Total Variation Processing of Images with Poisson Statistics

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Abstract. This paper deals with denoising of density images with bad Poisson statistics (low count rates), where the reconstruction of the major structures seems the only reasonable task. Obtaining the structures with sharp edges can also be a prerequisite for further processing, e.g. segmentation of objects.

A variety of approaches exists in the case of Gaussian noise, but only a few in the Poisson case. We propose some total variation (TV) based regularization techniques adapted to the case of Poisson data, which we derive from approximations of logarithmic a-posteriori probabilities. In order to guarantee sharp edges we avoid the smoothing of the total variation and use a dual approach for the numerical solution. We illustrate and test the feasibility of our approaches for data in positron emission tomography, namely reconstructions of cardiac structures with ^{18}F-FDG and $H_2\,^{15}O$ tracers, respectively.

Keywords: Denoising, Poisson noise, Total variation, Regularization techniques, Positron emission tomography, Segmentation.

1 Introduction

In this paper we shall discuss some approaches to denoising density images with Poisson statistics, with particular focus on cartoon reconstruction. The latter seems particularly reasonable for low count rates, where the effective SNR is too low to compute further details in the image. Moreover, appropriate cartoons are important for subsequent tasks such as segmentation of objects in the images or further quantitative analysis. For this sake we shall employ variational methods based on penalization by total variation (or related penalization functionals of ℓ^1 or L^1-type), which has become a standard approach for such tasks in the frequently investigated case of additive Gaussian noise.

Variational methods in the case of Gaussian noise [1], [2], can be written as minimizing an energy functional of the form

$$\frac{1}{2}\int_{\Omega}(u-f)^2\;d\mu\;+\;\alpha\,R(u)\quad\rightarrow\quad\min_{u}, \tag{1}$$

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in order to obtain a denoised version u of a given image f, where Ω is the image domain and α is a positive regularization parameter. The first, so-called data fidelity term, penalizes the deviation from the noisy image f and can be derived from the log-likelihood for the noise model (cf. [1], [2]). R is an energy functional that inserts a priori information about the favoured type of smoothness of solutions. The minimization (1) results in suppression of noise in u if R is smoothing, while u is fitted to f. The choice of the regularization term R is important for structure of solutions. Often functionals $R(u) = \int_\Omega |\nabla u|^p$ for $p > 1$ are used, where ∇ denotes the gradient and $|.|$ the Euclidean norm. The simplest choice $p = 2$ results in a scheme equivalent to a linear filter, which can be implemented very efficiently via a fast Fourier transform. However, such regularization approaches always lead to blurring of images, in particular they cannot yield results with sharp edges.

In order to preserve edges and obtain appropriate structures, we use an approach based on total variation (TV) as regularization functional. TV regularization was derived as a denoising technique in [3] and generalized to various other imaging tasks subsequently. The exact definition of TV [4] is

$$R(u) \;=\; |u|_{BV} \;:=\; \sup_{g \in C_0^\infty(\Omega,\mathbb{R}^d),\ ||g||_\infty \leq 1} \int_\Omega u \operatorname{div} g \;, \tag{2}$$

which is formally (true if u is sufficiently regular) $|u|_{BV} = \int_\Omega |\nabla u|$. The space of integrable functions with bounded (total) variation is denoted by $BV(\Omega)$ (cf. [4], [5]). The variational problem (1) with TV as regularization functional is the Rudin-Osher-Fatemi (ROF) model. The motivation for using TV is the effective suppression of noise and the realization of almost homogeneous regions with sharp edges. These features are particularly attractive for a posterior segmentation and quantitative evaluations on structures.

Images with Poisson statistics arise in various applications, e.g. in positron emission tomography (PET), in optical microscopy or in CCD cameras. In most of these cases, the raw data (positron or photon counts) are related to the images via some integral operator, which first needs to be (approximately) inverted in order to obtain an image. Recently, variational methods derived from Bayesian models have been combined with the reconstruction process [6],

$$\int_\Sigma (Ku - g \log Ku)\; d\mu \;+\; \alpha\, |u|_{BV} \quad \rightarrow \quad \min_{u \in BV(\Omega)} \;, \qquad u \geq 0 \;, \tag{3}$$

where g are the Poisson distributed raw data, Σ is the data domain and K is a linear operator that transforms the spatial distribution of desired object into sampled signals on the detectors. If $K = Id$, (3) becomes a denoising model, where g is the known noisy image. In the absence of regularization ($\alpha = 0$) in (3) the EM algorithm [7], [8], has become a standard reconstruction scheme in problems with incomplete data corrupted by Poisson noise, which is however difficult to be generalized to the regularized case. Robust iterative methods for minimizing (3) have been derived by the authors recently (cf. [9], [10], [11]), but in any case a minimization of (3) requires significant computational effort.

In order to obtain similar results faster, we investigate a natural alternative scheme, namely the postprocessing of reconstructions with EM methods based on a variational denoising model. This posterior denoising step needs to take into account that the reconstructed image still behaves like an image with Poisson noise and thus particular schemes need to be constructed.

In [12], a TV based variational model to denoise an image corrupted by Poisson noise is proposed,

$$\int_\Omega (u - f \log u)\, d\mu \;+\; \alpha\, |u|_{BV} \quad \to \quad \min_{u \in BV(\Omega)}\,, \qquad u \geq 0\,. \tag{4}$$

A particular complication of (4) compared to (1) is the strong nonlinearity in the data fidelity term and resulting issues in the computation of minimizers. Due to TV the variational problem (4) is non differentiable and the authors in [12] use an approximation of TV by differentiable functionals $\int_\Omega \sqrt{|\nabla u|^2 + \varepsilon}$ for any $\varepsilon > 0$. This approach leads to blurring of edges and due to an additional parameter dependence on ε such algorithms are even less robust. Here, we propose a robust algorithm for (4) without approximation of TV, i.e. we use (2) respectively a dual version. This allows to realize cartoon images with sharp edges. Moreover, we investigate a quadratic approximation of the fidelity term, which yields very similar results as (4) with a more straight-forward numerical solution.

The challenges of this work are that the Poisson based data-fidelity term in (4) and (3) is highly nonlinear and that the functionals to be minimized are non-differentiable. We propose robust minimization schemes using dual approaches for an appropriate treatment of the total variation. To illustrate the behavior of the proposed methods, we evaluate cardiac $H_2\,{}^{15}O$ and ${}^{18}F$-FDG PET measurements with low SNR.

2 Methods

2.1 EM and EM-TV Reconstruction

In this section we briefly discuss reconstruction methods based on Expectation-Maximization (EM, cf. [8]) and regularized EM methods. We consider the variational problem (3). A standard reconstruction scheme in the absence of regularization ($\alpha = 0$) is the EM method introduced by Shepp and Vardi [7],

$$u_{k+1} = u_k\, \frac{K^*}{K^* 1} \left(\frac{g}{K u_k} \right), \tag{5}$$

an approach which is reasonably easy to implement, K^* is the adjoint operator of K [2]. However, suitable reconstructions can only be obtained for good statistics, and hence either additional postprocessing by variational methods or additional regularization in the functional (cf. [13], [14]) is needed. For the latter we proposed in [9] and [10] a semi-implicit iteration scheme minimizing (3). This scheme can be realized as a nested two step iteration

$$\left\{\begin{array}{lcll} u_{k+\frac{1}{2}} & = & u_k \, \dfrac{K^*}{K^* 1} \left(\dfrac{g}{K u_k} \right) & \text{(EM step)} \\ u_{k+1} & = & u_{k+\frac{1}{2}} - \tilde{\alpha} \, u_k \, p_{k+1} & \text{(TV step)} \end{array}\right\} \tag{6}$$

with $p_{k+1} \in \partial |u_{k+1}|_{BV}$ and $\tilde{\alpha} := \frac{\alpha}{K^* 1}$, where ∂ denotes the subdifferential [2] and generalizes the notion of derivative. The first step in (6) is a single step of the EM algorithm (5). The more involved second step for TV correction in (6) can be realized by solving

$$u_{k+1} \;=\; \underset{u \in BV(\Omega)}{\arg\min} \left\{ \frac{1}{2} \int_\Omega \frac{(u - u_{k+\frac{1}{2}})^2}{u_k} \;+\; \tilde{\alpha} \, |u|_{BV} \right\} . \tag{7}$$

Inspecting the first order optimality condition confirms the equivalence of this minimization with the TV correction step in (6). Problem (7) is just a modified version of the ROF model, with weight $\frac{1}{u_k}$ in the fidelity term. This analogy creates the opportunity to carry over efficient numerical schemes known for the ROF model and actually to realize cartoon reconstructions with sharp edges. For a detailed analytical examination of EM-TV we refer to [11]. Since the coupled model needs several iteration steps and thus several solutions of (7), it becomes computationally rather involved. Therefore, we study a simple postprocessing strategy based on first computing a reconstruction of visually bad quality via a simple EM algorithm and postprocessing with total variation, which we expect to recover the major structures in the image at least for a certain range of statistics.

2.2 Denoising Images with Poisson Statistics

The straight-forward approach to denoising is based on maximizing the logarithmic a-posteriori probability, i.e. solving (4) (cf. [12]), whose optimality condition is given by

$$u \, (1 \;-\; \frac{f}{u} \;+\; \alpha \, p) \;=\; 0 \;, \qquad p \in \partial |u|_{BV} \; . \tag{8}$$

Since the reconstruction model (3) coincides in the case of K being the identity operator with (4), we can use the iteration scheme from the previous section, which simply results in (note that $u_{k+\frac{1}{2}} \;=\; f$ in this case)

$$u_{k+1} \;=\; f \;-\; \tilde{\alpha} \, u_k \, p_{k+1} \; . \tag{9}$$

As noticed above, we can realize this iteration step by solving the modified version of the ROF model (7). Note that (9) is a semi-implicit iteration scheme with respect to the optimality condition (8) and thus actually computes a denoised image in the Poisson case.

The iteration scheme (9) solves the denoising problem (4) by a sequence of modified ROF variational models. In this way one obtains an MAP estimate, but again at the price of high computational effort. Together with the reconstruction via the EM method, the effort is comparable to the incorporated EM-TV scheme.

Hence we introduce a further approximation of the denoising problem, which is based on a second order Taylor approximation of the data fidelity term in (4),

$$u \;=\; \underset{u \in BV(\Omega)}{\arg\min} \left\{ \frac{1}{2} \int_\Omega \frac{(u-f)^2}{f} \;+\; \tilde{\alpha}\, |u|_{BV} \right\} \;. \tag{10}$$

In this case we can compute the postprocessing by solving a single modified ROF model.

2.3 Computational Approach

Finally we briefly discuss the numerical solution of the minimization problem

$$u \;=\; \underset{u \in BV(\Omega)}{\arg\min} \left\{ \frac{1}{2} \int_\Omega \frac{(u-v)^2}{w} \;+\; \tilde{\alpha}\, |u|_{BV} \right\} \;, \tag{11}$$

which is the most general form of all schemes above with appropriate setting of v and the weight w. Most computational schemes for the ROF model can be adapted to this weighted modification, here we use a dual approach that does not need any smoothing of the total variation. Our approach is analogous to the one in [15], using a characterization of subgradients of total variations as divergences of vector fields with supremum norm less or equal one. We thus compute the primal variable from the optimality condition with $\tilde{g}$ to be determined as a minimizer of a dual problem

$$u \;=\; v \;-\; \tilde{\alpha}\, w \,\mathrm{div}\tilde{g} \;, \qquad \tilde{g} \;=\; \underset{g,\ \|g\|_\infty \leq 1}{\arg\min} \int_\Omega (\tilde{\alpha}\, w \,\mathrm{div} g - v)^2 \;. \tag{12}$$

This problem can be solved with projected gradient-type algorithms, we use

$$g^{n+1} \;=\; \frac{g^n + \tau\ \nabla(\tilde{\alpha}\, w \,\mathrm{div} g^n - v)}{1 + \tau\ |\nabla(\tilde{\alpha}\, w \,\mathrm{div} g^n - v)|} \;, \qquad 0 \;<\; \tau \;<\; \frac{1}{4\, \tilde{\alpha}\, w} \;, \tag{13}$$

with the damping parameter τ to ensure convergence of the algorithm.

3 Results

We illustrate our techniques at a simple synthetic object, see Fig. 1, and by evaluation of cardiac $H_2{}^{15}O$ and ^{18}F-FDG measurements obtained with positron emission tomography (PET) [16], [17]. In this modality, a specific radioactive tracer, binding to the molecules to be studied, is injected into blood circulation. $H_2{}^{15}O$ is used for the quantification of myocardial blood flow [18]. This quantification needs a segmentation of myocardial tissue, left and right ventricle [18], [19], which is extremely difficult to realize due to very low SNR of $H_2{}^{15}O$ data.

In order to obtain the tracer intensity in the left ventricle we take a fixed 2D layer in a suitable time frame, see Fig. 2. To illustrate the SNR issue we present reconstructions with the EM algorithm (A). As expected, the results

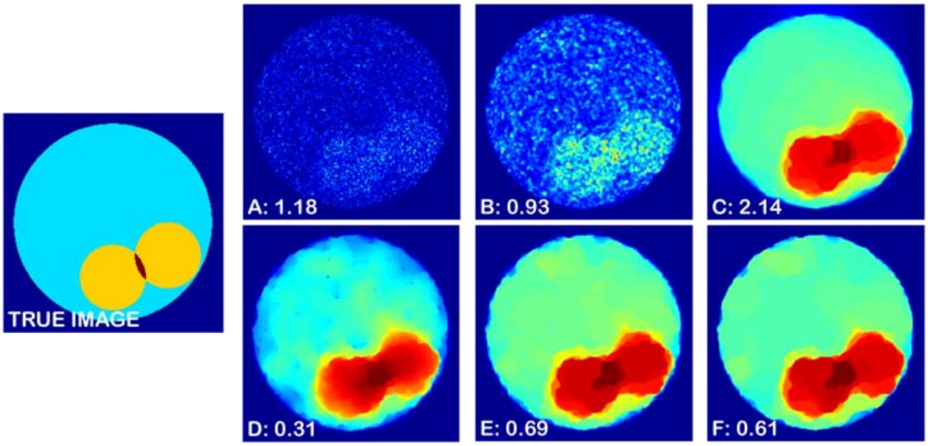

Fig. 1. Synthetic object: results of different reconstruction methods and Kullback-Leibler distances to true image. **A:** EM reconstruction, 18 its. **B:** EM, 18 its, with Gaussian smoothing any 5th step. **C:** A with standard ROF smoothing, (1) with (2). **D:** A with weighted ROF smoothing (10). **E:** A with iterative weighted ROF smoothing (9), 30 its. **F:** Nested EM-TV algorithm (6), 30 its.

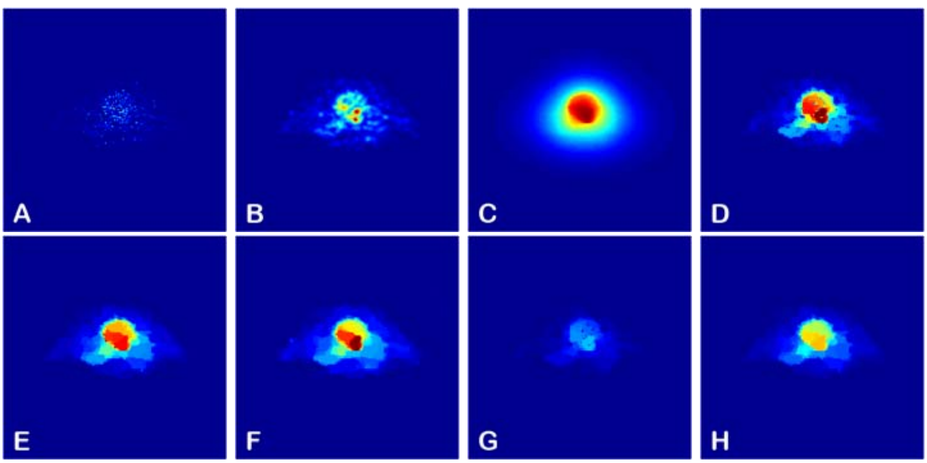

Fig. 2. Cardiac $H_2{}^{15}O$ PET measurements: tracer intensity results of different reconstruction methods in the left ventricle. **A-F:** as in Fig. 1, but all with 20 its. **G, H:** D, E scaled to maximum intensity of F.

suffer from unsatisfactory quality and are impossible to interpret. We hence take EM reconstrutions with Gaussian smoothing (B) as a reference. The next results (C - E) show different approaches of TV smoothing with the (weighted) ROF model. The result C demonstrates the approach with the standard ROF model, the result D is generated with the weighted ROF model (10) and E with

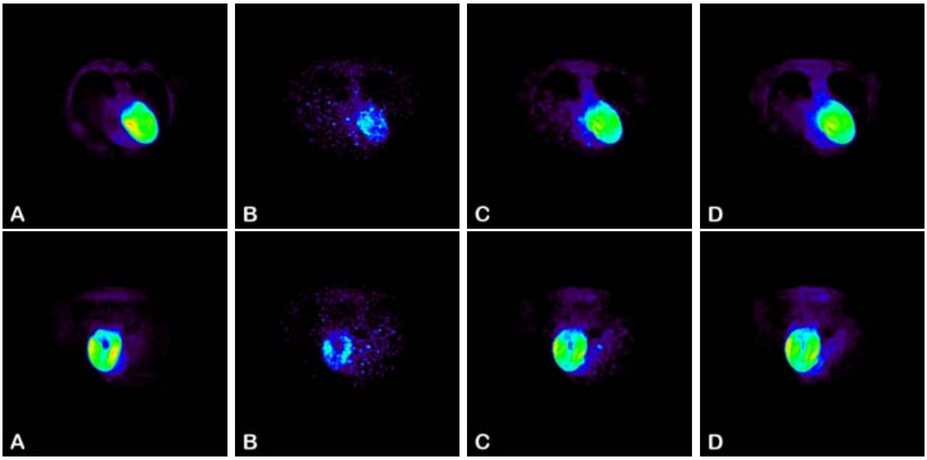

Fig. 3. Cardiac ^{18}F-FDG 3D PET measurements: reconstruction results from two different viewing angles (upper and bottom row). **A:** EM recontruction, 20 its, with Gaussian smoothing any 10th step after 20 minutes data acquisition. **B:** As A but after 5 seconds data acquisition. **C:** B with weighted ROF smoothing (10) and $\tilde{\alpha} = 0.3$. **D:** As C but with $\tilde{\alpha} = 0.5$.

the MAP estimate (4). The approach with the nested EM-TV algorithm (6) is presented in F. The reconstructions G and H are the same results as in D and E appropriate, but the images are scaled to the maximum intensity of F, such that a comparison is possible also for quantitative values. One observes that the results without or with standard smoothing are unsatisfactory, while all total variation techniques yield reasonable reconstructions of the structures. Quantitative values are usually more realistic for the nested EM-TV methods.

In Figure 3, we provide 3D postprocessing results generated with total variation using cardiac ^{18}F-FDG measurements. This tracer is an important radiopharmaceutical and is used for measuring glucose metabolism, e.g in brain, in heart or in cancer. For the illustration of a 3D data set, we take the projections of two fixed viewing angles. The EM reconstruction after a data acquisition of 20 minutes is shown in A as a ground truth for very high count rates. To simulate low count rates, we take the measurements after the first 5 seconds only. The corresponding EM reconstruction is illustrated in B. The results C and D show a postprocessing of B with the weighted ROF model (10) for two different regularization parameters $\tilde{\alpha} = 0.3$ and $\tilde{\alpha} = 0.5$. One observes that the major structures are well reconstructed by this approach also for low count rates.

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