The Synchronization Problem for Locally Strongly Transitive Automata*

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Abstract. The synchronization problem is investigated for a new class of deterministic automata called locally strongly transitive. An application to synchronizing colorings of aperiodic graphs with a cycle of prime length is also considered.

Keywords: Černý conjecture, road coloring problem, synchronizing automaton, rational series.

1 Introduction

The *synchronization problem* for a deterministic n-state automaton consists in the search of an input-sequence, called a *synchronizing* or *reset word*, such that the state attained by the automaton, when this sequence is read, does not depend on the initial state of the automaton itself. If such a sequence exists, the automaton is called *synchronizing*. If a synchronizing automaton is deterministic and complete, a well-known conjecture by Černý claims that it has a synchronizing word of length not larger than $(n-1)^2$ [7]. This conjecture has been shown to be true for several classes of automata (*cf.* [2,3,4,7,8,9,11,12,15,16,17,18,21]). The interested reader is refered to [13,21] for a historical survey of the Černý conjecture and to [6] for the study of the synchronization problem for unambiguous automata. In [8], the authors have studied the synchronization problem for a new class of automata called *strongly transitive*. An n-state automaton is said to be strongly transitive if it is equipped by a set of n words $\{w_0, \dots, w_{n-1}\}$, called *independent*, such that, for any pair of states s and t, there exists a word w_i, $0 \le i \le n-1$, such that $sw_i = t$. Interesting examples of strongly transitive automata are circular automata and transitive synchronizing automata. The main result of [8] is that any synchronizing strongly transitive n-state automaton has a synchronizing word of length not larger than $(n-2)(n+L-1)+1$,

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where L denotes the length of the longest word of an independent set of the automaton. As a straightforward corollary of this result, one can obtain the bound $2(n-2)(n-1)+1$ for the length of the shortest synchronizing word of any n-state synchronizing circular automaton.

In this paper, we consider a generalization of the notion of strong transitivity that we call *local strong transitivity*. An n-state automaton is said to be *locally strongly transitive* if it is equipped by a set of k words $W = \{w_0, \ldots, w_{k-1}\}$ and a set of k distinct states $R = \{q_0, \ldots, q_{k-1}\}$ such that, for all $s \in S$, $\{sw_0, \ldots, sw_{k-1}\} = \{q_0, \ldots, q_{k-1}\}$. The set W is still called *independent* while R is called the *range* of W. A typical example of such kind of automata is that of *one-cluster* automata, recently investigated in [4]. An automaton is called one-cluster if there exists a letter a such that the graph of the automaton has a unique cycle labelled by a power of a. Indeed, denoting by k the length of the cycle, one easily verifies that the words

$$a^{n-1}, a^{n-2}, \ldots, a^{n-k}$$

form an independent set of the automaton whose range is the set of vertices of the cycle. Another more general class of locally strongly transitive automata is that of *word connected* automata. Given a n-state automaton $\mathcal{A} = (S, A, \delta)$ and a word $u \in A^*$, $\mathcal{A}$ is called *u-connected* if there exists a state $q \in S$ such that, for every $s \in S$, there exists $\ell > 0$, with $su^\ell = q$. Define R and W respectively as:

$$R = \{q, qu, \ldots, qu^{k-1}\}, \quad W = \{u^i, u^{i+1}, \ldots, u^{i+k-1}\}, \tag{1}$$

where k is the least positive integer such that $qu^k = q$ and i is the least integer such that, for every $s \in S$, $su^i \in R$. Then one has that W is an independent set of $\mathcal{A}$ with range R.

In this paper, by developing the techniques of [8], we prove that any synchronizing locally strongly transitive n-state automaton has a synchronizing word of length not larger than

$$(k-1)(n+L)+\ell,$$

where k is the cardinality of an independent set W and L and ℓ denote respectively the maximal and the minimal length of the words of W. As a straightforward corollary of this result, we obtain that every n-state synchronizing u-connected automaton has a synchronizing word of length not larger than

$$(k-1)(n+(i+k-1)|u|)+i|u|,$$

where i and k are defined as in (1). In particular, if the automaton is one-cluster, the previous bound becomes $(2k-1)(n-1)$, where k is the length of the unique cycle of the graph of $\mathcal{A}$ labelled by a suitable letter of A.

Another result of this paper is related to the well-known *Road coloring problem*. This problem asks to determine whether any aperiodic and strongly connected graph, with all vertices of the same outdegree, (*AGW graph*, for short), has a synchronizing coloring. The problem was formulated in the context of Symbolic Dynamics by Adler, Goodwyn and Weiss and it is explicitly stated in [1].

In 2007, Trahtman has positively solved it in [19]. Recently Volkov has raised in [20] the problem of evaluating, for any AGW graph G, the minimal length of a reset word for a synchronizing coloring of G. This problem has been called *the Hybrid Černý–Road coloring problem.* It is worth to mention that Ananichev has found, for any $n \geq 2$, a AGW graph of n vertices such that the length of the shortest reset word for any synchronizing coloring of the graph is $(n-1)(n-2)+1$ (see [20]). By applying our main theorem and a result by O' Brien [14], we are able to obtain a partial answer to the Hybrid Černý–Road coloring problem. More precisely, we can prove that, given a AGW graph G of n vertices, without multiple edges, such that G has a simple cycle of prime length $p < n$, there exists a synchronizing coloring of G with a reset word of length not larger than $(2p-1)(n-1)$. Moreover, in the case $p = 2$, that is, if G contains a cycle of length 2, a similar result holds, even in presence of multiple edges. Indeed, for every graph of such kind, we can prove the existence of a synchronizing coloring with a reset word of length not larger than $5(n-1)$.

2 Preliminaries

We assume that the reader is familiar with the theory of automata and rational series. In this section we shortly recall a vocabulary of few terms and we fix the corresponding notation used in the paper.

Let A be a finite alphabet and let A^* be the free monoid of words over the alphabet A. The identity of A^* is called the *empty word* and is denoted by ϵ. The *length* of a word of A^* is the integer $|w|$ inductively defined by $|\epsilon| = 0$, $|wa| = |w| + 1$, $w \in A^*$, $a \in A$. For any finite set W of words of A^*, we denote by L_W and ℓ_W the lengths of the longest word and the shortest word in W respectively.

A finite automaton is a triple $\mathcal{A} = (S, A, \delta)$ where S is a finite set of elements called *states* and δ is a map

$$\delta : S \times A \longrightarrow S.$$

The map δ is called the *transition function* of $\mathcal{A}$. The canonical extension of the map δ to the set $S \times A^*$ is still denoted by δ. For any $u \in A^*$ and $s \in S$, the state $\delta(s, u)$ will be also denoted su. If P is a subset of S and u is a word of A^*, we denote by Pu and Pu^{-1} the sets:

$$Pu = \{su \mid s \in P\}, \quad Pu^{-1} = \{s \in S \ \mid \ su \in P\}.$$

If $\{sw \ : \ w \in A^*\} = S$, for all $s \in S$, $\mathcal{A}$ is *transitive.* If $n = \mathrm{Card}(S)$, we will say that $\mathcal{A}$ is a n-state automaton. A *synchronizing* or *reset* word of $\mathcal{A}$ is any word $u \in A^*$ such that $\mathrm{Card}(Su) = 1$. The state q such that $Su = \{q\}$ is called *reset state.* A *synchronizing* automaton is an automaton that has a reset word. The following conjecture has been raised in [7].

Černý Conjecture. *Each synchronizing n-state automaton has a reset word of length not larger than $(n-1)^2$.*

We recall that a *formal power series* with rational coefficients and non-commuting variables in A is a mapping of the free monoid A^* into $\mathbb{Q}$. A series $\mathcal{S} : A^* \to \mathbb{Q}$ is *rational* if there exists a triple (α, μ, β) where

- $\alpha \in \mathbb{Q}^{1\times n}$, $\beta \in \mathbb{Q}^{n\times 1}$ are a horizontal and a vertical vector respectively,
- $\mu : A^* \to \mathbb{Q}^{n\times n}$ is a morphism of the free monoid A^* in the multiplicative monoid $\mathbb{Q}^{n\times n}$ of matrices with coefficients in $\mathbb{Q}$,
- for every $u \in A^*$, $\mathcal{S}(u) = \alpha\mu(u)\beta$.

The triple (α, μ, β) is called *a representation* of $\mathcal{S}$ and the integer n is called its *dimension*. With a minor abuse of language, if no ambiguity arises, the number n will be also called the dimension of $\mathcal{S}$. Let $\mathcal{A} = (S, A, \delta)$ be any n-state automaton. One can associate with $\mathcal{A}$ a morphism

$$\varphi_{\mathcal{A}} : A^* \to \mathbb{Q}^{S\times S},$$

of the free monoid A^* in the multiplicative monoid $\mathbb{Q}^{S\times S}$ of matrices over the set of rational numbers, defined as: for any $u \in A^*$ and for any $s, t \in S$,

$$\varphi_{\mathcal{A}}(u)_{st} = \begin{cases} 1 & \text{if } t = su \\ 0 & \text{otherwise.} \end{cases}$$

Let R and K be subsets of S and consider the rational series $\mathcal{S}$ with linear representation $(\alpha, \varphi_{\mathcal{A}}, \beta)$, where, for every $s \in S$,

$$\alpha_s = \begin{cases} 1 & \text{if } s \in R, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta_s = \begin{cases} 1 & \text{if } s \in K, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that, for any $u \in A^*$, one has

$$\mathcal{S}(u) = \mathrm{Card}(Ku^{-1} \cap R). \tag{2}$$

The following well-known result (see [5,10]) extends to rational series a fundamental theorem by Moore and Conway on automata equivalence.

Theorem 1. *Let $\mathcal{S}_1$, $\mathcal{S}_2 : A^* \to \mathbb{Q}$ be two rational series with coefficients in $\mathbb{Q}$ of dimension n_1 and n_2 respectively. If, for every $u \in A^*$ such that $|u| \leq n_1+n_2-1$, $\mathcal{S}_1(u) = \mathcal{S}_2(u)$, the series $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal.*

The following result is a consequence of Theorem 1.

Lemma 1. *Let $\mathcal{A} = (S, A, \delta)$ be a synchronizing n-state automaton. Assume that R and K are subsets of S, and t is an integer such that $0 < t < \mathrm{Card}(R)$. Then there exists a word v such that*

$$|v| \leq n, \quad \mathrm{Card}(Kv^{-1} \cap R) \neq t\,.$$

Proof. Consider the series $\mathcal{S}_1$, $\mathcal{S}_2$ defined respectively by

$$\mathcal{S}_1(v) = \mathrm{Card}(Kv^{-1} \cap R), \quad \mathcal{S}_2(v) = t, \quad v \in A^*\,.$$

In view of (2), $\mathcal{S}_1$ is a rational series of dimension n. Moreover, $\mathcal{S}_2$ is a rational series of dimension 1. We have to prove that $\mathcal{S}_1(v) \neq \mathcal{S}_2(v)$ for some $v \in A^*$ such that $|v| \leq n$. In view of Theorem 1, it is sufficient to show that $\mathcal{S}_1 \neq \mathcal{S}_2$. Let u be a reset word of the automaton $\mathcal{A}$. Then $Ku^{-1} = S$ or $Ku^{-1} = \emptyset$, according to whether the corresponding reset state belongs to K or not. One derives, respectively, $\mathcal{S}_1(u) = \mathrm{Card}(R)$ or $\mathcal{S}_1(u) = 0$ and, in both cases, $\mathcal{S}_1(u) \neq t$. Thus, $\mathcal{S}_1 \neq \mathcal{S}_2$, and the statement follows. □

3 Locally Strongly Transitive Automata

In this section, we study the notion of local strong transitivity. We begin by introducing the following definition.

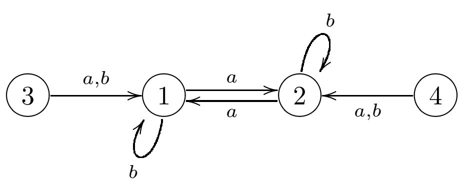
Definition 1. *Let $\mathcal{A} = (S, A, \delta)$ be an automaton. A set of k words $W = \{w_0, \ldots, w_{k-1}\}$ is called* independent *if there exist k distinct states $q_0, \ldots, q_{k-1}$ of $\mathcal{A}$ such that, for all $s \in S$,*

$$\{sw_0, \ldots, sw_{k-1}\} = \{q_0, \ldots, q_{k-1}\}.$$

The set $R = \{q_0, \ldots, q_{k-1}\}$ will be called the range *of W.*

An automaton is called *locally strongly transitive* if it has an independent set of words. The following example shows that local strong transitivity does not imply transitivity.

Example 1. Consider the 4-state automaton $\mathcal{A}$ over the alphabet $A = \{a, b\}$ defined by the following graph:



The automaton $\mathcal{A}$ is not transitive. On the other hand, one can easily check that the set $\{a, a^2\}$ is an independent set of $\mathcal{A}$ with range $R = \{1, 2\}$.

The following useful property easily follows from Definition 1.

Lemma 2. *Let $\mathcal{A}$ be an automaton and let W be an independent set of $\mathcal{A}$ with range R. Then, for every $u \in A^*$, the set uW is an independent set of $\mathcal{A}$ with range R.*

Proposition 1. *Let $\mathcal{A} = (S, A, \delta)$ be a n-state automaton and consider an independent set $W = \{w_0, \ldots, w_{k-1}\}$ of $\mathcal{A}$ with range R. Then, for every subset P of R, either*

$$\mathrm{Card}(Pw_i^{-1} \cap R) = \mathrm{Card}(P), \quad \textit{for all } i = 0, \ldots, k-1$$

or there exists j, $0 \leq j \leq k-1$, *such that*

$$\mathrm{Card}(Pw_j^{-1} \cap R) > \mathrm{Card}(P).$$

Proof. Because of Definition 1, for every $s \in S$ and $r \in R$, there exists exactly one word $w \in W$ such that $s \in \{r\}w^{-1}$. This implies that the sets $\{r\}w_i^{-1}$, $0 \leq i \leq k-1$, give a partition of S. Hence, for any $r \in R$, one has:

$$k \;=\; \mathrm{Card}(R) \;=\; \sum_{i=0}^{k-1} \mathrm{Card}(R \,\cap\, \{r\}w_i^{-1}). \tag{3}$$

Let P be a a subset of R. If P is empty then the statement is trivially true. If $P = \{p_1, \ldots, p_m\}$ is a set of $m \geq 1$ states, then one has:

$$\sum_{i=0}^{k-1} \mathrm{Card}(R \,\cap\, Pw_i^{-1}) \;=\; \sum_{i=0}^{k-1} \mathrm{Card}\left(\bigcup_{j=1}^{m} R \,\cap\, \{p_j\}w_i^{-1} \right).$$

Since $\mathcal{A}$ is deterministic, for any pair p_i, p_j of distinct states of P and for every $u \in A^*$, one has:

$$\{p_i\}u^{-1} \,\cap\, \{p_j\}u^{-1} \;=\; \emptyset,$$

so that the previous sum can be rewritten as:

$$\sum_{i=0}^{k-1} \sum_{j=1}^{m} \mathrm{Card}(R \,\cap\, \{p_j\}w_i^{-1}).$$

The latter equation together with (3) implies that

$$\sum_{i=0}^{k-1} \mathrm{Card}(Pw_i^{-1} \,\cap\, R) \;=\; k\,\mathrm{Card}(P).$$

The statement follows from the equation above. □

Corollary 1. *Let* $\mathcal{A} = (S, A, \delta)$ *be a synchronizing* n*-state automaton and let* W *be an independent set of* $\mathcal{A}$ *with range* R. *Let* P *be a proper and non empty subset of* R. *Then there exists a word* $w \in A^*$ *such that*

$$|w| \leq n + L_W, \quad \mathrm{Card}(Pw^{-1} \,\cap\, R) > \mathrm{Card}(P).$$

Proof. Let $W = \{w_0, \ldots, w_{k-1}\}$. We first prove that there exists a word $v \in A^*$ with $|v| \leq n$ such that

$$\mathrm{Card}(P(vw_0)^{-1} \,\cap\, R) \neq \mathrm{Card}(P). \tag{4}$$

If $\mathrm{Card}(Pw_0^{-1} \,\cap\, R) \neq \mathrm{Card}(P)$, take $v = \epsilon$. Now suppose that

$$\mathrm{Card}(Pw_0^{-1} \,\cap\, R) = \mathrm{Card}(P).$$

Since P is a proper and non-empty subset of R and since $\mathcal{A}$ is synchronizing, by applying Lemma 1 with $t = \text{Card}(P)$ and $K = Pw_0^{-1}$, one has that there exists a word $v \in A^*$ such that $|v| \leq n$ and $\text{Card}(P(vw_0)^{-1} \cap R) \neq \text{Card}(P)$. Thus take v that satisfies (4) and let $W' = \{vw_0, \ldots, vw_{k-1}\}$. By Lemma 2, W' is an independent set of $\mathcal{A}$ with range R and $L_{W'} \leq n + L_W$. Therefore, by Proposition 1, taking into account (4),

$$\text{Card}(P(vw)^{-1} \cap R) > \text{Card}(P),$$

for some $w \in W'$. The claim is thus proved. □

As a consequence of Corollary 1, the following theorem holds.

Theorem 2. *Let $\mathcal{A} = (S, A, \delta)$ be a synchronizing n-state automaton and let W be an independent set of $\mathcal{A}$ with range R. Then there exists a reset word for $\mathcal{A}$ of length not larger than*

$$(k-1)(n + L_W) + \ell_W,$$

where $k = \text{Card}(W)$.

Proof. Let $P = \{r\}$ where r is a given state of R. Starting from the set P, by iterated application of Corollary 1, one can find a word v such that

$$|v| \leq (k-1)(n + L_W), \quad Pv^{-1} \cap R = R.$$

Thus we have $Rv = P$. Let u be a word of W of minimal length. Because of Definition 1, we have $Su \subseteq R$ so that $Suv \subseteq Rv = P$. Therefore the word uv is the required word and the statement is proved. □

In the sequel of this section, we will present some results that can be obtained as straightforward corollaries of Theorem 2. We recall that an n-state automaton $\mathcal{A} = (S, A, \delta)$ is *strongly transitive* if there exists an independent set W of n words. Thus, in this case, S is the range of W. The notion of strong transitivity was introduced and studied in [8] where the following result has been proved.

Theorem 3. *Let $\mathcal{A} = (S, A, \delta)$ be a synchronizing strongly transitive n-state automaton with an independent set W. Then there exists a reset word w for $\mathcal{A}$ of length not larger than*

$$1 + (n-2)(n + L_W - 1). \tag{5}$$

Remark 1. Since a strongly transitive automaton $\mathcal{A}$ is also locally strongly transitive, by applying Theorem 2 to $\mathcal{A}$, we obtain the following upper bound on the length of w:

$$(n-1)(n + L_W) + \ell_W,$$

which is larger than that of (5). This gap is the consequence of the following three facts that depend upon the condition $S = R$. The quantity ℓ_W can be obviously deleted in the equation above. The reset word w of $\mathcal{A}$ is factorized as

$w = w_j w_{j-1} \cdots w_0$, where $j \leq n-2$ and, for every $i = 0, \ldots, j$, w_i is obtained by applying Corollary 1. The proof of Corollary 1 is based upon Lemma 1. Under the assumption $S = R$, one can see that the upper bound for the word defined in Lemma 1 is $n-1$ so that the corresponding upper bound of Corollary 1 can be lowered to $n + L_W - 1$. Finally, since $\mathcal{A}$ is synchronizing, the word w_0 can be chosen as a letter.

Let us now define a remarkable class of locally strongly transitive automata.

Definition 2. *Let $\mathcal{A} = (S, A, \delta)$ be an n-state automaton and let $u \in A^*$. Then $\mathcal{A}$ is called* u-connected *if there exists a state $q \in S$ such that, for every $s \in S$, there exists $k > 0$, such that $su^k = q$.*

Let $\mathcal{A}$ be a u-connected n-state automaton. Define the set R as:

$$R = \{q, qu, \ldots, qu^{k-1}\},$$

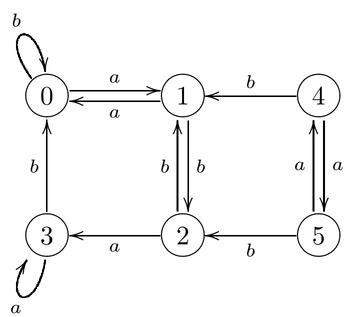
where k is the least positive integer such that $qu^k = q$. Let i be the least integer such that, for every $s \in S$, $su^i \in R$. Finally define the set W as:

$$W = \{u^i, u^{i+1}, \ldots, u^{i+k-1}\}.$$

One easily verifies that W is an independent set of $\mathcal{A}$ with range R and, moreover, $\ell_W = i|u|$, $L_W = (i + k - 1)|u|$.

Remark 2. We notice that, by definition of i, there exists a state s such that $s, su, \ldots, su^{i-1} \notin R$ and $su^i \in R$. This implies that the states $s, su, \ldots, su^{i-1}$ are pairwise distinct. Since moreover $\mathrm{Card}(R) = k$, one derives $i + k \leq n$, so that $L_W \leq (n-1)|u|$.

Example 2. Consider the following 6-state automaton $\mathcal{A}$:



Let $u = ab$ and $q = 0$. One can check that, for all $s \in S$, $su^k = q$, with $k \leq 2$. Since $qu^2 = q$, one has $R = \{0, 2\}$ and one can check that $i = 2$. Thus $W = \{u^2, u^3\}$ is an independent set of $\mathcal{A}$ with range R.

By Remark 2 and by applying Theorem 2, we have

Corollary 2. *Let $\mathcal{A} = (S, A, \delta)$ be a synchronizing n-state automaton. Suppose that $\mathcal{A}$ is u-connected with $u \in A^*$. Let i and k be integers defined as above. Then $\mathcal{A}$ has a reset word of length not larger than*

$$(k-1)(n+(i+k-1)|u|)+i|u|.$$

We say that an automaton $\mathcal{A}$ is *letter-connected* if it is a-connected for some letter $a \in A$. This notion is a natural generalization of that of circular automaton. Indeed, one easily verifies that $\mathcal{A}$ is a-connected if and only if it has a unique cycle labelled by a power of a. By Corollary 2, taking into account that $i+k \leq n$ one derives

Corollary 3. *A synchronizing a-connected n-state automaton, $a \in A$, has a reset word of length not larger than*

$$(2k-1)(n-1),$$

where k is the length of the unique cycle of $\mathcal{A}$ labelled by a power of a.

We remark that the tighter upper bound

$$i+(k-1)(n+k+i-2)$$

for the length of the shortest reset word of a synchronizing a-connected n-state automaton was established in [4].

4 On the Hybrid Černý–Road Coloring Problem

In the sequel, by using the word graph, we will term a finite, directed multigraph with all vertices of outdegree k. A *multiple edge* of a graph is a pair of edges with the same source and the same target. A graph is *aperiodic* if the greatest common divisor of the lengths of all cycles of the graph is 1. A *coloring* of a graph G is a labelling of its edges by letters of a k-letter alphabet that turns G into a complete and deterministic automaton. A coloring of G is *synchronizing* if it transforms G into a synchronizing automaton. The *Road coloring problem* asks the existence of a synchronizing coloring for every aperiodic and strongly connected graph. This problem was formulated in the context of Symbolic Dynamics by Adler, Goodwyn and Weiss and it is explicitly stated in [1]. In 2007, Trahtman has positively solved this problem in [19]. Recently Volkov has raised the following problem [20].

Hybrid Černý–Road coloring problem. *Let G be an aperiodic and strongly connected graph. What is the minimum length of a reset word for a synchronizing coloring of G?*

Now we present a partial answer to the previous problem. For this purpose, we recall the following theorem by O' Brien [14].

Theorem 4. *Let G be an aperiodic and strongly connected graph of n vertices, without multiple edges. Suppose G has a simple cycle C of prime length $p < n$. Then there exists a synchronizing coloring of G such that C is the unique cycle labelled by a power of a given letter a.*

Corollary 4. *Let G be an aperiodic and strongly connected graph of n vertices, without multiple edges. Suppose G has a simple cycle of prime length $p < n$. Then there exists a synchronizing coloring of G with a reset word of length $\leq (2p-1)(n-1)$.*

Proof. The statement follows by applying Theorem 4 and Corollary 3 to G. □

If G contains multiple edges, Theorem 4 cannot be applied so that neither Corollary 4 holds. However, in this case, if G has a cycle of length 2, a result akin to Corollary 4 can be proven. In order to prove this extension, by following [14], we recall some notions and results.

For the sake of simplicity, we assume that all vertices of the graph $G = (S, E)$ have outdegree 2. However, we notice that Proposition 2 stated below remains true also when the common outdegree of the vertices of G is larger.

We suppose that G has a cycle C of length 2 and call s_0, s_1 the vertices of C. A *C-tree* T of G is a subgraph of G that satisfies the following properties:

- the set of vertices of T is S and, for every vertex s of G, exactly one edge outgoing from s is an edge of T;
- C is a subgraph of T;
- for every vertex s of G, there exists a path from s to s_1.

Let T be a given C-tree of G. Define a map

$$C_T : S \longrightarrow \{0, 1\}$$

as follows: for every $s \in S$, $C_T(s) = 1$ (resp., $C_T(s) = 0$) if the length of the shortest path in T from s to s_1 is even (resp., odd).

Given a vertex $s \in S$, we say that s is *aperiodic* (with respect to T) if there exists an edge (s, t) of G such that $C_T(t) = C_T(s)$; otherwise the vertex is called *periodic* (with respect to T). One can easily prove that, since G is an aperiodic graph, for every C-tree T of G, there exists an aperiodic vertex.

Let $A = \{a, b\}$ be a binary alphabet and define a coloring of the edges of T as follows: for every edge $e = (s, t)$ of T, label e by the letter a if $C_T(s) = 1$ and by the letter b otherwise. Finally extend, in the obvious way, the latter coloring to the remaining edges of G in order to transform G into an automaton $\mathcal{A}$. We remark that with such a coloring, if $C_T(x) = 1$, then $C_T(xa) = 0$ and, if $C_T(x) = 0$, then $C_T(xb) = 1$. Moreover, if x is aperiodic, then $C_T(xa) = 0$ and $C_T(xb) = 1$. The following lemma can be proved easily.

Lemma 3. *Let x, y be states of $\mathcal{A}$. The following properties hold:*

1. *If $C_T(x) = 1$, then, for every $m \geq \lceil n/2 \rceil - 1$, $x(ab)^m = s_1$;*
2. *If $C_T(x) = 0$, then, for every $m \geq \lceil n/2 \rceil - 1$, $x(ba)^m = s_0$;*

3. *Either* $C_T(x(ab))^m = 0$ *for all* $m \geq 0$ *or* $x(ab)^{n-1} = s_1$.
4. *If* x *is aperiodic, then there exists* $\sigma \in \{a, b\}$ *such that* $C_T(x\sigma) = C_T(y\sigma)$.
5. *There is a word* u *such that* $xu = yu$, *with* $|u| \leq 2n - 2$.

Proof. Conditions 1 and 2 immediately follow from the definition of the coloring of G.

Let us prove Condition 3. Suppose that there exists $m \geq 0$ with $C_T(x(ab)^m) = 1$. Then, by Condition 1, $x(ab)^k = s_1$ for any $k \geq m + \lceil n/2 \rceil - 1$. Let k be the least non-negative integer such that $x(ab)^k = s_1$. The minimality of k implies that the states $x(ab)^i$, $0 \leq i \leq k$ are pairwise distinct. Consequently, $k + 1 \leq n$, so that $x(ab)^{n-1} = s_1(ab)^{n-k-1} = s_1$.

Now let us prove Condition 4. Since x is aperiodic, one has $C_T(xa) = 0$ and $C_T(xb) = 1$. Moreover, either $C_T(ya) = 0$ or $C_T(yb) = 1$, according to the value of $C_T(y)$. The conclusion follows. Let us prove Condition 5. We can find a word v such that $|v| \leq n - 2$ and at least one of the states xv, yv is aperiodic. By Condition 4, $C_T(xv\sigma) = C_T(yv\sigma)$ for some $\sigma = a, b$. Set

$$v' = \begin{cases} (ab)^{\lceil n/2 \rceil - 1} & \text{if } C_T(xv\sigma) = 1\,, \\ (ba)^{\lceil n/2 \rceil - 1} & \text{if } C_T(xv\sigma) = 0\,. \end{cases}$$

According to Conditions 1, 2 one has $xv\sigma v' = yv\sigma v' \in C$ so that the statement is verified with $u = v\sigma v'$. □

Proposition 2. *Let* G *be an aperiodic and strongly connected graph of* n *vertices with outdegree* 2. *Assume that* G *has a cycle of length two. Then there exists a synchronizing coloring of* G *with a reset word of length* $\leq 5(n-1)$.

Proof. Let C be the cycle of length two of G and let $\mathcal{A}$ be the automaton obtained from G by considering the coloring defined above. By Condition 3, $S(ab)^{n-1} \subseteq \{s_1\} \cup S_0$, where $S_0 = \{x \in S \mid C_T(x) = 0\}$. By Condition 2, $S_0(ba)^{\lceil n/2 \rceil - 1} = \{s_0\}$. Thus, the set $R = S(ab)^{n-1}(ba)^{\lceil n/2 \rceil - 1}$ contains at most 2 states. By Condition 5, there is a word u such that $|u| \leq 2n - 2$ and Ru is reduced to a singleton. We conclude that the word $w = (ab)^{n-1}(ba)^{\lceil n/2 \rceil - 1}u$ is a reset word. Moreover, $|w| \leq 5(n-1)$. □

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