$	$N-1-r$
$\mathbf{x}^* =$	$N-1-s$	$\underline{s}$	$\dots$	$\underline{s}$	$N-1-s$	$\underline{r}$

where $r \in \{0, 1, \dots, N-1\}$ and $r \neq s$. Therefore, if (i) $s > 0$ and (ii) $x^*_{n+2l} = N-1-s < x_{n+2l+1} = N-1-r$, i.e., $r < s$, then the s-block of $\pi(\mathbf{x})$ will contain the spiraling subsequence

$$n+2l-1, \dots, n+3, \dots, n+1, \dots, n, \dots, n+2, \dots, n+2l-2, \dots, n+2l.$$

The entry $n+2l+1$ will appear on the r-block (provided $n+2l+1 \leq L-1$), which is on the left of the s-block because $r < s$. If, on the other hand, (i) $s < L-1$ and (ii) $x^*_{n+2l} = N-1-s > x_{n+2l+1} = N-1-r$, i.e., $r > s$, then we obtain the mirrored, spiraling subsequence

$$n+2l, \dots, n+2l-2, \dots, n+2, \dots, n, \dots, n+1, \dots, n+3, \dots, n+2l-1,$$

with the entry $n+2l+1$ in a block on the right of the s-block (provided $n+2l+1 \leq L-1$).

The corresponding results for the first ($s = 0$) and last ($s = N - 1$) blocks follow readily from these general results.

R*3) If m and n belong to the s-block, $\sigma_s = +$, and $\Sigma_\sigma^m(\mathbf{x}) < \Sigma_\sigma^n(\mathbf{x})$ for $\mathbf{x} \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$, then

$$\Sigma_\sigma^m(\mathbf{x}) = (s, x_{m+1}, \ldots) < (s, x_{n+1}, \ldots) = \Sigma_\sigma^n(\mathbf{x}).$$

By the definition of lexicographical order, there are two possibilities: (i) $x_{m+1} < x_{n+1}$ or (ii) $x_{m+k} = x_{n+k}$ for $1 \leq k \leq l - 1$, $l \geq 2$, and $x_{m+l} < x_{n+l}$. In both cases,

$$\Sigma_\sigma^{m+1}(\mathbf{x}) = (x_{m+1}, \ldots) < (x_{n+1}, \ldots) = \Sigma_\sigma^{n+1}(\mathbf{x})$$

and, hence, the entry $m + 1$ appears on the left of $n + 1$ in $\pi(\mathbf{x})$.

If, on the other hand, m and n belong to the s-block, $\sigma_s = -$, and $\Sigma_\sigma^m(\mathbf{x}) < \Sigma_\sigma^n(\mathbf{x})$, then

$$\Sigma_\sigma^m(\mathbf{x}) = (s, x_{m+1}, \ldots) < (s, x_{n+1}, \ldots) = \Sigma_\sigma^n(\mathbf{x}).$$

As before, there are two possibilities: (i) $x_{m+1} < x_{n+1}$ and (ii) $x_{m+k} = x_{n+k}$ for $1 \leq k \leq l-1$, $l \geq 2$, and $x_{m+l} < x_{n+l}$. In both cases,

$$\Sigma_\sigma^{m+1}(\mathbf{x}) = (N - 1 - x_{m+1}, \ldots) > (N - 1 - x_{n+1}, \ldots) = \Sigma_\sigma^{n+1}(\mathbf{x})$$

and, hence, the entry $m + 1$ appears on the right of $n + 1$ in $\pi(\mathbf{x})$. □

Conditions R*1–R*3 are not only necessary for an ordinal pattern to be allowed for Σ_σ, $\sigma = (\sigma_0, \ldots, \sigma_{N-1})$, but also sufficient. Indeed, given the s-block decomposition of $\pi \in \mathcal{S}_L$ with each block satisfying the pertinent restrictions, then it is a simple matter to construct sequences $\mathbf{x} \in \{0, \ldots, N-1\}^{\mathbb{N}_0}$ of type π. Furthermore, it is obvious that all L-patterns with $L \leq N$ are allowed for Σ_σ.

Corollary 3 *If $\pi = \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle$ is allowed (correspondingly, forbidden) for Σ_σ, $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$, then $\pi_{mirrored} = \langle \pi_{L-1}, \pi_{L-2}, \ldots, \pi_0 \rangle$ is allowed (correspondingly, forbidden) for $\Sigma_{\sigma_{mirrored}}$, where*

$$\sigma_{mirrored} := (\sigma_{N-1}, \sigma_{N-2}, \ldots, \sigma_0).$$

In the particular case $\sigma = \sigma_{mirrored}$, it follows that π is allowed (correspondingly, forbidden) for Σ_σ, iff $\pi_{mirrored}$ is also allowed (correspondingly, forbidden) for Σ_σ. These statements hold also true if "forbidden pattern" is replaced by "root forbidden pattern."

Proof The s-block structure of an allowed ordinal pattern is preserved under the transformation $\pi \mapsto \pi_{\text{mirrored}}$. Indeed, monotone subsequences transform into monotone subsequences (in particular, increasing subsequences of the 0-block transform in decreasing subsequences of the $(N-1)$-block and vice versa), and spiraling subsequences go over to spiraling subsequences.

By the same token, mirrored outgrowth forbidden patterns for Σ_σ will be outgrowth forbidden patterns for $\Sigma_{\sigma_{\text{mirrored}}}$. It follows that $\pi \in \mathcal{S}_L$ is a root forbidden pattern for Σ_σ in the case $\sigma = \sigma_{\text{mirrored}}$, iff π_{mirrored} is also a root forbidden pattern for Σ_σ. □

Remark 1 If the first or last element of a monotone subsequence appearing in an s-block is assigned to the anterior or posterior block, respectively (if any), then the remaining subsequence preserves its increasing or decreasing character—or it becomes a single entry. If the leftmost or the rightmost element of a spiraling subsequence is assigned to the anterior or posterior block (if any), then the remaining subsequence preserves its spiraling character, eventually appearing also a new single entry in the same block. This implies that, when carrying out a decomposition of an ordinal L-pattern into s-blocks, $L \geq N$, we may assume without loss of generality that all s-blocks are non-empty.

For $\sigma_k = +, 0 \leq k \leq N-1$, we recover from Theorem 4 the restrictions fulfilled by the allowed patterns for Σ_N (Lemma 2). In the case $\sigma = (+,-)$, considered in Sect. 5.1.1, there are only two symbols and two blocks in the decomposition of the ordinal patterns. Restrictions R*1 and R*2 entail then that $\pi = \langle 2,1,0 \rangle$ is forbidden for $\Sigma_{(+,-)}$ (Lemma 6). Indeed, $\pi_0, \pi_1 = 2, 1$ cannot occur in the 0-block because it is a decreasing sequence (R*1), hence $\pi = \langle 2;1,0 \rangle$; but then the entry 2 should appear on the right of $\pi_1, \pi_2 = 1, 0$ in order to form a spiraling subsequence (R*2); the restriction R*3 is also violated.

The five root forbidden 4-patterns for the logistic map (hence, for Λ and $\Sigma_{(+,-)}$) were found graphically in Sect. 1.2, (1.38). We check here that they do fail to satisfy the restrictions R*1–R*3:

- $\langle 0;2,3,1 \rangle$ violates R*2; $\langle 0,2;3,1 \rangle$ and $\langle 0,2,3;1 \rangle$ violate R*3.
- $\langle 1;0,2,3 \rangle$ violates R*3; $\langle 1,0;2,3 \rangle$ and $\langle 1,0,2;3 \rangle$ violate R*1.
- $\langle 1;0,3,2 \rangle$ violates R*3; $\langle 1,0;3,2 \rangle$ and $\langle 1,0,3;2 \rangle$ violate R*1.
- $\langle 1;3,0,2 \rangle$ violates R*3; $\langle 1,0;3,2 \rangle$ and $\langle 1,0,3;2 \rangle$ violate R*1.
- $\langle 3;1,2,0 \rangle$ violates R*2; $\langle 3,1;2,0 \rangle$ violates R*3 and $\langle 3,1,2;0 \rangle$ violates R*1.

Exercise 8 Check that the allowed patterns for the logistic map, Fig. 1.7, comply with the restrictions (R*1)–(R*4).

Finally, let us prove that $\Sigma_{(+,-)}$ has root forbidden L-patterns for $L \geq 5$.

Theorem 5 *The patterns*

$$\pi = \langle 3, \ldots, L-2, 0, 1, 2, L-1 \rangle \in \mathcal{S}_L, \tag{5.22}$$

$L \geq 5$, *are root forbidden patterns for* $\Sigma_{(+,-)}$.

Proof Let us check that (5.22) is a forbidden pattern. First of all, $\pi_{L-5}, \pi_{L-4} = L-2, 0$ cannot belong to the 0-block because $\pi_{L-5}+1 = L-1$ is not on the left of $\pi_{L-4}+1 = 1$ (R*3). Hence

$$\pi = \langle 3, \ldots, L-2;0, 1, 2, L-1 \rangle\,.$$

But $\pi_{L-4}, \pi_{L-3}, \pi_{L-2} = 0, 1, 2$ is not a spiraling subsequence, hence it violates R*2.

Furthermore, we claim that (5.22) is a root forbidden pattern. Otherwise, see (3.12), (i) π would be an outgrowth pattern of group I, i.e., the $(L-1)$-pattern obtained from π after removing the entry $L-1$,

$$\langle 3, \ldots, L-2, 0, 1, 2 \rangle \in \mathcal{S}_{L-1}, \tag{5.23}$$

would be forbidden or (ii) π would be an outgrowth pattern of group II, i.e., the $(L-1)$-pattern obtained from π after removing the entry 0 and subtracting 1 from each remaining entry,

$$\langle 2, \ldots, L-3, 0, 1, L-2 \rangle \in \mathcal{S}_{L-1}, \tag{5.24}$$

would be forbidden. But (5.23) admits the s-block decompositions

$$\langle 3, \ldots, L-2, 0; 1, 2 \rangle \text{ and } \langle 3, \ldots, L-2, 0, 1; 2 \rangle ,$$

while (5.24) admits the decomposition

$$\langle 2, \ldots, L-3; 0, 1, L-2 \rangle .$$

□

Exercise 9 Consider the eight cylinder sets $C_{i_0 i_1 i_2}$ of $\{0, 1\}^{\mathbb{N}_0}$. Check that the sequences of these sets are of the following types under $\Sigma_{(+,-)}$:

(i) The sequences of C_{000} are of type $\langle 0, 1, 2 \rangle$.
(ii) The sequences of C_{001} are also of type $\langle 0, 1, 2 \rangle$.
(iii) The sequences $(0, 1, 0, 0, \ldots) \in C_{010}$ are of type $\langle 0, 1, 2 \rangle$, while the sequences $(0, 1, 0, 1, \ldots) \in C_{010}$ are of type $\langle 0, 2, 1 \rangle$.
(iv) The sequences of C_{011} are of type $\langle 0, 2, 1 \rangle$ or $\langle 2, 0, 1 \rangle$.
(v) The sequences of C_{100} are of type $\langle 2, 0, 1 \rangle$.
(vi) The sequences of C_{101} are of type $\langle 1, 0, 2 \rangle$ or $\langle 2, 0, 1 \rangle$.
(vii) The sequences of C_{110} are of type $\langle 1, 0, 2 \rangle$ or $\langle 1, 2, 0 \rangle$.
(viii) The sequences of C_{111} are of type $\langle 1, 2, 0 \rangle$.

Among the signed sawtooth maps, those with signatures of alternating signs (we call them *alternating signatures*) have the special property of being continuous. The tent map is one of the two possibilities for $N = 2$. The next theorem generalizes the result that the tent map has a forbidden pattern already for $L = 3$.

Theorem 6 *Let Σ_σ be a shift with alternating signature $\sigma = (\sigma_0, \ldots, \sigma_{N-1})$.*

1. *If N is even, then Σ_σ has forbidden L-patterns for $L \geq N + 1$.*
2. *If N is odd and $\sigma = (+, -, \ldots, -, +)$, then Σ_σ has forbidden L-patterns for $L \geq N + 1$.*

3. *If N is odd and $\sigma = (-, +, \ldots, +, -)$, then (i) all ordinal $(N+1)$-patterns are allowed for Σ_σ and (ii) Σ_σ has forbidden L-patterns for $L \geq N+2$.*

In cases 2 and 3, along with a forbidden pattern $\pi \in \mathcal{S}_L$, π_{mirrored} will also be a forbidden pattern (Corollary 3).

Proof Remember that if Σ_σ has a forbidden pattern of length L_0, then its outgrowth patterns provide forbidden L-patterns for every $L \geq L_0$. Hence, we need only to exhibit forbidden patterns of the minimal lengths claimed in each case of Theorem 6.

1. Let $N \geq 2$ be even. There are two possibilities: (a) $\sigma_0 = +$ and $\sigma_{N-1} = -$ and (b) $\sigma_0 = -$ and $\sigma_{N-1} = +$. Since the signatures of these cases are mirrored from each other, we need to consider only one of them (Corollary 3), say (b).

A forbidden pattern of length $L = N+1$ can be constructed attending to the positive signs of σ, together with the first and last negative signs, as follows. Take the entry $\pi_0 = 0$ for $\sigma_0 = -$,

$$\pi = \langle 0, \ldots \rangle ,$$

the decreasing subsequence $\pi_{2k-1}, \pi_{2k} = 2k, 2k-1$ for $\sigma_{2k-1} = +$, $1 \leq k \leq N/2 - 1$,

$$\pi = \langle 0, 2, 1, \ldots, 2k, 2k-1, \ldots, N-2, N-3, \ldots \rangle ,$$

and the increasing subsequence $\pi_{N-1}, \pi_N = N-1, N$ for $\sigma_{N-1} = -$,

$$\pi = \langle 0, 2, 1, \ldots, 2k, 2k-1, \ldots, N-2, N-3, N-1, N \rangle \in \mathcal{S}_{N+1}.$$

(For $N = 2$, $\pi = \langle 0, 1, 2 \rangle \in \mathcal{S}_3$.) Then R*3 requires a first semicolon between $\pi_0 = 0$ and $\pi_1 = 2$, a second semicolon between $\pi_1 = 2$ and $\pi_2 = 1, \ldots,$ and an $(N-1)$th semicolon (the maximal number allowed) between $\pi_{N-2} = N-3$ and $\pi_{N-1} = N-1$. Still the increasing subsequence $\pi_{N-1}, \pi_N = N-1, N$ in the last block ($\sigma_{N-1} = +$) violates R*1.

2. Let $N \geq 3$ be odd and $\sigma_0 = \sigma_{N-1} = +$. A forbidden pattern of length $L = N+1$ can then be constructed attending to positive signs of σ. Take the decreasing subsequence $\pi_0, \pi_1 = 1, 0$ for $\sigma_0 = +$,

$$\pi = \langle 1, 0, \ldots \rangle ,$$

the decreasing subsequence $\pi_{2k}, \pi_{2k+1} = 2k+1, 2k$ for $\sigma_{2k} = +$, $1 \leq k \leq (N-1)/2$,

$$\pi = \langle 1, 0, 3, 2, \ldots, 2k+1, 2k, \ldots, N-2, N-3, \ldots \rangle ,$$

and the increasing subsequence $\pi_{N-1}, \pi_N = N-1, N$ for $\sigma_{N-1} = +$,

$$\pi = \langle 1, 0, 3, 2, \ldots, 2k+1, 2k, \ldots, N-2, N-3, N-1, N \rangle \in \mathcal{S}_{N+1}.$$

(For $N = 3$, $\pi = \langle 1, 0, 2, 3\rangle \in \mathcal{S}_4$.) Then, R*3 requires a first semicolon between $\pi_0 = 1$ and $\pi_1 = 0$, a second semicolon between $\pi_1 = 0$ and $\pi_2 = 3, \ldots$, and an $(N-1)$th semicolon (the maximal number allowed) between $\pi_{N-2} = N-3$ and $\pi_{N-1} = N-1$. Hence we are left with the increasing subsequence $\pi_{N-1}, \pi_N = N-1, N$ in the last block ($\sigma_{N-1} = +$), what violates R*1.

3. Finally, let $N \geq 3$ be odd and $\sigma_0 = \sigma_{N-1} = -$.

(i) Let us prove that all ordinal $(N+1)$-patterns are allowed for $\Sigma_{(-,+,\ldots,+,-)}$. Given $\pi \in \mathcal{S}_{N+1}$, there are three possibilities: (a) $N = \pi_0$, (b) $N = \pi_n$ with $1 \leq n \leq N-1$, or (c) $N = \pi_N$. In the first case, π admits the allowed decomposition

$$\pi = \langle N, \pi_1; \pi_2; \ldots; \pi_k; \ldots; \pi_N\rangle .$$

In the second case, π admits the decomposition

$$\pi = \langle \pi_0; \pi_1; \ldots; \pi_{n-1}; N, \pi_{n+1}; \ldots; \pi_N\rangle$$

both if $\sigma_n = +$ or $\sigma_n = -$. In the third case, π admits the decomposition

$$\pi = \langle \pi_0;\pi_1; \ldots;\pi_k; \ldots;\pi_{N-1}, N\rangle .$$

(ii) A forbidden pattern of length $L = N+2$ can be constructed attending to the blocks with negative sign. Let first $N = 5 \bmod 4$, so that the central sign of σ is $\sigma_{(N-1)/2} = -$. Take the increasing subsequence $\pi_0, \pi_1 = 0, 1$ for $\sigma_0 = -$,

$$\pi = \langle 0, 1, \ldots\rangle ,$$

the decreasing subsequence $\pi_N, \pi_{N+1} = 3, 2$ for $\sigma_{N-1} = -$,

$$\pi = \langle 0, 1, \ldots, 3, 2\rangle ,$$

the increasing subsequence $\pi_2, \pi_3 = 4, 5$ for $\sigma_2 = -$,

$$\pi = \langle 0, 1, 4, 5, \ldots, 3, 2\rangle ,$$

the decreasing subsequence $\pi_{N-2}, \pi_{N-1} = 7, 6$ for $\sigma_{N-3} = -$,

$$\pi = \langle 0, 1, 4, 5, \ldots, 7, 6, 3, 2\rangle ,$$

and so on until arriving at the central block, $\sigma_{(N-1)/2} = -$, for which we take $\pi_{(N-1)/2},\pi_{(N+1)/2},\pi_{(N+3)/2} = N-1, N+1, N$,

$$\pi = \langle 0, 1, 4, 5, \ldots, N-1, N+1, N, \ldots, 7, 6, 3, 2 \rangle \in \mathcal{S}_{N+2}.$$

(For $N = 5$, $\pi = \langle 0, 1, 4, 6, 5, 3, 2 \rangle \in \mathcal{S}_7$.) Then, R*3 requires a first semicolon between $\pi_0 = 0$ and $\pi_1 = 1$, a second semicolon between $\pi_1 = 1$ and $\pi_2 = 4, \ldots$, an $((N+1)/2)$th semicolon between $\pi_{(N-1)/2} = N-1$ and $\pi_{(N+1)/2} = N+1$ or between $\pi_{(N+1)/2} = N+1$ and $\pi_{(N+3)/2} = N$ (since the central subsequence $N-1, N+1, N$ is not spiraling), ..., and an $(N-1)$th semicolon (the maximal number allowed) between $\pi_{N-1} = 6$ and $\pi_N = 3$. But the sequence $\pi_N, \pi_{N+1} = 3, 2$ in the last block ($\sigma_{N-1} = -$) violates R*3 because $\pi_N + 1 = 4$ is not on the right of $\pi_{N+1} + 1 = 3$.

In the case $N = 3 \bmod 4$, the central sign of σ is $\sigma_{(N-1)/2} = +$. The construction of a forbidden pattern of length $L = N+2$ follows the same assignment of entry pairs as before for σ_0, σ_{N-1}, σ_2, ..., $\sigma_{(N-3)/2}$, but takes $\pi_{(N+1)/2}, \pi_{(N+3)/2}, \pi_{(N+5)/2} = N+1, N, N-1$ for $\sigma_{(N+1)/2} = -$:

$$\pi = \langle 0, 1, 4, 5, \ldots, N-2, N+1, N, N-1, \ldots, 7, 6, 3, 2 \rangle \in \mathcal{S}_{N+2}.$$

(For $N = 3$, $\pi = \langle 0, 1, 4, 3, 2 \rangle \in \mathcal{S}_5$.) Then, R*3 requires a first semicolon between $\pi_0 = 0$ and $\pi_1 = 1$, a second semicolon between $\pi_1 = 1$ and $\pi_2 = 4$, ..., an $((N+1)/2)$th semicolon between $\pi_{(N-1)/2} = N-2$ and $\pi_{(N+1)/2} = N+1$ or between $\pi_{(N+1)/2} = N+1$ and $\pi_{(N+3)/2} = N$ (since the subsequence $\pi_{(N-1)/2}, \pi_{(N+1)/2}, \pi_{(N+3)/2} = N-2, N+1, N$ cannot belong to an s-block with positive sign because $\pi_{(N-1)/2}+1 = N-1$ is not on the left of $\pi_{(N+3)/2}+1 = N+1$), ..., and an $(N-1)$th semicolon (the maximal number allowed) between $\pi_{N-1} = 6$ and $\pi_N = 3$. But the sequence $\pi_N, \pi_{N+1} = 3, 2$ in the last block ($\sigma_{N-1} = -$) violates R*3 because $\pi_N + 1 = 4$ is not on the right of $\pi_{N+1} + 1 = 3$. □

A further signature with general features is $\sigma = (-, -, \ldots, -)$.

Theorem 7 *The shift* Σ_σ *with* $\sigma_0 = \cdots = \sigma_{N-1} = -$, $N \geq 2$, *has*

1. *allowed L-patterns for* $L \leq N+1$ *and*
2. *root forbidden L-patterns for* $L \geq N+2$.

Since $\sigma = (-, \ldots, -) = \sigma_{\text{mirrored}}$, the number of root forbidden patterns for Σ_σ will be even (Corollary 3).

Proof 1. We need to consider only the case $L = N+1$, since all L-patterns with $L \leq N$ are trivially allowed. Given $\pi \in \mathcal{S}_{N+1}$, there are three possibilities: (i) $N = \pi_0$, (ii) $N = \pi_n$ with $1 \leq n \leq N-1$, or (iii) $N = \pi_N$. The decompositions (i)

$$\pi = \langle N, \pi_1; \pi_2; \ldots; \pi_k; \ldots; \pi_N \rangle,$$

(ii)

$$\pi = \langle \pi_0;\, \pi_1; \ldots;\, \pi_{n-1};\, N, \pi_{n+1}; \ldots;\, \pi_N \rangle \quad \text{or} \quad \langle \pi_0;\, \pi_1; \ldots;\, \pi_{n-1}, N; \pi_{n+1}; \ldots;\, \pi_N \rangle$$

(since $\pi_{n-1}, \pi_n, \pi_{n+1} = \pi_{n-1}, N, \pi_{n+1}$ does not form a spiraling subsequence), and (iii)

$$\pi = \langle \pi_0; \pi_1; \ldots; \pi_k; \ldots; \pi_{N-1}, N \rangle ,$$

show that any $\pi \in \mathcal{S}_{N+1}$ is allowed for Σ_σ, $\sigma_0 = \cdots = \sigma_{N-1} = -$.

2. Consider

$$\pi = \langle 0, 1, 2, \ldots, N-1, N, N+1 \rangle \in \mathcal{S}_{N+2}.$$

Then R*2 requires a first semicolon between 0 and 1, a second semicolon between 1 and 2, and an $(N-1)$th semicolon (the maximal number allowed) between $N-2$ and $N-1$. This leads to a last block $\pi_{N-1}, \pi_N, \pi_{N+1} = N-1, N, N+1$, which is not a spiraling subsequence. Hence π is forbidden.

The assumption that π is not a root forbidden pattern leads to the fact that π is outgrowth of the forbidden pattern

$$\langle 0, 1, 2, \ldots, N-1, N \rangle \in \mathcal{S}_{N+1},$$

whether π belongs to group I or II (3.12). But clearly this pattern admits the decomposition

$$\langle 0;1;2;\ldots;N-1, N \rangle ,$$

with $N-1$ semicolons (the maximal number allowed). This contradiction shows that π is not an outgrowth forbidden pattern. Needless to say (Corollary 3),

$$\pi_{\text{mirrored}} = \langle N+1, N, N-1, \ldots, 2, 1, 0 \rangle$$

is also a root forbidden pattern. □

To conclude this chapter, we consider briefly the existence of *root* forbidden patterns for the signed shifts on $N \geq 3$ symbols. For $\sigma = (+, \ldots, +)$ and $\sigma = (-, \ldots, -)$ we know that there exist root forbidden patterns for every $L \geq N+2$ (Theorems 2 and 7, respectively). The structure of the forbidden ordinal patterns depends, of course, on the signature of the signed shift envisaged, thus the construction of root forbidden patterns can only be done, in general, on a case-by-case basis.

To illustrate this point, consider the signed shifts (with mixed signs) on three symbols. Because of the relation between the allowed/forbidden patterns for Σ_σ and $\Sigma_{\sigma_{\text{mirrored}}}$, only the following four cases are really distinct:

Case a: $\sigma = (+, +, -)$, Case b: $\sigma = (+, -, +)$,
Case c: $\sigma = (+, -, -)$, Case d: $\sigma = (-, +, -)$.

These four cases were studied in [17]. There it is proven that all the signed shifts (a)–(d) have root forbidden L-patterns for $L \geq 5$. Furthermore, $\Sigma_{(+,-,+)}$ has two (root) forbidden 4-patterns, $\Sigma_{(+,-,-)}$ has one (root) forbidden 4-pattern, while $\Sigma_{(+,+,-)}$, $\Sigma_{(-,+,-)}$ have no forbidden 4-patterns. Of course, the same holds for any map order isomorphic to those signed shifts, in particular for the corresponding signed saw-tooth maps.

Exercise 10 Check the following statements on root forbidden patterns for Σ_σ in the four cases a–d.

(a) The patterns

$$\pi = \langle 0, L-1, 2, 3, \ldots, L-2, 1 \rangle \in \mathcal{S}_L,$$

$L \geq 5$, are root forbidden patterns for $\Sigma_{(+,+,-)}$.

(b) The patterns

$$\pi = \langle L-2, 0, L-4, \ldots, 3, 1, 2, 4, \ldots, L-3, L-1 \rangle \in \mathcal{S}_L$$

if $L \geq 5$ is odd and

$$\pi = \langle L-1, L-3, \ldots, 3, 1, 2, 4, \ldots, L-4, 0, L-2 \rangle \in \mathcal{S}_L$$

if $L \geq 6$ is even, together with their corresponding mirrored patterns, are root forbidden patterns for $\Sigma_{(+,-,+)}$. (If $L = 5$, then $\pi = \langle 3, 0, 1, 2, 4 \rangle$; if $L = 6$, then $\pi = \langle 5, 3, 1, 2, 0, 4 \rangle$.)

(c) The patterns

$$\pi = \langle 2, 1, 0, 3, 4 \rangle \in \mathcal{S}_5,$$
$$\pi = \langle L-3, \ldots, 4, 2, 1, 0, 3, 5, \ldots, L-4, L-2, L-1 \rangle \in \mathcal{S}_L$$

for $L \geq 7$ odd, and

$$\pi = \langle L-1, L-2, L-4, \ldots, 4, 2, 1, 0, 3, 5, \ldots, L-3 \rangle \in \mathcal{S}_L$$

for $L \geq 6$ even, are root forbidden patterns for $\Sigma_{(+,-,-)}$. Although $\sigma = (+,-,-) \neq \sigma_{mirrored} = (-,-,+)$, the mirrored patterns of these patterns are also root forbidden patterns for $\Sigma_{(+,-,-)}$.

(d) The patterns

$$\pi = \langle 0, 1, 4, 3, 2 \rangle \in \mathcal{S}_5,$$
$$\pi = \langle 0, 1, L-1, L-2, \ldots, 3, 2, 4, \ldots, L-3 \rangle \in \mathcal{S}_L$$

if $L \geq 7$ is odd and

$$\pi = \langle 0, 1, L-1, L-2, \ldots, 4, 2, 3, \ldots, L-3 \rangle \in \mathcal{S}_L$$

if $L \geq 6$ is even, together with the corresponding mirrored patterns, are root forbidden patterns for $\Sigma_{(-,+,-)}$.

Exercise 11 Using signed sawtooth maps with alternating signature, construct a *continuous* map whose orbits realize all possible ordinal patterns (*hint*: the construction is similar to Fig. 4.2).