Chapter 11
Conclusion and Outlook

Ordinal (or permutation-based) analysis of dynamical systems originates from the properties of the order relations and order isomorphisms. Thereby it is assumed that the state space of the systems is equipped with a total ordering. The order relations among consecutive elements in the orbits of deterministic or random dynamical systems are then codified in the form of ordinal patterns. The ordinal patterns themselves—whether admissible or forbidden—together with other "higher level" tools based on them, like permutation entropy rates, discrete entropy, frequency or probability distributions, regularity parameters, build the main repertoire of ordinal analysis. Since the sort of properties addressed by ordinal analysis and captured by its tools are not the same as in the usual measure-theoretical and topological approaches, we proposed the term "permutation complexity" to distinguish them.

In the foregoing chapters we have reviewed the theoretical and practical aspects of ordinal analysis. Among the first ones, let us highlight the study of metric (Chap. 6) and topological (Chap. 7) permutation entropies, together with the relation to their standard counterparts. Among the applications, some of them are well established, like the estimation of entropy (Sect. 2.1), complexity analysis of time series (Sect. 2.2), or detection of determinism (Chap. 9). Others like the complexity analysis of spatially extended systems (Chap. 10) are still in an initial stage. An important message to keep regarding all ordinal pattern-based applications is their robustness against observational noise—an asset when analyzing real systems. In particular, deterministic generation is responsible for the persistence of forbidden patterns in very noise data, as shown in Sect. 9.1. Robustness makes ordinal analysis a practical tool.

The reader might be tempted to dismiss ordinal analysis of dynamics as an uninteresting equivalent to well-known symbolic dynamics. In fact, ordinal patterns of dynamical systems do maintain equivalent results with symbolic dynamics, such as the metric and topological entropies we discussed in Chaps. 6 and 7, respectively, but in other ways, there are major distinctions, which are just starting to be explored for permutations. For instance, the canonical tent map and the Bernoulli shift ($f(x) = 2x \mod 1$) are isomorphic under a conventional analysis and in symbolic dynamics are equivalent to an i.i.d. source of white bits. However, under permutation-based analysis, once the state is imbued with total ordering, the class of order isomorphisms is different. Both conventional symbolic dynamics, assuming a

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generating partition of a map, and ordinal analysis are useful discrete representations of what would otherwise be a dynamical system in continuous space. However, the symbolic dynamics which results from a conventional partitioning is not fundamentally distinguishable from a noisy system; both result in conventional information sources on a discrete alphabet with a positive Shannon entropy. By contrast, the ordinal analysis does show a fundamental distinction between deterministic chaos and noisy systems. With chaos there is a rich structure of forbidden patterns among the ordinal patterns of different length and a hierarchy of consequent derived forbidden patterns (Chap. 3), the nature of which is not shared with conventional symbolic dynamics. More closely impacting the present work, the number of allowed permutations can scale superexponentially, which is fundamentally faster than the exponential scaling which must eventually happen with a noise-free deterministic chaotic system.

As in any research field, work on theory and applications of ordinal analysis is in progress, meaning that the picture is far from complete. In the course of the exposition, we have pointed out different questions which are waiting for answers. I summarized next the most important ones.

One of the basic open problems refers to the relation between a map and the structure of its forbidden patterns. Some natural questions that arise in this context are the following:

- Understand how the allowed or forbidden ordinal patterns (especially the root patterns) depend on the map.
- Given a map, determine the length of its shortest forbidden pattern.
- Describe and/or enumerate (exactly or asymptotically) any of the above classes of ordinal patterns.
- Given a finite or infinite set of, say, root forbidden patterns, find a map with the corresponding ordinal pattern structure.
- More generally, characterize those hierarchies of ordinal patterns for which there exist maps realizing them.

Of course, some of these questions can be answered graphically for simple maps and short pattern lengths. What we seek though are general results, possibly emanating from the structure of periodic points. We reported partial successes along this line for the shifts (Chap. 4) and signed shifts (Chap. 5), but the general case seems exceedingly hard. Even the ordinal structure of a general subshift of finite type (order isomorphic to some piecewise linear maps) seems to be beyond the techniques used in those chapters. A list of more advanced research topics would include the relation of forbidden patterns with the kneading invariants of one-dimensional interval maps or, say, with the directional entropy of cellular automata.

Other interesting (albeit theoretical) problem is the exact relation between the original definition of permutation entropy by Bandt et al. [29], and the definition given in Chaps. 6 and 7. Technically, the difference boils down to the order of two limiting processes (ever longer ordinal patterns and ever finer partitions) in a double limit. In particular, the results of Sects. 6.2 and 6.3 show that both definitions of metric permutation entropy overlap for one-dimensional, piecewise ergodic maps, and numerical simulations advocate a more general coincidence. In any case, the usual computations, with an arithmetic precision fixed by default or by the numerical format chosen, implement our “Kolmogorov-like” approach to permutation entropy.

For practical applications, the numerical tools of the type we discussed in Chap. 9 serve as a way of distinguishing chaos-like dynamics from noise, at least in simulations. This may be useful in the detection of emergent “coherent structures” similar to low-dimensional chaos in what otherwise might be a high degree of freedom system which could be rather noise-like. We comment on the unique property of permutations having a discrete “algebraic” nature permitting some rapid computational methods, without the requirement of estimating a generating partition for each dynamics. We feel that the appropriate tools for analysis of the typically short *observed* time series will require more sophisticated statistical thinking and methods still, just as high-quality estimation of entropies from low-alphabet information sources can be a difficult problem despite the apparent simplicity of the definitions themselves.

In Chap. 9 we also showed that the forbidden pattern-based technique outperforms one of the standard methods for detecting statistical dependence. Similar conclusions were reached in the ordinal analysis of synchronization in [159], see Sect. 2.4. This exercise—comparing a pattern-based technique with the traditional methods—is missing in other applications of ordinal analysis to time series like entropy estimation or complexity study. If the applications refer to natural systems, then the possibilities are virtually unlimited. Real time series appeared only in Sect. 2.2 (“Permutation complexity”), where we considered biomedical data, a recurrent topic in the literature. But, of course, other kinds of real data have also been studied (see Sect. 2.2).

Apart from the future lines of research related to the above-mentioned open problems, other lines of research refer to more recent topics and other follow-up investigations. In Chap. 10 we showed that ordinal analysis provides quantitative tools for and insights into the dynamics of space–time dynamics. This brief account was meant as a corroboration of performances shown in other contexts, as well as a stimulus to further research. Clearly, a survey of permutation complexity in cellular automata and coupled map lattices is a broad field that will require time and ingenuity, especially in the unexplored dimensions 2 and higher. Add to this general networks of coupled map lattices, and you get a long-term research program! But the great challenge is the complexity analysis of physical systems. Simple models, like cellular automata and coupled map lattice, provide a bridge to this more ambitious objective, in that they model non-trivial physical phenomena while being amenable to discrete methods. The situation resembles the study of complex dynamical systems via symbolic dynamics—a quite remarkable technique. The author believes that the interplay between complex dynamical systems and discrete methods is a promising approach also in the case of physical systems. Chapter 10 reported on progress in this direction from the ordinal front. New chapters will follow.