Chapter 4
Ordinal Structure of the Shifts

Shift systems are dynamical systems which are used as universal models in information theory and stochastic processes. Besides they are interesting on its own because, in spite of their conceptual simplicity, they exhibit some of the intricacies of low-dimensional chaos, like sensitivity to initial conditions, strong mixing, and a dense set of periodic points.

In the last chapter we studied some general properties of the allowed and forbidden patterns associated with a dynamical system whose state space is linearly ordered. In this chapter we will be more specific and study the ordinal structure of the shift transformations. By ordinal structure we mean such properties as the length and number of the root forbidden patterns. Contrary to the generality of maps, we shall see that these issues can be ascertained with great detail for the shifts.

4.1 Ordinal Patterns and the Shift Maps

Let $E_N\colon [0,\,1] \to [0,\,1]$, $N \in \{2, 3, \ldots\}$, be the shift or sawtooth map

$$E_N\,(x) = Nx \pmod 1 \tag{4.1}$$

(Fig. 4.1). Observe that if

$$x = \sum_{n=0}^{\infty} x_n \cdot N^{-(n+1)} =: 0.x_0\,x_1 \ldots x_n \ldots,$$

$0 \leq x_n \leq N-1$, is an N-ary expansion of $x \in [0,\,1]$, then

$$Nx = \sum_{n=0}^{\infty} x_n \cdot N^{-n} = x_0 + \sum_{n=1}^{\infty} x_n \cdot N^{-n} = x_0\,.\,x_1\,x_2 \ldots x_{n+1} \ldots$$

and

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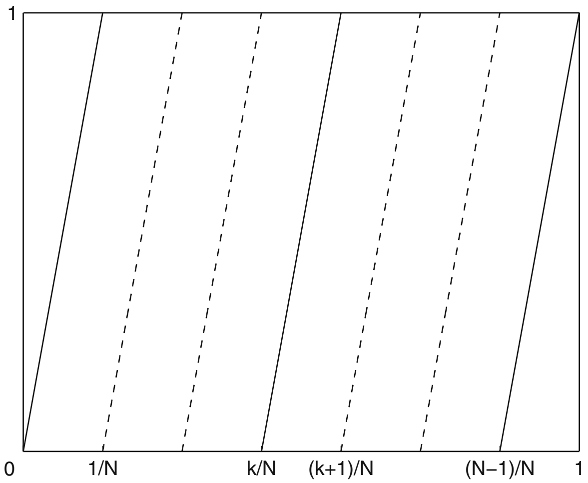

Fig. 4.1 The function $E_N(x) = Nx \bmod 1$. The figure shows only the first, the kth, and the last laps of the graph

$$E_N(0.x_0x_1\ldots x_n\ldots) = 0.x_1x_2\ldots x_{n+1}\ldots. \tag{4.2}$$

In other words, if we write $E_N(0.x_0x_1\ldots x_n\ldots) = 0.y_0y_1\ldots y_n\ldots$, then $y_n = x_{n+1}$ for $n \in \mathbb{N}_0$. This justifies the name "shift map" for E_N since it shifts the digits of the representation of x in base N, one position to the left (the first digit is deleted). Let us recall an N-ary expansion is not unique for some $x \in [0, 1]$ since

$$0.x_0\ldots x_{n-1}10^{\infty} = 0.x_0\ldots x_{n-1}0(N-1)^{\infty},$$

where (as in Sect. 1.1.2) the upper symbol "∞" stands for indefinite repetition. But the set of points $x \in [0, 1]$ whose N-ary expansion ends with 10^{∞} or $0(N-1)^{\infty}$ has zero Lebesgue measure, so such points can eventually be thought to have been removed from [0, 1].

If we identify now an N-ary expansion $0.x_0x_1\ldots x_n\ldots$ of $x \in [0, 1]$ with the one-sided sequence $(x_0, x_1, \ldots, x_n, \ldots) \in S^{\mathbb{N}_0}$, $S = \{0, \ldots, N-1\}$, then action (4.2) translates into the action of the one-sided shift Σ on $S^{\mathbb{N}_0}$. Formally, if $\phi_N : S^{\mathbb{N}_0} \to [0, 1]$ is the map defined by

$$\phi_N : (x_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} x_n N^{-(n+1)}, \tag{4.3}$$

then ϕ_N is an order isomorphism modulo 0 between E_N and the one-sided shift Σ on $S^{\mathbb{N}_0}$, i.e.,

$$\phi_N \circ \Sigma = E_N \circ \phi_N, \tag{4.4}$$

the order of $S^{\mathbb{N}_0}$ being given by the lexicographical rule:

$$\mathbf{x} < \mathbf{x}' \Leftrightarrow \begin{cases} x_0 < x'_0, \\ \text{or} \\ x_0 = x'_0, \ldots, x_{n-1} = x'_{n-1} \text{ and } x_n < x'_n \ (n \geq 1), \end{cases} \tag{4.5}$$

where $\mathbf{x} = (x_n)_{n\in\mathbb{N}_0}$ and $\mathbf{x}' = (x'_n)_{n\in\mathbb{N}_0}$. Observe that ϕ_N maps the cylinder set $C_{i_0\ldots i_n} = \{(x_n)_{n\in\mathbb{N}_0} : x_0 = i_0, \ldots, x_n = i_n\}$ $(i_0, \ldots, i_n \in S)$ to the interval

$$\left[\frac{i_0 N^n + \cdots + i_n}{N^{n+1}}, \frac{i_0 N^n + \cdots + i_n + 1}{N^{n+1}}\right].$$

Exercise 4 Let $\mathcal{B}$ be the Borel sigma-algebra on [0, 1], λ the corresponding Lebesgue measure, and E_N the sawtooth map (4.1). Prove that the dynamical system ([0, 1], $\mathcal{B}$, λ, E_N) and the $(\frac{1}{N}, \ldots, \frac{1}{N})$-Bernoulli one-sided shift are isomorphic (modulo 0).

Once we know that E_N and the one-sided shift Σ on N symbols are order-isomorphic (up to sets measure 0), it follows that they have the same forbidden patterns (see Sect. 3.4.1).

In general it is very difficult to work out the specifics of the forbidden patterns of a given map; the graphical methods can only help for small values of L. But we shall see next that the shifts and the signed shifts (to be defined in Chap. 5.) are an important exception. In particular, owing to the simple structure of one-sided and two-sided shifts, the structure of their admissible and forbidden patterns can be analyzed with great detail. By order isomorphy these conclusions will hold also for the sawtooth map family E_N (one-sided shifts), the baker map (two-sided shifts), and the logistic and symmetric tent maps (one-sided signed shifts), among others.

4.2 Forbidden Patterns for One-Sided Shifts

In Sect. 1.1.2 we saw that one-sided shifts Σ are continuous maps on the compact metric spaces $(\{0, 1, \ldots, N-1\}^{\mathbb{N}_0}, d)$, $N \geq 2$. Furthermore, if $\{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$ is lexicographically ordered (see (4.5)), then Σ is order-isomorphic (modulo 0) to E_N via map (4.3).

What is the structure of the allowed ordinal patterns for Σ? It is easy to convince oneself (see Example 7) that, given $\mathbf{x} = (x_0, \ldots, x_{L-1}, \ldots) \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$ of type $\pi \in \mathcal{S}_L$, π can be decomposed into at most N blocks (separated by semicolons),

$$\langle \pi_0, \ldots, \pi_{k_0-1}; \pi_{k_0}, \ldots, \pi_{k_0+k_1-1}; \ldots; \pi_{k_0+\cdots+k_{N-2}}, \ldots, \pi_{k_0+\cdots+k_{N-2}+k_{N-1}-1} \rangle, \tag{4.6}$$

where $k_s \geq 0$ is the number of times the symbol $s \in \{0, 1, \ldots, N-1\}$ appears in the segment $x_0^{L-1} = x_0, \ldots, x_{L-1}$ of $\mathbf{x}$ ($k_s = 0$ if none, with the corresponding block empty) and $k_0 + \cdots + k_{N-1} = L$. The entries $\pi_0, \ldots, \pi_{k_0-1}$ are the locations of the symbol 0 in x_0^{L-1}, the entries $\pi_{k_0}, \ldots, \pi_{k_0+k_1-1}$ are the locations of the symbol 1 in x_0^{L-1}, etc. For this reason, the first block will also be called the 0-block, and, in general, the $(s+1)$th block,

$$\pi_{k_0+\cdots+k_{s-1}}, \ldots, \pi_{k_0+\cdots+k_{s-1}+k_s-1}, \tag{4.7}$$

$1 \leq s \leq N-1$, will also be called the s-block. Decomposition (4.6) is sometimes called the *decomposition* of an allowed ordinal pattern $\pi \in \mathcal{S}_L$ *in s-blocks*.

A (finite) subsequence of components of π of the form $\pi_i, \ldots, \pi_i + 1, \ldots, \pi_i + 2, \ldots$ (respectively, $\pi_i, \ldots, \pi_i - 1, \ldots, \pi_i - 2, \ldots$) will be called an *increasing* (respectively, *decreasing*) *subsequence*. Increasing or decreasing subsequences will be collectively called *monotone*. Observe that we use these concepts in a restrictive way.

We will see next that from the fact that allowed patterns for the one-sided shift must be decomposable as in (4.6), it is possible to deduce their structure.

Lemma 2 *The blocks in decomposition (4.6) obey the following basic restrictions.*

R1 *The first (leftmost) block, $\pi_0, \ldots, \pi_{k_0-1}$, contains the locations of the 0's in x_0^{L-1}. Each 0-run (i.e., a segment of two or more consecutive 0's contained in or intersected by x_0^{L-1}), if any, contributes an increasing subsequence of the same length as the 0-run. Solitary symbols 0's in x_0^{L-1}, if any, contribute components to the first block that do not form monotone subsequences.*

R2 *The last (rightmost) block, $\pi_{k_0+\cdots+k_{N-2}}, \ldots, \pi_{k_0+\cdots+k_{N-2}+k_{N-1}-1}$, contains the locations of the $(N-1)$'s in x_0^{L-1}. Each $(N-1)$-run contained in or intersected by x_0^{L-1}, if any, contributes a decreasing subsequence of the same length as the $(N-1)$-run. Solitary symbols 1's in x_0^{L-1}, if any, contribute components to the last block that do not form monotone subsequences.*

R3 *Every intermediate block, $\pi_{k_0+\cdots+k_{j-1}}, \ldots, \pi_{k_0+\cdots+k_{j-1}+k_j-1}$, $1 \leq j \leq N-2$, contains the locations of the j's in x_0^{L-1}. Each j-run contained in or intersected by x_0^{L-1}, if any, contributes a subsequence of the same length as the j-run that is increasing if the run is followed by a symbol $> j$, or decreasing if the run is followed by a symbol $< j$. Isolated symbols j's in x_0^{L-1}, if any, contribute components to the corresponding block that do not form monotone subsequences.*

R4 *If the entries $\pi_m \leq L-2$ and $\pi_n \leq L-2$ belong to the same block of $\pi \in \mathcal{S}_L$ and π_m appears on the left of π_n (i.e., $0 \leq m < n \leq L-1$), then $\pi_m + 1$ appears also on the left of $\pi_n + 1$ (i.e., $\pi_m + 1 = \pi_{m'}$, $\pi_n + 1 = \pi_{n'}$ and $0 \leq m' < n' \leq L-1$), not necessarily in the same block.*

Proof **R1)** Consider a 0-run of length l in $\mathbf{x}$:

$i =$	$n-1$	n	$n+1$	$\dots$	$n+l-1$	$n+l$
$\mathbf{x} =$	a	0	0	$\dots$	0	b

with $0 \le n, n+l \le L$, and $a, b > 0$. Hence the 0-block of $\pi(\mathbf{x})$ contains the increasing subsequence

$$\dots, n, \dots, n+1, \dots, n+l-1, \dots,$$

The "…" stands for entries proceeding from other 0-runs in $\mathbf{x}$.

R2) Consider an $(N-1)$-run of length l in $\mathbf{x}$:

$i =$	$n-1$	n	$n+1$	$\dots$	$n+l-1$	$n+l$
$\mathbf{x} =$	c	$N-1$	$N-1$	$\dots$	$N-1$	d

with $0 \le n, n+l \le L$, and $c, d < N-1$. Hence the $(N-1)$-block of $\pi(\mathbf{x})$ contains the decreasing subsequence

$$\dots, n+l-1, \dots, n+1, \dots, n, \dots,$$

The "…" allows for entries proceeding from other $(N-1)$-runs in $\mathbf{x}$.

R3) This restriction follows similarly to R1 for s-runs, $0 < s < N-1$, terminated with $b > s$ (increasing subsequences), and similarly to R2 for s-runs terminated with $d < s$ (decreasing subsequences).

R4) Since π_m and π_n belong to the same block and $\Sigma^{\pi_m}(\mathbf{x}) < \Sigma^{\pi_n}(\mathbf{x})$ for some $\mathbf{x} \in \{0, 1, \dots, N-1\}^{\mathbb{N}_0}$, there exists $k \in \{0, 1, \dots, N-1\}$ such that

$$\Sigma^{\pi_m}(\mathbf{x}) = (k, x_{\pi_m+1} \dots) < (k, x_{\pi_n+1}, \dots) = \Sigma^{\pi_n}(\mathbf{x}).$$

By the definition of lexicographical order, there are two possibilities: (i) $x_{\pi_m+1} < x_{\pi_n+1}$ and (ii) $x_{\pi_m+\kappa} = x_{\pi_n+\kappa}$ for $1 \le \kappa \le l-1$, $l \ge 2$, and $x_{\pi_m+l} < x_{\pi_n+l}$. In both cases,

$$\Sigma^{\pi_m+1}(\mathbf{x}) = (x_{\pi_m+1} \dots) < (x_{\pi_n+1}, \dots) = \Sigma^{\pi_n+1}(\mathbf{x})$$

and, hence, the entry $\pi_m + 1$ appears on the left of $\pi_n + 1$. □

Example 7 Consider in $\{0, 1, 2\}^{\mathbb{N}_0}$ the sequence

$$\mathbf{x} = (2_0, 1_1, 1_2, 1_3, 2_4, 2_5, 0_6, 0_7, 1_8, 1_9, 0_{10}, 0_{11}, 2_{12}, 2_{13}, 2, 1, \dots), \tag{4.8}$$

where a_k indicates that the entry $a \in \{0, 1, 2\}$ is at place k. Then $\mathbf{x}$ defines the ordinal pattern

$$\pi = \langle 6, 10, 7, 11; 9, 8, 1, 2, 3; 5, 0, 4, 13, 12 \rangle \in \mathcal{S}_{14}.$$

The 0-block, $\pi_0^3 = 6, 10, 7, 11$, codifies the $k_0 = 4$ times the symbol 0 appears in x_0^{13}, grouped in two runs, x_6^7 and x_{10}^{11} (note the two increasing subsequences 6, 7 and 10, 11 in this block). The order results from

$$\Sigma_3^6(\mathbf{x}) = (0, 0, 1, \ldots) < \Sigma_3^{10}(\mathbf{x}) = (0, 0, 2, \ldots)$$
$$< \Sigma_3^7(\mathbf{x}) = (0, 1, 1, \ldots) < \Sigma_3^{11}(\mathbf{x}) = (0, 2, \ldots).$$

The 1-block, $\pi_4^8 = 9, 8, 1, 2, 3$, codifies the $k_1 = 5$ times the symbol 1 appears in x_0^{13}, grouped also in two runs: x_1^3, followed by the symbol $2 > 1$, and x_8^9, followed by the symbol $0 < 1$ (note the corresponding increasing subsequence 1, 2, 3, and decreasing subsequence 9, 8, in this block). The order results from

$$\Sigma_3^9(\mathbf{x}) = (1, 0, 0, \ldots) < \Sigma_3^8(\mathbf{x}) = (1, 1, 0, \ldots) < \Sigma_3^1(\mathbf{x}) = (1, 1, 1, \ldots) < \cdots$$

etc. Finally, the 2-block $\pi_9^{13} = 5, 0, 4, 13, 12$ codifies the $k_2 = 5$ appearances of the symbol 2 in x_0^{13}. The decreasing subsequences 5, 4 and 13, 12 come from the runs x_4^5 and x_{12}^{13}, respectively, where x_{12}^{13} is the intersection within x_0^{13} of a longer 2-run. The order results from

$$\Sigma_3^5(\mathbf{x}) = (2, 0, 0, \ldots) < \Sigma_3^0(\mathbf{x}) = (2, 1, 1, \ldots) < \Sigma_3^4(\mathbf{x}) = (2, 2, 0, \ldots) < \cdots .$$

The restriction R4 is easily checked to be fulfilled.

Observe that two sequences $\mathbf{x}, \mathbf{x}'$ with $x_0^{L-1} \neq x_0'^{L-1}$ may define the same ordinal L-pattern, while two sequences $\mathbf{y}, \mathbf{y}'$ with $y_0^{L-1} = y_0'^{L-1}$ may define different ordinal L-patterns (depending on $y_{L-2}, \ldots$ and $y'_{L-2}, \ldots$).

The restriction R4 implies some simple consequences for the relative locations of increasing and decreasing subsequences within the same block and their continuations (if any) outside the block.

Corollary 1 *In an allowed ordinal pattern $\pi \in \mathcal{S}_L$, the following relations among its components hold.*

(A) *If $\pi_i, \pi_i + 1, \ldots, \pi_i + l - 1$, $1 \leq l \leq L - 1$, is an increasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i + l < L$, then $\pi_i + l$ is on the right of $\pi_i + l - 1$ (i.e., $\pi_i + l - 1 = \pi_m$, $\pi_i + l = \pi_n$, and $m < n$).*

(B) *If $\pi_i, \pi_i - 1, \ldots, \pi_i - l + 1$, $1 \leq l \leq L - 1$, is a decreasing subsequence within the same block of $\pi \in \mathcal{S}_L$ with $\pi_i < L - 1$, then $\pi_i + 1$ is on the left of π_i (i.e., $\pi_i + 1 = \pi_j$ with $j < i$).*

(C) *If $\pi_i, \pi_i \pm 1, \ldots, \pi_i \pm l \mp 1$ and $\pi_j, \pi_j \pm 1, \ldots, \pi_j \pm h \mp 1$, $1 \leq l, h \leq L - 1$, are two subsequences with the same monotonicity (upper signs for increasing, lower signs for decreasing subsequences) within the same block of $\pi \in \mathcal{S}_L$, then they are fully separated or, if intertwined, then it may not happen that two or more entries of one of them are between two entries of the other.*

The proof is left as an easy exercise.

Theorem 1 *The one-sided shift on $N \geq 2$ symbols has no forbidden patterns of length $L \leq N + 1$.*

Proof If $L \leq N$ and $\pi = \langle \pi_0, \pi_1, \dots, \pi_{L-1} \rangle$, then any "point" $\mathbf{x} \in \{0, 1, \dots, N - 1\}^{\mathbb{N}_0}$ with $x_{\pi_n} = n$, $0 \leq n \leq L - 1 \leq N - 1$, is trivially of type π:

$$\Sigma^{\pi_0}(\mathbf{x}) = (0, \dots) < \Sigma^{\pi_1}(\mathbf{x}) = (1, \dots) < \dots < \Sigma^{\pi_{L-1}}(\mathbf{x}) = (L - 1, \dots).$$

Thus, suppose $L = N + 1$ and note if $\mathbf{x} = (x_0, x_1, x_2, \dots)$ is of type $\pi = \langle \pi_0, \pi_1, \dots, \pi_N \rangle$, then the sequence $\bar{\mathbf{x}} = (N - 1 - x_0, N - 1 - x_1, N - 1 - x_2, \dots)$ is of type $\pi_{\text{mirrored}} = \langle \pi_N, \pi_{N-1}, \dots, \pi_1, \pi_0 \rangle$.

Given $\pi = \langle \pi_0, \pi_1, \dots, \pi_N \rangle$, we can therefore assume, without loss of generality, that $\pi_0 < \pi_N$. Consider two cases.

- If $\pi_N \neq N$, then there is some $l \in \{1, 2, \dots, N - 1\}$ such that $\pi_l = N$. In this case, the point $\mathbf{x} = (x_0, x_1, \dots) \in \{0, 1, \dots, N - 1\}^{\mathbb{N}_0}$, where

$$x_{\pi_0} = 0,\ x_{\pi_1} = 1,\ \dots,\ x_{\pi_{l-1}} = l - 1,\ x_{\pi_l} = l - 1,\ x_{\pi_{l+1}} = l,\ \dots,$$
$$x_{\pi_{N-1}} = N - 2,\ x_{\pi_N} = N - 1,\ x_{N+1} = x_{N+2} = N - 1$$

is of type π. Indeed, it is enough to note that

$$\begin{aligned}\Sigma^{\pi_{l-1}}(\mathbf{x}) &= (l - 1, x_{\pi_{l-1}+1}, \dots) < (l - 1, N - 1, N - 1, \dots) \\ &= \Sigma^N(\mathbf{x}) = \Sigma^{\pi_l}(\mathbf{x}).\end{aligned}$$

- If $\pi_N = N$, let us first assume that $\pi_0 \neq 0$. Then there is $k \in \{1, 2, \dots, N-1\}$ such that $\pi_k + 1 = \pi_0$. In this case, the point $\mathbf{x} = (x_0, x_1, \dots) \in \{0, 1, \dots, N - 1\}^{\mathbb{N}_0}$, where

$$x_{\pi_0} = 0,\ \dots,\ x_{\pi_k} = k,\ x_{\pi_{k+1}} = k,\ x_{\pi_{k+2}} = k + 1,\ \dots,$$
$$x_{\pi_{N-1}} = N - 2,\ x_{\pi_N} = N - 1,\ x_{N+1} = N - 1$$

is of type π. This is clear because

$$\Sigma^{\pi_k}(\mathbf{x}) = (k, 0, \dots) < (k, x_{\pi_{k+1}+1}, \dots) = \Sigma^{\pi_{k+1}}(\mathbf{x}).$$

In the case that $\pi_0 = 0$, then there is $l \in \{1, 2, \dots, N - 1\}$ such that $\pi_l = N - 1$. Now the sequence $\mathbf{x} = (x_0, x_1, \dots) \in \{0, 1, \dots, N - 1\}^{\mathbb{N}_0}$, where

$$x_{\pi_0} = 0,\ x_{\pi_1} = 1,\ \dots,\ x_{\pi_{l-1}} = l - 1,\ x_{\pi_l} = l - 1,\ x_{\pi_{l+1}} = l,\ \dots,$$
$$x_{\pi_{N-1}} = N - 2,\ x_{\pi_N} = N - 1$$

is of type π, since

$$\Sigma^{\pi_{l-1}}(\mathbf{x}) = (l-1, x_{\pi_{l-1}+1}, \dots) \\ < (l-1, N-1, \dots) = \Sigma^{N-1}(\mathbf{x}) = \Sigma^{\pi_l}(\mathbf{x}). \square$$

Next we are going to show that the one-sided shift on N symbols has forbidden patterns (more specifically, forbidden *root* patterns) of any length $L \geq N+2$. In order to construct explicit instances, we need first to introduce some notation and definitions.

Consider a partition of the sequence $0, 1, \dots, L-1$ of the form

$$\overrightarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_2}, \dots, \overrightarrow{\mathbf{p}_d}, \dots, \overrightarrow{\mathbf{p}_D}, \tag{4.9}$$

where

$$\overrightarrow{\mathbf{p}_d} = e_d, e_d+1, \dots, e_d+h_d-1, \tag{4.10}$$

$1 \leq d \leq D$, $D \geq 2$, with (i) $h_d \geq 1$, $h_1 + \cdots + h_D = L$, (ii) $e_1 = 0$, $e_D + h_D - 1 = L-1$, and (iii) $e_d + h_d = e_{d+1}$ for $1 \leq d \leq D-1$, i.e., the *follower* of $\overrightarrow{\mathbf{p}_d}$, $e_d + h_d$, $d \leq D-1$, is the first element of p_{d+1}, namely, e_{d+1}. We call (4.9) a partition of $0, 1, \dots, L-1$ in D segments, (4.10) being an *increasing segment*, and denote by $\overleftarrow{\mathbf{p}_d}$ the *decreasing* or *reversed segment*

$$\overleftarrow{\mathbf{p}_d} = e_d + h_d - 1, \dots, e_d + 1, e_d.$$

We also call e_d the first element of $\overleftarrow{\mathbf{p}_d}$ and e_{d+1} the follower of $\overleftarrow{\mathbf{p}_d}$.

Since increasing and decreasing segments are nothing else but special cases of increasing and decreasing subsequences, respectively, the consequences (A)–(C) of restriction R4 apply as well. In the proof of the existence of forbidden root patterns below (Lemmas 3 and 4 and Theorem 2) we are going to use (A) and (B) in the following, particularized version (that will be also referred to as R4): *the follower (if any) of an increasing segment* $\overrightarrow{\mathbf{p}_n}$ *(correspondingly, decreasing segment* $\overleftarrow{\mathbf{p}_n}$*) in an allowed pattern* π *appears always to the right of* $\overrightarrow{\mathbf{p}_n}$ *(correspondingly, to the left of* $\overleftarrow{\mathbf{p}_n}$*).*

Definition 2 Consider partition (4.9) of $0, 1, \dots, L-1$ in segments.

1. We call

$$\pi = \langle \overrightarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_3}, \dots, \overleftarrow{\mathbf{p}_4}, \overleftarrow{\mathbf{p}_2} \rangle \quad \text{and} \quad \pi_{\text{mirrored}} = \langle \overrightarrow{\mathbf{p}_2}, \overrightarrow{\mathbf{p}_4}, \dots, \overleftarrow{\mathbf{p}_3}, \overleftarrow{\mathbf{p}_1} \rangle \tag{4.11}$$

   a tent *pattern of length L.*
2. We call

$$\pi = \langle \dots, \overleftarrow{\mathbf{p}_3}, \overleftarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_2}, \overrightarrow{\mathbf{p}_4}, \dots \rangle \quad \text{and} \quad \pi_{\text{mirrored}} = \langle \dots, \overleftarrow{\mathbf{p}_4}, \overleftarrow{\mathbf{p}_2}, \overrightarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_3}, \dots \rangle \tag{4.12}$$

   a spiraling *pattern of* length L.

Observe that the relation between partitions of $0, 1, \ldots, L-1$ in segments and spiraling patterns of length L is one-to-one except when $\overrightarrow{\mathbf{p}_1} = 0$ ($h_1 = 1$). In this case, $\overleftarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_2} = 0, 1, \ldots, e_2 + h_2 - 1$ can be taken for $\overrightarrow{\mathbf{p}_1'} := 0, 1, \ldots, e_2 + h_2 - 1$ ($h_1' = h_2 + 1$).

Lemma 3 *If $N \geq 2$ is the number of symbols and π is a tent pattern with D segments, then π is forbidden if and only if $D \geq N + 2$.*

Proof Consider the tent pattern $\pi = \langle \overrightarrow{\mathbf{p}_1}, \overrightarrow{\mathbf{p}_3}, \ldots, \overleftarrow{\mathbf{p}_4}, \overleftarrow{\mathbf{p}_2} \rangle$. To begin with, the last entry $h_1 - 1$ of $\overrightarrow{\mathbf{p}_1}$ and the first entry e_3 of $\overrightarrow{\mathbf{p}_3}$ may not be in the same block, otherwise the R4 would be violated ($e_2 = h_1$ should be on the left of $e_3 + 1$ if $h_3 \geq 2$ or on the left of e_4 if $h_3 = 1$). Thus we separate them with a first semicolon:

$$\pi = \langle \overrightarrow{\mathbf{p}_1}; \overrightarrow{\mathbf{p}_3}, \ldots, \overleftarrow{\mathbf{p}_4}, \overleftarrow{\mathbf{p}_2} \rangle.$$

Observe that the resulting leftmost block, $\overrightarrow{\mathbf{p}_1}$, complies with R1. Consider now the followers of $\overleftarrow{\mathbf{p}_2}$ and $\overleftarrow{\mathbf{p}_4}$ to conclude similarly that we need to separate these segments by a second semicolon:

$$\pi = \langle \overrightarrow{\mathbf{p}_1}; \overrightarrow{\mathbf{p}_3}, \ldots, \overleftarrow{\mathbf{p}_4}; \overleftarrow{\mathbf{p}_2} \rangle.$$

The resulting rightmost block satisfies R2.

The procedure continues along the same lines. In the kth step, R4 requires a kth semicolon between the segments $\overrightarrow{\mathbf{p}_k}$ and $\overrightarrow{\mathbf{p}_{k+2}}$, so that, if $D \geq N + 1$, the $(N-1)$th semicolon will separate $\overrightarrow{\mathbf{p}_{N-1}}$ and $\overrightarrow{\mathbf{p}_{N+1}}$. All these intermediary blocks trivially fulfill R3.

In the particular case $D = N + 1$, the "central" block $\overrightarrow{\mathbf{p}_N}\overleftarrow{\mathbf{p}_{N+1}}$ (N odd) or $\overrightarrow{\mathbf{p}_{N+1}}\overleftarrow{\mathbf{p}_N}$ (N even) complies with R3 and R4, and hence π is allowed. A further segment $\overrightarrow{\mathbf{p}_{N+2}}$ would require an Nth semicolon to separate $\overrightarrow{\mathbf{p}_N}$ and $\overrightarrow{\mathbf{p}_{N+1}}$ in order not to violate R4.

The proof for π_{mirrored} is completely analogous. □

Lemma 4 *If $N \geq 2$ is the number of symbols, π is a spiraling pattern with D segments, and $h_1 \geq 2$ (i.e., $\overrightarrow{\mathbf{p}_1} = 0, 1, \ldots$), then*

1. *π is forbidden if and only if (a) $D = N$ and $h_D \geq 2$ or (b) $D \geq N + 1$;*
2. *π is allowed if and only if (a′) $D < N$ or (b′) $D = N$ and $h_D = 1$.*

Part 2 of Lemma 4, which is the logical negation of part 1, has been explicitly formulated for further references.

Proof Consider the spiraling pattern (4.12). To begin with, the entries $h_1 - 1$ and $h_1 - 2$ of $\overleftarrow{\mathbf{p}_1} = h_1 - 1, \ldots, 1, 0$ may not be in the same block, otherwise R4 would be violated (e_2 should be on the left of $h_1 - 1$). Thus we separate them with a first semicolon:

$$\pi = \langle \ldots, \overleftarrow{\mathbf{p}_3}, h_1 - 1; h_1 - 2, \ldots, 1, 0, \overrightarrow{\mathbf{p}_2}, \overrightarrow{\mathbf{p}_4}, \ldots \rangle.$$

From here on, three possibilities can occur that we illustrate in a general step of even order. (i) If $\overrightarrow{\mathbf{p}_{2\nu}}$ consists of more than one element (i.e., $h_{2\nu} \geq 2$), then we apply R4 to $\overrightarrow{\mathbf{p}_{2\nu}}$ to conclude that we need a semicolon between $e_{2\nu} + h_{2\nu} - 2$ and $e_{2\nu} + h_2 - 1$ (since the follower of $\overrightarrow{\mathbf{p}_{2\nu}}$, i.e., the first entry of $\overleftarrow{\mathbf{p}_{2\nu+1}}$, is on the wrong side). (ii) If $\overrightarrow{\mathbf{p}_{2\nu}}$ consists of one element ($h_{2\nu} = 1$) and $\overrightarrow{\mathbf{p}_{2\nu-2}}$ consists of more than one element ($h_{2\nu-2} \geq 2$), then we apply R4 to the pair $\overrightarrow{\mathbf{p}_{2\nu}} = e_{2\nu}$ and $e_{2\nu-2} + h_{2\nu-2} - 1$, the last element of $\overrightarrow{\mathbf{p}_{2\nu-2}}$, which has been separated with a semicolon from the rest of elements in $\overrightarrow{\mathbf{p}_{2\nu-2}}$ two steps earlier. (iii) If both $\overrightarrow{\mathbf{p}_{2\nu}}$ and $\overrightarrow{\mathbf{p}_{2\nu-2}}$ consist of a single element ($h_{2\nu} = h_{2\nu-2} = 1$), apply R4 to the pair $\overrightarrow{\mathbf{p}_{2\nu-2}} = e_{2\nu-2} < \overrightarrow{\mathbf{p}_{2\nu}} = e_{2\nu}$ to infer the need for a semicolon separating them (since $e_{2\nu-2} + 1 = e_{2\nu-1}$, the first element of $\overleftarrow{\mathbf{p}_{2\nu-1}}$, is on the right of $e_{2\nu} + 1 = e_{2\nu+1}$, the first element of $\overleftarrow{\mathbf{p}_{2\nu+1}}$). As a general rule, we need one semicolon per segment $\overrightarrow{\mathbf{p}_{2\nu}}$ or $\overleftarrow{\mathbf{p}_{2\nu+1}}$ as long as there are still a posterior segment $\overleftarrow{\mathbf{p}_{2\nu+1}}$ or $\overrightarrow{\mathbf{p}_{2\nu+2}}$, respectively, on the "wrong" side. Note that all (intermediary) blocks ensued so far comply with R3.

Following this way, we run out of the $N-1$ semicolons we may use (corresponding to the N symbols), after having considered the segment $\overrightarrow{\mathbf{p}_{N-1}}$. Yet if $D = N$ and $h_N \geq 2$, then $\overrightarrow{\mathbf{p}_N}$ will violate R1 if N is odd or R2 if N is even. If $D \geq N + 1$, then the segment $\overrightarrow{\mathbf{p}_{N+1}}$ will be on the wrong side of $\overrightarrow{\mathbf{p}_N}$ and the pattern will not comply with R4.

The proof for π_{mirrored} is completely analogous. □

The constructive, stepwise procedure used in the proofs of Lemmas 3 and 4 can be used mutatis mutandis in general to decompose any ordinal pattern into well-formed (i.e., complying with R1–R4) blocks. For instance, one could start from the leftmost entry and move on rightward one entry at a time, inserting a semicolon between the current and the previous entry whenever necessary to enforce the restrictions R1–R4. Reciprocally, given a decomposition of an ordinal pattern π in s-blocks, one can easily construct a sequence $\mathbf{x} \in \{0, \ldots, N-1\}^{\mathbb{N}_0}$ of type π.

Theorem 2 *The following patterns of length $L \geq N + 2$, together with their corresponding mirrored patterns, are forbidden root patterns.*

1. *The tent patterns with $N + 2$ segments*

$$\langle 0, \overrightarrow{\mathbf{p}_3}, \ldots, \overrightarrow{\mathbf{p}_N}, L-1, \overleftarrow{\mathbf{p}_{N+1}}, \ldots, \overleftarrow{\mathbf{p}_2} \rangle \tag{4.13}$$

if N is odd or

$$\langle 0, \overrightarrow{\mathbf{p}_3}, \ldots, \overrightarrow{\mathbf{p}_{N+1}}, L-1, \overleftarrow{\mathbf{p}_N}, \ldots, \overleftarrow{\mathbf{p}_2} \rangle \tag{4.14}$$

if N is even. Here $\overrightarrow{\mathbf{p}_1} = 0$ and $\overrightarrow{\mathbf{p}_{N+2}} = L - 1$.

2. *The spiraling pattern with $N + 1$ segments*

$$\langle L-2, \overleftarrow{\mathbf{p}_{N-2}}, \ldots, \overleftarrow{\mathbf{p}_3}, 1, 0, \overrightarrow{\mathbf{p}_2}, \ldots, \overrightarrow{\mathbf{p}_{N-1}}, L-1 \rangle \tag{4.15}$$

if N is odd or

$$\langle L-1, \overleftarrow{\mathbf{p}_{N-1}}, \ldots, \overleftarrow{\mathbf{p}_3}, 1, 0, \overrightarrow{\mathbf{p}_2}, \ldots, \overrightarrow{\mathbf{p}_{N-2}}, L-2\rangle, \tag{4.16}$$

if N is even. Here $\overrightarrow{\mathbf{p}_1} = 0, 1$, $\overrightarrow{\mathbf{p}_N} = L-2$, *and* $\overrightarrow{\mathbf{p}_{N+1}} = L-1$.

3. *The spiraling pattern with N segments*

$$\langle L-1, L-2, \overleftarrow{\mathbf{p}_{N-2}}, \ldots, \overleftarrow{\mathbf{p}_3}, 1, 0, \overrightarrow{\mathbf{p}_2}, \ldots, \overrightarrow{\mathbf{p}_{N-1}}\rangle \tag{4.17}$$

if N is odd or

$$\langle \overleftarrow{\mathbf{p}_{N-1}}, \ldots, \overleftarrow{\mathbf{p}_3}, 1, 0, \overrightarrow{\mathbf{p}_2}, \ldots, \overrightarrow{\mathbf{p}_{N-2}}, L-2, L-1\rangle, \tag{4.18}$$

if N is even. Here $\overrightarrow{\mathbf{p}_1} = 0, 1$, *and* $\overrightarrow{\mathbf{p}_N} = L-2, L-1$.

Of course, cases 2 and 3 are related to the two possibilities in Lemma 4.

Proof First of all, remember from Sect. 3.4.2, (3.12), that given a forbidden pattern

$$\langle \pi_0, \ldots, \pi_{L-2}\rangle \in \mathcal{S}_{L-1},$$

its outgrowth patterns of length L have the form (*group I*)

$$\langle L-1, \pi_0, \ldots, \pi_{L-2}\rangle, \langle \pi_0, L-1, \ldots, \pi_{L-2}\rangle, \ldots, \langle \pi_0, \ldots, \pi_{L-2}, L-1\rangle$$

or the form (*group II*)

$$\langle 0, \pi_0+1, \ldots, \pi_{L-2}+1\rangle, \langle \pi_0+1, 0, \ldots, \pi_{L-2}+1\rangle, \ldots, \langle \pi_0+1, \ldots, \pi_{L-2}+1, 0\rangle.$$

1. This case is trivial. Any tent pattern made out of $N+2$ segments is forbidden according to Lemma 3. Moreover, since the entries $L-1$ and 0 in patterns (4.13) and (4.14) are segments on their own, the number of segments D of these tent patterns will fall below the threshold value $D = N+2$ once $L-1$ (group I) or 0 (group II) are deleted.

2. Only (4.15) will be considered here, the proof for (4.16) and their mirrored patterns being completely analogous. That (4.15) is forbidden follows readily from Lemma 4 (b). To prove that π is also a root pattern, we need to show that it is not the outgrowth of any forbidden pattern of shorter length.

There are two possibilities. Suppose first that π is an outgrowth forbidden pattern of group I. Deletion of the entry $L-1$ yields then the spiraling pattern

$$\langle L-2, \overleftarrow{\mathbf{p}_{N-2}}, \ldots, \overleftarrow{\mathbf{p}_3}, 1, 0, \overrightarrow{\mathbf{p}_2}, \ldots, \overrightarrow{\mathbf{p}_{N-1}}\rangle,$$

which is allowed on account of having N segments, $h_1 = 2$, and a last segment $\overrightarrow{\mathbf{p}_N} = L-2$ of length 1 (Lemma 4 (b′)).

Thus, suppose that π is an outgrowth forbidden pattern of group II. In this case, after removing the entry 0 and subtracting 1 from the remaining entries we are left with the pattern

$$\langle L-3, \overleftarrow{\mathbf{p}'_{N-2}}, \ldots, \overleftarrow{\mathbf{p}'_3}, 0, \overrightarrow{\mathbf{p}'_2}, \ldots, \overrightarrow{\mathbf{p}'_{N-1}}, L-2\rangle, \tag{4.19}$$

where $\overrightarrow{\mathbf{p}'_d} = e_d - 1, \ldots, e_d + h_d - 2$, $2 \leq d \leq N+1$. Since $\overrightarrow{\mathbf{p}'_1} = 0$ ($h'_1 = h_1 - 1 = 1$) and $\overrightarrow{\mathbf{p}'_2} = 1, \ldots$ ($h'_2 = h_2 \geq 1$), we can merge $\overrightarrow{\mathbf{p}'_1}$ and $\overrightarrow{\mathbf{p}'_2}$ into the new segment $\overrightarrow{\mathbf{p}''_1} := 0, 1, \ldots$, so that (4.19) is a spiraling pattern with $h''_1 \geq 2$ and the following N segments: $\overrightarrow{\mathbf{p}''_1}, \overrightarrow{\mathbf{p}'_3}, \ldots, \overrightarrow{\mathbf{p}'_{N-1}}, \overrightarrow{\mathbf{p}'_N} = L-3, \overrightarrow{\mathbf{p}_{N+1}} = L-2$. According to Lemma 4 (b′), the ordinal pattern (4.19) is allowed.

3. This case uses Lemma 4 (a)–(a′) instead. The proof proceeds similar to case 2. □

Example 8 For $N = 2n+1$, Theorem 2 provides the following six forbidden patterns of minimal length $L = N + 2$:

$$\begin{aligned}&\langle 0, 2, \ldots, 2n, 2n+2, 2n+1, \ldots, 3, 1\rangle,\\&\langle 2n+1, 2n-1, \ldots, 1, 0, 2, \ldots, 2n, 2n+2\rangle,\\&\langle 2n+2, 2n+1, \ldots, 1, 0, 2, \ldots, 2n-2, 2n\rangle,\end{aligned}$$

and their mirrored patterns. For $N = 2n$, the six forbidden patterns of minimal length $L = N + 2$ provided by Theorem 2 are

$$\begin{aligned}&\langle 0, 2, \ldots, 2n, 2n+1, \ldots, 3, 1\rangle,\\&\langle 2n+1, 2n-1, \ldots, 1, 0, 2, \ldots, 2n-2, 2n\rangle,\\&\langle 2n-1, 2n-3 \ldots, 1, 0, 2, \ldots, 2n, 2n+1\rangle,\end{aligned}$$

and their mirrored patterns. In particular, for $N = 2$ we obtain the following minimal-length forbidden patterns:

$$\begin{array}{ll}\langle 0, 2, 3, 1\rangle & \langle 1, 3, 2, 0\rangle,\\ \langle 3, 1, 0, 2\rangle & \langle 2, 0, 1, 3\rangle,\\ \langle 1, 0, 2, 3\rangle & \langle 3, 2, 0, 1\rangle.\end{array}$$

Needless to say, these are the six 4-patterns we got in (3.14) by graphical means.

It was proven in [76] that the shift Σ_N has exactly six root forbidden L-patterns for each $L \geq N + 2$, namely, those delivered by Theorem 2 after setting $\overrightarrow{\mathbf{p}_k} = k - 1$ (respectively, $\overrightarrow{\mathbf{p}_k} = k$) in those segments not explicitly given in the tent patterns (4.13) and (4.14) (respectively, in the spiraling patterns (4.15), (4.16), (4.17), and (4.18)).

Corollary 2 *For every $K \geq 2$ there are self-maps on the interval* $[0, 1]$ *without forbidden patterns of length $L \leq K$.*

Proof Let $E_N : [0, 1] \to [0, 1]$ be the shift map $x \mapsto Nx \pmod 1$, $N = 2, 3, \ldots$. We know that E_N and Σ have the same allowed and forbidden patterns because they are order isomorphic (see (4.4)). Therefore if $N + 1 \leq K$, then E_N has no forbidden patterns of length $L \leq K$ because of Theorem 1. □

It follows that *there exist n-dimensional interval maps without forbidden patterns*. For example, see Fig. 4.2, one can decompose [0, 1] in infinite many half-open intervals (of vanishing length), $[0, 1] = \cup_{N=2}^{\infty} I_N$ and define on each I_N a properly scaled version of E_N, $\tilde{E}_N : I_N \to I_N$. In $\mathbb{R}^2$ one can repeat the said decomposition along the 1-axis and define on $I_N \times [0, 1]$ the function $(\tilde{E}_N, \mathrm{Id})$, where Id denotes the identity. Proposition 4 shows that adding some natural assumption, like piecewise monotonicity, can make all the difference.

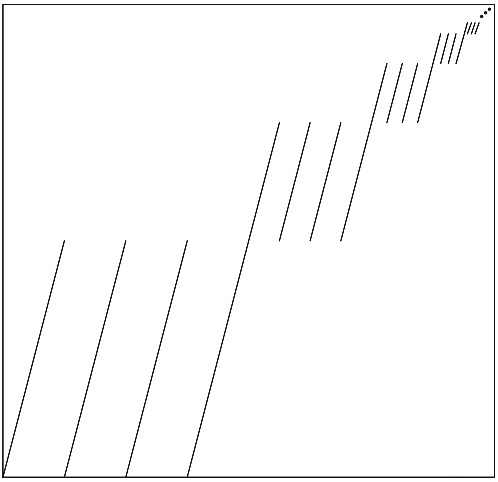

Fig. 4.2 A map with infinitely many monotonicity intervals and no forbidden patterns

4.3 Forbidden Patterns for Two-Sided Shifts

Consider now the bisequence space, $\{0, 1, \ldots, N-1\}^{\mathbb{Z}}$, equipped with the following lexicographical order. With the notation $\mathbf{x}^-$ for the *left sequence* $(x_{-n})_{n\in\mathbb{N}}$ of $\mathbf{x} \in \{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ and $\mathbf{x}^+$ for the *right sequence* $(x_n)_{n\in\mathbb{N}_0}$, we set

$$\mathbf{x} < \mathbf{x}' \Leftrightarrow \begin{cases} \mathbf{x}^+ < \mathbf{x}'^+ \\ \text{or} \\ \mathbf{x}^- < \mathbf{x}'^- \quad \text{if } \mathbf{x}^+ = \mathbf{x}'^+ \end{cases}, \tag{4.20}$$

where $\mathbf{x} = (\mathbf{x}^-, \mathbf{x}^+)$, $\mathbf{x}' = (\mathbf{x}'^-, \mathbf{x}'^+)$, and $<$ between right (respectively, left) sequences denote lexicographical order in $\{0, 1, \ldots, N-1\}^{\mathbb{N}_0}$ (respectively, $\{0, 1, \ldots, N-1\}^{\mathbb{N}}$). If we map $\{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ onto $[0, 1] \times [0, 1] \equiv [0, 1]^2$ via

$$(\mathbf{x}^-,\mathbf{x}^+)\mapsto\left(\sum_{n=1}^{\infty}x_{-n}N^{-n},\sum_{n=0}^{\infty}x_nN^{-(n+1)}\right), \tag{4.21}$$

we find that the lexicographical order (4.20) in $\{0,1,\ldots,N-1\}^{\mathbb{Z}}$ corresponds to the usual lexicographical order in $[0,1]^2$. In order for this map to be one-to-one, we have to dispose of the usual ambiguities in either direction.

In relation with the ordinal patterns defined by the orbits of two-sided sequences,

$$\begin{aligned}\Sigma^i(\mathbf{x}) &< \Sigma^j(\mathbf{x})\\ &\Leftrightarrow \begin{cases}(x_i,x_{i+1},\ldots)<(x_j,x_{j+1},\ldots)\\ \text{or}\\ (x_{i-1},x_{i-2},\ldots)<(x_{j-1},x_{j-2},\ldots) \text{ if } (x_i,x_{i+1},\ldots)=(x_j,x_{j+1},\ldots),\end{cases}\end{aligned}$$

where $i,j\geq 0$, $i\neq j$. It follows that the "exceptional" condition $(x_i,x_{i+1},\ldots)=(x_j,x_{j+1},\ldots)$ occurs if and only if $\Sigma^{|i-j|}(\mathbf{x}^+)=\mathbf{x}^+$, i.e., when the right sequence $\mathbf{x}^+$ of $\mathbf{x}\in\{0,1,\ldots,N-1\}^{\mathbb{Z}}$ is periodic from the entry $\min\{i,j\}$ on with period $p=|i-j|$.

Lemma 5 *One-sided and two-sided shifts on N symbols have the same admissible and forbidden ordinal patterns.*

Proof (i) Suppose that the one-sided sequence $\mathbf{x}^+\in\{0,1,\ldots,N-1\}^{\mathbb{N}_0}$ defines an ordinal L-pattern π, i.e.,

$$\Sigma^{\pi_0}(\mathbf{x}^+)<\Sigma^{\pi_1}(\mathbf{x}^+)<\cdots<\Sigma^{\pi_{L-1}}(\mathbf{x}^+).$$

Then, the two-sided sequences $\mathbf{x}=(\mathbf{x}^-,\mathbf{x}^+)$, with $\mathbf{x}^-\in\{0,1,\ldots,N-1\}^{\mathbb{N}}$ arbitrary, define the same ordinal pattern.

(ii) Suppose now that the two-sided sequence $\mathbf{x}=(\mathbf{x}^-,\mathbf{x}^+)\in\{0,1,\ldots,N-1\}^{\mathbb{Z}}$ defines an ordinal L-pattern π,

$$\Sigma^{\pi_0}(\mathbf{x})<\Sigma^{\pi_1}(\mathbf{x})<\cdots<\Sigma^{\pi_{L-1}}(\mathbf{x}). \tag{4.22}$$

If $\mathbf{x}^+$ is not eventually periodic, then (4.22) implies

$$\Sigma^{\pi_0}(\mathbf{x}^+)<\Sigma^{\pi_1}(\mathbf{x}^+)<\cdots<\Sigma^{\pi_{L-1}}(\mathbf{x}^+),$$

hence the pattern π is realized by the one-sided sequence $\mathbf{x}^+$. If $\mathbf{x}^+$ is eventually periodic, say

$$\mathbf{x}^+=(x_0,\ldots,x_{k-1},(x_k,\ldots,x_{k+p-1})^{\infty}),$$

i.e., $(\mathbf{x}^+)_{k+np}=(\mathbf{x}^+)_k$ for $k\geq 0$ and every $n\in\mathbb{N}$, then there are two subcases.

(ii-a) If $L\leq k+2p$, then the periodicity of $\mathbf{x}^+$ is not visible in the segment x_0^{L-1}, so the pattern π is realized by the one-sided pattern $\mathbf{x}^+$.

(ii-b) If $L = k+np+\nu$ with $n \geq 2$ and $\nu \geq 1$, then $\Sigma^{k+p+i}(\mathbf{x}) = \cdots = \Sigma^{k+np+i}(\mathbf{x})$ for $i = 0, \ldots, \nu - 1$, so their negative sequences $(\Sigma^{k+p+i}(\mathbf{x}))^-, \ldots, (\Sigma^{k+np+i}(\mathbf{x}))^-$ have to be compared before ordering them. In this case, the pattern π is realized by the one-sided sequence

$$\begin{aligned}\tilde{\mathbf{x}}^+ &= (x_0, \ldots, x_{k+np+\nu-1}, (\Sigma^{k+np+\nu-1}(\mathbf{x}))^-) \\ &= (x_0, \ldots, x_{k+np+\nu-1}, x_{k+np+\nu-2}, \ldots, x_0, x_{-1}, \ldots).\end{aligned}$$

From (i) and (ii) we deduce that one-sided and two-sided shifts on $N \geq 2$ symbols have the same admissible ordinal patterns, hence they have also the same forbidden patterns. □

As a corollary of Lemma 5, together with Theorems 1 and 2, we obtain the following result.

Theorem 3 *The two-sided shift on N symbols has no forbidden patterns of length $L \leq N + 1$ and has forbidden root patterns for $L \geq N + 2$.*

Example 9 Let $I^2 = [0, 1] \times [0, 1]$ endowed with the Lebesgue measure, and let $B : I^2 \to I^2$ be the *baker map*,

$$B(\xi, \eta) = \begin{cases} (2\xi, \frac{1}{2}\eta), & 0 \leq \xi < \frac{1}{2}, \\ (2\xi - 1, \frac{1}{2}\eta + \frac{1}{2}), & \frac{1}{2} \leq \eta \leq 1. \end{cases}$$

A generating partition of B is $A_0 = [0, \frac{1}{2}) \times [0, 1]$ and $A_1 = [\frac{1}{2}, 1] \times [0, 1]$. For Σ take the two-sided $\left(\frac{1}{2}, \frac{1}{2}\right)$-Bernoulli shift. Then B and Σ are isomorphic (mod 0) via the coding map $\Phi : I^2 \to \{0, 1\}^{\mathbb{Z}}$, given by

$$\Phi(\xi, \eta) = (\ldots, x_{-1}, x_0, x_1, \ldots),$$

where $x_n = i_n$ if $B^n(\xi, \eta) \in A_{i_n}$, $n \in \mathbb{Z}$. Since Φ preserves order (in fact, Φ is the inverse of the order-preserving map $(\mathbf{x}^-, \mathbf{x}^+) \mapsto (\sum_{n=0}^{\infty} x_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} x_n 2^{-n})$), we conclude that the baker transformation has no forbidden patterns of length ≤ 3. The forbidden 4-patterns of the baker map are the same as those of the one-sided shift, see (3.14).